THE RATIONALIZABILITY OF SURVEY RESPONSES

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ABSTRACT. We propose and study the concept of survey rationalizability, that we base on classical item response theories in psychology. Survey rationalizability involves positioning survey questions on a common scale such that, in the main case of attitudinal surveys, each respondent gives higher support to questions that are more aligned with her views. We first demonstrate that ideas from standard revealed preference analysis can be used to characterize when and how dichotomous surveys are rationalizable. We then show that these results readily extend to more general surveys. Furthermore, we investigate the identification of the models and extend the analysis in several directions.

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1. Introduction

Surveys are becoming increasingly popular in economics as tools for gathering data on various economic variables, such as subjective expectations, happiness, contingent valuations, beliefs, and political attitudes. Yet, the rationality foundations and the resulting empirical content of survey responses are not sufficiently understood from an economic point of view. This paper takes a first step in this direction by formalizing classical psychological theories of item response and then studying their empirical content using concepts inspired by traditional revealed preference analysis. Ultimately, this type of exercise is crucial for better assessing the external validity of survey information and for improving the structure of surveys in terms of informativeness and accuracy, aspects that ultimately lead to better decision-making and policies.

We start our analysis with attitudinal surveys to measure individuals' attitudes, opinions, beliefs, or preferences on a given topic. We use the classical treatment in psychology of item response theory with ideal points. Accordingly, we model rationalizability as the possibility of positioning survey questions on a common scale and describing the behavior of each individual by means of a strictly quasi-concave utility such that questions associated with larger utility receive larger support.[1] We commence by studying dichotomous surveys, which consist of yes/no questions. In this stylized case, rationalizability requires each individual to endorse questions with utility above a threshold level, and only those questions. Given the strict quasi-concavity of utilities, this must correspond to each individual endorsing questions located on an interval of the real line.

We show that the study of rationalizability can be built on the basis of an intuitive revelation principle. Consider an individual who endorses questions q_2 and q_3, but not question q_1. Given the interval nature of responses, the individual reveals that, within this triple, q_1 must occupy an extreme position within the triple. Following the logic of revealed preference analysis, a minimal rationalizability requirement should be that these revelations cannot be contradictory; if there are individuals that reveal two questions as extreme within a triple, it cannot be the case that another individual

[1]See Thurstone (1928) and especially the unfolding theory of Coombs (1964) for early developments, and see Tay and Ng (2018) and Van der Linden (2018) for more recent accounts.

reveals the third question as also extreme in the triple.[2] For obvious reasons, we call this property the Weak Axiom of Revealed Extremeness (WARE). However, in our first result, Proposition 1, we show that WARE is not sufficient for rationalizability. Fortunately, we are able to show that we can characterize rationalizability by expanding the revelations of extremeness to include indirect revelations. To illustrate how the expansion works, consider the responses of two individuals who share some endorsements but also have idiosyncratic endorsements. In this case, the response sets must correspond to two overlapping intervals on the real line. Intuitively, the union of these response sets must also correspond to an interval, which we can think of as the fictitious response of a fictitious individual. By recursively enriching the original survey responses with these fictitious sets of responses, we construct what we call the expanded survey. We show in Theorem 1 that a survey is rationalizable if and only if its expanded survey satisfies WARE. Moreover, Theorem 2 demonstrates that our method for testing rationalizability is computationally tractable.

We then show that our techniques for the dichotomous case prove useful in the study of more general settings. In particular, we consider the cases of polytomous surveys, where questions can be answered using one of multiple levels of endorsement, and probabilistic survey data, which allows for belief expressions or noisy responses. After adjusting our notion of survey rationalizability to consider ordered collections of thresholds, one for each endorsement level, and random thresholds, respectively, Theorems 3 and 4 establish the corresponding characterizations of rationalizability. Both results follow from a simple transformation of these richer datasets into analytically equivalent dichotomous datasets, allowing us to use the techniques developed for the proof of Theorem 1.

We then shift our focus to studying the identification of these models. Since they are essentially ordinal, the key step in the analysis is determining the order of questions and the location of individuals' ideal points. When the survey data is rationalizable, the extremeness revelations are key to learning the order of questions. In Proposition 2, we show that a richness condition guarantees that this order is unique. Regarding the location of individuals, Proposition 3 demonstrates that it is possible to establish bounds for them, with precise information being conveyed by probabilistic surveys, whereas

[2] Importantly, notice that we are required to work not with binary relations but with ternary relations, which substantially departs from the revealed preference literature. Additionally, another departure is that here we do not have menu variation, but individual variation.

polytomous surveys provide more information than dichotomous surveys. Furthermore, in Proposition 4, we show that all locations can be cardinally and uniquely identified when a specific parametric model is considered.

Finally, we revisit the two main assumptions in our analysis. The first assumption is the unidimensional location of questions and individuals. We argue that this assumption is both relevant and critical. It is relevant because a large number of theoretical settings and empirical applications in psychology, sociology, political science, and economics adhere to it. It is also critical because the multidimensional analysis can be addressed in a straightforward manner. Essentially, in a multidimensional model, survey rationalizability loses its empirical content; we show in Theorem 5 that every survey can be rationalized when already two dimensions are considered. The second assumption is related to our focus on attitudinal surveys. We briefly analyze surveys oriented toward the study of cumulative aptitudes, such as ability, health status, or the extent of a practice. A common approach in the psychology literature is to apply dominance item response theory, with the dichotomous version known as a Guttman scale (see Guttman (1944)). According to this approach, the difficulty of questions and the aptitude levels of individuals can be represented on the real line, with individuals responding positively to questions that are below their aptitude levels. We show in Theorem 6 that the study of rationalizability by Guttman scales is facilitated by their cumulative nature, reducing the problem to a more standard asymmetry check on a binary relation.

2. Related Literature

In economics, the empirical literature using surveys is large and rapidly growing. Here, our focus is on its theoretical and methodological counterpart. Bertrand and Mullainathan (2001) propose an econometric-based framework that accounts for errors in responses, enabling a meaningful interpretation of individuals subject to cognitive biases. Prelec (2004) proposes a scoring method for the elicitation of truthful subjective judgments based on answers that are more common than collectively predicted. Falk, Neuber, and Strack (2021) develop an individual-response model based on imperfect self-knowledge, where individuals' responses depend on a combination of private signals and the population mean. Manski (2004) and Stantcheva (2022) discuss several methodological issues in designing and interpreting survey studies. Benjamin, Guzman, Fleurbaey, Heffetz, and Kimball (2023) propose a methodology to uncover the informational content of self-reported well-being surveys, considering various potential response biases. Liu and Netzer (2023) study the use of response time to enhance the informational content of happiness survey responses, aiming to overcome the identification problems established in Bond and Lang (2019). Our contribution to this literature is to offer a theoretical framework that studies the rationality of survey responses based on classical accounts in psychology. We also show how the ordered structure of questions and individual attitudes or aptitudes is revealed. Finally, our probabilistic models of survey responses enable systematic consideration of various types of errors.

Our results can be related to various literatures across scientific disciplines, such as the use of sortability and seriation in archaeology, anthropology, biology, computer science, and psychology. The connection with all these fields is the so-called consecutive-ones property, or simply c1p, of a $0-1$ matrix. A matrix is c1p if there exists a permutation of its columns such that the ones in every row become consecutive.[3] Our results contribute to this literature by showing the usefulness of revealed preference techniques in studying c1p. Additionally, we demonstrate that our results apply well beyond dichotomous data.

An article in the literature on c1p matrices that is of special relevance to us is Tucker (1972). We leverage his results in the proof of our Theorem 1, namely that a $0-1$ matrix is c1p if and only if it does not contain the so-called asteroidal triples. In our language of surveys, these are triples of questions for which a cycle of individual responses exists with very specific features. We use techniques inspired by the revealed preference literature to decompose the study of c1p into a direct revelation principle, i.e., our notion of extreme questions, and its indirect concatenated implications, i.e., the revelations of the expanded survey. As a result, we offer a conceptually simple and tractable characterization of survey rationalizability. We believe that bringing Tucker's result to the economics literature in general, and to the study of survey rationalizability in particular, may be instrumental in future research. Additionally, our contribution shows how the use of revealed preference techniques can improve existing

[3]As an illustration of an application of c1p in a different field, consider the statistical archaeology problem discussed in Kendall (1969). Data is given by a set of graves and a set of objects that are or are not in these graves. C1p is used to determine whether graves and objects can be located on a time scale such that each grave contains only objects that were created, but were not obsolete, in the relevant period. See Coombs and Smith (1973) and Hubert (1974) for a connection between these techniques and survey responses, and Liiv (2010) for an overview of their use across different disciplines.

results, for example, by extending the study of $0-1$ matrices to non-dichotomous surveys, discussing the identification of these models, and considering other notions of rationalizability.

Our paper can also be related to the study of ternary relations. Huntington and Kline (1917) and Fishburn (1971) represent early treatments of abstract ternary relations, with the purpose of capturing the intermediateness notion on the real line. Most of our results are presented in terms of an extremeness relation, but they can be equivalently presented in terms of an intermediateness relation. Our results contribute to this literature both conceptually and technically. Conceptually, we apply ternary relations to a concrete economic problem and, instead of taking them as given, we derive them from data. Technically, we show which incomplete ternary relations can be extended into the ternary relation of a linear order. We achieve this by combining techniques from the c1p literature and the revealed preference literature.

Finally, our paper can be related to a number of applications, such as approval voting, homophily, text analysis, and more. We postpone the discussion to Section 3.1, once we have formally introduced our framework.

