### Semantics of HoTT

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## Why semantics?

- ► Soundness, completeness.
- ► Algebraic abstraction of syntactic constructions.
- Find (more general) syntactic proofs of facts about a given model.
- ▶ Justify the claim: "HoTT really is homotopy theory."

## Why functorial semantics?

### A general way of describing models:

- ▶ The syntax of a type theory  $\mathbb{T}$  forms a category with structure  $\mathbb{C}_{\mathbb{T}}$  (the *syntactic category*).
- ▶ A  $\mathbb{T}$ -model in a category  $\mathcal{C}$  is a structure-preserving functor  $\mathbb{C}_{\mathbb{T}} \to \mathcal{C}$ .
- $ightharpoonup \mathbb{C}_{\mathbb{T}}$  is *initial* among categories with said structure.
- ► Therefore a T-model in a category C is simply a choice of said structure in C.

## The syntactic category of HoTT

Defining the syntactic category  $\mathbb C$  only requires the *structural* rules of MLTT:

$$\begin{array}{cccc} & & & \frac{\Gamma \vdash A \text{ type}}{\vdash \Gamma, x : A \text{ ctxt}} & \text{EXT} \\ \\ & & \frac{\vdash \Gamma, x : A, \Delta \text{ ctxt}}{\Gamma, x : A, \Delta \vdash x : A} & \text{VAR} \\ \\ & & \frac{\Gamma, \Delta \vdash \mathcal{J} & \Gamma \vdash A \text{ type}}{\Gamma, x : A, \Delta \vdash \mathcal{J}} & \text{WEAK} \\ \\ & & \frac{\Gamma, x : A, \Delta \vdash \mathcal{J} & \Gamma \vdash a : A}{\Gamma, \Delta[a/x] \vdash \mathcal{J}[a/x]} & \text{SUBST} \end{array}$$

- ▶ The objects of  $\mathbb{C}$  are contexts  $\Gamma = x_1:A_1,\ldots,x_k:A_k$ .
- lacktriangle Morphisms  $\Delta \to x_1 : A_1, \dots, x_k : A_k$  are lists  $\tau = (t_1, \dots, t_k)$  of terms

$$\Delta \vdash t_1 : A_1$$
 
$$\vdots$$
 
$$\Delta \vdash t_k : A_k[t_1/x_1, \dots, t_{k-1}/x_{k-1}].$$

The identity morphism is just the list of variables  $(x_1, \ldots, x_k)$ , and composition is defined by substitution.

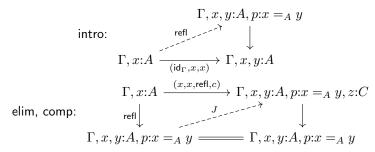
### Structure on $\mathbb C$

- ▶ The empty context is a terminal object.
- ▶ Given  $\Gamma \vdash A$  and a morphism  $\Delta \xrightarrow{\tau} \Gamma$ , substitution defines a

$$\begin{array}{c} \Delta, x \mathpunct{:}\! A[\tau] \xrightarrow{(\tau, x)} \Gamma, x \mathpunct{:}\! A \\ \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \\ \Delta \xrightarrow{\quad \tau \quad } \Gamma \end{array}$$

For  $\Gamma$  in  $\mathbb{C}$ , let  $\mathbb{C}/\!/\Gamma$  be the full category of  $\mathbb{C}_{/\Gamma}$  on the projections  $\Gamma, x : A \to \Gamma$ . Substitution defines pullback functors  $\mathbb{C}/\!/\Gamma \to \mathbb{C}/\!/\Gamma, x : A$ . The rules for  $\Sigma$ -types provide left adjoints to these functors, while those for  $\Pi$ -types provide right adjoints.

► The rules for Id-types give



▶ The rules for a type universe give, for every morphism  $\Gamma \xrightarrow{A} \mathcal{U}$ . a

$$\begin{array}{cccc} \Gamma, x{:}A & \xrightarrow{(A,x)} & X{:}\mathfrak{U}, x{:}X \\ & & \downarrow & & \downarrow \\ & \Gamma & \xrightarrow{A} & \mathfrak{U}. \end{array}$$
 choice of pullback square

#### Definition

A **universe** in a category  $\mathcal{C}$  is a map  $U' \to U$  along with for every

morphism 
$$X \xrightarrow{f} U$$
, a choice of pullback square  $X \xrightarrow{f} U$ 

## Digression: CCCs

- ▶ An object X of a category  $\mathfrak C$  is **squarable** if  $\forall Y$  in  $\mathfrak C$ , the product  $X \times Y$  exists in  $\mathfrak C$ ,
- $\Leftrightarrow$  the functor  $\mathcal{C}_{/X} \to \mathcal{C}$  from the slice category has a right adjoint.
- ▶ A squarable object X is **exponentiable** if (the right adjoint)  $\mathcal{C} \xrightarrow{X \times -} \mathcal{C}_{/X}$  has a (further) right adjoint  $X \Rightarrow -$ .
- ightharpoonup  $m {\it C}$  has finite products if it has a terminal object and every X in  $m {\it C}$  is squarable.
- $ightharpoonup {\mathcal C}$  is **cartesian closed** if it has a terminal object and every X in  ${\mathcal C}$  is exponentiable (thus also squarable).

