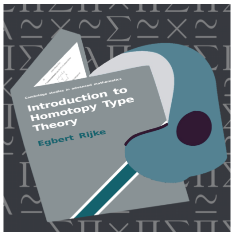

Worksheet 7 (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs

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1 $(\star)$

Consider two embeddings $f : A \hookrightarrow B$ and $g : B \hookrightarrow C$. Construct a function

$$\mathsf{is\text{-}equiv}(g \circ f) \to (\mathsf{is\text{-}equiv}(f) \times \mathsf{is\text{-}equiv}(g)) .$$

Suppose that $g \circ f$ is an equivalence. By the 3-for-2 property of equivalences, it suffices to prove that f is an equivalence. Define $\psi : B \to A$ by

$$\psi(b) :\equiv (g \circ f)^{-1} (g(b)).$$

For every $b : B$,

$$g(f(\psi(b))) = g(b).$$

Since g is an embedding, this implies that

$$f(\psi(b)) = b.$$

Moreover, $\psi(f(a)) = a$ for all $a : A$. Thus, ψ is an inverse of f.

2 $(\star\star)$

1. Let A be a type. Prove that the canonical map $\emptyset \xrightarrow{!_A} A$ is an embedding.
2. Let A and B be types. Prove that the inclusions $\mathsf{inl} : A \to A + B$ and $\mathsf{inr} : B \to A + B$ are embeddings.
3. Let A and B be types. Prove that $\mathsf{inl} : A \to A + B$ is an equivalence if and only if $B \simeq \emptyset$.

   Conclude that if both A and B are contractible, then $A + B$ is *not* contractible.

1. For every $x, y : \emptyset$, we must prove that

$$\mathsf{ap}_{!_A}(x, y) : (x = y) \to (!_A(x) = !_A(y))$$

is an equivalence. This follows directly from induction on $\emptyset$.

2. Let $x, y : A$ and consider the map

$$\mathsf{ap}_{\mathsf{inl}}(x, y) : (x = y) \to (\mathsf{inl}(x) = \mathsf{inl}(y))\,.$$

Recall from Lecture 7 the family

$$\mathsf{eq\text{-}id} : \prod_{s,t:A+B} (s = t) \to \mathsf{Eq}\text{+}_{A,B}(s, t)$$

of equivalences, defined by path induction. It is easy to check that $\mathsf{ap}_{\mathsf{inl}}(x, y)$ is a section of $\mathsf{eq\text{-}id}_{\mathsf{inl}(x),\mathsf{inl}(y)}$. Since the latter is an equivalence, it follows that $\mathsf{ap}_{\mathsf{inl}}(x, y)$ is actually an equivalence with inverse $\mathsf{eq\text{-}id}_{\mathsf{inl}(x),\mathsf{inl}(y)}$. This proves that inl is an embedding.

Similarly, for any $x, y : B$, the map $\mathsf{ap}_{\mathsf{inr}}(x, y)$ is an equivalence with inverse $\mathsf{eq\text{-}id}_{\mathsf{inr}(x),\mathsf{inr}(y)}$. Thus, inr is also an embedding.

3. Suppose that inl is an equivalence with inverse $\psi : A + B \to A$. Let $b : B$. Then $\mathsf{inl}(\psi(\mathsf{inr}(b))) = \mathsf{inr}(b)$. But recall that

$$(\mathsf{inl}(\psi(\mathsf{inr}(b))) = \mathsf{inr}(b)) \simeq \emptyset.$$

This gives us an element of $\emptyset$ and thus an element of $\neg B$.

Conversely, consider an equivalence $e : B \to \emptyset$. Define the function $\varphi : A + B \to A$ by

$$\begin{aligned} \varphi(\mathsf{inl}(a)) &:\equiv a \\ \varphi(\mathsf{inr}(b)) &:\equiv \mathsf{ind}_{\emptyset}(e(b)). \end{aligned}$$

It is easy to check that φ is an inverse of inl.

Now, suppose that both A and B are contractible. Also, suppose that $A + B$ is contractible. We have a commuting triangle

$$\begin{tikzcd} A \arrow[rr, "\mathsf{inl}"] \arrow[dr, "\simeq"'] & & A+B \arrow[dl, "\simeq"] \\ & \mathbb{1} & \end{tikzcd},$$

so that inl is an equivalence. This implies that $B \simeq \emptyset$. Since B is contractible, this gives us an element of $\emptyset$. Therefore, $A + B$ is not contractible.

3 $(\star\star)$

Consider a commuting triangle

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & X & \end{tikzcd} .$$

1. Suppose that g is an embedding. Prove that f is an embedding if and only if h is one.
2. Suppose that h is an equivalence. Prove that f is an embedding if and only if g is one.

