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Formulas for Computing the Tidal Accelerations Due to the Moon and the Sun[1]

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Abstract—A summary of formulas with which the tidal accelerations due to the moon and the sun can be computed at any given time for any point on the earth's surface, without reference to tables, is presented in this paper. These formulas are convenient for computer use.

Introduction—The basic formulas for the computation of the vertical and horizontal components of tidal acceleration, g_0 and h_0, on a rigid earth have been given by a number of authors. The analysis is given, for example, by *Doodson* [1921], *Schureman* [1924], *Pettit* [1954], and *Bartels* [1957]. A good account is also given by *Doodson and Warburg* [1941]. Schureman's manual was reissued as a revised edition in 1941, but in this paper references are given to the older edition in cases where a particular formula no longer appears in the new edition, or a result is less accurately given there. The essential first step in all these formulations is the expression of the effective tidal acceleration in terms of the zenith angle and the distance of the tide-producing body. From this point there are two main lines of development. Doodson, Schureman, and Bartels proceeded to develop the lunar and solar tides into their harmonic constituents, whereas Pettit gave formulas with which the tidal forces can be computed with the aid of tables from the *American Nautical Almanac*.

The author was recently engaged in programing g_0 for an electronic computer. The computer was to display g_0 as a function of time for any given place on the earth's surface, starting at any given epoch. For this purpose it seemed desirable to use a closed form for the expression for g_0, rather than its harmonic development, and to obviate the use of tables in the computation. The formulas of Schureman were cast into a form convenient for the purpose, and the resulting expressions were used in a g_0 program for an IBM 709 computer. In view of the usefulness of this program it appears to the author that a summary of the formulas used is of interest.

Theory—The symbols used in this discussion are

- a earth's equatorial radius (6.378270×10^8 cm)
- a' defined in equation (31)
- a_1' defined in equation (32)
- A ascending intersection of moon's orbit with the equator
- c mean distance between centers of the earth and the moon
- c_1 mean distance between centers of the earth and the sun (1.495000×10^{13} cm) [*Pettit*, 1954]
- C defined in equation (34)
- d distance between centers of the earth and the moon
- D distance between centers of the earth and the sun
- e eccentricity of the moon's orbit (0.054899720 [*Shureman*, 1924, p. 172]; 0.05490 [*Shureman*, 1941, p. 162])
- e_1 eccentricity of the earth's orbit
- g_0 vertical component of tidal acceleration due to the sun and the moon
- g_m vertical component of tidal acceleration due to the moon
- g_s vertical component of tidal acceleration due to the sun
- h mean longitude of the sun
- h_0 horizontal component of tidal acceleration due to the sun and the moon

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h_m horizontal component of tidal acceleration due to the moon
h_s horizontal component of tidal acceleration due to the sun
H height of point of observation above sea level
i inclination of the moon's orbit to the ecliptic
I inclination of the moon's orbit to the equator
l longitude of moon in its orbit reckoned from its ascending intersection with the equator
l_1 longitude of sun in the ecliptic reckoned from the vernal equinox
L terrestrial longitude of general point P on earth's surface
m ratio of mean motion of the sun to that of the moon (0.074804 [*Schureman*, 1941, p. 162)
M mass of moon
N longitude of the moon's ascending node in its orbit reckoned from the referred equinox ($N = \Omega\Upsilon'$ in Fig. 1)
p mean longitude of lunar perigee
p_1 mean longitude of solar perigee
P general point on the earth's surface
r distance from P to the center of the earth
s mean longitude of moon in its orbit reckoned from the referred equinox
S mass of sun
t hour angle of mean sun measured westward from the place of observations
t_0 Greenwich civil time measured in hours
T number of Julian centuries (36,525 days) from Greenwich mean noon on December 31, 1899
α defined in equations (15) and (16)
θ zenith angle of moon
λ terrestrial latitude of general point on earth's surface
μ Newton's gravitational constant
ν longitude in the celestial equator of its intersection A with the moon's orbit (side $A\Upsilon$ in Fig. 1)
ξ longitude in the moon's orbit of its ascending intersection with the celestial equator
σ mean longitude of moon in radians in its orbit reckoned from A
Υ vernal equinox
Υ' referred equinox
φ zenith angle of sun
χ right ascension of meridian of place of observations reckoned from A
χ_1 right ascension of meridian of place of observations reckoned from the vernal equinox
ω inclination of the earth's equator to the ecliptic = 23.452° [*Schureman* 1941, p. 162]
Ω moon's ascending node

