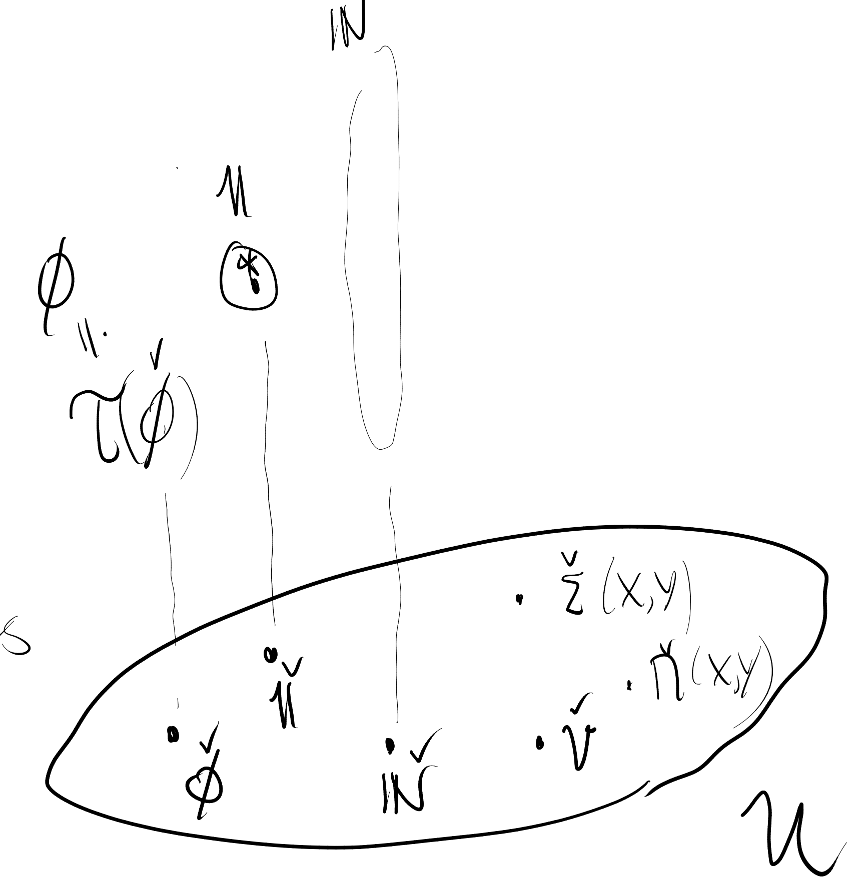
Outline



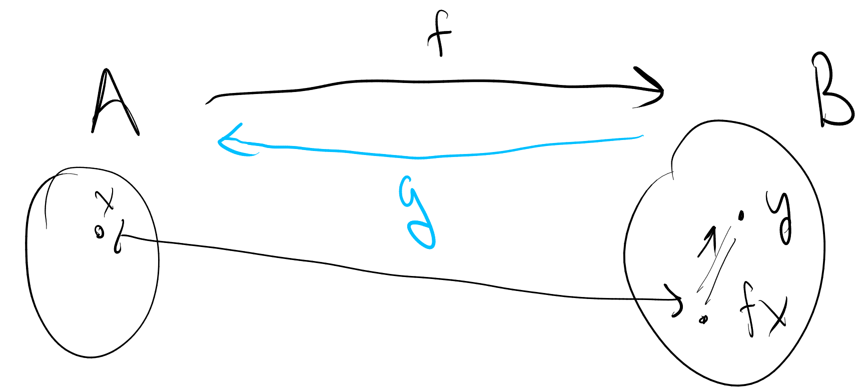
Last time: Universes $(\mathcal{U}, \mathcal{T})$

+ Propositions as types

Today Equivalences

- Homotopies
- Equivalences as bi-invertible maps
- Identifications in Σ-types
- Preview of:
  - function extensionality (Π)
  - univalence! $(\mathcal{U})$

Propositions as types interpretation of "$f : A \to B$ is surjective"



$\rightsquigarrow \quad \prod_{y:B} \sum_{x:A} f\,x = y$

by type theoretic choice, this amounts to

$$\sum_{g:B\to A} \underbrace{\prod_{y:B} f(g\,y) = y} \quad =: \text{is-split-surjective}(f) \quad = \text{sec}(f)$$

relation between $f\circ g$ and id_B

a priori weaker than $\underset{B\to B}{f\circ g = id_B}$

$\rightsquigarrow$ homotopy

In this particular case, we say

g is a section of f

/ f is a retraction of g.

