

# On Properties of Doeblin Coefficients

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**Abstract**—Doeblin coefficients are a classical tool to study the ergodicity of Markov chains. Propelled by recent works on contraction coefficients of strong data processing inequalities, we investigate whether Doeblin coefficients also exhibit some of the notable properties of canonical contraction coefficients. Specifically, we present various new structural and geometric properties of Doeblin coefficients. Then, by establishing an extremal coupling characterization, we show that Doeblin coefficients generalize the well-known total variation (TV) distance to a multi-way divergence, enabling us to measure the distance between multiple distributions rather than just two. We also demonstrate that Doeblin coefficients exhibit contraction properties over Bayesian networks similar to other canonical contraction coefficients. Finally, we discuss how Doeblin coefficients can be used to define a new rule for fusion of probability mass functions.

## I. INTRODUCTION

Recently, there has been a flurry of research activity on *strong data processing inequalities* (SDPIs), which provide quantitative bounds on the contraction of information in channels or Markov kernels as measured by various  $f$ -divergences. Specifically, such contraction of information is mathematically captured by so-called *contraction coefficients* and has been studied extensively, cf. [1]–[5]. Furthermore, the resulting ideas have been used in distributed estimation [6], differential privacy [7], [8], and secret key generation [9], among other applications. In [2], it was shown that contraction coefficients for Kullback-Leibler (KL) divergence of a given channel are defined by the extremal erasure probabilities so that an erasure channel dominates the given channel in the *less noisy* preorder sense [10]. This observation has been generalized for a broader class of  $f$ -divergences in [4], and to broader classes of dominating channels such as symmetric channels in [3]. In this paper, we study the extremal erasure probabilities so that an erasure channel dominates a given channel in the (output) *degradation* sense, cf. [3], [11], [12]. These extremal erasure probabilities are known as *Doeblin coefficients* [13], [14]. In particular, our objective is to illustrate that Doeblin coefficients share most of the nice properties of canonical contraction coefficients, such as their behavior over Bayesian networks.

Formally, let  $\mathbb{R}_{\text{sto}}^{n \times m}$  be the set of all  $n \times m$  row stochastic matrices, or equivalently, channels with input alphabet  $\mathcal{X}$  and output alphabet  $\mathcal{Y}$  such that  $|\mathcal{X}| = n$  and  $|\mathcal{Y}| = m$ . Then, for any channel  $W \in \mathbb{R}_{\text{sto}}^{n \times m}$ , its *Doeblin coefficient* of ergodicity  $\tau(W)$  is defined as

$$\tau(W) \triangleq \sum_{j=1}^m \min_{i \in \{1, \dots, n\}} W_{ij}. \quad (1)$$

Classically, the Doeblin coefficient was used in the study of ergodicity of Markov chains. Indeed, for a given channel (or Markov kernel)  $W$ , its Doeblin coefficient is the largest constant  $\alpha \in [0, 1]$  such that the *Doeblin minorization* condition holds [15], i.e., there exists a probability distribution  $\mu$  over  $\mathcal{Y}$  such that

$$W_{ij} \geq \alpha \mu_j \quad (2)$$

for all  $i \in \{1, \dots, n\}$  and  $j \in \{1, \dots, m\}$ . A Markov chain with a large Doeblin coefficient is said to be “well-minorized,” because it can be shown that the chain converges quickly to its invariant distribution.

