



Worksheet 12

HoTTEST Summer School 2022

The HoTTEST TAs , and
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1 (★)

For each of the following types, state how many unique¹ terms there are of that type, and list them. *Hint:* First figure out how many there are before defining them. Use Lemma 17.5.8 and think inductively!

(a)

$$\binom{\text{Fin } 3}{\text{Fin } 2}$$

(b)

$$\binom{\text{Fin } n}{\mathbb{1}}$$

(c)

$$\binom{\text{Fin } n}{\text{Fin } n + \mathbb{1}}$$

2 (★★)

Consider a type A .

- (a) We call a point $a : A$ *isolated* if the map $\text{const}_a : \mathbb{1} \rightarrow A$ is a decidable embedding. Construct an equivalence

$$\binom{A}{\mathbb{1}} \simeq \sum_{a:A} \text{is-isolated}(a).$$

- (b) Show that if A is a set, then $\binom{A}{\mathbb{1}} \simeq A$

- (c) Construct an equivalence

$$\binom{A}{\mathbb{1}} \simeq \left(\sum_{X:\mathcal{U}} (X + \mathbb{1}) \simeq A \right)$$

conclude that the map $X \mapsto X + \mathbb{1}$ on a univalent universe \mathcal{U} is 0-truncated.

¹Unique up to identity – assume univalence and function extensionality.

(d) More generally, construct an equivalence

$$\binom{A}{B} \simeq \sum_{X:\mathcal{U}_B} \sum_{Y:\mathcal{U}} X + Y \simeq A$$

3 (★★)

Given a type A , the type of **unordered pairs** in A is defined to be

$$\text{unordered-pairs}(A) := \sum_{X:BS_2} X \rightarrow A$$

(a) Construct an embedding

$$\binom{A}{\text{bool}} \hookrightarrow \text{unordered-pairs}(A)$$

Why does $\text{unordered-pairs}(A)$ have, in general, more elements than $\binom{A}{\text{bool}}$? Which elements of $\text{unordered-pairs}(A)$ are *not* in the image of this embedding?

(b) The type of homotopy commutative binary operations from A to B is defined as

$$\text{unordered-pairs}(A) \rightarrow B.$$

Show that if B is a set, then this type is equivalent to the type

$$\sum_{m:A \rightarrow A \rightarrow B} \prod_{x,y:A} m(x,y) = m(y,x).$$

4 (★★)

Consider the following claim.

$$\binom{\text{Fin}(n)}{\text{bool}} \simeq \sum_{k:\text{Fin}(n)} \text{Fin}(k) \quad (*)$$

Prove $(*)$ by induction on n , using the equivalences from Lemma 17.5.8 and the identities proved above. You should not need to unfold the definition of $\binom{A}{B}$ or explicitly construct any decidable embeddings.

5 (★★)

Given a finite type A , show that the following are equivalent:

(i) The type of all decidable equivalence relations on A

- (ii) The type of all surjective maps $A \rightarrow X$ into a finite type X
- (iii) The type of finite types X equipped with a family $Y : X \rightarrow \mathbf{Fin}$ of finite types, such that each $Y(x)$ is inhabited and equipped with an equivalence

$$e : \left(\sum_{x:X} Y(x) \right) \simeq A$$

- (iv) The type of all decidable partitions of A , i.e. the type of all $P : (A \rightarrow \mathbf{dProp}) \rightarrow \mathbf{dProp}$ such that each Q in P is inhabited, and such that for each $a : A$ the type of $Q : A \rightarrow \mathbf{dProp}$ such that $Q(a)$ holds and $P(Q)$ holds is contractible.