Outline

Last time: homotopies & equivalences (§9)

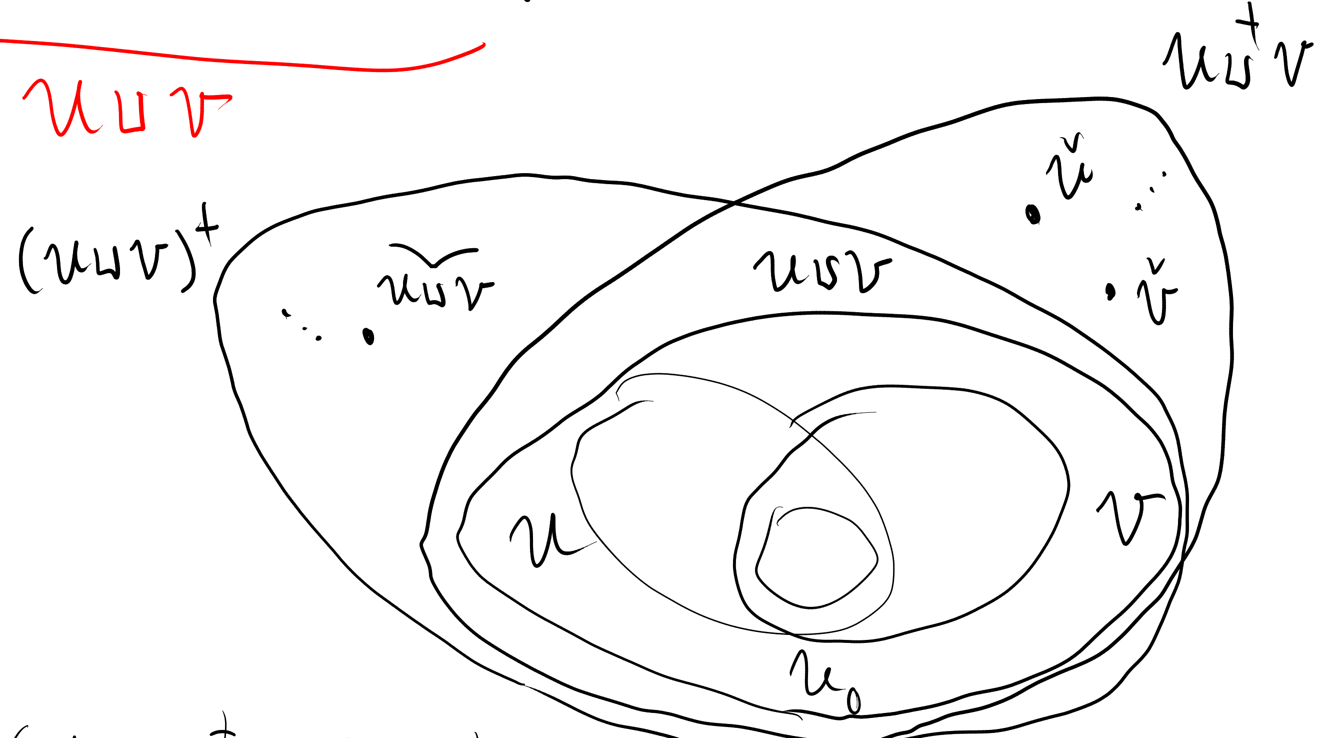
Today:
- A correction on joins of universes
- Contractible types & maps (§10)
  - Main result: equivs as contr. maps

    via: coherently invertible maps

A correction on joins of universes

In Lecture 4 I defined the join of $(\mathcal{U},\mathcal{T}_{\mathcal{U}}),(\mathcal{V},\mathcal{T}_{\mathcal{V}})$ as the universe obtained by reflecting 4 type families

$$\left.\begin{array}{l}\cancel{\vdash \mathcal{U}\ \mathrm{type}} \qquad\qquad \cancel{\vdash \mathcal{V}\ \mathrm{type}}\\ \underbrace{X:\mathcal{U}\vdash \mathcal{T}_{\mathcal{U}}(X)\ \mathrm{type} \qquad X:\mathcal{V}\vdash \mathcal{T}_{\mathcal{V}}(X)\ \mathrm{type}}_{\rightsquigarrow\ \mathcal{U}\sqcup\mathcal{V}}\end{array}\right\}\rightsquigarrow \mathcal{U}\sqcup^{+}\mathcal{V}$$



- In Agda, we use universe polymorphism to give types to type formers, e.g.,

$$+ : \mathcal{U}\to\mathcal{V}\to\mathcal{U}\sqcup\mathcal{V}$$

Want $\mathcal{U}\sqcup\mathcal{U}\doteq\mathcal{U}$

$\rightsquigarrow$ join semi-lattices w/ $+$, $(\mathcal{U}\sqcup\mathcal{V})^{+}=\mathcal{U}^{+}\sqcup\mathcal{V}^{+}$ etc.

