university of groningen

Languages and Machines

L5: Finite State Machines (Part 3)

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Languages and Their Machines

Regular	Finite State Machines (FSMs)
Context-free	Pushdown Machines
Context-sensitive	Linearly-bounded Machines
Decidable	Always-terminating Turing Machines
Semi-decidable	Turing Machines

Outline

The Many Faces of Regular Languages

Consider a **regular language** L over Σ.
By definition, L has an underlying *regular set*, a subset of Σ^*.

Each of the following is an equivalent characterization of L:

- a **regular expression**, built from $\emptyset$, ϵ, $a \in \Sigma$ using union, concatenation, power, and Kleene star;
- a **regular grammar** $G = (V, \Sigma, P, S)$, where every production rule is of the form $A \to \epsilon$ or $A \to a\,B$ (with $A, B \in V$, $a \in \Sigma$).
- a **DFSM** $M = (Q, \Sigma, \delta, q_0, F)$ where $\delta : Q \times \Sigma \to Q$, $q_0 \in Q$, $F \subseteq Q$. Variants of M can be non-deterministic, with ϵ-steps.

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Natural questions:

- Why do we identify multiple equivalent characterizations?
- Is every language a regular language? How to (dis)prove it?

There are Non Regular Sets

- Example 1:

$$\begin{aligned} L_1 &= \{a^n b^n \mid n \geq 1\} \\ &= \{ab, aabb, aaabbb \ldots\} \end{aligned}$$

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- Example 2:

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- Not accepted by a *finite* automaton:
  We cannot distinguish between **infinitely many** cases with **finitely many** states (unbounded vs bounded information)
- **Today**: A tool for proving that sets are not regular.

Example 1: $L_1 = \{a^n b^n \mid n \geq 1\}$ (1/2)

A proof by contradiction that L_1 is not regular:

- If L_1 were regular, it would be accepted by a DFSM M
- Machine M has a finite number of states, say k

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- By the pigeonhole principle, there is a state q_i that is visited more than once in processing the sequence of as:

$$[q_0]\ \overbrace{aaaaaaaa[q_i]aaaaa[q_i]aa}^{n}\ \overbrace{bbbbbbbbbbbbbbb}^{n}\ [q_f]$$

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- Let's split $a^n b^n$ into three pieces (u, v, w) according to q_i:

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- We could erase v and the obtained string would be accepted!

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- We could even insert extra copies of v (say, uv^3w), and the resulting string would be wrongly accepted too!

Example 2: $L_2 = \{a^{2^n} \mid n \geq 0\}$ (1/2)

- A DFSM M with k states, with $L(M) = L_2$ and start state q_0
- Let $n \gg k$ and consider the action of M on scanning string a^{2^n}
- By the pigeonhole principle, M must repeat a state q while scanning the first n symbols of a^{2^n}
- Now let i, j, m be such that $2^n = i + j + m$, with $0 < j \leq n$ and

$$\hat{\delta}(q_0, a^i) = q \qquad \hat{\delta}(q, a^j) = q \qquad \hat{\delta}(q, a^m) = q_f \in F$$

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- Now, given $\hat{\delta}(p, a^j) = p$, we could insert an extra a^j, to get a^{2^n+j}, and the resulting string would be erroneously accepted:

$$[q_0]\,\underbrace{aaaaaaaaaa}_{i}\,[q]\,\underbrace{aaaaa}_{j}\,[q]\,\underbrace{aaaaa}_{j}\,[q]\,\underbrace{aaaaaaaaaaaaaaaa}_{m}\,[q_f]$$

Indeed, we can derive $\hat{\delta}(q_0, a^{2^n+j}) = q_f \in F$.

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  Indeed, we can derive $\hat{\delta}(q_0, a^{2^n+j}) = q_f \in F$.

- But this is wrong, because $2^n + j$ is not a power of 2:

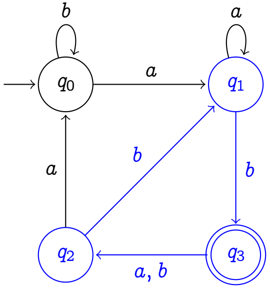
$$\begin{aligned} 2^n + j &\leq 2^n + n \\ 2^n + j &< 2^n + 2^n \\ &= 2^{n+1} \end{aligned}$$

  2^{n+1} is the next power of 2 greater than 2^n

Pumping Strings

Pump a string v: Construct new strings by repeating substrings in v.

