university of groningen

Program Correctness

Block 5

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March 9, 2023

Outline

Linear Search

We consider the following specification for computing the least natural number that satisfies some unspecified property $prop$:

$$
\begin{array}{l}
\textbf{var } k :\ \mathbb{N}; \\
\quad \{P :\ M = \text{Min}\,\{i \in \mathbb{N} \mid prop(i)\} < \infty\} \\
L \\
\quad \{Q :\ k = M\}
\end{array}
$$

- A bi-regular spec: the precondition is constant (it is independent from variable k) and the postcondition is of the form $x = X$.
- From now on, we use pre-regular preconditions anywhere in the annotation, without carrying it through every step.

Linear Search: Invariant

Notice:

$$P \equiv M = \text{Min}\,\{i \in \mathbb{N} \mid prop(i)\} < \infty$$

$$\{M < \infty \text{ (i.e. such an } M \text{ exists)}\}$$

$$\equiv M \in \mathbb{N} \ \wedge \ prop(M) \ \wedge \ \forall i \in \mathbb{N}(prop(i) \Rightarrow M \leq i)$$

0 We will iterate on k to inspect $prop(k)$, so we need a **while.**

1 Choose an invariant J and a guard B such that $J \wedge \neg B \Rightarrow Q$.

$$\boxed{\begin{array}{c} J : 0 \leq k \leq M \\ B : \neg prop(k) \end{array}}$$

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$$\boxed{\begin{array}{c} J : 0 \leq k \leq M \\ B : \neg prop(k) \end{array}}$$

$$\begin{aligned} J \wedge \neg B \equiv\ & 0 \leq k \leq M \wedge prop(k) \\ & \{prop(k) \wedge P\} \\ \Rightarrow\ & 0 \leq k \leq M \wedge M \leq k \\ \Rightarrow\ & Q :\ k = M \end{aligned}$$

Linear Search: Initialization & Variant

$$P : M = \text{Min}\,\{i \in \mathbb{N} \mid prop(i)\} < \infty\}$$
$$J : 0 \leq k \leq M$$
$$B : \neg prop(k)$$
$$Q : k = M$$

2 Initialization: Find a command T_0 such that $\{\textbf{true}\}\ T_0\ \{J\}$

Note that we use the precondition **true**, since P is pre-regular.

$\{\textbf{true}\}$
(* *use conjunct* $M \in \mathbb{N}$ *of* P *)
$\{0 \leq 0 \leq M\}$
$k := 0;$
$\{J :\ 0 \leq k \leq M\}$

3 Variant function: Choose a $vf \in \mathbb{Z}$ and prove $J \wedge B \Rightarrow vf \geq 0$

We choose $vf = M - k$. Clearly, $J \wedge B \Rightarrow M - k \geq 0$

Linear Search: Body of the Loop

$\{J \wedge B \wedge vf = V\}$

$\{J \wedge vf < V\}$

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  (* *definitions* J, B *and* vf *)
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(* *prepare* $k := k + 1$ *)

$\{0 \leq k + 1 \leq M \wedge M - (k + 1) < V\}$

$\{J \wedge vf < V\}$

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Linear Search: Conclusion

We derived the program fragment (*linear search algorithm*):

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\textbf{var}\ \ k:\ \mathbb{Z};\\
\quad \{P:\ \ M = \mathrm{Min}\ \{i \in \mathbb{N} \mid prop(i)\} < \infty\}\\
k := 0;\\
\quad \{J : 0 \le k \le M\}\\
\qquad (^*\ vf = M - k\ ^*)\\
\textbf{while}\ \neg prop(k)\ \textbf{do}\\
\quad k := k + 1;\\
\textbf{end};\\
\quad \{Q:\ \ k = M\}
\end{array}
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Application: Linear Search in an Array

Given an array a of length n and a value w, compute the smallest i such that $a[i] = w$.
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Note that $M \leq n$, so $M < \infty$.

(This is Specification (8.2) in the reader.)