Contractible types

We use contractibility to capture uniqueness / unique existence

$\rightsquigarrow$ (look up talk by Emily Riehl – Topos Colloquium)

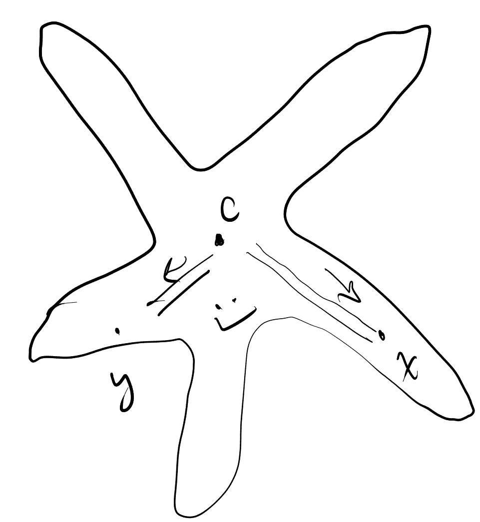
Def For any type A, let $\text{is-contr}(A) :\equiv \sum_{c:A} \prod_{x:A} c = x$

(props as types interpretation of

"there is a $c:A$ s.t. every $x:A$ is equal to c")

c is called the center of contraction

$C : \underset{(A \to A)}{\text{const}_c \sim \text{id}_A}$ is called the contraction



exercise:
If A is contr., then $x =_A y$ is as well.

Example 1 The unit type $\mathbb{1}$ is contr. w/ $c :\equiv *$, $C : \prod_{x:\mathbb{1}} * = x$

($\mathbb{1}$-induction)

Example 2 For a type A, w/ $a : A$,

the Σ-type $\sum_{x:A} a = x$ is contr. w/ center (a, refl_a)

Observation A is contr. iff $! = \mathrm{const}_* : A \to \mathbb{1}$ is an equiv.

"$\longrightarrow$" Have c, $C : \mathrm{const}_c \sim \mathrm{id}_A$. Then let $g :\equiv \mathrm{const}_c : \mathbb{1} \to A$

$$g \circ ! \doteq \mathrm{const}_c \overset{C}{\sim} \mathrm{id}_A \checkmark$$

$$\& \quad ! \circ g \doteq \mathrm{const}_* \underset{\mathbb{1}\to\mathbb{1}}{\sim} \mathrm{id}_{\mathbb{1}} \checkmark$$

"$\longleftarrow$" Have $g : \mathbb{1} \to A$ inverse of $!$, let $c :\equiv g\,*$, for any $x : A$

so $$c \doteq (g \circ !)(x) = x.$$

Singleton induction

Since contr. types are equiv. to $\mathbb{1}$, they have all the str. props as $\mathbb{1}$.

Def Let A be a type w/ $a : A$. We say A satisfies singleton induction if for every type family B over A the eval. map at a:

$$\text{ev-pt}_a : \left(\prod_{x:A} B(x) \right) \longrightarrow B(a)$$

has a section, i.e., we have

$$\text{ind-sing}_a : B(a) \to \prod_{x:A} B(x)$$

$$\text{comp-sing}_a : \text{ev-pt}_a \circ \text{ind-sing}_a \sim \text{id}_{B(a)}$$

Thm A is contr. iff we have $a : A$ s.t. A sat. singleton ind.(a)

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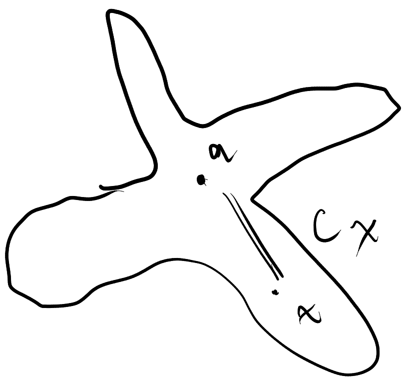
"$\longrightarrow$" We have (a, C) : let $C'(x) := (C\,a)^{-1} \cdot C\,x$

