university of groningen

Program Correctness

Block 8

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(based on slides by Arnold Meijster)

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Outline

A Digression on Counting

Exercise 9.7: Increasing & Descending
- Increasing & Descending
- The Roadmap: Triangle Case

Exercise 9.12: Ascending & Descending
- Ascending & Descending
- The Roadmap: Triangle Case

Exercise 9.14: Two Ascending Parameters
- Recurrence: Ascending Parameters
- The Roadmap: A Different Invariant

One-Dimensional Counting

- It is useful to compare the recurrence relations for 1D counting (in an array) and those for 2D counting.
- Consider the following spec:

const n : $\mathbb{N}$;
const a : **array** $[0..n)$ **of** $\mathbb{Z}$;
var x : $\mathbb{Z}$;
  $\{P : \textbf{true}\}$
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  $\{Q : x = \#\{i \mid a[i] = 42 \wedge i \in [0..n)\}\}$

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  and then we revise the postcondition to $Q : x = A(n)$.
  The invariant is $J : x = A(k) \wedge 0 \leq k \leq n$.

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  = & \text{(* } \textit{split domain: } i = k,\ a[k] \textit{ is defined} \text{ *)} \\
  & \mathrm{ord}(a[k] = 42) + \#\{i \mid a[i] = 42 \wedge i \in [0..k)\} \\
  = & \text{(* } \textit{definition } A(k) \text{ *)} \\
  & \mathrm{ord}(a[k] = 42) + A(k)
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  Hence, we have $A(0) = 0$ and
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- In the 2D case, we have $F(x, y)$ and work to express it in terms of $F(x+1, y)$ OR $F(x, y+1)$ OR $F(x-1, y)$ OR $F(x, y-1)$.
- Can you explain the difference?

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Exercise 9.7

Let $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ be a function increasing ($<$ / $<$) in x and descending ($\geq$ / $\leq$) in y:

$$x_0 < x_1 \Rightarrow g(x_0, y) < g(x_1, y)$$
$$y_0 \leq y_1 \Rightarrow g(x, y_0) \geq g(x, y_1)$$

Given $w \in \mathbb{Z}$ and $n \in \mathbb{N}$, specify and design a command to compute the number of pairs $(i, j) \in \mathbb{N}^2$ with

1. $g(i, j) = w$
2. $i + j < n$.

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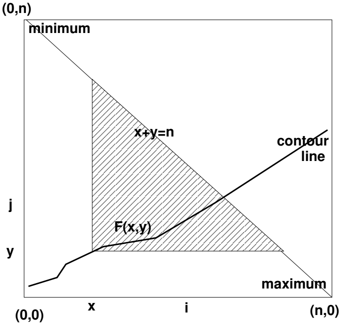
1. $g(i, j) = w$
2. $i + j < n$.

While condition (1) is as in previous examples, condition (2) constrains the "shrinking area": the rectangle becomes a triangle.

Exercise 9.7

In principle, we want to compute:

$$\#\{(i,j) \mid 0 \le i \ \wedge \ 0 \le j \ \wedge \ i+j < n \ \wedge \ g(i,j) = w\}$$



Let $F(x,y)$ be the number of points that we still need to count:

$$F(x,y) = \#\{(i,j) \mid x \le i \ \wedge \ y \le j \ \wedge \ i+j < n \ \wedge \ g(i,j) = w\}$$

Our goal is to compute $F(0,0)$.

Exercise 9.7

Given $F(x, y) = \#\{(i, j) \mid x \leq i \wedge y \leq j \wedge i + j < n \wedge g(i, j) = w\}$, we can specify the command T as follows:

const n : $\mathbb{N}$;
var z : $\mathbb{N}$;
$\{P : Z = F(0, 0)\}$
T
$\{Q : z = Z\}$

We reduce the triangle by maintaining the usual invariant:

$$\boxed{J : \; Z = z + F(x, y)}$$

Exercise 9.7

Given $F(x,y) = \#\{(i,j) \mid x \leq i \wedge y \leq j \wedge i+j < n \wedge g(i,j) = w\}$, it is easy to observe the base case:

$$x + y \geq n \Rightarrow F(x,y) = 0$$

We see how to reduce the size of the triangle by incrementing x:

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\begin{array}{ll}
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= & \{ \text{ definition } F \text{ } \} \\
 & \#\{(i,j) \mid x \leq i \ \wedge\ y \leq j \ \wedge\ i+j < n \ \wedge\ g(i,j) = w\} \\
= & \{ \textbf{ assume } x + y < n; \\
 & \quad \text{split non-empty domain; definition } F \text{ } \} \\
 & F(x+1,y) + \#\{j \mid j :\ y \leq j \ \wedge\ x + j < n \ \wedge\ g(x,j) = w\} \\
= & \{ \ x + j < n \text{ so } j < n - x \ \} \\
 & F(x+1,y) + \#\{j \mid j :\ y \leq j < n - x \ \wedge\ g(x,j) = w\} \\
= & \{ \ g(x,j) \text{ is descending in } j \text{ so } g(x,y) \text{ is maximal;} \\
 & \quad \textbf{assume } g(x,y) < w \text{, so } g(x,j) < w \text{ for all } j \geq y \ \} \\
 & F(x+1,y)
\end{array}
$$

Exercise 9.7

Next, we investigate what happens if we increment y:

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\begin{array}{ll}
 & F(x,y) \\
= & \{ \text{ definition } F \ \} \\
 & \#\{(i,j) \mid x \le i \wedge y \le j \wedge i+j < n \wedge g(i,j) = w\} \\
= & \{ \ \textbf{assume } x+y<n; \\
 & \quad \text{split non-empty domain; definition } F \ \} \\
 & F(x,y+1) + \#\{i \mid i:\ x \le i \wedge i+y < n \wedge g(i,y) = w\} \\
= & \{ \ g(i,y) \text{ is increasing in } i \text{ so } g(x,y) \text{ is minimal;} \\
 & \quad \textbf{assume } g(x,y) \ge w; \\
 & \quad \text{since } g(i,y) \text{ is increasing we have } g(x,y) > w \text{ for } x+1 \le i < n \ \} \\
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 & \quad \textbf{assume } g(x,y) \ge w; \\
 & \quad \text{since } g(i,y) \text{ is increasing we have } g(x,y) > w \text{ for } x + 1 \le i < n \text{ } \} \\
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In conclusion, $F(x,y)$ satisfies the following recursive equations:

$$
\boxed{
\begin{aligned}
x + y \ge n &\Rightarrow F(x,y) = 0 \\
x + y < n \wedge g(x,y) < w &\Rightarrow F(x,y) = F(x+1, y) \\
x + y < n \wedge g(x,y) \ge w &\Rightarrow F(x,y) = F(x, y+1) + \text{ord}(g(x,y) = w)
\end{aligned}
}
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Exercise 9.7: Guard & Initialization

We will iteratively reduce the remaining area by incrementing x or y.
We choose the guard $\boxed{B:\ x + y < n}$ such that $J \wedge \neg B \Rightarrow Q$.

