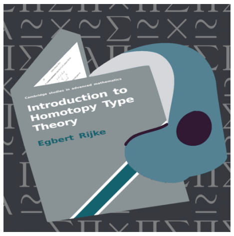

Worksheet 8

HoTTEST Summer School 2022

The HoTTEST TAs, and

2 August 2022

1 $(\star)$

Let A and B be types.

1. Suppose that both A and B are propositions. Prove that $A + B$ is a proposition if and only if $A \to \neg B$.

2. Let $k : \mathbb{T}$. Suppose that both A and B are $(k+2)$-types. Prove that $A + B$ is a $(k+2)$-type.

2 $(\star\star)$

Let $\mathcal{U}$ be a universe and A be a type. Consider a partial order

$$\_ \leq \_ \ : \ A \to A \to \mathsf{Prop}_{\mathcal{U}}$$

on A. Prove that A is a set.

3 $(\star\star)$

1. Let A be a type and B a set. Suppose that $f : A \to B$ is an injection in the sense that it has a term
$$c : \prod_{x,y:A} (f(x) = f(y)) \to (x = y)\,.$$
Prove that f is an embedding, so that A is a set.

2. Prove that the function $n \mapsto m + n$ is an embedding for every $m : \mathbb{N}$. Here, addition is defined by recursion on the *first* argument.

   Conclude that
$$(m \leq n) \simeq \sum_{k:\mathbb{N}} m + k = n$$
for every $m, n : \mathbb{N}$.

4 $(\star)$

Let A and B be types. Prove that the following are logically equivalent.

- Both A and B are contractible.
- The product $A \times B$ is contractible.

5 $(\star\star\star)$

Let A be a type and $a : A$. We say that a is an *isolated point* of A if it has a term

$$\tau : \prod_{x:A} (a = x) + (a \neq x)\,.$$

Suppose that a is isolated. Prove that $a = x$ is a proposition for all $x : A$. Conclude that $\mathsf{const}_a : \mathbb{1} \to A$ is an embedding.