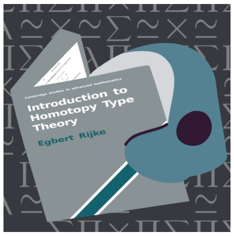

Worksheet 11 (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs

15 August 2022

1 (⋆)

Let A, B, C be types in a universe Type. Prove the following computation rules of $\mathsf{equiv\text{–}eq}$:

1. $\mathsf{equiv\text{–}eq}(\mathsf{refl}_A) = \mathsf{id}_A$;
2. for all paths $p : A = B$, $q : B = C$, we have $\mathsf{equiv\text{–}eq}(p \cdot q) = \mathsf{equiv\text{–}eq}(q) \circ \mathsf{equiv\text{–}eq}(p)$;
3. for any path $p : A = B$, we have $\mathsf{equiv\text{–}eq}(\bar{p}) = \mathsf{equiv\text{–}eq}(p)^{-1}$ where $(\bar{-})$ is path inversion.

(1) Since $\mathsf{equiv\text{–}eq}$ is defined by path induction, this equation holds definitionally.
(2) By inducting on both p and q, the goal becomes $\mathsf{equiv\text{–}eq}(\mathbf{refl}_A) = \mathbf{id}_A \circ \mathbf{id}_A$, which holds by function extensionality.
(3) By inducting on p, the goal becomes $\mathbf{id}_A = \mathbf{id}_A^{-1}$. It is straightforward to show that $\mathbf{id}_A$ is it's own inverse, so this holds.

Now suppose Type is a univalent universe. Do the analogous equations of (1–3) hold for $\mathsf{eq\text{–}equiv}$, the inverse of $\mathsf{equiv\text{–}eq}$?

Yes, they hold. For example, for any two equivalences $e : A \simeq B$ and $f : B \simeq C$, by (2) we have

$$\begin{aligned}
\mathsf{eq\text{–}equiv}(f \circ e) &= \mathsf{eq\text{–}equiv}(\mathsf{equiv\text{–}eq}(\mathsf{eq\text{–}equiv}(f)) \circ \mathsf{equiv\text{–}eq}(\mathsf{eq\text{–}equiv}(e))) \\
&= \mathsf{eq\text{–}equiv}(\mathsf{equiv\text{–}eq}(\mathsf{eq\text{–}equiv}(e) \cdot \mathsf{eq\text{–}equiv}(f)) \\
&= \mathsf{eq\text{–}equiv}(e) \cdot \mathsf{eq\text{–}equiv}(f)
\end{aligned}$$

using that $\mathsf{eq\text{–}equiv}$ is inverse to $\mathsf{equiv\text{–}eq}$.

2 (⋆)

Consider a type familly $P : \mathsf{Type} \to \mathsf{Type}$, and let $p : A = B$ in Type. We can form $\mathsf{ap}_P(p) : P(A) = P(B)$, and we can transport along p to get an equivalence $p_* : P(A) \simeq P(B)$.

(a) Show that $\mathsf{equiv\text{–}eq}(\mathsf{ap}_P(p)) = p_*$. When Type is univalent, deduce that

$$\mathsf{ap}_P(p) = \mathsf{eq\text{–}equiv}(p_*).$$

By induction on p, the goal becomes $\mathsf{equiv\text{–}eq}(\mathbf{refl}_{P(A)}) = \mathbf{id}_{P(A)}$, which holds definitionally. The other equation follows since $\mathsf{eq\text{–}equiv}$ is inverse to $\mathsf{equiv\text{–}eq}$.

(b) Let $A, B, C : \mathsf{Type}$. Using the universal property of propositional truncations, construct functions $\|A = B\| \to \|B = C\| \to \|A = C\|$ and $\|A = B\| \to \|B = A\|$ corresponding to composition of truncated paths and inversion of truncated paths, respectively.

