HoTT lecture 4 Q&A (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs

15th July 2022

Q3 "can it be rigorously proven that without universes we can't prove $0 \neq 1$?"

"live answered (yes)"

Q4 "Is this a Russel or Tarski style universe?"

"Tarski"

Q5 "The Sigma definition should be a sigma, right?"

"oh no, i see"

"Sigma-check is of the same type as Pi-check. That's the premise for Sigma formation"

"I think that it was right, the ingredients are a way to turn type failies into sigmas"

Q6 "For every type that we could define we would have to add new axiom of universes?"

"If the type can be built out of the type formers that have been given so far, then we don't need to add any rules. But if we would add W-types for example, or higher inductive types, then we also have to extend universes with those (if we wish)."

Q7 "Does the syntax $T(Y\,X)$ on the page that just got deleted (rules for Pi and Sigma), mean Y applied to X?"

> **"Y is something that takes terms of Tx to produce a term in the universe, and we then turn that into a type"**

> **"That makes sense, but we'll have to discuss after, I'm still not understanding how that relates to the syntax I wrote down."**

Q8 "I imagine we'll often be writing $A : \mathcal{U}$ now instead of A type' - does the underlying type theory nonetheless need to allow judgments like A type' to make it possible to define universes and the rules we've seen so far, or will/could we dispense with the A type' kind of judgment entirely?"

> **"This judgment is still necessary for stating the rules, I think"**

> **"Yes, it is still needed. For example, $\mathcal{U}$ itself will not be in $\mathcal{U}$, but it will be a type."**

Q9 "This reminds me a lot of how we write $A : Set$ or $A : Type$ in Agda, are Set and $Type$ universes in this sense or am I thinking about this wrong?"

> **"Yes, except that in Agda the terms of the universe are literally types, whereas here they are codes for types"**

Q10 "Do we also want to have a $\mathcal{U} : \mathcal{U}$?"

> **"No. We'll get to that"**

Q11 "Can there be a $X : \mathcal{U}$ such that $\tau(X)$ is judgementally equal to $\mathcal{U}$ itself?"

"We'll get to this soon!"

"The relevant paradox is called Girard's paradox"

Q12 "What does that funny T (tau?) mean?"

"It 'decodes' an element $X : \mathcal{U}$ to a type $\tau(X)$."

"That's the type family over $\mathcal{U}$ which records how $x : \mathcal{U}$ encodes types $\tau(x)$"

Q13 "What is a postulate? Is it like a theorem?"

"We assume that we are working in a type theory in which this is true"

"A postulate is a meta-theoretic axiom, whereas a Theorem is something you can prove within type theory."

Q14 "What does check mean?"

"It signifies the code for a type in a universe. In particular, it's a term belonging to a universe."

Q15 "why do we need to 'encode' types in universes? Why not just think of universes as classes of types satisfying certain closure conditions, and no encoding going on at all?"

"live answered"

"Martin Lof in his treatment of universes seems to get by without talk of encodings, if I recall correctly. (Though I should double check this.)"

Q16 "Does that mean Agda also internally uses encodings? Or does Agda use an alternative formalization that doesn't require encodings?"

"No way does Agda use encodings, from the general vibe of Agda"

"This is like, not a thing that you would do in implementations"

"i think the relevant notion is Tarski-style vs Russell-style universes?"

Q17 "doesnt that raise an ambiguity issue (U_0 , U^+ not being minimal)"

"If you write U_0, it suggests that you're working in a type theory where U_0 is at the bottom of the hierarchy, but we don't have to assume that this is always the case"

Q18 “Does the ‘open ended’ requirement mean that we will have no elimination rule (or induction rule) for universes?”

“Elimination rules are for inductive datatypes and the universe is not one of those (in general type theory)”

Q19 “Can we construct a type family $(n : \mathbb{N}) \to \mathcal{U}_n$? Will it be contained in $\mathcal{U}_\omega$?”

“That term might not be allowed in type theory”

“No, the universes are indexed by *external* natural numbers, not internal ones.”

Q20 “There’s no way to necessitate the existence of a $\mathcal{U}_\omega$?”

“live answered”

Q21 “Is this reflection principle related to the transfer principle of the hyperreal numbers? Or even more powerful?”