## Digression: LCCCs

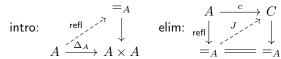
- ▶ A map  $X \xrightarrow{f} Y$  in  $\mathcal C$  is **squarable** if  $\forall Z \xrightarrow{g} Y$  in  $\mathcal C$ , the pullback  $\vdots \xrightarrow{h} \vdots \\ f^*g \downarrow \xrightarrow{\Box} \downarrow g$  exists in  $\mathcal C$ ,  $\vdots \xrightarrow{f} \vdots$
- $\Leftrightarrow$  the functor  $\mathcal{C}_{/X} \xrightarrow{f \circ -} \mathcal{C}_{/Y}$  has a right adjoint  $f^*$ .
- ▶ A squarable map  $X \xrightarrow{f} Y$  is **exponentiable** if  $\mathcal{C}_{/Y} \xrightarrow{f^*} \mathcal{C}_{/X}$  has a right adjoint  $\Pi_f$ .
- C is locally cartesian closed if all its maps are (squarable and) exponentiable,
- $\Leftrightarrow$  every slice category  $\mathcal{C}_{/X}$  is cartesian closed.

# Modeling dependent type theory in LCCCs "naïvely"

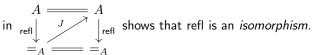
In the Set model of type theory,

- ightharpoonup Contexts are sets:  $\Gamma \in \operatorname{Set}$
- ▶ Types are indexed sets:  $\Gamma \vdash A$  corresponds to  $\{A_{\gamma} \in \text{Set} \mid \gamma \in \Gamma\}$  and  $(\Gamma, x:A) = \coprod_{\gamma \in \Gamma} A_{\gamma}$ . Equivalently, types are functions  $A \to \Gamma$ .
- $\begin{array}{c|c} & \Gamma, x : A \\ & \downarrow \\ & \Gamma \end{array} \text{ of the projection. }$
- ( $\Sigma$ -types) For  $B \to A \to \Gamma$ , we have  $(\Sigma_A B)_{\gamma} = \coprod_{a \in A_{\gamma}} B_a$
- ( $\Pi$ -types) For  $B \to A \to \Gamma$ , we have  $(\Pi_A B)_{\gamma} = \prod_{a \in A_{\gamma}} B_a$ .

► (Id types)



But in this interpretation, refl is also a type over  $=_A$ , so using elim



Consequently,  $=_A \to A \times A$  is the diagonal map  $\Delta_A$  of A, and hence is a monomorphism in Set.

**Corollary:** In the Set model, all types/contexts are 0-truncated (uniqueness of identity proofs).

So the Set model of type theory cannot be a model of HoTT. This argument also works in any LCCC.

# Models of type theory with $\Sigma, \Pi, \mathcal{U}$

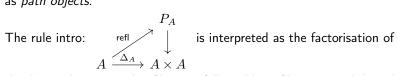
#### A LCCC C, with

- ► A chosen terminal object 1.
- ightharpoonup A universe  $U' \to U$ ,
- ▶ that is closed under  $\Sigma$ : the composite  $A \to B \to C$  of maps "in the universe" is in the universe,
- ▶ that is closed under  $\Pi$ : for any sequence  $C \to B \to A$  of maps in the universe, the exponential  $\Pi_B C \to A$  is in the universe.

(We'll ignore coherence conditions for today.)

## Models of Id-types in model categories

The first "homotopical" flavour in type theory is the observation (Awodey, Warren) that if types-in-context  $\Gamma \vdash A$  are interpreted as fibrations of a (Quillen) model category, then Id-types can be interpreted as path objects.



the diagonal as a trivial cofibration followed by a fibration, and the rules

elim, comp: 
$$P_A = P_A$$
 are given by the lifting of the trivial

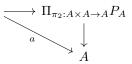
cofibration refl against the fibration  $C \to P_A$ .