Application: Linear Search in an Array

We instantiate $prop(i) \equiv (n \leq i \vee a[i] = w)$ in the linear search algorithm we derived. Note that $\neg prop(i) \equiv (i < n \wedge a[i] \neq w)$:

var $k : \mathbb{Z}$;
  $\{P : M = \text{Min}\,\{i \in \mathbb{N} \mid n \leq i \vee a[i] = w\}\}$
$k := 0$;
  $\{J : 0 \leq k \leq M\}$
    (* $vf = M - k$ *)
while $k < n \wedge a[k] \neq w$ **do** (* *short circuit evaluation* *)
  $k := k + 1$;
end;
  $\{Q : k = M\}$

Outline

Up to Here...

We have seen:

- The roadmap for designing repetitions
- Linear search (Ch 8.1)

Coming next:

- Binary search (Ch 8.2)
- The Dutch National Flag problem (Ch 8.4)
- Chapters 10 and 9

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We shall use additional keywords to express conditions within annotated linear proofs:

- **suppose**, **define**, **assume**, **introduce**, ...

Useful when deriving side conditions for recurrence relations.

Monotonic functions

It is convenient to distinguish ordered arrays.

Let V be an interval of $\mathbb{Z}$, and let $f : V \to \mathbb{R}$ be a function (a sequence of numbers).

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f is called monotonic if it has one of the above properties.

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Consider now an ascending array a, with lenght n.

Given a value w, compute the smallest index k such that $a[k] = w$.
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Q: Why is it relevant that a is ascending?

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To enforce the informal specification, we need an active finalization:
if $k < n \wedge a[k] \neq w$ **then** $k := n$ **end**;

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$\equiv$ {rewriting}

$Q :\; 0 \leq k \leq n \;\wedge\; a[k-1] < w \leq a[k]$

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$Q : \; 0 \le k \le n \;\land\; a[k-1] < w \le a[k]$

Note: The program will not inspect $a[-1]$ and $a[n]$. Indeed, $k = 0$ and $k = n$ are only needed to reason about boundary cases.

Binary Search: Invariant and Guard

Revising the postcondition, we obtain the following specification:

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0 We decide that we need a **while**-program:
We will inspect the array a iteratively for several indices.

1 Choose an invariant J and a guard B such that $J \wedge \neg B \Rightarrow Q$.
We use the heuristic split variable, with the new variable j:

$$J : 0 \leq j \leq k \leq n \ \wedge \ a[j-1] < w \leq a[k]$$
$$B : j \neq k$$

Clearly, $J \wedge \neg B \Rightarrow Q$.

Binary Search: Initialization & Variant

$$\boxed{\begin{array}{l} P: a \text{ is ascending; } a[-1] = -\infty \wedge a[n] = \infty \\ J: 0 \leq j \leq k \leq n \ \wedge \ a[j-1] < w \leq a[k] \\ B: j \neq k \end{array}}$$

2 Initialization: Because P is pre-regular, we can use **true** as precondition. We find a command T_0 such that $\{\textbf{true}\}\ T_0\ \{J\}$.

$$\begin{array}{l} \quad\{\textbf{true}\} \\ \qquad (^*\ n \in \mathbb{N}; \textit{calculus; use } P\ ^*) \\ \quad\{0 \leq 0 \leq n \leq n \wedge a[0-1] < w \leq a[n]\} \\ j := 0;\ k := n; \\ \quad\{J:\ 0 \leq j \leq k \leq n \wedge a[j-1] < w \leq a[k]\} \end{array}$$

3 Variant function: Choose a $vf \in \mathbb{Z}$ and prove $J \wedge B \Rightarrow vf \geq 0$. We choose $vf = k - j \in \mathbb{Z}$. Clearly, $J \wedge B \Rightarrow vf \geq 0$.

Binary Search: Body of the Loop

We will be working towards a body of the following form:

$\{J \wedge B \wedge vf = V\}$
S_0;
$\{J \wedge vf = V \wedge j \leq m < k\}$
if $a[m] < w$ **then**
$j := m + 1$;
else
$k := m$;
end
$\{J \wedge vf < V\}$

- Clearly, S_0 should involve an assignment to m, which is a point in the interval formed by j and k.
- Both '$m := j$' or '$m := k - 1$' are alternatives, but we would like to reduce by half the search area.
- Hence, we shall consider '$m := (j + k)$ **div** 2'.