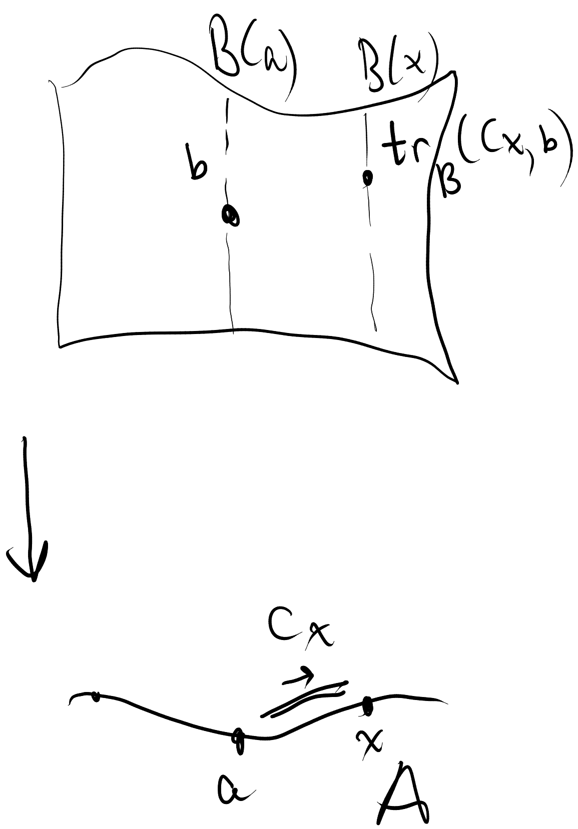


$(C\,a : a = a) \overset{?}{\Leftarrow} \mathrm{refl}_a$

$$C'(a) \doteq C\,a^{-1} \cdot C\,a = \mathrm{refl}_a .$$

WLOG assume $C_a = \mathrm{refl}_a$.

Let now B be given



Def. $s : B(a) \to \prod_{x:A} B(x)$

$$s\ b\ x := \mathrm{tr}_B(C\,x, b)$$

Then for any b, $s\ b\ a \doteq \mathrm{tr}_B(C\,a, b)$

$$= \mathrm{tr}_B(\mathrm{refl}_a, b) \doteq b \ \checkmark$$

"$\leftarrow$" Suppose A w/ $a : A$ sat. sing. ind.

of course, pick a as center.

to show $\prod_{x:A} a = x$

by singleton ind. applied to $B := x.\ a = x$

it suffices to give $\underset{\substack{\uparrow \\ \mathrm{refl}_a.}}{\_} : B(a) \doteq a = a$

$\square$

Fibers & contractible map

<u>**Def**</u> Let $f : A \to B$, $b : B$. The <u>fiber of f at b</u> is

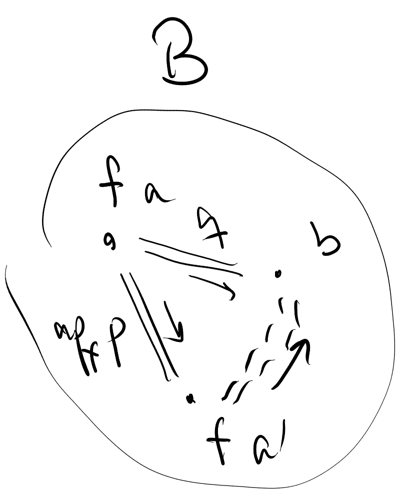
$$\mathrm{fib}_f(b) :\equiv \sum_{x:A} f\,x = b$$

(prop as types version of "b is in the image of f" / preimage of f at b.)

<u>**Obs**</u> For $p : a =_A a'$, we have $q : f\,a = b$

$$\mathrm{tr}_{x.\, f x = b}(p, q) = (\mathrm{ap}_f\, p)^{-1} \cdot q$$

(pf by path. ind)



<u>**Cor**</u> For all $(a, q), (a', q') : \mathrm{fib}_f(b)$ we have

$$\big((a, q) = (a', q')\big) \simeq \sum_{p : a = a'} \underbrace{q = \mathrm{ap}_f\, p \cdot q'}_{\begin{array}{c}\wr\wr \\ (\mathrm{ap}_f\, p)^{-1} \cdot q = q'\end{array}}$$

Ind. by path ind. w/ "refl"

<u>Def</u> A map $f : A \to B$ is contr. if all its fibers are,

i.e., we have elt in $\prod_{y:B} \text{is-contr}(\text{fib}_f\, y)$

<u>Thm</u> Any contr. map is an equiv.

