Hottest Summer school 2022 Lectures 10–12
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What you will learn:

- propositional truncations
- the univalence axiom
- univalent combinatorics
- How to express yourself in type theory
- How to think carefully about concepts and getting it right.
- How to spot univalence in your daily life.

Consider a map $f: A \to B$. If we want to say that f is surjective using the Curry-Howard interpretation, then we would express this in type theory as

$$\prod_{(b:B)} \sum_{(a:A)} f(a) = b$$

This asserts that "for all $b:B$ there is an $a:A$ equipped with an identification $f(a) = b$".

However, note that this captures something stronger than surjectivity: if f is surjective in this Curry-Howard sense, then we obtain

$g : B \to A$
$H : f \circ g \sim id$.

In order to properly express that f is surjective, we need a way to say that "for every $b:B$ there is an <u>unspecified</u> $a:A$ s.t. $f(a) = b$.

The Curry-Howard interpretation is <u>not</u> appropriate to express surjectivity.

Consider a type A. To count the elements of A is to obtain a number $n : \mathbb{N}$ and an equivalence

$$\mathrm{Fin}(n) \simeq A.$$

We define $\mathrm{count}(A) := \sum_{(n:\mathbb{N})} \mathrm{Fin}(n) \simeq A$

However, if A comes equipped with a counting $(n, e) : \mathrm{count}(A)$, then the type A inherits an ordering from the type $\mathrm{Fin}(n)$ via the equivalence e

So, types equipped with a counting are totally ordered finite types.

In order to express <u>only</u> that A is finite, we need a way to say that A has an unspecified counting.

Finally, we might want to say that a type A is inhabited, i.e. that it has an unspecified element. We would like that "A is inhabited" is a proposition in type theory.

We will introduce a new operation to type theory for this purpose:

The propositional truncation.

The propositional truncation operation is specified by its universal property.
Often in mathematics, when you introduce a new object, you first want to specify the characterizing features that you want your object to have. If you do this well, the specification will determine the object uniquely
Universal properties are a common way to give such specifications.

The propositional truncation of a type A should be a proposition P s.t.

- If A is inhabited, then P is true (the proposition $\mathbf{1}$?)
- If Q is a proposition s.t. $A \to Q$, then $P \to Q$. (so probably not $\mathbf{1}$)

Definition. A map $f : A \to P$ into a proposition P is said to be a propositional truncation of A if for every proposition Q the map

$$(P \to Q) \to (A \to Q)$$

given by $h \mapsto h \circ f$ is an equivalence.

Rmk. Since Q is a proposition, it follows that both $(P \to Q)$ and $(A \to Q)$ are propositions. So the map

$$(P \to Q) \to (A \to Q)$$

is an equivalence iff.

$$(A \to Q) \to (P \to Q)$$

The universal property of propositional truncations says that every map $g : A \to Q$ extends uniquely along f

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "g"] & \\ P \arrow[r, dashed] & Q \end{tikzcd}$$

Proposition. Consider $f : A \to P$ and $f' : A \to P'$, where P and P' are propositions. Consider the following three conditions:

(i) f is a propositional truncation

(ii) f' is a propositional truncation

(iii) P and P' are equivalent.

Proof. Let Q be a proposition, and spse $P \simeq P'$. Then we have a triangle

$$\begin{tikzcd} & (A \to Q) \arrow[dl, "- \circ f"'] \arrow[dr, "- \circ f'"] & \\ (P \to Q) \arrow[rr, "\simeq"] & & (P' \to Q) \end{tikzcd}$$

By the 3-for-2 property of equivs, it follows that the left map is an equiv iff the right map is an equiv. This shows that (iii) implies (ii) $\Leftrightarrow$ (i)

If (i) and (ii) hold, then $(P \to P') \simeq (A \to P')$ gives $P \to P'$ and $(P' \to P) \simeq (A \to P)$ gives $P' \to P$ $\square$

From now on we will assume that for each type A there is a proposition $\|A\|$ equipped with a map $\eta : A \to \|A\|$ (the <u>unit</u> of the propositional truncation) satisfying the universal property that

$$(\|A\| \to Q) \to (A \to Q)$$

is an equivalence for each proposition Q.

