university of groningen

Program Correctness

Block 7

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(based on slides by Arnold Meijster)

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2D Counting: Preview

- Problem: Deduce correct programs for counting certain elements of a given matrix (which represents a 2D function)
- Given an $n \times m$ matrix, the general case requires an iterative program that performs $n \times m$ comparisons.

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  For instance, counting occurrences of 4:

2	7	4	13	3
6	2	1	19	4
11	8	0	17	5
4	7	9	10	4

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0	2	4	7	10
1	2	4	8	11
2	3	5	9	13
4	4	6	17	19

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- We use recurrences to characterize a function $F(x, y)$, which defines (i) the rectangle's area and (ii) the entries to be counted.

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- We use recurrences to characterize a function $F(x, y)$, which defines (i) the rectangle's area and (ii) the entries to be counted.
- Clearly, different monotonicity assumptions entail:
  - different contour lines
  - different ways of approaching the recurrences
  (= different valid ways of reducing the rectangle)

Monotonic functions

Let $f : V \to \mathbb{R}$ be a function, where $V \subset \mathbb{Z}$ is a segment (interval).

We say f is

- ascending ($\leq$ / $\leq$): if $\forall i, j \in V : (i \leq j \Rightarrow f(i) \leq f(j))$
- increasing ($<$ / $<$): if $\forall i, j \in V : (i < j \Rightarrow f(i) < f(j))$

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- descending ($\leq$ / $\geq$): if $\forall i, j \in V : (i \leq j \Rightarrow f(i) \geq f(j))$
- decreasing ($<$ / $>$): if $\forall i, j \in V : (i < j \Rightarrow f(i) > f(j))$

f is called monotonic if it has one of the above properties.

Outline

Two dimensional (2D) counting

- Let $h : [0..m) \times [0..n) \rightarrow \mathbb{N}$ be a two-dimensional function.
- One can think of h as a landscape, where $h(x, y)$ denotes the altitude at location (x, y).
- Problem: Counting the number of grid points whose altitude stands below a value w.

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- Let $h : [0..m) \times [0..n) \to \mathbb{N}$ be a two-dimensional function.
- One can think of h as a landscape, where $h(x, y)$ denotes the altitude at location (x, y).
- Problem: Counting the number of grid points whose altitude stands below a value w.
- For the following grid and $w = 20$, we wish to establish $z = 70$.

1	**16**	25	22	**0**	**1**	**17**	20	**19**	29
9	22	**7**	**1**	**5**	**16**	**13**	**3**	**14**	24
12	**6**	**13**	**16**	**14**	20	**9**	**14**	**11**	**6**
16	**0**	**2**	**13**	**8**	**2**	**16**	**14**	**3**	**16**
25	**16**	20	27	**7**	**3**	**5**	27	24	22
23	23	**2**	29	**14**	26	26	**14**	**8**	**19**
25	**19**	**9**	**18**	29	20	27	**15**	**8**	**18**
27	20	27	**12**	21	**1**	**14**	**12**	**6**	26
16	**7**	**8**	**12**	**3**	**16**	**15**	**15**	**18**	**0**
13	**2**	**11**	29	**9**	23	**15**	24	**7**	**12**

Two dimensional (2D) counting

- Let $h : [0..m) \times [0..n) \rightarrow \mathbb{N}$ be a two-dimensional function.
- One can think of h as a landscape, where $h(x, y)$ denotes the altitude at location (x, y).
- We address the problem of counting the number of grid points whose altitude stands below a value w.

Consider the following pre-regular specification:

const $m,\ n,\ w :\ \mathbb{N}$;
var $z :\ \mathbb{Z}$;
$\quad \{P :\ Z = \#\{(i,j) \mid i,j :\ 0 \leq i < m \ \wedge\ 0 \leq j < n \ \wedge\ h(i,j) < w\}\ \}$
T;
$\quad \{Q :\ Z = z\}$

Two dimensional (2D) counting

Exercise 9.1 asks you to confirm that the program fragment satisfies the specification:

const $m,\ n,\ w:\ \mathbb{N};$
var $x,\ y,\ z:\ \mathbb{Z};$
$\{P:\ Z = \#\{(i,j) \mid i,j:\ 0 \le i < m \ \wedge\ 0 \le j < n \ \wedge\ h(i,j) < w\}\ \}$
$x := 0;$
$y := 0;$
$z := 0;$
while $y < n$ **do**
  if $x < m$ **then**
    $z := z + \mathrm{ord}(h(x,y) < w);$
    $x := x + 1;$
  else
    $x := 0;$
    $y := y + 1;$
  end;
end;
$\{Q:\ Z = z\}$

2D counting on monotonic functions

- Let $h : [0..m) \times [0..n) \to \mathbb{N}$ be a two-dimensional function, but now ascending ($\leq$ / $\leq$) in both its arguments:

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$$x_0 \leq x_1 \Rightarrow h(x_0, y) \leq h(x_1, y)$$
$$y_0 \leq y_1 \Rightarrow h(x, y_0) \leq h(x, y_1)$$

- Think of h as the slope of a landscape whose altitude increases (or stays stable) if one moves to the east or north (or northeast).

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- Think of h as the slope of a landscape whose altitude increases (or stays stable) if one moves to the east or north (or northeast).
- Example, from low height to high height:

7	13	14	25	25	27	29	29	32	33
6	11	12	23	24	25	27	29	32	32
6	9	12	22	22	23	27	29	30	30
6	9	10	20	20	23	25	25	27	28
6	9	10	18	20	21	21	23	25	25
6	7	10	16	16	19	21	22	23	23
5	5	8	14	15	17	19	21	21	23
5	5	6	12	12	15	16	17	18	19
5	5	6	10	12	14	15	16	17	19
3	5	6	8	9	9	9	10	11	13

(0,0)

2D counting on monotonic functions

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In the previous grid, with $w = 20$ we want to find $z = 59$ (in **bold**):

7	**13**	**14**	25	25	27	29	29	32	33
6	**11**	**12**	23	24	25	27	29	32	32
6	**9**	**12**	22	22	23	27	29	30	30
6	**9**	**10**	20	20	23	25	25	27	28
6	**9**	**10**	**18**	20	21	21	23	25	25
6	**7**	**10**	**16**	**16**	**19**	21	22	23	23
5	**5**	**8**	**14**	**15**	**17**	**19**	21	21	23
5	**5**	**6**	**12**	**12**	**15**	**16**	**17**	**18**	**19**
5	**5**	**6**	**10**	**12**	**14**	**15**	**16**	**17**	**19**
3	**5**	**6**	**8**	**9**	**9**	**9**	**10**	**11**	**13**

2D counting on monotonic functions

The value of Z depends on the contour line induced by w.

The contour line separates the grid points with altitude $< w$ from those with altitude $\geq w$. It may contain values $> w$.

Example:

7	13	14	25	25	27	29	29	32	33
6	11	12	23	24	25	27	29	32	32
6	9	12	22	22	23	27	29	30	30
6	9	10	20	20	23	25	25	27	28
6	9	10	18	20	21	21	23	25	25
6	7	10	16	16	19	21	22	23	23
5	5	8	14	15	17	19	21	21	23
5	5	6	12	12	15	16	17	18	19
5	5	6	10	12	14	15	16	17	19
3	5	6	8	9	9	9	10	11	13

(0,0)

Notice: $z = 59 =$

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Example:

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	6	9	10	20	20	23	25	25	27	28
	6	9	10	18	20	21	21	23	25	25
	6	7	10	16	16	19	21	22	23	23
	5	5	8	14	15	17	19	21	21	23
	5	5	6	12	12	15	16	17	18	19
	5	5	6	10	12	14	15	16	17	19
(0,0)	3	5	6	8	9	9	9	10	11	13

Notice: $z = 59 = 10 + 10 + 10 + 6 + 5 + 5 + 4 + 3 + 3 + 3$.

2D counting on monotonic functions

We derive a repetitive command that uses the contour line to guide the search, and maintains the invariant:

$$\boxed{J: \; Z = z + F(x, y)}$$

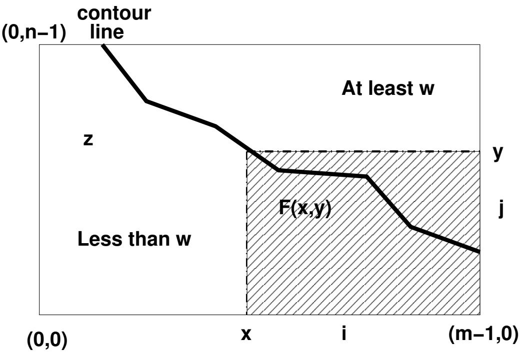
where

- z is the number of **already counted** grid points
- $F(x, y)$ denotes the points **still to be counted**, enclosed by an area called the shrinking rectangle

Maintaining $J: \ Z = z + F(x, y)$

Intuitively:

- At the beginning: $Z = F(0, n)$.