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$k := m$;
$\{0 \leq j \leq k \leq n \ \wedge \ a[j-1] < w \leq a[k] \ \wedge \ k - j < V\}$
end (* *collect branches; definitions* J *and* vf *)
$\{J \wedge vf < V\}$

Binary Search: Conclusion

const $n : \mathbb{N},\ w : \mathbb{Z},\ a :$ **array** $[0..n)$ **of** $\mathbb{R}$;
var $k,\ j,\ m : \mathbb{N}$;
  $\{P :\ a$ is ascending$\}$
$j := 0;\ k := n;$
  $\{J :\ 0 \leq j \leq k \leq n \wedge a[j-1] < w \leq a[k]\}$
    (* $vf = k - j$ *)
while $j \neq k$ **do**
  $m := (j + k)$ **div** 2;
  if $a[m] < w$ **then**
    $j := m + 1;$
  else
    $k := m;$
  end;
end;
  $\{k = \text{Min}\ \{i \in \mathbb{N} \mid i < n \Rightarrow w \leq a[i]\}\}$
if $k < n \wedge a[k] \neq w$ **then**
  $k := n;$
end;
  $\{Q :\ k = \text{Min}\ \{i \in \mathbb{N} \mid i < n \Rightarrow w = a[i]\}\}$

Outline

The Dutch National Flag problem (DNFP)

- A sorting problem introduced by Dijkstra.
- **Input**: An array of red, white, and blue balls.
  Output: The array re-arranged in a such way that balls of the same color are gathered together.

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  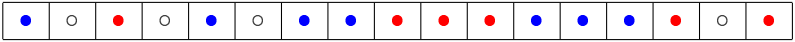

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  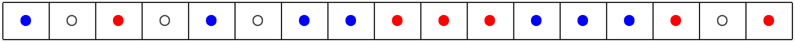

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- Notice: the array can only be modified by swapping two elements.
- We seek an efficient iterative procedure.
  As we will see, the choice of the invariant will be crucial.

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- We can swap elements, assuming the following specification:

$$\{0 \leq i = I < n \ \wedge \ 0 \leq j = J < n \ \wedge \ a[i] = X \ \wedge \ a[j] = Y\}$$

swap(i, j)

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- We look for a command that, after termination, ensures that there are indices r and b such that:

$$\begin{aligned}
&0 \le r \le w \le n \\
\land\ &(\forall i : 0 \le i < r,\ a[i] = 0) \\
\land\ &(\forall i : r \le i < w,\ a[i] = 1) \\
\land\ &(\forall i : w \le i < n,\ a[i] = 2)
\end{aligned}$$

Note: This postcondition allows for zero balls of each color.

DNFP: Invariant (1/3)

What is a good invariant for the required an iterative process?

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- At the beginning all balls are mixed; at the end, they are sorted. These should be two special cases of our invariant.
- It is natural to design an invariant that partitions the array into four segments: red, white, blue, and ‘mixed’ (unsorted).
  - At the beginning, the first three segments are empty and the mixed segment covers the entire array.
  - At the end, the mixed segment is empty.

DNFP: Invariant (2/3)

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red	unsorted	white	blue

0 ... n

unsorted	red	white	blue

- What (dis)advantages do you find in each of these options?

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& 0 \le r \le w \le b \le n \\
\wedge\ & (\forall i : 0 \le i < r,\ a[i] = 0) \\
\wedge\ & (\forall i : r \le i < w,\ a[i] = 1) \\
\wedge\ & (\forall i : b \le i < n,\ a[i] = 2)
\end{aligned}
$$

  The indices i such that $w \le i < b$ define the 'mixed' segment. Graphically:

0	r	w	b	n
red	white	unsorted	blue	

- With this invariant, the variant function is
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- With this invariant, the variant function is $vf = b - w$. The guard of the loop is $B : w < b$.

DNFP: Idea of the Body (1/3)

In the general case, we have:

						r			w				b				n
0	0	0	0	0	0	1	1	1	?	?	?	?	2	2	2	2	

We act depending on the color of the element at index w.