3. The dichotomous case

Let $\mathcal{Q} = \{1, \ldots, q, \ldots, Q\}$ be a set of questions related to a specific topic, such as capital punishment or immigration, on which individuals $\mathcal{N} = \{1, \ldots, n, \ldots, N\}$ are surveyed. We interpret question $q \in \mathcal{Q}$ as a request for the individuals to provide a graded evaluation or endorsement of the statement described in the question. We start by analyzing the case in which all questions in the survey are dichotomous, i.e., evaluations are yes/no, agree/disagree or positive/negative. Accordingly, a dichotomous survey (D-survey) is a collection of the form $\{Q_n\}_{n \in \mathcal{N}}$, with $Q_n \subseteq \mathcal{Q}$ describing the set of questions that are endorsed by individual n. Without loss of generality, we assume that all response sets are different and that any two questions are different in the sense that the subset of individuals endorsing them is not the same. We study when a D-survey can be rationalized in the following sense.

D-survey rationalizability. We say that $\{Q_n\}_{n \in \mathcal{N}}$ is rationalizable whenever there exist $[\{U_n, \tau_n\}_{n \in \mathcal{N}}, \{\mu_q\}_{q \in \mathcal{Q}}]$, with $U_n : \mathbb{R} \to \mathbb{R}$ being strictly quasi-concave and $\tau_n, \mu_q \in \mathbb{R}$, such that for every $n \in \mathcal{N}$ and $q \in \mathcal{Q}$, it holds that $q \in Q_n$ if and only if $U_n(\mu_q) \geq \tau_n$.

The notion of rationalizability contains three major assumptions: (i) views are to be located on the real line, (ii) utility functions are strictly quasi-concave, and (iii) the attitudinal nature of the survey. As for the first assumption, unidimensionality is a standard assumption in psychology, sociology, political science, and many economic settings of interest.[4] In any case, we extend our analysis to the multidimensional case in Section 6. Regarding the second assumption, once views are considered unidimensional, we believe that this is the most natural economic modeling of interest in the presence of trade-offs and it captures the notion of endorsement-by-proximity described in the psychology literature.[5] Note that since the model is essentially ordinal, all that is required is the assumption of single-peaked preferences over the ordered questions. We use strict quasi-concavity to capture this, as it simplifies the presentation of the more complex surveys studied later in the paper. Regarding the third assumption, we discuss in Section 6 the alternative approach of aptitude surveys, where the paradigmatic example is that of classical ability tests, in which an individual solves questions with a level of difficulty below their ability level.

Before continuing, note that the interpretation of the utility function depends on the application at hand. Some surveys contain questions where the utility function can be interpreted in line with classical economic terms, even though the question may be hypothetical and utility may result from the hypothetical contemplation of the question. In these situations, the threshold level may represent some internal expectations. In other types of surveys, the utility function may be interpreted as the internal value derived from supporting a specific statement, with rejected statements being those that entail high costs of departing from personal beliefs or views. The specific interpretation is not crucial to our study; the representation we adopt helps to formalize the idea of survey rationalizability in a language common to economists.

3.1. **Examples of the setting.** Our formal setting uses the language of attitudinal surveys on a given topic, which represents its most direct application. These surveys are extensively used in economics to gather information on a variety of topics and, similarly,

[4]In many settings, the order of questions is not explicitly known, and we show how it can be learned from the response data. Section 3.1 provides several examples and, in most of them, the order of questions is not necessarily known.

[5]It can be shown that rationalizability can be defined equivalently as the existence of $[\{\mu_n, \epsilon_n\}_{n\in\mathcal{N}}, \{\mu_q\}_{q\in\mathcal{Q}}]$ with μ_n representing the location or ideal view of the individual, such that for every $n \in \mathcal{N}$ and $q \in \mathcal{Q}$, it is $q \in Q_n$ if and only if $d(\mu_n, \mu_q) \leq \epsilon_n$, where d is a distance function.

widely used in other disciplines, such as psychology, to measure personality traits or attitudes. Examples of their use can be seen in Andrich (1988), where individuals are asked to endorse or reject a number of statements on capital punishment and Grable and Lytton (1999), that surveys the risk attitude of individuals.[6] However, our setting is also applicable well beyond the survey interpretation. Here, we provide some economic examples that show the potential applicability of our results.[7]

Activities. Behavioral traits, such as risk aversion, can be analyzed by gathering information on individual participation in various potentially risky activities, such as drug use or extreme sports. Similarly, altruism can be assessed by collecting data on participation in pro-social activities. This approach has a long tradition in psychology; an example is Fromme, Katz, and Rivet (1997), who designed and studied the cognitive appraisal of risky activities to assess risk preferences.[8] In this type of scenario, $\mathcal{N}$ is the set of individuals surveyed about their participation in a number of activities $\mathcal{Q}$.

Text analysis. Consider the particular use of language by various individuals or institutions on a given dimension, such as behavioral traits or political attitudes. The purpose is to order a set of linguistic expressions and, as a result, to categorize individuals by their language use. See Laver, Benoit, and Garry (2003) for an application to political texts to extract policy positions.[9] Using our terminology, Q_n represents the subset of linguistic expressions used by individual n.

Product Space and Recommendations. Many recommendation systems, such as those for movies or music bands, are based on arranging products and consumers within a

[6]The former includes statements/questions such as: (i) capital punishment is justified because it does act as a deterrent to crime, (ii) I think capital punishment is necessary but I wish it were not or (iii) until we find a more civilized way to prevent crime we must have capital punishment. The latter includes questions such as: (i) it's hard for me to pass up on a bargain, (ii) it is more important to be protected from rising consumer prices (inflation) than maintaining the safety of your money from loss and theft, or (iii) how/are you comfortable investing in stocks or stock mutual funds? Note that these questions have not a clear ordering.

[7]We use the dichotomous case, but the more general cases with Likert scales and probabilistic responses also apply to most of them.

[8]The authors gathered information on activities such as substance use (e.g., alcohol and marijuana), interpersonal behaviors (e.g., going on a blind date), and sports (e.g., rock climbing). Again, note that these activities do not have a clear ex-ante ranking of riskiness.

[9]They consider words such as 'drugs,' 'markets,' 'nation,' 'security,' or 'representation,' and order them in the ideological spectrum. See also the recent use of large language models to position political texts in several unidimensional ideological and policy spaces in Le Mens and Gallego (2024).

characteristic space, with consumers favoring products closer to them.[10] In some cases, objects may be arranged on a single scale, and understanding this can be instrumental in improving recommendation systems. This approach is rooted in classical psychology literature, where tastes are investigated by observing the acceptability or rejection of stimuli that can be ordered. See Pfaffmann (1960) for an early treatment, and consider, as discussed there, products with different salt concentrations as an example of ordered stimuli. Following this line of thought, Q_n represents the set of products liked by individual n.

Wishlisting. The previous example involves recommendations to other users, but one may also consider recommendations to oneself in the form of a wishlist. In this context, the wishlist Q_n represents the set of items for which an individual expresses interest or a positive attitude. See Manzini, Mariotti, and Ülkü (2024) for a sequential model of individual "wishlisting."

Attention studies. Another case that deviates from the traditional question-response paradigm of surveys involves the use of more implicit expressions of interest. This is common in attentional studies, where a number of individuals $\mathcal{N}$ is offered a number of stimuli $\mathcal{Q}$, and the analyst infers the subset of these stimuli that attract the individual's attention. Often, objects can be ordered, and individuals pay attention to an interval of stimuli. A classical paper in this field is Thomas (1973), which studies children's interest in faces drawn with different degrees of detail.[11]

Approval voting. The idea of allowing individuals to pick any number of options has been suggested as a voting mechanism. See Brams and Fishburn (1978) for an early analysis of approval voting. Here, Q_n represents the subset of candidates approved by individual n. Interestingly, Laslier (2009) and Alós-Ferrer and Buckenmaier (2019) have shown that equilibrium behavior in approval voting games may have the upper-contour-set structure analyzed in this paper.[12]

[10] The classical Hotelling setting is also based on this simple idea to ultimately model choice behavior.

[11] Eye-tracking techniques are used in economics to gather information on attention; see Lahey and Oxley (2016) for a review. Polonio, Di Guida and Coricelli (2015) employ seriation techniques to understand the linkage between saccades and strategies of play within a game.

[12] Núñez and Xefteris (2017) study approval voting under single-peaked preferences, which is related to our notion of endorsement-by-proximity. They show that every Nash-implementable welfare optimum can indeed be implemented by means of approval voting mechanisms.

Judgment aggregation. There is a set of individuals $\mathcal{N}$ making judgments on a set of logical propositions $\mathcal{Q}$. Dietrich and List (2010) study conditions that facilitate judgment aggregation; one of these conditions involves the case where propositions can be ordered on a line such that the accepted judgments of each individual form an interval.

Networks and homophily. Consider a set of individuals who have directed links to others. The question is whether the network can be explained by ordering individuals such that they form links with those most similar to them. For an application using Twitter data to gather political ideology, see Barberá (2014). We can understand this within our setting by considering $\mathcal{Q} = \mathcal{N}$.

Matching. Let $\mathcal{N}$ be a set of individuals and $\mathcal{Q}$ a set of objects (as in housing problems) or a set of different individuals (as in marriage problems). Individuals submit a set Q_n of acceptable elements. Typically, individuals have a common preference on $\mathcal{Q}$ determined, e.g., by quality. However, in some settings, individuals may have idiosyncratic single-peaked preferences on $\mathcal{Q}$, perhaps given by quasi-linear preferences on quality and price in the housing problem, or by affinity in the marriage problem (see Bade (2019)).

