

Worksheet 10 (Solved)

HoTTEST Summer School 2022

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1 (*)

Let A be a type. Show that

- (a) $|||A||| \leftrightarrow ||A||$
- (b) $\exists_{(x:A)} ||B(x)|| \leftrightarrow ||\Sigma_{(x:A)}B(x)||$
- (c) $\neg \neg \|A\| \leftrightarrow \neg \neg A$
- (d) is-decidable(A) \rightarrow ($||A|| \rightarrow A$)
- (a) For the left-to-right direction, we need to construct a map $f: \|\|A\|\| \to \|A\|$. Since the codomain is a proposition, is suffices, by the recursion principle for propositional truncation, to define f(|x|), where $x: \|A\|$. We simply let, f(|x|) = x. For the other direction, we simply map $x: \|A\|$ to $\|x\|: \|\|A\|\|$.
- (b) For the left-to-right direction, we need to define a map

$$f: \|\Sigma_{(x:A)} \|B(x)\|\| \to \|\Sigma_{(x:A)} B(x)\|$$

The codomain is a proposition, so it suffices to define $f(|x,b_x|)$ for x:A and $b_x:||B(x)||$ Now, the codomain is still a proposition, so we may apply the recursion principle again, this time to b_x . Thus, it suffices to define $f(|x,|b_x'||)$ for x:A and $b_x':B(x)$. We define $f(|x,|b_x'||)=|x,b_x'|$, and we are done. For the other direction, we need a map

$$g: \|\Sigma_{(x:A)}B(x)\| \to \|\Sigma_{(x:A)}\|B(x)\|\|$$

The recursion rule applies here too, so we only need to define $g(|x, b_x|)$ for x : A and $b_x : \Sigma_{(x:A)}B(x)$. We define $g(|x, b_x|) = |x, |b_x||$, and we are done.

(c) Let us spell out the types involved. What we need is a bi-implication

$$((||A|| \to \emptyset) \to \emptyset) \leftrightarrow ((A \to \emptyset) \to \emptyset)$$

For the left-to-right direction, we are given $p:((\|A\|\to\emptyset)\to\emptyset)$ and $q:A\to\emptyset$. The goal is to produce an element of type \emptyset . If we can construct an element of type $r:\|A\|\to\emptyset$, then $p(r):\emptyset$. Since \emptyset is a proposition, it suffices to define $r(\|a\|)$ for a:A. Defining $r(\|a\|):=q(a)$ does the job. The other direction is proved similarly.

(d) Recall, is-decidable(A) := $A + \neg A$. We want to define f(x,p) : A for x : is-decidable(A) and $p : \|A\|$. Let us do this by induction on x. When x is inl(a) with a : A, it's easy: we define f(inl(a),p) := a. When x is inr(b) with $b : \neg A$, we cannot directly produce an element of A. However, we can product an element of \emptyset , which gives us an element of A via \emptyset -recursion. We have $p : \|A\|$, which immediately gives us a term $q : \neg \neg \|A\|$. We apply part (c), and get a term $r : \neg \neg A$. Now, $r(b) : \emptyset$ and we are done.

2 (**)

Consider two maps $f: A \to P$ and $g: B \to Q$ into propositions P and Q.

- (a) Show that if f and g are propositional truncations, then $f \times g : A \times B \to P \times Q$ is also a propositional truncation
- (b) Conclude that $||A \times B|| \simeq ||A|| \times ||B||$
- (a) Let R be a proposition, We need to show that precomposition

$$-\circ (f\times g):(P\times Q\to R)\to (A\times B\to R)$$

is an equivalence. Since both sides are propositions, it suffices to construct a map of type $(A \times B \to R) \to (P \times Q \to R)$. To this end, let $F: A \times B \to R$, and let $(p,q): P \times Q$. We need to construct an element of R. Consider the type $(Q \to R)$. This is a proposition, so let us instantiate the universal property of f with it. We get

$$-\circ f:(P\to (Q\to R))\simeq (A\to (Q\to R))$$

If we can construct and element $h:A\to (Q\to R)$, we are done, since then $((-\circ f)^{-1}(h))(p,q):R$. In order to construct h, we introduce a:A and q':Q. Our goal is again to construct an element of R, but this time, we have managed to introduce a:A to our context! Let's repeat the procedure, this time using the universal property of g. We instantiate it with the proposition $(P\to R)$, which gives us an equivalence.

$$-\circ g:(Q\to (P\to R))\simeq (B\to (P\to R))$$

By the above, it again suffices to construct a function $\ell: B \to (P \to R)$, since $((-\circ g)^{-1}(\ell))(q,p): R$. We define ℓ by $\ell(b,p)=F(a,b)$ and we are done.

