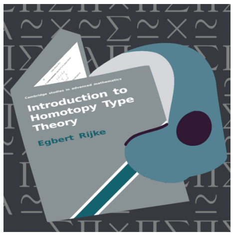

Worksheet 5 (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs

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1 $(\star)$

Consider a function $f : A \to B$. Recall that a *retraction* of f is a function $g : B \to A$ such that $g \circ f \sim \mathsf{id}_A$. Construct a function

$$\mathsf{retr}(f) \to \left(\prod_{a,a' : A} f(a) = f(a') \to a = a' \right).$$

This means that if f has a retraction, then it is an injection.

Let $\gamma : \mathsf{retr}(f)$. By Σ-induction, we may take $\gamma \doteq \langle g, r \rangle$, where $g : B \to A$ and $r : g \circ f \sim \mathsf{id}_A$. Let $a, a' : A$ and $p : f(a) = f(a')$. We must find a term of type $a = a'$. By the action of g on p, we have that $g(f(a)) = g(f(a'))$. We also have that

$$\begin{aligned} a &= g(f(a)) && (r(a)^{-1}) \\ g(f(a')) &= a'. && (r(a')) \end{aligned}$$

Finally, by transitivity of equality, we have an identity

$$r(a)^{-1} \cdot \mathsf{ap}_g(p) \cdot r(a') \ : \ a = a'.$$

2 $(\star\star)$

Consider a commuting triangle

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & X & \end{tikzcd}.$$

1. Suppose that h has a section $s : B \to A$. Prove that $f \circ s \sim g$ and that $\mathsf{sec}(f) \leftrightarrow \mathsf{sec}(g)$.
2. Suppose that g has a retraction $r : X \to B$. Prove that $r \circ f \sim h$ and that $\mathsf{retr}(f) \leftrightarrow \mathsf{retr}(h)$.
3. Prove that if any two of f, g, and h are equivalences, then so is the third.
4. Prove that any retraction or section of an equivalence is itself an equivalence.

1. **First, note that**
$$f(s(b)) = g(h(s(b))) = g(b)$$
for all $b : B$. This means that $f \circ s \sim g$.
Next, suppose that f has a section $s_f : X \to A$. Then
$$g(h(s_f(x))) = f(s_f(x)) = x$$
for all $x : X$. Hence $h \circ s_f$ is a section of g.
Conversely, suppose that g has a section $s_g : X \to B$. Then
$$f(s(s_g(x))) = g(s_g(x)) = x$$
for all $x : X$. Hence $s \circ s_g$ is a section of f.

2. **First, note that**
$$r(f(a)) = r(g(h(a))) = h(a)$$
for all $a : A$. This means that $r \circ f \sim h$.
Next, suppose that f has a retraction $r_f : X \to A$. Then
$$r_f(g(h(a))) = r_f(f(a)) = a$$
for all $a : A$. Hence $r_f \circ g$ is a retraction of h.
Conversely, suppose that h has a retraction $r_h : B \to A$. Then
$$r_h(r(f(a))) = r_h(h(a)) = a$$
for all $a : A$. Hence $r_h \circ r$ is a retraction of f.

3. **We have three cases to consider. For any equivalence e, let r_e and s_e denote its retraction and section, respectively.**
   - **Suppose that h and g are equivalences. Then f has a section by part (1) and a retraction by part (2). Thus, it's an equivalence.**
   - **Suppose that f and h are equivalences. Part (1) implies that g has a section. Moreover, since $f \circ s_h \sim g$ by part (1),**
   $$h(r_f(g(b))) = h(r_f(f(s_h(b)))) = h(s_h(b)) = b$$
   for all $b : B$. Thus, $h \circ r_f$ is a retraction of g, so that g is an equivalence.
   - **Suppose that f and g are equivalences. Part (2) implies that h has a retraction. Moreover, since $r_g \circ f \sim h$ by part (2),**
   $$h(s_f(g(b))) = r_g(f(s_f(g(b)))) = r_g(g(b)) = b$$
   for all $b : B$. Thus, $s_f \circ g$ is a section of h, so that h is an equivalence.

