Outline

Last time: homotopies & equivalences (§9)

Today:
- A correction on joins of universes
- Contractrible types & maps (§10)

  Main result: equivalences are contractible maps

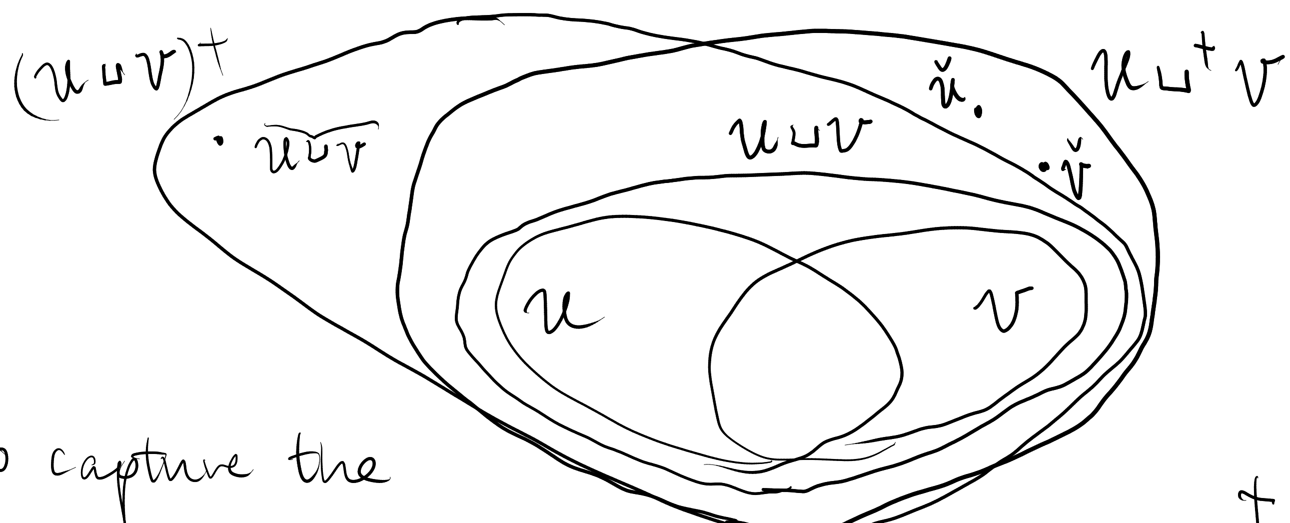
  via: coherently invertible map.

A correction regarding joins of universes

In Lecture 4 I said the join of $(\mathcal{U},\mathcal{T}_{\mathcal{U}}),(\mathcal{V},\mathcal{T}_{\mathcal{V}})$ was obtained by reflecting on the 4 type families

$$\left.\begin{array}{ll} \vdash \cancel{\mathcal{U}\ \mathsf{type}} & \vdash \cancel{\mathcal{V}\ \mathsf{type}} \\ \underbrace{X : \mathcal{U} \vdash \mathcal{T}_{\mathcal{U}}(X)\ \mathsf{type} \quad Y : \mathcal{V} \vdash \mathcal{T}_{\mathcal{V}}(Y)\ \mathsf{type}}_{\rightsquigarrow\ \mathcal{U}\sqcup\mathcal{V}} & \end{array}\right\} \rightsquigarrow \mathcal{U} \sqcup^{+} \mathcal{V}$$

but $\mathcal{U}\sqcup\mathcal{V}$ should only reflect the latter two.



In Agda, we use universe polymorphism to, among other things, give types to type formers, e.g.,

$$+ : \mathcal{U} \to \mathcal{V} \to \mathcal{U}\sqcup\mathcal{V}$$

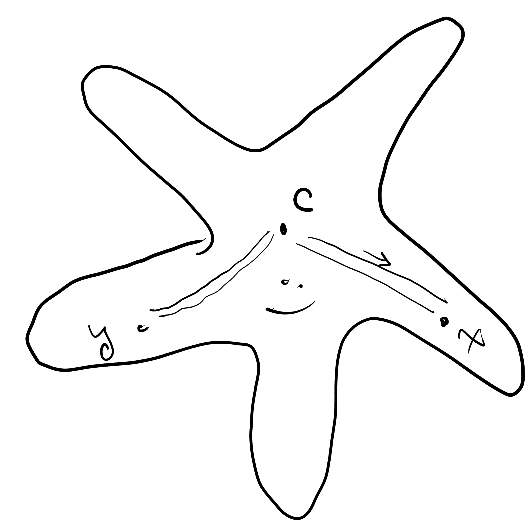
To capture the $\check{+}$ on $\mathcal{U}$, we'd need to say, but we haven't, that $\mathcal{U}\sqcup\mathcal{U} \doteq \mathcal{U}$.