(b) For any type C, the map λx . $|x|: C \to ||C||$ is a propositional truncation. By part (a), this implies that the canonical map $A \times B \to ||A|| \times ||B||$ is a propositional truncation. But so is the map λx . $|x|: A \times B \to ||A \times B||$. Hence we get, by the two-out-of-three property of propositional truncations (Proposition 14.1.4) that $||A|| \times ||B|| \simeq ||A \times B||$

3 (**)

Consider a map $f: A \to B$. Show that the following are equivalent:

- (i) f is an equivalence
- (ii) f is both surjective and an embedding

For the left-to-right direction, there is not much to prove. We know that all equivalences are embeddings, and clearly, since f is an equivalence, it must be surjective.

For the other direction, let us assume that f is both surjective and an embedding. Let us consider the family of propositions $P(b) := \mathsf{isContr}(\mathsf{fib}_f(b))$ over B. We are done if we can construct a section $\Pi_{(b:B)}P(b)$. Since f is surjective, we get, by Proposition 15.2.3, an equivalence

$$(\Pi_{(b:B)}P(b)) \simeq (\Pi_{(a:A)}P(f(a)))$$

Applying the inverse of the above equivalence, we only need to show that $fib_f(f(a))$ is contractible for each a:A. Clearly, $fib_f(f(a))$ is pointed (has an element) since $(a, refl_{f(a)}): fib_f(f(a))$. We now need to show that

$$(a, \mathsf{refl}_{f(a)}) =_{\mathsf{fib}_f(f(a))} (x, p) \tag{1}$$

for each x:A and p:f(x)=f(a). Since f is an embedding, we get $(\mathsf{ap}_f)^{-1}(p^{-1}):a=x$. This shows that the first components in (1) agree. We now need to show that the second components agree with respect to $(\mathsf{ap}_f)^{-1}(p^{-1})$. That is, we need to show that

$$\operatorname{tr}_{\lambda x. f(x) = f(a)}((\operatorname{ap}_f)^{-1}(p^{-1}), \operatorname{refl}_{f(a)}) = p$$

It is an easy lemma that the left-hand-side is equal to

$$(\mathsf{ap}_f((\mathsf{ap}_f)^{-1}(p^{-1})))^{-1} \cdot \mathsf{refl}_{f(a)}$$

 ap_f and ap_f^{-1} cancel out, an thus it is equal to

$$(p^{-1})^{-1} \cdot \mathsf{refl}_{f(a)} = p$$

and we are done.

4 $(\star \star \star)$

Prove **Lawvere's fixed point theorem:** For any two types A and B, if there is a surjective map $f: A \to B^A$, then for any $h: B \to B$, there (merely) exists an x: B such that h(x) = x. In other words, show that

$$\left(\exists_{(f:A\to(A\to B))} \text{is-surj}(f)\right) \to \left(\forall_{(h:B\to B)} \exists_{(b:B)} h(b) = b\right)$$

Let $x: (\exists_{(f:A \to (A \to B))} \text{is-surj}(f))$ and $h: B \to B$. The goal is to show that there merely exists an element b: B s.t. h(b) = b. This is a proposition, so we may assume x:=|f,p|, where $f: A \to B^A$ and p: is-surj(f). Given a: A, define, for ease of notation, $f_a: A \to B$ by $f_a = f(a)$. We get a function $F: A \to B$, given by

$$F(a) := h(f_a(a))$$

Surjectivity of f tells us that there is some a:A such that $f_a=F$. We claim that $f_a(a)$ is a fixed point. We need to show that $h(f_a(a))=f_a(a)$. Since $f_a=F$, it is enough to show that $h(f_a(a))=F(a)$. But this is precisely how we defined F, so we are done.

Disclaimer In the following exercises, we will use $\{0, ..., n\}$ to denote the elements of Fin_{n+1} , the finite type of n+1 elements.

5 (*)

- (a) Construct an equivalence $\operatorname{Fin}_{n^m} \simeq (\operatorname{Fin}_m \to \operatorname{Fin}_n)$. Conclude that if A and B are finite, then $(A \to B)$ is finite.
- (b) Construct an equivalence $\mathsf{Fin}_{n!} \simeq (\mathsf{Fin}_n \simeq \mathsf{Fin}_n)$. Conclude that if A is finite, then $A \simeq A$ is finite.
- (a) We proceed by induction on m. For m := 0, we need to show that