### A. Related Literature

As mentioned above, Doeblin minorization was originally developed to study the ergodicity of Markov chains. In [15], Doeblin characterized the conditions for weak ergodicity for possibly inhomogeneous Markov chains. In [16], Doeblin introduced coupling as a technique for proving uniform geometric rates of convergence of Markov chains to their stationary distributions in terms of *total variation (TV) distance* (see [17] for more details). The Doeblin minorization condition has also been studied in the context of information theory, where it was used to derive universal upper bounds on contraction coefficients for any  $f$ -divergence [1, Remark 3.2], [3, Section I-D]. In fact, Doeblin minorization is known to be equivalent to degradation by erasure channels (see [17, Theorem 3.1], [14, Lemma 5] and [18, Section IV-D] for more details). In the theory of Harris chains, this idea can be viewed as a specialization of the *regeneration* or Nummelin splitting technique [19], [20]. Moreover, this is why Doeblin coefficients can be characterized using degradation by erasure channels [13], [14]. As noted earlier, in this sense, Doeblin coefficients can be construed as extensions of the broader theory of contraction coefficients [21], [22, Definition 5.1], [5, mmmmmmmmmmm], which are characterized using domination by erasure channels under the less noisy preorder. Several properties of Doeblin coefficients have been explored in [13]. Finally, in a different vein, we will show that Doeblin coefficients have the flavor of *multi-way divergences*, which are measures of discrimination between two or more probability distributions. We refer readers to [23] and the references therein for a detailed overview of multi-way divergences and the references therein.

## B. Main Contributions

In this paper, we delve into various structural and geometric properties of Doeblin coefficients. Firstly, we collect several known properties and prove various new properties of Doeblin coefficients in Section II for the readers' convenience. For example, we show that the Doeblin coefficient has desirable properties that make it suitable for measuring some notion of "distance" between multiple distributions. In particular, it generalizes TV distance to a multi-way metric, which is an interesting observation in its own right. Secondly, we study the information contraction properties of Doeblin coefficients over Bayesian networks (or directed graphical models) in Section IV. To do this, we develop several extremal-coupling-based characterizations of Doeblin coefficients in Section III that generalize known maximal coupling and simultaneous coupling results for TV distance. Finally, on a more applied front, we briefly illustrate how Doeblin coefficients can be used to design a new notion of fusion or aggregation of probability mass functions (PMFs) in Section V. Specifically, we show that Doeblin coefficients allow us to develop an optimization-based approach for the aggregation of PMFs.

## II. PROPERTIES OF DOEBLIN COEFFICIENTS

In this section, we introduce several new properties of the Doeblin coefficient, starting with a brief review of the notation used throughout this work.

### A. Notation

We briefly collect some notation that is used throughout this work. For  $n \in \mathbb{N}$ , let  $[n]$  denote the set  $\{1, 2, \dots, n\}$ . Let  $\mathbf{1}$  denote a column vector with all entries equal to 1 of appropriate dimension. Let  $\mathcal{P}_n$  denote the  $(n-1)$ -dimensional probability simplex of row vectors in  $\mathbb{R}^n$ , i.e., the set  $\{P \in \mathbb{R}^n : P \geq 0 \text{ entry-wise and } P\mathbf{1} = 1\}$ . For an input alphabet  $\mathcal{X}$  and output alphabet  $\mathcal{Y}$  with  $|\mathcal{X}| = n$  and  $|\mathcal{Y}| = m$ , let  $X$  and  $Y$  be random variables taking values in  $\mathcal{X}$  and  $\mathcal{Y}$ , respectively, and let  $\mathbb{R}_{\text{sto}}^{n \times m}$  denote the set of all  $n \times m$  row stochastic matrices in  $\mathbb{R}^{n \times m}$ , which are conditional distributions of  $Y$  given  $X$ . For any matrix  $W \in \mathbb{R}_{\text{sto}}^{n \times m}$  such that  $W = [P_1^T P_2^T \dots P_n^T]^T$  is formed by stacking the row vectors  $P_1, P_2, \dots, P_n \in \mathcal{P}_m$ , we will interchangeably use the notation  $\tau(P_1, P_2, \dots, P_n)$  to denote the Doeblin coefficient  $\tau(W)$ .

### B. Properties

The following theorem summarizes various properties of Doeblin coefficients.

