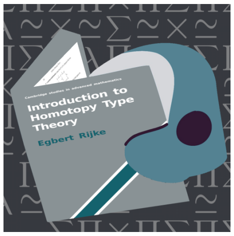

Worksheet 12 (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs

19 August 2022

1 $(\star)$

For each of the following types, state how many unique[1] terms there are of that type, and list them. *Hint:* First figure out how many there are before defining them. Use Lemma 17.5.8 and think inductively!

(a)

$$\binom{\mathsf{Fin}\ 3}{\mathsf{Fin}\ 2}$$

There are three such elements:

$$(\mathbf{Fin}\ 2, a),\ (\mathbf{Fin}\ 2, b),\ (\mathbf{Fin}\ 2, c) \quad : \quad \sum_{X : \mathcal{U}_{\mathsf{Fin}\ 2}} X \hookrightarrow_{\mathsf{d}} \mathbf{Fin}\ 3$$

$$\begin{aligned} a(0) &\doteq 0 \\ a(1) &\doteq 1 \\ b(0) &\doteq 0 \\ b(1) &\doteq 2 \\ c(0) &\doteq 1 \\ c(1) &\doteq 2 \end{aligned}$$

Everything else is equal to one of these three, according to the characterization of identity types of binomial types we obtained.

(b)

$$\binom{\mathsf{Fin}\ n}{\mathbb{1}}$$

There are n such elements:

$$(\mathbb{1}, \mathbf{const}_j) : \binom{\mathbf{Fin}\ n}{\mathbb{1}} \qquad \text{for each } j : \mathsf{Fin}\ n$$

[1] Unique up to identity – assume univalence and function extensionality.

(c)

$$\binom{\mathsf{Fin}\ n}{\mathsf{Fin}\ n + \mathbb{1}}$$

This type is empty. There cannot be an embedding $\textbf{Fin}(n)+\mathbb{1} \hookrightarrow \textbf{Fin}\ n$.

2 $(\star\star)$

Consider a type A.

(a) We call a point $a : A$ *isolated* if the map $\mathsf{const}_a : \mathbb{1} \to A$ is a decidable embedding. Construct an equivalence

$$\binom{A}{\mathbb{1}} \simeq \sum_{a:A} \mathsf{is\text{-}isolated}(a).$$

Given (a, σ) on the right-hand side, we can construct the element

$$E(a, \sigma) := (\mathbb{1}, (\textbf{const}_a, \sigma)) \colon \sum_{X:\mathcal{U}_{\mathbb{1}}} X \hookrightarrow_{\mathsf{d}} A.$$

This map is an equivalence: if (X, ψ) is any element of $\binom{A}{\mathbb{1}}$ then, since we know $\|X \simeq \textbf{Fin}\ 1\|$, it must be that X is a singleton, i.e. $X = \mathbb{1}$. So, if ψ is a map $\mathbb{1} \to A$, then by function extensionality $\psi = \textbf{const}_{\psi(\star)}$, so $(X, \psi) = E(a, \sigma)$ for some $a : A$. So the fibers of E are inhabited. We can readily check by function extensionality that E indeed has contractible fibers.

(b) Show that if A is a set, then $\binom{A}{\mathbb{1}} \simeq A$

This is a corollary of the previous part. This is because every point of a set is isolated: for any $a' : A$, the fiber

$$\textbf{fib}_{\textbf{const}_a}(a') \doteq \sum_{x:\mathbb{1}} \textbf{const}_a(x) = a'$$

is equivalent to the identity type $a = a'$, which is a decidable proposition by the hypothesis that A is a set. So, since $\textbf{is-isolated}$ is a proposition, we have

$$\binom{A}{\mathbb{1}} \simeq \sum_{a:A} \textbf{is-isolated}(a) \simeq \sum_{a:A} \mathbb{1} \simeq A$$

(c) Construct an equivalence

$$\binom{A}{\mathbb{1}} \simeq \left(\sum_{X:\mathcal{U}} (X+\mathbb{1}) \simeq A\right)$$

conclude that the map $X \mapsto X + \mathbb{1}$ on a univalent universe $\mathcal{U}$ is 0-truncated.