$$\mathsf{Fin}_1 \simeq (\mathsf{Fin}_0 \to \mathsf{Fin}_n)$$

Now, Fin_1 contains precisely 1 element, and so does $\mathsf{Fin}_0 \to \mathsf{Fin}_n$ since $\mathsf{Fin}_0 \simeq \emptyset$. Hence, the two types must be equivalent. For the inductive step, assume that $\mathsf{Fin}_{n^m} \simeq (\mathsf{Fin}_m \to \mathsf{Fin}_n)$ holds. The goal is to show that

$$\mathsf{Fin}_{n^{m+1}} \simeq (\mathsf{Fin}_{m+1} \to \mathsf{Fin}_n)$$

First, note that

$$\mathsf{Fin}_{n^{m+1}} = \mathsf{Fin}_{n^m \cdot n} = \mathsf{Fin}_{\underbrace{n^m + n^m + \dots + n^m}_{n \; \mathsf{times}}} \simeq \underbrace{\mathsf{Fin}_{n^m} + \dots + \mathsf{Fin}_{n^m}}_{n \; \mathsf{times}}$$

It is an easy lemma that for any type A, we have $\underbrace{A + \cdots + A}_{n \text{ times}} \simeq A \times \text{Fin}_n$.

Consequently, we get

$$\mathsf{Fin}_{n^{m+1}} \simeq \mathsf{Fin}_{n^m} \times \mathsf{Fin}_n$$

By the inductive hypothesis, we have

$$\mathsf{Fin}_{n^m} \times \mathsf{Fin}_n \simeq (\mathsf{Fin}_m \to \mathsf{Fin}_n) \times \mathsf{Fin}_n$$

Hence, it suffices to construct an equivalence

$$f: (\operatorname{Fin}_m \to \operatorname{Fin}_n) \times \operatorname{Fin}_n \simeq (\operatorname{Fin}_{m+1} \to \operatorname{Fin}_n)$$

We define f(g,x) by

$$f(g,x)(y) = \begin{cases} g(y) & \text{if } y < m \\ x & \text{otherwise} \end{cases}$$

and its inverse f^{-1} by

$$f^{-1}(g) = (g_m, g(m))$$

where $g_m : \operatorname{Fin}_m \to \operatorname{Fin}_n$ is the restriction of g to Fin_m . The fact that f and f^{-1} cancel out is immediate.

For the second part of the question, let us assume that A and B are finite types. We want to show that $(A \to B)$ is finite. This is a proposition, so we may assume that we have equivalences $A \simeq \operatorname{Fin}_m$ and $B \simeq \operatorname{Fin}_n$ for some $n, m : \mathbb{N}$. We then have

$$(A \to B) \simeq (\operatorname{Fin}_m \to \operatorname{Fin}_n) \simeq \operatorname{Fin}_{n^m}$$

and thus $(A \to B)$ is also a finite type.

(b) We proceed by induction on n. For n = 0, we immediately get

$$\mathsf{Fin}_{0!} = \mathsf{Fin}_1 \simeq (\mathsf{Fin}_0 \simeq \mathsf{Fin}_0)$$

since there is precisely one equivalence $\mathsf{Fin}_0 \simeq \mathsf{Fin}_0$ (recall, $\mathsf{Fin}_0 \simeq \emptyset$). Assume as inductive hypothesis that $\mathsf{Fin}_{n!} \simeq (\mathsf{Fin}_n \simeq \mathsf{Fin}_n)$. We need to construct an equivalence

$$\mathsf{Fin}_{(n+1)!} \simeq (\mathsf{Fin}_{n+1} \simeq \mathsf{Fin}_{n+1})$$

We have

$$\mathsf{Fin}_{(n+1)!} = \mathsf{Fin}_{n!} \times \mathsf{Fin}_{n+1}$$

using similar reasoning as in (a). Hence, by the inductive hypothesis, we have

$$\mathsf{Fin}_{(n+1)!} \simeq (\mathsf{Fin}_n \simeq \mathsf{Fin}_n) \times \mathsf{Fin}_{n+1}$$

Proving that

$$(\mathsf{Fin}_n \simeq \mathsf{Fin}_n) \times \mathsf{Fin}_{n+1} \simeq (\mathsf{Fin}_{n+1} \simeq \mathsf{Fin}_{n+1})$$

is now just straightforward combinatorics. The idea is that given an element $x: \mathsf{Fin}_{n+1}$, any equivalence $f: \mathsf{Fin}_n \simeq \mathsf{Fin}_n$ can be extended to an equivalence $f_x: \mathsf{Fin}_{n+1} \simeq \mathsf{Fin}_{n+1}$ by

$$f_x(y) = \begin{cases} f(y) & \text{if } y < x \\ n & \text{if } y = x \\ f(y-1) & \text{otherwise} \end{cases}$$

We won't do this here, but it's easy to verify that $(f,x) \mapsto f_x$ is an equivalence. Hence

$$\mathsf{Fin}_{(n+1)!} \simeq (\mathsf{Fin}_{n+1} \simeq \mathsf{Fin}_{n+1})$$

and we are done.

