

# Minimax optimal rates for Mondrian trees and forests

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March 14, 2018

## Abstract

Introduced by [Breiman \(2001\)](#), Random Forests are widely used as classification and regression algorithms. While being initially designed as batch algorithms, several variants have been proposed to handle online learning. One particular instance of such forests is the *Mondrian Forest* ([Lakshminarayanan et al., 2014, 2016](#)), whose trees are built using the so-called Mondrian process, therefore allowing to easily update their construction in a streaming fashion. In this paper, we study Mondrian Forests in a batch setting and prove their consistency assuming a proper tuning of the lifetime sequence. A thorough theoretical study of Mondrian partitions allows us to derive an upper bound for the risk of Mondrian Forests, which turns out to be the minimax optimal rate for both Lipschitz and twice differentiable regression functions. These results are actually the first to state that some particular random forests achieve minimax rates *in arbitrary dimension*, paving the way to a refined theoretical analysis and thus a deeper understanding of these black box algorithms.

## 1 Introduction

Originally introduced by [Breiman \(2001\)](#), *Random Forests* (RF) are state-of-the-art classification and regression algorithms that proceed by averaging the forecasts of a number of randomized decision trees grown in parallel. Despite their widespread use and remarkable success in practical applications, the theoretical properties of such algorithms are still not fully understood (for an overview of theoretical results on random forests, see [Biau and Scornet, 2016](#)). As a result of the complexity of the procedure, which combines sampling steps and feature selection, Breiman’s original algorithm has proved difficult to analyze. Consequently, most theoretical studies focus on modified and stylized versions of Random Forests.

Among these methods, *Purely Random Forests* (PRF) ([Breiman, 2000](#); [Biau et al., 2008](#); [Biau, 2012](#); [Genuer, 2012](#); [Arlot and Genuer, 2014](#)) that grow the individual trees independently of the sample, are particularly amenable to theoretical analysis. The consistency of such estimates (as well as other idealized RF procedures) was first obtained by [Biau et al. \(2008\)](#), as a byproduct of the consistency of individual tree estimates. A recent line of research ([Scornet et al., 2015](#); [Wager and Walther, 2015](#); [Mentch and Hooker, 2016](#); [Wager and Athey, 2017](#)) has sought to obtain some theoretical guarantees for RF variants that more closely resembled the algorithm used in practice. It should be noted, however, that most of these theoretical guarantees come at the price of assumptions either on the data structure or on the Random Forest algorithm itself, being thus still far from explaining the excellent empirical performance of Random Forests.

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Another aspect of the theoretical study of random forests is to quantify the performance guarantees by analyzing the bias/variance of simplified versions of Random Forests, such as PRF models (Genuer, 2012; Arlot and Genuer, 2014). In particular, Genuer (2012) shows that some PRF variants achieve the minimax rate for the estimation of a Lipschitz regression functions in dimension one. The bias-variance analysis is extended in Arlot and Genuer (2014), showing that PRF can also achieve minimax rates for  $\mathcal{C}^2$  regression functions in dimension one. The aforementioned rates of convergence are much more precise than mere consistency, and offer insights on the proper tuning of the procedure. Surprisingly, optimal rates are only obtained in the one-dimensional case (where decision trees reduce to histograms); only suboptimal rates are reached in the higher dimensional setting, where trees exhibit a more intricate recursive structure.

From a more practical perspective, an important limitation of the most commonly used RF algorithms, such as Breiman’s Random Forests (Breiman, 2001) and the Extra-Trees algorithm (Geurts et al., 2006), is that they are typically trained in a batch manner, using the whole dataset, available at once, to build the trees. In order to enable their use in situations when large amounts of data have to be incorporated in a streaming fashion, several online variants of the decision trees and random forests algorithms have been proposed (Domingos and Hulten, 2000; Saffari et al., 2009; Taddy et al., 2011; Denil et al., 2013, 2014).

Of particular interest in this article is the *Mondrian Forest* algorithm, an efficient and accurate online random forest classifier introduced by Lakshminarayanan et al. (2014) (see also Lakshminarayanan et al., 2016). This algorithm is based on the Mondrian process (Roy and Teh, 2009; Roy, 2011; Orbanz and Roy, 2015), a natural probability distribution on the set of recursive partitions of the unit cube  $[0, 1]^d$ . An appealing property of Mondrian processes is that they can be updated in an online fashion: in Lakshminarayanan et al. (2014), the use of the *conditional Mondrian* process enables to design an online algorithm which matches its batch counterpart: training the algorithm one data point at a time leads to the same randomized estimator than if trained on the whole dataset at once. The algorithm proposed in Lakshminarayanan et al. (2014) depends on a lifetime parameter  $\lambda$  that guides the complexity of the trees by stopping the tree building process. However, there are no theoretical insights to tune this parameter, which appears to be of great importance in Mondrian Trees and Forests.

We study in this paper the Mondrian Forests in a batch setting and provide theoretical guidance to tune the lifetime parameter. It turns out that allowing the lifetime parameter to depend on  $n$  at a proper rate results in the consistency of our proposed algorithm. Based on the detailed analysis of Mondrian partitions, we are able to derive the convergence rate of Mondrian Forests, which turns out to be the minimax rate for Lipschitz and twice differentiable functions *in arbitrary dimension*. To the best of our knowledge, such results have only been proved for very specific purely random forests, where the covariate space is of dimension one (Arlot and Genuer, 2014). Our analysis also sheds light on the benefits of Mondrian Forests compared to a single Mondrian Tree.

**Agenda.** This paper is organized as follows. In Section 2, we describe in details the setting we consider, and set the notations for trees and forests. Section 3 defines the Mondrian process introduced by Roy and Teh (2009) and describes the *Mondrian Forests* algorithm; Section 4 is devoted to the sharp properties established for Mondrian partitions that will be used throughout the rest of the paper to derive consistency and upper bounds which are minimax optimal. In Section 5, we prove statistical guarantees for Mondrian Forests, which provide us with a way to tune the lifetime parameter. We also state that Mondrian Forests achieve the minimax rate for regression and classification and stress the optimality of forests, compared to individual trees.

## 2 Setting and notations

We first explain the general setting of the paper and describe the notations related to the Mondrian tree structure. For the sake of conciseness, we consider the regression setting, and show how to extend the results to classification in Section 5 below.

**Setting.** We consider a regression framework, where the dataset  $\mathcal{D}_n = \{(X_1, Y_1), \dots, (X_n, Y_n)\}$  contains i.i.d.  $[0, 1]^d \times \mathbf{R}$ -valued random variables, distributed as the generic pair  $(X, Y)$ , with  $\mathbb{E}[Y^2] < \infty$ . This unknown distribution, characterized by the distribution  $\mu$  of  $X$  on  $[0, 1]^d$  and by the conditional distribution of  $Y|X$ , can be written as

$$Y = f(X) + \varepsilon, \quad (1)$$

where  $f(X) = \mathbb{E}[Y | X]$  is the conditional expectation of  $Y$  given  $X$ , and  $\varepsilon$  is a noise satisfying  $\mathbb{E}[\varepsilon|X] = 0$ . Our goal is to output a *randomized estimate*  $\hat{f}_n(\cdot, Z, \mathcal{D}_n) : [0, 1]^d \rightarrow \mathbf{R}$ , where  $Z$  is a random variable that accounts for the randomization procedure; to simplify notation, we will generally denote  $\hat{f}_n(x, Z) = \hat{f}_n(x, Z, \mathcal{D}_n)$ . The quality of a randomized estimate  $\hat{f}_n$  is measured by its quadratic risk

$$R(\hat{f}_n) = \mathbb{E}[(\hat{f}_n(X, Z, \mathcal{D}_n) - f(X))^2]$$

where the expectation is taken with respect to  $(X, Z, \mathcal{D}_n)$ . We say that a sequence  $(\hat{f}_n)_{n \geq 1}$  is *consistent* whenever  $R(\hat{f}_n) \rightarrow 0$  as  $n \rightarrow \infty$ .

**Trees and Forests.** Let  $M \geq 1$  be the number of trees in a forest. We let  $\hat{f}_n(x, Z_1), \dots, \hat{f}_n(x, Z_M)$  be the randomized tree estimates at point  $x$ , associated to the same randomized mechanism, where the  $Z_m$  are i.i.d. and correspond to the extra randomness introduced in the tree construction. Set  $Z^{(M)} = (Z_1, \dots, Z_M)$ . The *random forest estimate*  $\hat{f}_n^{(M)}(x, Z^{(M)})$  is then defined by taking the average over all tree estimates  $\hat{f}_n(x, Z_m)$ , namely

$$\hat{f}_n^{(M)}(x, Z^{(M)}) = \frac{1}{M} \sum_{m=1}^M \hat{f}_n(x, Z_m). \quad (2)$$

Let us now introduce some specific notations to describe the decision tree structure. A decision tree  $(T, \Sigma)$  is composed of the following components:

- A finite rooted ordered binary tree  $T$ , with nodes  $\mathcal{N}(T)$ , interior nodes  $\mathcal{N}^\circ(T)$  and leaves  $\mathcal{L}(T)$  (so that  $\mathcal{N}(T)$  is the disjoint union of  $\mathcal{N}^\circ(T)$  and  $\mathcal{L}(T)$ ). The nodes  $\mathbf{v} \in \mathcal{N}(T)$  are finite words on the alphabet  $\{0, 1\}$ , that is elements of the set  $\{0, 1\}^* = \bigcup_{n \geq 0} \{0, 1\}^n$ : the root  $\epsilon$  of  $T$  is the empty word, and for every interior  $\mathbf{v} \in \{0, 1\}^*$ , its left child is  $\mathbf{v}0$  (obtained by adding a 0 at the end of  $\mathbf{v}$ ) while its right child is  $\mathbf{v}1$  (obtained by adding a 1 at the end of  $\mathbf{v}$ ).
- A family of *splits*  $\Sigma = (\sigma_{\mathbf{v}})_{\mathbf{v} \in \mathcal{N}^\circ(T)}$  at each interior node, where each split  $\sigma_{\mathbf{v}} = (j_{\mathbf{v}}, s_{\mathbf{v}})$  is characterized by its split dimension  $j_{\mathbf{v}} \in \{1, \dots, d\}$  and its threshold  $s_{\mathbf{v}} \in [0, 1]$ .

Each randomized estimate  $\hat{f}_n(x, Z_m)$  relies on a decision tree  $(T, \Sigma)$ , the random variable  $Z_m$  being the random sampling of the tree structure  $T$  and of the splits  $(\sigma_{\mathbf{v}})$ . This sampling mechanism, based on the Mondrian process, is defined in Section 3.

We associate to  $\Pi = (T, \Sigma)$  a partition  $(C_{\mathbf{v}})_{\mathbf{v} \in \mathcal{L}(T)}$  of the unit cube  $[0, 1]^d$ , called a *tree partition* (or *guillotine partition*). For each node  $\mathbf{v} \in \mathcal{N}(T)$ , we define a hyper-rectangular region  $C_{\mathbf{v}}$  recursively:

- The cell associated to the root of  $T$  is  $[0, 1]^d$ ;
- For each  $\mathbf{v} \in \mathcal{N}^\circ(T)$ , we define

$$C_{\mathbf{v}0} := \{x \in C_{\mathbf{v}} : x_{j_{\mathbf{v}}} \leq s_{j_{\mathbf{v}}}\} \quad \text{and} \quad C_{\mathbf{v}1} := C_{\mathbf{v}} \setminus C_{\mathbf{v}0}.$$

The leaf cells  $(C_{\mathbf{v}})_{\mathbf{v} \in \mathcal{L}(T)}$  form a partition of  $[0, 1]^d$  by construction. In what follows, we will identify a tree with splits  $(T, \Sigma)$  with its associated tree partition, and a node  $\mathbf{v} \in \mathcal{N}(T)$  with the cell  $C_{\mathbf{v}} \subset [0, 1]^d$ . The Mondrian process, described in the next Section, defines a distribution over nested tree partitions, defined below.

**Definition 1** (Nested tree partitions). A tree partition  $\Pi' = (T', \Sigma')$  is a *refinement* of the tree partition  $\Pi = (T, \Sigma)$  if every leaf cell of  $\Pi'$  is contained in a leaf cell of  $\Pi$ . This is equivalent to the fact that  $T$  is a subtree of  $T'$  and, for every  $\mathbf{v} \in \mathcal{N}(T) \subseteq \mathcal{N}(T')$ ,  $\sigma_{\mathbf{v}} = \sigma'_{\mathbf{v}}$ .

A *nested tree partition* is a family  $(\Pi_t)_{t \geq 0}$  of tree partitions such that, for every  $t, t' \in \mathbf{R}^+$  with  $t \leq t'$ ,  $\Pi_{t'}$  is a refinement of  $\Pi_t$ . Such a family can be described as follows: let  $\mathbf{T}$  be the (in general infinite, and possibly complete) rooted binary tree, such that  $\mathcal{N}(\mathbf{T}) = \bigcup_{t \geq 0} \mathcal{N}(T_t) \subseteq \{0, 1\}^*$ . For each  $\mathbf{v} \in \mathcal{N}(T)$ , let  $\tau_{\mathbf{v}} = \inf\{t \geq 0 \mid \mathbf{v} \in \mathcal{N}(T_t)\} < \infty$  denote the *birth time* of the node  $\mathbf{v}$ . Additionally, let  $\sigma_{\mathbf{v}}$  be the value of the split  $\sigma_{\mathbf{v}, t}$  in  $\Pi_t$  for  $t > \tau_{\mathbf{v}}$  (which does not depend on  $t$  by the refinement property). Then,  $\Pi$  is completely characterized by  $\mathbf{T}$ ,  $\Sigma = (\sigma_{\mathbf{v}})_{\mathbf{v} \in \mathcal{N}(\mathbf{T})}$  and  $\mathfrak{T} = (\tau_{\mathbf{v}})_{\mathbf{v} \in \mathcal{N}(\mathbf{T})}$ .

The regression tree outputs a constant estimation of the label in each leaf cell  $C_{\mathbf{v}}$  using a simple averaging of the labels  $Y_i$  ( $1 \leq i \leq n$ ) such that  $X_i \in C_{\mathbf{v}}$ .

### 3 The Mondrian Forest algorithm

The Mondrian process is a distribution on (infinite) nested tree partitions of the unit cube  $[0, 1]^d$  introduced by Roy and Teh (2009). This distribution enables us to define the *Mondrian Forests* that average the forecasts of Mondrian Trees obtained by sampling from the Mondrian process distribution.

Given a rectangular box  $C = \prod_{j=1}^d [a_j, b_j] \subseteq \mathbf{R}^d$ , we denote  $|C| := \sum_{j=1}^d (b_j - a_j)$  its *linear dimension*. The Mondrian process distribution  $\text{MP}(C)$  is a distribution on nested tree partitions of  $C$ . To define it, we introduce the function  $\Phi_C$ , which maps any family of couples  $(e_{\mathbf{v}}^j, u_{\mathbf{v}}^j) \in \mathbf{R}^+ \times [0, 1]$  indexed by the coordinates  $j \in \{1, \dots, d\}$  and the nodes  $\mathbf{v} \in \{0, 1\}^*$  to a nested tree partition  $\Pi = \Phi_C((e_{\mathbf{v}}^j, u_{\mathbf{v}}^j)_{\mathbf{v}, j})$  of  $C$ . The splits  $\sigma_{\mathbf{v}} = (j_{\mathbf{v}}, s_{\mathbf{v}})$  and birth times  $\tau_{\mathbf{v}}$  of the nodes  $\mathbf{v} \in \{0, 1\}^*$  are defined recursively, starting from the root  $\epsilon$ :

- For the root node  $\epsilon$ , we let  $\tau_{\epsilon} = 0$  and  $C_{\epsilon} = C$ .
- At each node  $\mathbf{v} \in \{0, 1\}^*$ , given the labels of all its ancestors  $\mathbf{v}' \sqsubset \mathbf{v}$  (so that in particular  $\tau_{\mathbf{v}}$  and  $C_{\mathbf{v}}$  are determined), denote  $C_{\mathbf{v}} = \prod_{j=1}^d [a_{\mathbf{v}}^j, b_{\mathbf{v}}^j]$ . Then, select the split dimension  $j_{\mathbf{v}} \in \{1, \dots, d\}$  and its location  $s_{\mathbf{v}}$  as follows:

$$j_{\mathbf{v}} = \underset{1 \leq j \leq d}{\operatorname{argmin}} \frac{e_{\mathbf{v}}^j}{b_{\mathbf{v}}^j - a_{\mathbf{v}}^j}, \quad s_{\mathbf{v}} = a_{\mathbf{v}}^{j_{\mathbf{v}}} + (b_{\mathbf{v}}^{j_{\mathbf{v}}} - a_{\mathbf{v}}^{j_{\mathbf{v}}}) \cdot u_{\mathbf{v}}^{j_{\mathbf{v}}}, \quad (3)$$

where we break ties in the choice of  $j_{\mathbf{v}}$  e.g. by choosing the smallest index  $j$  in the argmin. The node  $\mathbf{v}$  is then split at time  $\tau_{\mathbf{v}} + e_{\mathbf{v}}^{j_{\mathbf{v}}} / (b_{\mathbf{v}}^{j_{\mathbf{v}}} - a_{\mathbf{v}}^{j_{\mathbf{v}}}) = \tau_{\mathbf{v}0} = \tau_{\mathbf{v}1}$ , we let  $C_{\mathbf{v}0} = \{x \in C_{\mathbf{v}} : x_{j_{\mathbf{v}}} \leq s_{\mathbf{v}}\}$ ,  $C_{\mathbf{v}1} = C_{\mathbf{v}} \setminus C_{\mathbf{v}0}$  and recursively apply the procedure to its children  $\mathbf{v}0$  and  $\mathbf{v}1$ .

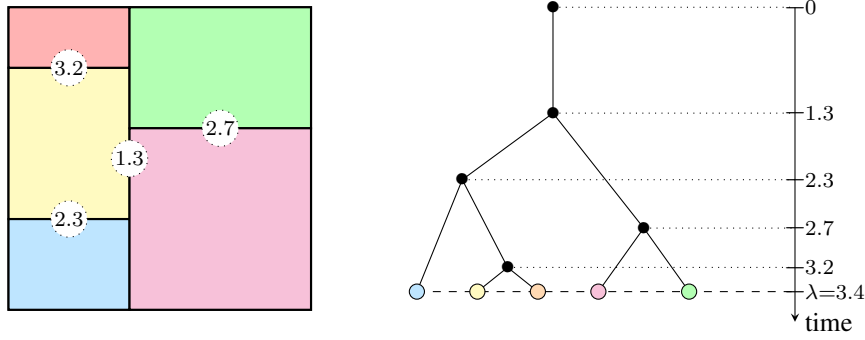


Figure 1: A Mondrian partition. The tree on the right-hand side (or equivalently the partition on the left-hand side) is grown sequentially, where the split times are indicated on the vertical time axis.

For each  $\lambda \in \mathbf{R}^+$ , the tree partition  $\Pi_\lambda = \Phi_{\lambda, C}((e_{\mathbf{v}}^j, u_{\mathbf{v}}^j)_{\mathbf{v}, j})$  is the *pruning of  $\Pi$  at time  $\lambda$* , obtained by removing all the splits in  $\Pi$  that occurred strictly after  $\lambda$ , so that the leaves of the tree are the maximal nodes (in the prefix order)  $\mathbf{v}$  such that  $\tau_{\mathbf{v}} \leq \lambda$ . Figure 1 presents a particular instance of Mondrian partition on a square box, with lifetime parameter  $\lambda = 3.4$ .

**Definition 2** (Mondrian process). Let  $(E_{\mathbf{v}}^j, U_{\mathbf{v}}^j)_{\mathbf{v}, j}$  be a family of independent random variables, with  $E_{\mathbf{v}}^j \sim \text{Exp}(1)$ ,  $U_{\mathbf{v}}^j \sim \mathcal{U}([0, 1])$ . The *Mondrian process*  $\text{MP}(C)$  on  $C$  is the distribution of the random nested tree partition  $\Phi_C((E_{\mathbf{v}}^j, U_{\mathbf{v}}^j)_{\mathbf{v}, j})$ . In addition, we denote  $\text{MP}(\lambda, C)$  the distribution of  $\Phi_{\lambda, C}((E_{\mathbf{v}}^j, U_{\mathbf{v}}^j)_{\mathbf{v}, j})$ .