3.2. **Revealed information: extremeness.** We start by discussing how survey responses reveal the order of questions, which will soon be proven to be the critical feature for survey rationalizability. First, recall that the upper contour set of a strictly quasi-concave utility function must correspond to an interval on the real line. Then, consider individual n, and suppose we observe $Q_n \cap \{q_1, q_2, q_3\} = \{q_2, q_3\}$. In other words, individual n endorses questions q_2 and q_3 but not question q_1. Given the interval nature of responses, this observation reveals that q_1 occupies an extreme position within the triple. When this is the case, we write $q_1 \mid \{q_2, q_3\}$ and say that q_1 has been revealed as extreme in the triple $\{q_1, q_2, q_3\}$.

For rationalizability, the revealed information across individuals should not be contradictory. We can then consider the following simple property.

Weak Axiom of Revealed Extremeness (WARE). We say that $\{Q_n\}_{n\in\mathcal{N}}$ satisfies WARE if, for every triple $\{q_1, q_2, q_3\}$ of questions, $q_1 \mid \{q_2, q_3\}$ and $q_2 \mid \{q_1, q_3\}$ imply that $q_3 \mid \{q_1, q_2\}$ cannot hold.

Having observed the revelation of $q_1 \mid \{q_2, q_3\}$ and $q_2 \mid \{q_1, q_3\}$, rationalization requires positioning q_1 and q_2 as extremes within the triple, which in turn necessitates that q_3 is not extreme. The following result shows that WARE is a necessary property for rationalizability, but not sufficient.

Proposition 1. *If $\{Q_n\}_{n\in\mathcal{N}}$ is rationalizable, then $\{Q_n\}_{n\in\mathcal{N}}$ satisfies WARE, but the converse is not necessarily true.*

Proof of Proposition 1: For the first part, suppose that $\{Q_n\}_{n\in\mathcal{N}}$ is rationalizable. Assume that $q_1 \mid \{q_2, q_3\}$ and $q_2 \mid \{q_1, q_3\}$ hold. From these extremeness revelations, it must be that $\{\mu_{q_1}, \mu_{q_2}\} = \{\min\{\mu_{q_1}, \mu_{q_2}, \mu_{q_3}\}, \max\{\mu_{q_1}, \mu_{q_2}, \mu_{q_3}\}\}$. Then, if we observe any (interval) response set containing both q_1 and q_2, it must also contain q_3, and WARE is satisfied.

For the second part, consider the following example of a D-survey: $\mathcal{N} = \{1, 2, 3\}$, $\mathcal{Q} = \{q_1, q_2, q_3, q_4\}$ and the response sets are $Q_1 = \{q_1, q_2\}$, $Q_2 = \{q_1, q_3\}$ and $Q_3 = \{q_1, q_4\}$. By construction, there are at most two extremeness revelations in every triple of alternatives, and WARE holds. However, the responses are not rationalizable; the information contained in Q_1 requires q_4 to be located as the maximum or minimum in the triple $\{q_1, q_2, q_4\}$. If it is a maximum (respectively, a minimum), the information contained in Q_2 requires q_4 to be also the maximum (respectively, the minimum), in the triple $\{q_1, q_3, q_4\}$ and, as a consequence, the maximum (respectively, the minimum) in the set $\{q_1, q_2, q_3, q_4\}$. However, the same argument can be applied to q_3 by considering response sets Q_1 and Q_3, and to q_2 by considering response sets Q_2 and Q_3. Since the three alternatives cannot be extreme in the set of four alternatives, a contradiction arises and the result follows. ■

3.3. **A characterization of rationalizability.** Proposition 1 shows that the absence of direct contradictions required by WARE is not sufficient for rationalizability. We now show that we can expand the information revealed by the survey via the consideration of indirect revelations of extremeness. In order to do so we need to consider pairs of overlapping responses, namely:

$$(Q_i, Q_j) \text{ such that } Q_i \cap Q_j \neq \emptyset,\ Q_i \setminus Q_j \neq \emptyset \text{ and } \ Q_j \setminus Q_i \neq \emptyset.$$

Recall that, under rationalizability, response sets correspond to intervals of the real line. Hence, the union of two overlapping response sets also corresponds to a (larger) interval.[13] If this larger set of responses does not belong to the original set of responses, it

[13]Notice that this statement does not necessarily apply if the responses are not overlapping: if two intervals are disjoint, the union may fail to be an interval. Also, notice that the intersection and

brings new information that can be added to the original survey. We then consider this principle recursively until no more novel unions of overlapping responses can be added. Denote $\mathcal{N}_0 = \mathcal{N}$ and consider a sequence of D-surveys $\{Q_n\}_{n\in\mathcal{N}_m}$, $m = 0, 1, \ldots, M$, such that for every $m < M$, the $(m+1)$-th D-survey is equal to the m-th D-survey with the addition of a (novel) union of two overlapping sets in the m-th D-survey. Given finiteness, this sequence ends up in a D-survey that admits no further addition, $\{Q_n\}_{n\in\mathcal{N}_M}$, that we call the expanded D-survey of $\{Q_n\}_{n\in\mathcal{N}}$. Importantly, notice that this survey is closed up to unions of overlapping responses.[14] We can now show that the asymmetry of the indirect revelations generated via union of overlapping responses is sufficient for rationalizability.

Theorem 1. *$\{Q_n\}_{n\in\mathcal{N}}$ is rationalizable if and only if $\{Q_n\}_{n\in\mathcal{N}_M}$ satisfies WARE.*

Proof of Theorem 1: We start by proving the only if part. Suppose that $\{Q_n\}_{n\in\mathcal{N}_0}$ is rationalizable. We now claim that if $\{Q_n\}_{n\in\mathcal{N}_m}$ is rationalizable, $\{Q_n\}_{n\in\mathcal{N}_{m+1}}$ is also rationalizable. To see this, suppose that $\{Q_n\}_{n\in\mathcal{N}_m}$ is rationalizable with parameters $[\{U_n, \tau_n\}_{n\in\mathcal{N}_m}, \{\mu_q\}_{q\in\mathcal{Q}}]$. Let $Q_i \cup Q_j$ be the novel response set added to form $\{Q_n\}_{n\in\mathcal{N}_{m+1}}$. This requires Q_i, Q_j to belong to $\{Q_n\}_{n\in\mathcal{N}_m}$ and to be overlapping. To construct the new rationalization, define the location of questions and the individual parameters of any individual other than the novel one as in the rationalization of $\{Q_n\}_{n\in\mathcal{N}_m}$; we then simply need to define parameters for the novel individual and prove the rationalization of her response set $Q_i \cup Q_j$. To do so, define $a = \min\{\mu_q : q \in Q_i \cup Q_j\}$, $b = \max\{\mu_q : q \in Q_i \cup Q_j\}$ and define the utility of the novel individual to be any strictly increasing utility function up to $\frac{a+b}{2}$ and strictly decreasing after this point, with the utility at a and b being equal to the threshold level. Given the original rationalizability and the overlapping nature of Q_i and Q_j, the interval $[a, b]$ contains exactly questions $Q_i \cup Q_j$, and rationalizability of this response set follows. Given finiteness, this inductive argument guarantees that the expanded D-survey is rationalizable and, by Proposition 1, it must satisfy WARE.

We now consider the if part. We start by showing that if the expanded D-survey satisfies WARE, the original survey does not contain asteroidal triples. These are triples of questions $\{q_1, q_2, q_3\}$ such that for every pair of questions $\{q_i, q_j\}$ in this triple, there exists a sequence of individuals $n_{ij}^1, \ldots, n_{ij}^K$ such that: (i) $q_i \in Q_{n_{ij}^1}$ and $q_j \in Q_{n_{ij}^K}$, (ii)

difference of overlapping intervals must also be intervals, but our revelation technique does not require including this information.

[14]The order in which information is added is irrelevant and the expanded D-survey is unique.

for every $k = 1, \ldots, K-1$, $Q_{n_{ij}^k} \cap Q_{n_{ij}^{k+1}} \neq \emptyset$, and (iii) the remaining alternative in the triple does not belong to the response set of any individual in this sequence. Suppose by contradiction that $\{q_1, q_2, q_3\}$ is an asteroidal triple. Consider first the response sets of n_{ij}^1 and n_{ij}^2. Given (ii), it must be either that one of these sets contains the other, or that they overlap. In the former case, the larger of these two response sets, denoted $\bar{Q}_{ij}^2$, belongs to the expanded D-survey (as it belongs to the survey), contains q_i but not q_o, and intersects $Q_{n_{ij}^3}$. In the latter case, since the expanded D-survey is closed under overlapping unions, $\bar{Q}_{ij}^2 = Q_{n_{ij}^1} \cup Q_{n_{ij}^2}$ must belong to the expanded D-survey, contains q_i but not q_o, and intersects $Q_{n_{ij}^3}$. Consider the response sets $\bar{Q}_{ij}^2$ and $Q_{n_{ij}^3}$. Again, it must be either the case that one of them contains the other or they overlap. With the same reasoning, we can identify $\bar{Q}_{ij}^3$ in the expanded D-survey, containing q_i but not q_o, and intersecting $Q_{n_{ij}^4}$. Proceed recursively until obtaining the response set $\bar{Q}_{n_{ij}^K}$. This set contains q_i and q_j, but not q_o. Since this argument applies to any combination in the triple, we have identified three (fictitious) individuals in the expanded D-survey violating WARE, a contradiction. Now, having proved that the original survey does not contain asteroidal triples, we can recur to Theorem 6 in Tucker (1972) to argue that the original survey must satisfy the consecutive ones property (c1p). This property requires the existence of a linear order $\succ$ over $\mathcal{Q}$ such that every response set Q_n is formed by consecutive questions in $\succ$. In the third and last step, we show that given that the original survey satisfies c1p, it is rationalizable. Given the finiteness of $\mathcal{Q}$, $\succ$ is representable and, hence, there exists a collection of real values $\{\mu_q\}_{q \in \mathcal{Q}}$ such that for every pair of questions $q, q' \in \mathcal{Q}$, it is $\mu_q > \mu_{q'} \Leftrightarrow q \succ q'$. For every $n \in \mathcal{N}$ such that $|Q_n| \geq 1$, consider any utility function $U_n(x)$ that is strictly increasing whenever $x \leq \frac{\min_{q \in Q_n} \mu_q + \max_{q \in Q_n} \mu_q}{2}$, and strictly decreasing otherwise. Moreover, we can select U_n to satisfy $U_n(\min_{q \in Q_n} \mu_q) = U_n(\max_{q \in Q_n} \mu_q) = 0$. This function is strictly quasi-concave. Set $\tau_n = 0$. Since Q_n is formed by consecutive questions in $\succ$, it is immediate that $Q_n = \{q : U_n(\mu_q) \geq \tau_n = 0\}$, proving rationalization. For every $n \in \mathcal{N}$ with $|Q_n| = 0$, we can rationalize responses by considering U_n to be strictly negative and strictly quasi-concave, and setting $\tau_n = 0$. This concludes the proof. ■