Lemma. Consider $f : A \to B$. Then we have $\|f\| : \|A\| \to \|B\|$.

Proof. We have $(\|A\| \to \|B\|) \simeq (A \to \|B\|)$

Note that $\eta \circ f : A \to \|B\|$. This gives $\|f\| : \|A\| \to \|B\|$.

Cor. If $A \simeq B$ then we get $\|A\| \leftrightarrow \|B\|$, and hence $\|A\| \simeq \|B\|$.

Logic in type theory.

Recall that if P and Q are propositions, then $P+Q$ is a proposition iff $P \to \neg Q$.

Proof. Let $x, y : P+Q$. WTS $x = y$.

$\mathrm{inl}(x) = \mathrm{inl}(y)$ since $x = y$ in P.

$\mathrm{inl}(x) = \mathrm{inr}(y)$ since $x : P$ and $y : Q$ gives $\emptyset$.

$\mathrm{inr}(x) = \mathrm{inl}(y)$ same

$\mathrm{inr}(x) = \mathrm{inr}(y)$ since $x = y$ in Q. $\square$

We see that $P+Q$ is not always a proposition.

Defn. Given $P, Q :$ Prop, we define the disjunction $P \vee Q := \| P + Q \|$.

Proposition. For any $P, Q, R :$ Prop we have

$$(P \vee Q \to R) \simeq (P \to R) \times (Q \to R)$$

Proof. $(P \vee Q \to R) \doteq (\| P + Q \| \to R) \simeq (P + Q \to R) \simeq$

$(P \to R) \times (Q \to R)$ (ex. 13.8). $\square$

Consider a type A and $P : A \to \mathrm{Prop}$.
Then $\sum_{(a:A)} P(a)$ is a proposition iff
$$\prod_{(x,y:A)} P(x) \to P(y) \to x = y.$$
Indeed $\left(\sum_{(a:A)} P(a)\right) \to A$ is an embedding.
So for $x, y : A$, $p : P(x)$, $q : P(y)$ we have
$$(x,p) = (y,q)$$
This implies $\sum_{(x:A)} P(x)$ is a proposition. □

By the above observation, $\sum_{(x:A)} P(x)$ is rarely a proposition. So, if we want to say that there exists an $x : A$ s.t $P(x)$ holds, then we must truncate.

Definition. Consider $P : A \to \mathrm{Prop}$. We define
$$\exists_{(x:A)} P(x) := \left\| \sum_{(x:A)} P(x) \right\|.$$

Proposition. Consider $P : A \to \mathrm{Prop}$, and $Q : \mathrm{Prop}$.
Then
$$\left(\exists_{(x:A)} P(x) \to Q\right) \simeq \left(\prod_{(x:A)} P(x) \to Q\right).$$

Proof. $(\exists_{(x:A)} P(x) \to Q)$

$$\doteq \left\| \sum_{(x:A)} P(x) \right\| \to Q$$

$$\simeq \left(\sum_{(x:A)} P(x) \right) \to Q$$

$$\simeq \prod_{(x:A)} (P(x) \to Q).$$

$\square$

Logical connective	interpretation
$\top$	$\mathbb{1}$
$\bot$	$\emptyset$
$P \Rightarrow Q$	$P \to Q$
$P \wedge Q$	$P \times Q$
$P \vee Q$	$\lVert P + Q \rVert$
$P \Leftrightarrow Q$	$(P \to Q) \times (Q \to P)$
$\exists_{(x:A)} P(x)$	$\lVert \sum_{(x:A)} P(x) \rVert$
$\forall_{(x:A)} P(x)$	$\prod_{(x:A)} P(x)$.

The image of a map.

