HoTT lecture 8 Q&A (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs

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Q1 I can imagine modeling these constructions by gluing cubes, simplices, n-cells... does it "matter" how we think about this gluing?

Cubes just have nice distinguished sides, but discs disc are homotopic to cubes, so that would work as well!

Not really! If you glue simplices together you call it a 'simplicial' thing, If you glue cubes together you call it a 'cubical' thing, if you glue spheres/balls together you call it a 'globular' thing (pronounced globe-ular), and these should all do the same thing.

Q2 Just to ground this a bit—can we think of these 'free' constructions as initial in something like the category of Kan complexes equipped with [the things needed to construct them]?

I'm not entirely sure how to make this precise off the top of my head (though I have a guess), but something to this effect will be true.

You probably want to say the word '∞-groupoid' rather than 'Kan-complex' since we're thinking of them as algebraic objects

A large part of it is that the description of these cells leads to an induction principle

For example, in a free group, maps from A to UG are equivalent to maps FA to G

Shulman and Lumsdaine have a paper that introduces a class of higher categorical objects to model HITs. It's not simple to describe.

Q3 could you call $\mathbb{S}^2$ an ∞-type?

That's right; it's not an n-type for any finite n

Q4 By identifying the two different paths on the points, we should also be able to create a Klein's bottle right? By identifying the paths in sort of opposite orientation.

Yup! You take a square, identify the left/right sides with the same orientation, then identify the top/bottom sides with opposite orientation

Q5 When thinking 'geometrically' I can't really imagine what makes $\mathbb{S}^2$ that much more complex than a torus. What does make it so complicated?

Great question! If you figure it out, be sure to let the rest of mathematicians know :P. A less snarky answer which you can google is the 'Hopf fibration'

The torus only has a nontrivial fundamental group, whereas the sphere has infinitely many nontrivial homotopy groups. In the type theory lingo: the torus is a 1-type, whereas the 2-sphere is not an n-type for any n (because that would imply that all homotopy groups above n would be trivial)

One way to see this is that you can turn the 2-plane (which is contractible) into a torus via a very nice and symmetric quotient map (take each component mod $\mathbb{Z}$). This means that the universal cover of the torus is contractible, and so it happens to be a very simple kind of space

It is actually very rare for spaces to be as simple as the torus. By default, if you give an arbitrary cell complex, you would expect it not to be a n-type for any n. So, yeah, thinking of $\mathbb{S}^2$ as the default case is the right way to go

oh so it's not the n-spheres ($n > 2$) being surprisingly complex, its the torus being surprisingly simple

Exactly!

Or alternatively, homotopy theory being surprisingly complex :p

Q6 Is it correct intuition that $\mathbb{S}^1 \times \mathbb{S}^1$ is the torus because you get a torus by taking a circle and adding a circle to every point on the circle?

Yup! That's a great way to see what's going on

Q7 How does this all connect with the informal intuition that propositions are just the things which can be true or false?

That propositions can't be maningfully proven in two different ways is the Main Idea of the definition of a proposition. Classically, you would then think that the only information that a proposition has is whether or not it can be proven at all

is-prop can be characterized as $\Pi_{x,y:A}(x = y)$ and a type is contractible iff it is a proposition and inhabited. The former statement really means that the only thing that matters about a proof of that proposition is that it merely exists.

classically, can't a proposition have more than the mere information that the proposition can be proven? E.g., some disjunctions $A \vee B$ can be proven in two different ways; one by proving A, and another by proving B.

That example would not be a proposition in the technical sense of 'proposition' that we are using

So not every proposition in the ordinary syntactic sense is a proposition in our new sense?

To reconcile this, we have 'propositional truncation', so classical logic becomes all propositional if the meaning of OR is the truncation of the sum type

So the answer is 'yes and no'. No if you do it naively, but yes if you do it right.

Chapter 3 of the HoTT book discusses this in detail

Are all propositions in the ordinary syntactic sense 'mere propositions' in the sense of the HoTT book?

Only if you define the transformation from propositional logic to HoTT as interpreting OR and THERE-EXISTS as being truncations. The HoTT book mentions that this is the correct way to do it in Chapter 3

Q8 "By default, if you give an arbitrary cell complex, you would expect it not to be a n-type for any n." Can this be made rigorous, e.g. probabilistically? Or should it be taken as a heuristic?

This is a topology intuition that I got from working in algebraic topology for several years. I am not aware of any analysis of cell complexes probabilistically

Q9 Is there some theorem that every proposition is isomorphic to (1 union 0)?

No. I assume that you mean that every proposition is either isomorphic to 1 or isomorphic to 0. That statement is equivalent to the law of excluded middle.

And $||1 \text{ or } 0||$**?**

could you give an example of a model where there are more than those 2 propositions?

If you have proven a proposition, you can use that proof to show that the proposition is equivalent to the unit type.

And disproving it allows you to prove that it's equivalent to the empty type

The type $||1 \text{ or } 0||$ **is contractible. It is not the case that every proposition is equivalent to** $||1 \text{ or } 0||$**, because that would imply that every proposition is contractible. We saw earlier in this lecture that the empty type is a proposition, but the empty type is not contractible.**

but i suspect you mean $||(P = \mathbf{0}) + (P = \mathbf{1})||$ **which i believe is 1. equivalent to LEM if P is restricted to propositions, and 2. inconsistent with univalence in general (not sure about that one)**

actually never mind it's obviously inconsistent with everything if P is not a prop. i was thinking of $(P = 0) + P$

Q10 A proven proposition is already a contractible

That's right! This is made precise by Emily's (iii) in the theorem we're currently proving

Q11 why don't we immediately prove $iii \Rightarrow iv$ using x (or y) instead of a?

There are many ways to prove $(iii) \to (iv)$. You can also do it directly

Q12 Does K in Agda also apply to other inductive types that are similar to the identity type in some way?

If you leave axiom-K on in Agda, then *every* type will be a set in the sense Emily is describing now

right, but the definition of is-set uses the identity type - it just seems like it should affect more since the identity type is not (necessarily) built-in in Agda (if you don't use the BUILTIN flag, which we don't)

that's right, K is more general (it affects the way agda is allowed to delete unification constraints of the form $a = a$)

you could define a "Trident x y z" type with a single constructor trefl : Trident x x x, then K would allow you to match any Trident x x x against trefl

Q13 Axiom K/'all loops are trivial' reminds me of a topological space being simply connected—is there any connection there?

simply connected (or 1-connected) means all homotopy groups ≤ 1 are trivial, while K/is-set means all homotopy groups ≥ 1 are trivial. so they agree on 1, but for example a sphere is simply connected but not a set, and the booleans are a non-simply connected set

That's right. Simply connected (or rather 1-connected) means that the 1-truncation of a type is contractible

In some conventional usage, 'simply connected' does not imply 'connected', so 1-connected is the better term

Q14 I wonder if there are any connections with the arithmetical hierarchy in computability theory. In that hierarchy, the Σ's and the Π's are intermixed. So I wonder if there are any ways that pis (products) get interspersed between the truncation levels?

If you have a Π-type, then the truncatedness level is that of the codomain, so this behavious is a bit different from what you would see in that hierarchy

Q16 Is this notion of subtype related to type refinement systems?

That's more of a PL concept and in our case, subtypes would not behave in that way (i.e. definitional subtyping) in an implementation of HoTT on a computer

okay, thanks. I know there's a relationship between type refinements (incl subtyping) and Σ-types, thought it might be showing up here, but good to know it's not

they probably express similar ideas (elements of type X *such that* ...) but behave in different ways