By part (a), it suffices to show that the right-hand side is equivalent to

$$\sum_{a:A} \textsf{is-isolated}(a).$$

Given $X : \mathcal{U}$ and an equivalence $e : (X+\mathbb{1}) \simeq A$, let's write a_0 for $e(\textsf{inr}\ \star)$. To see that $\textsf{const}_{a_0}$ is a decidable embedding, pick arbitrary $a : A$. Then we need to show that the fiber

$$\textsf{fib}_{\textsf{const}_{a_0}}(a) \doteq \sum_{x:\mathbb{1}} \textsf{const}_{a_0}(x) = a$$

is a decidable proposition. This follows because a_0 is the image of $\textsf{inr}(\star)$ under an equivalence: by the disjointedness of coproducts, $\textsf{inr}(\star)$ is an isolated point of $X + \mathbb{1}$. It follows that a_0 is an isolated point of A.
In the other direction: if we have an isolated point $a_0 : A$, then we know that $\textsf{fib}_{\textsf{const}_{a_0}}(a)$ is a decidable prop for each $a : A$. Then let's put

$$X \doteq \sum_{a:A} \neg\textsf{fib}_{\textsf{const}_{a_0}}(a)$$

Now define an equivalence $e : (X+\mathbb{1}) \simeq A$ by putting

$$e(\textsf{inl}(a, \_)) \doteq a$$
$$e(\textsf{inr}\ \star) \doteq a_0.$$

This is an equivalence: for any $a : A$, we have a term

$$\tau_a : \textsf{fib}_{\textsf{const}_{a_0}}(a) + \neg(\textsf{fib}_{\textsf{const}_{a_0}}(a)$$

since $\textsf{const}_{a_0}$ is decidable. If τ_a is $\textsf{inl}(p)$ then $a = a_0$ and $\textsf{inr}(\star)$ is the unique inhabitant of $\textsf{fib}_e(a)$. If τ_a is $\textsf{inr}(q)$, then

$$e(\textsf{inl}(a, q)) = a$$

and, moreoever, if there were another element of $X + \mathbb{1}$ mapped to a by e, then it would have to be $\textsf{inl}(a', q')$ for some $a' : A$ and $q' : \neg(\textsf{fib}_{\textsf{const}_{a_0}}(a'))$. But

$$a = e(\textsf{inl}(a', q')) \doteq a'$$

So $a = a'$. Since $\neg(\mathsf{fib}_{\mathsf{const}_{a_0}}(a))$ is a proposition, we also get that $q = q'$. So e has contractible fibers, and we're done.

(d) More generally, construct an equivalence

$$\binom{A}{B} \simeq \sum_{X:\mathcal{U}_B} \sum_{Y:\mathcal{U}} X + Y \simeq A$$

Given an $X : \mathcal{U}_B$ and a $Y : \mathcal{U}$ and an equivalence $e : X + Y \simeq A$, we construct an element of $\binom{A}{B}$ in the following way: X is, of course, the element of $\mathcal{U}_B$ we need, and then we obtain an embedding $\psi : X \hookrightarrow_{\mathsf{d}} A$ by composing the left injection map $X \hookrightarrow X + Y$ with the equivalence e. We can check that embeddings are preserved by composing with an equivalence, so this is an embedding. It is decidable because, given $a : A$, we can construct a term

$$\mathsf{fib}_\psi(a) + \neg\mathsf{fib}_\psi(a)$$

We do this by casing on $e^{-1}(a)$. If $e^{-1}(a)$ is $\mathsf{inl}(x)$ for some $x : X$, then, by construction, x is in $\mathsf{fib}_\psi(a)$. Otherwise, if $e^{-1}(a)$ is $\mathsf{inr}(y)$, then $e^{-1}(a)$ is not $\mathsf{inl}(x)$ for any $x : X$ and hence a is not $\psi(x)$ for any $x : X$, i.e. $\mathsf{fib}_\psi(a)$ is empty.

3 (⋆⋆)

Given a type A, the type of **unordered pairs** in A is defined to be

$$\mathsf{unordered\text{-}pairs}(A) :\equiv \sum_{X:BS_2} X \to A$$

(a) Construct an embedding

$$\binom{A}{\mathsf{bool}} \hookrightarrow \mathsf{unordered\text{-}pairs}(A)$$

Why does $\mathsf{unordered\text{-}pairs}(A)$ have, in general, more elements than $\binom{A}{\mathsf{bool}}$? Which elements of $\mathsf{unordered\text{-}pairs}(A)$ are *not* in the image of this embedding?