Sampling from  $\text{MP}(\lambda, C)$  can be done through the recursive procedure  $\text{SampleMondrian}(\lambda, C)$  of Algorithm 1.

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**Algorithm 1**  $\text{SampleMondrian}(\lambda, C)$  ; Sample a tree partition distributed as  $\text{MP}(\lambda, C)$ .

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- 1: **Parameters:** A rectangular box  $C \subset \mathbf{R}^d$  and a lifetime parameter  $\lambda > 0$ .
  - 2: **Call**  $\text{SplitCell}(C, \tau := 0, \lambda)$ .
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**Algorithm 2**  $\text{SplitCell}(C, \tau, \lambda)$  ; Recursively split a cell  $C$ , starting from time  $\tau$ , until  $\lambda$

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- 1: **Parameters:** A cell  $C = \prod_{1 \leq j \leq d} [a_j, b_j]$ , a starting time  $\tau$  and a lifetime parameter  $\lambda$ .
  - 2: **Sample** an exponential random variable  $E_C$  with intensity  $|C|$ .
  - 3: **if**  $\tau + E_C \leq \lambda$  **then**
  - 4:   **Draw** at random a split dimension  $J \in \{1, \dots, d\}$ , with  $\mathbb{P}(J = j) = (b_j - a_j)/|C|$ , and a split threshold  $s_J$  uniformly in  $[a_J, b_J]$ .
  - 5:   **Split**  $C$  along the split  $(J, s_J)$ . Let  $C_0$  and  $C_1$  be the resulting cells.
  - 6:   **Call**  $\text{SplitCell}(C_0, \tau + E_C, \lambda)$  and  $\text{SplitCell}(C_1, \tau + E_C, \lambda)$ .
  - 7: **else**
  - 8:   Do nothing.
  - 9: **end if**
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Indeed, for any cell  $C = \prod_{1 \leq j \leq d} [a_j, b_j]$ , if  $E_1, \dots, E_d$  are independent exponential random variables with intensities  $b_1 - a_1, \dots, b_d - a_d$ , then  $E_C = \min_{1 \leq j \leq d} E_j$  is distributed as  $\text{Exp}(\sum_{1 \leq j \leq d} (b_j - a_j)) \sim \text{Exp}(|C|)$ . Moreover, if  $J = \arg\min_{1 \leq j \leq d} E_j$ ,  $J$  and  $E$  are independent and  $\mathbb{P}[J = j] = (b_j - a_j)/|C|$ . These facts prove the equivalence between the definition of a Mondrian process from Definition 2 and the construction described in Algorithms 1 and 2.

*Remark 1.* Using the memoryless property of exponential random variables (if  $E \sim \text{Exp}(l)$  and  $\lambda > 0$ , the distribution of  $E - \lambda$  conditionally on  $\{E > \lambda\}$  is  $\text{Exp}(l)$ ) it is possible to efficiently sample  $\Pi_{\lambda'} \sim \text{MP}(\lambda', C)$  given its pruning at time  $\lambda \leq \lambda'$ :  $\Pi_\lambda \sim \text{MP}(\lambda, C)$ . This proves that the Mondrian process is Markovian.

Finally, the procedure to build the Mondrian Forest is as follows: grow randomized tree partitions  $\Pi_\lambda^{(1)}, \dots, \Pi_\lambda^{(M)}$ , fit each one with the dataset  $\mathcal{D}_n$  by averaging the labels falling into each leaf (predicting 0 if the leaf is empty), then combine the resulting Mondrian Tree estimates by averaging their predictions. In accordance with Equation (2), we let

$$\hat{f}_{\lambda,n}^{(M)}(x, Z^{(M)}) = \frac{1}{M} \sum_{m=1}^M \hat{f}_{\lambda,n}^{(m)}(x, Z_m), \quad (4)$$

be the Mondrian Forest estimate described above, where  $\hat{f}_{\lambda,n}^{(m)}(x, Z_m)$  denotes the Mondrian Tree parametrized by the random variable  $Z_m$ . Here, the variables  $Z_1, \dots, Z_M$  are independent and distributed as the generic random variable  $Z = (E_{\mathbf{v}}^j, U_{\mathbf{v}}^j)_{\mathbf{v},j}$  (see Definition 2).

## 4 Local and global properties of the Mondrian process

In this Section, we show that the properties of the Mondrian process enable to compute explicitly some local and global quantities related to the structure of Mondrian partitions. To do so, we will need the following two facts, exposed by Roy and Teh (2009).

**Fact 1** (Dimension 1). *For  $d = 1$ , the splits from a Mondrian process  $\Pi_\lambda \sim \text{MP}(\lambda, [0, 1])$  form a subset of  $[0, 1]$ , which is distributed as a Poisson point process of intensity  $\lambda dx$ .*

**Fact 2** (Restriction). *Let  $\Pi_\lambda \sim \text{MP}(\lambda, [0, 1]^d)$  be a Mondrian partition, and  $C = \prod_{j=1}^d [a_j, b_j] \subset [0, 1]^d$  be a box. Consider the restriction  $\Pi_\lambda|_C$  of  $\Pi_\lambda$  on  $C$ , i.e. the partition on  $C$  induced by the partition  $\Pi_\lambda$  of  $[0, 1]^d$ . Then  $\Pi_\lambda|_C \sim \text{MP}(\lambda, C)$ .*

Fact 1 deals with the one-dimensional case by making explicit the distribution of splits for Mondrian process, which follows a Poisson point process. The restriction property stated in Fact 2 is fundamental, and enables to precisely characterize the behavior of the Mondrian partitions.

Given any point  $x \in [0, 1]^d$ , the next Proposition 1 is a sharp result giving the exact distribution of the cell  $C_\lambda(x)$  containing  $x$  from the Mondrian partition. Such a characterization is typically unavailable for other randomized trees partitions involving a complex recursive structure.

**Proposition 1** (Cell distribution). *Let  $x \in [0, 1]^d$  and denote by*

$$C_\lambda(x) = \prod_{1 \leq j \leq d} [L_{j,\lambda}(x), R_{j,\lambda}(x)]$$

*the cell of containing  $x$  in a partition  $\Pi_\lambda \sim \text{MP}(\lambda, [0, 1]^d)$  (this cell corresponds to a leaf). Then, the distribution of  $C_\lambda(x)$  is characterized by the following properties :*

- (i)  $L_{1,\lambda}(x), R_{1,\lambda}(x), \dots, L_{d,\lambda}(x), R_{d,\lambda}(x)$  are independent ;
- (ii) For each  $j = 1, \dots, d$ ,  $L_{j,\lambda}(x)$  is distributed as  $(x - \lambda^{-1}E_{j,L}) \vee 0$  and  $R_{j,\lambda}(x)$  as  $(x + \lambda^{-1}E_{j,R}) \wedge 1$ , where  $E_{j,L}, E_{j,R} \sim \text{Exp}(1)$ .

The proof of Proposition 1 is given in Section 7.1 below. Figure 2 is a graphical representation of Proposition 1. A consequence of Proposition 1 is next Corollary 1, which gives a precise upper bound on cell diameters, which will help in providing the approximation error of the Mondrian Tree and Forest in Section 5.



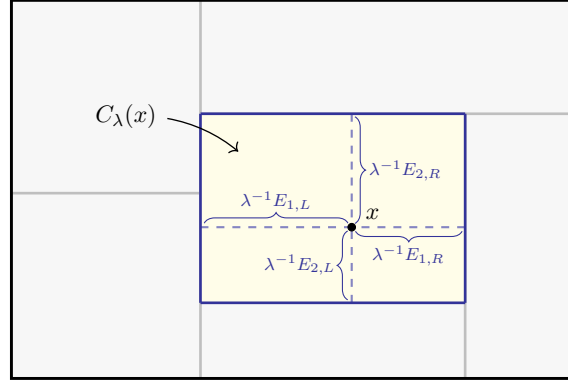


Figure 2: Cell distribution in a Mondrian partition. Proposition 1 specifies the distribution of distances between  $x$  and each side of the cell  $C_\lambda(x)$ : distances are depicted by dashed lines and their distributions correspond to braces. These distances are independent truncated exponential variables.

**Corollary 1** (Cell diameter). *Set  $\lambda > 0$ . Let  $x \in [0, 1]^d$ , and let  $D_\lambda(x)$  be the  $\ell^2$ -diameter of the cell  $C_\lambda(x)$  containing  $x$  in a Mondrian partition  $\Pi_\lambda \sim \text{MP}(\lambda, [0, 1]^d)$ . For every  $\delta > 0$ , we have*

$$\mathbb{P}(D_\lambda(x) \geq \delta) \leq d \left(1 + \frac{\lambda\delta}{\sqrt{d}}\right) \exp\left(-\frac{\lambda\delta}{\sqrt{d}}\right)$$

and

$$\mathbb{E}[D_\lambda(x)^2] \leq \frac{4d}{\lambda^2}.$$

In particular, if  $\lambda \rightarrow \infty$ , then  $D_\lambda(x) \rightarrow 0$  in probability.

To control the risk of the Mondrian Tree and Mondrian Forest, we need an upper bound on the number of cells in a Mondrian partition. Quite surprisingly, this quantity can be computed exactly, as shown in Proposition 2.

**Proposition 2** (Number of cells). *If  $K_\lambda$  denotes the number of cells in a Mondrian Tree partition  $\Pi_\lambda \sim \text{MP}(\lambda, [0, 1]^d)$ , we have  $\mathbb{E}[K_\lambda] = (1 + \lambda)^d$ .*

The proof of Proposition 2, which is given in Section 7.1 below, is technically involved. It relies on a coupling argument: we introduce a recursive modification of the construction of the Mondrian process which keeps the expected number of leaves unchanged, and for which this quantity can be computed directly using the Mondrian-Poisson equivalence in dimension one (Fact 1). A much simpler result is  $\mathbb{E}[K_\lambda] \leq (e(1 + \lambda))^d$ , which was previously proposed in Mourada et al. (2017). By contrast, Proposition 2 provides the *exact* value of this expectation, which removes a superfluous  $e^d$  factor. This significantly improves the dependency on  $d$  of the upper bounds stated in Theorems 2 and 3 below.

*Remark 2.* Proposition 2 naturally extends (with the same proof) to the more general case of a Mondrian process with finite measures with no atoms  $\nu_1, \dots, \nu_d$  on the sides  $C^1, \dots, C^d$  of a box  $C \subseteq \mathbb{R}^d$  (for a definition of the Mondrian process in this more general case, see Roy, 2011). In this case, we have  $\mathbb{E}[K_\lambda] = \prod_{1 \leq j \leq d} (1 + \nu_j(C^j))$ .

As illustrated in this Section, a remarkable fact with the Mondrian Forest is that the quantities of interest for the statistical analysis of the algorithm can be made explicit. In particular, we show that a Mondrian partition is balanced enough so that it contains  $O(\lambda^d)$  cells of diameter  $O(1/\lambda)$ , which is the minimal number of cells to cover  $[0, 1]^d$ .

## 5 Minimax theory for Mondrian Forests

This Section gathers a universal consistency result and sharp upper bounds for the Mondrian Trees and Forests. Section 5.1 states the universal consistency of the procedure, provided that the lifetime  $\lambda_n$  belongs to an appropriate range. Section 5.2 gives an upper bound valid for Mondrian Trees and Forests which turns out to be minimax optimal for Lipschitz regression functions, provided that  $\lambda_n$  is properly tuned. Finally, Section 5.3 shows that Mondrian Forests improve over Mondrian trees, for twice continuously differentiable regression functions. Results for classification are given in Section 5.4.

### 5.1 Consistency of Mondrian Forests

The consistency of the Mondrian Forest, described in Algorithm 1, is established in Theorem 1 below, provided a proper tuning of the lifetime parameter  $\lambda_n$ .

**Theorem 1** (Universal consistency). *Assume that  $\mathbb{E}[Y^2] < \infty$ . Let  $\lambda_n \rightarrow \infty$  such that  $\lambda_n^d/n \rightarrow 0$ . Then, Mondrian tree estimates (whose construction is described in Algorithm 1) with lifetime parameter  $\lambda_n$  are consistent. As a consequence, Mondrian Forests estimates with  $M \geq 1$  trees and lifetime parameter  $\lambda_n$  are consistent.*

The proof of Theorem 1 is given in Section 7.2. This consistency result is universal, in the sense that it makes no assumption on the joint distribution of  $(X, Y)$ , apart from the fact that  $\mathbb{E}[Y^2] < \infty$ , which is necessary to ensure that the quadratic risk is well-defined. This contrasts with several consistency results on Random Forests (see, e.g., Breiman, 2004; Biau, 2012) which assume that the density of  $X$  is bounded from below and above. The proof of Theorem 1 uses the properties of Mondrian partitions established in Section 4, in conjunction with general consistency results for histograms.

The only parameter in Mondrian Tree is the lifetime  $\lambda_n$ , which encodes the complexity of the trees. Requiring an assumption on this parameter is natural, and confirmed by the well-known fact that the tree-depth is an important tuning parameter for Random Forests (see, for instance, Biau and Scornet, 2016). However, Theorem 1 leaves open the question of a theoretically optimal tuning of  $\lambda_n$  under additional assumptions on the regression function  $f$ , which we address in the following sections.

### 5.2 Mondrian Trees and Forests are minimax over the class of Lipschitz functions

The bounds obtained in Corollary 1 and Proposition 2 are explicit and sharp in their dependency on  $\lambda$ . Based on these properties, we now establish a theoretical upper bound on the risk of Mondrian Trees, which gives the optimal theoretical tuning of the lifetime parameter  $\lambda_n$ . To pursue the analysis, we work under the following

**Assumption 1.** Assume that  $(X, Y)$  satisfies Equation (1) where  $Y$  satisfies  $\mathbb{E}(Y^2) < \infty$ , where  $\varepsilon$  is a real-valued random variable such that  $\mathbb{E}(\varepsilon | X) = 0$  and  $\text{Var}(\varepsilon | X) \leq \sigma^2 < \infty$  almost surely.

Theorem 2 states an upper bound on the risk of Mondrian Trees and Forests, which explicitly depends on the lifetime parameter  $\lambda$ . Selecting  $\lambda$  that minimizes this bound leads to a convergence rate which turns out to be minimax optimal over the class of Lipschitz functions (see e.g. Chapter I.3 in Nemirovski (2000) for details on minimax rates).

**Theorem 2.** Grant Assumption 1 and assume that  $f$  is  $L$ -Lipschitz. Let  $M \geq 1$ . The quadratic risk of the Mondrian Forest  $\hat{f}_{\lambda,n}^{(M)}$  with lifetime parameter  $\lambda > 0$  satisfies

$$\mathbb{E}[(\hat{f}_{\lambda,n}^{(M)}(X) - f(X))^2] \leq \frac{4dL^2}{\lambda^2} + \frac{(1+\lambda)^d}{n} (2\sigma^2 + 9\|f\|_\infty^2). \quad (5)$$



In particular, the choice  $\lambda := \lambda_n \asymp n^{1/(d+2)}$  gives

$$\mathbb{E}[(\hat{f}_{\lambda_n, n}^{(M)}(X) - f(X))^2] = O(n^{-2/(d+2)}), \quad (6)$$

which corresponds to the minimax rate over the class of Lipschitz functions.

The proof of Theorem 2 is given in Section 7.3. The core of the proof of Theorem 2 relies on the two new properties about Mondrian trees stated in Section 4. Corollary 1 allows to control the bias of Mondrian Trees (first term on the right-hand side of Equation 5), while Proposition 2 helps in controlling the variance of Mondrian Trees (second term on the right-hand side of Equation 5).

To the best of our knowledge, Theorem 2 is the first to prove that a purely random forest (Mondrian Forest in this case) can be minimax optimal *in arbitrary dimension*. Minimax optimal upper bounds are obtained for  $d = 1$  in Genuer (2012) and Arlot and Genuer (2014) for models of purely random forests such as Toy-PRF (where the individual partitions corresponded to random shifts of the regular partition of  $[0, 1]$  in  $k$  intervals) and PURF (Purely Uniformly Random Forests, where the partitions were obtained by drawing  $k$  random thresholds at random in  $[0, 1]$ ). However, for  $d = 1$ , tree partitions reduce to partitions of  $[0, 1]$  in intervals, and do not possess the recursive structure that appears in higher dimensions, which makes their analysis challenging. For this reason, the analysis of purely random forests for  $d > 1$  has typically produced sub-optimal results: for example, Biau (2012) exhibit a convergence rate for the centered random forests (a particular instance of PRF) which turns out to be much slower than the minimax rate for Lipschitz regression functions. A similar result was proved by Arlot and Genuer (2014), who studied the BPRF (Balanced Purely Random Forests algorithm, where all leaves are split, so that the resulting tree is complete), and obtained suboptimal rates. In our approach, the convenient properties of the Mondrian process enable to bypass the inherent difficulties met in previous attempts.

Theorem 2 provides theoretical guidance on the choice of the lifetime parameter, and suggests to set  $\lambda := \lambda_n \asymp n^{1/(d+2)}$ . Such an insight cannot be gleaned from an analysis that focuses only on consistency. Theorem 2 is valid for Mondrian Forests with any number of trees, and thus in particular for a Mondrian Tree (this is also true for Theorem 1). However, it is a well-known fact that forests often outperform single trees in practice (see, e.g., Fernández-Delgado et al., 2014). Section 5.3 proposes an explanation for this phenomenon, by considering  $\mathcal{C}^2$  regression functions.

### 5.3 Improved rates for Mondrian Forests compared to a single Mondrian Tree

The convergence rate stated in Theorem 2 for Lipschitz regression functions is valid for both trees and forests, and the risk bound does not depend on the number  $M$  of trees that compose the forest. In practice, however, it is observed that forests often outperform individual trees. In this section, we provide a result that illustrates the benefits of forests over trees. Assume that the regression function  $f$  is not only Lipschitz, but in fact twice continuously differentiable. As the counterexample in Lemma 1 below shows, single Mondrian trees do not benefit from this additional smoothness assumption, and achieve the same rate as in the Lipschitz case. This comes from the fact that the bias of trees is highly sub-optimal for such functions.

**Lemma 1.** *Grant Assumption 1 for the following simple one-dimensional regression model:*

$$Y = f(X) + \varepsilon,$$

where  $X \sim \mathcal{U}([0, 1])$ ,  $f : x \mapsto 1 + x$  and  $\varepsilon$  is independent of  $X$  with variance  $\sigma^2$ . Consider a single Mondrian Tree estimate  $\hat{f}_{\lambda, n}^{(1)}$ . Then, there exists a constant  $C_0 > 0$ , such that, for  $n \geq 18$ ,

$$\inf_{\lambda \in \mathbf{R}_+^*} \mathbb{E}[(\hat{f}_{\lambda, n}^{(1)}(X) - f(X))^2] \geq C_0 \wedge \frac{1}{4} \left( \frac{3\sigma^2}{n} \right)^{2/3}.$$

The proof of Lemma 1 is given in Section 7.4. Since the minimax rate over the class of  $\mathcal{C}^2$  functions in dimension 1 is  $O(n^{-4/5})$ , Lemma 1 proves that a single Mondrian Tree is not minimax optimal for the class of  $\mathcal{C}^2$  functions.

However, it turns out that large enough Mondrian Forests, which average Mondrian trees, are minimax optimal for  $\mathcal{C}^2$  functions. Therefore, Theorem 3 below highlights the benefits of a forest compared to a single tree.