Contractible types – intro

Contractibility is how we in HoTT/UF express uniqueness
(see talks by Emily Riehl)

Def For any type A, let $\text{is-contr}(A) :\equiv \sum_{c:A} \prod_{x:A} c = x$

(propositions as types interpretation of
"there is a c in A s.t. every x in A is equal to c")



c is called the <u>center of contraction</u> &

C : $\text{const}_c \sim \text{id}_A$ is called the <u>contraction</u>

Ex The unit type $\mathbb{1}$ is contractible w/

$c :\equiv *$, $\quad C :\equiv \text{ind}_{\mathbb{1}}^{x.* = x}(\text{refl}_*)$

Observation

A is contractible iff $! :\equiv \mathrm{const}_* : A \to \mathbb{1}$ is an equiv.

"$\longrightarrow$" Have $c : A$, $C : \mathrm{const}_c \sim \mathrm{id}_A$. Then w/ $g :\equiv \mathrm{const}_c : \mathbb{1} \to A$,

$$g \circ \mathrm{const}_* \doteq \mathrm{const}_c \sim \mathrm{id}_A$$

$$\& \quad \mathrm{const}_* \circ g \doteq \mathrm{const}_* \sim \mathrm{id}_{\mathbb{1}} \quad \checkmark$$

"$\longleftarrow$" If $g : \mathbb{1} \to A$ is an inverse of $!$, Let $c :\equiv g\,*$

so $\quad c \doteq (g \circ !)(*) \doteq (g \circ !)(x) = x$.

Ex 2 For any type A w/ $a : A$, the type $\Sigma_{x:A}\, a = x$ is contractible.

center $:\equiv (a, \mathrm{refl}_a)$, contraction by path induction.

Singleton induction

Since contractible types are equiv. to $\mathbb{1}$, they have all the str. of $\mathbb{1}$.

Def Let A be a type, $a : A$. We say A sat. <u>singleton induction</u> if for every type family B over A the eval. map

$$\text{ev-pt}_a : \left(\prod_{x:A} B(x)\right) \to B(a) \quad \text{has a section.}$$

I.e., we have: $\text{ind-sing}_a : B(a) \to \prod_{x:A} B(x)$

$$\text{comp-sing}_a : \text{ev-pt}_a \circ \text{ind-sing}_a \sim \text{id}_{B(a)}$$

Thm A is contr. iff we have $a : A$ s.t. A sat. singleton induction.

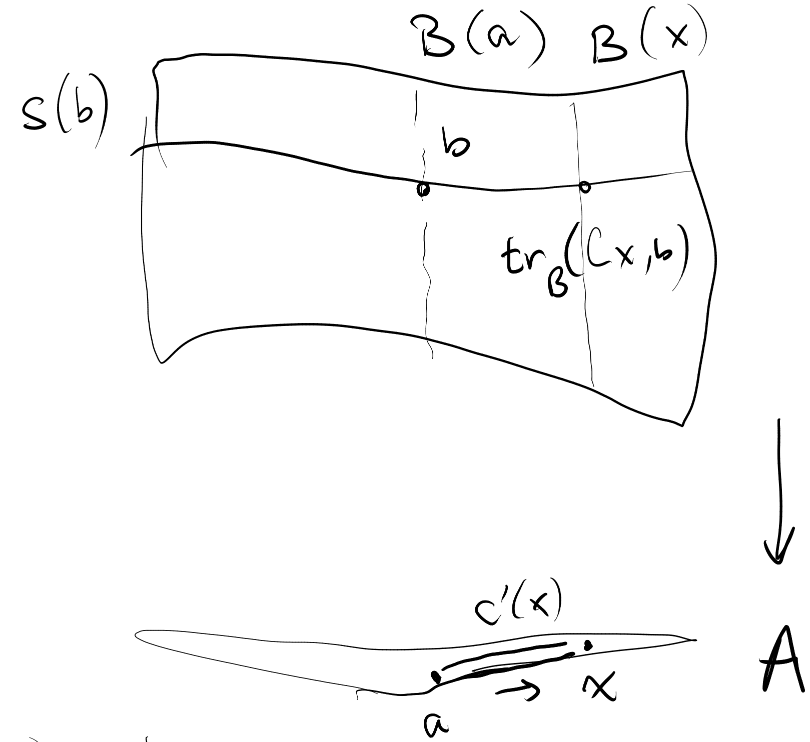
"$\to$": Have (a, C), let $C'(x) := C(a)^{-1} \cdot C(x)$ so $C'(a) = \text{refl}_a$.