Defn. Let $f : A \to B$. We define

$$\mathrm{im}(f) := \Sigma_{(b:B)} \| \mathrm{fib}_f(b) \|.$$

Question. What would happen if we defined the image of f to be

$$\Sigma_{(b:B)} \mathrm{fib}_f(b) \ ?$$

$$\begin{aligned}
\Sigma_{(b:B)} \mathrm{fib}_f(b) &\doteq \Sigma_{(b:B)} \Sigma_{(x:A)} f(x) = b \\
&\simeq \Sigma_{(x:A)} \Sigma_{(y:B)} f(x) = y \\
&\simeq A.
\end{aligned}$$

☹

We also define

$$g_f : A \to \mathrm{im}(f) \qquad g_f(a) := (f(a), \eta(a, \mathrm{refl}))$$
$$i_f : \mathrm{im}(f) \to B \qquad i_f(x) := \mathrm{pr}_1(x).$$

The map i_f is an embedding, because $\|\mathrm{fib}_f(y)\|$ is a proposition for each $y : B$.
The map g_f is surjective.
Defn. A map $f : A \to B$ is said to be surjective if it has an element

$$\text{is-surj}(f) := \prod_{b:B} \|\mathrm{fib}_f(b)\|.$$

g_f is surjective. Let $(b, x) : \mathrm{im}(f)$
$b : B$, $x : \|\mathrm{fib}_f(b)\|$.
We have $(\|\mathrm{fib}_f(b)\| \to \|\mathrm{fib}_{g_f}(b,x)\|) \simeq$

$$\mathrm{fib}_f(b) \to \|\mathrm{fib}_{g_f}(b,x)\|.$$

so by the universal property we may assume $a : A$, $p : f(a) = b$.

STS $\quad \mathrm{fib}_{q_f}(f(a), \eta(a, \mathrm{refl}))$

i.e. $\quad \mathrm{fib}_{q_f}(q_f(a))$ $\square$

Theorem. Let $f : A \to B$ be surjective. Let $P : B \to \mathrm{Prop}$. Then

$$\left(\prod_{(b:B)} P(b)\right) \xrightarrow{\simeq} \left(\prod_{(x:A)} P(f(x))\right)$$

Proof. It suffices to construct

$$\left(\prod_{(x:A)} P(f(x))\right) \to \prod_{(b:B)} P(b)).$$

Let $h : \prod_{(x:A)} P(f(x))$, $\quad b : B$.

then $\|\mathrm{fib}_f(b)\|$ holds. We're proving $P(b)$.
Note: $(\|\mathrm{fib}_f(b)\| \to P(b)) \xrightarrow{\simeq} (\mathrm{fib}_f(b) \to P(b))$

Let $a : A$, $p : f(a) = b$. Then we have $h(a) : P(f(a))$ and $\mathrm{tr}_P(p, h(a)) : P(b)$.

$\square$

Defn. We say that a type A is finite if it comes equipped with an elt of type
$\text{is-finite}(A) := \| \sum_{(n:\mathbb{N})} \text{Fin}(n) \simeq A \|$.

Rmk. (Thm. 16.3.3)
$\text{is-finite}(A) \simeq \sum_{(n:\mathbb{N})} \| \text{Fin}(n) \simeq A \|$

Defn. $\mathbb{F} := \sum_{(X:\mathcal{U}_0)} \text{is-finite}(X)$.

Also define $BS_n := \sum_{(X:\mathcal{U}_0)} \| \text{Fin}(n) \simeq X \|$.

(book writes $\mathbb{F}_n$ for BS_n).

Eg. Let A, B be finite. Then $A+B$ is finite.
Assume $H : \| \sum_{n:\mathbb{N}} \text{Fin}(n) \simeq A \|$, $K : \| \sum_{(n:\mathbb{N})} \text{Fin}(n) \simeq B \|$
We're proving a proposition, so assume
$(n, e) : \sum_{(n:\mathbb{N})} \text{Fin}(n) \simeq A$, $(m, f) \simeq \sum_{(n:\mathbb{N})} \text{Fin}(n) \simeq B$.

Claim. $\text{Fin}_{n+m} \simeq A + B$.

$\text{Fin}_{n+m} \simeq \text{Fin}_n + \text{Fin}_m \simeq A + B$ $\square$

The univalence axiom.

