Languages and Machines

L10: Decidability (Part I)

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May 21, 2025

Outline

Effective Computation

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- These formalisms are all equivalent — they embody the same notion of **effective computation**, from different angles (Effective as in: complete, mechanical, deterministic)
- Deterministic TMs are arguably closer to actual computers than the other formalisms

Church-Turing's Thesis

- While formalisms such as TMs, combinatory logic, λ-calculus, etc, are vastly dissimilar, they have a striking commonality
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- Of course TMs do not address all possible aspects of computation (say, interactivity or randomness) but they do capture a robust notion of effective computability

Universality

- **Programs as data**:
  TMs (but also all other models) are powerful enough that programs can be written to read/manipulate other programs (suitably encoded as data)
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- **Programs as data**:
  TMs (but also all other models) are powerful enough that programs can be written to read/manipulate other programs (suitably encoded as data)
  - Think: Compilers and interpreters
- TMs can interpret input strings as descriptions of other TMs (see next lecture!)
- A **universal machine** U is constructed to take an encoded description of another machine M and a string x as input. U can perform a step-by-step simulation of M on input x
- This is computers as we know them today!

Self-Reference

- A consequence of universality, and key to the discovery of uncomputable problems
- Observation: there are uncountably many **decision problems** but countably many TMs
- Extremely powerful: Gödel's incompleteness theorem, whose proof exploits self-reference
  (Idea: Construct the provable sentence “I am not provable”)

Some Terminology

Recall: A TM is **always terminating** (or **total**) if it halts on (accepts or rejects) all inputs

A language (set of strings) L is

- **recursive**
  if $L = L(M)$ for some always terminating TM M
- **recursively enumerable (r.e.)**
  if $L = L(M)$ for some TM M

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Alternatively, let P be a **property** of strings.

- P is decidable
  if the set of all strings having P is recursive: there is a total TM that
  accepts strings that have P and rejects those that don't
- P is semi-decidable
  if the set of strings having P is r.e.: there is a TM that
  accepts x if x has P and *rejects or loops if not*

Some Terminology

Recursive and **recursively enumerable** are best applied to sets, while **decidable** and **semi-decidable** to properties

- Property P is decidable $\Leftrightarrow$ Set $\{x \mid P(x)\}$ is recursive
- Set A is recursive $\Leftrightarrow$ "$x \in A$" is decidable

Similarly:

- Property P is semi-decidable $\Leftrightarrow$ Set $\{x \mid P(x)\}$ is r.e.
- Set A is r.e. $\Leftrightarrow$ "$x \in A$" is semi-decidable

Outline

Church-Turing's Thesis

Decision Problems

The Halting Problem

Problems and Languages

Decision problems

A question that expects an answer ‘yes’ or ‘no’, depending on some given **instance** (positive or negative).

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Examples:

1. Given a graph, is there a path between two of its nodes?
2. Is $n \in \mathbb{N}$ the difference between two prime numbers?
3. Given a CFG G and a string w, do we have $w \in L(G)$?
4. Given a CFG G, does $L(G)$ contain a palindrome?
5. Given a TM M and a string w, does it hold that $w \in L(M)$?
6. Given a program P, does the call of P with input I terminate?

Decidable and semi-decidable problems

A problem is

- **decidable** if there is a procedure (a program or TM) able to answer the question correctly in all cases
- **semi-decidable** if there is a procedure that
  - for every positive instance terminates with answer 'yes'
  - for every negative instance terminates with answer 'no' or loops

Examples

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   Given a program P, does the call of P with input I terminate?
   (not decidable)

Outline

Turing Machines, In Plaintext

From M to $R(M)$

- Define a **numbering function** n that maps each state q into a positive integer $n(q)$. Similarly for symbols in the tape alphabet and for the direction $d \in \{\mathtt{L}, \mathtt{R}\}$.
- The functions may clash: $n(q_0) = 1$, $n(0) = 1$, and $n(\mathtt{L}) = 1$.

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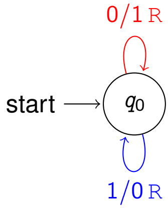
- The functions may clash: $n(q_0) = 1$, $n(0) = 1$, and $n(\mathtt{L}) = 1$.
- Let us write 1^k to denote $\underbrace{1\,1 \cdots 1}_{k \text{ times}}$. A transition $\delta(q, X) = [r, Y, d]$ is represented as:

$$0 0\, 1^{n(q)}\, 0\, 1^{n(X)}\, 0\, 1^{n(r)}\, 0\, 1^{n(Y)}\, 0\, 1^{n(d)}$$

- Given M, its string representation $R(M)$ corresponds to a sequence of encoded transitions, followed by '000'.
- Hence, assuming an input alphabet of bits, the string $R(M)w$ corresponds to the regular expression

$$\underbrace{(0(01^+)^5)^*\,000}_{R(M)}\;\underbrace{(0|1)^*}_{\text{input } w}$$

From M to $R(M)$: Tiny Example



- Encoding states, tape alphabet, directions:

$$n(q_0) = 1 \quad n(0) = 1,\ \ n(1) = 2,\ \ n(\mathtt{B}) = 3 \quad n(\mathtt{L}) = 1,\ \ n(\mathtt{R}) = 2$$

- $R(M)$:

$$\overbrace{00\underbrace{1}_{q_0}0\underbrace{1}_{0}0\underbrace{1}_{q_0}0\underbrace{11}_{1}0\underbrace{11}_{\mathtt{R}}}^{\delta(q_0,0)\,=\,[q_0,1,\mathtt{R}]}\ \overbrace{00\underbrace{1}_{q_0}0\underbrace{11}_{1}0\underbrace{1}_{q_0}0\underbrace{1}_{0}0\underbrace{11}_{\mathtt{R}}}^{\delta(q_0,1)\,=\,[q_0,0,\mathtt{R}]}000$$

The halting problem for TMs (1/3)

Theorem

The halting problem for TMs is undecidable.