So assume WLOG, already $C(a) = \text{refl}_a$.

Proof, cont'd

Def $s : B(a) \longrightarrow \prod_{x:A} B(x)$

by $s\ b\ x :\equiv tr_B(C(x), b)$



Then for every b, we have

$$s\ b\ a = tr_B(C(a), b) = tr_B(refl_a, b) \doteq b.$$

"$\longleftarrow$": Suppose we have $a : A$, s.t. A sat. singleton ind.

Let a be the center of contraction, and apply singleton to $B(x) := (a = x)$, so

$$\text{ind-sing}^B_a : a = a \to \prod_{x:A} a = x.$$

Apply to $refl_a$ to get contraction.

Fibers & contractible maps

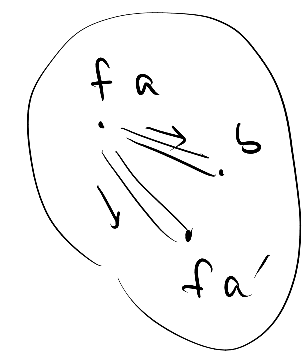
Def. Let $f : A \to B$, $b : B$. The <u>fiber of f at b</u> is

$$\mathrm{fib}_f(b) := \sum_{x:A} f\,x = b \qquad (\text{preimage})$$

Obs For all $p : a =_A a'$, $q : f\,a = b$ we have $\mathrm{tr}_{x.\, f\,x = b}(p, q) = (\mathrm{ap}_f\, p)^{-1} \cdot q$

Cor For all $(a,q), (a',q') : \mathrm{fib}_f\, b$ we have

$$\big((a,q) = (a',q')\big) \simeq \sum_{p : a = a'} \underbrace{q = \mathrm{ap}_f\, p \cdot q'}_{\cong\ (\mathrm{ap}_f\, p)^{-1} \cdot q = q'} \;=:\; \mathrm{Eq\text{-}fib}_f\big((a,q),(a',q')\big)$$



Induced by path ind & refl. of $\mathrm{Eq\text{-}fib}_f$.

Def A map $f : A \to B$ is <u>contractible</u> if all its fibers are, i.e., we have $\prod_{y:B} \text{is-contr}(\text{fib}_f(y))$

Thm Any contr. map is an equivalence. From the centers of contr. we get $\prod_{y:B} \text{fib}_f(y) \doteq \prod_{y:B} \sum_{x:A} fx = y$ by Π-choice, we get $g : B \to A$ & $G : f \circ g \sim \text{id}_B$

To show $g \circ f \sim \text{id}_A$, i.e., for all $x : A$, $(g \circ f)x = x$.

$f \circ g \circ f \overset{G \cdot f}{\sim} f$, so get $p : f(g(fx)) = f(x)$, $(g(fx), p) : \text{fib}_f(f(x))$

also, $(x, \text{refl}_{fx}) : \text{fib}_f(fx)$, so $(g(fx), p) = (x, \text{refl}_x)$

by contractibility of $\text{fib}_f(fx)$, so $g(fx) = x$ as desired

$\square$

Coh. inv. maps

Def Say $f : A \to B$ is coh. inv. if we have

$$\left.\begin{array}{l} \left.\begin{array}{l} g : B \to A \\ G : f \circ g \sim \mathrm{id}_B \\ H : g \circ f \sim \mathrm{id}_A \end{array}\right\} \text{has-inverse}(f) \\ K : G \cdot f \sim f \cdot H \quad (\text{as } f \circ g \circ f \sim f) \end{array}\right\} \text{is-coh-inv}(f)$$

Thm $\text{is-coh-inv}(f) \to \text{is-contr}(f)$ $\quad$ (i.e., $\prod_{y:B} \text{is-contr}(\text{fib}_f\, y)$)

center of contr: $(g\,y, G\,y)$. To show:

$\prod_{y:B} \prod_{x:A} \prod_{q : f(x) = y} (g\,y, G\,y) = (x, q)$ $\quad$ suffices: $\prod_{x:A} (g(f\,x), G(f\,x)) = (x, \mathrm{refl}_{f\,x})$

B_y = is fib's, give $H\,x : g(f\,x) = x$, $\underbrace{G(f\,x) = \mathrm{ap}_f(H\,x) \cdot \mathrm{refl}_{f\,x}}_{K\,x}$

□

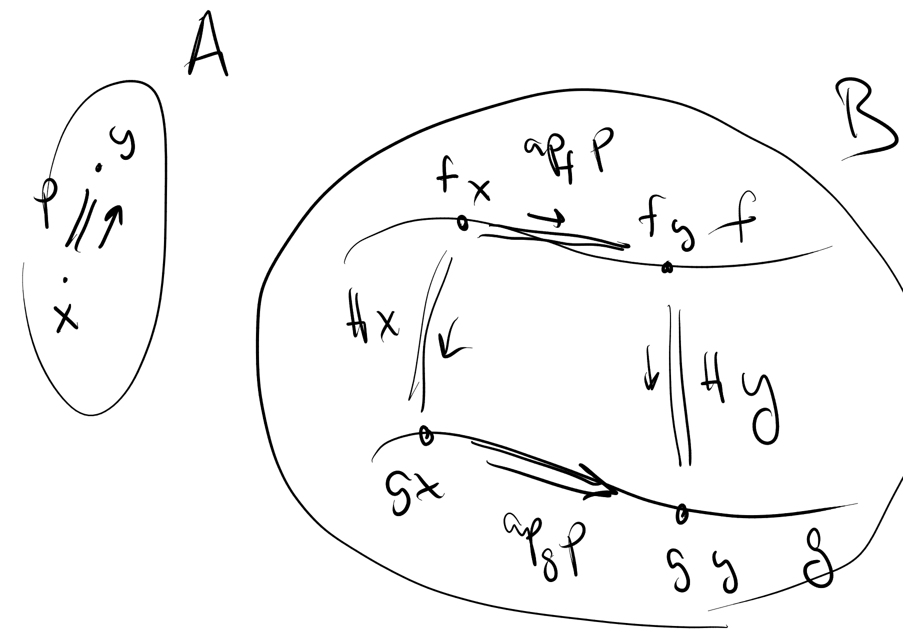
Final goal: has-inverse$(f) \to$ is-coh-inv(f)

Then:

$$\begin{tikzcd}
\text{is-equiv}(f) \arrow[r, bend left] & \text{has-inv}(f) \arrow[l, bend left] \arrow[d, dashed, shift right] & \\
& \text{is-coh-inv}(f) \arrow[u, shift right] \arrow[r] & \text{is-contr}(f) \arrow[ul]
\end{tikzcd}$$

<u>Tool</u> naturality squares of homotopies.

<u>Def</u> $f, g : A \to B$, $H : f \sim g$



$\text{nat-htpy}(H, p) : \text{ap}_f\, p \cdot H\, y = H\, x \cdot \text{ap}_g\, p$

By path ind on p, suffices to give

$\text{nat-htpy}(H, \text{refl}_x) : \text{refl}_{fx} \cdot H\, x = H\, x \cdot \text{refl}_{fx}$ $\square$

✓

<u>Special case</u> $h : A \to A$, $H : h \sim \mathrm{id}_A$, then

$$\begin{tikzcd} h(hx) \arrow[r, "\mathrm{ap}_h(Hx)"] \arrow[d, "H(hx)"'] & h(x) \arrow[d, "Hx"] \\ hx \arrow[r, "Hx"'] & x \end{tikzcd}$$

look at nat square at $Hx : hx = x$

So $\mathrm{ap}_h(Hx) = H(hx)$

<u>Now</u> Assume has-inverse(f), i.e., $g : B \to A$, $G : f \circ g \sim \mathrm{id}_B$, $H : g \circ f \sim \mathrm{id}_A$

We improve G to G' w/ $K : G' \cdot f \sim f \cdot H$

Let $y : B$, $G'y := \left(fg(y) \overset{G(fg(y))^{-1}}{=} fgfg(y) \overset{\mathrm{ap}_f(H(gy))}{=} fg(y) \overset{Gy}{=} y \right)$