We only construct the composition, as inversion is similar (and easier). Since function types between propositions are themselves propositions, by the universal property of truncations it suffices to give a function

$$(A = B) \to \|B = C\| \to \|A = C\|\,.$$

Let $p : A = B$. Then by the universal property again, it suffices to give a function

$$(B = C) \to \|A = C\|\,.$$

For this function, we give $q \mapsto |p \cdot q|$.
(Note that we could have given an alternative definition where we inducted on the path p.)

We will use the usual symbols for composition and inversion of truncated paths, since the operation is clear from the context.

Recall (or show) that in the family $X \mapsto (A = X) : \mathsf{Type} \to \mathsf{Type}$, a path $p : X = Y$ acts by post-composition: for any $q : A = X$, we have that $p_*(q) \doteq q \cdot p : (A = Y)$.

(c) Show that in the family $X \mapsto \|A = X\| : \mathsf{Type} \to \mathsf{Type}$, a path $p : X = Y$ acts by *truncated* post-composition: for any $q : \|A = X\|$, we have that

$$p_*(q) = q \cdot |p| : \|A = Y\|\,.$$

The equation lives in a proposition and therefore automatically holds (since all elements of a proposition are equal). An alternative, but longer, proof would proceed by unrolling the definition and using the dependent universal property of truncations.

3 $(\star\star)$

Assume Type is univalent.

1. Show that the type $\Sigma_{A:\mathsf{Type}}\mathsf{is\text{–}contr}A$ of all contractible types in Type is contractible;
2. Show that the universe of k-types

$$\mathsf{Type}^{\leq k} := \Sigma_{A:\mathsf{Type}}\mathsf{is\text{–}trunc}_k(A)$$

is a $(k+1)$-type, for any $k \geq -2$;

3. Show that the universe of propositions $\mathsf{Type}^{\leq -1}$ is not a proposition;
4. $(\star\star\star)$ Show that the universe of sets $\mathsf{Type}^{\leq 0}$ is not a set.

(This is exercise 17.1 from the HoTT intro book.)

(1) A type A is contractible if and only if the unique map $A \to \mathbb{1}$ is an equivalence. Thus by univalence, the type of all contractible types is equivalent to

$$\Sigma_{A:\mathsf{Type}}(A = \mathbb{1})$$

It is straightforward to construct a contraction of this type with center $(\mathbb{1}, \mathsf{refl}_{\mathbb{1}})$. (This was done in greater generality in exercise 4 from worksheet 3.)
(2) For $k \doteq -2$ we just showed that Type^{-2} is itself contractible, hence in particular a (-1)-type. Suppose $k > -2$. We need to show that the identity types of $\mathsf{Type}^{\leq k}$ are k-truncated. For $(X, p), (Y, q) : \mathsf{Type}^{\leq k}$, by characterization of paths in Σ-types, we have that

$$\big((X, p) =_{\mathsf{Type}^{\leq k}} (Y, q)\big) \simeq \big(\Sigma_{z:X=Y} z_*(p) = q\big) \simeq (X = Y)$$

where the last equivalence follows from the type $z_*(p) = q$ being contractible (since $\mathsf{is\text{–}trunc}_k(Y)$ is a proposition). The claim follows since we know that identity types between k-truncated types are themselves k-truncated.
(3) We need to produce two elements of $\mathsf{Type}^{\leq -1}$ which are not equal. Take $\emptyset, \mathbb{1} : \mathsf{Type}^{\leq -1}$. By the argument in the previous point, the identity types in $\mathsf{Type}^{\leq -1}$ are just the regular identity types in Type. Suppose we have a term $p : \mathbb{1} = \emptyset$ in $\mathsf{Type}^{\leq -1}$. Then we can transport the point $\star : \mathbb{1}$ along this path to get an element $p_*(\star) : \emptyset$. Thus we have shown that $(\mathbb{1} = \emptyset) \to \emptyset$, i.e., that $\mathbb{1} \neq \emptyset$, as required.
(4) We will show that $\mathbb{2} =_{\mathsf{Type}^{\leq 0}} \mathbb{2}$ is not a proposition. By the argument in (2), this type is equivalent to $\mathbb{2} = \mathbb{2}$. Applying univalence to the swap automorphism $\mathsf{swap} : \mathbb{2} \simeq \mathbb{2}$, we get a term $\mathsf{eq\text{–}equiv}(\mathsf{swap}) : \mathbb{2} = \mathbb{2}$. Now we argue that $\mathsf{eq\text{–}equiv}(\mathsf{swap}) \neq \mathsf{refl}_{\mathbb{2}}$, so that $\mathbb{2} = \mathbb{2}$ is not a proposition. By univalence, it suffices to show that $\mathsf{swap} \neq \mathsf{id}_{\mathbb{2}}$. By function extensionality, it suffices to show that $(\Pi_{x:\mathbb{2}}\mathsf{swap}(x) = x) \to \emptyset$. Suppose we have such a homotopy h, then we can evaluate at false to get a path $\mathsf{true} =_{\mathbb{2}} \mathsf{false}$. By exercise 3 from worksheet 5 we have that $\mathsf{true} \neq \mathsf{false}$, so we are done.