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\begin{array}{ll}
 & J \wedge \neg B \\
\equiv & \{ \text{ definition } J \text{ and } B \; \} \\
 & Z = z + F(x, y) \wedge \neg(x + y < n) \\
\equiv & \{ \text{ Logic } \} \\
 & Z = z + F(x, y) \wedge x + y \geq n \\
\Rightarrow & \{ \text{ base case recurrence: } F(x, y) = 0 \; \} \\
 & Q : Z = z
\end{array}
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The initialization is easy:

$\{P:\ Z = F(0, 0)\}$
(* *calculus* *)
$\{Z = 0 + F(0, 0)\}$
$z := 0;\ x := 0;\ y := 0;$
$\{J:\ Z = z + F(x, y)\}$

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z := 0; \; x := 0; \; y := 0; \\
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\end{array}
$$

Since we increment x or y as long as B holds, we choose the variant function $vf = n - x - y \in \mathbb{Z}$. Clearly, $J \wedge B \Rightarrow vf \geq 0$.

Exercise 9.7: Body of the Loop

$\{Z = z + F(x, y) \wedge x + y < n \ \wedge \ n - x - y = V\}$
if $g(x, y) < w$ **then**

$x := x + 1;$

else

$y := y + 1;$

end
$\{J \ \wedge \ vf < V\}$

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(* *logic; recurrence for* $F(x, y)$*: case* $x + y < n \wedge g(x, y) < w$ *)

$\{Z = z + F(x + 1, y) \ \wedge \ n - x - y = V\}$

$x := x + 1;$

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$z := z + \mathrm{ord}(g(x, y) = w);$

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(* *calculus; prepare* $x := x + 1$ *)

$\{Z = z + F(x + 1, y) \ \wedge \ n - (x + 1) - y < V\}$

$x := x + 1;$

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$z := z + \mathrm{ord}(g(x, y) = w);$

$y := y + 1;$

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$\{Z = z + F(x+1,y) \ \wedge\ n - (x+1) - y < V\}$

$x := x + 1;$

$\{Z = z + F(x,y) \ \wedge\ n - x - y < V\}$

else

$\{g(x,y) \geq w \ \wedge\ Z = z + F(x,y) \ \wedge\ x + y < n \ \wedge\ n - x - y = V\}$

$z := z + \text{ord}(g(x,y) = w);$

$y := y + 1;$

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(* *logic; recurrence for* $F(x, y)$*: case* $x + y < n \wedge g(x, y) < w$ *)

$\{Z = z + F(x + 1, y) \ \wedge \ n - x - y = V\}$

(* *calculus; prepare* $x := x + 1$ *)

$\{Z = z + F(x + 1, y) \ \wedge \ n - (x + 1) - y < V\}$

$x := x + 1;$

$\{Z = z + F(x, y) \ \wedge \ n - x - y < V\}$

else

$\{g(x, y) \geq w \ \wedge \ Z = z + F(x, y) \ \wedge \ x + y < n \ \wedge \ n - x - y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x + y < n \ \wedge \ g(x, y) \geq w$ *)

$\{Z = z + \mathrm{ord}(g(x, y) = w) + F(x, y + 1) \ \wedge \ n - x - y = V\}$

$z := z + \mathrm{ord}(g(x, y) = w);$

$y := y + 1;$

end

$\{J \ \wedge \ vf < V\}$

Exercise 9.7: Body of the Loop

$\{Z = z + F(x, y) \wedge x + y < n \ \wedge\ n - x - y = V\}$

if $g(x, y) < w$ **then**

$\{g(x, y) < w \ \wedge\ Z = z + F(x, y) \ \wedge\ x + y < n \ \wedge\ n - x - y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x + y < n \wedge g(x, y) < w$ *)

$\{Z = z + F(x + 1, y) \ \wedge\ n - x - y = V\}$

(* *calculus; prepare* $x := x + 1$ *)

$\{Z = z + F(x + 1, y) \ \wedge\ n - (x + 1) - y < V\}$

$x := x + 1;$

$\{Z = z + F(x, y) \ \wedge\ n - x - y < V\}$

else

$\{g(x, y) \geq w \ \wedge\ Z = z + F(x, y) \ \wedge\ x + y < n \ \wedge\ n - x - y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x + y < n \ \wedge\ g(x, y) \geq w$ *)

$\{Z = z + \text{ord}(g(x, y) = w) + F(x, y + 1) \ \wedge\ n - x - y = V\}$

$z := z + \text{ord}(g(x, y) = w);$

$\{Z = z + F(x, y + 1) \ \wedge\ n - x - y = V\}$

$y := y + 1;$

end

$\{J \ \wedge\ vf < V\}$

Exercise 9.7: Body of the Loop

$\{Z = z + F(x, y) \wedge x + y < n \ \wedge \ n - x - y = V\}$

if $g(x, y) < w$ **then**

$\{g(x, y) < w \ \wedge \ Z = z + F(x, y) \ \wedge \ x + y < n \ \wedge \ n - x - y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x + y < n \wedge g(x, y) < w$ *)

$\{Z = z + F(x + 1, y) \ \wedge \ n - x - y = V\}$

(* *calculus; prepare* $x := x + 1$ *)

$\{Z = z + F(x + 1, y) \ \wedge \ n - (x + 1) - y < V\}$

$x := x + 1;$

$\{Z = z + F(x, y) \ \wedge \ n - x - y < V\}$

else

$\{g(x, y) \geq w \ \wedge \ Z = z + F(x, y) \ \wedge \ x + y < n \ \wedge \ n - x - y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x + y < n \ \wedge \ g(x, y) \geq w$ *)

$\{Z = z + \text{ord}(g(x, y) = w) + F(x, y + 1) \ \wedge \ n - x - y = V\}$

$z := z + \text{ord}(g(x, y) = w);$

$\{Z = z + F(x, y + 1) \ \wedge \ n - x - y = V\}$

(* *calculus; prepare* $y := y + 1$ *)

$\{Z = z + F(x, y + 1) \ \wedge \ n - x - (y + 1) < V\}$

$y := y + 1;$

end

$\{J \ \wedge \ vf < V\}$

Exercise 9.7: Body of the Loop

$\{Z = z + F(x,y) \wedge x + y < n \ \wedge \ n - x - y = V\}$

if $g(x,y) < w$ **then**

$\{g(x,y) < w \ \wedge \ Z = z + F(x,y) \ \wedge \ x + y < n \ \wedge \ n - x - y = V\}$

(* *logic; recurrence for* $F(x,y)$*: case* $x + y < n \wedge g(x,y) < w$ *)

$\{Z = z + F(x+1,y) \ \wedge \ n - x - y = V\}$

(* *calculus; prepare* $x := x + 1$ *)

$\{Z = z + F(x+1,y) \ \wedge \ n - (x+1) - y < V\}$

$x := x + 1;$

$\{Z = z + F(x,y) \ \wedge \ n - x - y < V\}$

else

$\{g(x,y) \geq w \ \wedge \ Z = z + F(x,y) \ \wedge \ x + y < n \ \wedge \ n - x - y = V\}$

(* *logic; recurrence for* $F(x,y)$*: case* $x + y < n \ \wedge \ g(x,y) \geq w$ *)

$\{Z = z + \mathrm{ord}(g(x,y) = w) + F(x,y+1) \ \wedge \ n - x - y = V\}$

$z := z + \mathrm{ord}(g(x,y) = w);$

$\{Z = z + F(x,y+1) \ \wedge \ n - x - y = V\}$

(* *calculus; prepare* $y := y + 1$ *)