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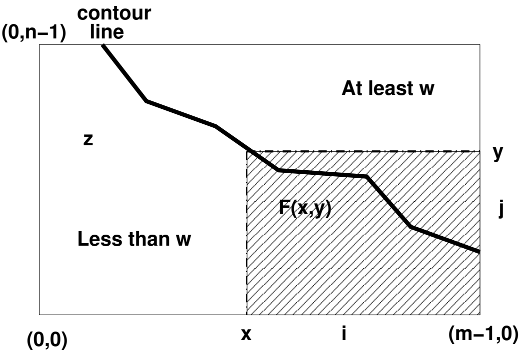

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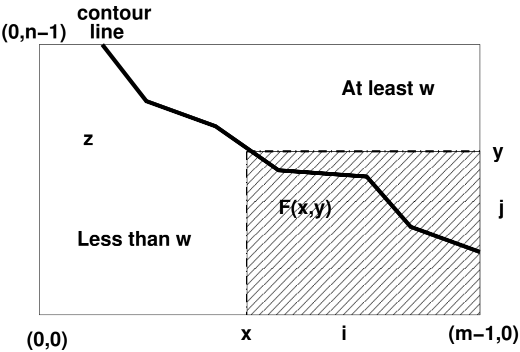

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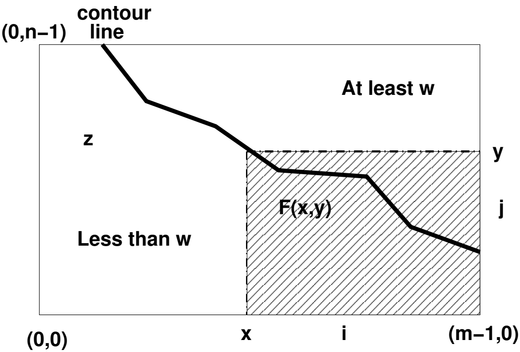

We define:

$$F(x,y) = \#\{(i,j) \mid i,j : \; x \le i < m \;\wedge\; 0 \le j < y \;\wedge\; h(i,j) < w\}$$

Maintaining $Z = z + F(x, y)$, Intuitively

The rectangle's definition:

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Intuitively:

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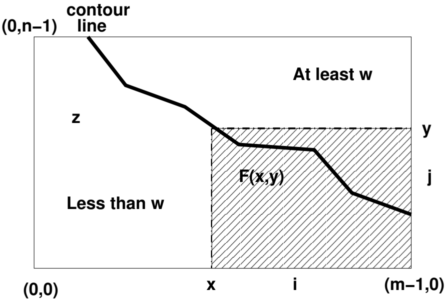
- At the end:
  $F(m, 0) = 0$ and $Z = z$.

Recurrence for $F(x, y)$



We characterize the rectangle $F(x, y)$ with a recurrence relation.
Side conditions relevant for counting:

- $x < m$ (and $m \leq x$)
- $y > 0$ (and $y \leq 0$)
- $h(x, y - 1) < w$ (and $h(x, y - 1) \geq w$)

Because $\#\emptyset = 0$, we have the base case:

$$m \leq x \vee y \leq 0 \Rightarrow F(x, y) = 0$$

Recurrence for $F(x, y)$

One way to reduce the rectangle is to increment x.
We exploit that h is ascending in y:

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$$
\begin{array}{ll}
 & F(x, y) \\
= & \quad \{ \text{ definition } F \ \} \\
 & \#\{(i,j) \mid i,j:\ x \le i < m \ \wedge\ 0 \le j < y \ \wedge\ h(i,j) < w\}
\end{array}
$$

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Conclusion:

$$x < m \ \wedge\ y > 0 \ \wedge\ h(x,y-1) < w \Rightarrow F(x,y) = F(x+1,y) + y$$

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Conclusion: $y > 0 \ \wedge\ h(x,y-1) \ge w \Rightarrow F(x,y) = F(x,y-1)$

Recurrence for $F(x, y)$

We conclude that

$$F(x, y) = \#\{(i, j) \mid i, j :\ x \leq i < m \wedge 0 \leq j < y \wedge h(i, j) < w\}$$

satisfies the following recursive equations:

$$\begin{array}{rcl} m \leq x \vee y \leq 0 & \Rightarrow & F(x, y) = 0 \\ x < m \wedge y > 0 \wedge h(x, y-1) < w & \Rightarrow & F(x, y) = y + F(x+1, y) \\ y > 0 \wedge h(x, y-1) \geq w & \Rightarrow & F(x, y) = F(x, y-1) \end{array}$$

2D counting: Guard & Invariant

We now rewrite the original specification to obtain:

const $m,\ n,\ w:\ \mathbb{N}$;
var $z:\ \mathbb{Z}$;
$\{P:\ Z = F(0, n)\}$
T;
$\{Q:\ Z = z\}$

2D counting: Guard & Invariant

We now rewrite the original specification to obtain:

const $m,\ n,\ w:\ \mathbb{N}$;
var $z:\ \mathbb{Z}$;
$\{P:\ Z = F(0, n)\}$
T;
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$$\begin{array}{ll} & J \wedge \neg B \\ \equiv & \{ \text{ definition } J \text{ and } B \ \} \\ & Z = z + F(x, y) \wedge \neg(x < m \wedge y > 0) \\ \equiv & \{ \text{ Logic; De Morgan } \} \\ & Z = z + F(x, y) \wedge (m \leq x \vee y \leq 0) \\ \Rightarrow & \{ \text{ base case recurrence: } F(x, y) = 0 \ \} \\ & Q : Z = z \end{array}$$

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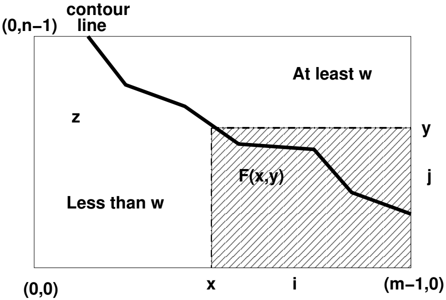

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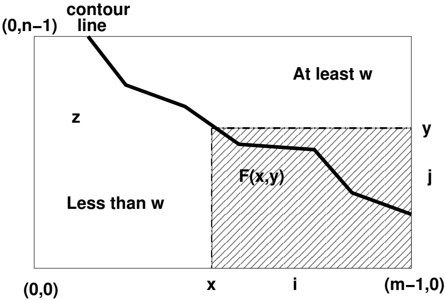

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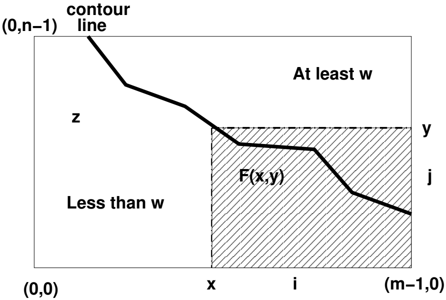

3 Variant function: We shrink the rectangle in South-Eastern direction, i.e. we increment x and decrement y.
It is then natural to choose $vf = y + m - x \in \mathbb{Z}$.
The guard is $x < m \wedge y > 0$, so clearly $J \wedge B \Rightarrow vf \geq 0$.

2D counting: Body of the Loop

$\{Z = z + F(x, y) \ \wedge \ x < m \ \wedge \ y > 0 \ \wedge \ y + m - x = V\}$

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(* *logic; recurrence for* $F(x, y)$*: case* $x < m \ \wedge\ y > 0 \ \wedge\ h(x, y - 1) < w$ *)

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(logic; recurrence for $F(x,y)$: case $x < m \ \wedge \ y > 0 \ \wedge \ h(x,y-1) < w$ *)*

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end (* *collect branches; definitions* J *and* vf *)

$\{J \wedge vf < V\}$

2D counting: Conclusion

```
const m, n, w : ℕ;
var x, y, z : ℤ;
   {P : Z = #{(i,j) ∈ [0..m) × [0..n) | h(i,j) < w} }
z := 0;
x := 0;
y := n;
   {J : Z = z + F(x,y)}
      (* vf : y + m − x *)
while x < m ∧ y > 0 do
   if h(x, y − 1) < w then
      z := y + z;
      x := x + 1;
    else
      y := y − 1;
   end;
end;
   {Q : z = Z}
```

2D counting: Conclusion

const $m,\ n,\ w :\ \mathbb{N}$;
var $x, y, z :\ \mathbb{Z}$;
  $\{P :\ Z = \#\{(i,j) \in [0..m) \times [0..n) \mid h(i,j) < w\}\ \}$
$z := 0$;
$x := 0$;
$y := n$;
  $\{J :\ Z = z + F(x,y)\}$
    (* $vf :\ y + m - x$ *)
while $x < m \wedge y > 0$ **do**
  if $h(x, y-1) < w$ **then**
    $z := y + z$;
    $x := x + 1$;
  else
    $y := y - 1$;
  end;
end;
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Note: Initially, $vf = m + n$, so the time complexity is $O(m+n)$, more efficient than the brute-force $O(m \cdot n)$ algorithm.

Outline

The Shrinking Area Method

- For counting, we use the invariant $J : Z = z + F(x, y)$.
  (A variation is needed for, e.g., minimization problems).
- Given a function $h(x, y)$, the method depends on the monotonicty properties of x and y.
- In turn, such properties define the contour line and its slope.
- The area $F(x, y)$ (and the way it is iteratively reduced) depends on this slope (and on the spec of the command).
- A recurrence relation for $F(x, y)$ must be determined.
  The side conditions of the recurrence capture the area we want to cover; they usually guide the conditionals in the command.
- The spec for counting may include a constraint on points (i, j).
  Such a constraint determines a section of the area; it typically appears as the guard of the loop.

We now explore variations of the method.

Different Functions and Contour Line

Our previous example, a function with two ascending parameters.
The slope of the contour line: $\searrow$

Example, with $w = 20$:

7	13	14	**25**	25	27	29	29	32	33
6	11	12	**23**	24	25	27	29	32	32
6	9	12	**22**	22	23	27	29	30	30
6	9	10	**20**	20	23	25	25	27	28
6	9	10	18	**20**	**21**	21	23	25	25
6	7	10	16	16	19	**21**	22	23	23
5	5	8	14	15	17	19	**21**	**21**	**23**
5	5	6	12	12	15	16	17	18	19
5	5	6	10	12	14	15	16	17	19
3	5	6	8	9	9	9	10	11	13

(0,0)

Different Functions & Contour Line (1/2)

Suppose a function $h : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Z}$ that is descending on x and ascending on y:

$$x_0 \leq x_1 \Rightarrow h(x_0, y) \geq h(x_1, y)$$
$$y_0 \leq y_1 \Rightarrow h(x, y_0) \leq h(x, y_1)$$

In this case, the slope is $\nearrow$.