In essence, Theorem 1 shows that rationalizability can be understood using an approach familiar in economics: (i) include all indirect revelations that can be derived from the data, and (ii) require them to be asymmetric. Note, however, that these revelations are not related to choice exercises but to attitudinal survey responses. They are not connected to the preference ordering of a decision-maker but to the positioning of questions/individuals on the corresponding attitudinal scale. Additionally, they are not related to a binary relation but to a ternary relation. Section 3.4 elaborates on the proof of Theorem 1 and the use of the c1p literature, and Section 3.5 argues that testing for rationalizability is computationally efficient.

3.4. **Discussion of the proof of Theorem 1.** The only if direction of the proof involves three steps. In step 1, we show that if the expanded D-survey satisfies WARE, the original survey lacks the so-called asteroidal triples, a concept introduced by Lekkerkerker and Boland (1962) in the study of interval graphs, and then applied by Tucker (1972) to the study of the c1p of $0-1$ matrices. In the language of survey responses, an asteroidal triple $\{q_1, q_2, q_3\}$ has the property that, for every pair of questions $\{q_i, q_j\}$ in the triple, we can find a sequence of individuals $n^1_{ij}, \ldots, n^K_{ij}$ such that: (i) $q_i \in Q_{n^1_{ij}}$ and $q_j \in Q_{n^K_{ij}}$, (ii) for every $k = 1, \ldots, K-1$, $Q_{n^k_{ij}} \cap Q_{n^{k+1}_{ij}} \neq \emptyset$, and (iii) the remaining alternative q_o in the triple does not belong to the response set of any individual in this sequence. We prove this implication by sequentially considering the individuals in one such sequence. Starting with the first two, the responses of these individuals must either be related by set inclusion or be overlapping. In any case, the union of responses must belong to the expanded D-survey, either because it is one original response or one obtained by overlapping. We can then proceed sequentially, considering this response set and that of the third individual. Once the last individual in the sequence is reached, we have identified a response set in the expanded D-survey that contains q_i and q_j but not q_o. This creates a contradiction with WARE. The key in this first step is the use of the expanded D-survey, that facilitates the analysis of the complicated sequences involved in an asteroidal triple, together with the revealed preference analysis involved in WARE. A violation of WARE involves the simplest possible asteroidal triple that one can conceive, and in this first step we show that this is the only one that must be checked for every triple of questions once we have properly presented the indirect revealed information contained in the expanded survey. A test based on asteroidal triples, on the other hand, would require to consider all possible cycles of individuals for each possible triple of questions. Step 1 is critical to understand survey rationalizability from the point of view of direct and indirect revelations.

The second part of the proof of Theorem 1 uses a classical result from the study of c1p in $0-1$ matrices, specifically Theorem 6 in Tucker (1972). This result demonstrates that the absence of asteroidal triples in a $0-1$ matrix ensures that the matrix satisfies c1p. By considering the incidence matrix of survey responses, we can view a D-survey as a matrix of 0s and 1s. Hence, we can use Tucker's result to argue that the D-survey satisfies c1p (i.e., we can find a linear order $\succ$ over the questions such that every response set is composed of consecutive questions).

The third step in our proof shows that any D-survey that satisfies c1p can be rationalized. This is done by considering any real representation of the order $\succ$ and by constructing a simple strictly quasi-concave utility function and threshold for each agent. This utility function is strictly increasing up to the middle point of the interval determined by the location of the minimal and the maximal questions that are positively responded, with zero utility (equal to the threshold) in these particular questions. The consecutive nature of responses guarantees that this is always a valid construction.

To conclude, we believe that the strategy of the proof is of interest for the following reasons. First, we bring classical results in the study of matrices and c1p into economics, and in particular into the study of surveys. Second, we show how the use of revealed preference techniques help to simplify, from both a conceptual and a computational point of view, existing algorithmic results for the matricial study of c1p. We critically show that c1p can be seen as the study of the simplest violation on concatenated information; in the language of surveys, this gives rise to the intuitive concepts of WARE and expanded D-survey.[15] Third, since the approach is based upon revealed information, the parameters of the model are constructively obtained; namely, the order of questions and the corresponding utilities and threshold levels of individuals are derived from the study of revealed information.

3.5. **Computational requirements.** In this section, we show that the conditions for rationalizability stated in Theorem 1 can be verified efficiently and present a simple method to do so. Rationalizability requires checking the asymmetry of the extremeness ternary relation derived from the expanded D-survey. We start by showing that the extremeness relation of the expanded D-survey can be obtained as a closure operation over the extremeness relation of the original survey. We then argue that we can compute the extremeness relation of the expanded survey by adapting the classical Floyd-Warshall algorithm, that is used to compute in polynomial time the transitive closure of a graph.

[15]This also provides an alternative, simplified, characterization of the classical c1p question, that may be of independent interest for researchers in other areas.

Let T be a generic ternary relation on a finite set X, and define the following operation, that we call overlapping:[16]

$$xT\{y, z_1\} \text{ and } xT\{y, z_2\} \text{ implies } xT\{z_1, z_2\},$$

Given T, the overlapping closure of T, denoted $\widehat{T}$, is the minimal ternary relation containing T and being closed under the overlapping operation. In the following result, we show that the extremeness relation of the expanded D-survey is the overlapping closure of the extremeness relation of the original survey. Formally, consider the sequence of D-surveys $\{Q_n\}_{n\in\mathcal{N}_m}$, $m = 0, 1, \ldots, M$, leading to the expanded D-survey. Denote by $|_m$ the extremeness revelations of survey $\{Q_n\}_{n\in\mathcal{N}_m}$.

Theorem 2. $|_M = \widehat{|_0}$.

Proof of Theorem 2: Consider two successive D-surveys $\{Q_n\}_{n\in\mathcal{N}_m}$ and $\{Q_n\}_{n\in\mathcal{N}_{m+1}}$ in the sequence, and assume that $Q_i \cup Q_j$, with $Q_i, Q_j \in \{Q_n\}_{n\in\mathcal{N}_m}$, is the novel response in $\{Q_n\}_{n\in\mathcal{N}_{m+1}}$. We claim that $|_{m+1}$ is composed by $|_m$ plus every overlapping $q \ |_m \ \{q', q_1\}, q \ |_m \ \{q', q_2\}$ in which $q \notin Q_i \cup Q_j$, $q_1 \in Q_i \setminus Q_j$ and $q_2 \in Q_j \setminus Q_i$. First, it is obvious that any such overlapping will be revealed by $Q_i \cup Q_j$ and, hence, present in $|_{m+1}$. Second, consider any revelation $q \ |_{m+1} \ \{q', q''\}$ that was not present in $|_m$. It must be the case that this revelation is generated by response set $Q_i \cup Q_j$, and for it to be novel, it must be $q \notin Q_i \cup Q_j$, $q' \in Q_i \setminus Q_j$ and $q'' \in Q_j \setminus Q_i$. But in such a case, since Q_i and Q_j are overlapping sets, there must exist $q^* \in Q_i \cap Q_j$, and the D-survey $\{Q_n\}_{n\in\mathcal{N}_m}$ must reveal $q \ |_m \ \{q^*, q'\}$ (via Q_i) and $q \ |_m \ \{q^*, q''\}$ (via Q_j), and hence the novel revelation can be seen as an overlapping of two original revelations. Using this recursively, we have shown that the extremeness revelations of the expanded D-survey can be seen as overlappings of the original extremeness relation and hence $|_0 \subseteq |_M \subseteq \widehat{|_0}$. We now claim that $|_M$ is closed under overlappings. To see this, suppose that we observe $q \ |_M \ \{q^*, q'\}$ and $q \ |_M \ \{q^*, q''\}$. Then, we know that there exists Q_i, Q_j in the expanded D-survey such that $q^* \in Q_i \cap Q_j$ and $q' \in Q_i$ and $q'' \in Q_j$. If $q'' \in Q_i$ or $q' \in Q_j$, we can use the corresponding set to claim that $q \ |_M \ \{q', q''\}$. If none is true, then Q_i and Q_j are overlapping sets and, since the expanded D-survey is closed under overlapping unions, $Q_i \cup Q_j$ belongs to the expanded D-survey, and we can use it to claim that $q \ |_M \ \{q', q''\}$. Since the overlapping closure is the minimal ternary

[16]The property is a type of transitivity and we use the term overlapping to relate it to the notion of overlapping sets.

relation containing $|_0$ and closed under overlapping, the following must hold $\widehat{|_0} \subseteq |_M$. This concludes the proof. ∎

Theorem 2 describes the nature of the information contained in the expanded D-survey. It shows that the (indirect) extremeness revelations contained in the expanded D-survey can be obtained by means of a closure, ternary, operation over the (direct) revelations of the original survey. Hence, a corollary of Theorem 2 is the following alternative characterization of survey rationalizability: a D-survey is rationalizable if and only if the overlapping closure of its extremeness revelations is asymmetric (satisfies WARE). This can be readily seen as a parallel to the classical choice result, which states that finite choice data is rationalizable if and only if the transitive closure of its preference revelations is asymmetric (satisfies WARP).

