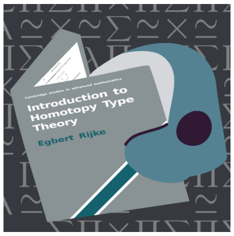

Worksheet 6 (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs

25 July 2022

1 $(\star)$

Prove that $\neg\mathsf{isContr}(\emptyset)$.

Recall that, for any type A,

$$\mathsf{isContr}(A) \doteq \sum_{c:A} \prod_{a:A} c = a.$$

So we have the projection map

$$\textstyle\sum_{c:\emptyset} \left(\prod_{a:\emptyset} c = a\right) \xrightarrow{\mathsf{pr}_1} \emptyset$$

Which is a map $\mathsf{isContr}(\emptyset) \to \emptyset$.

2 $(\star\star)$

Recall the *observational equality of natural numbers* $\mathsf{Eq}\text{-}\mathbb{N} \quad : \mathbb{N} \to \mathbb{N} \to \mathcal{U}$

$$\begin{aligned}
&\mathsf{Eq}\text{-}\mathbb{N}\ 0 && 0 && \doteq \mathbb{1} \\
&\mathsf{Eq}\text{-}\mathbb{N}\ (\mathsf{suc}\ m) && 0 && \doteq \emptyset \\
&\mathsf{Eq}\text{-}\mathbb{N}\ 0 && (\mathsf{suc}\ n) && \doteq \emptyset \\
&\mathsf{Eq}\text{-}\mathbb{N}\ (\mathsf{suc}\ m) && (\mathsf{suc}\ n) && \doteq \mathsf{Eq}\text{-}\mathbb{N}\ m\ n
\end{aligned}$$

Prove that, for every $n : \mathbb{N}$,

$$\mathsf{Eq}\text{-}\mathbb{N}\ n\ (\mathsf{suc}\ n) = \emptyset$$

Proceed by induction on n. For $n \doteq 0$, we have that $\mathsf{Eq}\text{-}\mathbb{N}\ 0\ 1$ is definitionally equal to $\emptyset$, so we use $\mathsf{refl}_\emptyset$. Then, for some n, assuming

$$\mathsf{ih} : (\mathsf{Eq}\text{-}\mathbb{N}\ n\ (\mathsf{suc}\ n)) = \emptyset$$

we have that $\mathsf{Eq}\text{-}\mathbb{N}\ (\mathsf{suc}\ n)\ (\mathsf{suc}(\mathsf{suc}\ n)) = \emptyset$ by the recursive definition of $\mathsf{Eq}\text{-}\mathbb{N}$.

In Lecture 4, we did most of the proof that

$$\mathsf{Eq}\text{-}\mathbb{N}\ m\ n \quad \simeq \quad m =_{\mathbb{N}} n.$$

Use this (and the fact proved above) to prove that $\neg(\mathsf{isContr}\ \mathbb{N})$.

Suppose

$$(c, \varphi) \colon \mathsf{isContr}\ \mathbb{N}.$$

So $c : \mathbb{N}$ **and** $\varphi : \prod_{n:\mathbb{N}} c = n$**. So** $\varphi(\mathsf{suc}\ c) : c = (\mathsf{suc}\ c)$**. But we have**

$$c = (\mathsf{suc}\ c) \quad \simeq \quad \mathsf{Eq}\text{-}\mathbb{N}\ c\ (\mathsf{suc}\ c)$$

We showed that the right-hand side is $\emptyset$**, and thus** $\mathsf{isContr}(\mathbb{N})$ **implies** $\emptyset$**.**

3 $(\star\star\star)$

Show that if A is contractible, then for any $x, y \colon A$, the identity type $x = y$ is also contractible.

Let $c : A$ **be the center of contraction. It will suffice to show that** $(x = y) \simeq \mathbb{1}$**. For the map** $f : (x = y) \to \mathbb{1}$**, we can just use the constant** $\star$ **map. For the other direction,** $g : \mathbb{1} \to (x = y)$**, we proceed by singleton induction and can therefore take** $x \doteq c \doteq y$ **and put**

$$g(\star) \doteq \mathsf{refl}_c.$$

To see that $g(f(p)) = p$ **for any** $p : x = y$**, we can proceed by path induction and assume** $x \doteq y$ **and** $p \doteq \mathsf{refl}_x$**. By singleton induction again, we can assume** $x \doteq c$**. So** $p \doteq \mathsf{refl}_c$**, which is the same as** $g(f(p))$**. The other direction is simpler: for any** $t : \mathbb{1}$**, we prove that** $f(g(t)) = t$ **by singleton induction, i.e. assuming** $t \doteq \star$**. But** $f(g(t)) \doteq \star$ **by definition, so** $\mathsf{refl}_\star : f(g(t)) = t$**.**

4 $(\star\star\star)$

Recall the first projection function

$$\mathsf{pr}_1 \quad : \quad \sum_{x:A} B(x) \to A$$

Show that pr_1 is an equivalence iff each $B(a)$ is contractible. *Hint: Use the results about identity types of* Σ *types we proved in a previous lecture.*

First the 'only if' direction: if pr_1 is an equivalence, that means it's a contractible map, i.e.