Let $z = ababbaaab$ be a string accepted by the following machine:



- z can be split as $z = \underbrace{a}_{u}\underbrace{bab}_{v}\underbrace{baaab}_{w}$
- Strings $a(bab)^i baaab$ are obtained by pumping $v = bab$ in z
  The pumped strings uv^iw are accepted (for all $i \geq 0$)

Outline

The Pumping Lemma: Essence

$$
\begin{aligned}
&\forall L : (L \text{ is regular} \Rightarrow \\
&\quad (\exists k \geq 1 : \forall z : z \in L \wedge |z| \geq k \Rightarrow \\
&\quad\quad (\exists u, v, w : z = uvw \wedge |uv| \leq k \wedge v \neq \epsilon \\
&\quad\quad\quad \wedge (\forall i \geq 0 : uv^i w \in L)))).
\end{aligned}
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Intuitively:

- k is the # of states of a DFSM accepting L (depends on L)
- $|z| \geq k$ ensures that at least one state is visited twice
- Reading u gets you to the repeated state, v gets you there again
- Since v traverses a loop, using $uv^i w$ we can omit it (if $i = 0$) or repeat it (if $i > 1$)

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$$\wedge (\forall i \geq 0 : uv^i w \in L)))).$$

Note the three conditions on u, v, w:

C1 $uv^i w \in L$

C2 $v \neq \epsilon$ (although u and w can be ϵ)

C3 $|uv| \leq k$

$L_1 = \{a^n b^n \mid n \geq 1\}$ is Not Regular (V1)

Claim. $L_1 = \{a^n b^n \mid n \geq 1\}$ is not regular.

Key steps in a proof **by contradiction**:

- Suppose, for the sake of contradiction, that L_1 is regular
- Then, the pumping lemma must apply to all suitably long $z \in L_1$
- Consider, in particular, $z = a^k b^k \in L_1$ (cf. $\exists k \geq 1 \ldots$)
- Let uvw be a decomposition of z (cf. $\exists u, v, w : z = uvw \ldots$). We consider different possibilities for $v \neq \epsilon$

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- Let uvw be a decomposition of z (cf. $\exists u, v, w : z = uvw \ldots$). We consider different possibilities for $v \neq \epsilon$ (C2):
  a) Suppose v contains only a's.
     $uvvw$ will have more a's than b's, violating (C1): contradiction
  b) Suppose v contains only b's.
     $uvvw$ will have more b's than a's, violating (C1): contradiction
  c) Suppose v contains both a's and b's.
     $uvvw$ may have the same number of a's and b's but in an incorrect sequence, as in $aaa\ ababab\ bbbbb$. This violates (C1).
- A contradiction is unavoidable in all cases: L_1 is not regular.

$L_1 = \{a^n b^n \mid n \geq 1\}$ is Not Regular (V2)

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Another proof by contradiction, now selecting z more carefully:

- Suppose, for the sake of contradiction, that L_1 is regular
- Then, the pumping lemma must apply to all suitably long $z \in L_1$
- Consider, in particular, $z = a^k b^k \in L_1$ (cf. $\exists k \geq 1 \ldots$).
  Let uvw be a decomposition of z
- Since $|uv| \leq k$ (C3) any split uv will contain only a's.
  Because $v \neq \epsilon$ (C2), v contains at least one a.
  Moreover, we should have $uw \in L_1$.
- However, the string uw contains less a's than z (i.e., those in v), while keeping the same number of b's
- Since $n_a(uw) < n_b(uw)$, we have that uw it cannot be in L_1.
  This contradicts the pumping lemma, and we conclude that L_1 is not regular.

Another Example

Claim. $L_3 = \{x \in \{a, b\}^* \mid n_a(x) = n_b(x)\}$ is not regular.