Importantly, Theorem 2 allows us to link the computational complexity of the rationalizability test with the algorithmic task of computing the overlapping closure relation of a ternary relation. We now show that this can be computed using a version of the classical Floyd-Warshall algorithm that is polynomial in time, and hence efficient. Consider the graph with $Q + Q(Q-1)$ nodes corresponding to all questions and pairs of questions. Connect every question q to every pair of questions $\{q', q''\}$, and associate to their edge a weight of 1 when $q \mid_0 \{q', q''\}$, and a weight of ∞ otherwise. We denote these weights by $[q, \{q', q''\}, 0]$. We then use the Floyd-Warshall algorithm over this (survey) graph, in order to identify the shortest path from q to $\{q', q''\}$, possibly using other nodes as intermediate. Denoting by $[q, \{q', q''\}, l]$ the weight of the shortest path from q to $\{q', q''\}$ possibly involving the use of intermediate questions $\{1, \ldots, l\}$ via overlapping, it is immediate that these values satisfy $[q, \{q', q''\}, l] = \min\{[q, \{q', q''\}, l-1], [q, \{q', l\}, l-1] + [q, \{q'', l\}, l-1]\}$. Iterate this process until l reaches Q, and note that edges with a finite value at the end of the process must correspond to the overlapping closure of $|_0$. Using the analogous reasoning to the standard result in which the Floyd-Warshall algorithm computes the transitive closure in polynomial order $\mathcal{O}(Q^3)$, the Q^2 order of the nodes of the survey graph (instead of Q in the transitive closure) shows that the computation of the overlapping closure has polynomial order $\mathcal{O}(Q^4)$. This proves that our tests for rationalizability are polynomial and, hence, computationally tractable.

4. Beyond the dichotomous case

In this section, we show how to exploit the structure of Theorem 1 when dealing with two related, but informationally richer, problems. The first one covers the Likert scale case in which individuals are allowed to use more than two labels in declaring their endorsement. The second analyzes the case of dichotomous surveys in which responses may be probabilistic.

4.1. **Polytomous surveys.** Given question $q \in \mathcal{Q}$, individual $n \in \mathcal{N}$ is now allowed to express strength of endorsement by using a label from $\mathcal{L} = \{0, 1, \ldots, L\}$, where $L > 0$. We write $\mathcal{L}^* = \mathcal{L} \setminus \{0\}$. Higher labels in the collection express stronger endorsement.[17] A polytomous survey or, simply, a P-survey is a map $P : \mathcal{N} \times \mathcal{Q} \to \mathcal{L}$, where $P(n, q) = l$ refers to the case where individual n assigns label l to question q. We consider the following notion of rationalizability, which reduces to the notion used for D-surveys when $L = 1$.

P-survey rationalizability. We say that P is rationalizable whenever there exist $[\{U_n, \{\tau_n^l\}_{l \in \mathcal{L}^*}\}_{n \in \mathcal{N}}, \{\mu_q\}_{q \in \mathcal{Q}}]$, with $U_n : \mathbb{R} \to \mathbb{R}$ being strictly quasi-concave, $\tau_n^1 < \cdots < \tau_n^L$, and $\mu_q \in \mathbb{R}$ such that, for every $n \in \mathcal{N}$, $q \in \mathcal{Q}$ and $l \in \mathcal{L}^*$, $P(n, q) = L$ if $U_n(\mu_q) \geq \tau_n^L$, $P(n, q) = l$, $0 < l < L$, if $\tau_n^{l+1} > U_n(\mu_q) \geq \tau_n^l$ and $P(n, q) = 0$ otherwise.

In words, the individual now uses different thresholds to determine her degree of endorsement. These thresholds naturally impose stronger requirements on stronger expressions of endorsement.

We now show that the rationalizability of a P-survey can be obtained building upon Theorem 1, by studying the rationalizability of an associated D-survey. Formally, given the response map P, define the D-survey

$$\{Q^P_{(n,l)}\}_{(n,l) \in \mathcal{N} \times \mathcal{L}^*} \text{ where } Q^P_{(n,l)} = \{q \in \mathcal{Q} : P(n, q) \geq l\}.$$

That is, the vector of responses of individual $n \in \mathcal{N}$ is used to construct $L - 1$ response sets. Each of these response sets contains cumulative information, e.g., response set $Q^P_{(n,l)}$ contains all the questions endorsed by individual n with intensity l or above. Note that, formally, $\{Q^P_{(n,l)}\}_{(n,l) \in \mathcal{N} \times \mathcal{L}^*}$ is a D-survey where every (n, l) can be understood as an individual. It is immediate that $\{Q^P_{(n,l)}\}_{(n,l) \in \mathcal{N} \times \mathcal{L}^*}$ provides the same information as P does. We need to define the corresponding expanded D-survey, that follows the

[17] For instance, strongly disagree, disagree, neutral, agree, strongly agree.

same logic than above, and that we denote by $\{Q_n^P\}_{n\in\mathcal{N}_M}$. Theorem 3 shows that rationalizability of P is equivalent to this expanded D-survey satisfying WARE.

Theorem 3. *P is rationalizable if and only if $\{Q_n^P\}_{n\in\mathcal{N}_M}$ satisfies WARE.*

Proof of Theorem 3: To see the only if part, suppose that P is rationalizable. Then, it must be so for some collection of parameters $[\{U_n, \{\tau_n^l\}_{l\in\mathcal{L}^*}\}_{n\in\mathcal{N}}, \{\mu_q\}_{q\in\mathcal{Q}}]$. Consider the D-survey $\{Q^P_{(n,l)}\}_{(n,l)\in\mathcal{N}\times\mathcal{L}^*}$. By associating to response set $Q^P_{(n,l)}$ the utility function U_n and the threshold τ_n^l, this D-survey must be rationalizable. We can then use the only if part of Theorem 1 and the expanded D-survey $\{Q_n^P\}_{n\in\mathcal{N}_M}$ must satisfy WARE.

Suppose now that $\{Q_n^P\}_{n\in\mathcal{N}_M}$ satisfies WARE. By using the if part of Theorem 1, the D-survey $\{Q^P_{(n,l)}\}_{(n,l)\in\mathcal{N}\times\mathcal{L}^*}$ must satisfy c1p. We now show it is rationalizable. Let $\succ$ be a linear order on $\mathcal{Q}$ that guarantees so and consider any set of values $\{\mu_q\}_{q\in\mathcal{Q}}$ such that $\mu_q > \mu_{q'} \Leftrightarrow q \succ q'$. Given individual $n \in \mathcal{N}$, let $l_n > 0$ be the highest label used by this individual. If this label is used to describe endorsement for at least two questions, define $\mu_n = \frac{\min_{q\in Q^P_{(n,l_n)}}\mu_q + \max_{q\in Q^P_{(n,l_n)}}\mu_q}{2}$ and set $U_n(\mu_n) = l_n$ and $U_n(\min_{q\in Q^P_{(n,l_n)}}\mu_q) = U_n(\max_{q\in Q^P_{(n,l_n)}}\mu_q) = l_n - 1$. All values in the interval $[\min_{q\in Q^P_{(n,l_n)}}, \max_{q\in Q^P_{(n,l_n)}}]$ are set by considering the piecewise linear function determined by these three values. If the label was used for only one question, set μ_n equal to the μ-value of that question, and $U_n(\mu_n) = l_n - 1$. After this, consider recursively the rest of the labels, starting with $l_n - 1$ down to label 1. For any label l, determine the questions $\min_{q\in Q^P_{(n,l)}}\mu_q$ and $\max_{q\in Q^P_{(n,l)}}\mu_q$. If any of them is different to those previously considered (i.e., whenever these questions have label l exactly), extend the piecewise linear function U_n by forcing it to adopt value $l-1$ in the location of these new questions. After label 1, extend the function beyond its extremes with any strictly monotone linear functions. Set thresholds $\tau_n^l = l - 1$. The function U_n is strictly quasi-concave, and since $Q^P_{(n,l)}$ is formed by a set of consecutive questions in $\succ$, it is immediate that questions in $Q^P_{(n,l)}$, and only these questions, have utility above threshold τ_n^l. Rationalization follows. ■

4.2. **Stochastic responses.** We now study the case where survey responses may be stochastic. This can be understood in terms of the variation of responses of the individuals of a given subgroup, like those given by different age or gender groups, or the case in which individual responses express a value in some continuum, such as probabilistic beliefs or expectations, or intensity of support. Stochastic models of responses are also instrumental in dealing with actual data, since they are a way to model and account for noise resulting from, e.g., mistakes due to fatigue or inattention.

Formally, a dichotomous survey with stochastic responses, or S-survey, is a map $S : \mathcal{N} \times \mathcal{Q} \to [0,1]$ where $S(n,q)$ describes the probability with which individual $n \in \mathcal{N}$ endorses question $q \in \mathcal{Q}$. Consider the following definition of rationalizability.