Let $x : A$, show

$$\begin{tikzcd} fgfgf(x) \arrow[r, "\mathrm{ap}_f(H(gf(x)))"] \arrow[d, "G(fgf(x))"'] & fgf(x) \arrow[d, "G(f(x))"] \\ fgf(x) \arrow[r, "\mathrm{ap}_f(Hx)"'] & f(x) \end{tikzcd}$$

but $H(gf(x)) = \mathrm{ap}_{gf}(Hx)$

so suffices

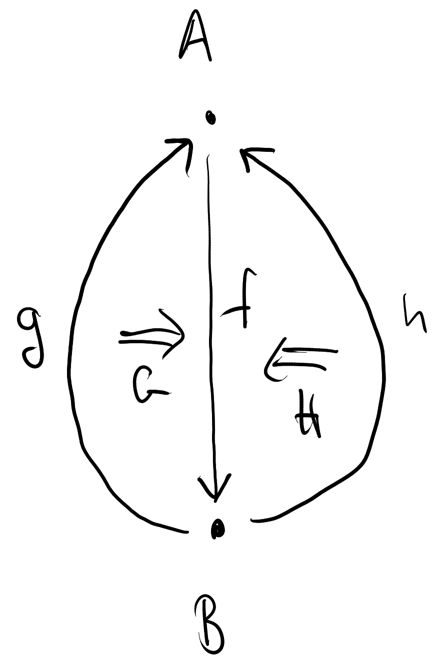
$$\begin{tikzcd} fgfgf(x) \arrow[r, "\mathrm{ap}_{fgf}(Hx)"] \arrow[d, "G(fgf(x))"'] & fgf(x) \arrow[d, "G(fx)"] \\ fgf(x) \arrow[r, "\mathrm{ap}_f(Hx)"'] & f(x) \end{tikzcd}$$

nat-htpy G

$\square$

Geom. intuition for why is-equiv(f) & is-coh-inv(f) are propositions:

assume f <u>is</u> invertible



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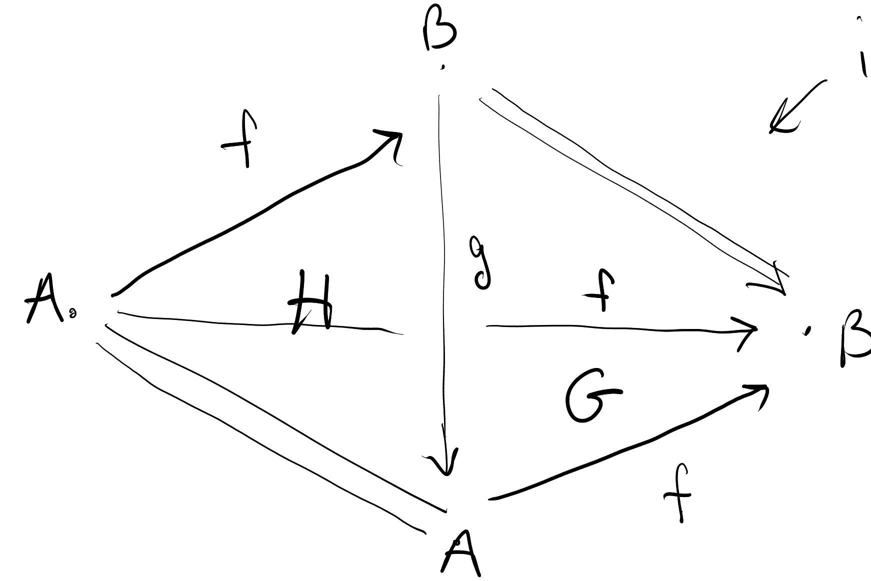
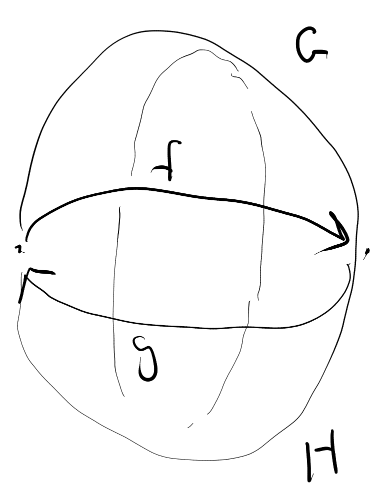


inside is K.

both contractible.

But



is a 2-sphere.