**Theorem 1 (Properties of Doeblin Coefficients):** Let  $W = [P_1^T P_2^T \dots P_n^T]^T \in \mathbb{R}_{\text{sto}}^{n \times m}$  be a channel formed by stacking the PMFs  $P_1, P_2, \dots, P_n \in \mathcal{P}_m$ . Then, the Doeblin coefficient  $\tau(W)$  of  $W$  satisfies the following properties:

- 1) (*Normalization*) We have  $0 \leq \tau(W) \leq 1$ , where  $\tau(W) = 1$  if and only if  $P_1 = P_2 = \dots = P_n$  (entry-wise), and  $\tau(W) = 0$  if and only if at least one of  $P_1(y), P_2(y), \dots, P_n(y)$  is zero for all  $y \in \mathcal{Y}$ .

- 2) (*n-way Metric*) The complement of the Doeblin coefficient  $\gamma(P_1, \dots, P_n) \triangleq 1 - \tau(P_1, \dots, P_n) \in [0, 1]$  is an  $n$ -way metric, i.e., it satisfies the following properties:

- a) (*Total Symmetry*) For any permutation  $\pi$  of  $[n]$ , we have

$$\gamma(P_{\pi(1)}, P_{\pi(2)}, \dots, P_{\pi(n)}) = \gamma(P_1, P_2, \dots, P_n).$$

- b) (*Positive Definiteness*)  $\gamma(P_1, \dots, P_n) = 0$  if and only if  $P_1 = P_2 = \dots = P_n$ .

- c) (*Polyhedron Inequality*) For any  $P_{n+1} \in \mathcal{P}_m$ ,  $\gamma(P_1, \dots, P_n)$  satisfies the inequality

$$(n-1)\gamma(P_1, \dots, P_n) \leq \sum_{i=1}^n \gamma(P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_{n+1}).$$

- 3) (*Concavity*) The map  $W \mapsto \tau(W)$  is concave.

- 4) (*Sub-multiplicativity [13]*) For channels  $V \in \mathbb{R}_{\text{sto}}^{n \times k}$  and  $W \in \mathbb{R}_{\text{sto}}^{k \times m}$ , the complement of the Doeblin coefficient is sub-multiplicative:

$$1 - \tau(VW) \leq (1 - \tau(V))(1 - \tau(W)).$$

- 5) (*Tensorization*) For channels  $V \in \mathbb{R}_{\text{sto}}^{n \times k}$  and  $W \in \mathbb{R}_{\text{sto}}^{k \times m}$ , we have

$$\tau(V \otimes W) = \tau(V)\tau(W),$$

where  $\otimes$  denotes the Kronecker product of matrices.

- 6) (*Upper Bound [1]*)  $\tau(W)$  satisfies the following bound:

$$\tau(W) \leq 1 - \eta_{\text{TV}}(W),$$

where  $\eta_{\text{TV}}(W) = \frac{1}{2} \max_{i,j \in [n]} \|P_i - P_j\|_1$  denotes the Dobrushin contraction coefficient for TV distance and  $\|\cdot\|_1$  is the  $\ell^1$ -norm.

- 7) (*Minimum Trace Characterization*)  $\tau(W)$  is the solution to the following optimization problem:

$$\tau(W) = \min_{P \in \mathbb{R}_{\text{sto}}^{m \times n}} \text{Tr}(PW),$$

where the minimum is over all possible  $m \times n$  row stochastic matrices  $P$ , and  $\text{Tr}(\cdot)$  denotes the trace of a matrix.

- 8) (*Optimal Estimator*) Consider a hidden random variable  $X$  that is uniformly distributed on  $\mathcal{X}$ , i.e.,  $X \sim \text{unif}(\mathcal{X})$ , and the fixed channel (or observation model)  $W = P_{Y|X} \in \mathbb{R}_{\text{sto}}^{n \times m}$ . Let  $\hat{X} \in \mathcal{X}$  denote any (possibly randomized) estimator of  $X$  based on the observation  $Y$ , which is defined by a kernel  $P_{\hat{X}|Y} \in \mathbb{R}_{\text{sto}}^{m \times n}$  such that  $X \rightarrow Y \rightarrow \hat{X}$  forms a Markov chain. Then, we have

$$\frac{\tau(P_{Y|X})}{n} = \min_{P_{\hat{X}|Y} \in \mathbb{R}_{\text{sto}}^{m \times n}} \mathbb{P}(\hat{X} = X),$$

where the minimum is computed over all estimators  $\hat{X}$ , or equivalently, over all kernels  $P_{\hat{X}|Y} \in \mathbb{R}_{\text{sto}}^{m \times n}$ , and  $\mathbb{P}(\cdot)$  denotes the probability law of  $(X, Y, \hat{X})$ .

