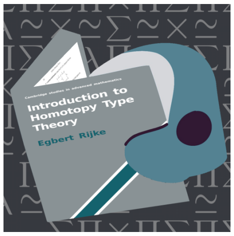

Worksheet 2 (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs

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1 $(\star)$

The type **suit** is a type generated by four constructors:

$$\begin{array}{rlrl} \textbf{Hearts}\colon & \textbf{suit} & \qquad \textbf{Diamonds}\colon & \textbf{suit} \\ \textbf{Clubs}\colon & \textbf{suit} & \qquad \textbf{Spades}\colon & \textbf{suit} \end{array}$$

Suppose we had a type family $s : \textbf{suit} \vdash P(s)$ type. What data do we need to supply in order to define a term of the following type?

$$\prod_{s:\textbf{suit}} P(s)$$

Four terms:

- $\textsf{pHearts}\colon P(\textbf{Hearts})$
- $\textsf{pDiamonds}\colon P(\textbf{Diamonds})$
- $\textsf{pClubs}\colon P(\textbf{Clubs})$
- $\textsf{pSpades}\colon P(\textbf{Spades})$

2 $(\star\star)$

Here's an informal description of a type:

$\mathbb{1}$ is a type generated by one constructor term, $\star\colon \mathbb{1}$.

Express this description formally as two inference rules, a 'Formation' rule, and an 'Introduction' rule (analogously to how we introduced $\textsf{bool}$ in lecture).

$$\frac{}{\vdash \mathbb{1}\ \textbf{type}} \qquad \frac{}{\vdash \star\colon \mathbb{1}}$$

Now, write the induction principle for $\mathbb{1}$, where $x : \mathbb{1} \vdash D$ type is some type family.

We have a term

$$\mathsf{ind}_{\mathbb{1}} \quad : \quad D(\star) \to \prod_{x:\mathbb{1}} D(x)$$

satisfying the computation rule:

$$\mathsf{ind}_{\mathbb{1}}(d, \star) \doteq d.$$

As inference rules:

$$\frac{\Gamma, x : \mathbb{1} \vdash D \ \textbf{type} \quad \Gamma \vdash d : D[\star/x]}{\Gamma, x : \mathbb{1} \vdash \mathsf{ind}_{\mathbb{1}}(d, x) : D} \ \mathbb{1}\textbf{-Elim}$$

$$\frac{\Gamma, x : \mathbb{1} \vdash D \ \textbf{type} \quad \Gamma \vdash d : D[\star/x]}{\Gamma \vdash \mathsf{ind}_{\mathbb{1}}(d, \star) \doteq d} \ \mathbb{1}\textbf{-Comp}$$

Instantiate this for the constant type family, i.e. D doesn't depend on $x : \mathbb{1}$ and is always a fixed type D. What do the elimination and computation rules for $\mathbb{1}$ say?

If d is some term of type D, then

$$\mathsf{ind}_{\mathbb{1}}(d) : \mathbb{1} \to D$$

is a term, and

$$\mathsf{ind}_{\mathbb{1}}(d, \star) \doteq d \quad : D$$

3 $(\star\star)$

Define a boolean-valued 'less than' operation on natural numbers:

$$<_2 \quad : \quad \mathbb{N} \to \mathbb{N} \to \mathsf{bool}$$

so that $(m <_2 n) \doteq \mathsf{true}$ when m is less than n, and false otherwise.

By the induction principle, we need to supply

$$0 <_2 - \quad : \mathbb{N} \to \mathsf{bool}$$

and, assuming some $m : \mathbb{N}$,

$$s(m) <_2 - \quad : \mathbb{N} \to \mathsf{bool}$$

We'll do each of these by induction, giving the four cases:

$$\begin{aligned} 0 <_2 0 &\doteq \mathsf{false} \\ 0 <_2 s(n) &\doteq \mathsf{true} \\ s(m) <_2 0 &\doteq \mathsf{false} \\ s(m) <_2 s(n) &\doteq (m <_2 n). \end{aligned}$$

This is adequate to define $<_2$**.**

Using this definition of add,

$$\begin{aligned} \mathsf{add}\ 0 \quad\ \ n &\doteq n \\ \mathsf{add}\ s(m)\ n &\doteq s(\mathsf{add}\ m\ n) \end{aligned}$$

