Languages and Machines

L1: Regular Languages

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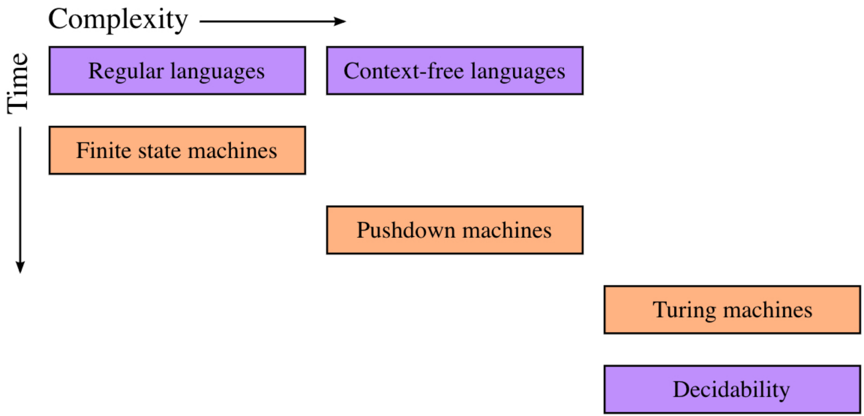
The Course At A Glance



Different Language Classes

	Expressiveness	Efficiency/Simplicity
Regular	↓	↑
Context-free		
Context-sensitive		
Decidable		
Semi-decidable		

Basic Notation

- $x \in X, \quad X \subseteq Y$
- $\forall x \in X : P(x)$, $\exists x \in X : P(x)$
- $R \subseteq X \times Y$ is a relation between X and Y
- $x\, R\, y \equiv (x, y) \in R$
- $G = (V, E)$, with $E \subseteq V \times V$ is a directed graph
- R^* is the reflexive, transitive closure of relation R

Induction

The theory:

- Basis: $0 \in \mathbb{N}$
- Inductive (or recursive) step: if $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$ too
- Closure: we only allow a finite number of steps ($\infty \notin \mathbb{N}$)

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The practice:

Given $f(n) = n(n+1)$ for all $n \in \mathbb{N}$, then $f(n)$ is even.

- Basis: for $n = 0$, we have that $f(n) = 0 \cdot 1 = 0$, which is even.
- Step: We must show that if $f(n)$ is even then $f(n+1)$ is even. Observe that

$$f(n+1) = (n+1)(n+2) = n(n+1) + 2(n+1) = f(n) + 2(n+1)$$

  Note: $f(n)$ is even (by IH) and $2(n+1)$ is also even (why?).
  Hence, $f(n+1)$ must be even too. This concludes the proof.

Induction is a **proof principle** and a tool for **defining mathematical objects**!

Strings and Languages

- Alphabet Σ: a finite set of indivisible elements ("letters")
- Σ^*: the set of strings over Σ, defined recursively
- Language: a subset of Σ^*

Examples:

- Given $\Sigma = \{a, b\}$, the empty string ϵ and non-empty strings such as ab, aaa, and $bbaba$ are all elements of Σ^*
- Length: $|bbaba| = 5$.
- Symbol counts: $n_a(bbaba) = 2$

Operations on Strings

- Given strings u and v, the string uv is their concatenation.
  An associative operation: $(uv)w = u(vw)$.
- Derived concepts: substring, prefix, suffix.
- Replication ("exponentiation"): a string concatenated with itself.
- Given a string u, its reversal u^R is u written backwards

Examples:

- Given $u = ab$ and $v = ba$, their concatenation is $uv = abba$
- Replication: $a^3 = aaa$, $(ab)^2 = abab$.
- Reversal: $(abb)^R = bba$

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  Therefore, $w^R = \epsilon$.

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  In this case, $|w| = n \geq 1$

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- Step:
  In this case, $|w| = n \geq 1$ and so $w = u\, a$, with $|u| = n - 1$.
  Therefore, u^R is defined and so $w^R = a\, u^R$.

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In what sense is this definition inductive?