<u>Pf</u> From centers of contr. get $\prod_{y:B} \text{fib}_f\, y \doteq \prod_{y:B} \sum_{x:A} fx = y$

by tt-choice, we get $g : B \to A$ & $G : f \circ g \sim \text{id}_B$ (sect. of f)

We want $g \circ f \sim \text{id}_A$, i.e., for all $x : A$, $(g \circ f)(x) = x$.

$f \circ g \circ f \overset{G \cdot f}{\sim} f$ we get $p : fgf(x) = f(x)$, $(gf(x), p) : \text{fib}_f(fx)$

also $(x, \text{refl}_{fx}) : \text{fib}_f(fx)$, so $(gf(x), p) = (x, \text{refl}_{fx})$

so get $gf(x) = x$, as desired $\square$.

- Coherently invertible maps

Def $f : A \to B$ is coh. inv if we have

$$\left.\begin{array}{l} \left.\begin{array}{l} g : B \to A \\ G : f \circ g \sim id_B \\ H : g \circ f \sim id_A \end{array}\right\} \text{has-inv}(f) \\ K : G \cdot f \sim f \cdot H \quad (\text{as htpies } f \circ g \circ f \sim f) \end{array}\right\} \text{is-coh-inv}(f)$$

Thm $\text{is-coh-inv}(f) \to \text{is-contr}(f)$ $\quad \left(\text{i.e., } \prod_{y:B} \text{is-contr}(\text{fib}_f\, y)\right)$

center at y : $(g\,y, G\,y)$ to show:

$$\prod_{y:B} \prod_{x:A} \prod_{q : fx = y} (g\,y, G\,y) =_{\text{fib}_f y} (x, q).$$

By path ind. it suffices to give

$$\prod_{x:A} (gf(x), G(fx)) = (x, \text{refl}_{fx})$$

By ='s in fib's, give for $x : A$

$$Hx : gf(x) = x, \quad G(fx) = \text{ap}_f(Hx) \cdot \text{refl}_{fx}$$

$$Kx \qquad \text{ap}_f(Hx) = (f \cdot H)(x)$$

$\square$

Final goal $\quad$ has-inv$(f) \to$ is-coh-inv(f)

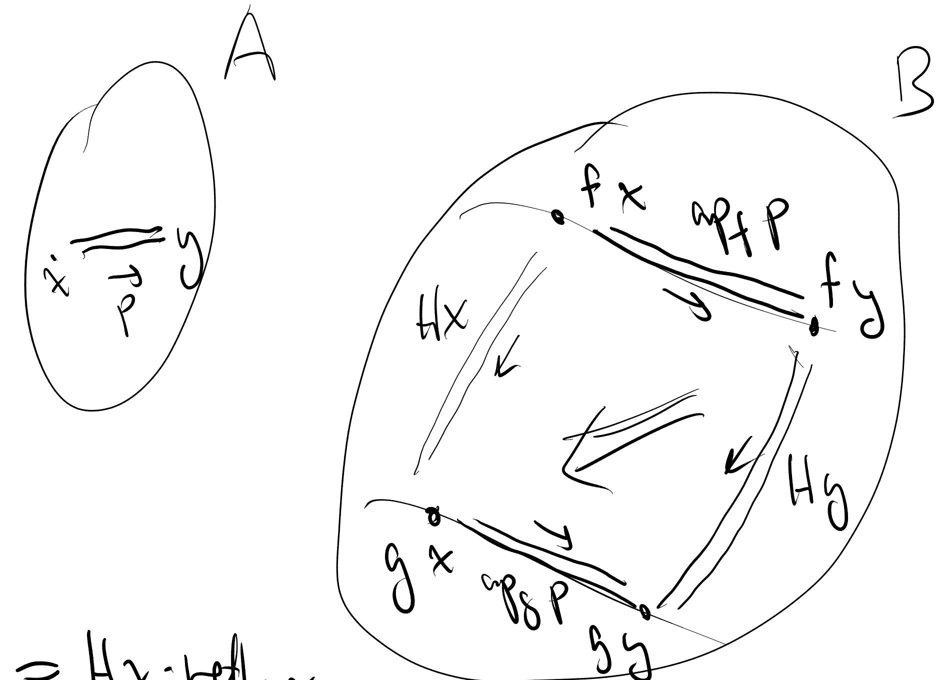
$$\begin{tikzcd}
 & \text{has-inv}(f) \arrow[dl, shift left] \arrow[dd, dashed] & \\
\text{is-equiv}(f) \arrow[ur, bend left] & & \text{is-contr}(f) \arrow[ul] \\
 & \text{is-coh-inv}(f) \arrow[ur, bend right] &
\end{tikzcd}$$