Compute the term

$$(\mathsf{add}\ s(s(0))\ s(s(0))) \ <_2\ (\mathsf{add}\ s(0)\ s(0))$$

to either true or false.

$$\begin{aligned} &(\mathsf{add}\ s(s(0))\ s(s(0))) \ <_2\ (\mathsf{add}\ s(0)\ s(0)) \\ &\doteq s(\mathsf{add}\ s(0)\ s(s(0))) \ <_2\ (\mathsf{add}\ s(0)\ s(0)) \\ &\doteq s(s(\mathsf{add}\ 0\ s(s(0)))) \ <_2\ (\mathsf{add}\ s(0)\ s(0)) \\ &\doteq s(s(s(s(0)))) \ <_2\ (\mathsf{add}\ s(0)\ s(0)) \\ &\doteq s(s(s(s(0)))) \ <_2\ s(\mathsf{add}\ 0\ s(0)) \\ &\doteq s(s(s(s(0)))) \ <_2\ s(s(0)) \\ &\doteq s(s(s(0))) \ <_2\ s(0) \\ &\doteq s(s(0)) \ <_2\ 0 \\ &\doteq \mathsf{false} \end{aligned}$$

4 (⋆⋆)

Fix some type X, and some type family $x : X, x' : X \vdash P(x, x')$ type. Write a term of type

$$\sum_{b:X} \prod_{a:X} P(a,b) \to \prod_{a:X} \sum_{b:X} P(a,b)$$

$$\lambda(b, f).\ \lambda a.(b, f\,a)$$

Here, f has type $\prod_{a:X} P(a,b)$.

What does this say, under our logical interpretation?

If there exists a b such that, for all a, $P(a,b)$ holds, then, for all a there exists a b such that $P(a,b)$ holds.

5 (⋆⋆⋆)

$\emptyset$ is the *empty type*: there are no terms of type $\emptyset$. It has the following induction principle:

For any type family $x : \emptyset \vdash Q(x)$ type, we have a term $\mathsf{ind}_\emptyset : \prod_{x:\emptyset} Q(x)$.

What does this say when $Q(x)$ is a fixed type Q?

For any type Q, there is a term $\mathsf{ind}_\emptyset : \emptyset \to Q$

Remember that we interpret types to be logical propositions, and terms/inhabitants to be proofs or witnesses of those propositions. What proposition does $\emptyset$ represent?

A false proposition: there are no proofs of $\emptyset$.

If P is some proposition, what is the logical meaning of $P \to \emptyset$?

P implies false, or P leads to absurdity: if there was a proof of P, we would have a proof of falsity.

We write $\neg P$ as an abbreviation for $P \to \emptyset$

Write a term of type $\neg\neg\mathbb{1}$.

$$\lambda f. f \star \quad : (\mathbb{1} \to \emptyset) \to \emptyset$$

Is there a term of type $\neg\neg\emptyset$? Why or why not?

There is no such term. Suppose we had such a term,

$$\textsf{absurd} : (\emptyset \to \emptyset) \to \emptyset$$

Then we can form the term

$$\textsf{ind}_{\emptyset} : \emptyset \to \emptyset$$

and apply $\textsf{absurd}$ **to it:**

$$\textsf{absurd}(\textsf{ind}_{\emptyset}) \quad : \quad \emptyset$$

so we've constructed a term of type $\emptyset$, which is impossible.

Let P and Q be types. We will write $P \leftrightarrow Q$ for the type $(P \to Q) \times (Q \to P)$. Use the fact that $\neg P$ is defined as the type $P \to \emptyset$ of functions from P to the empty type to give type theoretic proofs of the constructive tautologies

(i) $\neg(P \times \neg P)$

(ii) $\neg(P \leftrightarrow \neg P)$

(i) $\lambda((p, np) : P \times \neg P).\ np\ p$

(ii) $\lambda((ltr, rtl) : (P \to \neg P) \times (\neg P \to P)).ltr\ \phi\ \phi$, **where** $\phi \doteq rtl(\lambda(p : P).ltr\ p\ p))$