$\{Z = z + F(x,y+1) \ \wedge \ n - x - (y+1) < V\}$

$y := y + 1;$

$\{Z = z + F(x,y) \ \wedge \ n - x - y < V\}$

end

$\{J \ \wedge \ vf < V\}$

Exercise 9.7: Body of the Loop

$\{Z = z + F(x, y) \wedge x + y < n \ \wedge \ n - x - y = V\}$

if $g(x, y) < w$ **then**

$\{g(x, y) < w \ \wedge \ Z = z + F(x, y) \ \wedge \ x + y < n \ \wedge \ n - x - y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x + y < n \wedge g(x, y) < w$ *)

$\{Z = z + F(x + 1, y) \ \wedge \ n - x - y = V\}$

(* *calculus; prepare* $x := x + 1$ *)

$\{Z = z + F(x + 1, y) \ \wedge \ n - (x + 1) - y < V\}$

$x := x + 1;$

$\{Z = z + F(x, y) \ \wedge \ n - x - y < V\}$

else

$\{g(x, y) \geq w \ \wedge \ Z = z + F(x, y) \ \wedge \ x + y < n \ \wedge \ n - x - y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x + y < n \ \wedge \ g(x, y) \geq w$ *)

$\{Z = z + \mathrm{ord}(g(x, y) = w) + F(x, y + 1) \ \wedge \ n - x - y = V\}$

$z := z + \mathrm{ord}(g(x, y) = w);$

$\{Z = z + F(x, y + 1) \ \wedge \ n - x - y = V\}$

(* *calculus; prepare* $y := y + 1$ *)

$\{Z = z + F(x, y + 1) \ \wedge \ n - x - (y + 1) < V\}$

$y := y + 1;$

$\{Z = z + F(x, y) \ \wedge \ n - x - y < V\}$

end (* *collect branches; definitions* J*, and* vf *)

$\{J \ \wedge \ vf < V\}$

Exercise 9.7: Conclusion

const $n : \mathbb{N},\ w : \mathbb{Z}$;
var $x, y, z : \mathbb{Z}$;
  $\{P :\ \#\{(i,j) \mid 0 \leq i \ \wedge\ 0 \leq j \ \wedge\ i+j < n \ \wedge\ g(i,j) = w\}\ \}$
$z := 0$;
$x := 0$;
$y := 0$;
  $\{J :\ Z = z + \#\{(i,j) \mid x \leq i \ \wedge\ y \leq j \ \wedge\ i+j < n \ \wedge\ g(i,j) = w\}\ \}$
    (* $vf :\ n - x - y$ *)
while $x + y < n$ **do**
  if $g(x,y) < w$ **then**
    $x := x + 1$;
  else
    $z := z + \text{ord}(g(x,y) = w)$;
    $y := y + 1$;
  end;
end;
  $\{Q :\ z = Z\}$

Note: Initially, $vf = n$, so the algorithm has time complexity $O(n)$, which is much more efficient than an $O(m \cdot n)$ algorithm.

Outline

A Digression on Counting

Exercise 9.7: Increasing & Descending
- Increasing & Descending
- The Roadmap: Triangle Case

Exercise 9.12: Ascending & Descending
- Ascending & Descending
- The Roadmap: Triangle Case

Exercise 9.14: Two Ascending Parameters
- Recurrence: Ascending Parameters
- The Roadmap: A Different Invariant

Exercise 9.12

For all $i, j \in \mathbb{Z}$, the boolean function p satisfies the recursion:

$$\begin{array}{rcl} p(i,j) & \Rightarrow & p(i+1,j) \\ p(i,j+1) & \Rightarrow & p(i,j) \end{array}$$

We want to find a command T that satisfies the specification:

const $m: \mathbb{N}$;
var $z: \mathbb{Z}$;
$\{P: Z = \#\{(i,j) \mid 0 \leq i \wedge 0 \leq j \wedge i + 2 \cdot j < m \wedge p(i,j)\}\}$
T;
$\{Q: Z = z\}$

Exercise 9.12: Example

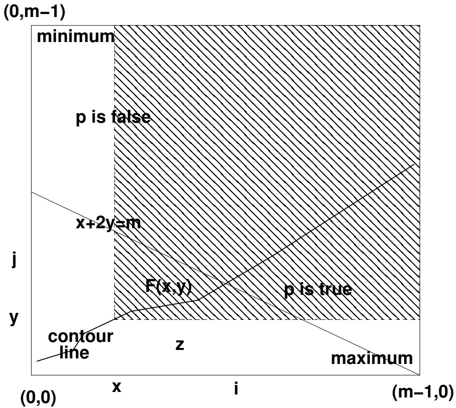
$$\begin{array}{rcl} p(i,j) & \Rightarrow & p(i+1,j) \\ p(i,j+1) & \Rightarrow & p(i,j) \end{array}$$

×	×	×	×	×	×	×	×	×	×
×	×	×	×	×	×	×	×	×	×
×	×	×	×	×	×	×	×	×	×
×	×	×	×	×	×	×	×	×	×
×	×	×	×	×	×	×	×	×	×
0	0	×	×	×	×	×	×	×	×
0	0	0	**1**	×	×	×	×	×	×
0	0	**1**	**1**	**1**	**1**	×	×	×	×
0	0	**1**	**1**	**1**	**1**	**1**	**1**	×	×
1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**

$$\#\{(i,j) \mid 0 \leq i \wedge 0 \leq j \;\wedge\; i + 2 \cdot j < m \;\wedge\; p(i,j)\} \;= 21$$

Note: p is ascending on x and descending on j.

Exercise 9.12



Let $F(x, y)$ be the number of points that we still need to process:

$$F(x, y) = \#\{(i, j) \mid x \leq i \wedge y \leq j \ \wedge \ i + 2 \cdot j < m \ \wedge \ p(i, j)\}$$

We store the already counted points in z, so we maintain the invariant:

$$\boxed{J : \ Z = z + F(x, y)}$$

Exercise 9.12

$$F(x, y) = \#\{(i, j) \mid x \leq i \wedge y \leq j \wedge i + 2 \cdot j < m \wedge p(i, j)\}$$

- We can rewrite the precondition P as $Z = F(0, 0)$.
- Hence, we start in $(x, y) = (0, 0)$ and will increment x and y.

Exercise 9.12

$$F(x, y) = \#\{(i, j) \mid x \leq i \wedge y \leq j \wedge i + 2 \cdot j < m \wedge p(i, j)\}$$

- We can rewrite the precondition P as $Z = F(0, 0)$.
- Hence, we start in $(x, y) = (0, 0)$ and will increment x and y.

We find a recurrence relation for $F(x, y)$.