**Theorem 3.** *Grant Assumption 1 and assume that  $X$  has a positive and  $C_p$ -Lipschitz density  $p$  w.r.t the Lebesgue measure on  $[0, 1]^d$  and that the regression function  $f$  is  $\mathcal{C}^2$  on  $[0, 1]^d$ . Let  $\hat{f}_{\lambda,n}^{(M)}$  be the Mondrian Forest estimate composed of  $M \geq 1$  trees, with lifetime parameter  $\lambda$ . Then, the following upper bound holds for every  $\varepsilon \in [0, 1/2)$ :*

$$\begin{aligned} \mathbb{E}[(\hat{f}_{\lambda,n}^{(M)}(X) - f(X))^2 \mid X \in [\varepsilon, 1 - \varepsilon]^d] &\leq \frac{8d\|\nabla f\|_\infty^2}{M\lambda^2} + \frac{2(1+\lambda)^d 2\sigma^2 + 9\|f\|_\infty^2}{n p_0(1-2\varepsilon)^d} \\ &+ \frac{72d\|\nabla f\|_\infty^2 p_1}{p_0(1-2\varepsilon)^d} \frac{e^{-\lambda\varepsilon}}{\lambda^3} + \frac{72d^3\|\nabla f\|_\infty^2 C_p^2 p_1^2}{p_0^4} \frac{1}{\lambda^4} + \frac{4d^2\|\nabla^2 f\|_\infty^2 p_1^2}{p_0^2} \frac{1}{\lambda^4}, \end{aligned} \quad (7)$$

where  $p_0 = \inf_{[0,1]^d} p$ ,  $p_1 = \sup_{[0,1]^d} p$ ,  $\|\nabla f\|_\infty = \sup_{x \in [0,1]^d} \|\nabla f(x)\|_2$  and  $\|\nabla^2 f\|_\infty = \sup_{x \in [0,1]^d} \|\nabla^2 f(x)\|_{\text{op}}$ , with  $\|\cdot\|_{\text{op}}$  the operator norm. In particular, the choices  $\lambda_n \asymp n^{1/(d+4)}$  and  $M_n \gtrsim n^{2/(d+4)}$  give

$$\mathbb{E}[(\hat{f}_{\lambda_n,n}^{(M_n)}(X) - f(X))^2 \mid X \in [\varepsilon, 1 - \varepsilon]^d] = O(n^{-4/(d+4)}), \quad (8)$$

which corresponds to the minimax rate over the set of  $\mathcal{C}^2$  functions. Besides, letting  $\lambda_n \asymp n^{1/(d+3)}$  and  $M_n \gtrsim n^{2/(d+3)}$  yields the following upper bound on the integrated risk of the Mondrian Forest estimate over the whole hypercube  $[0, 1]^d$ ,

$$\mathbb{E}[(\hat{f}_{\lambda_n,n}^{(M_n)}(X) - f(X))^2] = O(n^{-3/(d+3)}). \quad (9)$$

The proof of Theorem 3 is given in Section 7.5 below. It relies on an improved control of the bias, compared to what we did in Theorem 2 in the Lipschitz case: it exploits the knowledge of the distribution of the cell  $C_\lambda(x)$  given in Proposition 1 instead of merely the cell diameter given in Corollary 1 (which was enough for Theorem 2). The improved rate for Mondrian Forests compared to Mondrian trees comes from the fact that large enough forests smooth the decision function of single trees, which are discontinuous piecewise constant functions, and therefore cannot approximate smooth functions well enough. This was already noticed in Arlot and Genuer (2014) for purely random forests.

*Remark 3.* While Equation (8) gives the minimax minimax rate for  $\mathcal{C}^2$  function, it suffers from an unavoidable standard artifact, namely the boundary effect which affects local averaging estimates, such as kernel estimators, see Wasserman (2006) and Arlot and Genuer (2014). It is however possible to set  $\varepsilon = 0$  in Equation (7), which leads to the sub-optimal rate stated in (9).

Let us now consider, as a by-product of the analysis conducted for regression estimation, the setting of binary classification.

## 5.4 Results for binary classification

Assume that we are given a dataset  $\mathcal{D}_n = \{(X_1, Y_1), \dots, (X_n, Y_n)\}$  of i.i.d.  $[0, 1]^d \times \{0, 1\}$ -valued random variables, distributed as a generic pair  $(X, Y)$  and define  $\eta(x) = \mathbb{P}[Y = 1 \mid X = x]$ .

We define the Mondrian Forest classifier  $\hat{g}_{\lambda,n}^{(M)}$  as a plug-in estimator of the regression estimator. Namely, we introduce

$$\hat{g}_{\lambda,n}^{(M)}(x) = \mathbf{1}_{\{\hat{f}_{\lambda,n}^{(M)}(x) \geq 1/2\}}$$

for all  $x \in [0, 1]^d$ , where  $\hat{f}_{\lambda,n}^{(M)}$  is the Mondrian Forest estimate defined in the regression setting. The performance of  $\hat{g}_{\lambda,n}^{(M)}$  is assessed by the 0-1 classification error defined as

$$L(\hat{g}_{\lambda,n}^{(M)}) = \mathbb{P}[\hat{g}_{\lambda,n}^{(M)}(X) \neq Y], \quad (10)$$

where the probability is taken with respect to  $(X, Y, Z^{(M)}, \mathcal{D}_n)$ . Note that (10) is larger than the Bayes risk defined as

$$L(g^*) = \mathbb{P}[g^*(X) \neq Y],$$

where  $g^*(x) = \mathbf{1}_{\{\eta(x) \geq 1/2\}}$ . A general theorem (Theorem 6.5 in Devroye et al. (1996)) allows us to derive an upper bound on the distance between the classification risk of  $\hat{g}_{\lambda,n}^{(M)}$  and the Bayes risk, based on Theorem 2.

**Corollary 2.** *Let  $M \geq 1$  and assume that  $\eta$  is Lipschitz. Then, the Mondrian Forest classifier  $\hat{g}_{\lambda,n}^{(M)}$  with lifetime parameter  $\lambda_n \asymp n^{1/(d+2)}$  satisfies*

$$L(\hat{g}_{\lambda,n}^{(M)}) - L(g^*) = o(n^{-1/(d+2)}).$$

The rate of convergence  $o(n^{-1/(d+2)})$  for the error probability with a Lipschitz conditional probability  $\eta$  is optimal (Yang, 1999). We can also extend in the same way Theorem 3 to the context of classification. This is done in the next Corollary.

**Corollary 3.** *In the classification framework described in Section 5.2, assume that  $X$  has a positive and Lipschitz density  $p$  w.r.t the Lebesgue measure on  $[0, 1]^d$  and that the conditional probability  $\eta$  is  $\mathcal{C}^2$  on  $[0, 1]^d$ . Let  $\hat{g}_{\lambda,n}^{(M_n)}$  be the Mondrian Forest classifier composed of  $M_n \gtrsim n^{2/(d+4)}$  trees, with lifetime  $\lambda_n \asymp n^{1/(d+4)}$ . Then, for all  $\varepsilon \in [0, 1/2)$ ,*

$$\mathbb{P}[\hat{g}_{\lambda,n}^{(M_n)}(X) \neq Y | X \in [\varepsilon, 1 - \varepsilon]^d] - \mathbb{P}[g^*(X) \neq Y | X \in [\varepsilon, 1 - \varepsilon]^d] = o(n^{-2/(d+4)}). \quad (11)$$

This shows that Mondrian Forests achieve an improved rate compared to Mondrian trees for classification.

## 6 Conclusion

Despite their widespread use in practice, the theoretical understanding of Random Forests is still incomplete. In this work, we show that the Mondrian Forest, originally introduced to provide an efficient online algorithm, leads to an algorithm that is not only consistent, but in fact minimax optimal under nonparametric assumptions in arbitrary dimension. This is to the best of our knowledge, the first time such a result is obtained for a random forest method in arbitrary dimension. Besides, our analysis allows to illustrate improved rates for forests compared to individual trees. Mondrian partitions possess nice geometric properties, which we were able to control in a sharp and direct fashion, while previous approaches (Biau et al., 2008; Arlot and Genuer, 2014) require arguments that work conditionally on the structure of the tree. This suggests that Mondrian Forests can be viewed as an optimal variant of purely random forests, which could set a foundation for more sophisticated and theoretically sound random forest algorithms.

The optimal upper bound  $O(n^{-4/(d+4)})$  obtained in this paper is very slow when the number of features  $d$  is large. This comes from the well-known curse of dimensionality phenomenon, a problem affecting all fully nonparametric algorithms. A standard approach used in high-dimensional settings is to work under a sparsity assumption, where only  $s \ll d$  features are informative. A direction for future work could be to improve Mondrian Forests using a data-driven choice of the features along which the splits are performed, reminiscent of Extra-Trees (Geurts et al., 2006). From a theoretical perspective, it would be interesting to see how minimax rates obtained here can be combined with results on the ability of forests to select informative variables (see, for instance, Scornet et al., 2015).

## 7 Proofs

### 7.1 Proofs of Propositions 1 and 2 and of Corollary 1

*Proof of Proposition 1.* Let  $0 \leq a_1, \dots, a_n, b_1, \dots, b_n \leq 1$  be such that  $a_j \leq x_j \leq b_j$  for  $1 \leq j \leq d$ . Let  $A := \prod_{j=1}^d [a_j, b_j]$ . Note that the event

$$\{L_{1,\lambda}(x) \leq a_1, R_{1,\lambda}(x) \geq b_1, \dots, L_{d,\lambda}(x) \leq a_d, R_{d,\lambda}(x) \geq b_d\}$$

coincides — up to the negligible event that one of the splits of  $\Pi_\lambda$  occurs on coordinate  $j$  at  $a_j$  or  $b_j$  — with the event that  $\Pi_\lambda$  does not cut  $C$ , i.e. that the restriction  $\Pi_\lambda|_C$  of  $\Pi_\lambda$  to  $C$  contains no split. Now, by the restriction property of the Mondrian process (Fact 2),  $\Pi_\lambda|_C$  is distributed as  $\text{MP}(\lambda, A)$ ; in particular, the probability that  $\Pi_\lambda|_C$  contains no split is  $\exp(-\lambda|A|)$ . Hence, we have

$$\begin{aligned} \mathbb{P}(L_{1,\lambda}(x) \leq a_1, R_{1,\lambda}(x) \geq b_1, \dots, L_{d,\lambda}(x) \leq a_d, R_{d,\lambda}(x) \geq b_d) \\ = \exp(-\lambda(x - a_1)) \exp(-\lambda(b_1 - x)) \cdots \exp(-\lambda(x - a_d)) \exp(-\lambda(b_d - x)). \end{aligned} \quad (12)$$

In particular, setting  $a_j = b_j = x$  in (12) except for one  $a_j$  or  $b_j$ , and using that  $L_{\lambda,j}(x) \leq x$  and  $R_{\lambda,j}(x) \geq x$ , we obtain

$$\mathbb{P}(R_{j,\lambda}(x) \geq b_j) = \exp(-\lambda(b_j - x)) \quad \text{and} \quad \mathbb{P}(L_{j,\lambda}(x) \leq a_j) = \exp(-\lambda(x - a_j)). \quad (13)$$

Since clearly  $R_{j,\lambda}(x) \leq 1$  and  $L_{j,\lambda}(x) \geq 0$ , equation (13) implies (ii). Additionally, plugging equation (13) back into equation (12) shows that  $L_{1,\lambda}(x), R_{1,\lambda}(x), \dots, L_{d,\lambda}(x), R_{d,\lambda}(x)$  are independent, i.e. point (i). This completes the proof.  $\square$

*Proof of Corollary 1.* By Proposition 1,  $D_\lambda^1(x) = R_\lambda^1(x) - x_1 + x_1 - L_\lambda^1(x)$  is stochastically upper bounded by  $\lambda^{-1}(E_1 + E_2)$  with  $E_1, E_2$  two independent  $\text{Exp}(1)$  random variables, which is distributed as  $\text{Gamma}(2, \lambda)$ . This implies that, for every  $\delta > 0$ ,

$$\mathbb{P}(D_\lambda^1(x) \geq \delta) \leq (1 + \lambda\delta)e^{-\lambda\delta} \quad (14)$$

(with equality if  $\delta \leq x_1 \wedge (1 - x_1)$ ), and  $\mathbb{E}[D_\lambda^1(x)^2] \leq \lambda^{-2}(\mathbb{E}[E_1^2] + \mathbb{E}[E_2^2]) = \frac{4}{\lambda^2}$ . The bound (1) for the diameter  $D_\lambda(x) = \sqrt{\sum_{j=1}^d D_\lambda^j(x)^2}$  follows from the observation that

$$\mathbb{P}(D_\lambda(x) \geq \delta) \leq \mathbb{P}\left(\exists j : D_\lambda^j(x) \geq \frac{\delta}{\sqrt{d}}\right) \leq d \mathbb{P}\left(D_\lambda^1(x) \geq \frac{\delta}{\sqrt{d}}\right),$$

while the bound (1) is obtained by noting that  $\mathbb{E}[D_\lambda(x)^2] = d \mathbb{E}[D_\lambda^1(x)^2]$ .  $\square$

*Proof of Proposition 2.* At a high level, the idea of the proof is to modify the construction of the Mondrian partition (and hence, the distribution of the underlying process) without affecting the expected number of cells. More precisely, we show a recursively way to transform the Mondrian process that breaks the underlying independence structure but leaves  $\mathbb{E}[K_\lambda]$  unchanged, and which eventually leads to a random partition  $\tilde{\Pi}_\lambda$  for which this quantity can be computed directly and equals  $(1 + \lambda)^d$ .

We will in fact show the result for a general box  $C$  (not just the unit cube). The proof proceeds in two steps:

1. Define a modified process  $\tilde{\Pi}$ , and show that  $\mathbb{E}[\tilde{K}_\lambda] = \prod_{j=1}^d (1 + \lambda|C^j|)$ .
2. It remains to show that  $\mathbb{E}[K_\lambda] = \mathbb{E}[\tilde{K}_\lambda]$ . For this, it is sufficient to show that the distribution of the birth times  $\tau_v$  and  $\tilde{\tau}_v$  of the node  $v$  is the same for both processes. This is done by induction on  $v$ , by showing that the splits at one node of both processes have the same conditional distribution given the splits at previous nodes.

Let  $(E_v^j, U_v^j)_{v \in \{0,1\}^*, 1 \leq j \leq d}$  be a family of independent random variables with  $E_v^j \sim \text{Exp}(1)$  and  $U_v^j \sim \mathcal{U}([0, 1])$ . By definition,  $\Pi = \Phi_C((E_v^j, U_v^j)_{v,j})$  ( $\Phi_C$  being defined in Section 3) follows a Mondrian process distribution  $\text{MP}(C)$ . Denote for every node  $v \in \{0, 1\}^*$   $C_v$  the cell of  $v$ ,  $\tau_v$  its birth time, as well as its split time  $T_v$ , dimension  $J_v$ , and threshold  $S_v$  (note that  $T_v = \tau_{v0} = \tau_{v1}$ ). In addition, for every  $\lambda \in \mathbf{R}^+$ , denote  $\Pi_\lambda \sim \text{MP}(\lambda, C)$  the tree partition restricted to the time  $\lambda$ , and  $K_\lambda \in \mathbf{N} \cup \{+\infty\}$  its number of nodes.

**Construction of the modified process.** Now, consider the following modified nested partition of  $C$ , denoted  $\tilde{\Pi}$ , and defined through its split times, dimension and threshold  $\tilde{T}_v, \tilde{J}_v, \tilde{S}_v$  (which determine the birth times  $\tau_v$  and cells  $C_v$ ), and *current  $j$ -dimensional node*  $\mathbf{v}_j(v) \in \{0, 1\}^*$  ( $1 \leq j \leq d$ ) at each node  $v$ . First, for every  $j = 1, \dots, d$ , let  $\Pi'^j = \Phi_{C^j}((E_v^j, U_v^j)_{v \in \{0,1\}^*}) \sim \text{MP}(C^j)$  be the nested partition of the interval  $C^j$  determined by  $(E_v^j, U_v^j)_v$ ; its split times and thresholds are denoted  $(S_v'^j, T_v'^j)$ . Then,  $\tilde{\Pi}$  is defined recursively as follows:

- At the root node  $\epsilon$ , let  $\tilde{\tau}_\epsilon = 0$  and  $\tilde{C}_\epsilon = C$ , as well as  $\mathbf{v}_j(\epsilon) := \epsilon$  for  $j = 1, \dots, d$ .
- At any node  $v$ , given  $(\tau_{v'}, C_{v'}, \mathbf{v}_j(v'))_{v' \sqsubseteq v}$  (i.e., given  $(\tilde{J}_{v'}, \tilde{S}_{v'}, \tilde{T}_{v'})_{v' \sqsubseteq v}$ ) define:

$$\tilde{T}_v = \min_{1 \leq j \leq d} T_{\mathbf{v}_j(v)}'^j, \quad \tilde{J}_v := \operatorname{argmin}_{1 \leq j \leq d} T_{\mathbf{v}_j(v)}'^j, \quad \tilde{S}_v = S_{\mathbf{v}_j(v)}'^j \quad (15)$$

as well as

$$\mathbf{v}_j(va) = \begin{cases} \mathbf{v}_j(v)a & \text{if } j = \tilde{J}_v \\ \mathbf{v}_j(v) & \text{else.} \end{cases} \quad (16)$$

Finally, for every  $\lambda \in \mathbf{R}^+$ , define  $\tilde{\Pi}_\lambda$  and  $\tilde{K}_\lambda$  as before from  $\tilde{\Pi}$ . This construction is illustrated in Figure 3.

**Computation of  $\mathbb{E}[\tilde{K}_\lambda]$ .** Now, it can be seen that the partition  $\tilde{\Pi}_\lambda$  is a rectangular grid which is the “product” of the partitions  $\Pi'^j$  of the intervals  $C^j$ ,  $1 \leq j \leq d$ . Indeed, let  $x \in [0, 1]^d$ , and let  $\tilde{C}_\lambda(x)$  be the cell in  $\tilde{\Pi}_\lambda$  that contains  $x$ ; we need to show that  $\tilde{C}_\lambda(x) = \prod_{j=1}^d C_{\mathbf{v}_j(v)}'^j(x)$ , where  $C_{\mathbf{v}_j(v)}'^j(x)$  is the subinterval of  $C^j$  in the partition  $\Pi'^j$  that contains  $x_j$ . The proof proceeds in several steps:

- First, Equation (15) shows that, for every node  $v$ , we have  $\tilde{C}_v = \prod_{1 \leq j \leq d} C_{\mathbf{v}_j(v)}'^j$ , since the successive splits on the  $j$ -th coordinate of  $\tilde{C}_v$  are precisely the ones of  $C_{\mathbf{v}_j(v)}'^j$ .

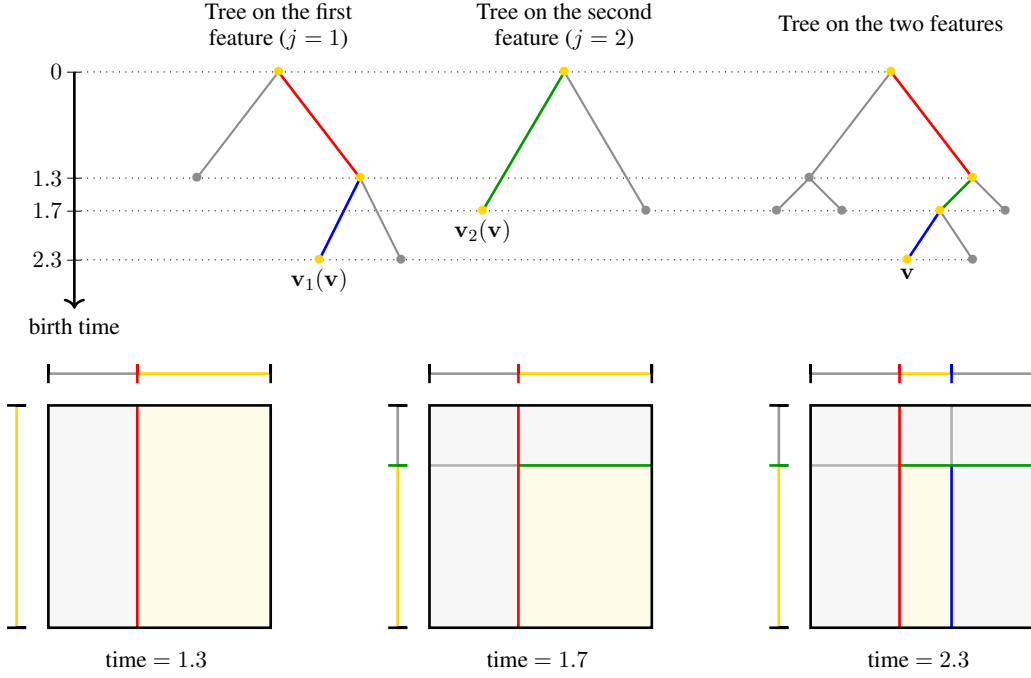


Figure 3: Modified construction in dimension two. At the top, from left to right: trees associated to partitions  $\Pi'^1$ ,  $\Pi'^2$  and  $\tilde{\Pi}$  respectively. At the bottom, from left to right: successive splits in  $\tilde{\Pi}$  leading to the leaf  $\mathbf{v}$  (depicted in yellow).