First case:

- If $a[w] = 1$ then

0	0	0	0	0	0	1	1	1	1	?	?	?	2	2	2	2

  and so it suffices to execute

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$$\begin{array}{|c|c|c|c|c|c|c|c|c||c|c|c|c||c|c|c|c|l} \multicolumn{6}{c}{} & \multicolumn{1}{c}{r} & \multicolumn{2}{c}{} & \multicolumn{1}{c}{w} & \multicolumn{3}{c}{} & \multicolumn{1}{c}{b} & \multicolumn{3}{c}{} & n \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & ? & ? & ? & ? & 2 & 2 & 2 & 2 & \\ \hline \end{array}$$

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First case:

- If $a[w] = 1$ then

$$\begin{array}{|c|c|c|c|c|c|c|c|c||c|c|c|c||c|c|c|c|} \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & ? & ? & ? & 2 & 2 & 2 & 2 \\ \hline \end{array}$$

  and so it suffices to execute

$$w := w + 1$$

DNFP: Idea of the Body (1/3)

In the general case, we have:

						r			w				b				n
0	0	0	0	0	0	1	1	1	?	?	?	?	2	2	2	2	

We act depending on the color of the element at index w.

First case:

- If $a[w] = 1$ then

0	0	0	0	0	0	1	1	1	1	?	?	?	2	2	2	2

  and so it suffices to execute

  $$w := w + 1$$

  This yields:

0	0	0	0	0	0	1	1	1	1	?	?	?	2	2	2	2

DNFP: Idea of the Body (2/3)

In the general case, we have:

						r			w				b				n
0	0	0	0	0	0	1	1	1	?	?	?	?	2	2	2	2	

We act depending on the color of the element at index w.

Second case:

- If $a[w] = 2$ then

0	0	0	0	0	0	1	1	1	2	?	?	?	2	2	2	2

  and so we execute

DNFP: Idea of the Body (2/3)

In the general case, we have:

						r			w				b				n
0	0	0	0	0	0	1	1	1	?	?	?	?	2	2	2	2	

We act depending on the color of the element at index w.

Second case:

- If $a[w] = 2$ then

0	0	0	0	0	0	1	1	1	2	?	?	?	2	2	2	2

  and so we execute

$$\text{swap}(b-1, w);$$
$$b := b - 1$$

DNFP: Idea of the Body (2/3)

In the general case, we have:

						r			w				b				n
0	0	0	0	0	0	1	1	1	?	?	?	?	2	2	2	2	

We act depending on the color of the element at index w.

Second case:

- If $a[w] = 2$ then

0	0	0	0	0	0	1	1	1	2	?	?	?	2	2	2	2

  and so we execute

  $$\text{swap}(b - 1, w);$$
  $$b := b - 1$$

  This yields:

0	0	0	0	0	0	1	1	1	?	?	?	2	2	2	2	2

DNFP: Idea of the Body (3/3)

In the general case, we have:

						r			w				b				n
0	0	0	0	0	0	1	1	1	?	?	?	?	2	2	2	2	

We act depending on the color of the element at index w.

Final case:

- If $a[w] = 0$ then

0	0	0	0	0	0	1	1	1	0	?	?	?	2	2	2	2

  and so we execute

DNFP: Idea of the Body (3/3)

In the general case, we have:

						r			w				b				n
0	0	0	0	0	0	1	1	1	?	?	?	?	2	2	2	2	

We act depending on the color of the element at index w.

Final case:

- If $a[w] = 0$ then

0	0	0	0	0	0	1	1	1	0	?	?	?	2	2	2	2

and so we execute

$$\text{swap}(r, w);$$
$$r := r + 1; \quad w := w + 1$$

This yields:

0	0	0	0	0	0	0	1	1	1	?	?	?	2	2	2	2

DNFP: Conclusion

```
const n : ℕ, a : array [0..n) of ℤ;
var r, w, b : ℤ;
   {P : (as discussed above)}
r := 0; w := 0; b := n;
   {J : (as discussed above)}
      (* vf = b − w *)
while w < b do
   if a[w] = 1 then
      w := w + 1;
    else
         if a[w] = 2 then
            swap(b − 1, w);
            b := b − 1;
          else
         if a[w] = 0 then
            swap(r, w);
            r := r + 1; w := w + 1;
         end;
      end;
   end
      {Q : (as discussed above)}
```

The End