Example, with $w = 7$:

20	19	16	15	14	12	10
18	17	12	11	10	9	**8**
15	12	10	9	8	**7**	**4**
13	12	8	8	**7**	**6**	**3**
11	10	8	**7**	**6**	**5**	**2**
10	9	8	**7**	**5**	**3**	**1**

(0,0)

Different Functions & Contour Line (2/2)

Now suppose a function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ that is increasing in x and descending in y:

$$x_0 < x_1 \Rightarrow g(x_0, y) < g(x_1, y)$$
$$y_0 \leq y_1 \Rightarrow g(x, y_0) \geq g(x, y_1)$$

In this case, the slope is $\nearrow$.

Example, with $w = 13$:

5	6	7	8	9	10	11
7	8	9	10	11	**13**	**16**
8	9	10	11	**13**	15	19
9	10	11	12	**16**	17	19
10	11	12	**13**	16	19	20
10	**13**	**14**	15	17	20	26

(0,0)

Outline

Exercise 9.9: Two Ascending Arguments

Let $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be a two-dimensional function that is ascending ($\leq$ / $\leq$) in both x and y:

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We want to find a command T that satisfies the specification:

const $m, \ n : \ \mathbb{N}$;
var $z : \ \mathbb{Z}$;
$\{P : \ Z = \#\{i \mid 0 \leq i < m \ \wedge \ (\exists j : 0 \leq j < n \ \wedge \ h(i,j) = 0)\} \}$
T;
$\{Q : \ Z = z\}$

Exercise 9.9: Two Ascending Arguments

const $m,\ n : \mathbb{N}$;
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T;

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Example:

-8	-2	-1	10	10	12	14	14	17	18
-9	-4	-3	8	9	10	12	14	17	17
-9	-6	-3	7	7	8	12	14	15	15
-9	-6	-5	5	5	8	10	10	12	13
-9	-6	-5	3	5	6	6	8	10	10
-9	-8	-5	1	**0**	4	6	7	8	8
-10	-10	-7	-1	**0**	**0**	4	6	6	8
-10	-10	-9	-3	-3	**0**	1	2	3	4
-10	-10	-9	-5	-3	**0**	**0**	1	2	4
-12	-10	-9	-7	-6	-6	-6	-5	-4	-2

$$\#\{i \mid 0 \leq i < m \wedge (\exists j : 0 \leq j < n \wedge h(i,j) = 0)\} =$$

Exercise 9.9: Two Ascending Arguments

const $m,\ n:\ \mathbb{N}$;
var $z:\ \mathbb{Z}$;

$\{P:\ Z = \#\{i \mid 0 \leq i < m\ \wedge\ (\exists j : 0 \leq j < n\ \wedge\ h(i,j) = 0)\}\ \}$

T;

$\{Q:\ Z = z\}$

Example:

-8	-2	-1	10	10	12	14	14	17	18
-9	-4	-3	8	9	10	12	14	17	17
-9	-6	-3	7	7	8	12	14	15	15
-9	-6	-5	5	5	8	10	10	12	13
-9	-6	-5	3	5	6	6	8	10	10
-9	-8	-5	1	**0**	4	6	7	8	8
-10	-10	-7	-1	**0**	**0**	4	6	6	8
-10	-10	-9	-3	-3	**0**	1	2	3	4
-10	-10	-9	-5	-3	**0**	**0**	1	2	4
-12	-10	-9	-7	-6	-6	-6	-5	-4	-2

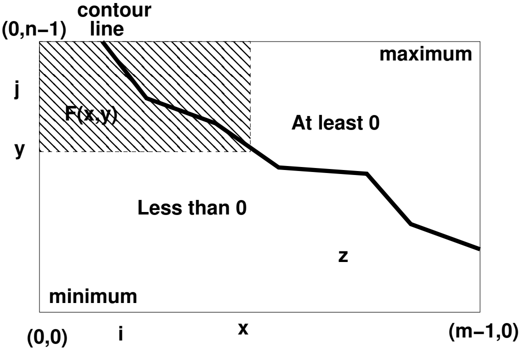
$$\#\{i \mid 0 \leq i < m \wedge (\exists j : 0 \leq j < n \wedge h(i,j) = 0)\} = 3$$

Exercise 9.9: Two Ascending Arguments

We stick to $J: \ Z = z + F(x, y)$, and solve the problem by following the contour line.

Intuitively:

- At the beginning: $Z = F(m, 0)$.



Exercise 9.9: Two Ascending Arguments

We stick to $J: \; Z = z + F(x, y)$, and solve the problem by following the contour line.

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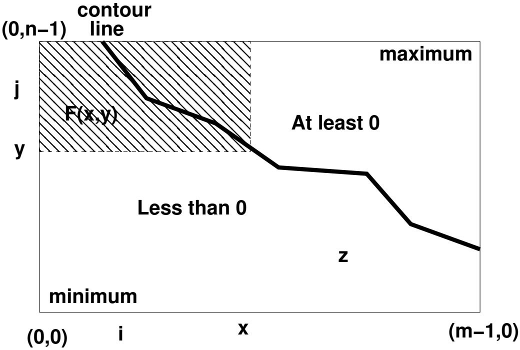
- At the beginning: $Z = F(m, 0)$.
- In the middle, reduce the rectangle: decrease x / increase y.



Exercise 9.9: Two Ascending Arguments

We stick to $J:\ Z = z + F(x, y)$, and solve the problem by following the contour line.

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- At the beginning: $Z = F(m, 0)$.
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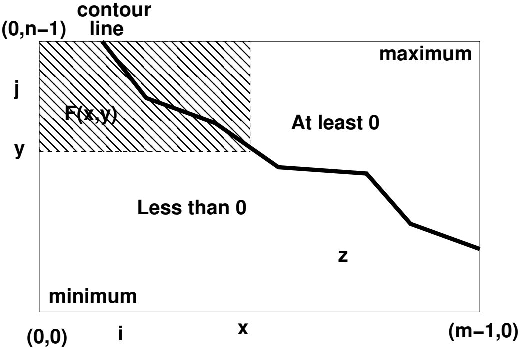
- At the end: $Z = z$ and $F(0, n) = 0$.



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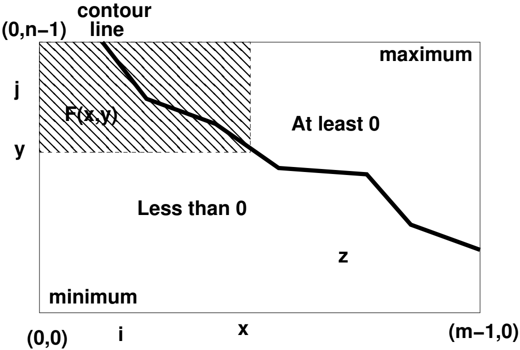
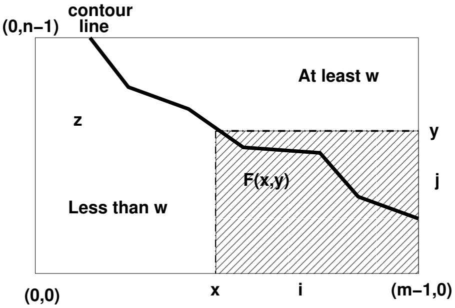


We define:

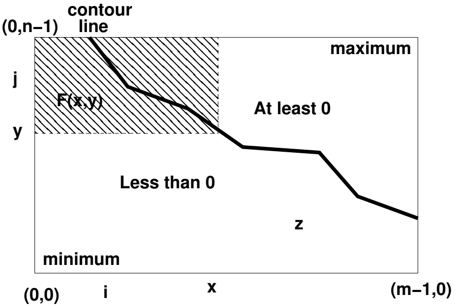
$$F(x, y) = \#\{i \mid 0 \leq i < x \ \wedge \ (\exists j : y \leq j < n \wedge h(i, j) = 0)\}$$

Comparison

Section 9.2:
From $F(0, n)$ to $F(m, 0)$ by incrementing x / decrementing y.



Exercise 9.9:
From $F(m, 0)$ to $F(0, n)$ by decrementing x / incrementing y.



Exercise 9.9: Two Ascending Arguments

We try to find a recurrence relation for

$$F(x, y) = \#\{i \mid 0 \leq i < x \wedge (\exists j : y \leq j < n \wedge h(i, j) = 0)\}$$

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We try to find a recurrence relation for

$$F(x, y) = \#\{i \mid 0 \leq i < x \wedge (\exists j : y \leq j < n \wedge h(i, j) = 0)\}$$

Relevant side conditions:

- $x > 0$ (and $x \leq 0$)
- $y < n$ (and $n \leq y$)
- $h(x - 1, y) \geq 0$ (and $h(x - 1, y) < 0$)

Exercise 9.9: Two Ascending Arguments

We try to find a recurrence relation for

$$F(x, y) = \#\{i \mid 0 \leq i < x \wedge (\exists j : y \leq j < n \wedge h(i, j) = 0)\}$$

Relevant side conditions:

- $x > 0$ (and $x \leq 0$)
- $y < n$ (and $n \leq y$)
- $h(x-1, y) \geq 0$ (and $h(x-1, y) < 0$)

We start with the base case. It is easy to see that (since $\#\emptyset = 0$):

$$x \leq 0 \vee n \leq y \Rightarrow F(x, y) = 0$$

Exercise 9.9: Two Ascending Arguments

We reduce the rectangle by decrementing x or incrementing y.

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We reduce the rectangle by decrementing x or incrementing y.
First we investigate what happens if we decrement x.