Key steps in a proof by contradiction:

- Suppose, for the sake of contradiction, that L_3 is regular
- The pumping lemma applies to all $z \in L_3$, in particular to $z = a^k b^k \in L_3$ (cf. $\exists k \geq 1 \ldots$)
- Note: by defining $u = w = \epsilon$ and $v = a^k b^k$, then the string z seems like a good candidate: $u\, v^i\, w \in L_3$.
- Here $|uv| \leq k$ (C3) is crucial. Any split uv will contain only a's. Thus, $u\, vv\, w \notin L_3$ (C1). This contradicts the pumping lemma, and we conclude that L_3 is not regular.

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The selection of z matters. Some strings are not good candidates:

- Take $z' = (ab)^k$. This string can be pumped by defining

$$u = \epsilon \qquad v = ab \qquad w = (ab)^{k-1}$$

You can check that $u\, v^i\, w \in L_3$, for any $i \geq 0$.

The Pumping Lemma (General Form)

Lemma (Pumping)

Let A be a regular set. Then the following property holds of A:

(P) *There exists $k \geq 0$ such that for any strings x, y, z with $xyz \in A$ and $|y| \geq k$, there exist strings u, v, w such that $y = uvw$, $v \neq \epsilon$, and for all $i \geq 0$, the string $xuv^iwz \in A$.*

Notice:

- Slightly more general than in the reader
  The string to be split (i.e., y) is "surrounded" by x, z
- *Intuition:* For any string in a regular set A we can find a non-null substring $v \neq \epsilon$ that can be freely "pumped" and the resulting string is still in A

Outline

The Pumping Lemma

The pumping lemma is used to prove that sets are non-regular.

Two strategies for devising a proof:

- Proofs **by contradiction** (as we have seen)
- Use the contrapositive form, and **play a game** (presented next) (Contrapositive means that to prove $Q \to P$ we prove $\neg P \to \neg Q$.)

Also: we could exploit the **closure properties** of regular languages to avoid using the pumping lemma.

The Pumping Lemma

Lemma (Pumping, **Standard Form** ($Q \to P$))

*Let A be a regular set (Q). **Then** the following property holds of A:*

(P) ***There exists*** *$k \geq 0$ such that **for any** strings x, y, z with $xyz \in A$ and $|y| \geq k$, **there exist** strings u, v, w such that $y = uvw$, $v \neq \epsilon$, and **for all** $i \geq 0$, the string $xuv^iwz \in A$.*

Notice the alternating sequence: $\exists$ - $\forall$ - $\exists$ - $\forall$.

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Notice the alternating sequence: $\exists$ - $\forall$ - $\exists$ - $\forall$.

Lemma (Pumping, **Contrapositive Form** ($\neg P \to \neg Q$))

Let A be a set of strings, which enjoys the following property:

($\neg P$) ***For all** $k \geq 0$ **there exist** strings x, y, z such that $xyz \in A$ and $|y| \geq k$, and **for all** u, v, w with $y = uvw$ and $v \neq \epsilon$, **there exists** an $i \geq 0$ such that the string $xuv^iwz \notin A$.*

Then, A is not regular ($\neg Q$).

Notice the (inverted) sequence: $\forall$ - $\exists$ - $\forall$ - $\exists$.

Games with The Pumping Lemma

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A game against the demon

The demon wants to show A is regular; you want to show it is not:

1. The demon picks k (what is his best strategy for choosing?)
2. You pick x, y, z such that $xyz \in A$ and $|y| \geq k$
3. The demon picks u, v, w such that $y = uvw$ and $v \neq \epsilon$
4. You pick $i \geq 0$

If $xuv^iwz \notin A$, you win. Otherwise, the demon wins if $xuv^iwz \in A$.

Playing with the demon: Example

We show that the language $L_4 = \{a^n b^m \mid n \geq m\}$ is not regular:

1. *The demon picks* k
2. *We pick* x, y, z *such that* $xyz \in L_4$ *and* $|y| \geq k$
   Let's pick $x = a^k$, $y = b^k$, and $z = \epsilon$. (We'll split only b's.)
3. *The demon picks* u, v, w *such that* $y = uvw$ *and* $v \neq \epsilon$
   Let $k = j + m + n$, with $m > 0$.
   The demon picks $u = b^j$, $v = b^m$, and $w = b^n$.
4. *We pick* $i \geq 0$
   No matter the demon's choice, we can take $i = 2$ to win:

$$\begin{aligned} xuv^2wz &= a^k \ b^j b^m b^m b^n \\ &= a^k \ b^{k+m} \end{aligned}$$

Clearly, $a^k \ b^{k+m} \notin L_4$.