- For the base case, because $\#\emptyset = 0$, it is easy to see that:

$$x + 2 \cdot y \geq m \Rightarrow F(x, y) = 0$$

- In the inductive cases, relevant conditions are:
  - $x + 2 \cdot y < m$
  - $p(x, y)$ (and $\neg p(x, y)$)

Exercise 9.12

First we investigate what happens if we increment x:

$$
\begin{array}{ll}
 & F(x,y) \\
= & \{ \text{ definition } F \text{ } \} \\
 & \#\{(i,j) \mid x \le i \ \wedge \ y \le j \ \wedge \ i+2\cdot j < m \ \wedge \ p(i,j)\} \\
= & \{ \text{ \textbf{assume} } x+2\cdot y < m; \\
 & \quad x \le i \equiv (x+1 \le i \vee i = x) \text{ } \} \\
 & \#\{(i,j) \mid x+1 \le i \ \wedge \ y \le j \ \wedge \ i+2\cdot j < m \ \wedge \ p(i,j)\} \\
 & + \ \#\{(x,j) \mid y \le j \ \wedge \ x+2\cdot j < m \ \wedge \ p(x,j)\} \\
= & \{ \text{ definition } F \text{ } \} \\
 & F(x+1,y) \ + \ \#\{(x,j) \mid y \le j \ \wedge \ x+2\cdot j < m \ \wedge \ p(x,j)\} \\
= & \{ \ p(x,j) \text{ is descending in } j \text{ so } p(x,y) \text{ is maximal;} \\
 & \quad \text{\textbf{assume} } \neg p(x,y)\text{, so } \neg p(x,j) \text{ for all } y \le j \text{ } \} \\
 & F(x+1,y) + \#\emptyset \\
= & \{ \text{ calculus } \} \\
 & F(x+1,y)
\end{array}
$$

This derivation proves:

$$x+2\cdot y < m \ \wedge \ \neg p(x,y) \Rightarrow F(x,y) = F(x+1,y)$$

Exercise 9.12

Next, we investigate what happens if we increment y:

$$
\begin{array}{ll}
 & F(x, y) \\
= & \{ \text{definition } F \} \\
 & \#\{(i,j) \mid x \le i \ \wedge\ y \le j \ \wedge\ i + 2 \cdot j < m \ \wedge\ p(i,j)\} \\
= & \{ \textbf{assume } x + 2 \cdot y < m;\ y \le j \equiv (y + 1 \le j \vee j = y) \} \\
 & \#\{(i,j) \mid x \le i \ \wedge\ y + 1 \le j \ \wedge\ i + 2 \cdot j < m \ \wedge\ p(i,j)\} \\
 & + \ \#\{(i,y) \mid x \le i \ \wedge\ i + 2 \cdot y < m \ \wedge\ p(i,y)\} \\
= & \{ \text{definition } F \} \\
 & F(x, y+1) \ + \ \#\{(i,y) \mid x \le i \ \wedge\ i + 2 \cdot y < m \ \wedge\ p(i,y)\} \\
= & \{ p(i,y) \text{ is ascending in } i \text{ so } p(x,y) \text{ is minimal;} \\
 & \quad \textbf{assume } p(x,y)\text{, so } p(i,y) \text{ for all } x \le i \} \\
 & F(x, y+1) \ + \ \#\{(i,y) \mid x \le i \ \wedge\ i + 2 \cdot y < m\} \\
= & \{ \text{calculus} \} \\
 & F(x, y+1) \ + \ \#\{(i,y) \mid x \le i < m - 2 \cdot y\} \\
= & \{ \text{size of half-open interval} \} \\
 & F(x, y+1) \ + \ m - 2 \cdot y - x
\end{array}
$$

This derivation proves:

$$x + 2 \cdot y < m \ \wedge\ p(x,y) \Rightarrow F(x,y) = F(x, y+1) + m - 2 \cdot y - x$$

Exercise 9.12

Summing up, given

$$F(x, y) = \#\{(i, j) \mid x \leq i \wedge y \leq j \wedge i + 2 \cdot j < m \wedge p(i, j)\}$$

we have the following recursive equations:

$$\boxed{\begin{array}{r} x + 2 \cdot y \geq m \Rightarrow F(x, y) = 0 \\ x + 2 \cdot y < m \wedge \neg p(x, y) \Rightarrow F(x, y) = F(x + 1, y) \\ x + 2 \cdot y < m \wedge p(x, y) \Rightarrow F(x, y) = F(x, y + 1) + k \end{array}}$$

where $k = m - 2 \cdot y - x$.

Exercise 9.12

We now rewrite the original specification to obtain:

const $m,\ n:\ \mathbb{N}$;
var $z:\ \mathbb{Z}$;
  $\{P:\ Z = F(0,0)\}$
T;
  $\{Q:\ Z = z\}$

0 We decide that we need a **while**-program: we will reduce the size of the remaining search area by incrementing x or y iteratively.

Exercise 9.12

We now rewrite the original specification to obtain:

const $m,\ n:\ \mathbb{N}$;
var $z:\ \mathbb{Z}$;
  $\{P:\ Z = F(0,0)\}$
T;
  $\{Q:\ Z = z\}$

0 We decide that we need a **while**-program: we will reduce the size of the remaining search area by incrementing x or y iteratively.

1 We introduce the variables $x,\ y : \mathbb{Z}$ and the invariant and guard

$$\boxed{\begin{array}{c} J : Z = z + F(x,y) \\ B : x + 2\cdot y < m \end{array}}$$

$$\begin{array}{ll} & J \wedge \neg B \\ \equiv & \{\text{ definition } J \text{ and } B\ \} \\ & Z = z + F(x,y) \wedge x + 2\cdot y \geq m \\ \Rightarrow & \{\text{ base case recurrence: } F(x,y) = 0\ \} \\ & Q : Z = z \end{array}$$

Exercise 9.12

2 Initialization:

$$\{P: \; Z = F(0,0)\}$$
(* *calculus* *)
$$\{Z = 0 + F(0,0)\}$$
$$z := 0; \; x := 0; \; y := 0;$$
$$\{J: \; Z = z + F(x,y)\}$$

We start with (x, y) in the South-West corner of the grid.

Exercise 9.12

2 Initialization:

$$\{P: \; Z = F(0,0)\}$$
(* *calculus* *)
$$\{Z = 0 + F(0,0)\}$$
$$z := 0; \; x := 0; \; y := 0;$$
$$\{J: \; Z = z + F(x,y)\}$$

We start with (x, y) in the South-West corner of the grid.

3 Variant function:
We shrink the search area in North-Eastern direction, i.e. we increment x and y while $x + 2 \cdot y < m$.
It is natural to choose $vf = m - x - 2 \cdot y \in \mathbb{Z}$.
Clearly $J \wedge B \Rightarrow vf \geq 0$.

Exercise 9.12: Body of the Loop

$\{Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

if $p(x, y)$ **then**

$y := y + 1;$

else

$x := x + 1;$

end

$\{J \wedge vf < V\}$

Exercise 9.12: Body of the Loop

$\{Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

if $p(x, y)$ **then**

$z := z + m - x - 2 \cdot y;$

$y := y + 1;$

else

$x := x + 1;$

end

$\{J \wedge vf < V\}$

Exercise 9.12: Body of the Loop

$\{Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

if $p(x, y)$ **then**

$\{p(x, y) \wedge Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

$z := z + m - x - 2 \cdot y;$

$y := y + 1;$

else

$x := x + 1;$

end

$\{J \wedge vf < V\}$

Exercise 9.12: Body of the Loop

$\{Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

if $p(x, y)$ **then**

$\{p(x, y) \wedge Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x + 2 \cdot y < m \wedge p(x, y)$ *)

$\{Z = z + m - x - 2 \cdot y + F(x, y + 1) \wedge m - x - 2 \cdot y = V\}$

$z := z + m - x - 2 \cdot y;$

$y := y + 1;$

else

$x := x + 1;$

end

$\{J \wedge vf < V\}$

Exercise 9.12: Body of the Loop

$\{Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

if $p(x, y)$ **then**

$\{p(x, y) \wedge Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x + 2 \cdot y < m \wedge p(x, y)$ *)