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\begin{array}{ll}
 & F(x, y) \\
= & \quad \{ \text{ definition } F \text{ } \} \\
 & \#\{i \mid 0 \leq i < x \wedge (\exists j : y \leq j < n \wedge h(i, j) = 0)\}
\end{array}
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= & \{ \textbf{ assume } x > 0\text{; so } 0 \leq i < x \equiv (0 \leq i < x - 1 \vee i = x - 1) \text{ } \}
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 & +
\end{array}
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& \#\{i \mid 0 \leq i < x - 1 \wedge (\exists j : y \leq j < n \wedge h(i, j) = 0)\} \\
& + \operatorname{ord}((\exists j : y \leq j < n \wedge h(x - 1, j) = 0))
\end{array}
$$

Exercise 9.9: Two Ascending Arguments

We reduce the rectangle by decrementing x or incrementing y.
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\begin{array}{ll}
 & F(x, y) \\
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 & + \operatorname{ord}((\exists j : y \leq j < n \wedge h(x - 1, j) = 0)) \\
= & \{ \text{ definition } F \ \} \\
 & F(x - 1, y) + \operatorname{ord}((\exists j : y \leq j < n \wedge h(x - 1, j) = 0))
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 & + \operatorname{ord}((\exists j : y \leq j < n \wedge h(x-1, j) = 0)) \\
= & \{ \text{definition } F \} \\
 & F(x-1, y) + \operatorname{ord}((\exists j : y \leq j < n \wedge h(x-1, j) = 0)) \\
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\begin{array}{ll}
 & F(x, y) \\
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 & \#\{i \mid 0 \leq i < x - 1 \wedge (\exists j : y \leq j < n \wedge h(i, j) = 0)\} \\
 & + \text{ord}((\exists j : y \leq j < n \wedge h(x - 1, j) = 0)) \\
= & \{ \text{ definition } F \text{ } \} \\
 & F(x - 1, y) + \text{ ord}((\exists j : y \leq j < n \wedge h(x - 1, j) = 0)) \\
= & \{ \text{ } h(x - 1, j) \text{ is ascending in } j \text{ so } h(x - 1, y) \text{ is}
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= & \quad \{ \text{ definition } F \ \} \\
 & F(x - 1, y) + \operatorname{ord}((\exists j : y \leq j < n \wedge h(x - 1, j) = 0)) \\
= & \quad \{ \ h(x - 1, j) \text{ is ascending in } j \text{ so } h(x - 1, y) \text{ is minimal;} \\
 & \qquad \textbf{assume}
\end{array}
$$

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$$
\begin{aligned}
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= & \quad \{ \text{ definition } F \text{ } \} \\
& F(x-1,y) + \operatorname{ord}((\exists j : y \leq j < n \wedge h(x-1,j) = 0)) \\
= & \quad \{ \text{ } h(x-1,j) \text{ is ascending in } j \text{ so } h(x-1,y) \text{ is minimal;} \\
& \quad\quad \textbf{assume } h(x-1,y) \geq 0,
\end{aligned}
$$

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\begin{array}{cl}
 & F(x, y) \\
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 & \#\{i \mid 0 \leq i < x \wedge (\exists j : y \leq j < n \wedge h(i, j) = 0)\} \\
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= & \{ \text{ } h(x - 1, j) \text{ is ascending in } j \text{ so } h(x - 1, y) \text{ is minimal;} \\
 & \quad \textbf{assume } h(x - 1, y) \geq 0\text{, so } h(x - 1, j) \geq 0 \text{ for all } y \leq j < n\text{;}
\end{array}
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& \quad \text{so}
\end{aligned}
$$

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& \quad \text{so } (\exists j : y \leq j < n \wedge h(x - 1, j) = 0) \equiv
\end{array}
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& \#\{i \mid 0 \leq i < x-1 \wedge (\exists j : y \leq j < n \wedge h(i,j) = 0)\} \\
& + \operatorname{ord}((\exists j : y \leq j < n \wedge h(x-1,j) = 0)) \\
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& \quad \textbf{assume } h(x-1,y) \geq 0\text{, so } h(x-1,j) \geq 0 \text{ for all } y \leq j < n; \\
& \quad \text{so } (\exists j : y \leq j < n \wedge h(x-1,j) = 0) \equiv (h(x-1,y) = 0) \ \} \\
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 & + \operatorname{ord}((\exists j : y \le j < n \wedge h(x-1,j) = 0)) \\
= & \{ \text{ definition } F \ \} \\
 & F(x-1,y) + \operatorname{ord}((\exists j : y \le j < n \wedge h(x-1,j) = 0)) \\
= & \{ \ h(x-1,j) \text{ is ascending in } j \text{ so } h(x-1,y) \text{ is minimal;} \\
 & \quad \textbf{assume } h(x-1,y) \ge 0\text{, so } h(x-1,j) \ge 0 \text{ for all } y \le j < n; \\
 & \quad \text{so } (\exists j : y \le j < n \wedge h(x-1,j) = 0) \equiv (h(x-1,y) = 0) \ \} \\
 & F(x-1,y) + \operatorname{ord}(h(x-1,y) = 0)
\end{array}
$$

This derivation proves:

$x > 0 \wedge h(x-1,y) \ge 0 \Rightarrow F(x,y) = F(x-1,y) + \operatorname{ord}(h(x-1,y) = 0)$

Exercise 9.9: Two Ascending Arguments

Next we investigate what happens if we increment y:

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 & \#\{i \mid 0 \le i < x \land (\exists j : y \le j < n \land h(i,j) = 0)\}
\end{array}
$$

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= & \{ \textbf{ assume } y < n\text{; so } y \le j < n \equiv (y+1 \le j < n \vee j = y) \text{ } \}
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= & \{\ \textbf{assume } y < n\text{; so } y \le j < n \equiv (y+1 \le j < n \vee j = y)\ \} \\
 & \#\{i \mid 0 \le i < x \wedge (h(i,y) = 0 \vee (\exists j : y+1 \le j < n \wedge h(i,j) = 0))\}
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 & \#\{i \mid 0 \le i < x \wedge (h(i,y) = 0 \vee (\exists j : y+1 \le j < n \wedge h(i,j) = 0))\} \\
= & \{ \textbf{ assume } x > 0;
\end{array}
$$

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 & F(x, y) \\
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= & \{ \textbf{ assume } y < n\text{; so } y \leq j < n \equiv (y+1 \leq j < n \vee j = y) \ \} \\
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= & \{ \textbf{ assume } x > 0\text{; } h(i,y) \text{ is ascending in } i \text{ so}
\end{array}
$$

Exercise 9.9: Two Ascending Arguments

Next we investigate what happens if we increment y:

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\begin{array}{ll}
 & F(x, y) \\
= & \{\ \text{definition } F\ \} \\
 & \#\{i \mid 0 \leq i < x \wedge (\exists j : y \leq j < n \wedge h(i, j) = 0)\} \\
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 & \quad \textbf{assume}
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 & F(x, y+1)
\end{array}
$$

This derivation proves:

$$x > 0 \wedge y < n \wedge h(x-1, y) < 0 \Rightarrow F(x,y) = F(x, y+1)$$

Exercise 9.9: Two Ascending Arguments

Given

$$F(x, y) = \#\{i \mid 0 \leq i < x \wedge (\exists j : y \leq j < n \wedge h(i, j) = 0)\}$$

we obtained the following recursive equations:

$$\begin{aligned}
x \leq 0 \vee n \leq y \quad &\Rightarrow \quad F(x, y) = 0 \\
x > 0 \wedge h(x-1, y) \geq 0 \quad &\Rightarrow \quad F(x, y) = b + F(x-1, y) \\
x > 0 \wedge y < n \wedge h(x-1, y) < 0 \quad &\Rightarrow \quad F(x, y) = F(x, y+1)
\end{aligned}$$

where $b = \text{ord}(h(x-1, y) = 0)$.

Exercise 9.9: Two Ascending Arguments

const $m,\ n:\ \mathbb{N}$;
var $z:\ \mathbb{Z}$;
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Exercise 9.9: Two Ascending Arguments

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0 We need a **while**-program to iteratively reduce the rectangle by decrementing x or incrementing y.

1 We introduce the variables $x, \ y : \mathbb{Z}$ and the invariant and guard

$$\boxed{\begin{array}{r} J : \ Z = z + F(x, y) \\ B : \ x > 0 \land y < n \end{array}}$$

$$\begin{array}{ll}
 & J \land \neg B \\
\equiv & \{ \text{ definition } J \text{ and } B \ \} \\
 & Z = z + F(x, y) \land \neg(x > 0 \land y < n) \\
\equiv & \{ \text{ Logic; De Morgan } \} \\
 & Z = z + F(x, y) \land (x \leq 0 \lor y \geq n) \\
\Rightarrow & \{ \text{ base case recurrence; } F(x, y) = 0 \ \} \\
 & Q : Z = z
\end{array}$$

Exercise 9.9: Two Ascending Arguments

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Exercise 9.9: Two Ascending Arguments

2 Initialization:

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(* *calculus* *)

$$\{Z = 0 + F(m,0)\}$$

Exercise 9.9: Two Ascending Arguments

2 Initialization:

$$\{P: \ Z = F(m,0)\}$$
(* *calculus* *)
$$\{Z = 0 + F(m,0)\}$$
$$z := 0;\ x := m;\ y := 0;$$

Exercise 9.9: Two Ascending Arguments

2 Initialization:

$$\{P:\ Z = F(m,0)\}$$

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$$\{J:\ Z = z + F(x,y)\}$$

Exercise 9.9: Two Ascending Arguments

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We start with (x, y) in the South-East corner of the grid.

Exercise 9.9: Two Ascending Arguments

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3 Variant function:

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We shrink the rectangle in North-Western direction, i.e. we decrement x and increment y.

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We shrink the rectangle in North-Western direction, i.e. we decrement x and increment y.
It is natural to choose $vf = x + n - y \in \mathbb{Z}$.