Homotopies

<u>Example</u> $\mathrm{neg} : \mathrm{bool} \to \mathrm{bool}$, $\quad \mathrm{neg}\ \mathrm{true} :\equiv \mathrm{false}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{neg}\ \mathrm{false} :\equiv \mathrm{true}$

Then $\mathrm{neg}(\mathrm{neg}\ \mathrm{true}) \doteq \mathrm{true}$
$\qquad\quad \mathrm{neg}(\mathrm{neg}\ \mathrm{false}) \doteq \mathrm{false}$

however, $b : \mathrm{bool} \vdash \mathrm{neg}(\mathrm{neg}\ b) \not\doteq b : \mathrm{bool}$

so ~~also~~ $\mathrm{neg} \circ \mathrm{neg} \not\doteq \mathrm{id}_{\mathrm{bool}}$

But we have $\_ : \prod_{b:\mathrm{bool}} (\mathrm{neg} \circ \mathrm{neg})\, b = b$ by bool-ind

i.e., $\mathrm{neg} \circ \mathrm{neg}$ & $\mathrm{id}_{\mathrm{bool}}$ pointwise equal!

Def Let $f, g : \Pi_{x:A} B(x)$

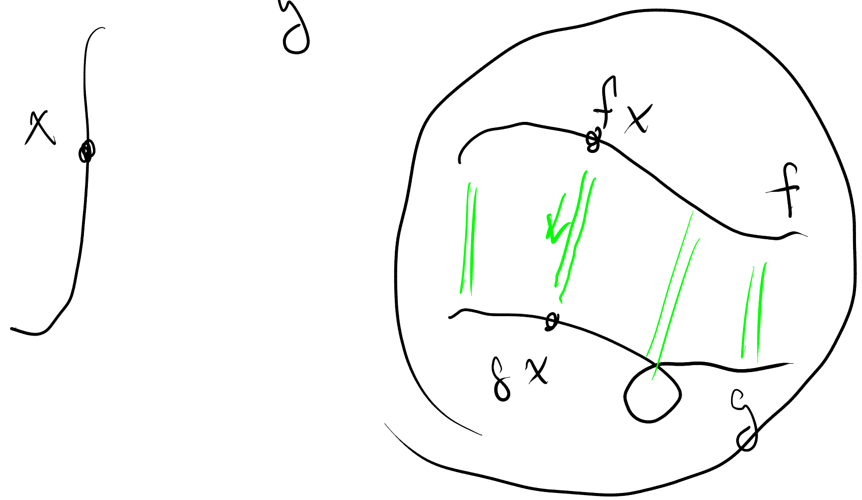
$$f \sim g := \Pi_{x:A}\, f x =_{B(x)} g x$$ the type of homotopies from f to g