$\{Z = z + m - x - 2 \cdot y + F(x, y + 1) \wedge m - x - 2 \cdot y = V\}$

$z := z + m - x - 2 \cdot y;$

$\{Z = z + F(x, y + 1) \wedge m - x - 2 \cdot y = V\}$

$y := y + 1;$

else

$x := x + 1;$

end

$\{J \wedge vf < V\}$

Exercise 9.12: Body of the Loop

$\{Z = z + F(x, y) \land x + 2 \cdot y < m \land m - x - 2 \cdot y = V\}$

if $p(x, y)$ **then**

$\{p(x, y) \land Z = z + F(x, y) \land x + 2 \cdot y < m \land m - x - 2 \cdot y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x + 2 \cdot y < m \land p(x, y)$ *)

$\{Z = z + m - x - 2 \cdot y + F(x, y + 1) \land m - x - 2 \cdot y = V\}$

$z := z + m - x - 2 \cdot y;$

$\{Z = z + F(x, y + 1) \land m - x - 2 \cdot y = V\}$

(* *calculus; prepare* $y := y + 1$ *)

$\{Z = z + F(x, y + 1) \land m - x - 2 \cdot (y + 1) < V\}$

$y := y + 1;$

else

$x := x + 1;$

end

$\{J \land vf < V\}$

Exercise 9.12: Body of the Loop

$\{Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

if $p(x, y)$ **then**

$\{p(x, y) \wedge Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x + 2 \cdot y < m \wedge p(x, y)$ *)

$\{Z = z + m - x - 2 \cdot y + F(x, y + 1) \wedge m - x - 2 \cdot y = V\}$

$z := z + m - x - 2 \cdot y;$

$\{Z = z + F(x, y + 1) \wedge m - x - 2 \cdot y = V\}$

(* *calculus; prepare* $y := y + 1$ *)

$\{Z = z + F(x, y + 1) \wedge m - x - 2 \cdot (y + 1) < V\}$

$y := y + 1;$

$\{Z = z + F(x, y) \wedge m - x - 2 \cdot y < V\}$

else

$x := x + 1;$

end

$\{J \wedge vf < V\}$

Exercise 9.12: Body of the Loop

$\{Z = z + F(x, y) \land x + 2 \cdot y < m \land m - x - 2 \cdot y = V\}$

if $p(x, y)$ **then**

$\{p(x, y) \land Z = z + F(x, y) \land x + 2 \cdot y < m \land m - x - 2 \cdot y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x + 2 \cdot y < m \land p(x, y)$ *)

$\{Z = z + m - x - 2 \cdot y + F(x, y + 1) \land m - x - 2 \cdot y = V\}$

$z := z + m - x - 2 \cdot y;$

$\{Z = z + F(x, y + 1) \land m - x - 2 \cdot y = V\}$

(* *calculus; prepare* $y := y + 1$ *)

$\{Z = z + F(x, y + 1) \land m - x - 2 \cdot (y + 1) < V\}$

$y := y + 1;$

$\{Z = z + F(x, y) \land m - x - 2 \cdot y < V\}$

else

$\{\neg p(x, y) \land Z = z + F(x, y) \land x + 2 \cdot y < m \land m - x - 2 \cdot y = V\}$

$x := x + 1;$

end

$\{J \land vf < V\}$

Exercise 9.12: Body of the Loop

$\{Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$
if $p(x, y)$ **then**
$\{p(x, y) \wedge Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$
(* *logic; recurrence for* $F(x, y)$*: case* $x + 2 \cdot y < m \wedge p(x, y)$ *)
$\{Z = z + m - x - 2 \cdot y + F(x, y + 1) \wedge m - x - 2 \cdot y = V\}$
$z := z + m - x - 2 \cdot y;$
$\{Z = z + F(x, y + 1) \wedge m - x - 2 \cdot y = V\}$
(* *calculus; prepare* $y := y + 1$ *)
$\{Z = z + F(x, y + 1) \wedge m - x - 2 \cdot (y + 1) < V\}$
$y := y + 1;$
$\{Z = z + F(x, y) \wedge m - x - 2 \cdot y < V\}$
else
$\{\neg p(x, y) \wedge Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$
(* *logic; recurrence for* $F(x, y)$*: case* $x + 2 \cdot y < m \wedge \neg p(x, y)$ *)
$\{Z = z + F(x + 1, y) \wedge m - x - 2 \cdot y = V\}$

$x := x + 1;$

end
$\{J \wedge vf < V\}$

Exercise 9.12: Body of the Loop

$\{Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

if $p(x, y)$ **then**

$\{p(x, y) \wedge Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x + 2 \cdot y < m \wedge p(x, y)$ *)

$\{Z = z + m - x - 2 \cdot y + F(x, y + 1) \wedge m - x - 2 \cdot y = V\}$

$z := z + m - x - 2 \cdot y;$

$\{Z = z + F(x, y + 1) \wedge m - x - 2 \cdot y = V\}$

(* *calculus; prepare* $y := y + 1$ *)

$\{Z = z + F(x, y + 1) \wedge m - x - 2 \cdot (y + 1) < V\}$

$y := y + 1;$

$\{Z = z + F(x, y) \wedge m - x - 2 \cdot y < V\}$

else

$\{\neg p(x, y) \wedge Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x + 2 \cdot y < m \wedge \neg p(x, y)$ *)

$\{Z = z + F(x + 1, y) \wedge m - x - 2 \cdot y = V\}$

(* *calculus; prepare* $x := x + 1$ *)

$\{Z = z + F(x + 1, y) \wedge m - (x + 1) - 2 \cdot y < V\}$

$x := x + 1;$

end

$\{J \wedge vf < V\}$

Exercise 9.12: Body of the Loop

$\{Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

if $p(x, y)$ **then**

$\{p(x, y) \wedge Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x + 2 \cdot y < m \wedge p(x, y)$ *)

$\{Z = z + m - x - 2 \cdot y + F(x, y + 1) \wedge m - x - 2 \cdot y = V\}$

$z := z + m - x - 2 \cdot y;$

$\{Z = z + F(x, y + 1) \wedge m - x - 2 \cdot y = V\}$

(* *calculus; prepare* $y := y + 1$ *)

$\{Z = z + F(x, y + 1) \wedge m - x - 2 \cdot (y + 1) < V\}$

$y := y + 1;$

$\{Z = z + F(x, y) \wedge m - x - 2 \cdot y < V\}$

else

$\{\neg p(x, y) \wedge Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x + 2 \cdot y < m \wedge \neg p(x, y)$ *)

$\{Z = z + F(x + 1, y) \wedge m - x - 2 \cdot y = V\}$

(* *calculus; prepare* $x := x + 1$ *)

$\{Z = z + F(x + 1, y) \wedge m - (x + 1) - 2 \cdot y < V\}$

$x := x + 1;$

$\{Z = z + F(x, y) \wedge m - x - 2 \cdot y < V\}$

end

$\{J \wedge vf < V\}$

Exercise 9.12: Body of the Loop

$\{Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

if $p(x, y)$ **then**

$\{p(x, y) \wedge Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x + 2 \cdot y < m \wedge p(x, y)$ *)

$\{Z = z + m - x - 2 \cdot y + F(x, y + 1) \wedge m - x - 2 \cdot y = V\}$

$z := z + m - x - 2 \cdot y;$

$\{Z = z + F(x, y + 1) \wedge m - x - 2 \cdot y = V\}$

(* *calculus; prepare* $y := y + 1$ *)