4. **Notice a retraction r_e and a section s_e of an equivalence $e : N \to M$ are commuting triangles $\mathsf{id}_N \sim r_e \circ e$ and $\mathsf{id}_M \sim e \circ s_e$, respectively.**

3 (⋆⋆)

Consider the type Bool, generated by

$$\begin{aligned} \mathtt{true} &: \mathsf{Bool} \\ \mathtt{false} &: \mathsf{Bool}. \end{aligned}$$

Define the type family $\mathsf{Eq\text{-}bool} : \mathsf{Bool} \to \mathsf{Bool} \to \mathcal{U}_0$ by

$$\begin{aligned} \mathsf{Eq\text{-}bool}(\mathtt{true}, \mathtt{true}) &:= \mathbb{1} \\ \mathsf{Eq\text{-}bool}(\mathtt{true}, \mathtt{false}) &:= \emptyset \\ \mathsf{Eq\text{-}bool}(\mathtt{false}, \mathtt{false}) &:= \mathbb{1} \\ \mathsf{Eq\text{-}bool}(\mathtt{false}, \mathtt{true}) &:= \emptyset. \end{aligned}$$

For every $b, b' : \mathsf{Bool}$, define $\varphi_{b,b'} : (b = b') \to \mathsf{Eq\text{-}bool}(b, b')$ by path induction. Prove that $\varphi_{b,b'}$ is an equivalence.

Let's construct an inverse $\psi_{b,b'}$ of $\varphi_{b,b'}$ by double induction on Bool:

$$\begin{aligned} \psi_{\mathtt{true},\mathtt{true}} &: \mathbb{1} \to (\mathtt{true} = \mathtt{true}) \\ \psi(*) &:= \mathsf{refl}_{\mathtt{true}} \end{aligned}$$

$$\begin{aligned} \psi_{\mathtt{true},\mathtt{false}} &: \emptyset \to (\mathtt{true} = \mathtt{false}) \\ \psi(x) &:= \emptyset\text{-}\mathsf{ind}(x) \end{aligned}$$

$$\begin{aligned} \psi_{\mathtt{false},\mathtt{false}} &: \mathbb{1} \to (\mathtt{false} = \mathtt{false}) \\ \psi(*) &:= \mathsf{refl}_{\mathtt{false}} \end{aligned}$$

$$\begin{aligned} \psi_{\mathtt{false},\mathtt{true}} &: \emptyset \to (\mathtt{false} = \mathtt{true}) \\ \psi(x) &:= \emptyset\text{-}\mathsf{ind}(x). \end{aligned}$$

By double induction on Bool, it's straightforward to check that $\varphi_{b,b'} \circ \psi_{b,b'} \sim \mathsf{id}_{\mathsf{Eq\text{-}bool}(b,b')}$. Moreover, by path induction on $b = b'$, it's straightforward to check that $\psi_{b,b'} \circ \varphi_{b,b'} \sim \mathsf{id}_{b=b'}$.

It is easy to show that $\neg(\mathbb{1} = \emptyset)$. As a consequence, we can prove that $\neg(b = \mathsf{neg\text{-}bool}(b))$ for every $b : \mathsf{Bool}$.

4 $(\star\star)$

Prove that for all $b : \mathsf{Bool}$,

$$\neg\mathsf{is\text{-}equiv}(\mathsf{const}_b).$$

Also, prove that

$$\mathsf{Bool} \not\simeq \mathbb{1}.$$

To prove that $\neg\mathsf{is\text{-}equiv}(\mathsf{const}_b)$, suppose that const_b is an equivalence. In particular, it has a retraction, hence is injective by Problem 1. Since

$$\mathsf{const}_b(\mathtt{true}) \doteq b \doteq \mathsf{const}_b(\mathtt{false}),$$

it follows that $\mathtt{true} = \mathtt{false}$. But we know that

$$\mathtt{true} \neq \mathsf{neg\text{-}bool}(\mathtt{true}),$$

where $\mathsf{neg\text{-}bool}(\mathtt{true}) \doteq \mathtt{false}$. This gives us an element of $\emptyset$, as required.