Ex $H : \mathrm{neg} \circ \mathrm{neg} \sim \mathrm{id}_{\mathrm{bool}}$

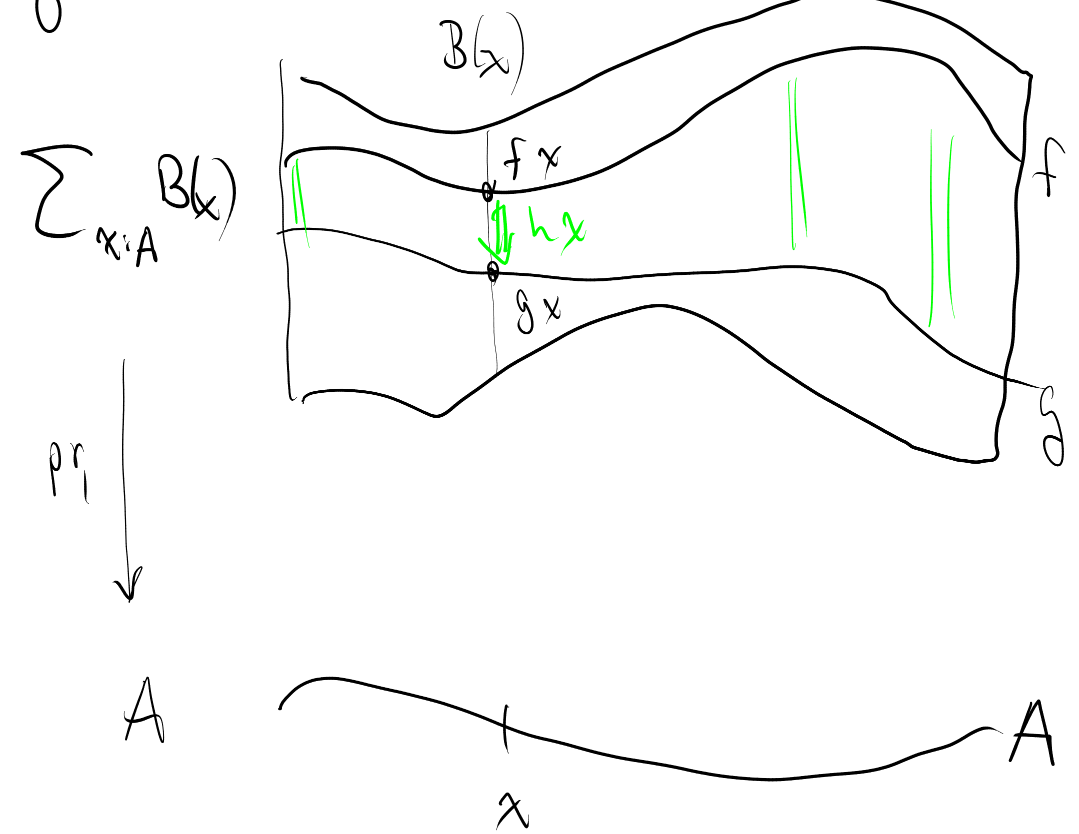
In pictures

- Special case where B doesn't depend on x:

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$



- In general:



Commutative diagrams

To express that a diagram of types & functions commutes, we use homotopies:

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & \arrow[d, phantom, "H"] & B \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

or

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "H"] & A' \arrow[d, "f'"] \\ B \arrow[r, "h"'] & B' \end{tikzcd}$$

$H : f \sim g \circ h$

$H : h \circ f \sim f' \circ g$

Homotopies of homotopies

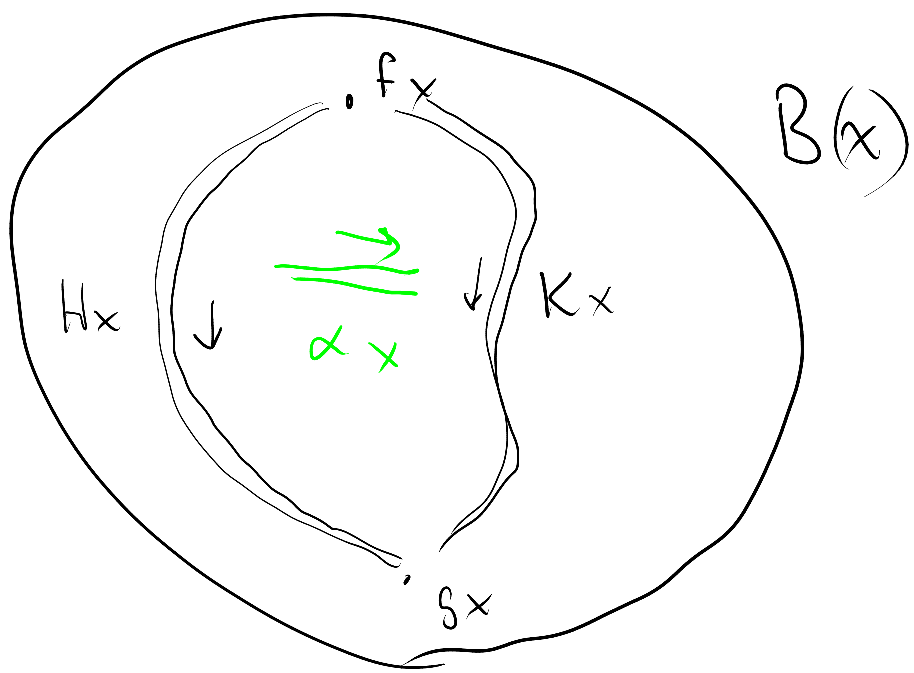
If $f, g : \Pi_{x:A} B(x)$

& $H, K : (f \sim g) \doteq \Pi_{x:A} f x =_{B(x)} g x$

we have $H \sim K \doteq \Pi_{x:A} H x =_{f x =_{B x} g x} K x$

$\alpha :$

for $x : A$



etc.