The proof is provided in Appendix A-A. Properties 1 and 3 in Theorem 1 are straightforward to prove (and property 1 is of course well-known). Property 4 was already proved in [13], but we provide a new proof of property 4. The upper bound in property 6 is also well-known, cf. [1, Remark III.2]. The rest of the properties are new to our knowledge. It is worth reiterating that, interestingly, the complement of the Doeblin coefficient  $\gamma(\cdot)$  acts as a multi-way metric, which allows us to measure “distance” between more than two distributions. In Theorem 2, we will establish that  $\gamma(\cdot)$  is actually a multi-way generalization of TV distance.

### III. EXTREMAL COUPLING CHARACTERIZATIONS

In this section, we develop several extremal-coupling-based characterizations of Doeblin coefficients that generalize known maximal coupling and simultaneous coupling results for TV distance. In particular, as mentioned earlier, our maximal coupling result will show that the complement of the Doeblin coefficient  $\gamma(P_1, \dots, P_n)$  is in fact a multi-way generalization of TV distance between multiple probability distributions.

#### A. Maximal Coupling

Recall that for random variables  $X$  and  $Y$  taking values in some common alphabet  $\mathcal{X}$  with probability distributions  $P$  and  $Q$ , respectively, a *coupling* of  $P$  and  $Q$  is a joint distribution  $P_{X,Y}$  on the product space  $\mathcal{X} \times \mathcal{X}$  such that the corresponding marginals distributions of  $X$  and  $Y$  are equal to  $P$  and  $Q$ , respectively. A *maximal coupling* of two distributions  $P$  and  $Q$  on  $\mathcal{X}$  is a coupling that maximizes the probability that the two random variables are equal. It is well-known that the TV distance between  $P$  and  $Q$  can be characterized by Dobrushin’s maximal coupling of  $P$  and  $Q$  [24, Prop. 4.7]

$$1 - \|P - Q\|_{\text{TV}} = \max_{P_{X,Y}: P_X=P, P_Y=Q} \mathbb{P}(X=Y), \quad (3)$$

where  $\|\cdot\|_{\text{TV}}$  denotes the TV distance, the maximum is over all couplings of  $P$  and  $Q$ , and  $\mathbb{P}(\cdot)$  is the probability law corresponding to the coupling  $P_{X,Y}$ . Moreover, under the maximal coupling, for any  $x \in \mathcal{X}$ ,

$$\mathbb{P}(X=Y=x) = \min\{P(x), Q(x)\}. \quad (4)$$

We will now generalize the result for the case where we have  $n$  distributions  $P_1, P_2, \dots, P_n \in \mathcal{P}_m$ . A coupling of random variables  $X_1, \dots, X_n$  distributed as  $P_1, \dots, P_n$  is defined as a joint distribution  $P_{X_1, \dots, X_n}$  that preserves the marginals. Specifically, under the maximal coupling of  $P_1, P_2, \dots, P_n$ , we will show that the measure of the “diagonal set” of  $\mathcal{X}^n$  equals  $\tau(P_1, \dots, P_n)$ .