Defn. For any two types A and B in $\mathcal{U}$ we define the map

$$\text{equiv-eq} : (A = B) \to (A \simeq B)$$

by $\text{equiv-eq}(\text{refl}) := \text{id}$.
The univalence axiom asserts that this map is an equivalence for all $A, B : \mathcal{U}$.

Remarks:
- The univalence axiom is a characterization of the identity type of a universe.
- By the fundamental theorem of identity types, the univalence axiom holds iff

$$\sum_{(B:\mathcal{U})} A \simeq B$$

is contractible for each $A : \mathcal{U}$.
- The univalence axiom is often popularized as the statement $(A = B) \simeq (A \simeq B)$

Q. Is this way of phrasing univalence equivalent to univalence?

A. Yes. If we have $(A = B) \simeq (A \simeq B)$ for all $A, B : \mathcal{U}$, then

$$\left(\sum_{(B:\mathcal{U})} A = B\right) \simeq \left(\sum_{(B:\mathcal{U})} A \simeq B\right)$$

so it follows that $\sum_{(B:\mathcal{U})} A \simeq B$ is contractible.

- The univalence axiom implies function extensionality (Voevodsky). For a proof, see Theorem 17.2.2.

Recall that $\mathrm{Prop}_{\mathcal{U}} := \sum_{(X:\mathcal{U})} \text{is-prop}(X)$.
The type is-prop (X) is itself a proposition, so it follows that the map

$$pr_1 : \left(\sum_{(X:\mathcal{U})} \text{is-prop}(X)\right) \to \mathcal{U}$$

is an embedding. So we have equivalences

$$\begin{aligned}
(P =_{\mathrm{Prop}} Q) &\simeq (pr_1(P) = pr_1(Q)) \\
&\simeq (pr_1(P) \simeq pr_1(Q)) \\
&\simeq (pr_1(P) \leftrightarrow pr_1(Q)) \\
&\text{``}\doteq (P \leftrightarrow Q)\text{''} \quad \text{by convention}
\end{aligned}$$

Thm. (Propositional extensionality). For any two propositions P and Q, the map

$$\text{iff-eq} : (P = Q) \longrightarrow (P \leftrightarrow Q)$$

given by $\text{iff-eq}(\text{refl}) := (\text{id}, \text{id})$ is an equivalence. $\square$

Cor. Consider two subtypes $P, Q : A \to \mathrm{Prop}$.
Then $(P = Q) \simeq \prod_{(a:A)} P(a) \leftrightarrow Q(a)$.
(subtypes are equal iff they contain the same elements.)

We can also show that $(A \to \mathrm{Prop}) \simeq \sum_{(X:\mathcal{U})} X \hookrightarrow A$.

Thm. For any $A : \mathcal{U}$, the map

$$\varphi : \left(\sum_{(X:\mathcal{U})} X \to A \right) \to (A \to \mathcal{U})$$

given by $(X, f) \mapsto \mathrm{fib}_f$ is an equivalence.

Proof. Define $\psi : (A \to \mathcal{U}) \to \sum_{(X:\mathcal{U})} X \to A$ by

$$\psi(B) :\equiv \left(\sum_{(a:A)} B(a),\ \mathrm{pr}_1 \right)$$

- $\varphi(\psi(B)) = B$.

Lemma. Let $B, C : A \to \mathcal{U}$. Then $(B = C) \simeq \prod_{(a:A)} B(a) \simeq C(a)$.

Proof.

$$\left(\sum_{(C:A\to\mathcal{U})} \prod_{(a:A)} B(a) \simeq C(a) \right) \simeq \prod_{(a:A)} \sum_{(X:\mathcal{U})} B(a) \simeq X$$

is a product of contractible types, so it is contractible. $\square$

$\varphi(\psi(B))(a) \equiv \mathrm{fib}_{\mathrm{pr}_1}(a) \simeq B(a)$ by exercise 10.7. ✓

- $\psi(\varphi(X,f)) = (X,f)$.

