Semantics of HoTT

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Why semantics?

- Soundness, completeness.
- Algebraic abstraction of syntactic constructions.
- Find (more general) syntactic proofs of facts about a given model.
- Justify the claim: "HoTT really *is* homotopy theory."

Why *functorial* semantics?

A general way of describing models:

- The syntax of a type theory $\mathbb{T}$ forms a category with structure $\mathbb{C}_{\mathbb{T}}$ (the *syntactic category*).
- A $\mathbb{T}$-model in a category $\mathcal{C}$ is a structure-preserving functor $\mathbb{C}_{\mathbb{T}} \to \mathcal{C}$.
- $\mathbb{C}_{\mathbb{T}}$ is *initial* among categories with said structure.
- Therefore a $\mathbb{T}$-model in a category $\mathcal{C}$ is simply a choice of said structure in $\mathcal{C}$.

The syntactic category of HoTT

Defining the syntactic category $\mathbb{C}$ only requires the *structural* rules of MLTT:

$$\frac{}{\vdash \diamond \ \mathrm{ctxt}}\ \texttt{EMP} \qquad \frac{\Gamma \vdash A \ \mathrm{type}}{\vdash \Gamma, x : A \ \mathrm{ctxt}}\ \texttt{EXT}$$

$$\frac{\vdash \Gamma, x : A, \Delta \ \mathrm{ctxt}}{\Gamma, x : A, \Delta \vdash x : A}\ \texttt{VAR}$$

$$\frac{\Gamma, \Delta \vdash \mathcal{J} \qquad \Gamma \vdash A \ \ \mathrm{type}}{\Gamma, x : A, \Delta \vdash \mathcal{J}}\ \texttt{WEAK}$$

$$\frac{\Gamma, x : A, \Delta \vdash \mathcal{J} \quad \Gamma \vdash a : A}{\Gamma, \Delta[a/x] \vdash \mathcal{J}[a/x]}\ \texttt{SUBST}$$

- The objects of $\mathbb{C}$ are contexts $\Gamma = x_1{:}A_1, \ldots, x_k{:}A_k$.
- Morphisms $\Delta \to x_1{:}A_1, \ldots, x_k{:}A_k$ are lists $\tau = (t_1, \ldots, t_k)$ of terms

$$\begin{gathered}
\Delta \vdash t_1 : A_1 \\
\vdots \\
\Delta \vdash t_k : A_k[t_1/x_1, \ldots, t_{k-1}/x_{k-1}].
\end{gathered}$$

The identity morphism is just the list of variables $(x_1, \ldots, x_k)$, and composition is defined by substitution.

Structure on $\mathbb{C}$

- The empty context is a terminal object.
- Given $\Gamma \vdash A$ and a morphism $\Delta \xrightarrow{\tau} \Gamma$, substitution defines a pullback square
$$\begin{tikzcd} \Delta, x{:}A[\tau] \arrow[r, "(\tau, x)"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma, x{:}A \arrow[d] \\ \Delta \arrow[r, "\tau"] & \Gamma \end{tikzcd}$$
- For Γ in $\mathbb{C}$, let $\mathbb{C}/\!/\Gamma$ be the full category of $\mathbb{C}_{/\Gamma}$ on the projections $\Gamma, x{:}A \to \Gamma$. Substitution defines pullback functors $\mathbb{C}/\!/\Gamma \to \mathbb{C}/\!/\Gamma, x{:}A$. The rules for Σ-types provide left adjoints to these functors, while those for Π-types provide right adjoints.