Exercise 9.9: Two Ascending Arguments

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\begin{array}{l}
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\qquad \text{(* calculus *)} \\
\quad \{Z = 0 + F(m, 0)\} \\
z := 0; \ x := m; \ y := 0; \\
\quad \{J: \ Z = z + F(x, y)\}
\end{array}
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3 Variant function:
We shrink the rectangle in North-Western direction, i.e. we decrement x and increment y.
It is natural to choose $vf = x + n - y \in \mathbb{Z}$.
The guard is $x > 0 \wedge y < n$, so clearly $J \wedge B \Rightarrow vf \geq 0$.

Exercise 9.9: Two Ascending Arguments

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$\{Z = z + F(x, y) \wedge x > 0 \wedge y < n \wedge x + n - y = V\}$

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(* *logic; recurrence for* $F(x, y)$*; case* $x > 0 \wedge h(x-1, y) \geq 0$ *)

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(* *logic; recurrence for* $F(x, y)$*; case* $x > 0 \wedge h(x-1, y) \geq 0$ *)

$\{Z = z + \text{ord}(h(x-1, y) = 0) + F(x-1, y) \wedge x + n - y = V\}$

$z := z + \text{ord}(h(x-1, y) = 0);$

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(* *calculus; prepare* $x := x - 1$ *)

$\{Z = z + F(x-1,y) \wedge x - 1 + n - y < V\}$

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$x := x - 1;$

$\{Z = z + F(x, y) \wedge x + n - y < V\}$

else

$\{h(x-1, y) < 0 \wedge Z = z + F(x, y) \wedge x > 0 \wedge y < n \wedge x + n - y = V\}$

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$x := x - 1;$

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Exercise 9.9: Two Ascending Arguments

$\{Z = z + F(x,y) \wedge x > 0 \wedge y < n \wedge x + n - y = V\}$

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$\{Z = z + F(x,y+1) \wedge x + n - (y+1) < V\}$

$y := y + 1;$

Exercise 9.9: Two Ascending Arguments

$\{Z = z + F(x, y) \land x > 0 \land y < n \land x + n - y = V\}$

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(* *logic; recurrence for* $F(x, y)$*; case* $x > 0 \land h(x-1, y) \geq 0$ *)

$\{Z = z + \text{ord}(h(x-1, y) = 0) + F(x-1, y) \land x + n - y = V\}$

$z := z + \text{ord}(h(x-1, y) = 0);$

$\{Z = z + F(x-1, y) \land x + n - y = V\}$

(* *calculus; prepare* $x := x - 1$ *)

$\{Z = z + F(x-1, y) \land x - 1 + n - y < V\}$

$x := x - 1;$

$\{Z = z + F(x, y) \land x + n - y < V\}$

else

$\{h(x-1, y) < 0 \land Z = z + F(x, y) \land x > 0 \land y < n \land x + n - y = V\}$

(* *logic; recurrence for* $F(x, y)$*; case* $x > 0 \land y < n \land h(x-1, y) < 0$ *)

$\{Z = z + F(x, y+1) \land x + n - y = V\}$

(* *calculus; prepare* $y := y + 1$ *)

$\{Z = z + F(x, y+1) \land x + n - (y+1) < V\}$

$y := y + 1;$

$\{Z = z + F(x, y) \land x + n - y < V\}$

Exercise 9.9: Two Ascending Arguments

$\{Z = z + F(x,y) \wedge x > 0 \wedge y < n \wedge x + n - y = V\}$

if $h(x-1,y) \geq 0$ **then**

$\{h(x-1,y) \geq 0 \wedge Z = z + F(x,y) \wedge x > 0 \wedge y < n \wedge x + n - y = V\}$

(* *logic; recurrence for* $F(x,y)$*; case* $x > 0 \wedge h(x-1,y) \geq 0$ *)

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$z := z + \text{ord}(h(x-1,y) = 0);$

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(* *calculus; prepare* $x := x - 1$ *)

$\{Z = z + F(x-1,y) \wedge x - 1 + n - y < V\}$

$x := x - 1;$

$\{Z = z + F(x,y) \wedge x + n - y < V\}$

else

$\{h(x-1,y) < 0 \wedge Z = z + F(x,y) \wedge x > 0 \wedge y < n \wedge x + n - y = V\}$

(* *logic; recurrence for* $F(x,y)$*; case* $x > 0 \wedge y < n \wedge h(x-1,y) < 0$ *)

$\{Z = z + F(x,y+1) \wedge x + n - y = V\}$

(* *calculus; prepare* $y := y + 1$ *)

$\{Z = z + F(x,y+1) \wedge x + n - (y+1) < V\}$

$y := y + 1;$

$\{Z = z + F(x,y) \wedge x + n - y < V\}$

end (* *collect branches; definitions* J *and* vf *)

$\{J \wedge vf < V\}$

Exercise 9.9: Two Ascending Arguments

const $m,\ n:\ \mathbb{N}$;
var $x, y, z:\ \mathbb{Z}$;
  $\{P:\ Z = \#\{i \mid 0 \leq i < m \wedge (\exists j : 0 \leq j < n \wedge h(i,j) = 0)\}\ \}$
$z := 0$;
$x := m$;
$y := 0$;
  $\{J:\ Z = z + \#\{i \mid 0 \leq i < x \wedge (\exists j : y \leq j < n \wedge h(i,j) = 0)\}\ \}$
    $(^*\ vf:\ x + n - y\ ^*)$
while $x > 0 \wedge y < n$ **do**
  if $h(x-1, y) \geq 0$ **then**
    $z := z + \text{ord}(h(x-1, y) = 0)$;
    $x := x - 1$;
  else
    $y := y + 1$;
  end;
end;
  $\{Q:\ z = Z\}$

Exercise 9.9: Two Ascending Arguments

const $m,\ n:\ \mathbb{N}$;
var $x, y, z:\ \mathbb{Z}$;
  $\{P:\ Z = \#\{i \mid 0 \leq i < m \wedge (\exists j : 0 \leq j < n \wedge h(i,j) = 0)\}\ \}$
$z := 0$;
$x := m$;
$y := 0$;
  $\{J:\ Z = z + \#\{i \mid 0 \leq i < x \wedge (\exists j : y \leq j < n \wedge h(i,j) = 0)\}\ \}$
    (* $vf:\ x + n - y$ *)
while $x > 0 \wedge y < n$ **do**
  if $h(x-1, y) \geq 0$ **then**
    $z := z + \mathrm{ord}(h(x-1, y) = 0)$;
    $x := x - 1$;
  else
    $y := y + 1$;
  end;
end;
  $\{Q:\ z = Z\}$

Note: The algorithm has time complexity $O(m+n)$, more efficient than the brute-force $O(m \cdot n)$ algorithm.

Outline

Exercise 9.4: Decreasing & Ascending

Let $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be a two-dimensional function, now decreasing in x and ascending in y:

Exercise 9.4: Decreasing & Ascending

Let $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be a two-dimensional function, now decreasing in x and ascending in y:

$$x_0 < x_1 \Rightarrow h(x_0, y) > h(x_1, y)$$
$$y_0 \leq y_1 \Rightarrow h(x, y_0) \leq h(x, y1)$$

Exercise 9.4: Decreasing & Ascending

Let $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be a two-dimensional function, now decreasing in x and ascending in y:

$$x_0 < x_1 \Rightarrow h(x_0, y) > h(x_1, y)$$
$$y_0 \leq y_1 \Rightarrow h(x, y_0) \leq h(x, y1)$$

We want to find a command T that satisfies the specification:

const $m,\ n :\ \mathbb{N};\ w :\ \mathbb{Z};$
var $z :\ \mathbb{Z};$
$\{P:\ Z = \#\{(i,j) \in [0..m) \times [0..n) \mid h(i,j) = w\}\}$
$T;$
$\{Q:\ Z = z\}$

Exercise 9.4: Decreasing & Ascending

const $m,\ n:\ \mathbb{N};\ w:\ \mathbb{Z};$
var $z:\ \mathbb{Z};$

$\{P:\ Z = \#\{(i,j) \in [0..m) \times [0..n) \mid h(i,j) = w\}\}$

$T;$

$\{Q:\ Z = z\}$

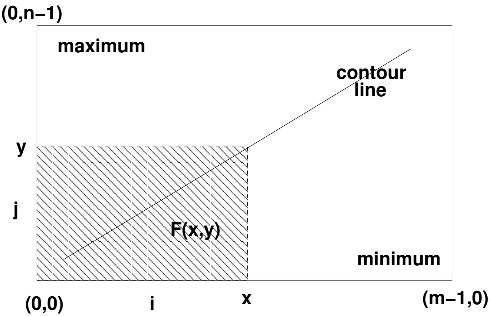
Example, with $w = 10$:

29	28	26	25	22	21	20	18	14	**10**
27	26	25	23	21	20	18	16	13	8
27	23	22	21	19	18	17	14	12	8
27	22	21	20	18	16	15	14	12	7
25	22	21	18	16	15	14	13	**10**	7
23	21	19	18	15	14	13	**10**	9	7
21	19	17	16	15	13	12	**10**	7	5
18	15	14	13	12	11	**10**	8	5	4
16	15	14	12	11	**10**	9	7	5	2
14	12	**10**	9	8	7	6	5	3	2

Exercise 9.4: Decreasing & Ascending

We keep $J: \; Z = z + F(x, y)$.

- At the beginning:
  $Z = F(m, n)$.



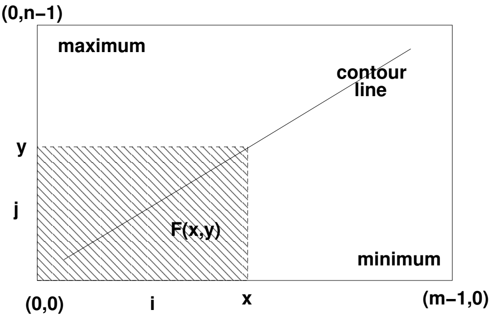
We define:

$$F(x, y) = \#\{(i, j) \mid 0 \leq i < x \; \wedge \; 0 \leq j < y \; \wedge \; h(i, j) = w\}$$

Exercise 9.4: Decreasing & Ascending

We keep $J:\ Z = z + F(x,y)$.