Lemma. For $(X,f), (Y,g) : \sum_{(Z:\mathcal{U})} Z \to A$, we have

$$\big((X,f) = (Y,g)\big) \simeq \sum_{(e : X \simeq Y)} f \sim g \circ e .$$

Proof. $$\sum_{(Y:\mathcal{U})} \sum_{(g:Y\to \mathcal{U})} \sum_{(e:X\simeq Y)} f \sim g\circ e$$

$$\simeq \sum_{(Y:\mathcal{U})} \sum_{(e:X\simeq Y)} \sum_{(g:Y\to A)} f \sim g\circ e$$

$$\simeq \sum_{g:X\to A} f \sim g \qquad \text{(use center } (X, \mathrm{id})\text{)}$$

$$\simeq \mathbf{1}. \qquad \square$$

$$\psi(\varphi(X,f)) \doteq \Big(\sum_{a:A} \mathrm{fib}_f(a), \mathrm{pr}_1 \Big)$$

$$\begin{tikzcd} X \arrow[rr, "x \mapsto (f(x),(x,\mathrm{refl}))", "\simeq \text{ by last lecture}"'] \arrow[dr, "f"'] & \circlearrowleft & \sum_{(a:A)} \mathrm{fib}_f(a) \arrow[dl, "\mathrm{pr}_1"] \\ & A & \end{tikzcd}$$

$\square$

Prp. The type $\sum_{(X:BS_2)} X$ is contractible.

Proof. By the univalence axiom it follows that
$$\sum_{(X:BS_2)} \mathrm{Fin}_2 \simeq X$$ is contractible.

Therefore it suffices to show that the
$$(\mathrm{Fin}(2) \simeq X) \longrightarrow X \qquad e \mapsto e(0)$$
is an equivalence. Since being an equivalence is a property, and since we assumed $\|\mathrm{Fin}(2) \simeq X\|$ we may assume $\alpha : \mathrm{Fin}(2) \simeq X$.

Therefore it suffices to show that $(\mathrm{Fin}_2 \simeq \mathrm{Fin}_2) \to \mathrm{Fin}_2$ given by $e \mapsto e(0)$ is an equivalence.

We define $f : \mathrm{Fin}_2 \to (\mathrm{Fin}_2 \simeq \mathrm{Fin}_2)$ by
$$f(0) :\equiv \mathrm{id}$$
$$f(1) :\equiv \mathrm{succ}$$
and this is the inverse of $\mathrm{ev}(0)$. $\square$

Cor. There is no dependent function $\prod_{(X:BS_2)} X$.

Proof. Since $(\mathrm{Fin}_2 = X) \simeq X$, we have

$$\prod_{(X:BS_2)} (\mathrm{Fin}_2 = X) \simeq \prod_{(X:BS_2)} X$$

Note that an element of the first type provides a proof is-contr(BS_2). However, BS_2 cannot be contractible, because $(\mathrm{Fin}_2 = \mathrm{Fin}_2) \simeq \mathrm{Fin}_2$, which is not contractible. $\square$

Cor. There can be no dependent function

$$\prod_{X:\mathcal{U}} \|X\| \to X.$$

Proof. If there is such a function, then we can restrict it to the 2-element types, i.e, we obtain

$\prod_{X:BS_2} \|X\| \to X$ which is equivalent to $\prod_{X:BS_2} X$. $\square$

We see that global choice is inconsistent with univalence.

However Voevodsky's simplicial model of univalence does satisfy the usual axiom of choice: for every set A and every family of sets B over A,

$$\left(\prod_{x:A} \|B(x)\|\right) \to \left\|\prod_{x:A} B(x)\right\|$$

Thm. There is no dependent function

$$\prod_{(X:\mathcal{U})} \text{is-decidable}(X).$$

Proof. If we had such a function, and $X : BS_2$, then we can use $\|X\|$ to conclude X.

Indeed, $\|A\| \to (A + \neg A) \to A$ since $\|A\| \to \neg\neg A$

Thus it follows that $\prod_{(x : BS_2)} X$, which is impossible.

$\square$