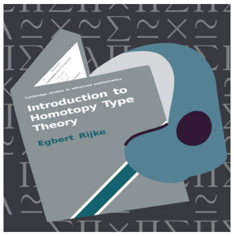


# Worksheet 3

HoTTEST Summer School 2022

The HoTTEST TAs, and

11 July 2022

## 1 $(\star)$

Let $A$ be a type, and $x, y, z : A$. Show that path inversion distributes over concatention, i.e., construct a term of the following type:

$$\Pi_{p:x=y}\Pi_{q:y=z}(p \cdot q)^{-1} = q^{-1} \cdot p^{-1}$$

## 2 $(\star\star)$

Let $A$ and $B$ be types. We can define the product type as $A \times B := \Sigma_A B$, where $B$ is considered a constant type family over $A$. The resulting elimination principle is:

$$\frac{\Gamma, z : A \times B \vdash D(z) \text{ type} \quad \Gamma, a : A, b : B \vdash d : D(a, b)}{\Gamma, z : A \times B \vdash \mathsf{ind}_{\times}(d, z) : D(z)} \times\text{−Elim}$$

Define the two projections $\mathsf{pr}_1 : A \times B \to A$ and $\mathsf{pr}_2 : A \times B \to B$ using the elimination principle.

Now use the elimination principle to give a term of the following type:

$$\Pi_{z:A \times B} z = (\mathsf{pr}_1 z, \mathsf{pr}_2 z)$$

## 3 $(\star)$

Let $A$ and $B$ be types. Give an informal construction of a term of the following type:

$$\Pi_{a,a':A}\Pi_{b,b':B}(a =_A a') \times (b =_B b') \to \big((a,b) =_{A\times B} (a',b')\big)$$

(We will later see that this map is an *equivalence*, and this map *characterises* paths in product types as pairs of paths between the components.)

## 4 $(\star\star)$

Let $A$ be a type and $a : A$. Show that $\mathsf{refl}_a$ is unique among paths starting at $a$ but with the other endpoint free. That is, for any $z : \Sigma_{x:A}(a = x)$ construct a term

$$(a, \mathsf{refl}_a) =_{\Sigma_{x:A}(a=x)} z$$

*(We emphasize that this does not mean that $\mathsf{refl}_a$ is unique as a loop at $a$!)*

*(This is Prop. 5.5.1 in Egbert's book.)*

## 5 $(\star\star\star)$

Let $A$ be a type. In the third lecture, a function $\mathsf{concat}$ of the following type was constructed:

$$\mathsf{concat} : \Pi_{x,y,z:A}(x = y) \to ((y = z) \to (x = z))$$

Call the above for $\mathsf{concat}_1$. Define two different terms $\mathsf{concat}_2$ and $\mathsf{concat}_3$ of the same type.

Choose your favorite numbers $i, j \in \{1, 2, 3\}$ $(i \neq j)$. Give a term of the following type:

$$\Pi_{x,y,z:A}\Pi_{p:x=y}\Pi_{q:y=z}\mathsf{concat}_i(p,q) = \mathsf{concat}_j(p,q)$$

# 6 $(\star\star\star)$

Let $A$ be a type with an element $a : A$. Can you construct a term of the following type? If not, what goes wrong?

$$\Pi_{p:(a=_A a)} p =_{(a=_A a)} \mathsf{refl}_a$$