HoTT lecture 5 Q&A (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs

20 July 2022

Q1 What does 'a priori weaker' mean here?

I think it means we are not assuming functional extensionality

It means that we won't always have that homotopy implies an equality of functions

But equality of functions implies homotopy?

and we are talking propositional equality right?

yes. consider the type

$$(f, g : (a : A) \to Ba) \to (p : (f = g)) \to (f \sim g)\,,$$

e.g. for all dependent functions f, g from A to Ba, and all identifications p from f to g, we have a homotopy $f \sim g$. But here we can do path induction, suppose $g \doteq f$ and $p \doteq \mathsf{refl}$, then our homotopy is given by refl_{fx} at every point in A

Q2 can we still fill the poll for providing feedback?

Please do so!

Q3 In what sense can two functions with equal values be different?

We are saying that it may not be provable that they are equal as elements of a function type $A \to B$.

Intensionally, meaning in the way they're defined. For examples, quicksort and bogosort have the same output on all inputs, but any complexity theorist will tell you they're not the same function.

Being definitionally not equal ie esay: you just give them different definitions. The question of, if you can prove that two formulas give the same outputs, then are the functions equal, is that of function extensionality

Extensionality is precisely saying that we don't care about how functions are defined, only about they're outputs

or their

Okay, so there would be a homotopy between quicksort and bogosort, but they are not equal. Thanks

I can give you the most standard counterexample of the failure of function extensionality in classical math:

Consider the finite field with two elements the the polynomial $(x-0)(x-1)$

As polynomials, this is distinct from the zero polynomial, despite having a root at every element

re: mergesort and bogosort, makes me wonder if there has been work done within HoTT in proving propositions about computational complexity of functions and types in HoTT

Don't know about HoTT, but look up Jan Hoffmann's and collaborators' work on RaML

cool ty

YW i saw him lecture on it a bit at another summer school a few years ago, those videos are online (look up OPLSS16)

Q4 If equality is thought of as a path between the objects, why not take homotopy to be the equality type between f and g in the dependent function type?

Good idea! We'll get to that

(that's what function extensionality says)

shh! no spoilers :)

it's for the people following along at home

Don't spoil it for people following along at home :)

Q5 Does our formalism so far guarantee that when we piece surjections together the associative law holds?

Composition is definitionally associative

But concatenation of the equality proofs is propositionally associative, so you have to prove it

And keep in mind that we've only defined split surjections so far

I guess I missed what the adjective 'split' means

I thought 'split surjections = surjections' for us?

Agreed, that's what page 1 of the notes seems to suggest

Maybe I will ask it in a separate question

Type theoretic choice states that surjections in the naive sense are the same as functions having a right inverse up to homotopy by naive sense you mean left cancellation, yes?

yes, for all b in B there exists a in A such that $fa = b$ (as a Pi-Sigma statement)

ah, the actually naive sense, not the categorically naive sense

Yes, the naivest of senses

Q6 Are split surjections the same as surjections? Put it another way, are split epi = epi (in whatever category we are working in)?

They are not :) For epis between sets, that is the axiom of choice. For surjections between general types, it's provably false that every epi splits!

What about ‘type theoretic axiom of choice‘?

There's a couple different formulations of the axiom of choice. The one Ulrik mentioned is allowed in HoTT. The one you need to prove that all epis split, is not (if I recall correctly)

The type theoretic axiom of choice is a terrible name IMO, it says that every split surjection is a split surjection. I like calling it the 'axiom of nonchoice' as a joke. The main point is that the Σ type which shows up there has *already made all the choices*

Q7 Just to be clear, all these laws are actually provable in regular type theory (e.g, Martin Lof type theory), without any fancy modern machinery beyond that?

Yes, and they were exercises in the last Agda problem set if you want to do it yourself :)

Yup: They follow from the laws on the identity types, they just need to be 'lifted under' the functions. So they're provable, just annoying

Q8 Is there a proof in cubical Agda that equalities are a groupoid?

Here's one: https://1lab.dev/1Lab.Path.Groupoid.html#types-are-groupoids

Q9 Is there any relation between the right and left inverses? (In set theory they must be equal.)

We should get to this in a bit!

In (1-)category theory too

Q10 Does g in sec have to be equal to h in retr?