Laws for homotopies

Fix $P := \prod_{x:A} B(x)$

Since homotopies are pointwise identity types, we can lift laws & operations on identity types pointwise to $\sim$:

refl-htpy : $\prod_{f:P} f \sim f$

inv-htpy : $\prod_{f,g:P} (f \sim g \to g \sim f)$, $H \mapsto H^{-1}$

concat-htpy : $\prod_{f,g,h:P} (f \sim g \to g \sim h \to f \sim h)$, $H \bullet K$

w/ laws

assoc-htpy : $(H \bullet K) \bullet L \sim H \bullet (K \bullet L)$

unit laws

inv laws

etc. ...

Whiskering

Def • If $f, g : A \to B$, $h : B \to C$

& $H : f \sim g$

then $h \cdot H : h \circ f \sim h \circ g$

$$h \cdot H :\equiv \lambda x.\, \mathrm{ap}_h(Hx)$$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "H\Downarrow"] & B \arrow[r, "h"'] & C \end{tikzcd}$$

• If $k : D \to A$, then

$$H \cdot k : f \circ k \sim g \circ k$$

$$H \cdot k = \lambda y{:}D.\, H(ky)$$

$$\begin{tikzcd} D \arrow[r, "k"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow H"] & B \end{tikzcd}$$

Bi-invertible maps

Def $f : A \to B$

- $\mathrm{sec}(f) := \sum_{g : B \to A} f \circ g \sim \mathrm{id}_B$ ("f is a split surjection") type of sections of f.
- $\mathrm{retr}(f) := \sum_{h : B \to A} h \circ f \sim \mathrm{id}_A$ type of retr.s of f.
- $\text{is-equiv}(f) := \mathrm{sec}(f) \times \mathrm{retr}(f)$ ("f is bi-invertible")

Ex $\mathrm{id}_A : A \to A$ for any A, $\mathrm{neg} : \mathrm{bool} \to \mathrm{bool}$, $\_ + k : \mathbb{Z} \to \mathbb{Z}$

Def $\text{has-inverse}(f) := \sum_{g : B \to A} (f \circ g \sim \mathrm{id}_B) \times (g \circ f \sim \mathrm{id}_A)$

Lemma: $\text{has-inverse}(f) \to \text{is-equiv}(f)$

Discussion

It turns out that is-equiv(f) is much better behaved than $\text{has-inverse}(f) \doteq \sum_{g:B\to A} (f\circ g \sim id_B) \times (g\circ f \sim id_A)$.

$$\text{has-inverse}(id_A) \doteq \sum_{g:B\to A} (g \sim id_A) \times (g \sim id_A)$$

$$\cong$$

$$(id_A \sim id_A)$$

(funext: $(f\sim g) \simeq (f = g)$.)

Prop If $f : A \to B$, then $\text{is-equiv}(f) \longrightarrow \text{has-inverse}(f)$.

By Σ-ind, $g, h : B \to A$, $G : f\circ g \sim id_B$, $H : h \circ f \sim id_A$

$K : g \sim h$, $\quad g \doteq id_A \circ g \overset{H^{-1}g}{\sim} h\circ f\circ g \overset{h \cdot G}{\sim} h \circ id_B \doteq h$.

now we get : $H' : g\circ f \sim id_A$: $\quad g\circ f \overset{K f}{\sim} h\circ f \overset{H}{\sim} id_A$ $\square$

Def A, B types, $A \simeq B := \sum_{f: A \to B} \text{is-equiv}(f)$

More examples of equivs:

$\mathbb{1} \times A \simeq A$

$A \times B \simeq B \times A$

$\emptyset \times A \simeq \emptyset$

$A + \emptyset \simeq A, \quad A \simeq \emptyset + A$

$A + B \simeq B + A$

$(A+B)+C \simeq A + (B+C)$

$(A+B) \times C \simeq A \times C + B \times C$

$A \times (B+C) \simeq A \times B + A \times C$

Some of these gen. to Σ-types:

$$\sum_{z : A+B} C(z) \simeq \left(\sum_{x:A} C(\mathrm{inl}\, x) \right) + \left(\sum_{y:B} C(\mathrm{inr}\, y) \right)$$

$$\sum_{x:A} (B(x) + C(x)) \simeq \left(\sum_{x:A} B(x) \right) + \left(\sum_{x:A} C(x) \right)$$