4 $(\star\star)$

Give an example of a type family $B : A \to \mathsf{Type}$ for which the implication

$$\neg\big(\Pi_{(a:A)} B(a)\big) \longrightarrow \big(\Sigma_{(a:A)} \neg B(a)\big)$$

is false. *(This is exercise 17.2 from the HoTT intro book.)*

From Corollary 17.4.2 in the book, we know that there is no dependent function $\Pi_{(X:BS_2)}X$. Taking $A := BS_2$ and B to be the natural map, the domain of the implication is therefore equivalent to $\mathbb{1}$. We now show that $\Sigma_{(X:BS_2)}\neg X$ is equivalent to $\emptyset$. It suffices to give a map $(\Sigma_{(X:BS_2)}\neg X) \to \emptyset$. Let $X : BS_2$ and suppose $X \to \emptyset$. Since we wish to construct an element of the proposition $\emptyset$, we may assume we have an actual identification $X = \mathbb{2}$. By path induction our hypothesis becomes a function $\mathsf{bool} \to \emptyset$. Thus by evaluating at e.g. $\mathsf{true} : \mathbb{2}$, we get a term of $\emptyset$, as required.

5 $(\star\star)$

Let $A : \mathsf{Type}$. The type $\mathsf{BAut}(A) := \Sigma_{X:\mathsf{Type}} \|A = X\|$ is called **the path component of A in Type**.

(a) Show that for $(X, p), (Y, q) : \mathsf{BAut}(A)$, we have $\big((X, p) =_{\mathsf{BAut}(A)} (Y, q)\big) \simeq (X = Y)$.

Using characterization of paths in Σ-types and transport of truncated paths, we have that

$$\big((X, p) =_{\mathsf{BAut}(A)} (Y, q)\big) \simeq \Sigma_{(z:X=Y)} p \cdot |z| = q.$$

The path $p \cdot |z| = q$ is a path in the proposition $\|A = Y\|$, hence inhabits a contractible type. Consequently, the right-hand side above is equivalent to $(X = Y)$, as desired.

Note that $(A, |\mathsf{refl}_A|) : \mathsf{BAut}(A)$, so that $\mathsf{BAut}(A)$ is pointed. Write pt for this base point, and denote $\mathsf{Aut}(A) := (A \simeq A)$.

(b) Assuming Type is univalent, deduce that $(\mathsf{pt} =_{\mathsf{BAut}(A)} \mathsf{pt}) \simeq \mathsf{Aut}(A)$.

From (a) we have that $(\mathsf{pt} = \mathsf{pt}) \simeq (A = A)$, and the right-hand side is equivalent to $(A \simeq A)$ by univalence.

Next, show that $\mathsf{BAut}(A)$ is **connected**:

(c) Show that for every $(X, p), (Y, q) : \mathsf{BAut}(A)$ we *merely* have a path:

$$\big\|(X, p) =_{\mathsf{BAut}(A)} (Y, q)\big\|.$$

By (a), we have that $\|(X,p) =_{\mathsf{BAut}(A)} (Y,q)\| \simeq \|X = Y\|$. An element of the right-hand side is given by $p \cdot \bar{q}$.