Tool: naturality squares of htpies.

<u>Def</u> $\quad f, g : A \to B, \quad H : f \sim g$

$\quad p : x =_A y$

$$\text{nat-htpy}(H, p) : ap_f\, p \cdot H y = H x \cdot ap_g\, p$$



Defn is by path ind.: it suffices to give $\quad \text{nat-htpy}(H, \text{refl}_x) : \text{refl}_{fx} \cdot H x = H x \cdot \text{refl}_{gx}$

✓

□

__Special case__ $h : A \to A$, $H : h \sim \mathrm{id}_A$

$$\begin{tikzcd} hh(x) \arrow[r, equal, "\mathrm{ap}_h(Hx)"] \arrow[d, equal, "H(hx)"'] & hx \arrow[d, equal, "Hx"] \\ hx \arrow[r, equal, "Hx"'] & x \end{tikzcd}$$

$x : A$, $\quad Hx : hx = x$

get $\quad \mathrm{ap}_h(Hx) = H(hx)$.

__Now__ Assume $f : A \to B$ has inv. $g : B \to A$, $G : fg \sim \mathrm{id}_B$
$H : gf \sim \mathrm{id}_A$

We improve G to $G' : fg \sim \mathrm{id}_B$ w/ $K : G' \cdot f \sim f \cdot H$.

Let $y : B$, $\quad G'y := \left(fg(y) \xrightarrow{(G(fg(y)))^{-1}} fgfg(y) \xrightarrow{\mathrm{ap}_f(H(gy))} fg(y) \xrightarrow{Gy} y \right)$

Let $x : A$ show

$$\begin{tikzcd} fgfgf(x) \arrow[r, equal, "\mathrm{ap}_f(H(gf(x)))"] \arrow[d, equal, "G(fgf(x))"'] & fgf(x) \arrow[d, equal, "G(fx)"] \\ fgf(x) \arrow[r, equal, "\mathrm{ap}_f(Hx)"'] & f(x) \end{tikzcd}$$

use

$$H(gf(x)) = \mathrm{ap}_{gf}(Hx) \; ; \qquad gf(x) \overset{Hx}{=} x$$

$$\begin{tikzcd} fgfgf(x) \arrow[r, equal, "\mathrm{ap}_{fgf}(Hx)"] \arrow[d, equal, "G(fgf(x))"'] \arrow[dr, phantom, "\text{nat-htpy}(G \cdot f, Hx)"] & fgf(x) \arrow[d, equal, "G(fx)"] \\ fgf(x) \arrow[r, equal, "\mathrm{ap}_f(Hx)"'] & f(x) \end{tikzcd}$$

