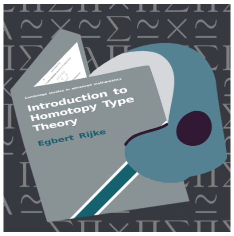

Worksheet 1 (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs

4th July 2022

1 ($\star$)

State the introduction and elimination rules for

1. $\times$-types
2. $\to$-types
3. $\prod$-types

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash (a, b) : A \times B} \qquad \frac{\Gamma \vdash x : A \times B}{\Gamma \vdash \mathbf{pr}_1(x) : A} \qquad \frac{\Gamma \vdash x : A \times B}{\Gamma \vdash \mathbf{pr}_2(x) : B}$$

$$\frac{\Gamma, x : A \vdash b : B}{\Gamma \vdash \lambda(x : A).b : A \to B} \qquad \frac{\Gamma \vdash f : A \to B \quad \Gamma \vdash a : A}{\Gamma \vdash f(x) : B}$$

$$\frac{\Gamma, x : A \vdash b : B(x)}{\Gamma \vdash \lambda(x : A).b : \prod_{x:A} B(x)} \qquad \frac{\Gamma \vdash f : \prod_{x:A} B(x) \quad \Gamma \vdash a : A}{\Gamma \vdash f(a) : B(a)}$$

2 ($\star$)

Fill in this proof tree:

$$\dfrac{\dfrac{\dfrac{}{a : A, b : B \vdash a : A} \qquad \dfrac{}{\boxed{a : A, b : B \vdash b : B}}}{\dfrac{\boxed{a : A, b : B} \vdash \boxed{(a, b)} : A \times B}{a : A \vdash \lambda(b : B).(a, b) : \boxed{B \to A \times B}}}}{\cdot \vdash \lambda(a : A).\lambda(b : B).(a, b) : A \to B \to A \times B}$$

3 (⋆⋆)

Write a proof tree ending with a term of type $A \times B \to B \times A$ in the empty context.

$$\dfrac{\dfrac{\dfrac{}{x : A \times B \vdash \mathbf{pr}_2(x) : B} \qquad \dfrac{}{x : A \times B \vdash \mathbf{pr}_1(x) : A}}{x : A \times B \vdash (\mathbf{pr}_2(x), \mathbf{pr}_1(x)) : B \times A}}{\cdot \vdash \lambda(x : A \times B).(\mathbf{pr}_2(x), \mathbf{pr}_1(x)) : A \times B \to B \times A}$$

4 (⋆⋆)

For problems 2 and 3, what is the *logical* content of the proof tree? That is, under the "types are theorem" interpretation of Curry-Howard, what theorems have we proven?

Let A and B be propositions. Then problem 2 proves the tautology

$$A \to B \to A \wedge B$$

and problem 3 proves the tautology

$$A \wedge B \to B \wedge A$$

Next, what is the *computational* content of the proof tree? That is, under the "programs are proofs" interpretation of Curry-Howard, what programs have we written?

Problem 2 implements the function that takes in a and outputs a function taking in b and outputting (a, b).

Problem 3 implements the function that takes in the tuple (a, b) as input, and outputs the tuple (b, a)

5 $(\star\star\star)$

Define the **swap** function $\sigma_{A,B}$ of type

$$\sigma_{A,B} : \left(\prod_{x:A}\prod_{y:B} C(x,y)\right) \to \left(\prod_{y:B}\prod_{x:A} C(x,y)\right)$$

and show that $\sigma_{B,A} \circ \sigma_{A,B}$ is (definitionally) equal to the identity.

$$\sigma_{A,B} \doteq \lambda\left(f : \prod_{x:A}\prod_{y:B} C(x,y)\right).\lambda(y:B).\lambda(x:A).f(x,y)$$

By our computation rules:

$$\begin{aligned}
(\sigma_{B,A} \circ \sigma_{A,B})(g) &\doteq \sigma_{B,A}(\sigma_{A,B}(g)) \\
&\doteq \sigma_{B,A}\left((\lambda f.\lambda(y:B).\lambda(x:A).f(x,y))\ \ (g)\right) \\
&\doteq \sigma_{B,A}(\lambda(y:B).\lambda(x:A).g(x,y)) \\
&\doteq \big(\lambda f.\lambda(x':A).\lambda(y':B).f(y',x')\big)\ (\lambda(y:B).\lambda(x:A).g(x,y)) \\
&\doteq \lambda(x':A).\lambda(y':B).\ \big((\lambda(y:B).\lambda(x:A).g(x,y))\ (y',x')\big) \\
&\doteq \lambda(x':A).\lambda(y':B).g(x',y') \\
&\doteq g
\end{aligned}$$

We first substitute the definition of $\sigma_{A,B}$, then apply this definition to g. Then we expand the definition of $\sigma_{B,A}$, and apply this definition to the previous result. Finally, we apply our $\lambda(y:B).\lambda(x:A).g(x,y)$ term to the inputs (y',x') to recover something which is g up to α-equivalence. Since every step was a definitional equality we see $(\sigma_{B,A} \circ \sigma_{A,B})(g) \doteq g$ for each input g, so that $\sigma_{B,A} \circ \sigma_{A,B}$ is definitionally the identity.