Laws & operations on $\simeq$s

refl-equiv : $A \simeq A$

inv-equiv : $A \simeq B \longrightarrow B \simeq A$ (first go to two-sided inverse g of $f : A \to B$ then to sec + retr.)

concat-equiv : $A \simeq B \to B \simeq C \to A \simeq C$ etc

Fun.ext. + Univalence

$P :\equiv \prod_{x:A} B(x)$

because $\sim$, $\simeq$ are refl. we get

$$\text{id-to-}\sim \; : \prod_{(f,g : \prod_{x:A} B(x))} \left(f =_P g \longrightarrow f \sim g \right)$$

$$\text{id-to-}\simeq \; : \prod_{A,B : \mathcal{U}} \left(A =_{\mathcal{U}} B \longrightarrow A \simeq B \right)$$

$$\text{funext} : \prod_{f,g : P} \text{is-equiv}(\text{id-to-}\sim \; f \; g), \quad \text{UA}^{\mathcal{U}} : \prod_{A,B : \mathcal{U}} \text{is-equiv}(\text{id-to-}\simeq \; A \; B)$$

Identifications in Σ-types

Fix A type, $x : A \vdash B(x)$ type

Want refl relation $z, z' : S \vdash \underset{\overset{\shortparallel}{\text{Eq-}\Sigma}}{R(z,z')}$ type

