Introduction to modalities

Motivation:

- modal logic

  $\Diamond$: "possibility"

  $\Box$: "necessity"

S4

$\Box A \longrightarrow \Box\Box A \qquad \Box A \longrightarrow A$ comonad

$\Diamond\Diamond A \longrightarrow \Diamond A \qquad A \longrightarrow \Diamond A$ monad

multiplication $\qquad$ unit

$$\dfrac{\dfrac{A \longrightarrow \Diamond A}{\Diamond A \longrightarrow \Diamond\Diamond A} \qquad \Diamond\Diamond A \longrightarrow \Diamond A}{\Diamond A \cong \Diamond\Diamond A}$$

- Moggi's monadic λ-calculus

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \eta(t) : TA} \qquad \frac{\Gamma \vdash s : TA \qquad \Gamma, x : A \vdash t : TB}{\Gamma \vdash (\mathrm{let}\ x := s\ \mathrm{in}\ t) : TB}$$

- model side effects
- not idempotent

Let $P_0 : U \longrightarrow \mathrm{Prop}$

be a predicate on a universe U.

We can form a <u>subuniverse</u>

$$U_0 = \underbrace{\{ A : U \mid P_0(A) \}}_{\sum_{A:U} P_0(A)} \hookrightarrow U$$

<u>Def</u>: We call a subuniverse $U_0 \hookrightarrow U$ <u>reflective</u>, if for all $A : U$

there exist (Σ) a type $\bigcirc A$ in U_0 together with a function

$$\eta : A \longrightarrow \bigcirc A \quad \text{s.th}$$

universal property

$$\forall (B : U_0)(f : A \to B) \underbrace{\exists! (\bar{f} : \bigcirc A \to B).\ \bar{f} \circ \eta = f}_{\mathrm{isContr}\left(\sum_{\bar{f} : \bigcirc A \to B} \bar{f} \circ \eta = f\right)}$$

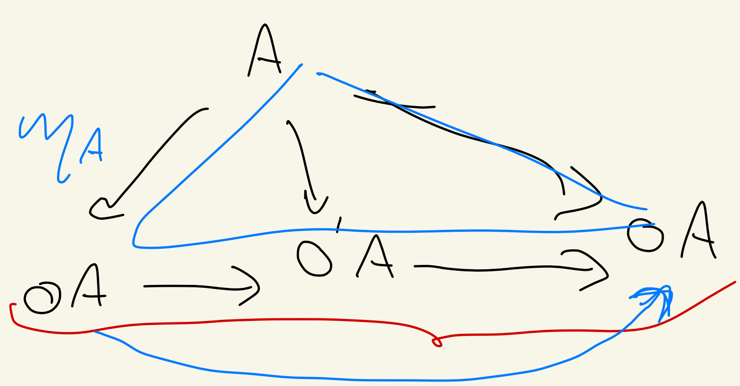
$$\begin{tikzcd}
A \arrow[d, "\eta_A"'] \arrow[dr, "f"] & \\
\bigcirc A \arrow[r, dashed, "\exists!\, \bar{f}"'] & B
\end{tikzcd}$$

Terminology

- $\bigcirc : \mathcal{U} \longrightarrow \mathcal{U}$ — modal operator
- $\eta_A : A \longrightarrow \bigcirc A$ — modal unit
- we call $A : \mathcal{U}$ a $\bigcirc$-modal type if η_A is an equivalence
- we call A $\bigcirc$-connected if $\bigcirc A \simeq 1$.

Lemma: The map $(A \xrightarrow{\eta_A} \bigcirc A)$ is determined up to unique equivalence by the UP.

$$\begin{tikzcd}
 & A \arrow[dl, "\eta_A"'] \arrow[dr, "\eta'_A"] & \\
\bigcirc A \arrow[rr, dashed, shift left] & & \bigcirc' A \arrow[ll, dashed, shift left]
\end{tikzcd}$$



Ex: truncation levels: for $n \geq -2$

$$\mathrm{istrunc}_n : \mathcal{U} \to \mathrm{Prop}$$

$$n\text{-Type} \hookrightarrow \mathcal{U}$$

Lemma: Given $A : \mathcal{U}$, $B : \mathcal{U}_\circ$

$$f, g : \bigcirc A \to B$$

we have $(f = g) \simeq (f \circ \eta = g \circ \eta)$

$$\begin{tikzcd}
A \arrow[d, "\eta"'] \arrow[dr, "f \circ \eta"] & \\
\bigcirc A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

Lemma: A type B is modal iff

$$\forall (A : \mathcal{U})\,(f : A \to B)\,\exists!\,(\bar{f} : \bigcirc A \to B)\ \bar{f} \circ \eta = f$$

$$\begin{tikzcd}
 & B \arrow[dl, "\eta"'] \arrow[dr, equal, "id"] & \\
\bigcirc B \arrow[rr, dashed, "\exists!\, \overline{id}"'] & & B
\end{tikzcd}$$

$\overline{id} \circ \eta = id$

$\eta \circ \overline{id} \overset{?}{=} id_{\bigcirc B}$

$\Leftrightarrow \eta \circ \overline{id} \cancel{\circ \eta} = \eta$

Theorem: TFAE for refl subunivs

$$U_0 \hookrightarrow U.$$

① U_0 is Σ-closed, i.e. for all $A : U_0$, $B : A \longrightarrow U_0$, the type $\sum_{a:A} B(a)$ is modal.