- The rules for Id-types give

intro:

$$
\begin{tikzcd}
 & \Gamma, x, y{:}A, p{:}x =_A y \arrow[d] \\
\Gamma, x{:}A \arrow[ur, dashed, "\mathsf{refl}"] \arrow[r, "(\mathsf{id}_\Gamma{,}x{,}x)"'] & \Gamma, x, y{:}A
\end{tikzcd}
$$

elim, comp:

$$
\begin{tikzcd}
\Gamma, x{:}A \arrow[r, "(x{,}x{,}\mathsf{refl}{,}c)"] \arrow[d, "\mathsf{refl}"'] & \Gamma, x, y{:}A, p{:}x =_A y, z{:}C \arrow[d] \\
\Gamma, x, y{:}A, p{:}x =_A y \arrow[ur, dashed, "J"] \arrow[r, equal] & \Gamma, x, y{:}A, p{:}x =_A y
\end{tikzcd}
$$

- The rules for a type universe give, for every morphism $\Gamma \xrightarrow{A} \mathcal{U}$, a choice of pullback square

$$\begin{tikzcd} \Gamma, x{:}A \arrow[r, "{(A,x)}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X{:}\mathcal{U}, x{:}X \arrow[d] \\ \Gamma \arrow[r, "A"] & \mathcal{U}. \end{tikzcd}$$

Definition

A **universe** in a category $\mathcal{C}$ is a map $U' \to U$ along with for every morphism $X \xrightarrow{f} U$, a choice of pullback square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U' \arrow[d] \\ X \arrow[r, "f"] & U. \end{tikzcd}$$

Digression: CCCs

- An object X of a category $\mathcal{C}$ is **squarable** if $\forall Y$ in $\mathcal{C}$, the product $X \times Y$ exists in $\mathcal{C}$,

$\Leftrightarrow$ the functor $\mathcal{C}_{/X} \to \mathcal{C}$ from the slice category has a right adjoint.

- A squarable object X is **exponentiable** if (the right adjoint) $\mathcal{C} \xrightarrow{X \times -} \mathcal{C}_{/X}$ has a (further) right adjoint $X \Rightarrow -$.
- $\mathcal{C}$ has finite products if it has a terminal object and every X in $\mathcal{C}$ is squarable.
- $\mathcal{C}$ is **cartesian closed** if it has a terminal object and every X in $\mathcal{C}$ is exponentiable (thus also squarable).

Digression: LCCCs

- A map $X \xrightarrow{f} Y$ in $\mathcal{C}$ is **squarable** if $\forall Z \xrightarrow{g} Y$ in $\mathcal{C}$, the pullback
$\begin{array}{ccc} \cdot & \xrightarrow{h} & \cdot \\ f^*g\downarrow & \lrcorner & \downarrow g \\ \cdot & \xrightarrow[f]{} & \cdot \end{array}$ exists in $\mathcal{C}$,

$\Leftrightarrow$ the functor $\mathcal{C}_{/X} \xrightarrow{f\circ -} \mathcal{C}_{/Y}$ has a right adjoint f^*.

- A squarable map $X \xrightarrow{f} Y$ is **exponentiable** if $\mathcal{C}_{/Y} \xrightarrow{f^*} \mathcal{C}_{/X}$ has a right adjoint Π_f.
- $\mathcal{C}$ is **locally cartesian closed** if all its maps are (squarable and) exponentiable,

$\Leftrightarrow$ every slice category $\mathcal{C}_{/X}$ is cartesian closed.

Modeling dependent type theory in LCCCs "naïvely"

In the Set model of type theory,

- Contexts are sets: $\Gamma \in \mathrm{Set}$
- Types are indexed sets: $\Gamma \vdash A$ corresponds to $\{A_\gamma \in \mathrm{Set} \mid \gamma \in \Gamma\}$ and $(\Gamma, x{:}A) = \coprod_{\gamma \in \Gamma} A_\gamma$. Equivalently, types are functions $A \to \Gamma$.
- Terms are sections $\begin{tikzcd} & \Gamma, x{:}A \arrow[d] \\ \Gamma \arrow[ur, "t"] \arrow[r, equal] & \Gamma \end{tikzcd}$ of the projection.
- (Σ-types) For $B \to A \to \Gamma$, we have $(\Sigma_A B)_\gamma = \coprod_{a \in A_\gamma} B_a$
- (Π-types) For $B \to A \to \Gamma$, we have $(\Pi_A B)_\gamma = \prod_{a \in A_\gamma} B_a$.