$\{Z = z + F(x, y + 1) \wedge m - x - 2 \cdot (y + 1) < V\}$

$y := y + 1;$

$\{Z = z + F(x, y) \wedge m - x - 2 \cdot y < V\}$

else

$\{\neg p(x, y) \wedge Z = z + F(x, y) \wedge x + 2 \cdot y < m \wedge m - x - 2 \cdot y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x + 2 \cdot y < m \wedge \neg p(x, y)$ *)

$\{Z = z + F(x + 1, y) \wedge m - x - 2 \cdot y = V\}$

(* *calculus; prepare* $x := x + 1$ *)

$\{Z = z + F(x + 1, y) \wedge m - (x + 1) - 2 \cdot y < V\}$

$x := x + 1;$

$\{Z = z + F(x, y) \wedge m - x - 2 \cdot y < V\}$

end (* *collect branches; definitions* J *and* vf *)

$\{J \wedge vf < V\}$

Exercise 9.12: Conclusion

const $m,\ n:\ \mathbb{N}$;
var $x, y, z:\ \mathbb{Z}$;
$\{P:\ Z = \#\{(i,j) \mid 0 \le i \wedge 0 \le j \wedge i + 2 \cdot j < m \wedge p(i,j)\}\ \}$
$z := 0$;
$x := 0$;
$y := 0$;
$\{J:\ Z = z + \#\{(i,j) \mid x \le i \wedge y \le j \wedge i + 2 \cdot j < m \wedge p(i,j)\}\ \}$
(* $vf:\ m - x - 2 \cdot y$ *)
while $x + 2 \cdot y < m$ **do**
if $p(x,y)$ **then**
$z := z + m - x - 2 \cdot y$;
$y := y + 1$;
else
$x := x + 1$;
end;
end;
$\{Q:\ z = Z\}$

Note: Initially, $vf = m$, so the algorithm has time complexity $O(m)$.

Outline

A Digression on Counting

Exercise 9.7: Increasing & Descending
- Increasing & Descending
- The Roadmap: Triangle Case

Exercise 9.12: Ascending & Descending
- Ascending & Descending
- The Roadmap: Triangle Case

Exercise 9.14: Two Ascending Parameters
- Recurrence: Ascending Parameters
- The Roadmap: A Different Invariant

Exercise 9.14: Both Ascending

The function $h(i, j)$ is ascending in both i and j.
Determine a command T that satisfies

const $m, n : \mathbb{N}^+$;
var $z : \mathbb{Z}$;
$\{P : Z = \text{Min}\ \{|h(i,j)| \ \mid\ i, j : 0 \leq i < m\ \wedge\ 0 \leq j < n\}\ \}$
T
$\{Q : Z = z\}$

Exercise 9.14: A Different Invariant

$$P : Z = \text{Min}\,\{|h(i,j)| \;\mid\; i,j : 0 \leq i < m \land 0 \leq j < n\}$$
$$Q : Z = z$$

Let $F(x, y) = \text{Min}\,\{|h(i,j)| \;\mid\; i,j : x \leq i < m \land 0 \leq j < y\}$.
We can rewrite the precondition as $P : \; Z = F(0, n)$ and iteratively increment x and decrement y.

In this case, the invariant to maintain is different:

Exercise 9.14: A Different Invariant

$$P : Z = \text{Min}\,\{|h(i,j)| \mid i,j : 0 \le i < m \wedge 0 \le j < n\}$$
$$Q : Z = z$$

Let $F(x,y) = \text{Min}\,\{|h(i,j)| \mid i,j : x \le i < m \wedge 0 \le j < y\}$.
We can rewrite the precondition as $P :\ Z = F(0,n)$ and iteratively increment x and decrement y.

In this case, the invariant to maintain is different:

$$J :\ Z = z \ \min\ F(x,y)$$

Because $\#\emptyset = 0$, it is easy to see that

$$x \ge m \vee y \le 0 \Rightarrow F(x,y) = \infty$$

Exercise 9.14

First we investigate what happens if we <u>increment x</u>:

$$
\begin{array}{ll}
& F(x, y) \\
= & \{ \text{ definition } F \ \} \\
& \text{Min } \{|h(i,j)| \mid i, j : x \le i < m \wedge 0 \le j < y\} \\
= & \{ \textbf{ assume } x < m; \\
& \quad \text{split non-empty domain; definition } F \ \} \\
& F(x+1, y) \min \text{Min } \{|h(x,j)| \mid j : 0 \le j < y\} \\
= & \{ \ h(x,j) \text{ is ascending in } j \text{ so } h(x, y-1) \text{ is maximal;} \\
& \quad \textbf{assume } h(x, y-1) < 0, \\
& \quad \text{so } h(x,j) < 0 \text{ for all } j < y; \\
& \quad \text{so } |h(x, y-1)| = -h(x, y-1) \text{ is } \underline{\text{minimal}} \ \} \\
& F(x+1, y) \min (-h(x, y-1))
\end{array}
$$

This derivation proves:

$$x < m \wedge h(x, y-1) < 0 \quad \Rightarrow F(x, y) = F(x+1, y) \min (-h(x, y-1))$$

Exercise 9.14

Next, we investigate what happens if we decrement y:

$$
\begin{array}{ll}
 & F(x,y) \\
= & \quad \{\text{ definition } F\ \} \\
 & \text{Min}\ \{|h(i,j)| \mid i,j : x \leq i < m \wedge 0 \leq j < y\} \\
= & \quad \{\ \textbf{assume } y > 0; \\
 & \quad\ \ \text{split non-empty domain; definition } F\ \} \\
 & F(x,y-1) \min \text{Min}\ \{h(i,y-1) \mid x \leq i < m\} \\
= & \quad \{\ h(i,y-1) \text{ is ascending in } i \text{ so } h(x,y-1) \text{ is minimal;} \\
 & \quad\ \ \textbf{assume } h(x,y-1) \geq 0; \\
 & \quad\ \ \text{so } |h(x,y-1)| = h(x,y-1) \text{ is minimal}\} \\
 & F(x,y-1) \min h(x,y-1)
\end{array}
$$

This derivation proves:

$$y > 0 \wedge h(x,y-1) \geq 0 \quad \Rightarrow \quad F(x,y) = F(x,y-1) \min h(x,y-1)$$

Exercise 9.14

Summing up, we have:

$$\boxed{\begin{aligned} x \geq m \vee y \leq 0 &\Rightarrow F(x,y) = \infty \\ x < m \wedge h(x,y-1) < 0 &\Rightarrow F(x,y) = F(x+1,y)\ \min\ (-h(x,y-1)) \\ y > 0 \wedge h(x,y-1) \geq 0 &\Rightarrow F(x,y) = F(x,y-1)\ \min\ h(x,y-1) \end{aligned}}$$

Exercise 9.14

We now turn to the derivation of the program:

const $m,\ n:\ \mathbb{N}$;
var $z:\ \mathbb{Z}$;
  $\{P:\ Z = F(0, n)\}$
T;
  $\{Q:\ Z = z\}$

1 We introduce the variables $x,\ y : \mathbb{Z}$ and the invariant and guard

$$\boxed{\begin{array}{l} J : Z = z \ \min\ F(x,y) \\ B : x < m \wedge y > 0 \end{array}}$$

$$\begin{array}{ll}
 & J \wedge \neg B \\
\equiv & \{\text{ definition } J \text{ and } B\ \} \\
 & Z = z \min F(x,y) \wedge (x \geq m \vee y \leq 0) \\
\Rightarrow & \{\text{ base case recurrence: } F(x,y) = \infty\ \} \\
 & Q : Z = z
\end{array}$$

Exercise 9.14

2 Initialization:

$$\{P:\ Z = F(0,n)\}$$

(* *calculus* *)

$$\{Z = \infty \min F(0,n)\}$$

$$z := \infty;\ x := 0;\ y := n;$$

$$\{J:\ Z = z \min F(x,y)\}$$

So, we start with (x, y) in the North-West corner of the grid.