Observe that $\mathcal{U}_{\mathsf{bool}}$, the type of types $X : \mathcal{U}$ such that $\|X \simeq \mathsf{bool}\|$ is the same thing as BS_2. So we can say

$$\mathsf{unordered\text{-}pairs}(A) = \sum_{X:\mathcal{U}_{\mathsf{bool}}} X \to A.$$

Since $\binom{A}{\mathsf{bool}}$ is defined as

$$\sum_{X:\mathcal{U}_{\mathsf{bool}}} X \hookrightarrow_{\mathsf{d}} A$$

we can define a function $\binom{A}{\mathsf{bool}} \to \mathsf{unordered\text{-}pairs}(A)$ by sending each (X, ψ) to itself ("forgetting" that ψ is a decidable embedding). This is an embedding, by function extensionality.
The unordered pairs which are in $\binom{A}{\mathsf{bool}}$ are the ones whose two components are *distinct*. However, $\mathsf{unordered\text{-}pairs}(A)$ additionally contains unordered pairs whose components are the same. For example,

$$(\mathsf{bool}, \mathsf{const}_7) : \mathsf{unordered\text{-}pairs}(\mathbb{N})$$

because bool is a 2-element type and $\mathsf{const}_7 : \mathsf{bool} \to \mathbb{N}$. This encodes the unordered pair whose two components are both 7. But const_7 is not an embedding – the fiber of 7 is bool itself, which is not a proposition – so this is not an element of $\binom{\mathbb{N}}{\mathsf{bool}}$.

(b) The type of homotopy commutative binary operations from A to B is defined as

$$\mathsf{unordered\text{-}pairs}(A) \to B.$$

Show that if B is a set, then this type is equivalent to the type

$$\sum_{m:A\to A\to B} \prod_{x,y:A} m(x,y) = m(y,x).$$

Begin by observing that

$$\left(\sum_{X:BS_2} X \to A\right) \to B \qquad \simeq \qquad \prod_{X:\mathcal{U}} \|X \simeq \textsf{Fin}\ 2\| \to (X \to A) \to B$$

To define something of the right-hand-side type, we write a function which takes in a $X : \mathcal{U}$ and then produce a function $\|X \simeq \textsf{Fin}\ 2\| \to (X \to A) \to B$. Since B is a set, then by Theorem 14.4.6, it suffices to define a function

$$q : (X \simeq \textsf{Fin}\ 2) \to (X \to A) \to B$$

which is constant in its first argument: $q(e) = q(e')$ for all e, e'.
So, if we have a commutative binary function m, we define the corresponding function unordered-pairs$(A) \to B$ as follows: given $X : \mathcal{U}$ and $e : X \simeq \textsf{Fin}\ 2$ and $\psi : X \to A$, we define $q(e, \psi)$ to be $m(\psi(e^{-1}(0)), \psi(e^{-1}(1)))$. This is constant in e: if $d : X \simeq \textsf{Fin}\ 2$, then, by function extensionality, either

$$e = d \qquad \text{or} \qquad e^{-1}(0) = d^{-1}(1) \text{ and } e^{-1}(1) = d^{-1}(0)$$

Intuitively: X only has two elements, so two different isomorphisms between X and Fin 2 must be the same, up to a permutation of Fin 2. But this implies that $q(d) = q(e)$: if $d = e$ then this follows immediately. Otherwise, if d and e are equal up to the nontrivial permutation of Fin 2, then

$$\begin{aligned} q(e, \psi) &\doteq m(\psi(e^{-1}(0)), \psi(e^{-1}(1))) \\ &= m(\psi(e^{-1}(1)), \psi(e^{-1}(0))) && (m \text{ commutative}) \\ &= m(\psi(d^{-1}(0)), \psi(d^{-1}(1))) && (\text{above}) \\ &\doteq q(d, \psi). \end{aligned}$$

So by function extensionality, $q(d) = q(e)$. So we've satisfied Theorem 14.4.6, so we have our homotopy commutative operation.
Conversely, if we have M : unordered-pairs$(A) \to B$, then obtain $m : A \to A \to B$ by putting

$$m(a, a') \doteq M(\textsf{bool}, \lambda b.\text{if } b \text{ then } a \text{ else } a')$$

where $(\lambda b.\text{if } b \text{ then } a \text{ else } a')$ is the function bool $\to A$ sending true to a and false to a'. Check that this m is commutative: when calculating $m(a', a)$ instead, we form the element

$$(\textsf{bool}, \lambda b.\text{if } b \text{ then } a' \text{ else } a) \quad : \textsf{unordered-pairs}(A)$$

But elements $(X, \psi), (Y, \mu)$ of unordered-pairs(A) are equal if we have an equivalence $e : X \simeq Y$ such that $\psi \sim \mu \circ e$. Notice that the two elements we have constructed are indeed equal, by the nontrivial equivalence bool $\simeq$ bool.