S-survey rationalizability. We say that S is rationalizable whenever there exist $[\{U_n, F_n\}_{n\in\mathcal{N}}, \{\mu_q\}_{q\in\mathcal{Q}}]$, with $U_n : \mathbb{R} \to \mathbb{R}$ being strictly quasi-concave, $F_n : \mathbb{R} \to [0,1]$ being a cumulative distribution function (CDF) governing the realization of thresholds τ_n, and $\mu_q \in \mathbb{R}$ such that, for every $n \in \mathcal{N}$ and $q \in \mathcal{Q}$, $S(n,q) = F_n(U_n(\mu_q))$.

A question is endorsed whenever the utility of the view represented by this question is above a threshold but now, these thresholds are realized according to a distribution F_n. Hence, endorsement happens probabilistically, and the probability of individual n endorsing question q corresponds to the mass of thresholds below $U_n(\mu_q)$, i.e., $F_n(U_n(\mu_q))$. This is a convenient account of stochastic responses that aligns well with the models studied in this paper and can easily accommodate mistakes at the time of responding. Consider, e.g., an individual who with a large probability considers her true threshold, and with small probability trembles and considers a smaller or a larger threshold. This leads the individual to occasionally endorse questions with an associated utility below her true threshold and to reject questions that are above it.

We now argue that we can rely again on the techniques developed for the analysis of D-surveys. For every $n \in \mathcal{N}$, consider the (at most) Q response sets defined by $Q^S_{(n,q)} = \{q' \in \mathcal{Q} : S(n,q') \geq S(n,q)\}$, and the corresponding D-survey $\{Q^S_{(n,q)}\}_{(n,q)\in\mathcal{N}\times\mathcal{Q}}$. That is, the information of individual $n \in \mathcal{N}$ appears at most Q times, describing response sets that contain the questions chosen with at least as much probability as a given question. Denote the corresponding expanded D-survey by $\{Q^S_n\}_{n\in\mathcal{N}_M}$. Theorem 4 shows that rationalizability of S is equivalent to this expanded D-survey satisfying WARE.

Theorem 4. *S is rationalizable if and only if $\{Q^S_n\}_{n\in\mathcal{N}_M}$ satisfies WARE.*

Proof of Theorem 4: The proof is analogous to that of Theorem 3, so we only mention the most significant differences. Notice that from S we can construct a set of at most $Q \times N$ labels, and define the P-survey with the response map P having the property that, for every $n, n' \in \mathcal{N}$ and $q, q' \in \mathcal{Q}$, $P(n,q) \geq P(n',q')$ if and only if $S(n,q) \geq S(n',q')$. This allows us to construct the values μ_q and the functions U_n exactly as in Theorem 3. For every τ_n such that there exists $q \in \mathcal{Q}$ with $U_n(\mu_q) = \tau_n$, set $F_n(\tau_n) = S(n,q)$. Now, for every individual $n \in \mathcal{N}$, a larger threshold corresponds to a question with a larger utility and given the construction and the assumption that $P(n,q) \geq P(n,q')$ if and only if $S(n,q) \geq S(n,q')$, a larger utility corresponds to a larger probability of endorsement. We can then extend these values to a CDF F_n over the reals, and rationalization follows. ■

5. Identification

In this section, we discuss the identification of the models studied. In order to do so, we impose a richness condition, according to which for every triple of consecutive questions $\mu_{q_1} > \mu_{q_2} > \mu_{q_3}$, there are two individuals overlapping on this triple, i.e., $Q_{n_1} \cap \{q_1, q_2, q_3\} = \{q_1, q_2\}$ and $Q_{n_2} \cap \{q_1, q_2, q_3\} = \{q_2, q_3\}$.[18] To present results across different types of surveys, we make the following assumptions: for every individual $n \in \mathcal{N}$: (i) $\emptyset \neq Q_n \neq \mathcal{Q}$, and (ii) $S(n,q) = S(n,q') \Rightarrow P(n,q) = P(n,q') \Rightarrow [q \in Q_n$ if and only $q' \in Q_n]$. That is, (i) implies that we eliminate from the analysis the trivial individuals that endorse everything or nothing, and (ii) connects the response behavior in S-surveys, P-surveys and D-surveys.

As discussed above, the models are essentially ordinal, so we need to study the identification of the linear order $\succ$ corresponding to $\{\mu_q\}_{q \in Q}$. Later, we analyze what we can learn about the location of individuals. Finally, we show that a parametric version of the model allows for cardinal identification results.

5.1. **Identification of questions.** We start by showing that $\succ$ is fully identified. The reason for this is that, under the richness assumption, the extremeness revelations can be shown to be complete. Having the two extreme questions for every triple, the linear order is fully identified (up to inversion).[19]

Proposition 2. *Under the Richness assumption, if a survey is rationalizable, it is so for a unique linear order $\succ$ (up to inversion). This linear order is fully described by the extremeness revelations.*

Proof of Proposition 2: Let the survey be rationalizable. Suppose without loss of generality that it is so for the linear order $Q \succ Q-1 \succ \cdots \succ 1$. We now claim that for

[18]Notice that this requirement can be achieved with $Q-1$ individuals. If every triple involves different pairs of individuals, then $2(Q-1)$ individuals would be needed.

[19]Since this result holds for all types of surveys, we do not specify the type in the formulation of the result. Also, we simplify the exposition of the proof by showing the key steps using only the D-survey notation.

every triple of different questions in this sequence satisfying $i > j > k$, the expanded D-survey must reveal $i \mid \{j, k\}$ and $k \mid \{i, j\}$, i.e., that i, k are extreme in the triple (or, in a terminology that simplifies the exposition, that j must be the intermediate question in the triple). We prove this claim recursively, over the difference $i - k$. The result is trivially true for $i - k = 2$ because in this case the triple must be formed by three consecutive questions and the richness condition guarantees that two individuals exist revealing the intermediateness of j. Suppose now that the result is true up to some value $i - k = t \geq 2$ and consider a triple of questions such that $i - k = t + 1$. Consider question $j^* \neq j$ such that $i > j^* > k$, which must exist because $i - k = t \geq 2$. Let us assume that $j^* > j$ (the other case is dual and omitted). Given the recursive step, we know that $i \mid \{j, j^*\}$ and $j \mid \{i, j^*\}$, i.e., j^* has been revealed intermediate in $\{i, j^*, j\}$. We also know that $k \mid \{j, j^*\}$ and $j^* \mid \{j, k\}$, i.e., j has been revealed intermediate in $\{j^*, j, k\}$. Therefore, there exist four individuals $m_1, m_2, m_3, m_4 \in \mathcal{N}_M$ such that $Q_{m_1} \cap \{i, j^*, j\} = \{i, j^*\}$, $Q_{m_2} \cap \{i, j^*, j\} = \{j^*, j\}$, $Q_{m_3} \cap \{j^*, j, k\} = \{j^*, j\}$, and $Q_{m_4} \cap \{j^*, j, k\} = \{j, k\}$. Suppose first that $i \in Q_{m_3}$. Then, by rationalizability, since $j^* \notin Q_{m_4}$, it must be that $i \notin Q_{m_4}$ and, as a result, individuals m_3 and m_4 reveal that $k \mid \{i, j\}$ and $i \mid \{j, k\}$, as desired. Suppose now that $i \notin Q_{m_3}$. In this case, notice that by rationalizability, $k \notin Q_{m_1}$ (because it does not contain $j > k$) and, by assumption, $k \notin Q_{m_3}$. Since Q_{m_1} and Q_{m_3} are overlapping, the response $Q_{m_1} \cup Q_{m_3}$ belongs to the expanded D-survey and $[Q_{m_1} \cup Q_{m_3}] \cap \{i, j, k\} = \{i, j\}$. Then, responses $[Q_{m_1} \cup Q_{m_3}]$ and Q_{m_4} reveal that $k \mid \{i, j\}$ and $i \mid \{j, k\}$, as desired. All intermediate revelations of $\succ$ have been obtained through the extremeness revelations. Finally, to show uniqueness, consider any other order $\succ'$ different than $\succ$ and its inverse. Hence, there must be three questions $i > j > k$ where the intermediate one is different to j. Without loss of generality, let i be the intermediate question in $\succ'$. But then, we know that there is a response in the expanded D-survey such that $Q_m \cap \{i, j, k\} = \{j, k\}$, and this is not rationalizable for $\succ'$. If the expanded D-survey is not rationalizable with $\succ'$, the proof of Theorem 1 guarantees that the original survey cannot be rationalizable with $\succ'$, concluding the proof. ■

Proposition 2 shows that, under the richness condition, the order of questions is identified by the extreme revelations. Notice that this idea can also be used even in the absence of the richness condition. In essence, partial identification of the order of questions is obtained: any linear order defined by an extremeness relation that contains the one revealed by the expanded D-survey is plausible.[20]

5.2. **Identification of individuals.** We now discuss what can be potentially learnt about a specific individual $n \in \mathcal{N}$. As previously discussed, the model is basically ordinal. Accordingly, here we care about the identification of the location of the ideal view of the individual with respect to $\succ$.[21] We show that S-surveys are more informative in this respect than P-surveys, which in turn are more informative than D-surveys. For any individual $n \in \mathcal{N}$, denote by $\underline{q}_n^\alpha$ and $\overline{q}_n^\alpha$, with $\alpha \in \{\text{D,P,S}\}$, the minimum and maximum questions (according to μ_q or $\succ$) among: (i) the endorsed ones when $\alpha = \text{D}$, (ii) those with the highest strength of endorsement when $\alpha = \text{P}$, and (iii) those with the highest probability of endorsement when $\alpha = \text{S}$. Then, denote by $\alpha(n)$ the open interval of $\succ$ defined by the question immediately below $\underline{q}_n^\alpha$ and the one immediately above $\overline{q}_n^\alpha$ in $\succ$.[22] Importantly, in S-surveys whenever the CDF over the threshold is atomless, these two questions must either coincide or be consecutive, implying that the identification is almost complete.[23]

Proposition 3. *Under the Richness assumption, the ideal view of individual $n \in \mathcal{N}$ belongs to $S(n) \subseteq P(n) \subseteq D(n)$, and no further identification is possible.*

Proof of Proposition 3: Let the survey be rationalizable and $\succ$ be the unique linear order, up to inversion, guaranteeing so. Consider $n \in \mathcal{N}$ and an α-survey. First, we claim that the ideal view must belong to the interval $\alpha(n)$. Suppose by way of contradiction that this is not the case. We analyze the case in which the ideal view occupies a position equal or above the question immediately above $\overline{q}_n^\alpha$, and omit the other, dual case. In this case, notice that the question $\overline{q}_n^\alpha$ achieves a higher utility than the question above, contradicting the quasi-concavity of U_n. Second, we claim that

[20]The combination of Theorems 1 and 2 can be seen as a result in the spirit of the classical result by Szpijlran (1930), for the case of an extremeness ternary relation.