$$\mathsf{isContr}(\mathsf{fib}_{\mathsf{pr}_1}(a))$$

for each $a : A$. Then, given any $a : A$ and $b, b' : B(a)$, we have

$$((a, b), \mathsf{refl}_a) : \mathsf{fib}_{\mathsf{pr}_1}(a) \qquad \text{and} \qquad ((a, b'), \mathsf{refl}_a) : \mathsf{fib}_{\mathsf{pr}_1}(a).$$

Since $\mathsf{fib}_{\mathsf{pr}_1}(a)$ is contractible, we have

$$((a, b), \mathsf{refl}_a) = ((a, b'), \mathsf{refl}_a).$$

By the characterization of identity types of fibers (Prop. 10.3.3), this is equivalent to

$$\Sigma_{q:(a,b)=(a,b')}(\mathsf{refl}_a = \mathsf{ap}_{\mathsf{pr}_1}(q) \cdot \mathsf{refl}_a).$$

Using the characterization of euqality on Σ types, we can show that this type is equivalent to

$$\Sigma_{q:a=a}(\mathsf{tr}_B(q, b) = b') \times (\mathsf{refl}_a = q).$$

This data implies that $b = b'$, because by the third component we may substitute refl_a for q in $\mathsf{tr}_B(q, b) = b'$, and $\mathsf{tr}_B(\mathsf{refl}, b) = b$. Thus we have that $B(a)$ is contractible.
In the other direction, suppose each $B(a)$ is contractible with center c_a. Then define an inverse $q : A \to \sum_{x:A} B(x)$ by

$$q(a) \doteq (a, c_a).$$

Then $\mathsf{pr}_1(q(a)) \doteq \mathsf{pr}_1(a, c_a) \doteq a$, so q is a section of pr_1. Furthermore, for any $(a, b) : \sum_{x:A} B(x)$, we have a proof that $b = c_a$, so, again using that identities of pairs are equivalent to pairs of identities, we get

$$(a, c_a) = (a, b).$$

The left-hand side is exactly $q(\mathsf{pr}_1(a, b))$, as desired.

Show that for any $a : A$, the map

$$\lambda((x, y), p).\mathsf{tr}_B(p, y) \colon \mathsf{fib}_{\mathsf{pr}_1}(a) \to B(a)$$

is an equivalence.

Call this map k. Now, pick some $b : B(a)$ and observe that $((a, b), \mathsf{refl}_a) : \mathsf{fib}_{\mathsf{pr}_1}(a)$ and that

$$k((a, b), \mathsf{refl}_a) \quad \doteq \quad \mathsf{tr}_B(\mathsf{refl}_a, b) \quad \doteq \quad b$$

So now pick some $((x, y), p) : \mathsf{fib}_{\mathsf{pr}_1}(b)$ and

$$q : \mathsf{tr}_B(p, y) = b.$$

Recall (again) from Lecture 5 that

$$(x, y) = (a, b) \quad \simeq \quad \sum_{p:x=a} \mathsf{tr}_B(p, y) = b$$

So, since we have $p : x = a$ and $q : \mathsf{tr}_B(p, y) = b$, we get a proof $r : (x, y) = (a, b)$. Finally, by the observations we made in lecture, to prove

$$((x, y), p) = ((a, b), \mathsf{refl}_a)$$

we just need to have our proof $r : (x, y) = (a, b)$ and then show

$$p = \mathsf{ap}_{\mathsf{pr}_1} r \cdot \mathsf{refl}_a.$$

But we can check that $\mathsf{ap}_{\mathsf{pr}_1} r$ gives us p, and $p \cdot \mathsf{refl}_a = p$.

5 $(\star\star)$

Construct for any map $f : A \to B$ an equivalence

$$e : A \simeq \sum_{y:B} \mathsf{fib}_f(y)$$

with a homotopy $H : f \sim \mathsf{pr}_1 \circ e$ witnessing that the triangle

$$\begin{tikzcd} A \arrow[rr, "e"] \arrow[dr, "f"'] & & \sum_{y:B} \mathsf{fib}_f(y) \arrow[dl, "\mathsf{pr}_1"] \\ & B & \end{tikzcd}$$

commutes.

The natural definition for e is

$$e(a) \doteq (f(a), (a, \mathsf{refl}_{f(a)}))$$

The inverse for e is given by

$$e^{-1}(y, (x, p)) \doteq x.$$

Then $e^{-1}(e(a)) \doteq a$, so we have $e^{-1} \circ e \sim \mathsf{id}_A$. For the other composition, take any $x : A$. To prove that for any $y : B$ and $p : f(x) = y$

$$(y, (x, p)) = (f(x), (x, \mathsf{refl}_{f(x)}))$$

we use path induction. Therefore, we can take $y \doteq f(x)$ and $p \doteq \mathsf{refl}_{f(a)}$, and then we have that

$$e(e^{-1}(y, (x, p))) = (y, (x, p))$$

as desired.
The homotopy H is given by

$$H(x) \quad \doteq \quad \mathsf{refl}_{f(x)} \quad : f(x) = \mathsf{pr}_1(e(x))$$