- (Id types)

intro:
$$\begin{tikzcd} & =_A \arrow[d] \\ A \arrow[ur, dashed, "\mathsf{refl}"] \arrow[r, "\Delta_A"] & A \times A \end{tikzcd}$$
elim:
$$\begin{tikzcd} A \arrow[r, "c"] \arrow[d, "\mathsf{refl}"'] & C \arrow[d] \\ =_A \arrow[ur, dashed, "J"] \arrow[r, equal] & =_A \end{tikzcd}$$

But in this interpretation, refl is also a type over $=_A$, so using elim

in
$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\mathsf{refl}"'] & A \arrow[d, "\mathsf{refl}"] \\ =_A \arrow[ur, "J"] \arrow[r, equal] & =_A \end{tikzcd}$$
shows that refl is an *isomorphism*.

Consequently, $=_A \to A \times A$ is the diagonal map Δ_A of A, and hence is a monomorphism in Set.

Corollary: In the Set model, all types/contexts are 0-truncated (uniqueness of identity proofs).

So the Set model of type theory cannot be a model of HoTT. This argument also works in any LCCC.

Models of type theory with $\Sigma, \Pi, \mathcal{U}$

A LCCC $\mathcal{C}$, with

- A chosen terminal object 1.
- A universe $U' \to U$,
- that is closed under Σ: the composite $A \to B \to C$ of maps "in the universe" is in the universe,
- that is closed under Π: for any sequence $C \to B \to A$ of maps in the universe, the exponential $\Pi_B C \to A$ is in the universe.

(We'll ignore coherence conditions for today.)

Models of Id-types in model categories

The first "homotopical" flavour in type theory is the observation (Awodey, Warren) that if types-in-context $\Gamma \vdash A$ are interpreted as *fibrations* of a (Quillen) model category, then Id-types can be interpreted as *path objects*.

The rule intro:

$$\begin{tikzcd} & P_A \arrow[d] \\ A \arrow[ur, "\mathsf{refl}"] \arrow[r, "\Delta_A"'] & A \times A \end{tikzcd}$$

is interpreted as the factorisation of the diagonal as a trivial cofibration followed by a fibration, and the rules

elim, comp:

$$\begin{tikzcd} A \arrow[r, "c"] \arrow[d, "\mathsf{refl}"'] & C \arrow[d] \\ P_A \arrow[r, equal] \arrow[ur, dashed, "J"] & P_A \end{tikzcd}$$

are given by the lifting of the trivial cofibration refl against the fibration $C \to P_A$.

Example: Contractibility

Syntactically, the type $\Sigma_{a:A}\Pi_{x:A} a =_A x$ is equivalent to the type $\Sigma_{a:A}(\lambda x.a \sim \mathsf{id}_A)$ of deformation retracts of A onto the terminal object 1 (the empty context).

Semantically, consider a path object $P_A \to A \times A$. Then a section $1 \to \Sigma_{A\to 1}\Pi_{\pi_2 : A\times A \to A} P_A$ corresponds to a section $1 \xrightarrow{a} A$ and a commuting triangle

$$\begin{tikzcd} 1 \arrow[r] \arrow[dr, "a"'] & \Pi_{\pi_2 : A\times A\to A} P_A \arrow[d] \\ & A \end{tikzcd}$$

which, by adjointness, corresponds to a triangle

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "a\times A"'] & P_A \arrow[d] \\ & A \times A. \end{tikzcd}$$

But this right homotopy describes a deformation retract of A onto 1.

The simplicial model of univalence

It turns out that there is a very nice model category in which we can interpret HoTT (Voevodsky). This is the model category sSet of simplicial sets with the usual "Kan-Quillen" model structure. For a suitable cardinal κ

- There is a κ-small (each fibre is a κ-small set) map of simplicial sets $U' \to U$ that classifies (is a universe of) κ-small fibrations,
- that is closed under Σ, Π and Id,
- such that U is fibrant and $U' \to U$ is a κ-small fibration,
- and such that $U' \to U$ satisfies the univalence axiom.

