Languages and Machines

L10: Decidability (Part I)

Jorge A. Pérez

Bernoulli Institute for Mathematics, Computer Science, and AI
University of Groningen, Groningen, the Netherlands

May 25, 2023

Outline

Church-Turing's Thesis

Decision Problems

The Halting Problem

Problems and Languages

Effective Computation

- How to distinguish between the computable and non-computable?

Effective Computation

- How to distinguish between the computable and non-computable?
- Mathematicians approached the notion via several formalisms:

Effective Computation

- How to distinguish between the computable and non-computable?
- Mathematicians approached the notion via several formalisms:
  - Turing machines (TMs)
  - Combinatory logic
  - The λ-calculus (functional programming!)

  Then ‘computable’ would mean computable by, e.g., a TM.

Effective Computation

- How to distinguish between the computable and non-computable?
- Mathematicians approached the notion via several formalisms:
  - Turing machines (TMs)
  - Combinatory logic
  - The λ-calculus (functional programming!)

  Then 'computable' would mean computable by, e.g., a TM.
- These formalisms are all equivalent — they embody the same notion of **effective computation**, from different angles (Effective as in: complete, mechanical, deterministic)

Effective Computation

- How to distinguish between the computable and non-computable?
- Mathematicians approached the notion via several formalisms:
  - Turing machines (TMs)
  - Combinatory logic
  - The λ-calculus (functional programming!)

  Then 'computable' would mean computable by, e.g., a TM.
- These formalisms are all equivalent — they embody the same notion of **effective computation**, from different angles (Effective as in: complete, mechanical, deterministic)
- Deterministic TMs are arguably closer to actual computers than the other formalisms

Church-Turing's Thesis

- While formalisms such as TMs, combinatory logic, λ-calculus, etc, are vastly dissimilar, they have a striking commonality
- **Effective computability**: they capture our intuition about what it means to be effectively computable — no more and no less

Church-Turing's Thesis

- While formalisms such as TMs, combinatory logic, λ-calculus, etc, are vastly dissimilar, they have a striking commonality
- **Effective computability**: they capture our intuition about what it means to be effectively computable — no more and no less
- **Church-Turing's thesis**: computability is not just TMs, nor Java, nor the λ-calculus, but the 'common spirit' they embody
- Not a theorem, but an observation (perhaps unsurprising today)

Church-Turing's Thesis

- While formalisms such as TMs, combinatory logic, λ-calculus, etc, are vastly dissimilar, they have a striking commonality
- **Effective computability**: they capture our intuition about what it means to be effectively computable — no more and no less
- **Church-Turing's thesis**: computability is not just TMs, nor Java, nor the λ-calculus, but the 'common spirit' they embody
- Not a theorem, but an observation (perhaps unsurprising today)
- TMs were a key step towards acceptance of the Church-Turing's thesis: they were the first readily programmable model
- Of course TMs do not address all possible aspects of computation (say,

Church-Turing's Thesis

- While formalisms such as TMs, combinatory logic, λ-calculus, etc, are vastly dissimilar, they have a striking commonality
- **Effective computability**: they capture our intuition about what it means to be effectively computable — no more and no less
- **Church-Turing's thesis**: computability is not just TMs, nor Java, nor the λ-calculus, but the 'common spirit' they embody
- Not a theorem, but an observation (perhaps unsurprising today)
- TMs were a key step towards acceptance of the Church-Turing's thesis: they were the first readily programmable model
- Of course TMs do not address all possible aspects of computation (say, interactivity or randomness)

Church-Turing's Thesis

- While formalisms such as TMs, combinatory logic, λ-calculus, etc, are vastly dissimilar, they have a striking commonality
- **Effective computability**: they capture our intuition about what it means to be effectively computable — no more and no less
- **Church-Turing's thesis**: computability is not just TMs, nor Java, nor the λ-calculus, but the 'common spirit' they embody
- Not a theorem, but an observation (perhaps unsurprising today)
- TMs were a key step towards acceptance of the Church-Turing's thesis: they were the first readily programmable model
- Of course TMs do not address all possible aspects of computation (say, interactivity or randomness) but they do capture a robust notion of effective computability

Universality

- **Programs as data**:
  TMs (but also all other models) are powerful enough that programs can be written to read/manipulate other programs (suitably encoded as data)
  - Think: Compilers and interpreters

Universality

- **Programs as data**:
  TMs (but also all other models) are powerful enough that programs can be written to read/manipulate other programs (suitably encoded as data)
  - Think: Compilers and interpreters
- TMs can interpret input strings as descriptions of other TMs (see next lecture!)
- A **universal machine** U is constructed to take an encoded description of another machine M and a string x as input. U can perform a step-by-step simulation of M on input x
- This is computers as we know them today!