## Example: Contractibility

Syntactically, the type  $\Sigma_{a:A}\Pi_{x:A}a=_Ax$  is equivalent to the type  $\Sigma_{a:A}(\lambda x.a\sim \mathrm{id}_A)$  of deformation retracts of A onto the terminal object 1 (the empty context).

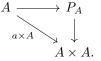
Semantically, consider a path object  $P_A \to A \times A$ . Then a section  $1 \to \Sigma_{A \to 1} \Pi_{\pi_2: A \times A \to A} P_A$  corresponds to a section  $1 \xrightarrow{a} A$  and a

commuting triangle



which, by adjointness,

corresponds to a triangle



But this right homotopy

describes a deformation retract of A onto 1.

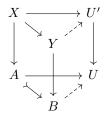
## The simplicial model of univalence

It turns out that there is a very nice model category in which we can interpret HoTT (Voevodsky). This is the model category  ${\rm sSet}$  of simplicial sets with the usual "Kan-Quillen" model structure. For a suitable cardinal  $\kappa$ 

- There is a  $\kappa$ -small (each fibre is a  $\kappa$ -small set) map of simplicial sets  $U' \to U$  that classifies (is a universe of)  $\kappa$ -small fibrations,
- ▶ that is closed under  $\Sigma$ ,  $\Pi$  and Id,
- such that U is fibrant and  $U' \to U$  is a  $\kappa$ -small fibration.
- ightharpoonup and such that  $U' \to U$  satisfies the univalence axiom.

(The construction of the fibration  $U' \to U$  is technical.)

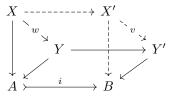
▶ in order to show that every small fibration is a pullback of  $U' \to U$ , it suffices that in every solid diagram of pullback squares as below, where  $A \rightarrowtail B$  is a monomorphism, and where the vertical arrows are small fibrations, the dashed part exists and is a pullback.



▶ Once this is done, showing that *U* is fibrant turns out to reduce to requiring that the dashed part of the following pullback square exist whenever the solid part does, where vertical arrows are fibrations, and *i* is a trivial cofibration (the "fibration extension property").



Finally, univalence comes down to the "equivalence extension property": given a cofibration i and weak equivalence w in the solid diagram below, where all vertical maps are fibrations and the front square is a pullback, the dashed part of the diagram exists, where v is a weak equivalence, vertical maps are fibrations, and all squares are pullbacks.



## Models in higher topoi

The model category sSet presents the  $\infty$ -topos of  $\infty$ -groupoids (spaces), and the model of HoTT in sSet does interpret types as arbitrary  $\infty$ -groupoids.

In fact, it is possible to extend this to an interpretation of HoTT in any Grothendieck  $\infty$ -topos.

## Type-theoretic model topoi

A type-theoretic model topos (Shulman) is a category  ${\mathcal E}$ 

- that is a 1-topos,
- with a model structure that is right proper, simplicial, combinatorial and whose cofibrations are exactly the monomorphisms,
- that is simplicially locally cartesian closed,
- that has a suitable "notion of fibred structure" that classifies all fibrations.

The key fact (Shulman) is that every type-theoretic model topos models type theory with  $\Sigma$ ,  $\Pi$ , Id, univalent universes, W-types, pushouts, truncations, etc.

## Model categories of simplicial presheaves

Given a small category A, the category  $\mathrm{Sp}A = [A^{op}, \mathrm{sSet}]$  of simplicial presheaves on A has an **injective** model structure, whose weak equivalences and cofibrations are the pointwise weak equivalences and cofibrations of simplicial presheaves.

Every Grothendieck  $\infty$ -topos can be presented by a left exact left Bousfield localisation of one of these model categories. So it would be nice if they were type-theoretic model topoi.

However, the fibrations of this model structure are difficult to characterise (unlike those of  $\mathrm{sSet}$ ), so constructing a universal univalent fibration is non-trivial.

## Injective fibrations

Given a square  $C \longrightarrow X \\ \downarrow p \\ \downarrow p$  of simplicial presheaves on A such that i is a  $D \longrightarrow Y$ 

pointwise cofibration and p is a pointwise fibration, there exist lifts  $C_a \to X_a$  for all a that do not fit into a natural transformation, but into a homotopy-coherent natural transformation.

There is an object  $\mathbf{C}(X)$  that classifies homotopy-coherent natural transformations into X, with a map  $X \to \mathbf{C}(X)$ . It so happens that X is injectively fibrant just when it is equipped with a retraction  $\mathbf{C}(X) \to X$ .

Since retractions are *structure*, this turns out to be a suitable "notion of fibred structure" that makes  $\mathrm{Sp}A$  with the injective model structure a type-theoretic model topos.