- At the beginning:
  $Z = F(m,n)$.
- In the middle, reduce the rectangle:
  decrease x / decrease y.



We define:

$$F(x,y) = \#\{(i,j) \mid 0 \le i < x \ \wedge \ 0 \le j < y \ \wedge \ h(i,j) = w\}$$

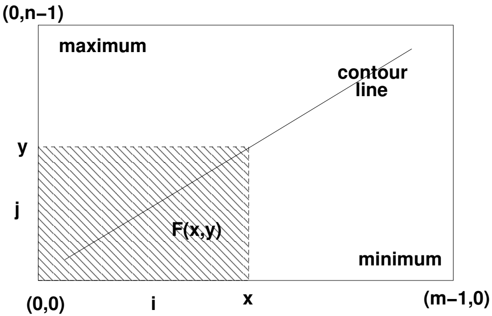
We find a recurrence for $F(x,y)$. Because $\#\emptyset = 0$, the base case is:

$$x \le 0 \vee y \le 0 \Rightarrow F(x,y) = 0$$

Exercise 9.4: Decreasing & Ascending

We keep $J: \; Z = z + F(x,y)$.

- At the beginning: $Z = F(m,n)$.
- In the middle, reduce the rectangle: decrease x / decrease y.
- At the end: $Z = z$ and $F(0,0) = 0$.



We define:

$$F(x,y) = \#\{(i,j) \mid 0 \le i < x \;\wedge\; 0 \le j < y \;\wedge\; h(i,j) = w\}$$

We find a recurrence for $F(x,y)$. Because $\#\emptyset = 0$, the base case is:

$$x \le 0 \vee y \le 0 \Rightarrow F(x,y) = 0$$

Exercise 9.4: Decreasing & Ascending

$$F(x, y) = \#\{(i, j) \mid 0 \leq i < x \wedge 0 \leq j < y \wedge h(i, j) = w\}$$

We can reduce the rectangle by decrementing x or decrementing y.
We first investigate a decrement to x:

Exercise 9.4: Decreasing & Ascending

$$F(x, y) = \#\{(i, j) \mid 0 \le i < x \land 0 \le j < y \land h(i, j) = w\}$$

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Exercise 9.4: Decreasing & Ascending

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 & F(x, y) \\
= & \{\ \text{definition } F\ \} \\
 & \#\{(i, j) \mid 0 \le i < x \wedge 0 \le j < y \wedge h(i, j) = w\} \\
= & \{\ \textbf{assume } x > 0\text{; so } 0 \le i < x \equiv (0 \le i < x - 1 \vee i = x - 1)\ \}
\end{array}
$$

Exercise 9.4: Decreasing & Ascending

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$$\begin{aligned}
& F(x, y) \\
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& \#\{(i, j) \mid 0 \leq i < x \wedge 0 \leq j < y \wedge h(i, j) = w\} \\
= \quad & \{ \ \textbf{assume } x > 0\text{; so } 0 \leq i < x \equiv (0 \leq i < x - 1 \vee i = x - 1) \ \} \\
& \#\{(i, j) \mid i, j :\ 0 \leq i < x - 1 \wedge 0 \leq j < y \wedge h(i, j) = w\} + \\
& \#\{(x - 1, j) \mid j :\ 0 \leq j < y \wedge h(x - 1, j) = w\}
\end{aligned}$$

Exercise 9.4: Decreasing & Ascending

$$F(x, y) = \#\{(i, j) \mid 0 \leq i < x \wedge 0 \leq j < y \wedge h(i, j) = w\}$$

We can reduce the rectangle by decrementing x or decrementing y.
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 & F(x, y) \\
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 & \#\{(i, j) \mid 0 \leq i < x \wedge 0 \leq j < y \wedge h(i, j) = w\} \\
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= & \{\text{ definition } F \text{ }\} \\
 & F(x - 1, y) + \#\{(x - 1, j) \mid j :\ 0 \leq j < y \wedge h(x - 1, j) = w\}
\end{array}
$$

Exercise 9.4: Decreasing & Ascending

$$F(x,y) = \#\{(i,j) \mid 0 \leq i < x \wedge 0 \leq j < y \wedge h(i,j) = w\}$$

We can reduce the rectangle by decrementing x or decrementing y.
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 & F(x,y) \\
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 & \#\{(i,j) \mid 0 \leq i < x \wedge 0 \leq j < y \wedge h(i,j) = w\} \\
= & \quad \{ \textbf{ assume } x > 0\text{; so } 0 \leq i < x \equiv (0 \leq i < x-1 \vee i = x-1) \ \} \\
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= & \quad \{ \text{ definition } F \ \} \\
 & F(x-1,y) + \#\{(x-1,j) \mid j :\ 0 \leq j < y \wedge h(x-1,j) = w\} \\
= & \quad \{ \textbf{ assume } y > 0; \\
\end{array}
$$

Exercise 9.4: Decreasing & Ascending

$$F(x,y) = \#\{(i,j) \mid 0 \le i < x \wedge 0 \le j < y \wedge h(i,j) = w\}$$

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 & \#\{(i,j) \mid 0 \le i < x \wedge 0 \le j < y \wedge h(i,j) = w\} \\
= & \{\ \textbf{assume } x > 0\text{; so } 0 \le i < x \equiv (0 \le i < x-1 \vee i = x-1)\ \} \\
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 & \#\{(x-1,j) \mid j:\ 0 \le j < y \wedge h(x-1,j) = w\} \\
= & \{\text{ definition } F\ \} \\
 & F(x-1,y) + \#\{(x-1,j) \mid j:\ 0 \le j < y \wedge h(x-1,j) = w\} \\
= & \{\ \textbf{assume } y > 0\text{; } h(x-1,j) \text{ is ascending in } j\text{, so}
\end{array}
$$

Exercise 9.4: Decreasing & Ascending

$$F(x, y) = \#\{(i, j) \mid 0 \le i < x \land 0 \le j < y \land h(i, j) = w\}$$

We can reduce the rectangle by decrementing x or decrementing y.
We first investigate a decrement to x:

$F(x, y)$
$=$ { definition F }
$\#\{(i, j) \mid 0 \le i < x \land 0 \le j < y \land h(i, j) = w\}$
$=$ { **assume** $x > 0$; so $0 \le i < x \equiv (0 \le i < x - 1 \lor i = x - 1)$ }
$\#\{(i, j) \mid i, j :\ 0 \le i < x - 1 \land 0 \le j < y \land h(i, j) = w\} +$
$\#\{(x - 1, j) \mid j :\ 0 \le j < y \land h(x - 1, j) = w\}$
$=$ { definition F }
$F(x - 1, y) + \#\{(x - 1, j) \mid j :\ 0 \le j < y \land h(x - 1, j) = w\}$
$=$ { **assume** $y > 0$; $h(x - 1, j)$ is ascending in j, so $h(x - 1, y - 1)$ is maximal;

Exercise 9.4: Decreasing & Ascending

$$F(x, y) = \#\{(i, j) \mid 0 \le i < x \land 0 \le j < y \land h(i, j) = w\}$$

We can reduce the rectangle by decrementing x or decrementing y.
We first investigate a decrement to x:

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$\#\{(i, j) \mid 0 \le i < x \land 0 \le j < y \land h(i, j) = w\}$

$=$ { **assume** $x > 0$; so $0 \le i < x \equiv (0 \le i < x - 1 \lor i = x - 1)$ }

$\#\{(i, j) \mid i, j :\ 0 \le i < x - 1 \land 0 \le j < y \land h(i, j) = w\} +$
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$F(x - 1, y) + \#\{(x - 1, j) \mid j :\ 0 \le j < y \land h(x - 1, j) = w\}$

$=$ { **assume** $y > 0$; $h(x - 1, j)$ is ascending in j, so $h(x - 1, y - 1)$ is maximal;
assume

Exercise 9.4: Decreasing & Ascending

$$F(x, y) = \#\{(i, j) \mid 0 \le i < x \land 0 \le j < y \land h(i, j) = w\}$$

We can reduce the rectangle by decrementing x or decrementing y.
We first investigate a decrement to x:

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= & \quad \{ \text{ definition } F \ \} \\
 & F(x - 1, y) + \#\{(x - 1, j) \mid j : \ 0 \le j < y \land h(x - 1, j) = w\} \\
= & \quad \{ \ \textbf{assume } y > 0\text{; } h(x - 1, j) \text{ is ascending in } j\text{, so } h(x - 1, y - 1) \text{ is maximal;} \\
 & \qquad \textbf{assume } h(x - 1, y - 1) < w,
\end{array}
$$

Exercise 9.4: Decreasing & Ascending

$$F(x, y) = \#\{(i, j) \mid 0 \le i < x \land 0 \le j < y \land h(i, j) = w\}$$

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= & \quad \{ \ \textbf{assume } y > 0\text{; } h(x - 1, j) \text{ is ascending in } j\text{, so } h(x - 1, y - 1) \text{ is maximal;} \\
 & \quad\ \ \textbf{assume } h(x - 1, y - 1) < w\text{, so } h(x - 1, j) < w \text{ for all } j \le y - 1 \ \}
\end{array}
$$

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$\#\{(x - 1, j) \mid j :\ 0 \leq j < y \wedge h(x - 1, j) = w\}$

$=$ { definition F }

$F(x - 1, y) + \#\{(x - 1, j) \mid j :\ 0 \leq j < y \wedge h(x - 1, j) = w\}$

$=$ { **assume** $y > 0$; $h(x - 1, j)$ is ascending in j, so $h(x - 1, y - 1)$ is maximal;
assume $h(x - 1, y - 1) < w$, so $h(x - 1, j) < w$ for all $j \leq y - 1$ }

$F(x - 1, y) + 0$

$=$ { calculus }

$F(x - 1, y)$

Exercise 9.4: Decreasing & Ascending

$$F(x, y) = \#\{(i, j) \mid 0 \le i < x \wedge 0 \le j < y \wedge h(i, j) = w\}$$

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& \#\{(x - 1, j) \mid j :\ 0 \le j < y \wedge h(x - 1, j) = w\} \\
= & \{\text{ definition } F \text{ }\} \\
& F(x - 1, y) + \#\{(x - 1, j) \mid j :\ 0 \le j < y \wedge h(x - 1, j) = w\} \\
= & \{\textbf{ assume } y > 0\text{; } h(x - 1, j) \text{ is ascending in } j\text{, so } h(x - 1, y - 1) \text{ is maximal;} \\
& \quad \textbf{assume } h(x - 1, y - 1) < w\text{, so } h(x - 1, j) < w \text{ for all } j \le y - 1 \text{ }\} \\
& F(x - 1, y) + 0 \\
= & \{\text{ calculus }\} \\
& F(x - 1, y)
\end{array}$$

This derivation proves:
$x > 0 \wedge y > 0 \wedge h(x - 1, y - 1) < w \Rightarrow F(x, y) = F(x - 1, y)$

Exercise 9.4: Decreasing & Ascending

Next, we investigate what happens if we decrement y.