Self-Reference

- A consequence of universality, and key to the discovery of uncomputable problems
- Observation: there are uncountably many **decision problems** but countably many TMs
- Extremely powerful: Gödel's incompleteness theorem, whose proof exploits self-reference
  (Idea: Construct the provable sentence "I am not provable")

Some Terminology

Recall: A TM is **always terminating** (or **total**) if it halts on (accepts or rejects) all inputs

A language (set of strings) L is

- **recursive**
  if $L = L(M)$ for some always terminating TM M
- **recursively enumerable (r.e.)**
  if $L = L(M)$ for some TM M

Some Terminology

Recall: A TM is **always terminating** (or **total**) if it halts on (accepts or rejects) all inputs

A language (set of strings) L is

- **recursive**
  if $L = L(M)$ for some always terminating TM M
- **recursively enumerable (r.e.)**
  if $L = L(M)$ for some TM M

Alternatively, let P be a **property** of strings.

- P is decidable
  if the set of all strings having P is recursive: there is a total TM that
  accepts strings that have P and rejects those that don't
- P is semi-decidable
  if the set of strings having P is r.e.: there is a TM that
  accepts x if x has P and *rejects or loops if not*

Some Terminology

Recursive and **recursively enumerable** are best applied to sets, while **decidable** and **semi-decidable** to properties

- Property P is decidable $\Leftrightarrow$ Set $\{x \mid P(x)\}$ is recursive
- Set A is recursive $\Leftrightarrow$ "$x \in A$" is decidable

Similarly:

- Property P is semi-decidable $\Leftrightarrow$ Set $\{x \mid P(x)\}$ is r.e.
- Set A is r.e. $\Leftrightarrow$ "$x \in A$" is semi-decidable

Outline

Church-Turing's Thesis

Decision Problems

The Halting Problem

Problems and Languages

Decision problems

A question that expects an answer ‘yes’ or ‘no’, depending on some given **instance** (positive or negative).

We would like to have procedures (programs, TMs) to answer correctly this question in all cases.

Decision problems

A question that expects an answer ‘yes’ or ‘no’, depending on some given **instance** (positive or negative).

We would like to have procedures (programs, TMs) to answer correctly this question in all cases.

Examples:

1. Is a natural n the difference between two prime numbers?
2. Given a graph, is there a path between two of its nodes?
3. Given a CFG G and a string w, do we have $w \in L(G)$?
4. Given a CFG G, does $L(G)$ contain a palindrome?
5. Given a TM M and a string w, does it hold that $w \in L(M)$?
6. Given a program P, does the call of P with input I terminate?

Decidable and semi-decidable problems

A problem is

- **decidable** if there is a procedure (a program or TM) able to answer the question correctly in all cases
- **semi-decidable** if there is a procedure that
  - for every positive instance terminates with answer ‘yes’
  - for every negative instance terminates with answer ‘no’ or loops

Examples

1. Is a natural n the difference between two prime numbers?
2. Given a graph, is there a path between two of its nodes?
3. Given a CFG G and a string w, do we have $w \in L(G)$?
4. Given a CFG G, does $L(G)$ contain a palindrome?
5. Given a TM M and a string w, does it hold that $w \in L(M)$?
6. **The halting problem:**
   Given a program P, does the call of P with input I terminate?

Examples

1. Is a natural n the difference between two prime numbers?
2. Given a graph, is there a path between two of its nodes?
(decidable)
3. Given a CFG G and a string w, do we have $w \in L(G)$?
(decidable)
4. Given a CFG G, does $L(G)$ contain a palindrome?
5. Given a TM M and a string w, does it hold that $w \in L(M)$?
6. **The halting problem:**
Given a program P, does the call of P with input I terminate?

