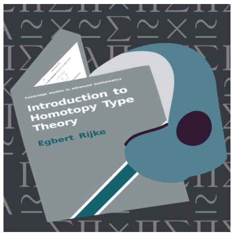

Worksheet 5

HoTTEST Summer School 2022

The HoTTEST TAs, and

21 July 2022

1 $(\star)$

Consider a function $f : A \to B$. Recall that a *retraction* of f is a function $g : B \to A$ such that $g \circ f \sim \mathsf{id}_A$. Construct a function

$$\mathsf{retr}(f) \to \left(\prod_{a,a' : A} f(a) = f(a') \to a = a' \right).$$

This means that if f has a retraction, then it is an injection.

2 (★★)

Consider a commuting triangle

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & X & \end{tikzcd} .$$

1. Suppose that h has a section $s : B \to A$. Prove that $f \circ s \sim g$ and that $\mathsf{sec}(f) \leftrightarrow \mathsf{sec}(g)$.
2. Suppose that g has a retraction $r : X \to B$. Prove that $r \circ f \sim h$ and that $\mathsf{retr}(f) \leftrightarrow \mathsf{retr}(h)$.
3. Prove that if any two of f, g, and h are equivalences, then so is the third.
4. Prove that any retraction or section of an equivalence is itself an equivalence.

Suppose that C is a type and that the canonical map $C \to \mathbb{1}$ is an equivalence. Consider a commuting diagram of the form

$$\begin{tikzcd} D \arrow[r, "s"'] \arrow[rr, bend left, "\mathsf{id}_D"] & C \arrow[r, "r"'] & D \end{tikzcd} .$$

Here, we call D a *retract* of C. Note that the diagram

$$\begin{tikzcd} D \arrow[r] \arrow[dr] & C \arrow[r] \arrow[d] & D \arrow[dl] \\ & \mathbb{1} & \end{tikzcd}$$

commutes. Therefore, by part (1), the map $D \to \mathbb{1}$ has a section. By part (2), it also has a retraction. Hence it is an equivalence. (This proves that a retract of a *contractible* type, to be defined in Lecture 6, is itself contractible.)

3 (⋆⋆)

Consider the type Bool, generated by

$$\begin{aligned} \texttt{true} &: \mathsf{Bool} \\ \texttt{false} &: \mathsf{Bool}. \end{aligned}$$

Define the type family $\mathsf{Eq\text{-}bool} : \mathsf{Bool} \to \mathsf{Bool} \to \mathcal{U}_0$ by

$$\begin{aligned} \mathsf{Eq\text{-}bool}(\texttt{true}, \texttt{true}) &:\equiv \mathbb{1} \\ \mathsf{Eq\text{-}bool}(\texttt{true}, \texttt{false}) &:\equiv \emptyset \\ \mathsf{Eq\text{-}bool}(\texttt{false}, \texttt{false}) &:\equiv \mathbb{1} \\ \mathsf{Eq\text{-}bool}(\texttt{false}, \texttt{true}) &:\equiv \emptyset. \end{aligned}$$

For every $b, b' : \mathsf{Bool}$, define $\varphi_{b,b'} : (b = b') \to \mathsf{Eq\text{-}bool}(b, b')$ by path induction. Prove that $\varphi_{b,b'}$ is an equivalence.

It is easy to show that $\neg(\mathbb{1} = \emptyset)$. As a consequence, we can prove that $\neg(b = \mathsf{neg\text{-}bool}(b))$ for every $b : \mathsf{Bool}$.

4 (⋆⋆)

Prove that for all $b : \mathsf{Bool}$,

$$\neg\mathsf{is\text{-}equiv}(\mathsf{const}_b).$$

Also, prove that

$$\mathsf{Bool} \not\simeq \mathbb{1}.$$

5 $(\star)$

Let A be a type and B be a type family over A. For each $x, y : A$, construct an inverse of the function

$$\mathsf{inv}_{x,y} : (x = y) \to (y = x)\,.$$

Further, for each $p : x = y$, construct an inverse of the function

$$\mathsf{tr}_B(p) : B(x) \to B(y).$$

6 $(\star)$

Let $f, g : A \to B$ and $H : f \sim g$. Prove that $\mathsf{is\text{-}equiv}(f) \leftrightarrow \mathsf{is\text{-}equiv}(g)$.

7 $(\star\star)$

Suppose that $e, e' : A \to B$ are equivalences and that $H : e \sim e'$. Let s and s' denote the sections of e and e', respectively. Prove that s and s' are homotopic.