$\square$

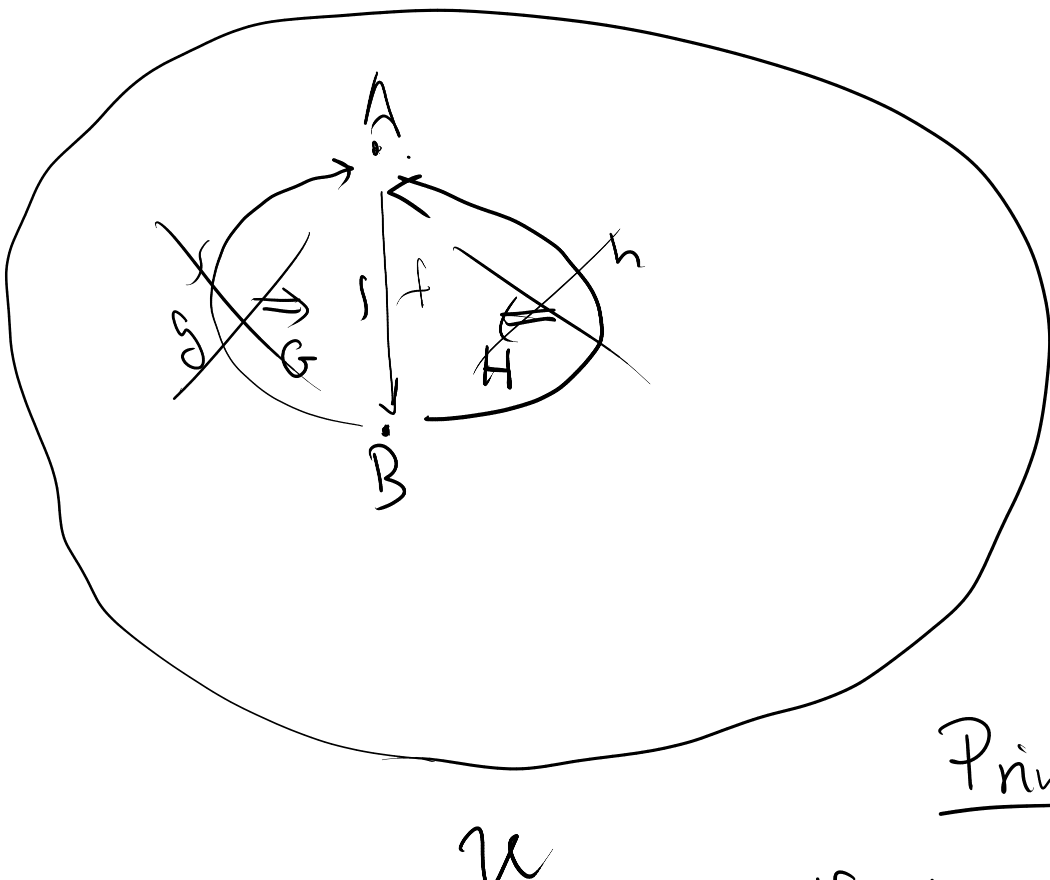
Geom intuition for why is-equiv(f) & is-coh-inv(f) are proposition

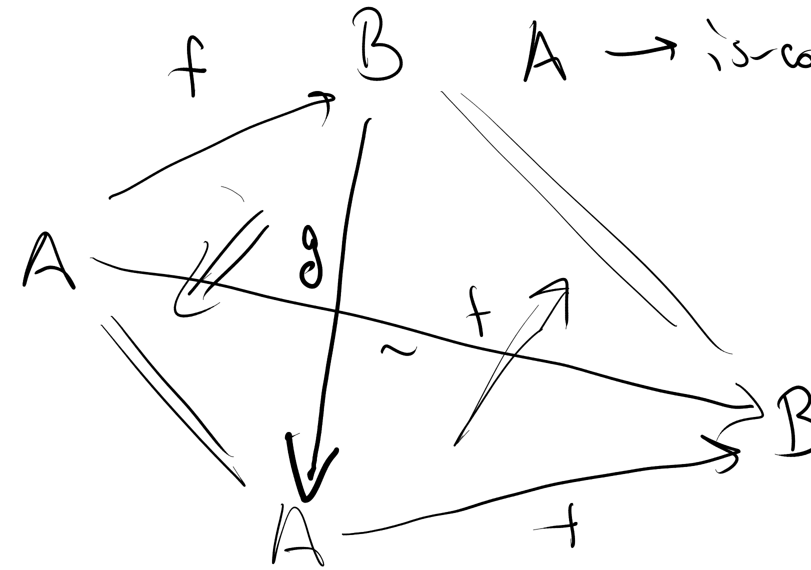
$\left(\text{is-prop}(A) := \prod_{x,y:A} x=y\right)$

$\Updownarrow$

$A \to \text{is-contr}(A)$

assume f is an equiv.





Principle of contracting away:

If A is contr., B type fam/A,

$B(c) \xrightarrow{\sim} \sum_{x:A} B(x)$, special case $\sum_{x:A} \sum_{p:a=x} B(x,p) \simeq B(a, \mathrm{refl}_a)$

$\left(\Sigma_{x:A}\, a=x\right)$