Referring to *Schureman* [1941, p. 13], we see that, if the fifth power of the moon's parallax (which could only contribute less than 0.05 per cent of the total tide-producing force) is ignored, the vertical component (upwards) of the lunar tidal force per unit mass at a point P on the earth's surface is

$$g_m = \frac{\mu M r}{d^3}(3\cos^2\theta - 1) + \frac{3}{2}\frac{\mu M r^2}{d^4}(5\cos^3\theta - 3\cos\theta) \tag{1}$$

To the same order of accuracy the horizontal component is

$$h_m = \frac{3}{2}\frac{\mu M r}{d^3}\sin 2\theta + \frac{3}{2}\frac{\mu M r^2}{d^4}(5\cos^2\theta - 1)\sin\theta \tag{2}$$

The expressions for the components of tidal acceleration due to the sun are similar, the terms depending on the fourth power of the sun's parallax being negligible. Thus

$$g_s = \frac{\mu S r}{D^3}(3\cos^2\varphi - 1) \tag{3}$$

$$h_s = \frac{3}{2}\frac{\mu S r}{D^3}\sin 2\theta\,\varphi \tag{4}$$

$$g_0 = g_m + g_s \tag{5}$$

and

$$h_0 = h_m + h_s \tag{6}$$

In order to express g_0, h_0 as functions of the time for any given point P (given latitude λ and longitude L), it is necessary to obtain θ, φ, d, and D as functions of time, and r as a function of latitude (and altitude). *Schureman*

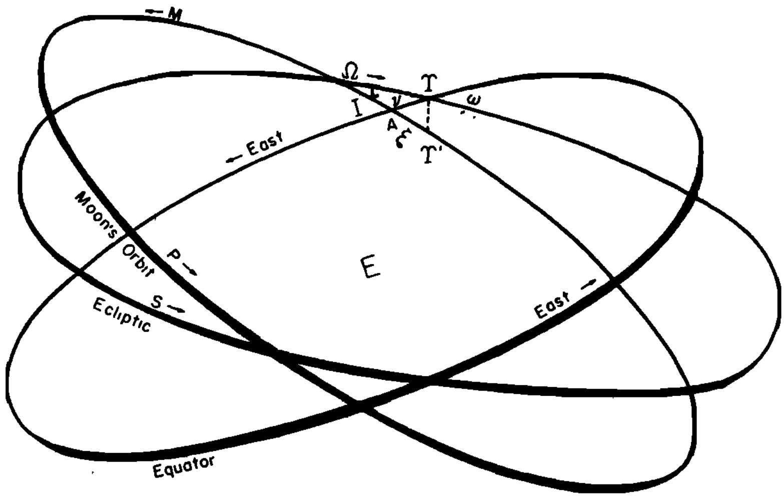

FIG. 1—Orbital parameters.

[1924, p. 30, equation 81] derives the relation*

$$\cos\theta = \sin\lambda \sin I \sin l + \cos\lambda\,[\cos^2 \tfrac{1}{2}I \cos(l-\chi) + \sin^2 \tfrac{1}{2}I \cos(l+\chi)] \qquad (7)$$

A similar relation holds for the sun's zenith angle φ:

$$\cos\varphi = \sin\lambda \sin\omega \sin l_1 + \cos\lambda\,[\cos^2 \tfrac{1}{2}\omega \cos(l_1-\chi_1) + \sin^2 \tfrac{1}{2}\omega \cos(l_1+\chi_1)] \qquad (8)$$