It remains to verify that these two operations are inverse, but this is routine.

4 $(\star\star)$

Consider the following claim.

$$\binom{\mathsf{Fin}(n)}{\mathsf{bool}} \simeq \sum_{k:\mathsf{Fin}(n)} \mathsf{Fin}(k) \tag{*}$$

Prove (*) by induction on n, using the equivalences from Lemma 17.5.8 and the identities proved above. You should not need to unfold the definition of $\binom{A}{B}$ or explicitly construct any decidable embeddings.

Start with $n \doteq 0$. The Lemma tells us that

$$\binom{\emptyset}{B + \mathbb{1}} \simeq \emptyset$$

so, since $\mathsf{bool} = \mathbb{1} + \mathbb{1}$, the left-hand side is equivalent to $\emptyset$. Since $\mathsf{Fin}(0) \doteq \emptyset$, the right hand side is also empty.
Now suppose

$$\binom{\mathsf{Fin}(n)}{\mathsf{bool}} \simeq \sum_{k:\mathsf{Fin}(n)} \mathsf{Fin}(k)$$

for some $n : \mathbb{N}$. Again, using the Lemma and basic identities of finite sets, we have

$$\binom{\mathsf{Fin}(\mathsf{suc}\ n)}{\mathsf{bool}} = \binom{\mathsf{Fin}(n) + \mathbb{1}}{\mathbb{1} + \mathbb{1}} \simeq \binom{\mathsf{Fin}(n)}{\mathbb{1}} + \binom{\mathsf{Fin}(n)}{\mathsf{bool}}$$

By Question 2b above and the fact that $\mathsf{Fin}(n)$ is a set, we know $\binom{\mathsf{Fin}\ n}{\mathbb{1}} \simeq \mathsf{Fin}\ n$. Applying the inductive hypothesis, we now have

$$\binom{\mathsf{Fin}(\mathsf{suc}\ n)}{\mathsf{bool}} \simeq (\mathsf{Fin}\ n) + \sum_{k:\mathsf{Fin}(n)} \mathsf{Fin}(k)$$

The right-hand side can easily be seen to be equivalent to $\sum_{k:\mathsf{Fin}(\mathsf{suc}\ n)} \mathsf{Fin}(k)$, as desired.

5 $(\star\star)$

Given a finite type A, show that the following are equivalent:

(i) The type of all decidable equivalence relations on A

(ii) The type of all surjective maps $A \to X$ into a finite type X

(iii) The type of finite types X equipped with a family $Y : X \to \mathsf{Fin}$ of finite types, such that each $Y(x)$ is inhabited and equipped with an equivalence

$$e : \left(\sum_{x:X} Y(x)\right) \simeq A$$

(iv) The type of all decidable partitions of A, i.e. the type of all $P : (A \to \mathsf{dProp}) \to \mathsf{dProp}$ such that each Q in P is inhabited, and such that for each $a : A$ the type of $Q : A \to \mathsf{dProp}$ such that $Q(a)$ holds and $P(Q)$ holds is contractible.

First, let's show (i) is equivalent to (ii). If we have a surjective map $f : A \to X$ into a finite X, we define a decidable equivalence relation $R : A \to A \to \mathsf{dProp}$ by

$$R_f(a, a') \doteq f(a) = f(a')$$

This is evidently an equivalence relation, and a decidable one because X is finite, i.e. a set. Conversely, given a decidable equivalence relation R, we define X_R as the image of R as a map from A to $A \to \mathsf{dProp}$

$$X_R \doteq \sum_{Q:A\to\mathsf{dProp}} \left\| \sum_{a:A} R(a) = Q \right\|$$

The surjection from A to X_R sends a to $(R(a), |\mathsf{refl}|)$. It is a surjection by construction: given $(Q, k) : X_R$, we have k as a proof that there merely exists some $a : A$ such that $R(a) = Q$. We can also check that X is finite. To see that these two operations are mutually inverse, we can check that X_{R_f} is equivalent to X: if $Q : A \to \mathsf{dProp}$ is equal to $R_f(a)$ for some $a : A$, then $Q(a')$ holds iff $f(a') = f(a)$. So Q corresponds to the element $f(a) : X$. And for each $x : X$, there is an $a : A$ such that $f(a) = x$, and so x corresponds to $(R(a), |\mathsf{refl}|) : X_R$. Similarly, if we start with R, then form X_R and f, and then obtain the relation R_f, it will be equal to the R we started with.
Next, (ii) is equivalent to (iii). Given surjective $f : A \to X$ as in (ii), we define