Using an almost identical argument to that in the second part of (a); we get that $A \simeq A$ is finite for any finite type A.

$6 \quad (\star \star \star)$

Consider a map $f: X \to Y$, and suppose that X is finite.

- (a) For y: Y, define $\mathsf{inlm}_f(y) := \exists_{x:X} (f(x) = y)$. Show that, if type the Y has decidable equality, then inlm_f is decidable.
- (b) Suppose that f is surjective. Show that the following two statements are equivalent:
 - (i) The type Y has decidable equality
 - (ii) The type Y is finite

Hint for (i) \Longrightarrow (ii): Induct on the size of X. If $f: X \simeq \mathsf{Fin}_{n+1} \to Y$, consider its restriction $f_n: \mathsf{Fin}_n \to Y$. Use (a) to do a case distinction on whether or not $\mathsf{inIm}_{f_n}(f(n))$ holds.

Note that since X is finite, we have an element of type $e': ||X \simeq \mathsf{Fin}_n||$ for some $n: \mathbb{N}$. For all problems here, we are concerned with proving *propositions*, so we may use the recursion rule on e' to get an element $e: X \simeq \mathsf{Fin}_n$. It is an easy lemma that

- $f: X \to Y$ has decidable image (question (a)) iff $f \circ e^{-1}: \operatorname{Fin}_n \to Y$ has decidable image
- $f: X \to Y$ is surjective (question (b)) iff $f \circ e^{-1}: \mathsf{Fin}_n \to Y$ is surjective

Therefore, let's be informal and pretend that $f: \mathbf{Fin}_n \to Y$ in what follows (even though, under the hood, we are working with $f \circ e^{-1}$) and forget about X altogether.

- (a) Assume Y has decidable equality and let y:Y. The goal is to show that for any $f:\operatorname{Fin}_n\to Y$, we have a term of type $(\operatorname{inlm}_f(y)+\neg\operatorname{inlm}_f(y))$. We proceed by induction on n, the size of the domain of f. For n=0, the lemma is trivial, since the image of a map $\operatorname{Fin}_0\to Y$ is empty. Assume that the statment holds for all maps $\operatorname{Fin}_n\to Y$. Let us prove it for $f:\operatorname{Fin}_{n+1}\to Y$. Let $f_n:\operatorname{Fin}_n\to Y$ be the restriction of f to Fin_n . Using the induction hypothesis on f_n , we get a term of type $(\operatorname{inlm}_{f_n}(y)+\neg\operatorname{inlm}_{f_n}(y))$. This allows us to do a case distinction:
 - If y is in the image of f_n , it must be in the image of f.
 - If not, consider instead the two cases y = f(n) and $y \neq f(n)$, which we get from the decidable equality of Y.
 - If y = f(n), then y is in the image of f.
 - If not, y cannot be in the image of f, since $y \neq f(n)$ and $y \neq f_n(x)$ for any $x : Fin_n$.

This covers all cases, and hence we have a term of type $(\mathsf{inlm}_f(y) + \neg \mathsf{inlm}_f(y))$.

(b)

(i) \Longrightarrow (ii) Assume Y has decidable equality. We proceed by induction on n. For n=0, the problem is trivial, since surjectivity of f implies that Y is empty. For the inductive step, let $f_n : \operatorname{Fin}_n \to Y$ be the restriction of $f : \operatorname{Fin}_{n+1} \to Y$ to Fin_n . Using (a), we may consider two cases:

Case 1: f(n) is in the image of f_n . In this case, f_n is surjective and thus Y is finite by the induction hypothesis.

Case 2: f(n) is not in the image of f_n . In this case, one can construct an equivalence

$$e: Y \simeq \operatorname{im}(f_n) + \mathbb{1}$$

defined (informally) by

$$e(y) = \begin{cases} & \mathbf{inr}(\star) \ \mathbf{if} \ y = f(n) \\ & \mathbf{inl}(y) \ \mathbf{otherwise} \end{cases}$$

Note that $\operatorname{im}(f_n)$ inherits decidable equality from Y. Now, since f_n surjects onto its own image, we may conclude that $\operatorname{im}(f_n)$ is finite. Furthermore, $\mathbb{1}$ is finite, and thus Y is finite.

(ii) \implies (i) Since having decidable equality is a proposition and Y is finite, we may assume that $Y \simeq \operatorname{Fin}_n$ for some n. But then we are done, since Fin_n has decidable equality.