Schureman [1941, p. 19] gave for the longitude of the moon in its orbit

$$l = \sigma + 2e\sin(s-p) + \frac{5}{4}e^2 \sin 2(s-p) + \frac{15}{4}me\sin(s-2h+p) + \frac{11}{8}m^2 \sin 2(s-h) \qquad (9)$$

and (p. 162) the following expressions for s, p, h:

$$s = 270^\circ\, 26'\, 14.72'' + (1336\text{ rev.} + 1{,}108{,}411.20'')T + 9.09''T^2 + 0.0068''T^3 \qquad (10)$$

$$p = 334^\circ\, 19'\, 40.87'' + (11\text{ rev.} + 392{,}515.94'')T - 37.24''T^2 - 0.045''T^3 \qquad (11)$$

$$h = 279^\circ\, 41'\, 48.04'' + 129{,}602{,}768.13''T + 1.089''T^2 \qquad (12)$$

These expressions may be compared with those given by *Bartels* [1957, p. 747]. Bartels' formulas are equivalent to

$$s = 270^\circ\, 26'\, 11.72'' + (1336\text{ rev.} + 1{,}108{,}406.05'')T + 7.128''T^2 + 0.0072''T^3 \qquad (10')$$

$$p = 334^\circ\, 19'\, 46.42'' + (11\text{ rev.} + 392{,}522.51'')T - 37.15''T^2 - 0.036''T^3 \qquad (11')$$

$$h = 279^\circ\, 41'\, 48.05'' + 129{,}602{,}768.11''T + 1.080''T^2 \qquad (12')$$

σ is given by the relation

$$\sigma = s - \xi \qquad (13)$$

With reference to Figure 1, a little elementary spherical trigonometry shows ξ to be given by

$$\xi = N - \sin^{-1}(\sin\omega \sin N/\sin I) \qquad (14)$$

In order to render the inverse sine in this formula

* This relation is not given in *Schureman* [1941], where the development of the tidal forces has been rearranged.

unique, we also apply a cosine formula to the spherical triangle $\Omega A \Upsilon$. Denoting the side ΩA by α, we then have

$$\cos \alpha = \cos N \cos \nu + \sin N \sin \nu \cos \omega \quad (15)$$

where ν is the side $A\Upsilon$ (Fig. 1) and is the longitude in the celestial equator of its intersection A with the moon's orbit; ν is given by equation (21) below, while $\sin \alpha$ is given, as above, by

$$\sin \alpha = \sin \omega \sin N/\sin I \quad (16)$$

From the values of $\sin \alpha$ and $\cos \alpha$ we compute $\tan (\alpha/2)$ from the formula

$$\tan (\alpha/2) = \sin \alpha/(1 + \cos \alpha) \quad (17)$$

Now since α lies in the interval $(0, 2\pi)$, $\alpha/2$ lies in $(0, \pi)$ and hence when α is computed as

$$\alpha = 2 \tan^{-1} [\sin \alpha/(1 + \cos \alpha)] \quad (18)$$

its value is uniquely determined.

The longitude N of the moon's node is given by *Schureman* [1941, p. 162]

$$N = 259° \, 10' \, 57.12'' - (5 \text{ rev.} + 482{,}912.63'')T + 7.58''T^2 + 0.008''T^3 \quad (19)$$

Bartels [1957, p. 747] gives a formula which is equivalent to

$$N = 259° \, 10' \, 59.81'' - (5 \text{ rev.} + 482{,}911.24'')T + 7.48''T^2 + 0.007''T^3 \quad (19')$$

The inclination I of the moon's orbit to the equator is given by

$$\cos I = \cos \omega \cos i - \sin \omega \sin i \cos N \quad (20)$$

I is always positive and varies between about 18° and 28°. Also ν is given in terms of I, N by the relation

$$\nu = \sin^{-1} [\sin i \sin N/\sin I] \quad (21)$$

and here the inverse sine is unique, since we always have $-15° < \nu < 15°$. *Schureman* [1941, p. 162] gives

$$i = 5.145° \quad (22)$$

The angle χ in (7) is given by

$$\chi = t + h - \nu \quad (23)$$

For a point P on the earth's surface with longitude L, the value of t is

$$t = 15(t_0 - 12) - L \quad (24)$$

expressed in degrees.