“I think this should be discussed on discord afterwards, since reflection is such a general phenomenon”

Q22 "So the universes are cumulative or there is a distinct copy of U_0 in U_1?"

"It depends on the type theory"

"The axioms do not necessitate this in either way"

Q23 "if $n > m$, and we have U_n (the nth successor universe of U_0) and U_m, is $U_n \sqcup U_m$ the same as U_n?"

"This depends on the type theory as well!"

"We'll probably go with Agda's answer, which is yes :)"

"You could have a non-total ordering of universes, in which case the max would be strictly above both"

"should be U_m right? the larger of the two"

"oh never mind"

Q24 "Is there an intepretation of universes in term of logic? If propositions are types, what are universe?"

"These can be viewed as the collection of all propositions. In categorical semantics, universes play the role of (sub)object classifiers."

Q25 "What is a generalized inductive type?"

"So not the specific inductives we talked about like Nat, but the ability to make "data" declerations"

"See this post: http://liamoc.net/posts/2015-09-10-girards-paradox.html**"**

"So you need $SET : Set$ **where** $set : (X : Set) \to (X \to SET) \to SET$

"thanks!"

Q26 "Does a successor universe U^+ always have the number of inhabitants of U plus one (namely the encoding of U)?"

"We have an "inclusion" map from the codes of U to those of U^+, and U^+ also has a code for U."

"Right, but can I throw in a bunch of new type encodings into U^+ and still have it be a successor of U?"

"U^+ contains a lot more than just everything in this inclusion and U in general, it contains for example sigma types over U."

"Yes, that's what we mean by open-ended. We don't give any upper bound on U^+"

Q27 "Why do we use 1 and 0 as true and false? Couldn't we use bool itself?"

"Well we need false to be empty to get the contradicion. True can get sent to any inhabited type"

"Through the types-as-propositions paradigm, true corresponds to 1 and false to 0."

"There is no connection between the boolean vlues and cannonical representatives of them as types, I would argue"

Q28 "Is there any practical difference when you choose the 'true' type to be any other inhabited type, like $\mathbb{N}$, as opposed to the unit type?"

"No, you can use any inhabited type!"

"Not really — you can choose any type that you have a handy inhabitant of :)"

"Wonderful thank you :)"

Q29 "I guess this differs from what Coq does, where types and terms are kind of interchangeable? like we can't have functions that basically return types as values w/o some notion of a universe?"

"Coq does the same but simpler"

"Just a sequence of universes, like the successor universes we have seen, I think (it just hides them in '$Type$ type')"

"Yeah, formal type theory has to do a few things differently for theoretical niceness"

Q30 "What is a motive"

"live answered"

Q31 "So if we previously wrote $x : N \vdash Cxtype$ then we can equivalently say $\vdash C : N \to U$?

> "In the left judgement there is no guarantee that all Cx are actually encoded in U"

> "* in that particular U. Of course, there is a universe which contains all Cx"

> "In Agda, this is automatic, because $(x : N) \to Cx$ is just inside the base universe?"

> "Since the Cx are arbitrary type expressions here, we do not automatically know that they have codes in the base universe. Not all types in Agda are in the base universe, either. For example, the base universe is not contained in itself."

Q32 "so is the necessity for universes in Eq-N just the formal constraint that the right hand of a .= needs to be a term, and we want it instead to be a type, so we just say "actually this is just a term i've quietly defined that indicates the type 1 and we'll treat it that way even though it's actually a term referring to 1 rather than 1 itself"?"

> "We do use universes to treat types as terms in our system. (This may not answer your exact question.)"

> "Yes, if you phrase it like that anything is 'just' a formal constraint, but there needs to be a non-paradoxical way of forming such type families."

"I'm not saying "just" to be diminutive, and I'm not even going as far as paradoxical reasoning yet, I'm asking about syntax. Like, if we write down Eq-N as Ulrik has given it, it's not enough to say "we can't do that without universes because it's paradoxical" — for one thing, that presupposes that the definitions we're working with aren't themselves paradoxical, which I hope is true but relying on that feels circular :-) Idk, a lot of the trouble I'm having with intuition comes not from the abstract paradoxes but from uncertainty about how to tell if something is even syntactically well-formed."