Examples

1. Is a natural n the difference between two prime numbers?
   (semi-decidable)
2. Given a graph, is there a path between two of its nodes?
   (decidable)
3. Given a CFG G and a string w, do we have $w \in L(G)$?
   (decidable)
4. Given a CFG G, does $L(G)$ contain a palindrome?
5. Given a TM M and a string w, does it hold that $w \in L(M)$?
6. **The halting problem:**
   Given a program P, does the call of P with input I terminate?

Examples

1. Is a natural n the difference between two prime numbers?
   (semi-decidable)
2. Given a graph, is there a path between two of its nodes?
   (decidable)
3. Given a CFG G and a string w, do we have $w \in L(G)$?
   (decidable)
4. Given a CFG G, does $L(G)$ contain a palindrome?
   (not decidable)
5. Given a TM M and a string w, does it hold that $w \in L(M)$?
   (not decidable)
6. **The halting problem:**
   Given a program P, does the call of P with input I terminate?
   (not decidable)

Outline

Church-Turing's Thesis

Decision Problems

The Halting Problem

Problems and Languages

The halting problem for TMs (1/3)

Theorem

The halting problem for TMs is undecidable.

Idea for a proof by contradiction.

1. Assume there is a TM H that solves the halting problem.

A string is accepted by H if

- the input consists of two strings, $R(M)$ and w.
  $R(M)$ is the **representation** of a TM M, and w is the input to M
- the computation of M with input w halts.

Otherwise, H rejects the input.

The halting problem for TMs (1/3)

Theorem

The halting problem for TMs is undecidable.

Idea for a proof by contradiction.

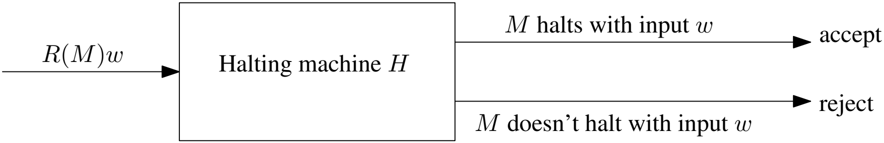
1. Assume there is a TM H that solves the halting problem.

A string is accepted by H if

- the input consists of two strings, $R(M)$ and w.
  $R(M)$ is the **representation** of a TM M, and w is the input to M
- the computation of M with input w halts.

Otherwise, H rejects the input.

Graphically:



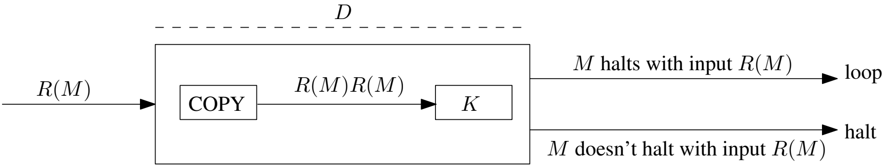
The halting problem for TMs (2/3)

2. Modify H to build another TM, called K: the computations of K are the same as H, but K loops indefinitely whenever H terminates in an accepting state, i.e., whenever M halts on w.

The halting problem for TMs (2/3)

2. Modify H to build another TM, called K: the computations of K are the same as H, but K loops indefinitely whenever H terminates in an accepting state, i.e., whenever M halts on w.

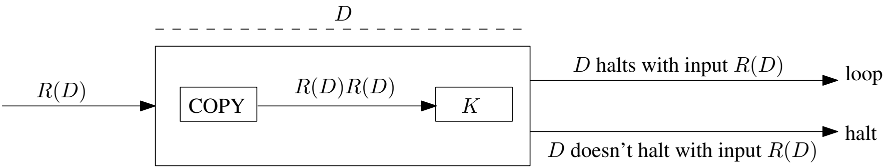
3. Combine K with a "copy machine" to build another TM, called D, with $D(M) = K(M, M)$:



If the call $D(M)$ terminates, then the call $M(M)$ won't terminate

The halting problem for TMs (3/3)

4. The input to D may be the representation of any TM, even D itself. Adapting the diagram in the previous slide:



Thus, $D(D)$ terminates iff $D(D)$ doesn't terminate.
A contradiction, derived from the assumption that there is a machine H that solves the halting problem.

The halting problem, without input

A seemingly simpler problem, which is also undecidable.

Given a program P <u>without input</u>, is there a program Q that can decide whether or not P terminates?