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& \#\{(i, j) \mid 0 \leq i < x \wedge 0 \leq j < y \wedge h(i, j) = w\}
\end{aligned}
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& \#\{(i,j) \mid i,j:\ 0 \le i < x \wedge 0 \le j < y-1 \wedge h(i,j) = w\} + \\
& \#\{(i,y-1) \mid i:\ 0 \le i < x \wedge h(i,y-1) = w\}
\end{aligned}
$$

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& F(x, y) \\
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& \#\{(i,j) \mid 0 \leq i < x \wedge 0 \leq j < y \wedge h(i,j) = w\} \\
= & \{ \textbf{ assume } y > 0\text{; so } 0 \leq j < y \equiv (0 \leq j < y-1 \vee j = y-1) \} \\
& \#\{(i,j) \mid i,j:\ 0 \leq i < x \wedge 0 \leq j < y-1 \wedge h(i,j) = w\} + \\
& \#\{(i,y-1) \mid i:\ 0 \leq i < x \wedge h(i,y-1) = w\} \\
= & \{ \text{ definition } F \} \\
& F(x, y-1) + \#\{(i,y-1) \mid i:\ 0 \leq i < x \wedge h(i,y-1) = w\}
\end{aligned}
$$

Exercise 9.4: Decreasing & Ascending

Next, we investigate what happens if we decrement y.

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$= \{$ definition F $\}$
$\#\{(i, j) \mid 0 \leq i < x \wedge 0 \leq j < y \wedge h(i, j) = w\}$
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& \quad \textbf{assume } h(x-1, y-1) \geq w\text{, so } h(i, y-1) > w \text{ for all } 0 \leq i < x-1 \text{ } \}
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& F(x,y-1) + \mathrm{ord}(h(x-1,y-1) = w)
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This derivation proves:

$$
\begin{aligned}
& x > 0 \ \wedge \ y > 0 \ \wedge \ h(x-1,y-1) \ge w \Rightarrow \\
& \qquad F(x,y) = F(x,y-1) + \mathrm{ord}(h(x-1,y-1) = w)
\end{aligned}
$$

Exercise 9.4: Decreasing & Ascending

Given

$$F(x, y) = \#\{(i, j) \mid 0 \leq i < x \ \wedge \ 0 \leq j < y \ \wedge \ h(i, j) = w\}$$

we obtained the following recursive equations:

$$\begin{aligned}
x \leq 0 \vee y \leq 0 \quad &\Rightarrow \quad F(x, y) = 0 \\
x > 0 \ \wedge \ y > 0 \ \wedge \ h(x-1, y-1) < w \quad &\Rightarrow \quad F(x, y) = F(x-1, y) \\
x > 0 \ \wedge \ y > 0 \ \wedge \ h(x-1, y-1) \geq w \quad &\Rightarrow \quad F(x, y) = b + F(x, y-1)
\end{aligned}$$

where $b = \text{ord}(h(x-1, y-1) = w)$.

Exercise 9.4: Decreasing & Ascending

We now rewrite the original specification to obtain:

const $m,\ n:\ \mathbb{N};\ w:\ \mathbb{Z}$;
var $z:\ \mathbb{Z}$;
$\quad\{P:\ Z = F(m,n)\}$
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1 We introduce the variables $x,\ y:\mathbb{Z}$, the invariant, and guard:

$$\boxed{\begin{array}{c} J: Z = z + F(x,y) \\ B: x > 0 \wedge y > 0 \end{array}}$$

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$$\begin{array}{ll}
& J \wedge \neg B\\
\equiv & \{\text{ definition } J \text{ and } B\ \}\\
& Z = z + F(x,y) \wedge \neg(x > 0 \wedge y > 0)\\
\equiv & \{\text{ Logic; De Morgan }\}\\
& Z = z + F(x,y) \wedge (x \leq 0 \vee y \leq 0)\\
\Rightarrow & \{\text{ base case recurrence: } F(x,y) = 0\ \}\\
& Q : Z = z
\end{array}$$

Exercise 9.4: Decreasing & Ascending

2 Initialization:

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We start with (x, y) in the North-East corner of the grid.

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3 Variant function:
We shrink the rectangle in South-Western direction: we decrement x and decrement y.
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The guard is $x > 0 \wedge y > 0$, so clearly $J \wedge B \Rightarrow vf \geq 0$.

Exercise 9.4: Decreasing & Ascending

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$\{Z = z + F(x, y) \land x > 0 \land y > 0 \land x + y = V\}$

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Exercise 9.4: Decreasing & Ascending

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if $h(x-1, y-1) < w$ **then**

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$\{Z = z + F(x, y) \wedge x + y < V\}$

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$\{h(x-1, y-1) \geq w \wedge Z = z + F(x, y) \wedge x > 0 \wedge y > 0 \wedge x + y = V\}$

(* *logic; recurrence for* $F(x, y)$; *case* $x > 0 \wedge y > 0 \wedge h(x-1, y-1) \geq w$ *)

$\{Z = z + \text{ord}(h(x-1, y-1) = w) + F(x, y-1) \wedge x + y = V\}$

$z := z + \text{ord}(h(x-1, y-1) = w);$

$\{Z = z + F(x, y-1) \wedge x + y = V\}$

(* *calculus; prepare* $y := y - 1$ *)

$\{Z = z + F(x, y-1) \wedge x + y - 1 < V\}$

Exercise 9.4: Decreasing & Ascending

$\{Z = z + F(x,y) \wedge x > 0 \wedge y > 0 \wedge x + y = V\}$

if $h(x-1, y-1) < w$ **then**

$\{h(x-1, y-1) < w \wedge Z = z + F(x,y) \wedge x > 0 \wedge y > 0 \wedge x + y = V\}$

(* *logic; recurrence for* $F(x,y)$*; case* $x > 0 \wedge y > 0 \wedge h(x-1, y-1) < w$ *)

$\{Z = z + F(x-1, y) \wedge x + y = V\}$

(* *calculus; prepare* $x := x - 1$ *)

$\{Z = z + F(x-1, y) \wedge x - 1 + y < V\}$

$x := x - 1;$

$\{Z = z + F(x,y) \wedge x + y < V\}$

else

$\{h(x-1, y-1) \geq w \wedge Z = z + F(x,y) \wedge x > 0 \wedge y > 0 \wedge x + y = V\}$

(* *logic; recurrence for* $F(x,y)$*; case* $x > 0 \wedge y > 0 \wedge h(x-1, y-1) \geq w$ *)

$\{Z = z + \text{ord}(h(x-1, y-1) = w) + F(x, y-1) \wedge x + y = V\}$

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Exercise 9.4: Decreasing & Ascending

$\{Z = z + F(x, y) \wedge x > 0 \wedge y > 0 \wedge x + y = V\}$

if $h(x-1, y-1) < w$ **then**

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Exercise 9.4: Decreasing & Ascending

$\{Z = z + F(x,y) \wedge x > 0 \wedge y > 0 \wedge x + y = V\}$
if $h(x-1, y-1) < w$ **then**
$\{h(x-1,y-1) < w \wedge Z = z + F(x,y) \wedge x > 0 \wedge y > 0 \wedge x + y = V\}$
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$\{Z = z + F(x,y-1) \wedge x + y - 1 < V\}$
$y := y - 1;$
$\{Z = z + F(x,y) \wedge x + y < V\}$
end (* *collect branches; definitions* J *and* vf *)
$\{J \wedge vf < V\}$

Exercise 9.4: Decreasing & Ascending

const $m,\ n,\ w:\ \mathbb{N}$;
var $x, y, z:\ \mathbb{Z}$;
  $\{P:\ Z = \#\{(i,j) \in [0..m) \times [0..n) \mid h(i,j) = w\}\ \}$
$z := 0$;
$x := m$;
$y := n$;
  $\{J:\ Z = z + \#\{(i,j) \in [0..x) \times [0..y) \mid h(i,j) = w\}\ \}$
    (* $vf:\ x + y$ *)
while $x > 0 \wedge y > 0$ **do**
  if $h(x-1, y-1) < w$ **then**
    $x := x - 1$;
  else
    $z := y + \mathrm{ord}(h(x-1, y-1) = w)$;
    $y := y - 1$;
  end;
end;
  $\{Q:\ z = Z\}$

Exercise 9.4: Decreasing & Ascending

const $m,\ n,\ w :\ \mathbb{N}$;
var $x, y, z :\ \mathbb{Z}$;
  $\{P :\ Z = \#\{(i,j) \in [0..m) \times [0..n) \mid h(i,j) = w\}\ \}$
$z := 0$;
$x := m$;
$y := n$;
  $\{J :\ Z = z + \#\{(i,j) \in [0..x) \times [0..y) \mid h(i,j) = w\}\ \}$
    (* $vf :\ x + y$ *)
while $x > 0 \wedge y > 0$ **do**
  if $h(x-1, y-1) < w$ **then**
    $x := x - 1$;
  else
    $z := y + \mathrm{ord}(h(x-1, y-1) = w)$;
    $y := y - 1$;
  end;
end;
  $\{Q :\ z = Z\}$

Note: Because $vf = m + n$ the algorithm has time complexity $O(m+n)$, much more efficient than the brute-force $O(m \cdot n)$ algorithm.