Note that for every $x, y : A$, we have a commuting square

$$\begin{tikzcd} x = y \arrow[r, "\mathsf{ap}_h(x,y)"] \arrow[d, "\mathsf{ap}_f(x,y)"'] & h(x) = h(y) \arrow[d, "\mathsf{ap}_g(h(x),h(y))"] \\ f(x) = f(y) \arrow[r, "\simeq"'] & g(h(x)) = g(h(y)) \end{tikzcd} .$$

1. **We have that $\mathsf{ap}_g(h(x), h(y))$ is an equivalence. Suppose that f is an embedding. Then $\mathsf{ap}_h(x, y)$ is an equivalence, and thus h is an embedding.**

   Conversely, suppose that h is an embedding. Then $\mathsf{ap}_f(x, y)$ is an equivalence, and thus f is an embedding.

2. **Suppose that f is an embedding. Then $\mathsf{ap}_{g \circ h}(x, y)$ is an equivalence. Further, the triangle**

$$\begin{tikzcd} B \arrow[rr, "h^{-1}"] \arrow[dr, "g"'] & & A \arrow[dl, "g \circ h"] \\ & X & \end{tikzcd}$$

   commutes, and h^{-1} is an embedding because it's an equivalence. By part (1), it follows that g is an embedding.

   Conversely, suppose that g is an embedding. Then $\mathsf{ap}_f(x, y)$ is an equivalence. As h is also an embedding, so is f.

4 $(\star\star)$

Let A, B, and C be types and let $f : A \to C$ and $g : B \to C$ be maps. Prove that the following are logically equivalent.

1. The map $[f, g] : A + B \to C$ is an embedding.
2. Both f and g are embeddings, and $f(a) \neq g(b)$ for all $a : A$ and $b : B$.

Suppose that $[f, g]$ is an embedding. Then $f \doteq [f, g] \circ \mathsf{inl}$ is an embedding as the composite of two embeddings. Likewise, g is an embedding. Let $a : A$ and $b : B$ and suppose that $f(a) = g(b)$. Since $[f, g]$ is an embedding,

$$\mathsf{inl}(a) = \mathsf{inr}(b).$$

But $(\mathsf{inl}(a) = \mathsf{inr}(b)) \simeq \emptyset$, and thus $f(a) \neq g(b)$.

Conversely, suppose that both f and g are embeddings and that

$$\tau : \prod_{a:A} \prod_{b:B} f(a) \neq g(b).$$

We must show that

$$\mathsf{ap}_{[f,g]}(s, t) : (s = t) \to ([f, g]\,(s) = [f, g]\,(t))$$

is an equivalence for all $s, t : A + B$. Notice that the diagrams

$$\begin{tikzcd}
\mathsf{inl}(a) = \mathsf{inl}(a') \arrow[r, "\mathsf{ap}_{\mathsf{inl}}(a,a')^{-1}"'] \arrow[rr, bend left, "\mathsf{ap}_{[f,g]}(\mathsf{inl}(a),\mathsf{inl}(a'))"] & a = a' \arrow[r, "\mathsf{ap}_f(a,a')"'] & f(a) = f(a')
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{inr}(b) = \mathsf{inr}(b') \arrow[r, "\mathsf{ap}_{\mathsf{inr}}(b,b')^{-1}"] \arrow[rr, bend right, "\mathsf{ap}_{[f,g]}(\mathsf{inr}(b),\mathsf{inr}(b'))"'] & b = b' \arrow[r, "\mathsf{ap}_g(b.b')"] & g(b) = g(b')
\end{tikzcd}$$

commute. Define $\psi_{s,t} : ([f, g]\,(s) = [f, g]\,(t)) \to (s = t)$ by double induction on $A + B$:

$$\begin{aligned}
\psi_{\mathsf{inl}(a),\mathsf{inl}(a')}(p) &:= \mathsf{ap}_{\mathsf{inl}}(\mathsf{ap}_f(a, a')^{-1}(p)) \\
\psi_{\mathsf{inl}(a),\mathsf{inr}(b)}(p) &:= \mathsf{ind}_{\emptyset}(\tau_{a,b}(p)) \\
\psi_{\mathsf{inr}(b),\mathsf{inr}(b')}(p) &:= \mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_g(b, b')^{-1}(p)) \\
\psi_{\mathsf{inr}(b),\mathsf{inl}(a)}(p) &:= \mathsf{ind}_{\emptyset}(\tau_{a,b}(p^{-1})).
\end{aligned}$$

By double induction on $A + B$, it's easy to prove that $\psi_{s,t}$ is an inverse of $\mathsf{ap}_{[f,g]}(s, t)$.