1. Assume Q does indeed exist, and is an always terminating program with boolean output.
2. Hence, $Q(P)$ terminates iff the call P terminates.
3. Define a "linker" L: a program that calls program P_i with input I. That is, $L(P_i, I) = P_i(I)$.
4. $L(P_i, I)$ is a program without input, for any P_i and I.
5. Thus, $Q(L(P_i, I))$ terminates iff the call $P_i(I)$ terminates
6. Define a program Q' such that $Q'(P_i, I) = Q(L(P_i, I))$
7. Q' would decide the halting problem—a contradiction

Outline

Church-Turing's Thesis

Decision Problems

The Halting Problem

Problems and Languages

Problems and languages

As mentioned earlier:

- Languages can be recursive or recursively enumerable
- Decision problems can be decidable or semi-decidable

We can relate problems and languages:

- Given a decision problem P, we can define a language L_P that consists of its positive instances.
- We need a function $encode$ that transforms problem instances into a suitable alphabet.
- This way, the decision problem P is reduced to the problem of constructing a TM that accepts the language L_P.

Taking Stock

- Effective computation and Church-Turing's thesis
- Universality and self-reference
- A language (set of strings) is recursive or recursively enumerable
- A property can be decidable or semi-decidable
- Decision problems
- Accepting a language is a decision problem; every decision problem corresponds to a language (via an encoding function)
- The halting problem is not decidable, even without input

Taking Stock

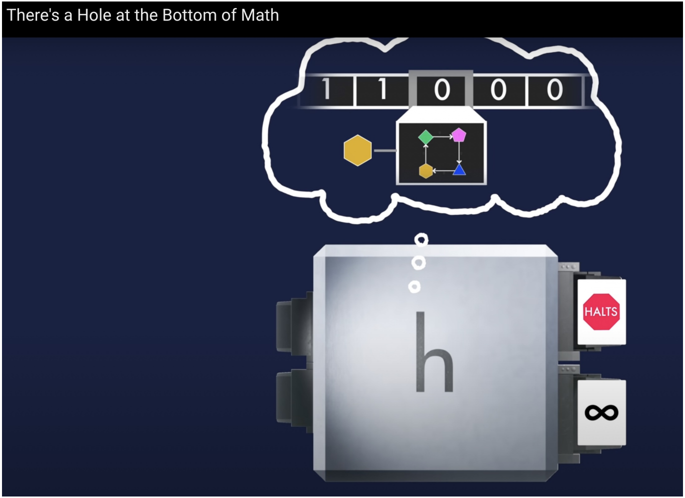
- Effective computation and Church-Turing's thesis
- Universality and self-reference
- A language (set of strings) is recursive or recursively enumerable
- A property can be decidable or semi-decidable
- Decision problems
- Accepting a language is a decision problem; every decision problem corresponds to a language (via an encoding function)
- The halting problem is not decidable, even without input

Next lecture:

- A Universal Turing machine
- Acceptance of the empty string (the blank tape problem)
- Undecidability results

A Suggestion

You Can't Prove Everything That's True



https://www.youtube.com/watch?v=HeQX2HjkcNo

The Universal TM

- A Universal TM can read representations for TMs and their inputs, and simulate running a TM on its input
- $TM0$: a very simple class of TMs with acceptance by termination and bits as input alphabet
- Given M, we write $R(M)$ to denote its representation
- M terminates on input w iff
  the UTM terminates on input $R(M)w$
- We need to define/establish R and UTM

From M to $R(M)$

- Define a **numbering function** n that maps each state q into a positive integer $n(q)$
- Define numbering functions also for symbols in the tape alphabet and directions L and R
- Mappings may clash, as in $n(q_0) = 1$, $n(0) = 1$, and $n(L) = 1$.

From M to $R(M)$

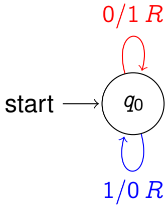
- Define a **numbering function** n that maps each state q into a positive integer $n(q)$
- Define numbering functions also for symbols in the tape alphabet and directions L and R
- Mappings may clash, as in $n(q_0) = 1$, $n(0) = 1$, and $n(L) = 1$.
- Let $1^k = \underbrace{1\,1\,\cdots\,1}_{k \text{ times}}$. A transition $\delta(q, X) = [r, Y, d]$:

$$0\,0\,1^{n(q)}\,0\,1^{n(X)}\,0\,1^{n(r)}\,0\,1^{n(Y)}\,0\,1^{n(d)}$$

- Given M, its representation $R(M)$ corresponds to a sequence of encoded transitions, followed by '000'.
- Given an input alphabet of bits, $R(M)w$ corresponds to the regular expression

$$(0(01^{+})^{5})^{*}\,000\,(0|1)^{*}$$

From M to $R(M)$: Example



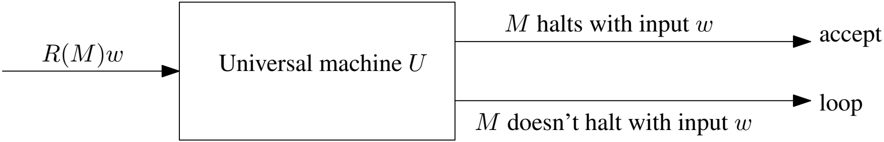
- Encoding states, tape alphabet, directions:

$$n(q_0) = 1 \quad n(0) = 1, n(1) = 2, n(B) = 3 \quad n(L) = 1, n(R) = 2$$

- $R(M)$:

$$\overbrace{00\underbrace{1}_{q_0}0\underbrace{1}_{0}0\underbrace{1}_{q_0}0\underbrace{11}_{1}0\underbrace{11}_{R}}^{\delta(q_0,0)=[q_0,1,R]}\overbrace{00\underbrace{1}_{q_0}0\underbrace{11}_{1}0\underbrace{1}_{q_0}0\underbrace{1}_{0}0\underbrace{11}_{R}}^{\delta(q_0,1)=[q_0,0,R]}000$$

A Universal TM



Theorem

Consider Turing's halting language:

$$L_H = \{R(M)w \mid R(M) \textit{ represents a TM } M \textit{ and } M \textit{ halts with input } w\}$$

We have that L_H is recursively enumerable.

Proof (Sketch).

It is possible to give a deterministic, three-tape machine U that accepts L_H, simulating the transitions of M—see next. □

Running the UTM

A deterministic, 3-tape TM simulating TMs with a binary alphabet:

1. Check the format of the input; enter into an infinite loop if invalid.
2. Move input to tape 2
3. Write 1 on tape 3 — state 1 should always be the start state
4. Simulate the machine by repeating the following:
    i. Find a transition based on the state (tape 3) and the current symbol (tape 2)
    ii. If no transition is found, terminate
    iii. Otherwise, if a transition is found: change the state (tape 3), change the symbol (tape 2), and move the head (tape 2).

The reader explains how to handle a non-binary alphabet; this requires a fourth tape.

Decidable $\neq$ Semi-decidable

Recall:

- **Theorem**. If L is recursive (decidable) then L is recursively enumerable (semi-decidable).
- **Theorem**. If L is recursive, then $\overline{L}$ is also recursive.

As already seen, the UTM terminates for input u iff $u \in L_H$, where L_H is Turing's halting language (which the UTM accepts precisely).

- **Theorem**. Language L_H is recursively enumerable.
- **Theorem**. Language $\overline{L_H}$ is not recursively enumerable.
  Similar to the proof of undecidability of the halting problem.
- **Theorem**. Language L_H is not recursive.
  If L_H were recursive, $\overline{L_H}$ would be recursive too (closure properties), and therefore recursively enumerable: contradiction.

Gödel's Incompleteness Theorem

- A proof system is sound if all theorems are true, i.e., it is not possible to prove a false sentence
- A proof system is complete if all true sentences are theorems of the system

Gödel's Result: No reasonable formal system for number theory is complete—can prove all true sentences

In the following:

- The language of number theory
- Peano arithmetic (a proof system for number theory)
- Sketch of Gödel's proof

The language of number theory

A language for expressing properties of the naturals $\mathbb{N} = \{0, 1, \ldots\}$.

- variables $x, y, z, \ldots$ ranging over $\mathbb{N}$
- operator symbols $+$ and $\cdot$, and
- constant symbols 0 and 1 (identities for $+$ and $\cdot$)
- relation symbol $=$ (symbols such as $<, \leq, >, \geq$ are definable)
- quantifiers $\forall$, $\exists$, and propositional operators $\vee$, $\wedge$, $\neg$, $\Rightarrow$, etc.
- parentheses

The language can define concepts such as "y divides x", "x is odd", and bit-manipulation formulas (cf. TM encodings)

A formula without free (unquantified) variables is called a sentence. Sentences have a well-defined truth value.