- Second, it follows from Equation (15) that  $\tilde{T}_{\mathbf{v}} = \min_{1 \leq j \leq d} T_{\mathbf{v}_j(\mathbf{v})}^{lj}$ ; in addition, since the cell  $C_{\mathbf{v}}$  is formed when its last split is performed,  $\tilde{\tau}_{\mathbf{v}} = \max_{1 \leq j \leq d} \tau_{\mathbf{v}_j(\mathbf{v})}^{lj}$ .
- Now, let  $\tilde{\mathbf{v}}$  be the node such that  $\tilde{C}_{\tilde{\mathbf{v}}} = \tilde{C}_{\lambda}(x)$ , and  $\mathbf{v}^{lj}$  be such that  $C_{\mathbf{v}^{lj}}^{lj} = C_{\lambda}^{lj}(x_j)$ . By the first point, it suffices to show that  $\mathbf{v}_j(\tilde{\mathbf{v}}) = \mathbf{v}_j'$  for  $j = 1, \dots, d$ .
- Observe that  $\tilde{\mathbf{v}}$  (resp.  $\mathbf{v}_j'$ ) is characterized by the fact that  $x \in \tilde{C}_{\tilde{\mathbf{v}}}$  and  $\tilde{\tau}_{\tilde{\mathbf{v}}} \leq \lambda < \tilde{T}_{\tilde{\mathbf{v}}}$  (resp.  $x_j \in C_{\mathbf{v}_j'}^{lj}$  and  $\tau_{\mathbf{v}_j'}^{lj} \leq \lambda < T_{\mathbf{v}_j'}^{lj}$ ). But since  $\tilde{C}_{\tilde{\mathbf{v}}} = \prod_{1 \leq j \leq d} C_{\mathbf{v}_j(\tilde{\mathbf{v}})}^{lj}$  (first point),  $x \in \tilde{C}_{\tilde{\mathbf{v}}}$  implies  $x_j \in C_{\mathbf{v}_j(\tilde{\mathbf{v}})}^{lj}$ . Likewise, since  $\tilde{\tau}_{\tilde{\mathbf{v}}} = \max_{1 \leq j \leq d} \tau_{\mathbf{v}_j(\tilde{\mathbf{v}})}^{lj}$  and  $\tilde{T}_{\tilde{\mathbf{v}}} = \min_{1 \leq j \leq d} T_{\mathbf{v}_j(\tilde{\mathbf{v}})}^{lj}$  (second point),  $\tilde{\tau}_{\tilde{\mathbf{v}}} \leq \lambda < \tilde{T}_{\tilde{\mathbf{v}}}$  implies  $\tau_{\mathbf{v}_j(\tilde{\mathbf{v}})}^{lj} \leq \lambda < T_{\mathbf{v}_j(\tilde{\mathbf{v}})}^{lj}$ . Since these properties characterize  $\mathbf{v}^{lj}$ , we have  $\mathbf{v}_j(\tilde{\mathbf{v}}) = \mathbf{v}_j'$ , which concludes the proof.

Hence, the partition  $\tilde{\Pi}_{\lambda}$  is the product of the partitions  $\Pi'^j = \Phi_{C^j}((E_{\mathbf{v}}^j, U_{\mathbf{v}}^j)_{\lambda})$  of the intervals  $C^j$ ,  $1 \leq j \leq d$ , which are independent Mondrians distributed as  $\text{MP}(\lambda, C^j)$ . By Fact 1, the partition defined by a Mondrian  $\text{MP}(\lambda, C^j)$  is distributed as the one formed by the intervals defined by a Poisson point process on  $C^j$  of intensity  $\lambda$ , so that the expected number of cells in such a partition is  $1 + \lambda|C^j|$ . Since  $\tilde{\Pi}_{\lambda}$  is a “product” of such independent partitions, we have:

$$\mathbb{E}[\tilde{K}_{\lambda}] = \prod_{j=1}^d (1 + \lambda|C^j|). \quad (17)$$

**Equality of  $\mathbb{E}[K_{\lambda}]$  and  $\mathbb{E}[\tilde{K}_{\lambda}]$ .** In order to establish Proposition 2, it is thus sufficient to prove that  $\mathbb{E}[K_{\lambda}] = \mathbb{E}[\tilde{K}_{\lambda}]$ . First, note that, since the number of cells in a partition is one plus the



number of splits (as each split increases the number of cells by one)

$$K_\lambda = 1 + \sum_{\mathbf{v} \in \{0,1\}^*} \mathbf{1}(T_{\mathbf{v}} \leq \lambda)$$

so that

$$\mathbb{E}[K_\lambda] = 1 + \sum_{\mathbf{v} \in \{0,1\}^*} \mathbb{P}(T_{\mathbf{v}} \leq \lambda) \quad (18)$$

and, likewise,

$$\mathbb{E}[\tilde{K}_\lambda] = 1 + \sum_{\mathbf{v} \in \{0,1\}^*} \mathbb{P}(\tilde{T}_{\mathbf{v}} \leq \lambda). \quad (19)$$

Therefore, it suffices to show that  $\mathbb{P}(T_{\mathbf{v}} \leq \lambda) = \mathbb{P}(\tilde{T}_{\mathbf{v}} \leq \lambda)$  for every  $\mathbf{v} \in \{0,1\}^*$  and  $\lambda \geq 0$ , i.e. that  $T_{\mathbf{v}}$  and  $\tilde{T}_{\mathbf{v}}$  have the same distribution for every  $\mathbf{v}$ .

In order to establish this, we show that, for every  $\mathbf{v} \in \{0,1\}^*$ , the conditional distribution of  $(\tilde{T}_{\mathbf{v}}, \tilde{J}_{\mathbf{v}}, \tilde{S}_{\mathbf{v}})$  given  $\tilde{\mathcal{F}}_{\mathbf{v}} = \sigma((\tilde{T}_{\mathbf{v}'}, \tilde{J}_{\mathbf{v}'}, \tilde{S}_{\mathbf{v}'}), \mathbf{v}' \sqsubset \mathbf{v})$  has the same form as the conditional distribution of  $(T_{\mathbf{v}}, J_{\mathbf{v}}, S_{\mathbf{v}})$  given  $\mathcal{F}_{\mathbf{v}} = \sigma((T_{\mathbf{v}'}, J_{\mathbf{v}'}, S_{\mathbf{v}'}), \mathbf{v}' \sqsubset \mathbf{v})$ , in the sense that there exists a family of conditional distributions  $(\Psi_{\mathbf{v}})_{\mathbf{v}}$  such that, for every  $\mathbf{v}$ , the conditional distribution of  $(T_{\mathbf{v}}, J_{\mathbf{v}}, S_{\mathbf{v}})$  given  $\mathcal{F}_{\mathbf{v}} = \sigma((T_{\mathbf{v}'}, J_{\mathbf{v}'}, S_{\mathbf{v}'}), \mathbf{v}' \sqsubset \mathbf{v})$  is  $\Psi_{\mathbf{v}}(\cdot | (T_{\mathbf{v}'}, J_{\mathbf{v}'}, S_{\mathbf{v}'}), \mathbf{v}' \sqsubset \mathbf{v})$  and the conditional distribution of  $(\tilde{T}_{\mathbf{v}}, \tilde{J}_{\mathbf{v}}, \tilde{S}_{\mathbf{v}})$  given  $\tilde{\mathcal{F}}_{\mathbf{v}} = \sigma((\tilde{T}_{\mathbf{v}'}, \tilde{J}_{\mathbf{v}'}, \tilde{S}_{\mathbf{v}'}), \mathbf{v}' \sqsubset \mathbf{v})$  is  $\Psi_{\mathbf{v}}(\cdot | (\tilde{T}_{\mathbf{v}'}, \tilde{J}_{\mathbf{v}'}, \tilde{S}_{\mathbf{v}'}), \mathbf{v}' \sqsubset \mathbf{v})$ .

First, recall that the variables  $(E_{\mathbf{v}'}^j, U_{\mathbf{v}'}^j)_{\mathbf{v}' \in \{0,1\}^*, 1 \leq j \leq d}$  are independent, so that  $(E_{\mathbf{v}}^j, U_{\mathbf{v}}^j)_{1 \leq j \leq d}$  is independent from  $\mathcal{F}_{\mathbf{v}} \subseteq \sigma((E_{\mathbf{v}'}^j, U_{\mathbf{v}'}^j)_{\mathbf{v}' \sqsubset \mathbf{v}, 1 \leq j \leq d})$ . As a result, conditionally on  $\mathcal{F}_{\mathbf{v}}$ , the  $E_{\mathbf{v}}^j, U_{\mathbf{v}}^j$ ,  $1 \leq j \leq d$  are independent variables with  $E_{\mathbf{v}}^j \sim \text{Exp}(1)$  and  $U_{\mathbf{v}}^j \sim \mathcal{U}([0,1])$ . Also, recall that if  $T_1, \dots, T_d$  are independent exponential random variables of intensities  $\lambda_1, \dots, \lambda_d$ , and if  $T = \min_{1 \leq j \leq d} T_j$  and  $J = \text{argmin}_{1 \leq j \leq d} T_j$ , then  $\mathbb{P}(J = j) = \lambda_j / \sum_{j'=1}^d \lambda_{j'}$ ,  $T \sim \text{Exp}(\sum_{j=1}^d \lambda_j)$  and  $J$  and  $T$  are independent. Hence, conditionally on  $\mathcal{F}_{\mathbf{v}}$ ,  $T_{\mathbf{v}} - \tau_{\mathbf{v}} = \min_{1 \leq j \leq d} E_{\mathbf{v}}^j / |C_{\mathbf{v}}^j| \sim \text{Exp}(\sum_{j=1}^d |C_{\mathbf{v}}^j|) = \text{Exp}(|C_{\mathbf{v}}|)$ ,  $J_{\mathbf{v}} := \text{argmin}_{1 \leq j \leq d} E_{\mathbf{v}}^j / |C_{\mathbf{v}}^j|$  equals  $j$  with probability  $|C_{\mathbf{v}}^j| / |C_{\mathbf{v}}|$ ,  $T_{\mathbf{v}}, J_{\mathbf{v}}$  are independent and  $(S_{\mathbf{v}} | T_{\mathbf{v}}, J_{\mathbf{v}}) \sim \mathcal{U}(C_{\mathbf{v}}^{J_{\mathbf{v}}})$ .

Now consider the conditional distribution of  $(\tilde{T}_{\mathbf{v}}, \tilde{J}_{\mathbf{v}}, \tilde{S}_{\mathbf{v}})$  given  $\tilde{\mathcal{F}}_{\mathbf{v}}$ . Let  $(\mathbf{v}_v)_{v \in \mathbb{N}}$  be a path in  $\{0,1\}^*$  from the root:  $\mathbf{v}_0 := \epsilon$ ,  $\mathbf{v}_{v+1}$  is a child of  $\mathbf{v}_v$  for  $v \in \mathbb{N}$ , and  $\mathbf{v}_v \sqsubset \mathbf{v}$  for  $0 \leq v \leq \text{depth}(\mathbf{v})$ . Define for  $v \in \mathbb{N}$ ,  $E_v^j = E_{\mathbf{v}_v}^j$  and  $U_v^j = U_{\mathbf{v}_v}^j$  if  $\mathbf{v}_{v+1}$  is the left child of  $\mathbf{v}_v$ , and  $1 - U_v^j$  otherwise. Then, the variables  $(E_v^j, U_v^j)_{v \in \mathbb{N}, 1 \leq j \leq d}$  are independent, with  $E_v^j \sim \text{Exp}(1)$ ,  $U_v^j \sim \mathcal{U}([0,1])$ , so that the hypotheses of Technical Lemma 1 apply. In addition, note that, with the notations of Technical Lemma 1, a simple induction shows that  $\tilde{J}_v = \tilde{J}_{\mathbf{v}_v}$ ,  $\tilde{T}_v = \tilde{T}_{\mathbf{v}_v}$ ,  $\tilde{U}_v = \tilde{U}_{\mathbf{v}_v}$  and  $L_v^j = |\tilde{C}_{\mathbf{v}_v}^j|$ , so that  $\tilde{\mathcal{F}}_v = \tilde{\mathcal{F}}_{\mathbf{v}_v}$ . Applying Technical Lemma 1 for  $v = \text{depth}(\mathbf{v})$  (so that  $\mathbf{v}_v = \mathbf{v}$ ) therefore gives the following: conditionally on  $\tilde{\mathcal{F}}_{\mathbf{v}}$ ,  $\tilde{T}_{\mathbf{v}}, \tilde{J}_{\mathbf{v}}, \tilde{U}_{\mathbf{v}}$  are independent,  $\tilde{T}_{\mathbf{v}} - \tilde{\tau}_{\mathbf{v}} \sim \text{Exp}(|\tilde{C}_{\mathbf{v}}^j|)$ ,  $\mathbb{P}(\tilde{J}_{\mathbf{v}} = j | \tilde{\mathcal{F}}_{\mathbf{v}}) = |\tilde{C}_{\mathbf{v}}^j| / (\sum_{j'=1}^d |\tilde{C}_{\mathbf{v}}^{j'}|)$  and  $\tilde{U}_{\mathbf{v}} \sim \mathcal{U}([0,1])$ , so that  $(\tilde{S}_{\mathbf{v}} | \tilde{\mathcal{F}}_{\mathbf{v}}, \tilde{T}_{\mathbf{v}}, \tilde{J}_{\mathbf{v}}) \sim \mathcal{U}(\tilde{C}_{\mathbf{v}}^{\tilde{J}_{\mathbf{v}}})$ .

Hence, we have proven that, for every  $\mathbf{v} \in \{0,1\}^*$ , the conditional distribution of  $(T_{\mathbf{v}}, J_{\mathbf{v}}, S_{\mathbf{v}})$  given  $\mathcal{F}_{\mathbf{v}} = \sigma((T_{\mathbf{v}'}, J_{\mathbf{v}'}, S_{\mathbf{v}'}), \mathbf{v}' \sqsubset \mathbf{v})$  has the same form as that of  $(T_{\mathbf{v}}, J_{\mathbf{v}}, S_{\mathbf{v}})$  given  $\tilde{\mathcal{F}}_{\mathbf{v}} = \sigma((\tilde{T}_{\mathbf{v}'}, \tilde{J}_{\mathbf{v}'}, \tilde{S}_{\mathbf{v}'}), \mathbf{v}' \sqsubset \mathbf{v})$ . By induction on  $\mathbf{v}$ , since  $\mathcal{F}_{\epsilon} = \tilde{\mathcal{F}}_{\epsilon}$  is the trivial  $\sigma$ -algebra, this implies that the distribution of  $T_{\mathbf{v}}$  is the same as that of  $\tilde{T}_{\mathbf{v}}$  for every  $\mathbf{v}$ . Plugging this into Equations (18) and (19) and combining it with (17) completes the proof of Proposition 2.  $\square$

**Technical Lemma 1.** Let  $(E_v^j, U_v^j)_{v \in \mathbb{N}^*, 1 \leq j \leq d}$  be a family of independent random variables, with  $U_v^j \sim \mathcal{U}([0,1])$  and  $E_v^j \sim \text{Exp}(1)$ . Let  $a_1, \dots, a_d > 0$ . For  $1 \leq j \leq d$ , define the sequence  $(T_v^j, L_v^j)_{v \in \mathbb{N}}$  as follows:

- $L_0^j = a_j$ ,  $T_0^j = \frac{E_0^j}{a_j}$ ,

- for  $v \in \mathbf{N}$ ,  $L_{v+1}^j = U_v^j L_v^j$ ,  $T_{v+1}^j = T_v^j + \frac{E_{v+1}^j}{L_{v+1}^j}$ .

Define recursively the variables  $\tilde{V}_v^j$  ( $v \in \mathbf{N}$ ,  $1 \leq j \leq d$ ) as well as  $\tilde{J}_v, \tilde{T}_v, \tilde{U}_v$  ( $v \in \mathbf{N}$ ) as follows:

- $\tilde{V}_0^j = 0$  for  $j = 1, \dots, d$ .
- for  $v \in \mathbf{N}$ , given  $\tilde{V}_v^j$  ( $1 \leq j \leq d$ ), denoting  $\tilde{T}_v^j = T_{\tilde{V}_v^j}^j$  and  $\tilde{U}_v^j = U_{\tilde{V}_v^j}^j$ , set

$$\tilde{J}_v = \underset{1 \leq j \leq d}{\operatorname{argmin}} \tilde{T}_v^j, \quad \tilde{T}_v = \min_{1 \leq j \leq d} \tilde{T}_v^j = \tilde{T}_v^{\tilde{J}_v}, \quad \tilde{U}_v = \tilde{U}_v^{\tilde{J}_v}, \quad \tilde{V}_{v+1}^j = \tilde{V}_v^j + \mathbf{1}(\tilde{J}_v = j). \quad (20)$$

Then, the conditional distribution of  $(\tilde{J}_v, \tilde{T}_v, \tilde{U}_v)$  given  $\mathcal{F}_v = \sigma((\tilde{J}_{v'}, \tilde{T}_{v'}, \tilde{U}_{v'}), 0 \leq v' < v)$  is the following (denoting  $\tilde{L}_v^j = L_{\tilde{V}_v^j}^j$ ):  $\tilde{J}_v, \tilde{T}_v, \tilde{U}_v$  are independent,  $\mathbb{P}(\tilde{J}_v = j \mid \mathcal{F}_v) = \tilde{L}_v^j / (\sum_{j'=1}^d \tilde{L}_v^{j'})$ ,  $\tilde{T}_v - \tilde{T}_{v-1} \sim \operatorname{Exp}(\sum_{j=1}^d \tilde{L}_v^j)$  (with the convention  $\tilde{T}_{-1} = 0$ ) and  $\tilde{U}_v \sim \mathcal{U}([0, 1])$ .

*Proof of Technical Lemma 1.* We show by induction on  $v \in \mathbf{N}$  the following property: conditionally on  $\mathcal{F}_v$ ,  $(\tilde{T}_v^j, \tilde{U}_v^j)_{1 \leq j \leq d}$  are independent,  $\tilde{T}_v^j - \tilde{T}_{v-1} \sim \operatorname{Exp}(\tilde{L}_v^j)$  and  $\tilde{U}_v^j \sim \mathcal{U}([0, 1])$ .

**Initialization** For  $v = 0$  (with  $\mathcal{F}_0$  the trivial  $\sigma$ -algebra), since  $\tilde{V}_0^j = 0$  we have  $\tilde{T}_0^j = E_0^j / a_j \sim \operatorname{Exp}(a_j) = \operatorname{Exp}(L_0^j)$ ,  $\tilde{U}_0^j = U_0^j \sim \mathcal{U}([0, 1])$  and these random variables are independent.