Likewise, to prove that $\mathsf{Bool} \not\simeq \mathbb{1}$, suppose that we have an equivalence $e : \mathsf{Bool} \to \mathbb{1}$ with retraction $r_e : \mathbb{1} \to \mathsf{Bool}$. By induction on $\mathbb{1}$, it's easy to check that $r_e(x) = r_e(*)$ for all $x : \mathbb{1}$. Then

$$\mathtt{true} = r_e(e(\mathtt{true})) = r_e(*) = r_e(e(\mathtt{false})) = \mathtt{false}.$$

Again, this gives us an element of $\emptyset$.

5 $(\star)$

Let A be a type and B be a type family over A. For each $x, y : A$, construct an inverse of the function

$$\mathsf{inv}_{x,y} : (x = y) \to (y = x)\,.$$

Further, for each $p : x = y$, construct an inverse of the function

$$\mathsf{tr}_B(p) : B(x) \to B(y).$$

We may define these inverses by path induction on $x = y$. Specifically, define $\mathsf{inv}_{x,y}^{-1} : (y = x) \to (x = y)$ by

$$\mathsf{inv}_{x,x}^{-1}(\mathsf{refl}_x) :\equiv \mathsf{refl}_x$$

and define $\mathsf{tr}_B(p)^{-1} : B(y) \to B(x)$ by

$$\mathsf{tr}_B(\mathsf{refl}_x)^{-1} :\equiv \mathsf{id}_{B(x)}.$$

Our definition of $\mathsf{tr}_B(p)^{-1}$ gives us a homotopy

$$\mathsf{tr}_B(p)^{-1} \sim \mathsf{tr}_B(p^{-1})$$

of functions $B(y) \to B(x)$.

6 $(\star)$

Let $f, g : A \to B$ and $H : f \sim g$. Prove that $\mathsf{is\text{-}equiv}(f) \leftrightarrow \mathsf{is\text{-}equiv}(g)$.

To define a function $\mathsf{is\text{-}equiv}(f) \to \mathsf{is\text{-}equiv}(g)$, let $\langle h, t_h, k, t_k \rangle : \underbrace{\mathsf{sec}(f) \times \mathsf{retr}(f)}_{\mathsf{is\text{-}equiv}(f)}$.

Note that for all $b : B$ and $a : A$,

$$\begin{aligned} g(h(b)) &= f(h(b)) && (H(h(b))^{-1}) \\ &= b && (t_h(b)) \end{aligned}$$

$$\begin{aligned} k(g(a)) &= k(f(a)) && (\mathsf{ap}_k(H(a)^{-1})) \\ &= a. && (t_k(a)) \end{aligned}$$

Thus, h is a section of g, and k is a retraction of g. Now we may take

$$\Big\langle h,\ \lambda b.H(h(b))^{-1} \cdot t_h(b),\ k,\ \lambda a.\mathsf{ap}_k(H(a)^{-1}) \cdot t_k(a) \Big\rangle \ : \ \overbrace{\mathsf{sec}(g) \times \mathsf{retr}(g)}^{\mathsf{is\text{-}equiv}(g)}$$

as our element of $\mathsf{is\text{-}equiv}(g)$. The function in the other direction is similar.

7 $(\star\star)$

Suppose that $e, e' : A \to B$ are equivalences and that $H : e \sim e'$. Let s and s' denote the sections of e and e', respectively. Prove that s and s' are homotopic.

Recall that any section of an equivalence is also a retraction of it. Therefore, we see that

$$s(b) = s'(e'(s(b))) = s'(e(s(b))) = s'(b)$$

for all $b : B$.