Th($\mathbb{N}$): the set of all true sentences. The decision problem for number theory is to decide whether a sentence is in Th($\mathbb{N}$).

Peano Arithmetic (PA)

A proof system for number theory.

Consists of axioms (basic assumptions) + rules of inference (applied mechanically to derive theorems from the axioms)
Write $\varphi(x)$ to denote a formula with free variable x

- Axioms from first-order logic (propositional formulas, quantifiers, equality) but also from number theory (successor, identities, induction axiom)
- Inference rules:

$$\frac{\varphi \quad \varphi \Rightarrow \psi}{\psi} \qquad \frac{\varphi}{\forall x\, \varphi}$$

A proof of φ_n is a sequence $\varphi_0, \ldots, \varphi_n$ of formulas s.t. each φ_i either is an axiom or follows from earlier formulas by an inference rule. A sentence is a theorem if it has a proof.

PA is sound: the set of theorems of PA is a subset of Th($\mathbb{N}$).
Gödel's remarkable result is that PA is not complete.

Incompleteness Theorem

Gödel proved incompleteness by constructing a sentence of number theory φ that asserts its own unprovability:

$$\varphi \text{ is true} \iff \varphi \text{ is not provable}$$

The construction of φ is interesting, as it captures self-reference as present in TMs and programming languages.

Incompleteness Theorem - Proof Sketch

A proof approach due to Turing: In a proof system such as PA one can show that

1. The set of theorems (provable sentences) is r.e., but
2. The set of true sentences $\mathrm{Th}(\mathbb{N})$ is not r.e.

Therefore, the two sets cannot be equal, and the proof system cannot be complete.

It is relatively easy to show (1), but proving (2) is much harder.

$\text{Th}(\mathbb{N})$ is not r.e - Proof Sketch

A reduction from $\overline{L_H}$ to $\text{Th}(\mathbb{N})$.

- Given $R(M)w$, we produce a sentence γ in the language of number theory such that $R(M)w \in \overline{L_H} \iff \gamma \in \text{Th}(\mathbb{N})$. Thus, M doesn't halt on $w \iff \gamma$ is true.
- Intuitively, γ uses number theory to say "M doesn't halt on w".
- Construct a formula $\text{VALCOMP}_{M,w}(y)$ that says that y represents a valid computation history of M on input w.
- Hence, $\text{VALCOMP}_{M,w}(y)$ says that y represents a sequence of configurations of M, written $\alpha_0, \ldots, \alpha_N$, such that α_0 is the start configuration (with w) and α_N is a halt configuration.
- The desired formula is then $\gamma = \neg\exists y\, \text{VALCOMP}_{M,w}(y)$.

Taking Stock (2)

This lecture:

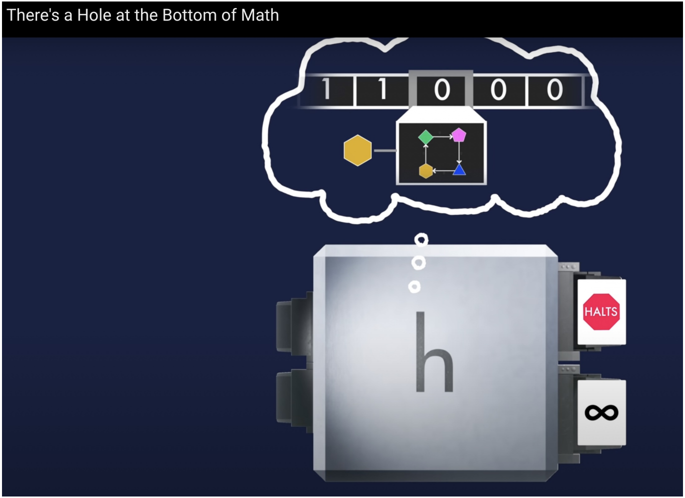
- A Universal Turing machine
- Undecidability results
- Acceptance of the empty string (the blank tape problem)
- Incompleteness of arithmetic

Next Lecture

- Unrestricted and Context-Sensitive Grammars/Languages
- Course Evaluation

A Suggestion

You Can't Prove Everything That's True



https://www.youtube.com/watch?v=HeQX2HjkcNo