"Of course you can't guarantee that there is no paradox, but at least you can avoid the type-in-type one: You can't define Eq: N-¿N-¿Type to a type of all types. How else would you define this type valued function? Maybe you can find a way to define such functions if the only values are Empty and Unit, but in general such an observational equality could be required to be of arbitrary homotopical complexity"

"Ok but, like I said I'm not asking about paradoxical reasoning. Paradoxes don't matter until you have at least some syntactically valid string of symbols; I'm not asking why Ind-N could lead to paradoxes without universes, I'm asking why (if?) Ind-N as written is illegal syntactically given the rules we knew prior to today"

Q33 "Why is observational equality important?"

"It is actually crucial for type checking (assuming that observational means definitional)"

"For non-dependent type theory, you can do logic without computation"

"But as soon as you have even something like type operators, you can mix types and terms and have reductions on the RHS of a colon"

"I think this refers to Eq-N. One of the motivations for defining that was to prove that $0 \neq 1$."

"Yes, I've meant Eq-N"

"Yes, then, we will eventually use it to charictarise equality on N"

“Via what is known as an encode-decode argument”

Q34 “is the equality of proofs ’judgemental’? should we be worried about removing homotopic behavior if we say things about the equalities of terms/proofs of P?”

“There are always two distinct notions of equality”

“In particular, you can consider proposition equality between any two terms of the same type”

“And you get all of the homotopical behaviour there”

“(On the type thory side)”

Q35 “Every proposition corresponds to a type, but not vice versa, right?”

“What does it mean to prove the theorem Nat?”

“it’s an isomorphism, so yes”

“It’s not an isomorphism between propositions and types though, becuase of astra’s point.”

“if you interpret a type as its propositional truncation, then to prove Nat would be to prove that Nat is inhabited”

“Nat represents a proposition, like if you have P and $P \to P$ then you have P”

“induction comes from polymorphism”

“not sure what you mean by that, Raul”

"Nat is defined as $P \to (P \to P) \to P$, that's all it needs"

"so that's a proposition"

"another example, Booleans would be forall P, $P \to P \to P$ and false would be forall P, P"

"that's a possible encoding of inductive types. it's not the one we're using though"

"But that's the one for Curry-Howard"

Q36 "Was Howard's paper on classical arithmetic or intuitionistic arithmetic?"

"I assume the latter"

Q37 "Just to clarify, we are defining 0/0 to be 0 right?"

"No, this is not an operation"

"This is the logical statement that a divides b"

"It is true that 0 divides 0"

"It is false, that 2 divides 3"

"0 divides 0 has infinitely many proofs, in fact"

"So if you tried to extract alpha, then you would not be able to do so uniquely in that case"

"Not sure I understand, sorry"

"$a \mid b$ is defined as a sigma type, not a natural number"

"I understand that part. It doesn't answer my question though"

"Maybe I'm baby brained but normally 5 / 0 is not defined, and would thus (IMO) have no proof. Don't we have to extend our notion of division to get what you said?"

"$0 \mid 5$ is the type of pairs (alpha, p) such that p is a proof that (alpha * p = 5)"

"There is no such pair, but we can consider the collection of such pairs logically"

"Sorry (alpha * 0 = 5)"

"I understand that but I don't see how it answers my specific question. Can you please address my quesiton literally? sorry"

"The answer is that we *are not considering a notion of division at all*"

"We ae thinking about a notion of *divisibility*"

"I don't understand how you can have one without the other, sorry. Let's talk about it after, I cant pay attention to the lecture and chat"

"I think it's the difference between the statements '5 / 0 is a number' and '5 is a multiple of 0'. Usually 'a / b is a number' works the same as 'a is a multiple of b', except when b is zero"

"So to be more direct, we're not defining 0/0 as anything, but we *are* stating that 0 is a multiple of 0 (by any number)"

Q38 "I am curious in a proposition/type such as $(k \mid d)$ in what sense are we able to retrieve d ? I think of the Axiom of Choice."

"Here $(k \mid n)$ means there is an actual witness d given, so you can retrieve it."

“Good question! That’s a sigma type so you can use the projection.”

“0 divides 0 has infinitely many proofs”

Q40 “So, for example, what if we had a proposition ’there exists an element of a subset’

“That depends on your definition of subset. You can always nest Sigma types to impose further conditions”

“That should correspond to a sigma type since our subsets are always definable (by a type family).”

Q42 “Does this proof of the axiom of choice require universes?”

“no”

“That’s right, it does not”