5 $(\star\star)$

1. Let $f, g : \prod_{x:A} B(x) \to C(x)$. Construct a function

$$\left(\prod_{x:A} f(x) \sim g(x)\right) \to (\mathsf{tot}(f) \sim \mathsf{tot}(g)) .$$

2. Let $f : \prod_{x:A} B(x) \to C(x)$ and $g : \prod_{x:A} C(x) \to D(x)$. Construct a homotopy

$$\mathsf{tot}(\lambda x. g(x) \circ f(x)) \sim \mathsf{tot}(g) \circ \mathsf{tot}(f).$$

3. For any type family B over A, construct a homotopy

$$\mathsf{tot}(\lambda x. \mathsf{id}_{B(x)}) \sim \mathsf{id}_{\sum_{x:A} B(x)}.$$

4. Let $a : A$ and let B be a type family over A. Prove that if $B(x)$ is a retract of $a = x$ for each $x : A$, then $(a = x) \simeq B(x)$ for each $x : A$.

5. Let $f : \prod_{x:A} (a = x) \to B(x)$. Prove that if each $f(x)$ has a section, then f is a family of equivalences.

   As a consequence, for any function $k : X \to Y$, if

$$\mathsf{ap}_k(x, y) : (x = y) \to (k(x) = k(y))$$

   has a section for every $x, y : X$, then k is an embedding.

1. **Let** $H : \prod_{x:A} f(x) \sim g(x)$**. For each** $(x,y) : \sum_{x:A} B(x)$**, we have a term**

$$\mathsf{pair}^{=}(\mathsf{refl}_x, H_x(y)) \ : \ \mathsf{tot}(f)(x,y) = \mathsf{tot}(g)(x,y).$$

2. **For each** $(x,y) : \sum_{x:A} B(x)$**, we have a term**

$$\mathsf{refl}_{(x,g(x,f(x,y)))} \ : \ \mathsf{tot}(\lambda x.g(x) \circ f(x))(x,y) = (\mathsf{tot}(g) \circ \mathsf{tot}(f))\,(x,y).$$

3. **For each** $(x,y) : \sum_{x:A} B(x)$**, we have a term**

$$\mathsf{refl}_{(x,y)} \ : \ \mathsf{tot}(\lambda x.\mathsf{id}_{B(x)})(x,y) = \mathsf{id}_{\sum_{x:A} B(x)}(x,y).$$

4. **For each** $x : A$**, suppose that we have maps** $B(x) \xrightarrow{s_x} (a = x)$ **and** $(a = x) \xrightarrow{r_x} B(x)$ **such that** $r_x \circ s_x \sim \mathsf{id}_{B(x)}$**. Let us show that** $\lambda x.r_x$ **is a family of equivalences. By combining parts (1), (2), and (3), we get a commuting diagram**

$$\begin{tikzcd} \sum_{x:A} B(x) \arrow[r, "\mathsf{tot}(\lambda x.s_x)"] \arrow[rr, bend right, "\mathsf{id}_{\sum_{x:A} B(x)}"'] & \sum_{x:A} a = x \arrow[r, "\mathsf{tot}(\lambda x.r_x)"] & \sum_{x:A} B(x) \end{tikzcd} .$$

Moreover, $\sum_{x:A} a = x$ **is contractible. As a retract of a contractible type is itself contractible, we see that** $\mathsf{tot}(\lambda x.r_x)$ **is an equivalence. Hence** $\lambda x.r_x$ **is a family of equivalences.**

5. **This follows immediately from our proof of part (4).**

6 $(\star\star\star)$

We say that a map $f : A \to B$ is *path-split* if

1. f has a section and
2. the map $\mathsf{ap}_f(x, y) : (x = y) \to (f(x) = f(y))$ has a section for each $x, y : A$.

Prove that a map $f : A \to B$ is an equivalence if and only if it is path-split.

Suppose that f is an equivalence. Then f has a section. It's also an embedding, so that $\mathsf{ap}_f(x, y)$ has a section.

Conversely, suppose that f is path-split with section $s_f : B \to A$. By Problem 5, f is an embedding. Let $b : B$ and note that $f(s_f(b)) = b$, so that $\mathsf{fib}_f(b)$ is inhabited. To see that $\mathsf{fib}_f(b)$ is contractible, we must show that any two elements $(a, p), (a', u) : \mathsf{fib}_f(b)$ of the fiber of f over b are equal. Since f is an embedding, we have a term

$$q :\equiv \mathsf{ap}_f(a, a')^{-1}(p \cdot u^{-1}) \ : \ a = a',$$

with

$$\mathsf{tr}_{f(x)=b}(q, p) = \mathsf{ap}_f(a, a')(q)^{-1} \cdot p = \left(p \cdot u^{-1}\right)^{-1} \cdot p = u.$$

It follows that

$$(a, p) = (a', u)\,.$$