Proof by contradiction (Idea).

1. Assume there is a TM H that solves the halting problem.

A string is accepted by H if

- the input consists of two strings, $R(M)$ and w.
- the computation of M with input w halts.

Otherwise, H rejects the input.

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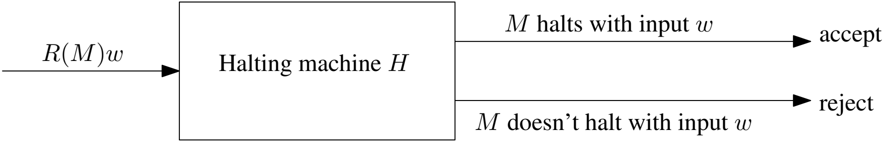
Graphically:



The halting problem for TMs (2/3)

2. Modify H to build another TM, called K: the computations of K are the same as H, but K loops indefinitely whenever H terminates in an accepting state, i.e., whenever M halts on w.

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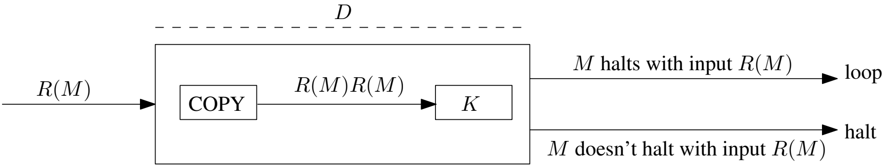
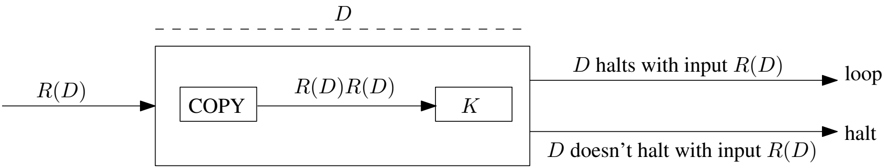
3. Combine K with a "copy machine" to build another TM, called D, with $D(M) = K(M, M)$, as follows:



If the call $D(M)$ terminates, then the call $M(M)$ won't terminate

The halting problem for TMs (3/3)

4. The input to D may be the representation of any TM, even D itself. Adapting the diagram in the previous slide:



Thus, $D(D)$ terminates iff $D(D)$ doesn't terminate.
A contradiction, derived from the assumption that there is a machine H that solves the halting problem.

The halting problem, without input

A seemingly simpler problem, which is also undecidable.

Given a program P without input, is there a program Q that can decide whether or not P terminates?

1. Assume Q does indeed exist, and is an always terminating program with boolean output.
2. Hence, $Q(P)$ terminates iff the call P terminates.
3. Define a “linker” L: a program that calls program P_i with input I. That is, $L(P_i, I) = P_i(I)$.
4. $L(P_i, I)$ is a program without input, for any P_i and I.
5. Thus, $Q(L(P_i, I))$ terminates iff the call $P_i(I)$ terminates
6. Define a program Q' such that $Q'(P_i, I) = Q(L(P_i, I))$
7. Q' would decide the halting problem—a contradiction

Outline

Problems and languages

As mentioned earlier:

- Languages can be recursive or recursively enumerable
- Decision problems can be decidable or semi-decidable

We can relate problems and languages:

- Given a decision problem P, we can define a language L_P that consists of its positive instances.
- We need a function $encode$ that transforms problem instances into a suitable alphabet.
- This way, the decision problem P is reduced to the problem of constructing a TM that accepts the language L_P.

Taking Stock

- Effective computation and Church-Turing's thesis
- Universality and self-reference
- A language (set of strings) is recursive or recursively enumerable
- A property can be decidable or semi-decidable
- Decision problems
- Accepting a language is a decision problem; every decision problem corresponds to a language (via an encoding function)
- The halting problem is not decidable, even without input

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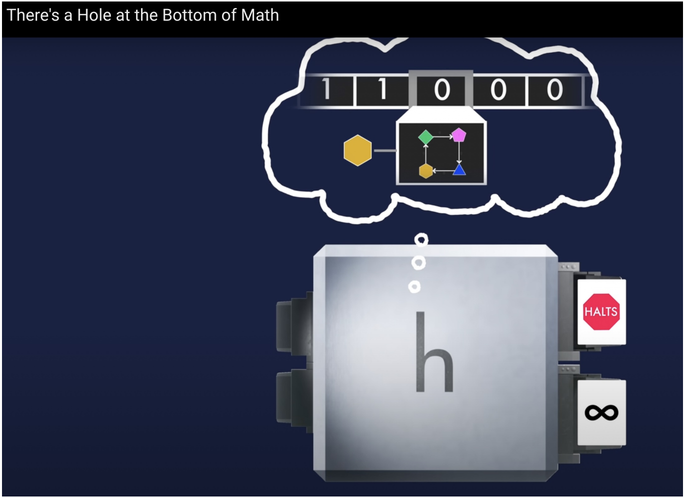
Next lecture:

- A Universal Turing machine
- Acceptance of the empty string (the blank tape problem)
- Undecidability results

A Suggestion

You Can't Prove Everything That's True



https://www.youtube.com/watch?v=HeQX2HjkcNo