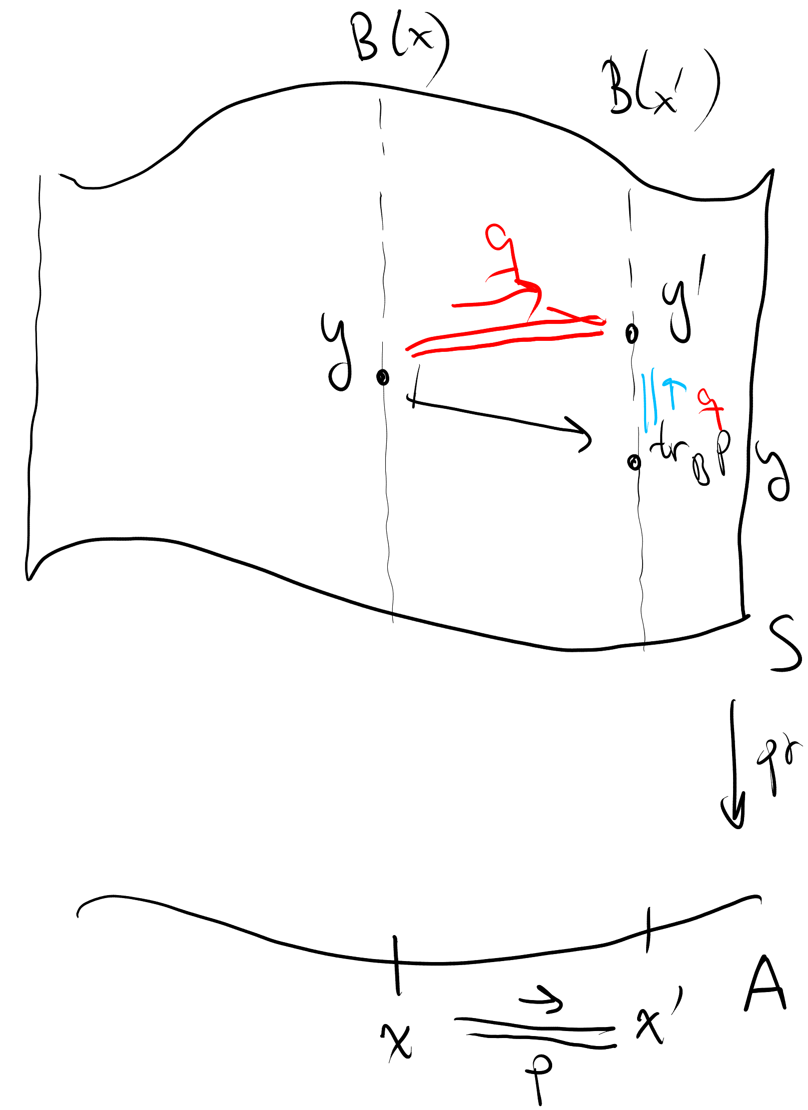
assume

$z \doteq (x, y)$

$z' \doteq (x', y')$

By ap_{pr_1}, we should have $p : x =_A x'$

Recall: $tr_B : \prod_{\{x,x'\}} \left(x = x' \to B(x) \to B(x') \right)$



$S :\equiv \sum_{x:A} B(x)$

__Def__ $\left(y \overset{B}{\underset{p}{=}} y' \right) :\equiv \left(tr_B\, p\, y =_{B(x')} y' \right)$ (type of identifications of y w/ y' over p)

__Def__ $\text{Eq-}\Sigma(z, z') :\equiv \sum_{p : pr_1 z = pr_1 z'} \left(pr_2 z \overset{B}{\underset{p}{=}} pr_2 z' \right)$

$\underline{\text{Def}}$ $\mathrm{Eq}\text{-}\Sigma(z, z') :\equiv \sum_{p : pr_1 z = pr_1 z'} \left(pr_2 z =^B_p pr_2 z' \right)$

$$\Vert\cdot$$

$$tr_B\, p\,(pr_2 z) = pr_2 z'$$

$\underline{\text{Def}}$ $\mathrm{refl}\text{-}\mathrm{Eq}\text{-}\Sigma : \prod_{z : S} \mathrm{Eq}\text{-}\Sigma(z, z)$

$$:\equiv \lambda z. \left(\mathrm{refl}_{pr_1 z}, \mathrm{refl}_{pr_2 z} \right)$$

$\underline{\text{Def}}$ $\mathrm{pair}\text{-}\mathrm{eq} : \prod_{z, z' : S} \left(z = z' \longrightarrow \mathrm{Eq}\text{-}\Sigma(z, z') \right)$

by path induction.

$\underline{\text{Def}}$ $\mathrm{eq}\text{-}\mathrm{pair} : \prod_{z, z' : S} \left(\mathrm{Eq}\text{-}\Sigma(z, z') \longrightarrow z = z' \right)$

use repeated Σ-ind. to get $\begin{pmatrix} x, x' : A,\ y : B(x) \\ y' : B(x'),\ p : x =_A x' \\ q : tr_B(p, y) = y' \end{pmatrix}$, path ind. on q.

get to $\begin{pmatrix} x, x' : A,\ y : B(x) \\ p : x = x',\ \text{goal} : (x, y) = (x', tr_B(p, y)) \end{pmatrix}$, path ind. on p. $\begin{pmatrix} x : A,\ y : B(x) \\ \text{goal} : (x, y) \overset{\mathrm{refl}}{=} (x, y) \end{pmatrix}$ $\square$

<u>Claim</u> 1) $\prod_{z,z' : S} \prod_{w : Eq\text{-}\Sigma(z,z')} \text{pair-eq}(\text{eq-pair } w) = w$

by Σ-ind. (repeated), path ind. 2x, $(\text{refl}, \text{refl}) = (\text{refl}, \text{refl})$

use refl. ✓

2) $\prod_{z,z' : S} \prod_{r : z =_S z'} \text{eq-pair}(\text{pair-eq } r) = r$

by path ind, goal: $\text{refl}_z = \text{refl}_z$

use refl. ✓

<u>Cor</u> For all $z, z' : S$ $\quad (z =_S z') \simeq Eq\text{-}\Sigma(z, z')$

Q&A

$$\mathrm{Monoid}^{\mathcal{U}} := \sum_{(X:\mathcal{U})} \sum_{(s:\,\text{is-set}\,X)} \sum_{(e:X)} \sum_{(\mu : X \to X \to X)} \cdots$$

$\mathrm{UA}^{\mathcal{U}} : \prod_{A,B:\mathcal{U}} \text{is-equiv}(\text{id-to-}\simeq\, A\, B)$

Q: connections UA ↔ Yoneda:

- $=_A \;:\; A \longrightarrow (A \longrightarrow \mathcal{U})$ embedding.

- (Rezk completeness)

- For all $A:\mathcal{U}$, $\sum_{B:\mathcal{U}} (A \simeq B) \xrightarrow{\sim} \mathbb{1}$