[21]Note that an individual with strictly increasing (decreasing) utility, and hence with no peak on the reals, is indistinguishable in ordinal terms with respect to an individual with a peak above the maximum (below the minimum) of all questions according to $\succ$.

[22]If one of these two questions does not exist, the position of the ideal view can be seen as unbounded in that specific direction.

[23]This follows immediately from the fact that, given strict quasi-concavity and the atomless of the CDF, only the closest question at each side of the ideal point may come with the highest endorsement probability (up to copies of these questions).

$S(n) \subseteq P(n) \subseteq D(n)$. To see this, consider first $q \in S(n)$. By construction, q achieves the highest endorsement probability in $\mathcal{Q}$ and, given our basic assumptions, there exists at least one question q' with strictly less endorsement probability. Question q must belong to the equivalence class of questions with the highest label in the P-survey, and this level must be different than zero. It then follows that $S(n) \subseteq P(n)$. Consider now a question $q \in P(n)$. By construction, q achieves the highest label in $\mathcal{Q}$ and, given our basic assumptions, this label is strictly above zero. Question q must belong to the response set in the D-survey and $P(n) \subseteq D(n)$. Third, we claim that no further identification is possible. We analyze the case of D-surveys, with the other two being analogous. Consider the set Q_n. Modify the selection of the ideal point in the proof of Theorem 1 to occupy any position in $D(n)$. If this position is between $\overline{q}_n^\alpha$ and $\underline{q}_n^\alpha$, set the utility function to be piecewise linear, with value 1 for the ideal view and 0 for these two questions. Otherwise, it must be either above $\overline{q}_n^\alpha$ or below $\underline{q}_n^\alpha$. Consider the first case (the second being omitted as it is analogous). Then, set the utility function to be piecewise linear, with value 1 for the ideal view, 0 for $\underline{q}_n^\alpha$, and -1 for the question immediately above $\underline{q}_n^\alpha$ (if no question exists there, any strictly decreasing function works). This alternative representation also provides rationalizability and completes the proof. ∎

5.3. **Exponential responses.** In our non-parametric models, the identification of questions and ideal points is ordinal. We now introduce a stochastic parametric model that illustrates how cardinal identification of the location of questions and ideal points, $\{\mu_q\}_{q \in \mathcal{Q}}$ and $\{\mu_n\}_{n \in \mathcal{N}}$ can be achieved. The model postulates that the individual endorsement probability is of the exponential form on the distance between the locations of the question and the ideal point of the individual, $S(n,q) = e^{-\frac{|\mu_n - \mu_q|}{\sigma_n}}$.[24] Intuitively, the probability of endorsing a question is 1 when its location coincides with that of the ideal point, decays with the distance between them, and is zero in the limit. It is modulated by a single parameter $\sigma_n > 0$ with a natural interpretation. Smaller values of σ_n accelerate the decay, representing individuals with a less tendency to endorse

[24]The literature in psychology offers various probabilistic models of responses based on the notion of endorsement-by-proximity. See, e.g., Davison (1977). The present model is a convenient version of Andrich (1988) and Hoijtink (1990).

views away from their ideal point. Notice that the exponential endorsement probability could be alternatively presented via our stochastic framework by way of a strictly quasi-concave utility and a random threshold.[25]

We start by normalizing the location of any two questions, that we denote by 0_q and 1_q, to $\mu_{0_q} = 0$ and $\mu_{1_q} = 1$, respectively.[26]

Proposition 4. *Let $Q \geq 4$. Under the Richness assumption, in an S-survey with exponential responses, $\{\mu_q\}_{q\in\mathcal{Q}\setminus\{0_q,1_q\}}$ and $\{\mu_n\}_{n\in\mathcal{N}}$ are cardinally identified.*

Proof of Proposition 4: Consider an S-survey with exponential responses. By Proposition 2, we know that $\succ$ is fully identified, and we set it to be $1_q \succ 0_q$. Given the atomless nature of the exponential model, for every $n \in \mathcal{N}$, the interval $S(n)$ contains either one or two questions. As in the proof of Proposition 3, denote these two questions by $\overline{q}_n^S$ and $\underline{q}_n^S$, and it must be either $\overline{q}_n^S \succ \underline{q}_n^S$ or $\overline{q}_n^S = \underline{q}_n^S \equiv q_n^S$. We now construct a map $h_n : \mathcal{Q} \to \{0,1\}$ for each individual such that $h_n(q) = 1$ if and only if $\mu_q \geq \mu_n$ and $h_n(q) = 0$ otherwise. To do so, we consider two cases. If $|S(n)| = 2$, the exponential model of responses guarantees that μ_n must occupy a position strictly between these two questions, and we can assign $h_n(q) = 1$ to $\overline{q}_n^S$ and to every question above this one, according to $\succ$, and zero otherwise. If $|S(n)| = 1$, the map can be defined by $h_n(q) = 1$ whenever $q \succ q_n^S$ and $h_n(q) = 0$ whenever $q_n^S \succ q$. However, the relative location of q_n^S is yet undetermined. To do so, we consider another individual n' for which $h_{n'}(q_n^S)$ is known, and two questions q', q'' for which $h_n(q'), h_n(q''), h_{n'}(q'), h_{n'}(q'')$ are all known. It is immediate to see that the exponential model requires that

$$\frac{(-1)^{h_n(q_n^S)}\log S(n,q_n^S) - (-1)^{h_n(q'')}\log S(n,q'')}{(-1)^{h_n(q')}\log S(n,q') - (-1)^{h_n(q'')}\log S(n,q'')} = \frac{\mu_{q_n^S} - \mu_{q''}}{\mu_{q'} - \mu_{q''}} =$$

$$= \frac{(-1)^{h_{n'}(q_n^S)}\log S(n',q_n^S) - (-1)^{h_{n'}(q'')}\log S(n',q'')}{(-1)^{h_{n'}(q')}\log S(n',q') - (-1)^{h_{n'}(q'')}\log S(n',q'')}.$$

Hence, $h_n(q_n^S)$ can be determined.[27] Given the complete map h_n, notice now that for every q other than 0_q and 1_q it is

$$\frac{(-1)^{h_n(q)}\log S(n,q) - (-1)^{h_n(0_q)}\log S(n,0_q)}{(-1)^{h_n(1_q)}\log S(n,1_q) - (-1)^{h_n(0_q)}\log S(n,0_q)} = \mu_q,$$

[25] It suffices to set $U_n(x) = e^{-\frac{|\mu_n - x|}{\sigma_n}}$ and F_n the CDF of the uniform distribution in $[0,1]$.

[26] Recall that the inverse of a linear order is equivalent to the linear order for our purposes, so the order of these two alternatives is without loss of generality.

[27] Notice that in the extreme case in which $S(n,q_n^S) = 1$, it must be $\mu_n = \mu_{q_n^S}$ and $h_n(q_n^S)$ can be freely assigned.

$$\frac{-(-1)^{h_n(0_q)}\log S(n,0_q)}{(-1)^{h_n(1_q)}\log S(n,1_q)-(-1)^{h_n(0_q)}\log S(n,0_q)}=\mu_n.$$

All locations are fully determined, concluding the proof. ■

6. Variations

In this section we come back to the two main assumptions in our notion of D-survey rationalizability; the unidimensional location of questions and individuals, and the attitudinal type of surveys.

6.1. **Multidimensionality.** We now allow for the possibility that questions and individual ideal points belong to a multidimensional space. Given the result below, it suffices to consider the bidimensional case in which $U_n : \mathbb{R}^2 \to \mathbb{R}$ and $\mu_q \in \mathbb{R}^2$.