**Theorem 2 (Maximal Coupling):** For any random variables  $X_1, \dots, X_n \in \mathcal{Y}$  distributed according to the PMFs  $P_1, \dots, P_n \in \mathcal{P}_m$ , respectively, let  $W \in \mathbb{R}_{\text{sto}}^{n \times m}$  be a channel defined by  $W = [P_1^T \ P_2^T \ \dots \ P_n^T]^T$ . Then, we have

$$\tau(W) = 1 - \gamma(W) = \max_{P_{X_1, \dots, X_n}: P_{X_1}=P_1, \dots, P_{X_n}=P_n} \mathbb{P}(X_1 = \dots = X_n),$$

where the maximum is over all couplings of  $P_1, \dots, P_n$ , and  $\mathbb{P}(\cdot)$  denotes the probability law corresponding to a coupling. Moreover, under the maximal coupling, for any  $x \in \mathcal{Y}$

$$\mathbb{P}(X_1 = \dots = X_n = x) = \min\{P_1(x), \dots, P_n(x)\}.$$

The proof is provided in Appendix A-B. Theorem 2 illustrates that  $\gamma(\cdot)$  is a generalization of TV distance and for the special case of  $n = 2$ ,  $\gamma(\cdot)$  reduces to the TV distance. Moreover, by property 2 in Theorem 1,  $\gamma(\cdot)$  is a  $n$ -way metric just like TV distance is a metric between two distributions.

In [1], it was shown that the TV distance could be used to define *DeGroot distance* [25], which is a measure of Bayes statistical information. We will now generalize the DeGroot distance to the case when we have multiple probability distributions. We show that we get two different notions of statistical information: min-DeGroot and max-DeGroot distance.

**1) Min-DeGroot Distance:** Given any prior PMF  $\lambda \in \mathcal{P}_n$ , consider a hidden random variable  $X \in \mathcal{X}$  and an observed random variable  $Y \in \mathcal{Y}$  such that  $|\mathcal{X}| = n$ ,  $|\mathcal{Y}| = m$ , and

$$X \sim \lambda \text{ and } P_{Y|X=x_i} = P_i \quad (5)$$

for  $i \in [n]$ , where  $P_1, \dots, P_n \in \mathcal{P}_m$  determine the channel (or observation model)  $P_{Y|X}$ , and we let  $\mathcal{X} = \{x_1, \dots, x_n\}$  without loss of generality. Let  $\hat{X} \in \mathcal{X}$  denote any (possibly randomized) estimator of  $X$  based on  $Y$ , which is defined by a kernel  $P_{\hat{X}|Y} \in \mathbb{R}_{\text{sto}}^{m \times n}$  such that  $X \rightarrow Y \rightarrow \hat{X}$  forms a Markov chain. Suppose the goal is to minimize the *Bayes risk* for the loss function  $l(X, \hat{X}) = \mathbb{1}_{\hat{X} \neq X}$ , where  $\mathbb{1}$  denotes the indicator function. When we have no observations related to  $X$ , the Bayes optimal estimator is to choose the least likely  $x \in \mathcal{X}$ , i.e.,  $\hat{X}^* = x_{i^*}$  with  $i^* = \arg \min_{i \in [n]} \lambda_i$ , and the corresponding Bayes risk  $R_\lambda^*$  is

$$R_\lambda^* = \min\{\lambda_1, \dots, \lambda_n\}. \quad (6)$$

On the other hand, when  $Y$  is observed, the Bayes optimal estimator  $\hat{X}^*$  is the solution to the following problem:

$$\hat{X}^* = \arg \min_{P_{\hat{X}|Y} \in \mathbb{R}_{\text{sto}}^{m \times n}} \mathbb{P}(\hat{X} = X), \quad (7)$$

where the minimum is over all possible estimators. In statistical decision theory, it is well-known that for an observation model  $P_{Y|X} \in \mathbb{R}_{\text{sto}}^{n \times m}$ , prior PMF  $P_X \in \mathcal{P}_n$ , loss function  $l: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ , and randomized estimator  $P_{\hat{X}|Y} \in \mathbb{R}_{\text{sto}}^{m \times n}$ , the risk function is defined as