The type $\mathsf{BAut}(\mathbb{2}) \doteq BS_2$ is also called *the universe of* 2*-element sets.*

(d) By combining the previous points and exercise 5 from worksheet 10, show that

$$(\mathsf{pt} =_{\mathsf{BAut}(\mathbb{2})} \mathsf{pt}) \simeq \mathbb{2}.$$

Letting $n :\equiv 2$ in exercise 5(b) from worksheet 10, we get an equivalence $\mathbb{2} \simeq \mathsf{Aut}(\mathbb{2})$. Thus from point (b) above, we deduce that

$$(\mathsf{pt} =_{\mathsf{BAut}(\mathbb{2})} \mathsf{pt}) \simeq \mathsf{Aut}(\mathbb{2}) \simeq \mathbb{2}.$$

6 $(\star\star\star)$

Let Type be a univalent universe, and consider $A : \mathsf{Type}$. Recall (or show!) the *type-theoretic Yoneda lemma (Theorem 13.3.3):* for any $P : A \to \mathsf{Type}$ and $a : A$ we have an equivalence

$$\big(\Pi_{b:A}(a = b) \to P(b)\big) \simeq P(a).$$

(a) Suppose $\Sigma_{a:A}P(a)$ is contractible. Show that you then get an equivalence

$$\big(\Pi_{b:A}(a = b) \simeq P(b)\big) \simeq P(a).$$

By Theorem 11.1.3, a family of maps $\Pi_{b:A}(a = b) \to P(b)$ is a family of equivalences if and only if the induced map $\Sigma_{b:A}(a = b) \to \Sigma_{b:A}P(b)$ on total spaces is an equivalence. Since $\Sigma_{b:A}(a = b)$ is contractible, the latter always holds whenever $\Sigma_{b:A}P(b)$ is contractible.

(b) Show that the identity type, seen as a function $\mathsf{Id} : A \to (A \to \mathsf{Type})$ is an embedding. *(This is exercise 17.5 from the HoTT intro book, and it is due to Escardó.)*

We need to show that the fibres of Id are propositions. Given a family $P : A \to \textsf{\textbf{Type}}$, we give the fibre of Id over P the following name:

$$\mathsf{is\text{--}representable}(P) := \Sigma_{a:A}\mathsf{Id}(a) = P.$$

Thus we need to show that $\mathsf{is\text{--}representable}(P)$ is a proposition for all families P. It suffices to show that $\mathsf{is\text{--}representable}(P) \to \mathsf{Contr}(\mathsf{is\text{--}representable}(P))$ since then any two elements of $\mathsf{is\text{--}representable}(P)$ must be equal. Note that $(\mathsf{Id}(a) = P) \simeq (\Pi_{b:A}(a = b) \simeq P(b))$ by univalence and function extensionality. Thus an element of $\mathsf{is\text{--}representable}(P)$ corresponds to a natural equivalence $\Pi_{b:A}(a = b) \simeq P(b)$ for some $a : A$. Assuming such a natural equivalence η, then η gives an equivalence of total spaces

$$(\star) \qquad \Sigma_{b:A}(a = b) \simeq \Sigma_{b:A}P(b).$$

and since the left-hand side is contractible, so is the right-hand side. Finally, we have the following chain of equivalences:

$$\begin{aligned} P(b) &\simeq \Pi_{c:A}(b = c) \to P(c) && \textbf{(type-theoretic Yoneda lemma)} \\ &\simeq \Pi_{c:A}(b = c) \simeq P(c) && (\Sigma_A P \textbf{ is contractible}) \\ &\simeq \mathsf{Id}(b) = P \end{aligned}$$

Thus the right-hand side of equation $(\star)$ is equivalent to $\mathsf{is\text{--}representable}(P)$, which implies $\mathsf{is\text{--}representable}(P)$ is contractible, as required.