university of groningen

Program Correctness

Block 5

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Outline

Linear Search

Binary Search in Ordered Sequences

- Massaging the Postcondition
- Roadmap

The Dutch National Flag problem

Linear Search

We consider the following specification for computing the least natural number that satisfies some unspecified property $prop$:

$$
\begin{array}{l}
\textbf{var } k : \mathbb{N}; \\
\quad \{P: \; M = \text{Min}\,\{i \in \mathbb{N} \mid prop(i)\} < \infty\} \\
L \\
\quad \{Q: \; k = M\}
\end{array}
$$

- A bi-regular spec: the precondition is constant (it is independent from variable k) and the postcondition is of the form $x = X$.
- From now on, we use pre-regular preconditions anywhere in the annotation, without carrying it through every step.

Linear Search: Invariant

Notice:

$$P \equiv M = \text{Min}\,\{i \in \mathbb{N} \mid prop(i)\} < \infty$$

$$\{M < \infty \ \text{(i.e. such an } M \text{ exists)}\}$$

$$\equiv M \in \mathbb{N} \ \wedge \ prop(M) \ \wedge \ \forall i \in \mathbb{N}(prop(i) \Rightarrow M \leq i)$$

0 We will iterate on k to inspect $prop(k)$, so we need a **while.**

1 Choose an invariant J and a guard B such that $J \wedge \neg B \Rightarrow Q$.

$$\boxed{\begin{array}{c} J : 0 \leq k \leq M \\ B : \neg prop(k) \end{array}}$$

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1 Choose an invariant J and a guard B such that $J \wedge \neg B \Rightarrow Q$.

$J : 0 \leq k \leq M$
$B : \neg prop(k)$

$$J \wedge \neg B \equiv \ 0 \leq k \leq M \wedge prop(k)$$

$$\{prop(k) \wedge P\}$$

$$\Rightarrow \ 0 \leq k \leq M \wedge M \leq k$$

$$\Rightarrow \ Q : \ k = M$$

Linear Search: Initialization & Variant

$$P : M = \text{Min}\,\{i \in \mathbb{N} \mid prop(i)\} < \infty\}$$
$$J : 0 \leq k \leq M$$
$$B : \neg prop(k)$$
$$Q : k = M$$

2 Initialization: Find a command T_0 such that $\{\textbf{true}\}\ T_0\ \{J\}$

Note that we use the precondition **true**, since P is pre-regular.

$\{\textbf{true}\}$
(* *use conjunct* $M \in \mathbb{N}$ *of* P *)
$\{0 \leq 0 \leq M\}$
$k := 0;$
$\{J :\ 0 \leq k \leq M\}$

3 Variant function: Choose a $vf \in \mathbb{Z}$ and prove $J \wedge B \Rightarrow vf \geq 0$

We choose $vf = M - k$. Clearly, $J \wedge B \Rightarrow M - k \geq 0$

Linear Search: Body of the Loop

$\{J \wedge B \wedge vf = V\}$

$\{J \wedge vf < V\}$

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$\{J \wedge B \wedge vf = V\}$
  (* *definitions* J, B *and* vf *)
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$\{0 \leq k < M \wedge M - k = V\}$

(* *prepare* $k := k + 1$ *)

$\{0 \leq k + 1 \leq M \wedge M - (k + 1) < V\}$

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$\{J \wedge vf < V\}$

Linear Search: Conclusion

We derived the program fragment (*linear search algorithm*):

var $k : \mathbb{Z}$;
  $\{P : \; M = \text{Min}\,\{i \in \mathbb{N} \mid prop(i)\} < \infty\}$
$k := 0$;
  $\{J : 0 \leq k \leq M\}$
    (* $vf = M - k$ *)
while $\neg prop(k)$ **do**
  $k := k + 1$;
end;
  $\{Q : \; k = M\}$

Application: Linear Search in an Array

Given an array a of length n and a value w, compute the smallest i such that $a[i] = w$.
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var $k :\ \mathbb{N}$;
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Note that $M \leq n$, so $M < \infty$.

