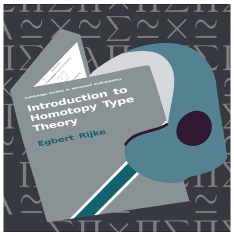

Worksheet 6

HoTTEST Summer School 2022

The HoTTEST TAs , and

25 July 2022

1 ($\star$)

Prove that $\neg \mathsf{isContr}(\emptyset)$.

2 ($\star\star$)

Recall the *observational equality of natural numbers* $\mathsf{Eq}\text{-}\mathbb{N} : \mathbb{N} \to \mathbb{N} \to \mathcal{U}$

$$\begin{aligned}
&\mathsf{Eq}\text{-}\mathbb{N}\ 0 && 0 && \doteq \mathbb{1} \\
&\mathsf{Eq}\text{-}\mathbb{N}\ (\mathsf{suc}\ m) && 0 && \doteq \emptyset \\
&\mathsf{Eq}\text{-}\mathbb{N}\ 0 && (\mathsf{suc}\ n) && \doteq \emptyset \\
&\mathsf{Eq}\text{-}\mathbb{N}\ (\mathsf{suc}\ m) && (\mathsf{suc}\ n) && \doteq \mathsf{Eq}\text{-}\mathbb{N}\ m\ n
\end{aligned}$$

Prove that, for every $n : \mathbb{N}$,

$$\mathsf{Eq}\text{-}\mathbb{N}\ n\ (\mathsf{suc}\ n) = \emptyset$$

In Lecture 4, we did most of the proof that

$$\mathsf{Eq}\text{-}\mathbb{N}\ m\ n \quad \simeq \quad m =_{\mathbb{N}} n.$$

Use this (and the fact proved above) to prove that $\neg(\mathsf{isContr}\ \mathbb{N})$.

3 $(\star\star\star)$

Show that if A is contractible, then for any $x, y \colon A$, the identity type $x = y$ is also contractible.

4 $(\star\star\star)$

Recall the first projection function

$$\mathsf{pr}_1 \quad : \quad \sum_{x:A} B(x) \to A$$

Show that pr_1 is an equivalence iff each $B(a)$ is contractible. *Hint: Use the results about identity types of Σ types we proved in a previous lecture.*

Show that for any $a : A$, the map

$$\lambda((x, y), p).\mathsf{tr}_B(p, y) \colon \mathsf{fib}_{\mathsf{pr}_1}(a) \to B(a)$$

is an equivalence.

5 $(\star\star)$

Construct for any map $f : A \to B$ an equivalence

$$e : A \simeq \sum_{y:B} \mathsf{fib}_f(y)$$

with a homotopy $H : f \sim \mathsf{pr}_1 \circ e$ witnessing that the triangle

$$\begin{tikzcd} A \arrow[rr, "e"] \arrow[dr, "f"'] & & \sum_{y:B} \mathsf{fib}_f(y) \arrow[dl, "\mathsf{pr}_1"] \\ & B & \end{tikzcd}$$

commutes.