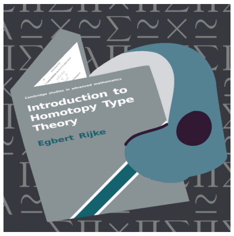

Worksheet 8 (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs

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1 $(\star)$

Let A and B be types.

1. Suppose that both A and B are propositions. Prove that $A+B$ is a proposition if and only if $A \to \neg B$.
2. Let $k : \mathbb{T}$. Suppose that both A and B are $(k+2)$-types. Prove that $A+B$ is a $(k+2)$-type.

1. Suppose that $A+B$ is a proposition. Let $a : A$ and $b : B$. We have that $\mathsf{inl}(a) = \mathsf{inr}(b)$, which gives us an element of $\emptyset$.

Conversely, suppose that $A \to \neg B$. By double induction on $A+B$, we have that $s = t$ for all $s, t : A+B$ because both A and B are propositions.

2. We want to prove that for all $s, t : A+B$, the type $s = t$ is a $(k+1)$-type. By induction on $A+B$, it suffices to show that each of

$$\begin{aligned} \mathsf{inl}(a) &= \mathsf{inl}(a') \\ \mathsf{inr}(b) &= \mathsf{inr}(b') \\ \mathsf{inl}(a) &= \mathsf{inr}(b) \\ \mathsf{inr}(b) &= \mathsf{inl}(a) \end{aligned}$$

is a $(k+1)$-type. The first two types are $(k+1)$-truncated because both A and B are $(k+2)$-truncated. The last two types are $(k+1)$-truncated because the empty type is.

2 $(\star\star)$

Let $\mathcal{U}$ be a universe and A be a type. Consider a partial order

$$\_ \leq \_ \ : \ A \to A \to \mathsf{Prop}_{\mathcal{U}}$$

on A. Prove that A is a set.

It suffices to find a reflexive binary relation

$$R : A \to A \to \mathcal{U}$$

on A such that for all $x, y : A$,

- **$R(x, y)$ is a proposition and**
- **$R(x, y) \to (x = y)$.**

Take

$$R(x, y) \ :\equiv \ (x \leq y) \times (y \leq x)\,.$$

Then R is reflexive because $\leq$ is reflexive. Also, since the Cartesian product of two propositions is again a proposition, we see that $R(x, y)$ is a proposition. Finally, since $\leq$ is anti-symmetric, we have a map $R(x, y) \to (x = y)$.

3 $(\star\star)$

1. Let A be a type and B a set. Suppose that $f : A \to B$ is an injection in the sense that it has a term
$$c : \prod_{x,y:A} (f(x) = f(y)) \to (x = y)\,.$$
Prove that f is an embedding, so that A is a set.

2. Prove that the function $n \mapsto m + n$ is an embedding for every $m : \mathbb{N}$. Here, addition is defined by recursion on the *first* argument.
Conclude that
$$(m \leq n) \simeq \sum_{k:\mathbb{N}} m + k = n$$
for every $m, n : \mathbb{N}$.

1. For every $x, y : A$, the map $c_{x,y}$ is a section of $\mathsf{ap}_f(x, y)$ because the codomain $f(x) = f(y)$ of the latter is a proposition. Hence f is an embedding.

2. To see that $n \mapsto m + n$ is an embedding, proceed by induction on $\mathbb{N}$. The base case amounts to proving that $\mathsf{id}_{\mathbb{N}}$ is an embedding, which is clear. For the induction step, form a commuting diagram

$$\begin{tikzcd}
& \mathsf{Eq}_{\mathbb{N}}(m+x, m+y) & \\
m + x = m + y \arrow[ur, "\simeq"] \arrow[rr, "\mathsf{ap}_S(m+x,m+y)"] & & S(m+x) = S(m+y) \arrow[ul, "\simeq"'] \\
& x = y \arrow[ul, "\mathsf{ap}_{n\mapsto m+n}(x,y)"] \arrow[ur, "\mathsf{ap}_{n\mapsto S(m)+n}(x,y)"'] &
\end{tikzcd}$$

for each $x, y : \mathbb{N}$. By three-for-two together with our induction hypothesis, we deduce that $\mathsf{ap}_{n\mapsto S(m)+n}(x, y)$ is an equivalence.