(This is Specification (8.2) in the reader.)

Application: Linear Search in an Array

We instantiate $prop(i) \equiv (n \leq i \vee a[i] = w)$ in the linear search algorithm we derived. Note that $\neg prop(i) \equiv (i < n \wedge a[i] \neq w)$:

var $k : \mathbb{Z}$;
  $\{P : \; M = \text{Min}\, \{i \in \mathbb{N} \mid n \leq i \vee a[i] = w\}\}$
$k := 0$;
  $\{J : 0 \leq k \leq M\}$
    (* $vf = M - k$ *)
while $k < n \wedge a[k] \neq w$ **do**   (* *short circuit evaluation* *)
  $k := k + 1$;
end;
  $\{Q : \; k = M\}$

Outline

Up to Here...

We have seen:

- The roadmap for designing repetitions
- Linear search (Ch 8.1)

Coming next:

- Binary search (Ch 8.2)
- The Dutch National Flag problem (Ch 8.4)
- Chapters 10 and 9

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We shall use additional keywords to express conditions within annotated linear proofs:

- **suppose**, **define**, **assume**, **introduce**, ...

Useful when deriving side conditions for recurrence relations.

Monotonic functions

It is convenient to distinguish ordered arrays.

Let V be an interval of $\mathbb{Z}$, and let $f : V \to \mathbb{R}$ be a function (a sequence of numbers).

We say f is

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f is called monotonic if it has one of the above properties.

Binary Search

Consider now an ascending array a, with lenght n.

Given a value w, compute the smallest index k such that $a[k] = w$.
If such an index does not exist, the result should be $k = n$.

Q: Why is it relevant that a is ascending?

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A: If a is ascending and w occurs in a then
the smallest k s.t. $a[k] = w$ is also the smallest k s.t. $w \leq a[k]$.
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To enforce the informal specification, we need an active finalization:
if $k < n \wedge a[k] \neq w$ **then** $k := n$ **end**;

Massaging the Postcondition

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0 \le k \le n \ \land\ (k = n \lor w \le a[k]) \ \land\ (k = 0 \lor a[k-1] < w) \\
\equiv \quad \{\textbf{define: } a[-1] = -\infty \text{ and } a[n] = \infty\} \\
0 \le k \le n \ \land\ w \le a[k] \ \land\ a[k-1] < w \\
\equiv \quad \{\text{rewriting}\} \\
Q :\ 0 \le k \le n \ \land\ a[k-1] < w \le a[k]
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$\equiv$ {**define:** $a[-1] = -\infty$ and $a[n] = \infty$}

$0 \leq k \leq n \;\wedge\; w \leq a[k] \;\wedge\; a[k-1] < w$

$\equiv$ {rewriting}

$Q :\; 0 \leq k \leq n \;\wedge\; a[k-1] < w \leq a[k]$

Note: The program will not inspect $a[-1]$ and $a[n]$. Indeed, $k = 0$ and $k = n$ are only needed to reason about boundary cases.

Binary Search: Invariant and Guard

We obtained the following specification:

const $n : \mathbb{N}$, $w : \mathbb{Z}$, $a :$ **array** $[0..n)$ **of** $\mathbb{R}$;
var $k : \mathbb{N}$;
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0. We decide that we need a **while**-program:
   We will inspect the array a iteratively for several indices.
1. Choose an invariant J and a guard B such that $J \wedge \neg B \Rightarrow Q$.
   We use the heuristic split variable, with the new variable j:

$$\boxed{\begin{array}{l} J : 0 \leq j \leq k \leq n \ \wedge \ a[j-1] < w \leq a[k] \\ B : j \neq k \end{array}}$$

Clearly, $J \wedge \neg B \Rightarrow Q$.

Binary Search: Initialization & Variant

P : a is ascending; $a[-1] = -\infty \wedge a[n] = \infty$

$J : 0 \leq j \leq k \leq n \;\wedge\; a[j-1] < w \leq a[k]$

$B : j \neq k$

2 Initialization: Because P is pre-regular, we can use **true** as precondition. We find a command T_0 such that $\{\textbf{true}\}\ T_0\ \{J\}$.