PL is Necessary but not Sufficient

- The lemma gives a **necessary condition** for regularity, i.e. a condition that necessarily holds for all regular languages.
- But it does not give a **sufficient condition**, i.e., a condition that suffices to guarantee that the language is regular.

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  language is regular $\Rightarrow$ C (i.e., C is a necessary condition)

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- There exist non regular sets that satisfy the hypotheses of the pumping lemma
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- In the game formulation, each non regular set determines a different game
- There exist non regular sets that satisfy the hypotheses of the pumping lemma
- To show that a set is regular, we must construct a corresponding FSM or a regular expression
- Another characterization of regular languages, not covered here: the Myhill-Nerode theorem (related to Section 3.8)

Outline

Closure Properties

Closure properties state that when one (or several) languages are in a given class C, then certain related languages are also in C.

Useful to define new languages on C (say, the regular languages) building upon existing/known ones.

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Regular languages are closed under the following operations:

- Union, concatenation, Kleene star
- Complement (recall: the complement $\overline{L}$ of L is $\Sigma^* \setminus L$).
- Intersection
- Reversal (i.e., if $a_1 a_2 \ldots a_{n-1} a_n \in L$ then $a_n a_{n-1} \ldots a_2 a_1 \in L$).

Closure Properties of Reg. Languages

- *Union, concatenation, Kleene star*:
  Immediate by considering regular expressions

Closure Properties of Reg. Languages

- *Union, concatenation, Kleene star*:
  Immediate by considering regular expressions
- *Complement*:
  Obtain a DFSA M of the given language, and define another DFSA M' in which the accepting states of M become non-accepting states of M', and vice versa

Closure Properties of Reg. Languages

- *Intersection*: Two possibilities
  1) Use the law $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_1}}$ (and reduce to cases above)
  2) Define a construction that runs two DFSAs in "parallel"
  Given $M_i = (Q_i, \Sigma, \delta_i, q_i, F_i)$ (with $i \in \{1, 2\}$), define
  $M = (Q_1 \times Q_2, \Sigma, \delta, (q_1, q_2), F_1 \times F_2)$, where
  $\delta((p, q), a) = (\delta_1(p, a), \delta_2(q, a))$.

Closure Properties of Reg. Languages

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  $\delta((p, q), a) = (\delta_1(p, a), \delta_2(q, a))$.
- Reversal: Given a machine M and create a machine M^R by:
  - reversing all arcs in the state diagram;
  - making the start state the only accepting state;
  - creating a new starting state, with ϵ-transitions to the accepting states

Application of Closure Properties

Claim. If L is a non-regular language, then $\overline{L}$ is non-regular too.

Idea for a proof by contradiction:

- Suppose, for the sake of contradiction, that $\overline{L}$ is regular.
- By the closure properties, $\overline{\overline{L}}$ should be regular as well.
- But $\overline{\overline{L}} = L$, which contradicts our assumption.
- We conclude that $\overline{L}$ must be non-regular.

Using Closure Properties Instead of PL

Claim. Language $L_3 = \{x \in \{a, b\}^* \mid n_a(x) = n_b(x)\}$ is non-regular.

- Suppose, for the sake of contradiction, that L_3 is regular.
- We know a^*b^* is a regular set
- Then the set $L_3 \cap a^*b^*$ would be regular, because of the closure property under intersection
- But $L_3 \cap a^*b^* = \{a^n b^n \mid n \geq 0\}$ is not regular: contradiction.

Taking Stock

Properties of regular languages:

- Pumping lemma:
  A technique to prove that languages are not regular
- Closure properties:
  Conditions to obtain new regular languages from the operation of existing ones

Next Lecture

- FSM minimization
  Not yet in the reader, but you can check the slides, and/or Chapter 14 in Kozen's book ('Automata and Computability').