**Inductive step** Let  $v \in \mathbf{N}$ , and assume the property is true up to step  $v$ . Conditionally on  $\mathcal{F}_{v+1}$ , i.e. on  $\mathcal{F}_v, \tilde{T}_v, \tilde{J}_v, \tilde{U}_v$ , we have:

- for  $j \neq \tilde{J}_v$ , the variables  $\tilde{T}_{v+1}^j - \tilde{T}_{v-1} = \tilde{T}_v^j - \tilde{T}_{v-1}$  are independent  $\operatorname{Exp}(\tilde{L}_v^j) = \operatorname{Exp}(\tilde{L}_{v+1}^j)$  random variables (when conditioned only on  $\mathcal{F}_v$ , by the induction hypothesis), conditioned on  $\tilde{T}_{v+1}^j - \tilde{T}_{v-1} \geq \tilde{T}_v - \tilde{T}_{v-1}$ , so by the memory-less property of exponential random variables  $\tilde{T}_{v+1}^j - \tilde{T}_v = (\tilde{T}_{v+1}^j - \tilde{T}_{v-1}) - (\tilde{T}_v - \tilde{T}_{v-1}) \sim \operatorname{Exp}(\tilde{L}_{v+1}^j)$  (and those variables are independent).
- for  $j \neq \tilde{J}_v$ , the variables  $\tilde{U}_{v+1}^j = \tilde{U}_v^j$  are independent  $\mathcal{U}([0, 1])$  random variables (conditionally on  $\mathcal{F}_v$ ), conditioned on the independent variables  $\tilde{T}_v, \tilde{J}_v, \tilde{U}_v$ , so they remain independent  $\mathcal{U}([0, 1])$  random variables.
- $(\tilde{T}_{v+1}^{\tilde{J}_v} - \tilde{T}_v, \tilde{U}_{v+1}^{\tilde{J}_v}) = (E_{\tilde{V}_{v+1}^{\tilde{J}_v}}^{\tilde{J}_v} / \tilde{L}_{v+1}^{\tilde{J}_v}, U_{\tilde{V}_{v+1}^{\tilde{J}_v}}^{\tilde{J}_v})$  is distributed, conditionally on  $\mathcal{F}_{v+1}$ , i.e. on  $\tilde{J}_v, \tilde{T}_v, \tilde{V}_{v+1}^{\tilde{J}_v}, \tilde{L}_{v+1}^{\tilde{J}_v}$ , as  $\operatorname{Exp}(\tilde{L}_{v+1}^{\tilde{J}_v}) \otimes \mathcal{U}([0, 1])$ , and independent of  $(\tilde{T}_{v+1}^j, \tilde{U}_{v+1}^j)_{j \neq \tilde{J}_v}$ .

This completes the proof by induction.

Let  $v \in \mathbf{N}$ . We have established that, conditionally on  $\mathcal{F}_v$ , the variables  $(\tilde{T}_v^j, \tilde{U}_v^j)_{1 \leq j \leq d}$  are independent, with  $\tilde{T}_v^j - \tilde{T}_{v-1} \sim \operatorname{Exp}(\tilde{L}_v^j)$  and  $\tilde{U}_v^j \sim \mathcal{U}([0, 1])$ . In particular, conditionally on  $\mathcal{F}_v$ ,  $\tilde{U}_v$  is independent from  $(\tilde{J}_v, \tilde{T}_v)$ ,  $\tilde{U}_v \sim \mathcal{U}([0, 1])$ , and (by the property of the minimum of independent exponential random variables)  $\tilde{J}_v$  is independent of  $\tilde{T}_v$ ,  $\tilde{T}_v \sim \operatorname{Exp}(\sum_{j=1}^d \tilde{L}_v^j)$  and  $\mathbb{P}(\tilde{J}_v = j \mid \mathcal{F}_v) = \tilde{L}_v^j / (\sum_{j'=1}^d \tilde{L}_v^{j'})$ . This concludes the proof.  $\square$

## 7.2 Proof of Theorem 1: Consistency of Mondrian Forests

Recall that a Mondrian Forest estimate with lifetime parameter  $\lambda$ , is defined, for all  $x \in [0, 1]^d$ , by

$$\hat{f}_{\lambda, n}^{(M)}(x, Z^{(M)}) = \frac{1}{M} \sum_{m=1}^M \hat{f}_{\lambda, n}(x, Z_m). \quad (21)$$

where  $(\hat{f}_{\lambda,n}(\cdot, Z_m))$  is a Mondrian Tree estimate, grown independently of the dataset  $\mathcal{D}_n$  with the extra randomness  $Z_m$ . First, note that, by Jensen's inequality,

$$\begin{aligned} R(\hat{f}_{\lambda,n}^{(M)}) &= \mathbb{E}_{(X, Z^{(M)})}[(\hat{f}_{\lambda,n}^{(M)}(X, Z^{(M)}) - f(X))^2] \\ &\leq \frac{1}{M} \sum_{m=1}^M \mathbb{E}_{(X, Z_m)}[(\hat{f}_{\lambda,n}(X, Z_m) - f(X))^2] \\ &\leq \mathbb{E}_{(X, Z_1)}[(\hat{f}_{\lambda,n}(X, Z_1) - f(X))^2], \end{aligned}$$

since each Mondrian tree has the same distribution. Therefore, it is sufficient to prove that a single Mondrian tree is consistent. Now, since Mondrian partitions are independent of the data set  $\mathcal{D}_n$ , we can apply Theorem 4.2 in Györfi et al. (2002), which states that a Mondrian tree estimate is consistent if

- (i)  $D_\lambda(X) \rightarrow 0$  in probability, as  $n \rightarrow \infty$ ,
- (ii)  $K_\lambda/n \rightarrow \infty$  in probability, as  $n \rightarrow \infty$ ,

where  $D_\lambda(X)$  is the diameter of the cell of the Mondrian tree that contains  $X$ , and  $K_\lambda$  is the number of cells in the Mondrian tree. Note that the initial assumptions in Theorem 4.2 in Györfi et al. (2002) contains deterministic convergence, but can be relaxed to convergences in probability by a close inspection of the proof. In the sequel, we prove that an individual Mondrian tree satisfies (i) and (ii) which will conclude the proof. To prove (i), just note that, according to Corollary 1,

$$\mathbb{E}[D_\lambda(X)^2] = \mathbb{E}[\mathbb{E}[D_\lambda(X)^2 | X]] \leq \frac{4d}{\lambda^2},$$

which tends to zero, since  $\lambda = \lambda_n \rightarrow \infty$ , as  $n \rightarrow \infty$ . Thus, (i) condition holds. Now, to prove (ii), observe that

$$\mathbb{E}\left[\frac{K_\lambda}{n}\right] = \frac{(1 + \lambda)^d}{n},$$

which tends to zero since  $\lambda_n^d/n \rightarrow 0$  by assumption, as  $n \rightarrow \infty$ .

### 7.3 Proof of Theorem 2: Minimax rates for Mondrian Forests in regression

Recall that the Mondrian Forest estimate at  $x$  is given by

$$\hat{f}_{\lambda,n}^{(M)}(x) = \frac{1}{M} \sum_{m=1}^M \hat{f}_{\lambda,n}^{(m)}(x).$$

By the convexity of the function  $y \mapsto (y - f(x))^2$  for any  $x \in [0, 1]^d$ , we have

$$R(\hat{f}_{\lambda,n}^{(M)}) \leq \frac{1}{M} \sum_{m=1}^M R(\hat{f}_{\lambda,n}^{(m)}) = R(\hat{f}_{\lambda,n}^{(1)}),$$

since the random trees classifiers  $\hat{f}_{\lambda,n}^{(m)}$  ( $1 \leq m \leq M$ ) have the same distribution. Hence, it suffices to prove Theorem 2 for a single tree: in the following, we assume that  $M = 1$ , and consider the random estimator  $\hat{f}_{\lambda,n}^{(1)}$  associated to a tree partition  $\Pi_\lambda \sim \text{MP}(\lambda, [0, 1]^d)$ . Note that the following analysis is done for any fixed  $\lambda$ . We will allow  $\lambda$  to depend on  $n$  at the end of the proof.

We now establish a *bias-variance* decomposition of the risk of a Mondrian tree, akin to the one stated for purely random forests by [Genuer \(2012\)](#). Denote  $\bar{f}_\lambda^{(1)}(x) := \mathbb{E}[f(X)|X \in C_\lambda(x)]$  (which only depends on the random partition  $\Pi_\lambda$ ) for every  $x$  in the support of  $\mu$ . Note that, given  $\Pi_\lambda$ ,  $\bar{f}_\lambda^{(1)}$  is the orthogonal projection of  $f$  in  $L^2([0, 1]^d, \mu)$  on the subspace of functions constant on the cells of  $\Pi_\lambda$ . Since, given  $\mathcal{D}_n$ ,  $\hat{f}_{\lambda,n}^{(1)}$  belongs to this subspace, we have conditionally on  $\Pi_\lambda, \mathcal{D}_n$ :

$$\mathbb{E}_X[(f(X) - \hat{f}_{\lambda,n}^{(1)}(X))^2] = \mathbb{E}_X[(f(X) - \bar{f}_\lambda^{(1)}(X))^2] + \mathbb{E}_X[(\bar{f}_\lambda^{(1)}(X) - \hat{f}_{\lambda,n}^{(1)}(X))^2],$$

which gives the following decomposition of the risk of  $\hat{f}_{\lambda,n}^{(1)}$  by taking the expectation over  $\Pi_\lambda, \mathcal{D}_n$ :

$$R(\hat{f}_{\lambda,n}^{(1)}) = \mathbb{E}[(f(X) - \bar{f}_\lambda^{(1)}(X))^2] + \mathbb{E}[(\bar{f}_\lambda^{(1)}(X) - \hat{f}_{\lambda,n}^{(1)}(X))^2]. \quad (22)$$

The first term of the sum, the *bias*, measures how close  $f$  is to its best approximation  $\bar{f}_\lambda^{(1)}$  that is constant on the leaves of  $\Pi_\lambda$  (on average over  $\Pi_\lambda$ ). The second term, the *variance*, measures how well the expected value  $\bar{f}_\lambda^{(1)}(x) = \mathbb{E}[f(X) | X \in C_\lambda(x)]$  (i.e. the optimal label on the leaf  $C_\lambda(x)$ ) is estimated by the empirical average  $\hat{f}_{\lambda,n}^{(1)}(x)$  (on average over the sample  $\mathcal{D}_n$  and the partition  $\Pi_\lambda$ ).

Note that our bias-variance decomposition (22) holds for the estimation risk *integrated over the hypercube*  $[0, 1]^d$ , and not for the point-wise estimation risk. This is because in general, we have  $\mathbb{E}_{\mathcal{D}_n}[\hat{f}_{\lambda,n}^{(1)}(x)] \neq \bar{f}_\lambda^{(1)}(x)$ : indeed, the cell  $C_\lambda(x)$  may contain no data point in  $\mathcal{D}_n$ , in which case the estimate  $\hat{f}_{\lambda,n}^{(1)}(x)$  equals 0. It seems that a similar difficulty occurs for the decomposition in [Genuer \(2012\)](#); [Arlot and Genuer \(2014\)](#), which should only hold for the integrated risk.

**Bias term.** For each  $x \in [0, 1]^d$  in the support of  $\mu$ , we have

$$\begin{aligned} |f(x) - \bar{f}_\lambda^{(1)}(x)| &= \left| \frac{1}{\mu(C_\lambda(x))} \int_{C_\lambda(x)} (f(x) - f(z))\mu(dz) \right| \\ &\leq \sup_{z \in C_\lambda(x)} |f(x) - f(z)| \\ &\leq L \sup_{z \in C_\lambda(x)} \|x - z\|_2 \quad (\text{since } f \text{ is } L\text{-Lipschitz}) \\ &= LD_\lambda(x), \end{aligned}$$

where  $D_\lambda(x)$  is the  $\ell^2$ -diameter of  $C_\lambda(x)$ . By Corollary 1, this implies

$$\mathbb{E}[(f(x) - \bar{f}_\lambda^{(1)}(x))^2] \leq L^2 \mathbb{E}[D_\lambda(x)^2] \leq \frac{4dL^2}{\lambda^2}. \quad (23)$$

Integrating the bound (23) with respect to  $\mu$  yields the following bound on the integrated bias:

$$\mathbb{E}[(f(X) - \bar{f}_\lambda^{(1)}(X))^2] \leq \frac{4dL^2}{\lambda^2}. \quad (24)$$

**Variance term.** In order to bound the variance term, we make use of Proposition 2 in [Arlot and Genuer \(2014\)](#): if  $\Pi$  is a random tree partition of the unit cube in  $k$  cells (with  $k \in \mathbb{N}^*$  deterministic) formed independently of the training data  $\mathcal{D}_n$ , we have

$$\mathbb{E}[(\bar{f}_\Pi(X) - \hat{f}_\Pi(X))^2] \leq \frac{k}{n} (2\sigma^2 + 9\|f\|_\infty^2). \quad (25)$$

Note that the Proposition 2 in [Arlot and Genuer \(2014\)](#), stated in the case where the noise variance is constant, can be relaxed to lead to inequality (25), where the noise variance is just upper-bounded, based on Proposition 1 in [Arlot \(2008\)](#). For every  $k \in \mathbf{N}^*$ , applying the upper bound (25) to the random partition  $\Pi_\lambda \sim \text{MP}(\lambda, [0, 1]^d)$  conditionally on the event  $\{K_\lambda = k\}$ , and summing over  $k$ , we get

$$\begin{aligned} \mathbb{E}[(\bar{f}_\lambda^{(1)}(X) - \hat{f}_{\lambda,n}^{(1)}(X))^2] &= \sum_{k=1}^{\infty} \mathbb{P}(K_\lambda = k) \mathbb{E}[(\bar{f}_\lambda^{(1)}(X) - \hat{f}_{\lambda,n}^{(1)}(X))^2 | K_\lambda = k] \\ &\leq \sum_{k=1}^{\infty} \mathbb{P}(K_\lambda = k) \frac{k}{n} (2\sigma^2 + 9\|f\|_\infty^2) \\ &= \frac{\mathbb{E}[K_\lambda]}{n} (2\sigma^2 + 9\|f\|_\infty^2). \end{aligned}$$

Then, applying Proposition 2 gives an upper bound of the variance term:

$$\mathbb{E}[(\bar{f}_\lambda^{(1)}(X) - \hat{f}_{\lambda,n}^{(1)}(X))^2] \leq \frac{(1+\lambda)^d}{n} (2\sigma^2 + 9\|f\|_\infty^2). \quad (26)$$

Combining the bounds (24) and (26) yields

$$R(\hat{f}_{\lambda,n}^{(1)}) \leq \frac{4dL^2}{\lambda^2} + \frac{(1+\lambda)^d}{n} (2\sigma^2 + 9\|f\|_\infty^2),$$

which concludes the proof.

#### 7.4 Proof of Lemma 1

Let  $\Pi_\lambda^{(1)}$  be the Mondrian partition of  $[0, 1]$  used to construct the randomized estimator  $\hat{f}_{\lambda,n}^{(1)}$ . Denote by  $\bar{f}_\lambda^{(1)}$  the random function  $\bar{f}_\lambda^{(1)}(x) = \mathbb{E}_X[f(X) | X \in C_\lambda(x)]$ , and define  $\tilde{f}_\lambda(x) = \mathbb{E}[\bar{f}_\lambda^{(1)}(x)]$  (which is deterministic). For the seek of clarity, we will drop the exponent “(1)” in all notations, keeping in mind that we consider only one particular Mondrian partition, whose associated Mondrian Tree estimate is denoted by  $\hat{f}_{\lambda,n}$ . Recall the bias-variance decomposition (22) for Mondrian trees:

$$R(\hat{f}_{\lambda,n}^{(1)}) = \mathbb{E}[(f(X) - \bar{f}_\lambda(X))^2] + \mathbb{E}[(\bar{f}_\lambda(X) - \tilde{f}_\lambda(X))^2]. \quad (27)$$

We will provide lower bounds for the first term (the bias, depending on  $\lambda$ ) and the second (the variance, depending on both  $\lambda$  and  $n$ ), which will lead to the stated lower bound on the risk, valid for every value of  $\lambda$ .

**Lower bound on the bias.** As we will see, the point-wise bias  $\mathbb{E}[(\bar{f}_\lambda(x) - f(x))^2]$  can be computed explicitly given our assumptions. Let  $x \in [0, 1]$ . Since  $\tilde{f}_\lambda(x) = \mathbb{E}[\bar{f}_\lambda(x)]$ , we have

$$\mathbb{E}[(\bar{f}_\lambda(x) - f(x))^2] = \text{Var}(\bar{f}_\lambda(x)) + (\tilde{f}_\lambda(x) - f(x))^2. \quad (28)$$

By Proposition 1, the cell of  $x$  in  $\Pi_\lambda$  can be written as  $C_\lambda(x) = [L_\lambda(x), R_\lambda(x)]$ , with  $L_\lambda(x) = (x - \lambda^{-1}E_L) \vee 0$  and  $R_\lambda(x) = (x + \lambda^{-1}E_R) \wedge 1$ , where  $E_L, E_R$  are two independent  $\text{Exp}(1)$  random variables. Now, since  $X \sim \mathcal{U}([0, 1])$  and  $f(u) = 1 + u$ ,

$$\bar{f}_\lambda(x) = \frac{1}{R_\lambda(x) - L_\lambda(x)} \int_{L_\lambda(x)}^{R_\lambda(x)} (1 + u) du = 1 + \frac{L_\lambda(x) + R_\lambda(x)}{2}.$$

Since  $L_\lambda(x)$  and  $R_\lambda(x)$  are independent, we have

$$\text{Var}(\bar{f}_\lambda(x)) = \frac{\text{Var}(L_\lambda(x)) + \text{Var}(R_\lambda(x))}{4}.$$

In addition,

$$\text{Var}(R_\lambda(x)) = \text{Var}((x + \lambda^{-1}E_R) \wedge 1) = \text{Var}(x + \lambda^{-1}[E_R \wedge \lambda(1-x)]) = \lambda^{-2}\text{Var}(E_R \wedge [\lambda(1-x)])$$

Now, if  $E \sim \text{Exp}(1)$  and  $a \geq 0$ , we have

$$\begin{aligned} \mathbb{E}[E \wedge a] &= \int_0^a ue^{-u}du + a\mathbb{P}(E \geq a) = 1 - e^{-a} \\ \mathbb{E}[(E \wedge a)^2] &= \int_0^a u^2e^{-u}du + a^2\mathbb{P}(E \geq a) = 2(1 - (a+1)e^{-a}), \end{aligned} \quad (29)$$

so that

$$\text{Var}(E \wedge a) = \mathbb{E}[(E \wedge a)^2] - \mathbb{E}[E \wedge a]^2 = 1 - 2ae^{-a} - e^{-2a}.$$

The formula above gives the variances of  $R_\lambda(x)$  and  $L_\lambda(x)$  respectively:

$$\begin{aligned} \text{Var}(R_\lambda(x)) &= \lambda^{-2}(1 - 2\lambda(1-x)e^{-\lambda(1-x)} - e^{-2\lambda(1-x)}) \\ \text{Var}(L_\lambda(x)) &= \lambda^{-2}(1 - 2\lambda xe^{-\lambda x} - e^{-2\lambda x}), \end{aligned}$$

and thus

$$\text{Var}(\bar{f}_\lambda(x)) = \frac{1}{4\lambda^2}(2 - 2\lambda xe^{-\lambda x} - 2\lambda(1-x)e^{-\lambda(1-x)} - e^{-2\lambda x} - e^{-2\lambda(1-x)}). \quad (30)$$

In addition, the formula (29) yields

$$\begin{aligned} \mathbb{E}[R_\lambda(x)] &= x + \lambda^{-1}(1 - e^{-\lambda(1-x)}) \\ \mathbb{E}[L_\lambda(x)] &= x - \lambda^{-1}(1 - e^{-\lambda x}), \end{aligned}$$

and thus

$$\tilde{f}_\lambda(x) = 1 + \frac{\mathbb{E}[L_\lambda(x)] + \mathbb{E}[R_\lambda(x)]}{2} = 1 + x + \frac{1}{2\lambda}(e^{-\lambda x} - e^{-\lambda(1-x)}). \quad (31)$$

Combining (30) and (31) with the decomposition (28) gives

$$\mathbb{E}[(\bar{f}_\lambda(x) - f(x))^2] = \frac{1}{2\lambda^2} \left( 1 - \lambda xe^{-\lambda x} - \lambda(1-x)e^{-\lambda(1-x)} - e^{-\lambda} \right). \quad (32)$$

Integrating over  $X$ , we obtain

$$\begin{aligned} \mathbb{E}[(\bar{f}_\lambda(X) - f(X))^2] &= \frac{1}{2\lambda^2} \left( 1 - \int_0^1 \lambda xe^{-\lambda x} dx - \int_0^1 \lambda(1-x)e^{-\lambda(1-x)} dx - e^{-\lambda} \right) \\ &= \frac{1}{2\lambda^2} \left( 1 - 2 \times \frac{1}{\lambda} (1 - (\lambda+1)e^{-\lambda}) - e^{-\lambda} \right) \\ &= \frac{1}{2\lambda^2} \left( 1 - \frac{2}{\lambda} + e^{-\lambda} + \frac{2}{\lambda}e^{-\lambda} \right). \end{aligned} \quad (33)$$

Now, note that the bias  $\mathbb{E}[(\bar{f}_\lambda(X) - f(X))^2]$  is positive for  $\lambda \in \mathbf{R}_+^*$  (indeed, it is nonnegative, and non-zero since  $f$  is not piecewise constant). In addition, the expression (33) shows that it is continuous in  $\lambda$  on  $\mathbf{R}_+^*$ , and that it admits a limit  $\frac{1}{12}$  as  $\lambda \rightarrow 0$  (using the fact that  $e^{-\lambda} = 1 - \lambda + \frac{\lambda^2}{2} - \frac{\lambda^3}{6} + o(\lambda^3)$ ). Hence, the function  $\lambda \mapsto \mathbb{E}[(\bar{f}_\lambda(X) - f(X))^2]$  is positive and continuous on  $\mathbf{R}_+$ , so that it admits a minimum  $C_1 > 0$  on the compact interval  $[0, 6]$ . In addition, the expression (33) shows that for  $\lambda \geq 6$ , we have

$$\mathbb{E}[(\bar{f}_\lambda(X) - f(X))^2] \geq \frac{1}{2\lambda^2} \left( 1 - \frac{2}{6} \right) = \frac{1}{3\lambda^2}. \quad (34)$$



**First lower bound on the variance.** We now turn to the task of bounding the variance from below. In order to avoid restrictive conditions on  $\lambda$ , we will provide two separate lower bounds, valid in two different regimes.