Outline

Exercise 9.7

Let $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ be a function increasing ($<$ / $<$) in x and descending ($\geq$ / $\leq$) in y:

$$x_0 < x_1 \Rightarrow g(x_0, y) < g(x_1, y)$$
$$y_0 \leq y_1 \Rightarrow g(x, y_0) \geq g(x, y_1)$$

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Given $n \in \mathbb{N}$ and $w \in \mathbb{Z}$, specify and design a command to compute the number of pairs $(i, j) \in \mathbb{N}^2$ with

1. $g(i, j) = w$
2. $i + j < n$.

Exercise 9.7

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1. $g(i, j) = w$
2. $i + j < n$.

While condition (1) is as in previous examples, condition (2) constrains the "shrinking area": the rectangle becomes a triangle.

Exercise 9.7

In principle, we want to compute:

$$\#\{(i,j) \mid 0 \leq i \ \wedge\ 0 \leq j \ \wedge\ i+j < n \ \wedge\ g(i,j) = w\}$$

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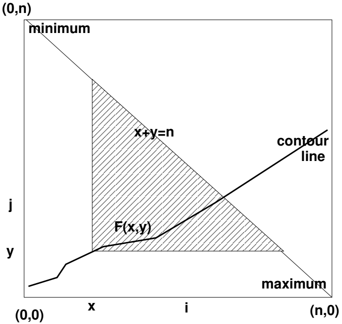

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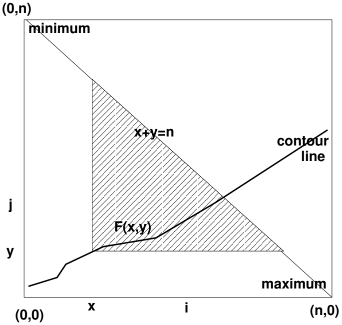

Let $F(x,y)$ be the number of points that we still need to count:

$$F(x,y) = \#\{(i,j) \mid x \leq i \ \wedge \ y \leq j \ \wedge \ i+j < n \ \wedge \ g(i,j) = w\}$$

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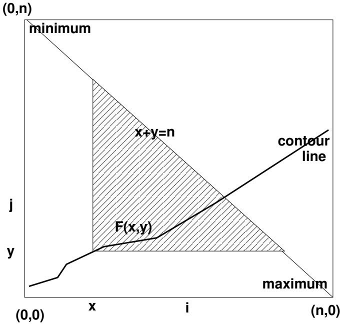

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Clearly, we want to compute $F(0,0)$.

Exercise 9.7

Given $F(x, y) = \#\{(i, j) \mid x \leq i \wedge y \leq j \wedge i + j < n \wedge g(i, j) = w\}$, we can specify the command T as follows:

const n : $\mathbb{N}$;
var z : $\mathbb{N}$;
  $\{P : Z = F(0, 0)\}$
T
  $\{Q : z = Z\}$

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const n : $\mathbb{N}$;
var z : $\mathbb{N}$;
  $\{P : Z = F(0, 0)\}$
T
  $\{Q : z = Z\}$

We reduce the triangle by maintaining the usual invariant:

$$\boxed{J : \; Z = z + F(x, y)}$$

Exercise 9.7

Given $F(x, y) = \#\{(i, j) \mid x \leq i \wedge y \leq j \wedge i + j < n \wedge g(i, j) = w\}$, it is easy to observe the base case:

$$x + y \geq n \Rightarrow F(x, y) = 0$$

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We see how to reduce the size of the triangle by <u>incrementing x</u>:

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$$\begin{array}{ll} & F(x, y) \\ = & \quad \{ \text{ definition } F \text{ } \} \\ & \#\{(i, j) \mid x \leq i \ \wedge \ y \leq j \ \wedge \ i + j < n \ \wedge \ g(i, j) = w\} \end{array}$$

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 & \#\{(i, j) \mid x \leq i \ \wedge \ y \leq j \ \wedge \ i + j < n \ \wedge \ g(i, j) = w\} \\
= & \{ \ \textbf{assume } x + y < n; \\
 & \quad \text{split non-empty domain; definition } F \ \} \\
 & F(x + 1, y) + \#\{j \mid j : \ y \leq j \ \wedge \ x + j < n \ \wedge \ g(x, j) = w\}
\end{array}$$

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= & \quad \{ \ x + j < n \text{ so } j < n - x \ \}
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= & \{ \ x+j<n \text{ so } j < n-x \ \} \\
 & F(x+1,y) + \#\{j \mid j : \ y \le j < n-x \ \wedge \ g(x,j) = w\} \\
= & \{ \ g(x,j) \text{ is descending in } j \text{ so } g(x,y) \text{ is}
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Given $F(x, y) = \#\{(i, j) \mid x \leq i \wedge y \leq j \wedge i + j < n \wedge g(i, j) = w\}$, it is easy to observe the base case:

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& F(x, y + 1) + \#\{i \mid i : \ x \le i \wedge i + y < n \wedge g(i, y) = w\}
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= & \{ \text{ } g(i,y) \text{ is increasing in } i \text{ so } g(x,y) \text{ is minimal;}
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& F(x,y+1) + \#\{i \mid i:\ x \le i \wedge i+y < n \wedge g(i,y) = w\} \\
= \quad & \{ \ g(i,y) \text{ is increasing in } i \text{ so } g(x,y) \text{ is minimal}; \\
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Next, we investigate what happens if we increment y:

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\end{array}
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In conclusion, $F(x,y)$ satisfies the following recursive equations:

$$
\boxed{
\begin{array}{rcl}
x + y \ge n & \Rightarrow & F(x,y) = 0 \\
x + y < n \wedge g(x,y) < w & \Rightarrow & F(x,y) = F(x+1, y) \\
x + y < n \wedge g(x,y) \ge w & \Rightarrow & F(x,y) = F(x, y+1) + \mathrm{ord}(g(x,y) = w)
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We will iteratively reduce the remaining area by incrementing x or y.

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Since we increment x or y as long as B holds, we choose the variant function $vf = n - x - y \in \mathbb{Z}$. Clearly, $J \wedge B \Rightarrow vf \geq 0$.

Exercise 9.7: Body of the Loop

$\{Z = z + F(x, y) \wedge x + y < n \ \wedge \ n - x - y = V\}$

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$\{Z = z + F(x,y) \wedge x+y<n \;\wedge\; n-x-y = V\}$

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$\{Z = z + F(x+1,y) \;\wedge\; n-x-y = V\}$

(* *calculus; prepare* $x := x+1$ *)

$\{Z = z + F(x+1,y) \;\wedge\; n-(x+1)-y < V\}$

$x := x+1;$

$\{Z = z + F(x,y) \;\wedge\; n-x-y < V\}$

else

$\{g(x,y) \geq w \;\wedge\; Z = z + F(x,y) \;\wedge\; x+y<n \;\wedge\; n-x-y = V\}$

(* *logic; recurrence for* $F(x,y)$*: case* $x+y<n \;\wedge\; g(x,y) \geq w$ *)

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$z := z + \mathrm{ord}(g(x,y) = w);$

$\{Z = z + F(x,y+1) \;\wedge\; n-x-y = V\}$

(* *calculus; prepare* $y := y+1$ *)

$\{Z = z + F(x,y+1) \;\wedge\; n-x-(y+1) < V\}$

$y := y+1;$

$\{Z = z + F(x,y) \;\wedge\; n-x-y < V\}$

Exercise 9.7: Body of the Loop

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(* *logic; recurrence for* $F(x, y)$*: case* $x + y < n \ \wedge\ g(x, y) \geq w$ *)

$\{Z = z + \mathrm{ord}(g(x, y) = w) + F(x, y + 1) \ \wedge\ n - x - y = V\}$

$z := z + \mathrm{ord}(g(x, y) = w);$

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(* *calculus; prepare* $y := y + 1$ *)

$\{Z = z + F(x, y + 1) \ \wedge\ n - x - (y + 1) < V\}$

$y := y + 1;$

$\{Z = z + F(x, y) \ \wedge\ n - x - y < V\}$

end (* *collect branches; definitions* J*, and* vf *)

$\{J \ \wedge\ vf < V\}$

Exercise 9.7: Conclusion

const $n : \mathbb{N},\ w : \mathbb{Z};$
var $x, y, z : \mathbb{Z};$
$\{P:\ \#\{(i,j) \mid 0 \le i \ \wedge\ 0 \le j \ \wedge\ i + j < n \ \wedge\ g(i,j) = w\}\ \}$
$z := 0;$
$x := 0;$
$y := 0;$
$\{J:\ Z = z + \#\{(i,j) \mid x \le i \ \wedge\ y \le j \ \wedge\ i + j < n \ \wedge\ g(i,j) = w\}\ \}$
(* $vf:\ n - x - y$ *)
while $x + y < n$ **do**
if $g(x, y) < w$ **then**
$x := x + 1;$
else
$z := z + \mathrm{ord}(g(x, y) = w);$
$y := y + 1;$
end;
end;
$\{Q:\ z = Z\}$

Note: Initially, $vf = n$, so the algorithm has time complexity $O(n)$, which is much more efficient than the brute-force $O(m \cdot n)$ algorithm.

The End