$\{\textbf{true}\}$

(* $n \in \mathbb{N}$; *calculus; use* P *)

$\{0 \leq 0 \leq n \leq n \wedge a[0-1] < w \leq a[n]\}$

$j := 0;\ k := n;$

$\{J:\ 0 \leq j \leq k \leq n \wedge a[j-1] < w \leq a[k]\}$

3 Variant function: Choose a $vf \in \mathbb{Z}$ and prove $J \wedge B \Rightarrow vf \geq 0$. We choose $vf = k - j \in \mathbb{Z}$. Clearly, $J \wedge B \Rightarrow vf \geq 0$.

Binary Search: Body of the Loop

We will be working towards a body of the following form:

$$\{J \wedge B \wedge vf = V\}$$

S_0;

$$\{J \wedge vf = V \wedge j \leq m < k\}$$

if $a[m] < w$ **then**

$j := m + 1$;

else

$k := m$;

end

$$\{J \wedge vf < V\}$$

- Clearly, S_0 should involve an assignment to m, which is a point in the interval formed by j and k.
- Both '$m := j$' or '$m := k - 1$' are alternatives, but we would like to reduce by half the search area.
- Hence, we shall consider '$m := (j + k)$ **div** 2'.

Binary Search: Body of the Loop

$\{J \wedge B \wedge vf = V\}$

$\{J \wedge vf < V\}$

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$(^*\, (j \leq k \wedge j < k) \equiv (j + j \leq j + k \wedge j + k < k + k) \equiv 2 \cdot j \leq j + k < 2 \cdot k\, ^*)$
$\{0 \leq j \leq (j+k)$ **div** $2 < k \leq n \;\wedge\; a[j-1] < w \leq a[k] \;\wedge\; k - j = V\}$

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$\{0 \leq j \leq k \leq n \ \wedge \ a[j-1] < w \leq a[k] \ \wedge \ k - j < V\}$
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end (* *collect branches; definitions* J *and* vf *)

$\{J \wedge vf < V\}$

Binary Search: Conclusion

const $n : \mathbb{N},\ w : \mathbb{Z},\ a :$ **array** $[0..n)$ **of** $\mathbb{R}$;
var $k,\ j,\ m : \mathbb{N}$;
  $\{P :\ a$ is ascending$\}$
$j := 0;\ k := n;$
  $\{J :\ 0 \leq j \leq k \leq n \wedge a[j-1] < w \leq a[k]\}$
    (* $vf = k - j$ *)
while $j \neq k$ **do**
  $m := (j + k)$ **div** 2;
  if $a[m] < w$ **then**
    $j := m + 1;$
  else
    $k := m;$
  end;
end;
  $\{k = \text{Min}\ \{i \in \mathbb{N} \mid i < n \Rightarrow w \leq a[i]\}\}$
if $k < n \wedge a[k] \neq w$ **then**
  $k := n;$
end;
  $\{Q :\ k = \text{Min}\ \{i \in \mathbb{N} \mid i < n \Rightarrow w = a[i]\}\}$

Outline

The Dutch Flag problem (Preview)

- A sorting problem introduced by Dijkstra.
- **Input**: An array of red, white, and blue balls.
  Output: The array re-arranged in a such way that balls of the same color are gathered together.

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  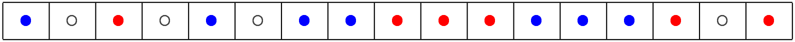

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  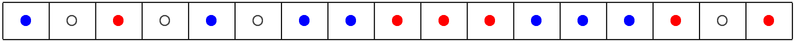

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- Notice: the array can be altered, but only by *swapping* two elements.
- We seek an efficient iterative procedure.
  As we will see, the choice of the invariant will be crucial.

The End

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- Today: Linear and binary search.
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- Next time:
  The Dutch flag problem, Longest positive subsequence (LPS), Exercises from Chapter 10.
- More tutorials starting this week!