We'll get to this now :)

Not in the definition of a bi-invertible map; you can prove that they are equal, but that proof constains information

Q11 $\mathsf{has\text{-}inv}(\mathsf{f})$ and $\mathsf{is\text{-}equiv}(\mathsf{f})$ are logically equivalent? I.e. they have maps going both directions?

Yes

But just not equivalent in this stronger sense

Correct, not in general

Q12 If we can have h, a right sided inverse for f, so $h \circ f = \mathsf{id}_A$, how does that work if f doesn't hit every point in B? Is it a partial function? Or am I missing something

We don't have partial functions. Every function is total

Never mind, I just realized that it will still work normally, bc we still have to define for h what it does on the elements of B that f does not hit. Thanks for answering Jacob

Q13 Do proofs of these equivalences typically involve universes, or can they be done more easily?

They can be done just be re-arranging data

great, that is what I suspected

For example, for $A \times B \simeq B \times A$, the functions just swap the components of the pairs and the homotopies are done by splitting the argument into a pair

Q14 Is there a logical/proposition interpretation of homotopy and equivalence?

They are, privided axioms, the notions of equality for functions and types

By Curry-Howard, doesn't $f \sim g$ just become 'f and g are pointwise equal' (using the definition on the top of slide 4)? Would that be a 'proposition interpretation'?

Q15 Why can't we do exponentials if we can do products (without function extensionality?

Why can't we do exponentials if we can do products (without function extensionality?

We have function types in type theory, but in a more general context, not every category with products is cartesian closed

Forgot to say this earlier, thx

Q16 where can I get a proof of the argument that there are models of MLTT with only sets?

I like Martin Hofmann's "Syntax and Semantics of Dependent Types", which explicitly defines the set model (as a cwf)

My STLC project formally specifies what it means to be a model of STLC and constructs such a model: https://github.com/FrozenWinters/stlc

Great. I guess I was hoping for something that presupposes nothing about computer science, but rather a purely mathematical argument.

Q17 Is funext also an axiom or does it follow from UA?

live answered

Q18 What happens to the set interpretation of Hott if we assume UA?

It gets a little weird. Basically, univalence for sets says that sets X and Y are equal to each other ($X = Y$) if we can construct a bijection between them

It goes away — it is not possible to construct an interpretation of MLTT+UA into sets, only of MLTT+funext. Well, I should cover my bases: It is not possible to construct a *nontrivial* model into sets

Interesting, thank you!

You can have a model of MLTT + 'there is a univalent universe of propositions' in the category of sets! But generally UA means 'there are infinitely many univalent universes of *arbitrary types*'

In particular, in the standard interpretation of types as sets univalence is false! So it's a genuinely new idea.

Right, I think that without UA, we don't know about the homotopic structure of universes, but suddenly, with it, we have the existence of types that are not sets. For example, the universe of sets is a groupoid

Q19 In the language of MLTT: Does taking UA as an axiom mean that we have an introduction rule for terms of the type of UA?

Yes. We're postulating a term ua, of the type Ulrik wrote down

Q20 Why are we defining equality separately for Σ again? Is it because it's more useful than the standard equality?

We are charictarising equality in Sigma types and we don't have to assume any axioms to get this charictarisation

We're characterizing the identity type in Sigma types to make it easier to prove that pairs are equal

I see, thanks

A *big part* of HoTT is computing what the identity types of types 'work out to be'. Like, we know that the identity type of Σ is the identity type of Σ, but can we say any more? And in this case, we can! We've actually already done so for other types before: We've done it for the natural numbers and the booleans

I see

In the future we'll be able to piece together these characterisations of identity types to compute that, e.g., the identity type in the type of groups (relative to some universe) is the type of group isomorphisms

hm interesting

Q21 Can we talk about function extensionality in CT?

What is the notion of pointwise equality in CT is the first question that we should ask

Maybe we can work in concrete categories, not sure

Q22 I'm interested how the Univalence Axiom might be related to the Yoneda lemma? Could he please preview any connection?

Hmm nothing obvious comes to mind, but probably some connection

There are a few versions of the Yoneda lemma in HoTT. This may be a good discussion on Discord!

this exists https://www.math.lmu.de/ petrakis/Univalence.pdf

Nice!

Q23 Will we do a similar identification thing for Pi types? or is that function extensionality?

Yep, that's function extensionality