Our first lower bound on the variance, valid for  $\lambda \leq n/3$ , controls the error of estimation of the optimal labels in nonempty cells. It depends on  $\sigma^2$ , and is of order  $\Theta(\sigma^2 \frac{\lambda}{n})$ . We use a general bound on the variance of regressograms (Arlot and Genuer, 2014, Proposition 2) (note that while this result is stated for a fixed number of cells, it can be adapted to a random number of cells by conditioning on  $K_\lambda = k$  and then by averaging):

$$\mathbb{E} \left[ (\hat{f}_{\lambda,n}(X) - \tilde{f}_\lambda(X))^2 \right] \geq \frac{\sigma^2}{n} \left( \mathbb{E}[K_\lambda] - 2\mathbb{E}_{\Pi_\lambda} \left[ \sum_{\mathbf{v} \in \mathcal{L}(\Pi_\lambda)} \exp(-n\mathbb{P}(X \in C_{\mathbf{v}})) \right] \right). \quad (35)$$

Now, recall that the splits defining  $\Pi_\lambda$  form a Poisson point process on  $[0, 1]$  of intensity  $\lambda dx$  (Fact 1). In particular, the splits can be described as follows. Let  $(E_k)_{k \geq 1}$  be an i.i.d. sequence of  $\text{Exp}(1)$  random variables, and  $S_p := \sum_{k=1}^p E_k$  for  $p \geq 0$ . Then, the (ordered) splits in  $\Pi_\lambda$  have the same distribution as  $(\lambda^{-1}S_1, \dots, \lambda^{-1}S_{K_\lambda-1})$ , where  $K_\lambda := 1 + \sup\{p \geq 0 : S_p \leq \lambda\}$ . In addition, the probability that  $X \sim \mathcal{U}([0, 1])$  falls in the cell  $[\lambda^{-1}S_{k-1}, \lambda^{-1}S_k \wedge 1)$  ( $1 \leq k \leq K_\lambda$ ) is  $\lambda^{-1}(S_k \wedge 1 - S_{k-1})$ , so that

$$\begin{aligned} \mathbb{E} \left[ \sum_{\mathbf{v} \in \mathcal{L}(\Pi_\lambda)} \exp(-n\mathbb{P}(X \in C_{\mathbf{v}})) \right] &= \mathbb{E} \left[ \sum_{k=1}^{K_\lambda-1} e^{-n\lambda^{-1}(S_k - S_{k-1})} + e^{-n(1 - \lambda^{-1}S_{K_\lambda-1})} \right] \\ &\leq \mathbb{E} \left[ \sum_{k=1}^{\infty} \mathbf{1}(S_k \leq \lambda) e^{-n\lambda^{-1}E_k} \right] + 1 \end{aligned} \quad (36)$$

$$= \sum_{k=1}^{\infty} \mathbb{E}[\mathbf{1}(S_k \leq \lambda)] \mathbb{E}[e^{-n\lambda^{-1}E_k}] + 1 \quad (37)$$

$$\begin{aligned} &= \sum_{k=1}^{\infty} \mathbb{E}[\mathbf{1}(S_k \leq \lambda)] \cdot \int_0^\infty e^{-n\lambda^{-1}u} e^{-u} du + 1 \\ &= \frac{\lambda}{n + \lambda} \mathbb{E} \left[ \sum_{k=1}^{\infty} \mathbf{1}(S_k \leq \lambda) \right] + 1 \\ &= \frac{\lambda}{n + \lambda} \mathbb{E}[K_\lambda] + 1 \\ &= \frac{\lambda}{n + \lambda} (1 + \lambda) + 1 \end{aligned} \quad (38)$$

where (37) comes from the fact that  $E_k$  and  $S_{k-1}$  are independent. Plugging Equation (38) in the lower bound (35) yields

$$\mathbb{E} \left[ (\hat{f}_{\lambda,n}(X) - \tilde{f}_\lambda(X))^2 \right] \geq \frac{\sigma^2}{n} \left( (1 + \lambda) - 2(1 + \lambda) \frac{\lambda}{n + \lambda} - 2 \right) = \frac{\sigma^2}{n} \left( (1 + \lambda) \frac{n - \lambda}{n + \lambda} - 2 \right).$$

Now, assume that  $6 \leq \lambda \leq \frac{n}{3}$ . Since

$$(1 + \lambda) \frac{n - \lambda}{n + \lambda} - 2 \underset{(\lambda \leq n/3)}{\geq} (1 + \lambda) \frac{n - n/3}{n + n/3} - 2 = (1 + \lambda) \frac{1}{2} - 2 \underset{(\lambda \geq 6)}{\geq} \frac{\lambda}{4},$$

the above lower bound implies, for  $6 \leq \lambda \leq \frac{n}{3}$ ,

$$\mathbb{E} \left[ (\hat{f}_{\lambda,n}(X) - \tilde{f}_\lambda(X))^2 \right] \geq \frac{\sigma^2 \lambda}{4n}. \quad (39)$$

**Second lower bound on the variance.** The lower bound (39) is only valid for  $\lambda \leq n/3$ ; as  $\lambda$  becomes of order  $n$  or larger, the previous bound becomes vacuous. We now provide another lower bound on the variance, valid when  $\lambda \geq n/3$ , by considering the contribution of empty cells to the variance.

Let  $\mathbf{v} \in \mathcal{L}(\Pi_\lambda)$ . If  $C_{\mathbf{v}}$  contains no sample point from  $\mathcal{D}_n$ , then for  $x \in C_{\mathbf{v}}$ :  $\widehat{f}_{\lambda,n}(x) = 0$  and thus  $(\widehat{f}_{\lambda,n}(x) - \bar{f}_\lambda(x))^2 = \bar{f}_\lambda(x)^2 \geq 1$ . Hence, the variance term is lower bounded as follows, denoting  $N_n(C)$  the number of  $1 \leq i \leq n$  such that  $X_i \in C$  and  $N_{\lambda,n}(x) = N_n(C_\lambda(x))$ :

$$\begin{aligned} \mathbb{E}[(\widehat{f}_{\lambda,n}(X) - \bar{f}_\lambda(X))^2] &\geq \mathbb{P}(N_{\lambda,n}(X) = 0) \\ &= \mathbb{E} \left[ \sum_{\mathbf{v} \in \mathcal{L}(\Pi_\lambda)} \mathbb{P}(X \in C_{\mathbf{v}}) \mathbb{P}(N_n(C_{\mathbf{v}}) = 0) \right] \\ &= \mathbb{E} \left[ \sum_{\mathbf{v} \in \mathcal{L}(\Pi_\lambda)} \mathbb{P}(X \in C_{\mathbf{v}}) (1 - \mathbb{P}(X \in C_{\mathbf{v}}))^n \right] \\ &\geq \mathbb{E} \left[ \left( \sum_{\mathbf{v} \in \mathcal{L}(\Pi_\lambda)} \mathbb{P}(X \in C_{\mathbf{v}}) (1 - \mathbb{P}(X \in C_{\mathbf{v}})) \right)^n \right] \end{aligned} \quad (40)$$

$$\geq \mathbb{E} \left[ \sum_{\mathbf{v} \in \mathcal{L}(\Pi_\lambda)} \mathbb{P}(X \in C_{\mathbf{v}}) (1 - \mathbb{P}(X \in C_{\mathbf{v}}))^n \right] \quad (41)$$

$$= \left( 1 - \mathbb{E} \left[ \sum_{\mathbf{v} \in \mathcal{L}(\Pi_\lambda)} \mathbb{P}(X \in C_{\mathbf{v}})^2 \right] \right)^n \quad (42)$$

where (40) and (41) come from Jensen's inequality applied to the convex function  $x \mapsto x^n$ . Now, using the notations defined above, we have

$$\begin{aligned} \mathbb{E} \left[ \sum_{\mathbf{v} \in \Pi_\lambda} \mathbb{P}(X \in C_{\mathbf{v}})^2 \right] &\leq \mathbb{E} \left[ \sum_{k=1}^{K_\lambda} (\lambda^{-1} E_k)^2 \right] \\ &= \lambda^{-2} \mathbb{E} \left[ \sum_{k=1}^{\infty} \mathbf{1}(S_{k-1} \leq \lambda) E_k^2 \right] \\ &= \lambda^{-2} \mathbb{E} \left[ \sum_{k=1}^{\infty} \mathbf{1}(S_{k-1} \leq \lambda) \mathbb{E}[E_k^2 | S_{k-1}] \right] \\ &= 2\lambda^{-2} \mathbb{E} \left[ \sum_{k=1}^{\infty} \mathbf{1}(S_{k-1} \leq \lambda) \right] \end{aligned} \quad (43)$$

$$\begin{aligned} &= 2\lambda^{-2} \mathbb{E}[K_\lambda] \\ &= \frac{2(\lambda + 1)}{\lambda^2}, \end{aligned} \quad (44)$$

where the equality  $\mathbb{E}[E_k^2 | S_{k-1}] = 2$  (used in Equation (43)) comes from the fact that  $E_k \sim \text{Exp}(1)$  is independent of  $S_{k-1}$ .

The bounds (42) and (44) imply that, if  $2(\lambda + 1)/\lambda^2 \leq 1$ , then

$$\mathbb{E}[(\widehat{f}_{\lambda,n}(X) - \bar{f}_\lambda(X))^2] \geq \left( 1 - \frac{2(\lambda + 1)}{\lambda^2} \right)^n. \quad (45)$$

Now, assume that  $n \geq 18$  and  $\lambda \geq \frac{n}{3} \geq 6$ . Then

$$\frac{2(\lambda+1)}{\lambda^2} \leq 2 \cdot \frac{3}{n} \left(1 + \frac{3}{n}\right) \leq 2 \cdot \frac{3}{n} \left(1 + \frac{3}{18}\right) = \frac{7}{n} \underset{(n \geq 18)}{\leq} 1,$$

so that, using the inequality  $(1-x)^m \geq 1-mx$  for  $m \geq 0$  and  $x \in \mathbf{R}$ ,

$$\left(1 - \frac{2(\lambda+1)}{\lambda^2}\right)^{n/8} \geq \left(1 - \frac{7}{n}\right)^{n/8} \geq 1 - \frac{n}{8} \cdot \frac{7}{n} = \frac{1}{8}.$$

Combining the above inequality with Equation (45) gives, letting  $C_2 := 1/8^8$ ,

$$\mathbb{E}[(\widehat{f}_{\lambda,n}(X) - \bar{f}_\lambda(X))^2] \geq C_2. \quad (46)$$

**Summing up.** Assume that  $n \geq 18$ . Recall the bias-variance decomposition (27) of the risk  $R(\widehat{f}_{\lambda,n})$  of the Mondrian tree.

- If  $\lambda \leq 6$ , then we saw that the bias (and hence the risk) is larger than  $C_1$ ;
- If  $\lambda \geq \frac{n}{3}$ , Equation (45) implies that the variance (and hence the risk) is larger than  $C_2$ ;
- If  $6 \leq \lambda \leq \frac{n}{3}$ , Equations (34) (bias term) and (39) (variance term) imply that

$$R(\widehat{f}_{\lambda,n}) \geq \frac{1}{3\lambda^2} + \frac{\sigma^2\lambda}{4n}.$$

In particular,

$$\inf_{\lambda \in \mathbf{R}^+} R(\widehat{f}_{\lambda,n}) \geq C_1 \wedge C_2 \wedge \inf_{\lambda \in \mathbf{R}^+} \left( \frac{1}{3\lambda^2} + \frac{\sigma^2\lambda}{4n} \right) = C_0 \wedge \frac{1}{4} \left( \frac{3\sigma^2}{n} \right)^{2/3} \quad (47)$$

where we let  $C_0 = C_1 \wedge C_2$ .

## 7.5 Proof of Theorem 3: Minimax rates for Mondrian Forests over the class $\mathcal{C}^2$

We first prove Theorem 3 assuming that  $X$  has a uniform density over the hypercube  $[0, 1]^d$ . The proof is then extended to match the assumption of a positive and Lipschitz density function for  $X$ .

Consider a finite Mondrian Forest

$$\widehat{f}_{\lambda,n}^{(M)}(X) = \frac{1}{M} \sum_{m=1}^M \widehat{f}_{\lambda,n}^{(m)}(X),$$

and denote by, for all  $1 \leq m \leq M$ ,  $\bar{f}_\lambda^{(m)}$  the random function  $\bar{f}_\lambda^{(m)}(x) = \mathbb{E}_X [f(X) | X \in C_\lambda^{(m)}(x)]$ .

Also, let  $\tilde{f}_\lambda(x) = \mathbb{E}_{\Pi_\lambda} [\bar{f}_\lambda^{(m)}(x)]$ , which is deterministic and does not depend on  $m$ . We have

$$\begin{aligned} & \mathbb{E}[(\widehat{f}_{\lambda,n}^{(M)}(X) - f(X))^2] \\ & \leq 2 \mathbb{E}\left[\left(\frac{1}{M} \sum_{m=1}^M \widehat{f}_{\lambda,n}^{(m)}(X) - \frac{1}{M} \sum_{m=1}^M \bar{f}_\lambda^{(m)}(X)\right)^2\right] + 2 \mathbb{E}\left[\left(\frac{1}{M} \sum_{m=1}^M \bar{f}_\lambda^{(m)}(X) - f(X)\right)^2\right] \\ & \leq 2 \mathbb{E}\left[\left(\frac{1}{M} \sum_{m=1}^M \widehat{f}_{\lambda,n}^{(m)}(X) - \frac{1}{M} \sum_{m=1}^M \bar{f}_\lambda^{(m)}(X)\right)^2\right] + 2 \mathbb{E}\left[\left(\frac{1}{M} \sum_{m=1}^M \bar{f}_\lambda^{(m)}(X) - \tilde{f}_\lambda(X)\right)^2\right] \\ & \quad + 2 \mathbb{E}[(\tilde{f}_\lambda(X) - f(X))^2]. \end{aligned} \quad (48)$$

Note that, by Jensen's inequality,

$$\begin{aligned} \mathbb{E}\left[\left(\frac{1}{M} \sum_{m=1}^M \widehat{f}_{\lambda,n}^{(m)}(X) - \frac{1}{M} \sum_{m=1}^M \bar{f}_{\lambda}^{(m)}(X)\right)^2\right] &\leq \frac{1}{M} \sum_{m=1}^M \mathbb{E}\left[(\widehat{f}_{\lambda,n}^{(m)}(X) - \bar{f}_{\lambda}^{(m)}(X))^2\right] \\ &\leq \mathbb{E}\left[(\widehat{f}_{\lambda,n}^{(1)}(X) - \bar{f}_{\lambda}^{(1)}(X))^2\right]. \end{aligned} \quad (49)$$

Since, for all  $1 \leq m \leq M$ ,  $\mathbb{E}_{\Pi_{\lambda}}[\bar{f}_{\lambda}^{(m)}(X)] = \tilde{f}_{\lambda}(X)$ , we have

$$\mathbb{E}\left[\left(\frac{1}{M} \sum_{m=1}^M \bar{f}_{\lambda}^{(m)}(X) - \tilde{f}_{\lambda}(X)\right)^2\right] = \frac{\mathbb{E}_X[\text{Var}_{\Pi_{\lambda}}[\bar{f}_{\lambda}^{(1)}(X)]]}{M}. \quad (50)$$

Combining equations (48), (49) and (50), we have

$$\begin{aligned} &\mathbb{E}\left[(\widehat{f}_{\lambda,n}^{(M)}(X) - f(X))^2\right] \\ &\leq 2\mathbb{E}\left[(\widehat{f}_{\lambda,n}^{(1)}(X) - \bar{f}_{\lambda}^{(1)}(X))^2\right] + 2\frac{\mathbb{E}_X[\text{Var}_{\Pi_{\lambda}}[\bar{f}_{\lambda}^{(1)}(X)]]}{M} + 2\mathbb{E}\left[(\tilde{f}_{\lambda}(X) - f(X))^2\right]. \end{aligned}$$

Since  $f$  is  $G$ -Lipschitz with  $G := \sup_{x \in [0,1]^d} \|\nabla f(x)\|$ , we have for all  $x \in [0,1]^d$ , recalling that  $D_{\lambda}(x)$  denotes the diameter of  $C_{\lambda}(x)$ ,

$$\begin{aligned} \text{Var}_{\Pi_{\lambda}}(\bar{f}_{\lambda}^{(1)}(x)) &\leq \mathbb{E}_{\Pi_{\lambda}}\left[(\bar{f}_{\lambda}^{(1)}(x) - f(x))^2\right] \\ &\leq G^2 \mathbb{E}_{\Pi_{\lambda}}[D_{\lambda}(x)^2] \\ &\leq \frac{4dG^2}{\lambda^2} \quad (\text{by Lemma 1}). \end{aligned}$$

Consequently, taking the expectation with respect to  $X$ ,

$$\mathbb{E}\left[(\widehat{f}_{\lambda,n}^{(M)}(X) - f(X))^2\right] \leq \frac{8dG^2}{M\lambda^2} + 2\mathbb{E}\left[(\widehat{f}_{\lambda,n}^{(1)}(X) - \bar{f}_{\lambda}^{(1)}(X))^2\right] + 2\mathbb{E}\left[(\tilde{f}_{\lambda}(X) - f(X))^2\right].$$

The same upper bound also holds conditional on  $X \in [\varepsilon, 1 - \varepsilon]^d$ ,

$$\begin{aligned} \mathbb{E}\left[(\widehat{f}_{\lambda,n}^{(M)}(X) - f(X))^2 | X \in [\varepsilon, 1 - \varepsilon]^d\right] &\leq \frac{8dG^2}{M\lambda^2} + 2\mathbb{E}\left[(\widehat{f}_{\lambda,n}^{(1)}(X) - \bar{f}_{\lambda}^{(1)}(X))^2 | X \in [\varepsilon, 1 - \varepsilon]^d\right] \\ &\quad + 2\mathbb{E}\left[(\tilde{f}_{\lambda}(X) - f(X))^2 | X \in [\varepsilon, 1 - \varepsilon]^d\right]. \end{aligned} \quad (51)$$

### 7.5.1 First case: $X$ is uniform over $[0,1]^d$

In the sequel, we assume that  $X$  is uniformly distributed over  $[0,1]^d$ . By the exact same argument developed in the proof of Theorem 2, the variance term is upper-bounded by

$$\mathbb{E}\left[(\bar{f}_{\lambda}^{(1)}(X) - \widehat{f}_{\lambda,n}^{(1)}(X))^2\right] \leq \frac{(1+\lambda)^d}{n} (2\sigma^2 + 9\|f\|_{\infty}^2).$$

Hence, the conditional variance in the decomposition (48) satisfies

$$\begin{aligned} \mathbb{E}\left[(\bar{f}_{\lambda}^{(1)}(X) - \widehat{f}_{\lambda,n}^{(1)}(X))^2 | X \in [\varepsilon, 1 - \varepsilon]^d\right] &\leq \mathbb{E}[(\bar{f}_{\lambda}^{(1)}(X) - \widehat{f}_{\lambda,n}^{(1)}(X))^2 | \mathbb{P}(X \in [\varepsilon, 1 - \varepsilon]^d)]^{-1} \\ &\leq \frac{(1+\lambda)^d}{n} (2\sigma^2 + 9\|f\|_{\infty}^2) (1 - 2\varepsilon)^{-d}. \end{aligned} \quad (52)$$

It now remains to control the bias of the infinite Mondrian Forest estimate, namely

$$\mathbb{E}[(\tilde{f}_{\lambda}(X) - f(X))^2 | X \in [\varepsilon, 1 - \varepsilon]^d] = \int_{B_{\varepsilon}} (\tilde{f}_{\lambda}(x) - f(x))^2 dx, \quad (53)$$

where  $B_{\varepsilon} = [\varepsilon, 1 - \varepsilon]^d$ .