It follows that
$$\mathsf{fib}_{j\mapsto m+j}(n) \;\doteq\; \sum_{k:\mathbb{N}} m + k = n$$
is a proposition for every $m, n : \mathbb{N}$. Further, it's easy to check that $m \leq n$ is a proposition by double induction on $\mathbb{N}$. It's also easy to check that
$$(m \leq n) \;\leftrightarrow\; \sum_{k:\mathbb{N}} m + k = n$$
by double induction on $\mathbb{N}$. This is a logical equivalence of propositions, which gives us an equivalence of types.

4 $(\star)$

Let A and B be types. Prove that the following are logically equivalent.

- Both A and B are contractible.
- The product $A \times B$ is contractible.

Suppose that both A and B are contractible with centers of contraction a_0 and b_0, respectively. By our characterization of identity types of Σ-types, it's easy to check that $A \times B$ is contractible with center of contraction (a_0, b_0).

Conversely, suppose that $A \times B$ is contractible with center of contraction s. Note that for every $b : B$,

$$(s = (\mathsf{pr}_1(s), b)) \to (\mathsf{pr}_2(s) = b) .$$

Thus, B is contractible with center of contraction $\mathsf{pr}_2(s)$. Similarly, A is contractible with center of contraction $\mathsf{pr}_1(s)$.

5 $(\star\star\star)$

Let A be a type and $a : A$. We say that a is an *isolated point* of A if it has a term

$$\tau : \prod_{x:A} (a = x) + (a \neq x) \, .$$

Suppose that a is isolated. Prove that $a = x$ is a proposition for all $x : A$. Conclude that $\mathsf{const}_a : \mathbb{1} \to A$ is an embedding.

To begin, define four functions

$$\begin{aligned}
&B : \prod_{x:A} ((a = x) + (a \neq x)) \to \mathcal{U}_0 \\
&B(x, \mathsf{inl}(p)) :\equiv \mathbb{1} \\
&B(x, \mathsf{inr}(q)) :\equiv \emptyset
\end{aligned}$$

$$\begin{aligned}
&f : \prod_{d:(a=a)+(a\neq a)} \sum_{z:a=a} \mathsf{inl}(z) = d \\
&f(\mathsf{inl}(p)) :\equiv \big(p, \mathsf{refl}_{\mathsf{inl}(p)}\big) \\
&f(\mathsf{inr}(q)) :\equiv \mathsf{ind}_{\emptyset}(q(\mathsf{refl}_a))
\end{aligned}$$

$$\begin{aligned}
&\varphi : \prod_{x:A} (a = x) \to B(x, \tau(x)) \\
&\varphi_a(\mathsf{refl}_a) :\equiv \mathsf{tr}_{B(a,-)}(\mathsf{pr}_2(f(\tau_a)), *)
\end{aligned}$$

$$\begin{aligned}
&\psi : \prod_{x:A} \prod_{d:(a=x)+(a\neq x)} B(x, d) \to (a = x) \\
&\psi_x(\mathsf{inl}(p), b) :\equiv p \\
&\psi_x(\mathsf{inr}(q), b) :\equiv \mathsf{ind}_{\emptyset}(b).
\end{aligned}$$

Now, we want to prove that the total space

$$\sum_{x:A} B(x, \tau(x))$$

is contractible with center $(a, \varphi_a(\mathsf{refl}_a))$. It's easy to check that $B(x,t)$ is a proposition for all $x : A$ and $t : (a = x) + (a \neq x)$. Moreover, for every $(x, b) : \sum_{x:A} B(x, \tau(x))$, we have a term $\psi_{x,\tau_x}(b) : a = x$. This implies that $(a, \varphi_a(\mathsf{refl}_a)) = (x, b)$.

By the fundamental theorem of identity types, it follows that φ is a family of equivalences, so that $a = x$ is a proposition for every $x : A$.

Finally, for each $x : A$, the fiber of const_a over x is equivalent to $a = x$ and thus is a proposition. It follows that const_a is an embedding.