Exercise 9.14

2 Initialization:

$$\{P:\ Z = F(0,n)\}$$
(* *calculus* *)
$$\{Z = \infty \min F(0,n)\}$$
$$z := \infty;\ x := 0;\ y := n;$$
$$\{J:\ Z = z \min F(x,y)\}$$

So, we start with (x, y) in the North-West corner of the grid.

3 Variant function:
We shrink the area by incrementing x and decrementing y.
It is then natural to choose $\boxed{vf = m - x + y \in \mathbb{Z}}$.
Clearly $J \wedge B \Rightarrow vf \geq 0$.

Exercise 9.14: Body of the Loop

$\{Z = z \min F(x, y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

if $h(x, y - 1) < 0$ **then**

$x := x + 1;$

else

$y := y - 1;$

end

$\{J \wedge vf < V\}$

Exercise 9.14: Body of the Loop

$\{Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

if $h(x, y-1) < 0$ **then**

$z := z\ \min\ (-h(x, y-1));$

$x := x + 1;$

else

$z := z\ \min\ h(x, y-1);$

$y := y - 1;$

end

$\{J \wedge vf < V\}$

Exercise 9.14: Body of the Loop

$\{Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

if $h(x, y-1) < 0$ **then**

$\{h(x,y-1) < 0 \wedge Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

$z := z \ \min\ (-h(x, y-1));$

$x := x + 1;$

else

$z := z \ \min\ \ h(x, y-1);$

$y := y - 1;$

end

$\{J \wedge vf < V\}$

Exercise 9.14: Body of the Loop

$\{Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

if $h(x, y-1) < 0$ **then**

$\{h(x,y-1) < 0 \wedge Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

(* *logic; recurrence for* $F(x,y)$*: case* $x < m \wedge h(x,y-1) < 0$ *)

$\{Z = z \ \min \ (-h(x,y-1)) \min F(x+1,y) \wedge m - x + y = V\}$

$z := z \ \min \ (-h(x, y-1));$

$x := x + 1;$

else

$z := z \ \min \ h(x, y-1);$

$y := y - 1;$

end

$\{J \wedge vf < V\}$

Exercise 9.14: Body of the Loop

$\{Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

if $h(x, y-1) < 0$ **then**

$\{h(x,y-1) < 0 \wedge Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

(* *logic; recurrence for* $F(x,y)$*: case* $x < m \wedge h(x,y-1) < 0$ *)

$\{Z = z \ \min\ (-h(x,y-1)) \min F(x+1,y) \wedge m - x + y = V\}$

$z := z \ \min\ (-h(x,y-1));$

$\{Z = z \min F(x+1,y) \wedge m - x + y = V\}$

$x := x + 1;$

else

$z := z \ \min\ h(x,y-1);$

$y := y - 1;$

end

$\{J \wedge vf < V\}$

Exercise 9.14: Body of the Loop

$\{Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

if $h(x, y-1) < 0$ **then**

$\{h(x,y-1) < 0 \wedge Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

(* *logic; recurrence for* $F(x,y)$*: case* $x < m \wedge h(x,y-1) < 0$ *)

$\{Z = z \ \min \ (-h(x,y-1)) \min F(x+1,y) \wedge m - x + y = V\}$

$z := z \ \min \ (-h(x,y-1));$

$\{Z = z \min F(x+1,y) \wedge m - x + y = V\}$

(* *calculus; prepare* $x := x + 1$ *)

$\{Z = z \ \min \ F(x+1,y) \wedge m - (x+1) + y < V\}$

$x := x + 1;$

else

$z := z \ \min \ h(x,y-1);$

$y := y - 1;$

end

$\{J \wedge vf < V\}$

Exercise 9.14: Body of the Loop

$\{Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

if $h(x, y-1) < 0$ **then**

$\{h(x,y-1) < 0 \wedge Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

(* *logic; recurrence for* $F(x,y)$*: case* $x < m \wedge h(x,y-1) < 0$ *)

$\{Z = z \ \min\ (-h(x,y-1)) \min F(x+1,y) \wedge m - x + y = V\}$

$z := z \ \min\ (-h(x,y-1))$;

$\{Z = z \min F(x+1,y) \wedge m - x + y = V\}$

(* *calculus; prepare* $x := x + 1$ *)

$\{Z = z \ \min\ F(x+1,y) \wedge m - (x+1) + y < V\}$

$x := x + 1$;

$\{Z = z \ \min\ F(x,y) \wedge m - x + y < V\}$

else

$z := z \ \min\ h(x, y-1)$;

$y := y - 1$;

end

$\{J \wedge vf < V\}$

Exercise 9.14: Body of the Loop

$\{Z = z \min F(x, y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

if $h(x, y-1) < 0$ **then**

$\{h(x, y-1) < 0 \wedge Z = z \min F(x, y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x < m \wedge h(x, y-1) < 0$ *)

$\{Z = z \ \min\ (-h(x, y-1)) \min F(x+1, y) \wedge m - x + y = V\}$

$z := z \ \min\ (-h(x, y-1));$

$\{Z = z \min F(x+1, y) \wedge m - x + y = V\}$

(* *calculus; prepare* $x := x + 1$ *)

$\{Z = z \ \min\ F(x+1, y) \wedge m - (x+1) + y < V\}$

$x := x + 1;$

$\{Z = z \ \min\ F(x, y) \wedge m - x + y < V\}$

else

$\{h(x, y-1) \geq 0 \wedge Z = z \min F(x, y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

$z := z \ \min\ h(x, y-1);$

$y := y - 1;$

end

$\{J \wedge vf < V\}$

Exercise 9.14: Body of the Loop

$\{Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

if $h(x, y-1) < 0$ **then**

$\{h(x,y-1) < 0 \wedge Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

(* *logic; recurrence for* $F(x,y)$*: case* $x < m \wedge h(x,y-1) < 0$ *)

$\{Z = z\ \min\ (-h(x,y-1)) \min F(x+1,y) \wedge m - x + y = V\}$

$z := z\ \min\ (-h(x,y-1));$

$\{Z = z \min F(x+1,y) \wedge m - x + y = V\}$

(* *calculus; prepare* $x := x+1$ *)

$\{Z = z\ \min\ F(x+1,y) \wedge m - (x+1) + y < V\}$

$x := x+1;$

$\{Z = z\ \min\ F(x,y) \wedge m - x + y < V\}$

else

$\{h(x,y-1) \geq 0 \wedge Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

(* *logic; recurrence for* $F(x,y)$*: case* $y > 0 \wedge h(x,y-1) \geq 0$ *)