② U_0 "admits unique dependent elimination", i.e. given $A : U$, $B : \bigcirc A \longrightarrow U_0$, $f : \prod_{a:A} B(\eta_A)$, there ex a unique $\tilde{f} : \prod_{a:\bigcirc A} B(a)$ such that $\tilde{f} \circ \eta = f$

$$\prod(a:A)\ \tilde{f}(\eta a) = f a$$

$$\begin{tikzcd}
 & \sum_{a:\bigcirc A} B_a \arrow[d, "pr_1"] \\
A \arrow[ur, "\langle \eta, f \rangle"] \arrow[r, "\eta"'] & \bigcirc A \arrow[u, dashed, bend left, "\langle id, \tilde{f} \rangle"]
\end{tikzcd}$$

$\langle \eta, f \rangle (a) = (\eta(a), f(a))$

$\langle id, \tilde{f} \rangle (a) = (a, \tilde{f} a)$

Proof: If U is Σ-closed ~~then~~ &

$$A : U \quad B : \bigcirc A \to U_\bigcirc \,,\; f : \prod_{a:A} B(\eta_A)$$

$$g = \langle \eta, f \rangle \qquad \bar{g} \circ \eta = g$$

$$\begin{tikzcd}
 & \Sigma_a B_a \arrow[d, "pr_1"] \\
A \arrow[r, "\eta"'] \arrow[ur, "g"] & \bigcirc A \arrow[u, dashed, bend left, "\bar{g}"]
\end{tikzcd}$$

$$pr_1 \circ \bar{g} \overset{?}{=} 1$$

$$pr_1 \circ \bar{g} \circ \eta = pr_1 \circ g = \eta$$

Conversely . . . . $\square$

Def: A *modality* is a refl subuniverse satisfying the conditions of the theorem.

Examples

① Trucation

$$n\text{-Type} \hookrightarrow U$$

Proof of reflectivity:

We have to show that

$$\begin{tikzcd} & A \arrow[dl, "|-|_n"'] \arrow[dr, "f"] & \\ \|A\|_n \arrow[rr, dashed, "\bar{f}"'] & & B \end{tikzcd}$$

n-type

Proof by Induction:

- (-2) is trivial $\|A\|_{-2} = 1$
- (-1) is prop truncation

$$\|A\|_{-1} = \forall (Q : \mathrm{Prop}).(A \to Q) \to Q$$

<u>Impredicative</u>

or join construction

IS: Assume $n\text{-Trunc} \hookrightarrow \mathcal{U}$ is reflective ~~let $A : (n+1)\text{-Trunc}$~~

Define

$$\begin{tikzcd}
R_n : A \arrow[rr] \arrow[d, two heads] \arrow[drr, dashed] & & \mathcal{U}^A \\
\{f : A \to \mathcal{U} \mid \exists_{a}\, R_n(a) = f\} \arrow[rr, hook] & & ((n)\text{-Type})^A \arrow[u, hook]
\end{tikzcd}$$

$$R_n(a) = \lambda b. \|a = b\|_n$$

$$\simeq \sum_{f : A \to \mathcal{U}} \|\mathrm{fib}_{R_n} f\|_{-1}$$

Open and closed modalities

Let $Q : \mathrm{Prop}$

$$\mathrm{Op}_Q, \mathrm{Cl}_Q : \mathcal{U} \longrightarrow \mathcal{U}$$

$$\mathrm{Op}_Q(A) = A^Q$$

$$\mathrm{Cl}_Q(A) = A * Q$$

"join"
"~~suspension~~"

$$\begin{tikzcd}
A \times Q \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & Q \arrow[d] \\
A \arrow[r] & A * Q \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

lex modalities

$\top, \bot : \mathrm{Prop}$

Localization & nullification.

If $f : B \longrightarrow \overset{1}{A}$, call $X : \mathcal{U}$ f-local, if $\begin{pmatrix} X^A \xrightarrow{X^f} X^B \\ h \longmapsto h \circ f \end{pmatrix}$ is an equivalence.

$\{\rho\text{-local-types}\} \hookrightarrow U$

- is always reflective
- Σ-closed whenever $A = 1$

$B \longrightarrow 1$

Cohesive HoTT — purely topological

- nullify at $\mathbb{R}$ (Dedekind reals)

$\Rightarrow$ shape modality

in cohesive toposes

$\mathcal{E}$ topos $\qquad j : \Omega \longrightarrow \Omega$

$$\begin{tikzcd}
 & \mathcal{E} & \\
sh_j(\mathcal{E}) \arrow[ur] \arrow[r, hook] & sep_j(\mathcal{E}) \arrow[u] &
\end{tikzcd}$$

quasi-topos

$\mathcal{E}$ a topos $\quad \bigcirc : \mathcal{E} \longrightarrow \mathcal{E}$

$$\begin{tikzcd}
& \mathcal{E} \arrow[dl] \arrow[d] \\
sh_j(\mathcal{E}) \arrow[r, hook] & sep_j(\mathcal{E})
\end{tikzcd}$$