**Expression for  $\tilde{f}_\lambda$ .** Denote by  $C_\lambda(x)$  the cell of  $x \in [0, 1]^d$  in  $\Pi_\lambda \sim \text{MP}(\lambda, [0, 1]^d)$ . We have

$$\begin{aligned}\tilde{f}_\lambda(x) &= \mathbb{E} \left[ \frac{1}{\text{vol } C_\lambda(x)} \int_{[0,1]^d} f(z) \mathbf{1}(z \in C_\lambda(x)) dz \right] \\ &= \int_{[0,1]^d} f(z) \mathbb{E} \left[ \frac{\mathbf{1}(z \in C_\lambda(x))}{\text{vol } C_\lambda(x)} \right] dz \\ &= \int_{[0,1]^d} f(z) F_\lambda(x, z) dz\end{aligned}\tag{54}$$

where we defined

$$F_\lambda(x, z) = \mathbb{E} \left[ \frac{\mathbf{1}(z \in C_\lambda(x))}{\text{vol } C_\lambda(x)} \right].\tag{55}$$

**Computation of  $F_\lambda(x, z)$ .** Let  $C(x, z) = \prod_{1 \leq j \leq d} [x_j \wedge z_j, x_j \vee z_j] \subseteq [0, 1]^d$  be the smallest box containing both  $x$  and  $z$ . Note that  $z \in C_\lambda(x)$  if and only if  $\Pi_\lambda$  does not cut  $C(x, z)$ . Thus, when  $z \in C_\lambda(x)$ ,  $C(x, z) \subseteq C_\lambda(x)$ , so that  $C_\lambda(x') = C_\lambda(x)$  for each  $x' \in C(x, z)$ ; we denote this cell  $C_\lambda(C(x, z))$ .

The above reasoning shows that  $F_\lambda(x, z) = F_\lambda(C(x, z))$ , where for each box  $C \subseteq [0, 1]^d$  we define

$$F_\lambda(C) = \mathbb{E} \left[ \frac{\mathbf{1}(\Pi_\lambda \not\cap C)}{\text{vol } C_\lambda(C)} \right],\tag{56}$$

where by convention, the term in the expectation is null if  $\Pi_\lambda$  intersects  $C$  (in that case,  $C_\lambda(C)$ , which is the unique cell of  $\Pi_\lambda$  that contains  $C$ , is not defined and neither is the denominator in 56). In particular, this shows that  $F_\lambda(x, z)$  only depends on  $C(x, z)$ , *i.e.* it is symmetric in  $x_j, z_j$  for each  $1 \leq j \leq d$ . We can now write:

$$F_\lambda(C) = \mathbb{P}(\Pi_\lambda \not\cap C) \mathbb{E} \left[ \frac{1}{\text{vol } C_\lambda(C)} \mid \Pi_\lambda \not\cap C \right]\tag{57}$$

Let  $C = \prod_{1 \leq j \leq d} [a_j, b_j]$  and  $a = (a_1, \dots, a_d) \in [0, 1]^d$ . Note that  $\Pi_\lambda \not\cap C$  is equivalent to  $R_{\lambda,j}(a) \geq b_j$  for  $j = 1, \dots, d$ , *i.e.*, denoting  $R_{\lambda,j}(a) = (a_j + \lambda^{-1} E_{j,R}) \wedge 1$  with  $E_{j,R} \sim \text{Exp}(1)$  (by Proposition 1), to  $E_{j,R} \geq \lambda(b_j - a_j)$ . By the memory-less property of the exponential distribution, the distribution of  $E_{j,R} - \lambda(b_j - a_j)$  conditionally on  $E_{j,R} \geq \lambda(b_j - a_j)$  is  $\text{Exp}(1)$ . As a result (using the independence of the exponential random variables drawn for each side, see Proposition 1), conditionally on  $\Pi_\lambda \not\cap C$ , the distribution of  $C_\lambda(C)$  is the following:

The coordinates  $L_{\lambda,1}(C), \dots, L_{\lambda,d}(C), R_{\lambda,1}(C), \dots, R_{\lambda,d}(C)$  are independent, with  $a_j - L_{\lambda,j}(C) = \lambda^{-1} E_{j,L} \wedge a_j$  and  $R_{\lambda,j}(C) - b_j = \lambda^{-1} E_{j,R} \wedge (1 - b_j)$  ( $E_{j,L}, E_{j,R} \sim \text{Exp}(1)$ ).

This enables us to compute  $F_\lambda(C)$  from equation (57): using the above and the fact that

$$\mathbb{P}(\Pi_\lambda \not\cap C) = \exp(-\lambda|C|),$$

we get

$$\begin{aligned}F_\lambda(C) &= \exp(-\lambda|C|) \mathbb{E} \left[ \prod_{1 \leq j \leq d} (R_{\lambda,j}(C) - L_{\lambda,j}(C))^{-1} \mid \Pi_\lambda \not\cap C \right] \\ &= \exp(-\lambda|C|) \prod_{1 \leq j \leq d} \mathbb{E} \left[ ((b_j - a_j) + \lambda^{-1} E_{j,L} \wedge a_j + \lambda^{-1} E_{j,R} \wedge (1 - b_j))^{-1} \right] \\ &= \lambda^d \exp(-\lambda|C|) \prod_{1 \leq j \leq d} \mathbb{E} \left[ (\lambda(b_j - a_j) + E_{j,L} \wedge \lambda a_j + E_{j,R} \wedge \lambda(1 - b_j))^{-1} \right].\end{aligned}$$

Applying the previous equality to  $C = C(x, z)$ , and recalling that  $|C(x, z)| = \|x - z\|_1$  and  $b_j - a_j = |x_j - z_j|$ , we get

$$F_\lambda(x, z) = \lambda^d \exp(-\lambda \|x - z\|_1) \prod_{1 \leq j \leq d} \mathbb{E} \left[ \left\{ \lambda |x_j - z_j| + E_{j,L} \wedge \lambda(x_j \wedge z_j) + E_{j,R} \wedge \lambda(1 - (x_j \vee z_j)) \right\}^{-1} \right]. \quad (58)$$

**Bias of  $\tilde{f}(x)$ .** Assume  $f \in \mathcal{C}^2([0, 1]^d)$ , with  $\|\nabla^2 f\| \leq C_2$ . We have, for every  $x \in [0, 1]^d$  and  $h$  such that  $x + h \in [0, 1]^d$  (where  $\|\cdot\|$  denotes the Euclidean norm), by a Taylor expansion:

$$|f(x + h) - f(x) - \nabla f(x) \cdot h| \leq \frac{C_2}{2} \|h\|^2 \quad (59)$$

Now, by the triangle inequality,

$$\begin{aligned} & \left| \left| \int_{[0,1]^d} (f(z) - f(x)) F_\lambda(x, z) dz \right| - \left| \int_{[0,1]^d} (\nabla f(x) \cdot (z - x)) F_\lambda(x, z) dz \right| \right| \\ & \leq \left| \int_{[0,1]^d} (f(z) - f(x) - \nabla f(x) \cdot (z - x)) F_\lambda(x, z) dz \right| \\ & \leq C_2 \int_{[0,1]^d} \frac{1}{2} \|z - x\|^2 F_\lambda(x, z) dz. \end{aligned}$$

Since  $\int F_\lambda(x, z) dz = 1$ , recalling the expression (54) we obtain

$$\begin{aligned} |\tilde{f}_\lambda(x) - f(x)| &= \left| \int_{[0,1]^d} (f(z) - f(x)) F_\lambda(x, z) dz \right| \\ &\leq \underbrace{\left| \nabla f(x) \cdot \int_{[0,1]^d} (z - x) F_\lambda(x, z) dz \right|}_{:=A} + \underbrace{C_2 \int_{[0,1]^d} \frac{1}{2} \|z - x\|^2 F_\lambda(x, z) dz}_{:=B} \end{aligned}$$

According to Technical Lemma 2 (see Section 7.5.3), we have

$$\|A\|^2 = \left\| \int_{[0,1]^d} (z - x) F_\lambda(x, z) dz \right\|^2 \leq \frac{9}{\lambda^2} \sum_{j=1}^d e^{-\lambda[x_j \wedge (1-x_j)]},$$

and

$$B = \int_{[0,1]^d} \frac{1}{2} \|z - x\|^2 F_\lambda(x, z) dz \leq \frac{d}{\lambda^2}.$$

Hence, we obtain, for each  $x \in [0, 1]^d$

$$\begin{aligned} \left| \tilde{f}_\lambda(x) - f(x) \right|^2 &\leq (|\nabla f(x) \cdot A| + C_2 B)^2 \\ &\leq 2 \left( |\nabla f(x) \cdot A|^2 + C_2^2 B^2 \right) \\ &\leq 18G^2 \lambda^{-2} \sum_{j=1}^d e^{-\lambda[x_j \wedge (1-x_j)]} + 2C_2^2 d^2 \lambda^{-4}, \end{aligned} \quad (60)$$



where  $G := \sup_{x \in [0,1]^d} \|\nabla f(x)\|$  (which is finite since  $f$  is  $\mathcal{C}^2$ ). Integrating over  $\mathcal{U}([\varepsilon, 1 - \varepsilon])$  we get

$$\mathbb{E} \left[ \left( \tilde{f}_\lambda(X) - f(X) \right)^2 | X \in [\varepsilon, 1 - \varepsilon]^d \right] \leq 18G^2 d(1 - 2\varepsilon)^{-d} \lambda^{-2} \psi_\varepsilon(\lambda) + 2C_2^2 d^2 \lambda^{-4} \quad (61)$$

where

$$\psi_\varepsilon(\lambda) := \int_\varepsilon^{1-\varepsilon} e^{-\lambda[u \wedge (1-u)]} du = 2 \int_\varepsilon^{1/2} e^{-\lambda u} du = \frac{2}{\lambda} \left( e^{-\lambda \varepsilon} - e^{-\lambda/2} \right) \leq \frac{2e^{-\lambda \varepsilon}}{\lambda}.$$

Finally, using inequalities (51), (52) and (61), we obtain

$$\begin{aligned} \mathbb{E}[(\tilde{f}_{\lambda,n}^{(M)}(X) - f(X))^2 | X \in [\varepsilon, 1 - \varepsilon]^d] &\leq \frac{8dG^2}{M\lambda^2} + \frac{2(1+\lambda)^d}{n} (2\sigma^2 + 9\|f\|_\infty^2) (1 - 2\varepsilon)^{-d} \\ &\quad + 72G^2 d(1 - 2\varepsilon)^{-d} \lambda^{-3} e^{-\lambda \varepsilon} + 4C_2^2 d^2 \lambda^{-4}. \end{aligned} \quad (62)$$

When  $\varepsilon > 0$  is fixed, the risk of the Mondrian Forest satisfies

$$\mathbb{E}[(\tilde{f}_{\lambda,n}^{(M)}(X) - f(X))^2 | X \in [\varepsilon, 1 - \varepsilon]^d] \leq O\left(\frac{\lambda^d}{n}\right) + O\left(\frac{1}{\lambda^4}\right) + O\left(\frac{1}{M\lambda^2}\right).$$

Optimizing this bound by setting  $\lambda_n \asymp n^{1/(d+4)}$  and  $M_n \gtrsim n^{2/(d+4)}$ , we obtain the minimax risk rate for a  $\mathcal{C}^2$  regression function:

$$\mathbb{E}[(\tilde{f}_{\lambda_n,n}^{(M_n)}(X) - f(X))^2 | X \in [\varepsilon, 1 - \varepsilon]^d] = O\left(n^{-4/(d+4)}\right). \quad (63)$$

Note that Equation (62) also provides an upper bound on the integrated risk on the whole hypercube  $[0, 1]^d$  by setting  $\varepsilon = 0$ , which leads to

$$\mathbb{E}[(\tilde{f}_{\lambda,n}^{(M)}(X) - f(X))^2] = O\left(\frac{\lambda^d}{n}\right) + O\left(\frac{1}{\lambda^3}\right) + O\left(\frac{1}{M\lambda^2}\right),$$

and results in a suboptimal rate of consistency

$$\mathbb{E}[(\tilde{f}_{\lambda_n,n}^{(M_n)}(X) - f(X))^2] = O\left(n^{-3/(d+3)}\right),$$

letting  $\lambda_n \asymp n^{1/(d+3)}$  and  $M_n \gtrsim n^{1/(d+3)}$ . This concludes the first part of the proof.

### 7.5.2 Second case: $X$ has a positive Lipschitz density

Here, we show how the assumption that  $X$  is uniformly distributed can be relaxed. From now on, we assume that the distribution  $\mu$  of  $X$  has a positive density  $p : [0, 1]^d \rightarrow \mathbf{R}_+^*$  which is  $C_p$ -Lipschitz. We denote  $p_0 = \inf_{[0,1]^d} p$  and  $p_1 = \sup_{[0,1]^d} p$ , both of which are positive and finite by compactness of  $[0, 1]^d$ . Like in the uniform case, the most difficult part of the proof is to control the bias term. Here, we have

$$\begin{aligned} \tilde{f}_\lambda(x) &= \mathbb{E} \left[ \frac{1}{\mu(C_\lambda(x))} \int_{[0,1]^d} f(z) p(z) \mathbf{1}(z \in C_\lambda(x)) dz \right] \\ &= \int_{[0,1]^d} f(z) \mathbb{E} \left[ \frac{p(z) \mathbf{1}(z \in C_\lambda(x))}{\mu(C_\lambda(x))} \right] dz \\ &= \int_{[0,1]^d} f(z) F_{p,\lambda}(x, z) dz \end{aligned}$$

where we defined

$$F_{p,\lambda}(x, z) = \mathbb{E} \left[ \frac{p(z) \mathbf{1}(z \in C_\lambda(x))}{\mu(C_\lambda(x))} \right]. \quad (64)$$

In particular,  $\int_{[0,1]^d} F_{p,\lambda}(x, z) dz = 1$  for any  $x \in [0, 1]^d$ . Note that since  $|f(z) - f(x) - \nabla f(x) \cdot (z - x)| \leq \frac{1}{2} C_2 \|z - x\|^2$ , we have

$$\begin{aligned} |\tilde{f}_\lambda(x) - f(x)| &= \left| \int_{[0,1]^d} F_{p,\lambda}(x, z) (f(z) - f(x)) dz \right| \\ &\leq \left| \nabla f(x) \cdot \int_{[0,1]^d} (z - x) F_{p,\lambda}(x, z) dz \right| + \frac{C_2}{2} \int_{[0,1]^d} \|z - x\|^2 F_{p,\lambda}(x, z) dz \end{aligned} \quad (65)$$

It remains to bound the above term as  $O(\lambda^{-2})$ , for each  $x \in B_\varepsilon := [\varepsilon, 1 - \varepsilon]$ . For the second term, note that since  $p \leq p_1$  and  $\mu \geq p_0 \text{ vol}$  (since  $p \geq p_0$ ), we have

$$F_{p,\lambda}(x, z) \leq \frac{p_1}{p_0} F_\lambda(x, z) \quad (66)$$

so that

$$\int_{[0,1]^d} \frac{1}{2} \|z - x\|^2 F_{p,\lambda}(x, z) dz \leq \frac{p_1}{p_0} \int_{[0,1]^d} \frac{1}{2} \|z - x\|^2 F_\lambda(x, z) dz \leq \frac{p_1 d}{p_0 \lambda^2}$$

where the second bound results from Technical Lemma 2.

Hence, it remains to control  $\int_{[0,1]^d} (z - x) F_{p,\lambda}(x, z) dz$ . We will again relate this quantity to the one obtained for a uniform density  $p \equiv 1$ , which was already controlled before. However, this time the crude bound (66) is no longer sufficient, since we need the first order terms to compensate. Rather, we will show that  $F_{p,\lambda}(x, z) = (1 + O(\|x - z\|)) F_\lambda(x, z)$ . First, by the exact same argument used for  $p \equiv 1$ , we have

$$\begin{aligned} F_{p,\lambda}(x, z) &= \exp(-\lambda \|x - z\|_1) p(z) \\ &\quad \times \mathbb{E} \left[ \left\{ \int_{(x_1 \wedge z_1 - \lambda^{-1} E_L^1) \vee 0}^{(x_1 \vee z_1 + \lambda^{-1} E_R^1) \wedge 1} \cdots \int_{(x_d \wedge z_d - \lambda^{-1} E_L^d) \vee 0}^{(x_d \vee z_d + \lambda^{-1} E_R^d) \wedge 1} p(y_1, \dots, y_d) dy_1 \dots dy_d \right\}^{-1} \right] \\ &= \exp(-\lambda \|x - z\|_1) \mathbb{E} \left[ \left\{ \int_{C_\lambda(x, z)} \frac{p(y)}{p(z)} dy \right\}^{-1} \right] \end{aligned} \quad (67)$$

where

$$C_\lambda(x, z) := \prod_{j=1}^d \left[ (x_j \wedge z_j - \lambda^{-1} E_L^j) \vee 0, (x_j \vee z_j + \lambda^{-1} E_R^j) \wedge 1 \right] \quad (68)$$

with  $E_L^1, E_R^1, \dots, E_L^d, E_R^d$  i.i.d.  $\text{Exp}(1)$  random variables.