$\{Z = z\ \min\ h(x,y-1) \min F(x,y-1) \wedge m - x + y = V\}$

$z := z\ \min\ h(x,y-1);$

$y := y-1;$

end

$\{J \wedge vf < V\}$

Exercise 9.14: Body of the Loop

$\{Z = z \min F(x, y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

if $h(x, y-1) < 0$ **then**

$\{h(x, y-1) < 0 \wedge Z = z \min F(x, y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x < m \wedge h(x, y-1) < 0$ *)

$\{Z = z \ \min\ (-h(x, y-1)) \min F(x+1, y) \wedge m - x + y = V\}$

$z := z \ \min\ (-h(x, y-1));$

$\{Z = z \min F(x+1, y) \wedge m - x + y = V\}$

(* *calculus; prepare* $x := x + 1$ *)

$\{Z = z \ \min\ \ F(x+1, y) \wedge m - (x+1) + y < V\}$

$x := x + 1;$

$\{Z = z \ \min\ \ F(x, y) \wedge m - x + y < V\}$

else

$\{h(x, y-1) \geq 0 \wedge Z = z \min F(x, y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $y > 0 \wedge h(x, y-1) \geq 0$ *)

$\{Z = z \ \min\ \ h(x, y-1) \min F(x, y-1) \wedge m - x + y = V\}$

$z := z \ \min\ \ h(x, y-1);$

$\{Z = z \ \min\ \ F(x, y-1) \wedge m - x + y = V\}$

$y := y - 1;$

end

$\{J \wedge vf < V\}$

Exercise 9.14: Body of the Loop

$\{Z = z \min F(x, y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

if $h(x, y-1) < 0$ **then**

$\{h(x, y-1) < 0 \wedge Z = z \min F(x, y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $x < m \wedge h(x, y-1) < 0$ *)

$\{Z = z \ \min\ (-h(x, y-1)) \min F(x+1, y) \wedge m - x + y = V\}$

$z := z \ \min\ (-h(x, y-1));$

$\{Z = z \min F(x+1, y) \wedge m - x + y = V\}$

(* *calculus; prepare* $x := x + 1$ *)

$\{Z = z \ \min\ \ F(x+1, y) \wedge m - (x+1) + y < V\}$

$x := x + 1;$

$\{Z = z \ \min\ \ F(x, y) \wedge m - x + y < V\}$

else

$\{h(x, y-1) \geq 0 \wedge Z = z \min F(x, y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

(* *logic; recurrence for* $F(x, y)$*: case* $y > 0 \wedge h(x, y-1) \geq 0$ *)

$\{Z = z \ \min\ \ h(x, y-1) \min F(x, y-1) \wedge m - x + y = V\}$

$z := z \ \min\ \ h(x, y-1);$

$\{Z = z \ \min\ \ F(x, y-1) \wedge m - x + y = V\}$

(* *calculus; prepare* $y := y - 1$ *)

$\{Z = z \ \min\ \ F(x, y-1) \wedge m - x + y - 1 < V\}$

$y := y - 1;$

end

$\{J \wedge vf < V\}$

Exercise 9.14: Body of the Loop

$\{Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

if $h(x, y-1) < 0$ **then**

$\{h(x, y-1) < 0 \wedge Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

(* *logic; recurrence for* $F(x,y)$*: case* $x < m \wedge h(x,y-1) < 0$ *)

$\{Z = z \ \min\ (-h(x,y-1)) \min F(x+1,y) \wedge m - x + y = V\}$

$z := z \ \min\ (-h(x,y-1));$

$\{Z = z \min F(x+1,y) \wedge m - x + y = V\}$

(* *calculus; prepare* $x := x + 1$ *)

$\{Z = z \ \min\ F(x+1,y) \wedge m - (x+1) + y < V\}$

$x := x + 1;$

$\{Z = z \ \min\ F(x,y) \wedge m - x + y < V\}$

else

$\{h(x, y-1) \geq 0 \wedge Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$

(* *logic; recurrence for* $F(x,y)$*: case* $y > 0 \wedge h(x,y-1) \geq 0$ *)

$\{Z = z \ \min\ h(x,y-1) \min F(x,y-1) \wedge m - x + y = V\}$

$z := z \ \min\ h(x,y-1);$

$\{Z = z \ \min\ F(x,y-1) \wedge m - x + y = V\}$

(* *calculus; prepare* $y := y - 1$ *)

$\{Z = z \ \min\ F(x,y-1) \wedge m - x + y - 1 < V\}$

$y := y - 1;$

$\{Z = z \ \min\ F(x,y) \wedge m - x + y < V\}$

end

$\{J \wedge vf < V\}$

Exercise 9.14: Body of the Loop

$\{Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$
if $h(x, y-1) < 0$ **then**
$\{h(x,y-1) < 0 \wedge Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$
(* *logic; recurrence for* $F(x,y)$: *case* $x < m \wedge h(x,y-1) < 0$ *)
$\{Z = z\ \min\ (-h(x,y-1)) \min F(x+1,y) \wedge m - x + y = V\}$
$z := z\ \min\ (-h(x,y-1));$
$\{Z = z \min F(x+1,y) \wedge m - x + y = V\}$
(* *calculus; prepare* $x := x + 1$ *)
$\{Z = z\ \min\ F(x+1,y) \wedge m - (x+1) + y < V\}$
$x := x + 1;$
$\{Z = z\ \min\ F(x,y) \wedge m - x + y < V\}$
else
$\{h(x,y-1) \geq 0 \wedge Z = z \min F(x,y) \wedge x < m \wedge y > 0 \wedge m - x + y = V\}$
(* *logic; recurrence for* $F(x,y)$: *case* $y > 0 \wedge h(x,y-1) \geq 0$ *)
$\{Z = z\ \min\ h(x,y-1) \min F(x,y-1) \wedge m - x + y = V\}$
$z := z\ \min\ h(x,y-1);$
$\{Z = z\ \min\ F(x,y-1) \wedge m - x + y = V\}$
(* *calculus; prepare* $y := y - 1$ *)
$\{Z = z\ \min\ F(x,y-1) \wedge m - x + y - 1 < V\}$
$y := y - 1;$
$\{Z = z\ \min\ F(x,y) \wedge m - x + y < V\}$
end (* *collect branches; definitions* J *and* vf *)
$\{J \wedge vf < V\}$

Exercise 9.14: Conclusion

const $m, n : \mathbb{N}^+$;
var $x, y, z : \mathbb{Z}$;
  $\{P : Z = \text{Min}\,\{|h(i,j)| \mid i,j : 0 \leq i < m \wedge 0 \leq j < n\}$
$z := \infty;\ x := 0;\ y := n;$
  $\{J : Z = z \,\min\, \text{Min}\,\{|h(i,j)| \mid i,j : x \leq i < m \wedge 0 \leq j < y\}\,\}$
    (* $vf : m - x + y$ *)
while $x < m \wedge y > 0$ **do**
  if $h(x, y-1) < 0$ **then**
    $z := z \,\min\, (-h(x, y-1));$
    $x := x + 1;$
  else
    $z := z \,\min\, h(x, y-1);$
    $y := y - 1;$
  end;
end;
  $\{Q : z = Z\}$

The End

- Important: please attend the remaining tutorials - tomorrow and next week.
- Please make sure to respond to the student evaluation (you should get an email soon)
- Last but not least

The End

- Important: please attend the remaining tutorials - tomorrow and next week.
- Please make sure to respond to the student evaluation (you should get an email soon)
- Last but not least: Thanks for your attention!