$$Y(x) \doteq \mathsf{fib}_f(x).$$

This is a subtype of a finite type, hence finite as well. The surjectivity guarantees that each $Y(x)$ is inhabited, and then the equivalence

$$\sum_{x:X} \mathsf{fib}_f(x) \simeq A$$

is obtained by sending $(x, (a, p))$ to a – check that this defines an equivalence. Now, going the other direction: given the data of (iii), define $f : A \to X$ as the composition

$$A \xrightarrow{e^{-1}} \sum_{x:X} Y(x) \xrightarrow{\mathsf{pr}_1} X$$

This is surjective: because each $Y(x)$ is inhabited, the fibers of pr_1 are inhabited, so it's surjective; and e^{-1} is an equivalence, hence also surjective; and surjections are closed under composition. Now, to see that these processes are inverse, start with the surjection f from (ii) and observe that if $Y(x)$ is defined as $\mathsf{fib}_f(x)$, then, for any $a : A$, e^{-1} sends a to some $(x,(a,p))$ where $p : f(a) = x$, so then $\mathsf{pr}_1(e^{-1}(a)) = f(a)$, and thus the surjection we obtain is equal to the f we began with. For the other compositon, start with the data of (iii), then define f as $\mathsf{pr}_1 \circ e^{-1}$, and then define $Y(x)$ as

$$Y(x) \doteq \mathsf{fib}_{\mathsf{pr}_1 \circ e^{-1}}(x)$$

. But we can calculate:

$$\begin{aligned}
\mathsf{fib}_{\mathsf{pr}_1 \circ e^{-1}}(x) &\doteq \sum_{a:A} \mathsf{pr}_1(e^{-1}(a)) = x \\
&\simeq \sum_{x':X} \sum_{y':Y(x')} \mathsf{pr}_1(e^{-1}(e(x',y'))) = x && (e) \\
&\simeq \sum_{x':X} \sum_{y':Y(x')} \mathsf{pr}_1(x',y') = x \\
&\doteq \sum_{x':X} \sum_{y':Y(x')} x' = x \\
&\simeq Y(x)
\end{aligned}$$

So, by univalence, we get that this definition of Y gives us the Y we started with. Finally, we show that (ii) is equivalent to (iv). Given a surjection $f : A \to X$, let's define for each $x : X$ the map $F_x : A \to \mathsf{dProp}$ by

$$F_x(a) \doteq (f(a) = x)$$

This is a decidable proposition, because X is a set. Now, note that, since dProp has decidable equality, so too does $A \to \mathsf{dProp}$, and thus we can define $P : (A \to \mathsf{dProp}) \to \mathsf{dProp}$ by

$$P(Q) \doteq \exists_{x:X} Q = F_x$$

To prove that "each Q in P is inhabited", i.e.

$$P(Q) \to \exists_{a:A} Q(a)$$

we proceed as follows: if $P(Q)$ holds, then since we're proving a proposition, we use the universal property of the truncation to obtain the $x : X$ such that $Q = F_x$. Appealing to the surjectivity of f and again using the universal property, obtain an $a : A$ such that $f(a) = x$. It then follows that $Q(a)$ holds, by definition of F_x. And then finally we can check that, for each $a : A$ and each Q, if $P(Q)$ and $Q(a)$ both hold, then $Q = F_x$ for some x in the fiber of a, i.e. the type of Q such that $P(Q)$ and $Q(a)$ hold is contractible with center F_x.

Conversely, given the decidable partition P, we define the X called for in (ii) to be

$$X_P \doteq \sum_{Q:A\to\mathsf{dProp}} P(Q).$$

And then we define a surjection from A to X_P by sending each $a : A$ to the uniquely-determined $Q : A \to \mathsf{dProp}$ such that $P(Q)$ and $Q(a)$. This is a surjection because of the requirement that each Q in P is inhabited. We can readily check that X must be finite as well. We can also reason by function extensionality that these two operations are inverse of each other.