**A first upper bound on  $|F_{p,\lambda}(x, z) - F_\lambda(x, z)|$ .** Now, since  $p$  is  $C_p$ -Lipschitz and lower bounded by  $p_0$ , we have for every  $y \in C_\lambda(x, z)$ ,

$$\left| \frac{p(y)}{p(z)} - 1 \right| = \frac{|p(y) - p(z)|}{p(z)} \leq \frac{C_p}{p_0} \|y - z\| \leq \frac{C_p}{p_0} \text{diam } C_\lambda(x, z), \quad (69)$$

so that

$$1 - \frac{C_p}{p_0} \text{diam } C_\lambda(x, z) \leq \frac{p(y)}{p(z)} \leq 1 + \frac{C_p}{p_0} \text{diam } C_\lambda(x, z),$$

and thus, by integrating over  $C_\lambda(x, z)$ , and by recalling that  $p(y)/p(z) \geq p_0/p_1$ ,

$$\begin{aligned} \left\{ 1 + \frac{C_p}{p_0} \text{diam } C_\lambda(x, z) \right\}^{-1} \text{vol } C_\lambda(x, z)^{-1} &\leq \left\{ \int_{C_\lambda(x, z)} \frac{p(y)}{p(z)} dy \right\}^{-1} \\ &\leq \left\{ \left[ 1 - \frac{C_p}{p_0} \text{diam } C_\lambda(x, z) \right] \vee \frac{p_0}{p_1} \right\}^{-1} \text{vol } C_\lambda(x, z)^{-1}. \end{aligned} \quad (70)$$

In addition, since  $(1 + u)^{-1} \geq 1 - u$  for  $u \geq 0$ , so that

$$\left\{ 1 + \frac{C_p}{p_0} \text{diam } C_\lambda(x, z) \right\}^{-1} \geq 1 - \frac{C_p}{p_0} \text{diam } C_\lambda(x, z),$$

and since, setting  $a := \left[ 1 - \frac{C_p}{p_0} \text{diam } C_\lambda(x, z) \right] \vee \frac{p_0}{p_1} \in (0, 1]$ , we have

$$a^{-1} - 1 = \frac{1 - a}{a} \leq \frac{(C_p/p_0) \text{diam } C_\lambda(x, z)}{p_0/p_1} = \frac{p_1 C_p}{p_0^2} \text{diam } C_\lambda(x, z),$$

Equation (70) implies that

$$\begin{aligned} -\frac{C_p}{p_0} \text{diam } C_\lambda(x, z) \text{vol } C_\lambda(x, z)^{-1} &\leq \left\{ \int_{C_\lambda(x, z)} \frac{p(y)}{p(z)} dy \right\}^{-1} - \text{vol } C_\lambda(x, z)^{-1} \\ &\leq \frac{p_1 C_p}{p_0^2} \text{diam } C_\lambda(x, z) \text{vol } C_\lambda(x, z)^{-1}. \end{aligned}$$

By taking the expectation over  $C_\lambda(x, z)$ , and recalling the identity (67), this gives

$$\begin{aligned} -\frac{C_p}{p_0} \mathbb{E} [\text{diam } C_\lambda(x, z) \text{vol } C_\lambda(x, z)^{-1}] &\leq \exp(\lambda \|x - z\|_1) (F_{p, \lambda}(x, z) - F_\lambda(x, z)) \\ &\leq \frac{p_1 C_p}{p_0^2} \mathbb{E} [\text{diam } C_\lambda(x, z) \text{vol } C_\lambda(x, z)^{-1}] \end{aligned}$$

and hence

$$|F_{p, \lambda}(x, z) - F_\lambda(x, z)| \leq \frac{p_1 C_p}{p_0^2} \exp(-\lambda \|x - z\|_1) \mathbb{E} [\text{diam } C_\lambda(x, z) \text{vol } C_\lambda(x, z)^{-1}]. \quad (71)$$

**Control of  $\mathbb{E} [\text{diam } C_\lambda(x, z) \text{vol } C_\lambda(x, z)^{-1}]$ .** Let  $C_\lambda^j(x, z) := [(x_j \wedge z_j - \lambda^{-1} E_L^j) \vee 0, (x_j \vee z_j + \lambda^{-1} E_R^j) \wedge 1]$ , and  $|C_\lambda^j(x, z)| = (x_j \vee z_j + \lambda^{-1} E_R^j) \wedge 1 - (x_j \wedge z_j - \lambda^{-1} E_L^j) \vee 0$  be its length.

We have, using the triangular inequality,  $\text{diam } C_\lambda(x, z) \leq \text{diam}_{\ell^1} C_\lambda(x, z)$ , so that

$$\begin{aligned}
\mathbb{E} [\text{diam } C_\lambda(x, z) \text{vol } C_\lambda(x, z)^{-1}] &\leq \mathbb{E} \left[ \sum_{j=1}^d |C_\lambda^j(x, z)| \text{vol } C_\lambda(x, z)^{-1} \right] \\
&= \sum_{j=1}^d \mathbb{E} \left[ |C_\lambda^j(x, z)| \prod_{l=1}^d |C_\lambda^l(x, z)|^{-1} \right] \\
&= \sum_{j=1}^d \mathbb{E} \left[ \prod_{l \neq j} |C_\lambda^l(x, z)|^{-1} \right] \\
&\leq \sum_{j=1}^d \mathbb{E} [|C_\lambda^j(x, z)|] \mathbb{E} [|C_\lambda^j(x, z)|^{-1}] \mathbb{E} \left[ \prod_{l \neq j} |C_\lambda^l(x, z)|^{-1} \right] \quad (72)
\end{aligned}$$

$$= \sum_{j=1}^d \mathbb{E} [|C_\lambda^j(x, z)|] \times \mathbb{E} \left[ \prod_{l=1}^d |C_\lambda^l(x, z)|^{-1} \right] \quad (73)$$

$$= \mathbb{E} [\text{diam}_{\ell^1} C_\lambda(x, z)] \times \exp(\lambda \|x - z\|_1) F_\lambda(x, z) \quad (74)$$

where inequality (72) relies on the fact that, for any positive real variable  $X$ ,  $\mathbb{E}[X]^{-1} \leq \mathbb{E}[X^{-1}]$  by convexity of the inverse function, and thus  $\mathbb{E}[X] \mathbb{E}[X^{-1}] \geq 1$  (here  $X = |C_\lambda^j(x, z)|$ ), while Equation (73) is a consequence of the independence of  $|C_\lambda^1(x, z)|, \dots, |C_\lambda^d(x, z)|$ . Multiplying both sides of (74) by  $\exp(-\lambda \|x - z\|_1)$  yields

$$\exp(-\lambda \|x - z\|_1) \mathbb{E} [\text{diam } C_\lambda(x, z) \text{vol } C_\lambda(x, z)^{-1}] \leq \mathbb{E} [\text{diam}_{\ell^1} C_\lambda(x, z)] F_\lambda(x, z). \quad (75)$$

In addition,

$$\begin{aligned}
\mathbb{E} [\text{diam}_{\ell^1} C_\lambda(x, z)] &= \sum_{j=1}^d \mathbb{E} [|C_\lambda^j(x, z)|] \\
&\leq \sum_{j=1}^d \mathbb{E} [|x_j - z_j| + \lambda^{-1} (E_R^j + E_L^j)] \\
&= \|x - z\|_1 + \frac{2d}{\lambda}. \quad (76)
\end{aligned}$$

Finally, combining the bounds (71), (75) and (76) gives

$$|F_{p,\lambda}(x, z) - F_\lambda(x, z)| \leq \frac{p_1 C_p}{p_0^2} \left[ \|x - z\|_1 + \frac{2d}{\lambda} \right] F_\lambda(x, z). \quad (77)$$

**Control of the bias.** From (77), we can control  $\int_{[0,1]^d} (z - x) F_{p,\lambda}(x, z) dz$  by approximating  $F_{p,\lambda}$  by  $F_\lambda$ . Indeed, we have

$$\left\| \int_{[0,1]^d} (z - x) F_{p,\lambda}(x, z) dz - \int_{[0,1]^d} (z - x) F_\lambda(x, z) dz \right\| \leq \int_{[0,1]^d} \|z - x\| |F_{p,\lambda}(x, z) - F_\lambda(x, z)| dz, \quad (78)$$

with

$$\begin{aligned}
& \int_{[0,1]^d} \|z - x\| |F_{p,\lambda}(x, z) - F_\lambda(x, z)| dz \\
& \leq \frac{p_1 C_p}{p_0^2} \int_{[0,1]^d} \|z - x\| \left[ \|x - z\|_1 + \frac{2d}{\lambda} \right] F_\lambda(x, z) dz \quad (\text{by (77)}) \\
& \leq \frac{p_1 C_p}{p_0^2} \sqrt{d} \int_{[0,1]^d} \|z - x\|^2 F_\lambda(x, z) dz + \frac{p_1 C_p}{p_0^2} \frac{2d}{\lambda} \int_{[0,1]^d} \|z - x\| F_\lambda(x, z) dz \\
& \leq \frac{p_1 C_p}{p_0^2} \frac{d\sqrt{d}}{\lambda^2} + \frac{p_1 C_p}{p_0^2} \frac{2d}{\lambda} \sqrt{\int_{[0,1]^d} \|z - x\|^2 F_\lambda(x, z) dz} \quad (\text{by Cauchy-Schwarz}) \\
& \leq \frac{p_1 C_p}{p_0^2} \frac{d\sqrt{d}}{\lambda^2} + \frac{p_1 C_p}{p_0^2} \frac{2d}{\lambda} \sqrt{\frac{d}{\lambda^2}} \\
& = \frac{p_1 C_p}{p_0^2} \frac{3d\sqrt{d}}{\lambda^2}, \tag{79}
\end{aligned}$$

where we used several times the inequalities  $\|v\| \leq \|v\|_1 \leq \sqrt{d}\|v\|$ . Using the inequalities (78) and (79), together with Technical Lemma 2, we obtain

$$\begin{aligned}
& \left\| \int_{[0,1]^d} (z - x) F_{p,\lambda}(x, z) dz \right\|^2 \\
& \leq 2 \left\| \int_{[0,1]^d} (z - x) F_\lambda(x, z) dz \right\|^2 + 2 \left( \int_{[0,1]^d} \|z - x\| |F_{p,\lambda}(x, z) - F_\lambda(x, z)| dz \right)^2 \\
& \leq \frac{18}{\lambda^2} \sum_{j=1}^d e^{-\lambda x_j \wedge (1-x_j)} + 2 \left( \frac{p_1 C_p}{p_0^2} \frac{3d\sqrt{d}}{\lambda^2} \right)^2.
\end{aligned}$$

Now, using inequality (65), the bias term satisfies

$$\begin{aligned}
|\tilde{f}_\lambda(x) - f(x)|^2 & \leq 2 \left| \nabla f(x) \cdot \int_{[0,1]^d} (z - x) F_{p,\lambda}(x, z) dz \right|^2 + 2C_2^2 \left( \int_{[0,1]^d} \frac{1}{2} \|z - x\|^2 F_{p,\lambda}(x, z) dz \right)^2 \\
& \leq 2G^2 \left\| \int_{[0,1]^d} (z - x) F_{p,\lambda}(x, z) dz \right\|^2 + 2C_2^2 \left( \int_{[0,1]^d} \frac{1}{2} \|z - x\|^2 F_{p,\lambda}(x, z) dz \right)^2 \\
& \leq \frac{36G^2}{\lambda^2} \sum_{j=1}^d e^{-\lambda x_j \wedge (1-x_j)} + \frac{36G^2 d^3}{\lambda^4} \left( \frac{p_1 C_p}{p_0^2} \right)^2 + \frac{2C_2^2}{\lambda^4} \left( \frac{p_1 d}{p_0} \right)^2. \tag{80}
\end{aligned}$$

where  $G := \sup_{x \in [0,1]^d} \|\nabla f(x)\|$ . As before, integrating the previous inequality, and recalling that the variance term satisfies

$$\mathbb{E} \left[ (\tilde{f}_{\lambda,n}^{(1)}(X) - \tilde{f}_\lambda^{(1)}(X))^2 | X \in [\varepsilon, 1 - \varepsilon]^d \right] \leq \frac{(1 + \lambda)^d 2\sigma^2 + 9\|f\|_\infty^2}{n p_0 (1 - 2\varepsilon)^d},$$

we finally obtain, using inequality (51),

$$\begin{aligned}
\mathbb{E}[(\hat{f}_{\lambda,n}^{(M)}(X) - f(X))^2 | X \in [\varepsilon, 1 - \varepsilon]^d] & \leq \frac{8dG^2}{M\lambda^2} + \frac{2(1 + \lambda)^d 2\sigma^2 + 9\|f\|_\infty^2}{n p_0 (1 - 2\varepsilon)^d} \\
& \quad + \frac{72G^2 d p_1}{p_0 (1 - 2\varepsilon)^d} \frac{e^{-\lambda\varepsilon}}{\lambda^3} + \frac{72G^2 d^3}{\lambda^4} \left( \frac{p_1 C_p}{p_0^2} \right)^2 + \frac{4C_2^2 d^2 p_1^2}{p_0^2} \frac{1}{\lambda^4}. \tag{81}
\end{aligned}$$

In particular, if  $\varepsilon \in (0, \frac{1}{2})$  is fixed, the upper bound (81) gives

$$\mathbb{E}[(\hat{f}_{\lambda,n}^{(M)}(X) - f(X))^2 | X \in [\varepsilon, 1 - \varepsilon]^d] \leq O\left(\frac{\lambda^d}{n}\right) + O\left(\frac{1}{\lambda^4}\right) + O\left(\frac{1}{M\lambda^2}\right);$$

while for the integrated risk over the whole hypercube, the bound (81) with  $\varepsilon = 0$  leads to

$$\mathbb{E}[(\hat{f}_{\lambda,n}^{(M)}(X) - f(X))^2] \leq O\left(\frac{\lambda^d}{n}\right) + O\left(\frac{1}{\lambda^3}\right) + O\left(\frac{1}{M\lambda^2}\right).$$

The rate being the same as in the uniform case, the same conclusions follow.

### 7.5.3 Technical Lemma 2

**Technical Lemma 2.** For all  $x \in [0, 1]^d$ ,

$$\left\| \int_{[0,1]^d} (z - x) F_\lambda(x, z) dz \right\|^2 \leq \frac{9}{\lambda^2} \sum_{j=1}^d e^{-\lambda[x_j \wedge (1-x_j)]}$$

and

$$\int_{[0,1]^d} \frac{1}{2} \|z - x\|^2 F_\lambda(x, z) dz \leq \frac{d}{\lambda^2}.$$

*Proof.* According to Equation (58), we have

$$F_\lambda(x, z) = \lambda^d \exp(-\lambda \|x - z\|_1) \prod_{1 \leq j \leq d} G_\lambda(x_j, z_j) \quad (82)$$

where we defined, for  $u, v \in [0, 1]$ ,

$$\begin{aligned} G_\lambda(u, v) &= \mathbb{E} \left[ (\lambda |u - v| + E_1 \wedge \lambda(u \wedge v) + E_2 \wedge \lambda(1 - u \vee v))^{-1} \right] \\ &= H(\lambda |u - v|, \lambda u \wedge v, \lambda(1 - u \vee v)) \end{aligned}$$

with  $E_1, E_2$  two independent  $\text{Exp}(1)$  random variables, and  $H : (\mathbf{R}_+^*)^3 \rightarrow \mathbf{R}$  the function defined by

$$H(a, b_1, b_2) = \mathbb{E} \left[ (a + E_1 \wedge b_1 + E_2 \wedge b_2)^{-1} \right];$$

also, let

$$H(a) = \mathbb{E} \left[ (a + E_1 + E_2)^{-1} \right].$$

Denote

$$\begin{aligned} A &= \int_{[0,1]^d} (z - x) F_\lambda(x, z) dz \\ B &= \int_{[0,1]^d} \frac{1}{2} \|z - x\|^2 F_\lambda(x, z) dz. \end{aligned}$$

Since  $1 = \int F_\lambda^{(1)}(u, v) dv = \int \lambda \exp(-\lambda |u - v|) G_\lambda(u, v) dv$ , applying Fubini's theorem we obtain

$$A_j = \Phi_\lambda^1(x_j) \quad \text{and} \quad B = \sum_{j=1}^d \Phi_\lambda^2(x_j) \quad (83)$$



where we define for  $u \in [0, 1]$  and  $k \in \mathbf{N}$

$$\Phi_\lambda^k(u) = \int_0^1 \lambda \exp(-\lambda|u-v|) G_\lambda(u, v) \frac{(v-u)^k}{k!} dv. \quad (84)$$

Observe that

$$\Phi_\lambda^k(u) = \lambda^{-k} \int_{-\lambda u}^{\lambda(1-u)} \frac{v^k}{k!} \exp(-|v|) H(|v|, \lambda u + v \wedge 0, \lambda(1-u) - v \vee 0) dv.$$

We will show that  $\Phi_\lambda^k(u) = O(\lambda^{-2})$  for  $k = 1, 2$ . First, write

$$\lambda \Phi_\lambda^1(u) = - \int_0^{\lambda u} v e^{-v} H(v, \lambda u - v, \lambda(1-u)) dv + \int_0^{\lambda(1-u)} v e^{-v} H(v, \lambda u, \lambda(1-u) - v) dv$$

Now, let  $\beta := \lambda \frac{u \wedge (1-u)}{2}$ . We have

$$\begin{aligned} & \lambda \Phi_\lambda^1(u) - \int_0^\beta v e^{-v} [H(v, \lambda u, \lambda(1-u) - v) - H(v, \lambda u - v, \lambda(1-u))] dv \\ &= - \underbrace{\int_\beta^{\lambda u} v e^{-v} H(v, \lambda u - v, \lambda(1-u)) dv}_{:= I_1 \geq 0} + \underbrace{\int_\beta^{\lambda(1-u)} v e^{-v} H(v, \lambda u, \lambda(1-u) - v) dv}_{:= I_2 \geq 0} \end{aligned}$$

so that the left-hand side of the above equation is between  $-I_1 \leq 0$  and  $I_2 \geq 0$ , and thus its absolute value is bounded by  $|I_1| \vee |I_2|$ . Now, note that, since  $H(v, \cdot, \cdot) \leq v^{-1}$ , we have

$$|I_2| \leq \int_\beta^\infty v e^{-v} v^{-1} dv = e^{-\beta}$$

and similarly  $|I_1| \leq e^{-\beta}$ , so that

$$\left| \lambda \Phi_\lambda^1(u) - \underbrace{\int_0^\beta v e^{-v} [H(v, \lambda u, \lambda(1-u) - v) - H(v, \lambda u - v, \lambda(1-u))] dv}_{:= I_3} \right| \leq e^{-\beta} \quad (85)$$

It now remains to bound  $|I_3|$ . For that purpose, note that since  $H$  is decreasing in its second and third argument, we have

$$\begin{aligned} H(v) - H(v, \lambda u - v, \lambda(1-u)) &\leq H(v, \lambda u, \lambda(1-u) - v) - H(v, \lambda u - v, \lambda(1-u)) \\ &\leq H(v, \lambda u, \lambda(1-u) - v) - H(v) \end{aligned}$$

which implies, since the above lower bound is non-positive and the upper bound nonnegative,

$$\begin{aligned} & |H(v, \lambda u, \lambda(1-u) - v) - H(v, \lambda u - v, \lambda(1-u))| \\ &\leq \max(|H(v, \lambda u, \lambda(1-u) - v) - H(v)|, |H(v) - H(v, \lambda u - v, \lambda(1-u))|). \end{aligned}$$

Besides, since  $(a + E_1 \wedge b_1 + E_2 \wedge b_2)^{-1} \leq (a + E_1 + E_2)^{-1} + a^{-1}(\mathbf{1}\{E_1 \geq b_1\} + \mathbf{1}\{E_2 \geq b_2\})$ ,

$$H(a, b_1, b_2) - H(a) \leq a^{-1}(e^{-b_1} + e^{-b_2}), \quad (86)$$

for all  $a, b_1, b_2$ . Since  $\lambda u - v \geq \beta$  and  $\lambda(1-u) - v \geq \beta$  for  $v \in [0, \beta]$ , we have

$$|H(v) - H(v, \lambda u - v, \lambda(1-u))|, |H(v) - H(v, \lambda u, \lambda(1-u) - v)| \leq 2v^{-1}e^{-\beta}$$

so that for  $v \in [0, \beta]$

$$|H(v, \lambda u, \lambda(1-u) - v) - H(v, \lambda u - v, \lambda(1-u))| \leq 2v^{-1}e^{-\beta}$$

and hence

$$\begin{aligned} |I_3| &\leq \int_0^\beta v e^{-v} |H(v, \lambda u, \lambda(1-u) - v) - H(v, \lambda u - v, \lambda(1-u))| dv \\ &\leq \int_0^\beta v e^{-v} 2v^{-1} e^{-\beta} dv \\ &\leq 2e^{-\beta} \int_0^\infty e^{-v} dv \\ &= 2e^{-\beta} \end{aligned} \tag{87}$$

Combining Equations (85) and (87) yields:

$$|\Phi_\lambda^1(u)| \leq \frac{3}{\lambda} e^{-\lambda[u \wedge (1-u)]/2} \tag{88}$$

that is,

$$\left\| \int_{[0,1]^d} (z-x) F_\lambda(x, z) dz \right\|^2 = \sum_{j=1}^d (\Phi_\lambda^1(x_j))^2 \leq \frac{9}{\lambda^2} \sum_{j=1}^d e^{-\lambda[x_j \wedge (1-x_j)]}.$$

Furthermore,

$$\begin{aligned} 0 \leq \Phi_\lambda^2(u) &= \lambda^{-2} \int_{-\lambda u}^{\lambda(1-u)} \frac{v^2}{2} e^{-|v|} H(|v|, \lambda u + v \wedge 0, \lambda(1-u) - v \vee 0) dv \\ &\leq \lambda^{-2} \int_0^\infty v^2 e^{-v} v^{-1} dv \\ &= \lambda^{-2} \end{aligned}$$

so that

$$0 \leq \Phi_\lambda^2(u) \leq \frac{1}{\lambda^2},$$

which proves the second inequality by summing over  $j = 1, \dots, d$ . □

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