(The construction of the fibration $U' \to U$ is technical.)

- in order to show that every small fibration is a pullback of $U' \to U$, it suffices that in every solid diagram of pullback squares as below, where $A \rightarrowtail B$ is a monomorphism, and where the vertical arrows are small fibrations, the dashed part exists and is a pullback.

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] \arrow[dd] & & U' \arrow[dd] \\
& Y \arrow[ur, dashed] \arrow[dd] & \\
A \arrow[rr] \arrow[dr, tail] & & U \\
& B \arrow[ur, dashed] &
\end{tikzcd}$$

- Once this is done, showing that U is fibrant turns out to reduce to requiring that the dashed part of the following pullback square exist whenever the solid part does, where vertical arrows are fibrations, and i is a trivial cofibration (the "fibration extension property").

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, dashed] \\ A \arrow[r, tail, "i", "\sim"'] & B \end{tikzcd}$$

- Finally, univalence comes down to the "equivalence extension property": given a cofibration i and weak equivalence w in the solid diagram below, where all vertical maps are fibrations and the front square is a pullback, the dashed part of the diagram exists, where v is a weak equivalence, vertical maps are fibrations, and all squares are pullbacks.

$$\begin{tikzcd}
X \arrow[rr, dashed] \arrow[dr, "w"] \arrow[dd] & & X' \arrow[dr, dashed, "v"] \arrow[dd, dashed] & \\
& Y \arrow[rr] \arrow[dl] & & Y' \arrow[dl] \\
A \arrow[rr, tail, "i"] & & B &
\end{tikzcd}$$

Models in higher topoi

The model category sSet presents the ∞-topos of ∞-groupoids (spaces), and the model of HoTT in sSet does interpret types as arbitrary ∞-groupoids.

In fact, it is possible to extend this to an interpretation of HoTT in any Grothendieck ∞-topos.

Type-theoretic model topoi

A type-theoretic model topos (Shulman) is a category $\mathcal{E}$

- that is a 1-topos,
- with a model structure that is right proper, simplicial, combinatorial and whose cofibrations are exactly the monomorphisms,
- that is simplicially locally cartesian closed,
- that has a suitable "notion of fibred structure" that classifies all fibrations.

The key fact (Shulman) is that every type-theoretic model topos models type theory with Σ, Π, Id, univalent universes, W-types, pushouts, truncations, etc.

Model categories of simplicial presheaves

Given a small category A, the category $\mathrm{Sp}A = [A^{op}, \mathrm{sSet}]$ of simplicial presheaves on A has an **injective** model structure, whose weak equivalences and cofibrations are the pointwise weak equivalences and cofibrations of simplicial presheaves.

Every Grothendieck ∞-topos can be presented by a left exact left Bousfield localisation of one of these model categories. So it would be nice if they were type-theoretic model topoi.

However, the fibrations of this model structure are difficult to characterise (unlike those of sSet), so constructing a universal univalent fibration is non-trivial.

Injective fibrations

Given a square $\begin{array}{ccc} C & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ D & \longrightarrow & Y \end{array}$ of simplicial presheaves on A such that i is a pointwise cofibration and p is a pointwise fibration, there exist lifts $C_a \to X_a$ for all a that do not fit into a natural transformation, but into a *homotopy-coherent* natural transformation.

There is an object $\mathbf{C}(X)$ that classifies homotopy-coherent natural transformations into X, with a map $X \to \mathbf{C}(X)$. It so happens that X is injectively fibrant just when it is equipped with a retraction $\mathbf{C}(X) \to X$.

Since retractions are *structure*, this turns out to be a suitable "notion of fibred structure" that makes $\mathrm{Sp}A$ with the injective model structure a type-theoretic model topos.