Theorem 5. *In the bidimensional case, every $\{Q_n\}_{n\in\mathcal{N}}$ is rationalizable.*

Proof of Theorem 5: Consider the collection of D-survey responses $\{Q_n\}_{n\in\mathcal{N}}$. We construct a rationalization of these responses. Locate all questions in different positions of the circumference of radius one centered at the origin, in any way that is desired. Consider Q_n. If $|Q_n| = 0$, rationalization is trivial by considering Euclidean preferences centered in the origin, with utility 0 at the origin and $\tau_n = 0$. If $|Q_n| = 1$, rationalization is trivial, by considering Euclidean preferences centered in the only question endorsed, with utility 0 at that point, and $\tau_n = 0$. If $|Q_n| \geq 3$, consider the convex hull formed by (the location) of all questions that belong to Q_n; we denote this set by $CH(Q_n)$. Given that all questions in Q are in different points of the unit circumference, it is evident that $q \in CH(Q_n) \cap Q$ if and only if $q \in Q_n$; every question in $Q \setminus Q_n$ lies outside this convex hull. We construct the utility function U_n as follows. Select any point in the interior of $CH(Q_n)$ and denote it by μ_n. Then,

- $U_n(\mu_n) = 0$.
- For every y in the frontier of $CH(Q_n)$, set $U_n(y_1, y_2) = -1$.
- For every x outside $CH(Q_n)$, let y be the unique point in the frontier of $CH(Q_n)$ such that $y = \lambda x + (1-\lambda)\mu_n$. We can then set $U_n(x) = \frac{-1}{\lambda}$.
- For every x in the interior of $CH(Q_n)$, let y be the unique point in the frontier of $CH(Q_n)$ such that $x = \lambda y + (1-\lambda)\mu_n$. We can then set $U_n(x) = -\lambda$.

By construction, all indifference curves have the same shape (expansions or contractions of the polygon created by the set of positive responses of the individual), with utility normalized to 0 and -1 for the ideal point and the polygon indifference curve. This is obviously a strictly quasi-concave utility function. By setting $\tau_n = -1$, this rationalizes the responses of the individual.

If $|Q_n| = 2$, select a third point in the circumference that does not correspond to the location of any question, and consider the triangle formed by these three points. Then, proceed to construct the utility function and the threshold as per the above reasoning. This again rationalizes the set of responses. □

Theorem 5 shows that the multidimensional case has no empirical power. The intuition is that already with two dimensions there can be an infinite number of extreme questions, which imposes no constraint in the responses. This is in sharp contrast with the unidimensional case, in which there can only be two extreme questions.

The indifference curves in the construction of the proof are polygons. The same result would go through even if we strengthen the rationalization concept to require strictly convex contour sets, because for every polygon created by a finite number of points in the unit circumference, we can create a strictly convex superset of this polygon that intersects with the unit circumference in exactly the same points. Hence, any non-parametric analysis of significance with more than one dimension becomes trivial. Naturally, one may consider parametric versions of the model in which the structure of the utility functions is constrained. A natural one is that of Euclidean preferences. In this case, it can be seen that not every collection of responses can be rationalized. For instance, consider the problem with 4 individuals and 2^4 alternatives in which each of the alternatives is endorsed by a different subset of individuals. For rationalizability with Euclidean preferences to be possible, there should exist four circles in the plane forming a Venn diagram, which is known not to be possible due to the Euler's relation on the number of faces, edges and vertices in a plane graph (see e.g. Ruskey, Savage and Wagon (2006)). Hence, the question arises on what are the properties of survey responses that can be rationalized by way of Euclidean preferences using a given number of dimensions. Relatedly, another interesting question is, given survey responses, what is the minimal number of dimensions that allow for Euclidean rationalizability. Bogomolnaia and Laslier (2007) study this issue but in the context of an analyst knowing the preference rankings of all individuals.[28] Importantly, note that our setting with

[28]Ballester and Haeringer (2011) study the single-peaked unidimensional case when, again, the analyst knows the preference rankings of all individuals. Kalandrakis (2010) studies the multidimensional case under voting, that, effectively, corresponds to partial information on preferences.

data based on survey responses reflects the case where information is much coarser, in that only one upper-contour set per individual is known.

6.2. **Rationalizability of aptitude surveys.** We now briefly study the rationalizability of surveys measuring aptitudes, rather than attitudes. These are surveys with the aim of studying the absolute, cumulative, extent of a given variable, as it is often the case in the study of ability (say, the mathematical skills of pupils), health status (say, the extent of mobility for the aged), social functionality (say, the social skills of psychiatric patients), or the extent of a practice (say, the level of religiosity of citizens). Here we focus on the dichotomous case, which is usually known as Guttman-scale and, accordingly, we refer to it as Guttman-rationalizability.[29]

Guttman-rationalizability. We say that $\{Q_n\}_{n\in\mathcal{N}}$ is Guttman-rationalizable whenever there exist $[\{\mu_n\}_{n\in\mathcal{N}}, \{\mu_q\}_{q\in\mathcal{Q}}]$ with $\mu_n, \mu_q \in \mathbb{R}$, such that for every $n \in \mathcal{N}$ and $q \in \mathcal{Q}$, it is $q \in Q_n$ if and only if $\mu_n \geq \mu_q$.

In Guttman-rationalizability, a question (or problem, or task) q is passed whenever its location falls below the location of the individual, $\mu_q \leq \mu_n$. Notice that Guttman-rationalizability is a particular case of D-survey rationalizability since we could alternatively rationalize responses by constructing a strictly quasi-concave utility function U_n and a threshold τ_n such that the individual endorses every question located below the ideal point, and only these.[30] Hence, property WARE is necessary for Guttman-rationalizability, but not sufficient. We now provide a necessary and sufficient condition. The cumulative nature of Guttman-rationalizability facilitates the survey revelation exercise; it allows us to analyze the responses by way of the transitive closure of a binary relation, instead of the ternary relation we adopted above.

Consider a pair of questions $q_1, q_2 \in \mathcal{Q}$ and an individual $n \in \mathcal{N}$ such that $Q_n \cap \{q_1, q_2\} = q_2$. It is then evident that the aptitude of individual n separates the complexity of both questions, where the complexity of q_1 must be above that of q_2. We then say that q_1 has been revealed to be more complex than question q_2, and write

[29] The extensions to non-dichotomous and probabilistic responses follow analogously to the cases studied above.

[30] For instance, consider any utility function such that $\lim_{x\to-\infty} U_n(x) = 1$, increasing up to $U_n(\mu_n) > 1$, and decreasing afterwards with $U_n(\mu_n + \epsilon) < 1$ for any sufficiently small $\epsilon > 0$ such that there is no question in the interval $(\mu_n, \mu_n + \epsilon)$, and $\tau_n = 1$. This is strictly quasi-concave and leads to positive answers until the ideal point of the individual, and negative ones afterwards.

$q_1 \succ_{mc} q_2$. The standard asymmetry condition on the transitive closure of $\succ_{mc}$ is enough for Guttman-rationalizability.

Theorem 6. *$\{Q_n\}_{n\in\mathcal{N}}$ is Guttman-rationalizable if and only if the transitive closure of $\succ_{mc}$ is asymmetric.*

Proof of Theorem 6: Suppose first that $\{Q_n\}_{n\in\mathcal{N}}$ is Guttman-rationalizable. If $q_t \succ_{mc} q_{t-1}$ for every $t = 1, \ldots, T$, we know that there exist individuals n_t, $t = 1, \ldots, T$ such that $\mu_{q_T} > \mu_{n_T} \geq \mu_{q_{T-1}} > \mu_{n_{T-1}} \geq \mu_{q_{T-2}} > \cdots \geq \mu_{q_1} > \mu_{n_1} \geq \mu_{q_0}$ and hence, $\mu_{q_T} > \mu_{q_0}$. Then, it is evident that $q_0 \not\succ_{mc} q_T$.

Suppose now that the transitive closure of $\succ_{mc}$ is asymmetric. Given the finiteness of $\mathcal{Q}$, we can find values $\{\mu_q\}_{q\in\mathcal{Q}}$ such that $q \succ_{mc} q'$ implies $\mu_q > \mu_{q'}$. If an individual responds negatively to all questions set $\mu_n < \min_{q\in\mathcal{Q}} \mu_q$; otherwise, set $\mu_n = \max_{q\in Q_n} \mu_q$. We claim that this constitutes a valid Guttman-rationalization. Suppose, by way of contradiction, that this is not the case. Notice that the construction guarantees that every question such that $\mu_q > \mu_n$ is negatively responded. The contradiction must involve an individual $n \in \mathcal{N}$ and a question $q \in \mathcal{Q}$ such that $\mu_n \geq \mu_q$ but $q \notin Q_n$. However, our construction guarantees that n responds positively to at least one question, and the definition of μ_n guarantees that there is a question q' with $\mu_{q'} > \mu_q$ and $q' \in Q_n$. This would make n to reveal $q \succ_{mc} q'$, which leads to $\mu_q > \mu_{q'}$, a contradiction, concludig the proof. ■

7. Discussion

This paper represents a first attempt in the study of the rationality foundations of survey responses, setting the basis for the theoretical and empirical treatment of other important issues. For example, another direction of interest represents the combination of surveys with more standard economic data to enhance the information revelation process, in line with the recent suggestions of Caplin (2021), Gerasimou (2021) and Almås, Attanasio, and Jervis (2023).

Also, our framework remains valid in the presence of non-truthful responses and, indeed, it provides grounds to empirically study the nature of the potentially existing deviations. Notice first that a respondent trying to appear, say, more moderate would merely present a set of responses with the form of an interval, but shifted towards the moderate part of the scale. Hence, our techniques can still be used in the presence of non-truthful responses, providing correct information on the order of questions, and on the self-presentation of the individuals (that could deviate from her internal views due to conformism, shame, etc). As it is common in the survey literature, we could then compare survey responses run under different conditions (e.g., different degrees of confidentiality or anonymity, different informational structures on the social or specific support that questions may have, or different incentives, etc), or use implicit information (such as response times or attention) to identify better individuals' actual views. The ordered nature of questions obtained in the analysis would facilitate this. Moreover, we can shed light on which variables of the model, e.g., the threshold level or the individual utilities, are affected and how. Alternatively, another interesting approach would entail comparing survey responses with relevant choice data (e.g., risky activities versus lottery choices).

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