Operations on Languages

- Operations on strings can be lifted to languages (sets of strings)
- Concatenation of languages X and Y:

$$XY = \{uv \mid u \in X, v \in Y\}$$

  X^n denotes the concatenation of X with itself n times
  We define X^0 as $\{\epsilon\}$.
- The **Kleene star** of a set X, written X^*:

$$X^* = \bigcup_{i=0}^{\infty} X^i$$

- The derived operator $+$, defined as: $X^+ = XX^*$

Operations on Languages

Examples:

- If $L = \{aa, bb\}$, $M = \{c, d\}$ then $LM = \{aac, aad, bbc, bbd\}$
- Powers:
  $\{a, b, ab\}^2 = \{aa, ab, aab, ba, bb, bab, aba, abb, abab\}$
- Kleene star:

$$\begin{aligned}\{a, b\}^* &= \{\epsilon\} \cup \{a, b\} \cup \{aa, ab, ba, bb\} \cup \{aaa, \ldots\} \cup \cdots \\ &= \{\epsilon, a, b, aa, ab, ba, bb, aaa, \ldots\}\end{aligned}$$

- Reversal: $\{ab, cd\}^R = \{ba, dc\}$

Regular Sets / Languages

- Recursively defined over an alphabet Σ from
  - $\emptyset$
  - $\{\epsilon\}$
  - $\{a\}$ for all $a \in \Sigma$

  by applying union, concatenation, and Kleene star.

Regular Expressions: A notation to denote regular languages

- Example: The regular expression

$$\mathtt{a}^*(\mathtt{c} \mid \mathtt{d})\,\mathtt{b}^*$$

  denotes the regular set

$$\{a\}^*(\{c\} \cup \{d\})\{b\}^*$$

- The regular expression of a set is not unique

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Exercise

Give a regular expression L over $\Sigma = \{a, b, c\}$ that contains every string not containing the substring "ab".

- Strings that do not contain a's are clearly acceptable:

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- Strings with two groups of a's:

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Exercise

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We have seen that:

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$$(\mathtt{b} \mid \mathtt{c})^*\mathtt{a}\mathtt{a}^*[\epsilon \mid \mathtt{c}(\mathtt{b} \mid \mathtt{c})^*] \subseteq L$$
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Continuing this line of reasoning we see that

$$L = (\mathtt{b} \mid \mathtt{c})^*(\epsilon \mid [\mathtt{a}\mathtt{a}^*\mathtt{c}(\mathtt{b} \mid \mathtt{c})^*]^*\mathtt{a}\mathtt{a}^*[\epsilon \mid \mathtt{c}(\mathtt{b} \mid \mathtt{c})^*])$$

Proofs

- Q: When is a proof correct (enough)?
- A: When it convinces the reader!

Essential elements:

- What do you know?
- What do you want to prove?
- How are you going to prove it?
- The actual, step-by-step, proof—the proof method!

  Example: If we have A, then because of B we also have C. Now, because of C and D, we also have E.
- Conclusion! Finally, we see that we must indeed have Z.

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- To prove: $x = 0$

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Suppose $x < 0$*: picking* $y = 1$ *suffices to infer that* $0 \leq x$*. Hence,* $x \not< 0$*.*
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Proof:

- First consider $x < 0$. If we pick $y = 1$ then $y > 0$ and we should also have $0 \leq x$. This is clearly contradictory, so $x \not< 0$.
- If $x > 0$ would hold then picking $y = x$ would give us $y > 0$, and so $x < y$ would lead to the contradiction $x < x$. We thus conclude that $x \not> 0$.
- Clearly, we must now have $x = 0$. Indeed we see that if $x = 0$, then $0 \leq x < y$ holds for all $y > 0$.

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Proof:
We proceed by case analysis on x.
We consider the three cases $x < 0$, $x > 0$, and $x = 0$, and show that only $x = 0$ can be true:

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Q.E.D. ✓

Proofs: Some Hints

What proof method/technique should you use?

- Direct proof difficult $\rightarrow$ Proof by contradiction
- Equivalence or set equality $\rightarrow$ Split into two implications
- Recursive definition $\rightarrow$ Proof by induction
- General case too hard $\rightarrow$ Case analysis
- Show something is *not* true $\rightarrow$ Contradiction + counter example

Preview: Context-Free Languages

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- Give a regular expression for $L = \{a^k b^k \mid k \in \mathbb{N}\}$
- Impossible! The expression $\mathtt{a}^*\mathtt{b}^*$ does *not* work.
- Consider the grammar G given by

$$S \quad \to \quad \epsilon \mid aSb$$

- To show that $aabb \in L(G)$, we can write the derivation

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

- Equivalently, we can draw the corresponding *derivation tree*.

Taking Stock

- Basic notations
- Regular languages and regular notations
- Proofs
- There are non-regular languages:
  Context-free languages to the rescue!