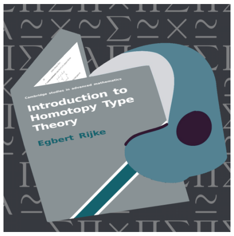

Worksheet 7

HoTTEST Summer School 2022

The HoTTEST TAs, and

30 July 2022

1 $(\star)$

Consider two embeddings $f : A \hookrightarrow B$ and $g : B \hookrightarrow C$. Construct a function

$$\mathsf{is\text{-}equiv}(g \circ f) \to (\mathsf{is\text{-}equiv}(f) \times \mathsf{is\text{-}equiv}(g))\,.$$

2 $(\star\star)$

1. Let A be a type. Prove that the canonical map $\emptyset \xrightarrow{!_A} A$ is an embedding.
2. Let A and B be types. Prove that the inclusions $\mathsf{inl} : A \to A + B$ and $\mathsf{inr} : B \to A + B$ are embeddings.
3. Let A and B be types. Prove that $\mathsf{inl} : A \to A + B$ is an equivalence if and only if $B \simeq \emptyset$.

   Conclude that if both A and B are contractible, then $A + B$ is *not* contractible.

3 (⋆⋆)

Consider a commuting triangle

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & X & \end{tikzcd}.$$

1. Suppose that g is an embedding. Prove that f is an embedding if and only if h is one.
2. Suppose that h is an equivalence. Prove that f is an embedding if and only if g is one.

4 (⋆⋆)

Let A, B, and C be types and let $f : A \to C$ and $g : B \to C$ be maps. Prove that the following are logically equivalent.

1. The map $[f, g] : A + B \to C$ is an embedding.
2. Both f and g are embeddings, and $f(a) \neq g(b)$ for all $a : A$ and $b : B$.

5 $(\star\star)$

1. Let $f, g : \prod_{x:A} B(x) \to C(x)$. Construct a function

$$\left(\prod_{x:A} f(x) \sim g(x)\right) \to (\mathsf{tot}(f) \sim \mathsf{tot}(g)) .$$

2. Let $f : \prod_{x:A} B(x) \to C(x)$ and $g : \prod_{x:A} C(x) \to D(x)$. Construct a homotopy

$$\mathsf{tot}(\lambda x.g(x) \circ f(x)) \sim \mathsf{tot}(g) \circ \mathsf{tot}(f).$$

3. For any type family B over A, construct a homotopy

$$\mathsf{tot}(\lambda x.\mathsf{id}_{B(x)}) \sim \mathsf{id}_{\sum_{x:A} B(x)}.$$

4. Let $a : A$ and let B be a type family over A. Prove that if $B(x)$ is a retract of $a = x$ for each $x : A$, then $(a = x) \simeq B(x)$ for each $x : A$.

5. Let $f : \prod_{x:A} (a = x) \to B(x)$. Prove that if each $f(x)$ has a section, then f is a family of equivalences.

   As a consequence, for any function $k : X \to Y$, if

$$\mathsf{ap}_k(x, y) : (x = y) \to (k(x) = k(y))$$

   has a section for each $x, y : X$, then k is an embedding.

6 $(\star\star\star)$

We say that a map $f : A \to B$ is *path-split* if

1. f has a section and
2. the map $\mathsf{ap}_f(x, y) : (x = y) \to (f(x) = f(y))$ has a section for each $x, y : A$.

Prove that a map $f : A \to B$ is an equivalence if and only if it is path-split.