Equations (9) to (24) enable us to determine the moon's zenith angle from equation (7).

Turning now to equation (8) for the sun's zenith angle, we see that the sun's longitude l_1 is given by

$$l_1 = h + 2e_1 \sin (h - p_1) \quad (25)$$

According to *Schureman* [1941, p. 162] p_1 is given by

$$p_1 = 281° \, 13' \, 15.0'' + 6{,}189.03''T + 1.63''T^2 + 0.012''T^3 \quad (26)$$

and e_1 is given* by *Schureman* [1924, p. 172] as

$$e_1 = 0.01675104 - 0.00004180T - 0.000000126T^2 \quad (27)$$

Bartels [1957, p. 747] gave an almost identical expression for p_1:

$$p_1 = 281° \, 13' \, 14.99'' + 6188.47''T + 1.62''T^2 + 0.011''T^3 \quad (26')$$

The quantity χ_1 is given by

$$\chi_1 = t + h \quad (28)$$

Equations (25) to (28) suffice to determine the sun's zenith angle from equation (8).

Referring to equations (1) to (4) we see that if we use the known values of μ, M, S, that is [*Pettit*, 1954],

$\mu = 6.670 \times 10^{-8}$ cgs units
$M = 7.3537 \times 10^{25}$ grams
$S = 1.993 \times 10^{33}$ grams

the tidal forces are determined if we know d, the distance between the centers of the earth and moon, and D, the distance between the centers of the earth and sun. Both quantities

* *Schureman* [1941, p. 162] merely gives $e_1 = 0.01675$, epoch Jan. 1, 1900.

are variable, being given by the relations [*Schureman*, 1924, pp. 55 and 172]

$$1/d = 1/c + a'e \cos(s - p) + a'e^2 \cos 2(s - p) + (15/8)a'me \cos(s - 2h + p) + a'm^2 \cos 2(s - h) \quad (29)$$

$$1/D = 1/c_1 + a_1'e_1 \cos(h - p_1) \quad (30)$$

Here c = mean distance between the centers of the earth and the moon = 3.84402×10^{10} cm. This figure is derived from *Schureman's* [1941, p. 162] value c = 238,857 miles. Also

$$a' = 1/[c(1 - e^2)] \quad (31)$$

a_1' is given by the formula analogous to (31):

$$a_1' = 1/[c_1(1 - e_1^2)]^* \quad (32)$$

Equations (29) to (32) now enable us to determine the tidal forces at any given point at distance r, say, from the center of the earth. For points on the earth's surface it is convenient to make use of the known shape of the earth and to express r in terms of the height above sea level and the latitude. Assuming the earth to be an ellipsoid with parameters as adopted by *Lecar and others* [1959], we have

$$r = Ca + H \quad (33)$$

where C is given by

$$C^2 = 1/(1 + 0.006738 \sin^2 \lambda) \quad (34)$$

Equations (1) to (34) determine the tidal acceleration at any point on the earth's surface. The (unprimed) equations have been checked by computing a number of cases (using an IBM 709 computer) and comparing the results with computations based on *Pettit's* [1954] paper, and also with computations (unpublished) by Pettit on S.W.A.C. (an electronic computer at the University of California). In every case agreement to within a fraction of a microgal was obtained. To this order of accuracy it is immaterial whether equations (10′), (11′), (12′), (19′), (26′) or the unprimed equivalents are used. Furthermore, in the actual program, values of a and C based on the Hayford spheroid model of the earth [*Hayford*, 1910] were used, and here again adoption of the later values given in this paper has no effect on the order of accuracy stated above.

* Equations (29) and (30) are also given by *Schureman* [1941, pp. 20 and 39] but with $a' = 1/c$, $a'_1 = 1/c_1$. Essentially, this means that e^2, e_1^2 have been neglected in comparison with unity.

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