$$\begin{aligned} R_{P_X}(P_{Y|X}, l, P_{\hat{X}|Y}) &\triangleq \mathbb{E}[l(X, \hat{X})] \\ &= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \sum_{\hat{x} \in \mathcal{X}} l(x, \hat{x}) P_X(x) P_{Y|X}(y|x) P_{\hat{X}|Y}(\hat{x}|y) \\ &= \text{Tr}(L^T \text{diag}(P_X) P_{Y|X} P_{\hat{X}|Y}), \end{aligned} \quad (8)$$

where  $L \in \mathbb{R}^{n \times n}$  is the matrix representation of the loss function  $l: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ , i.e., the  $(i, j)$ th entry of  $L$  is  $l(x_i, \hat{x}_j)$  for  $i, j \in [n]$  and  $x_i, \hat{x}_j \in \mathcal{X}$ . By our earlier choice of loss function, we have  $L = I$ , where  $I$  is identity matrix. Moreover,

$P_X = \lambda$  and let  $\Lambda = \text{diag}(\lambda)$  as the diagonal matrix with  $\lambda$  along its principal diagonal. This gives the Bayes risk

$$\begin{aligned} R_\lambda^*(P_{Y|X}) &= \min_{P_{\hat{X}|Y} \in \mathbb{R}_{\text{sto}}^{m \times n}} \mathbb{P}(\hat{X} = X) \\ &= \min_{P_{\hat{X}|Y} \in \mathbb{R}_{\text{sto}}^{m \times n}} \text{tr}(\Lambda P_{Y|X} P_{\hat{X}|Y}) \\ &= \sum_{y \in \mathcal{Y}} \min\{\lambda_1 P_{Y|X}(y|x_1), \dots, \lambda_n P_{Y|X}(y|x_n)\}. \end{aligned} \quad (9)$$

Now define the *min-DeGroot distance* as  $\tilde{\tau}_{\min}(\lambda, P_{Y|X}) \triangleq R_\lambda^* - R_\lambda^*(P_{Y|X})$ , which gives

$$\begin{aligned} \tilde{\tau}_{\min}(\lambda, P_{Y|X}) &= \min\{\lambda_1, \dots, \lambda_n\} \\ &\quad - \sum_{y \in \mathcal{Y}} \min\{\lambda_1 P_{Y|X}(y|x_1), \dots, \lambda_n P_{Y|X}(y|x_n)\}. \end{aligned} \quad (10)$$

Note that  $\tilde{\tau}(P_{Y|X}) \geq 0$  is a measure of the statistical information obtained about  $X$  after observing  $Y$ .

2) *Max-DeGroot Distance*: Consider the same setting as Section III-A1, where we are given a pair of random variables  $(X, Y)$  distributed as in (5). This time, fix the loss function as  $l(X, \hat{X}) = \mathbb{1}_{\hat{X} \neq X}$ . Then, the Bayes risk  $R_\lambda^*$  in estimating  $X$  when we do not observe  $Y$  is

$$R_\lambda^* = 1 - \max\{\lambda_1, \dots, \lambda_n\}. \quad (11)$$

Now when  $Y$  is observed, since  $L = \mathbf{1}\mathbf{1}^T - I$  and  $\text{diag}(P_X) = \Lambda$  in the context of (8), the Bayes risk  $R_\lambda^*(P_{Y|X})$  is given by

$$\begin{aligned} R_\lambda^*(P_{Y|X}) &= \min_{P_{\hat{X}|Y} \in \mathbb{R}_{\text{sto}}^{m \times n}} \text{Tr}((\mathbf{1}\mathbf{1}^T - I)\Lambda P_{Y|X} P_{\hat{X}|Y}) \\ &= \text{Tr}(\Lambda P_{Y|X} \mathbf{1}\mathbf{1}^T) - \max_{P_{\hat{X}|Y} \in \mathbb{R}_{\text{sto}}^{m \times n}} \text{Tr}(\Lambda P_{Y|X} P_{\hat{X}|Y}) \\ &= 1 - \sum_{y \in \mathcal{Y}} \max\{\lambda_1 P_{Y|X}(y|x_1), \dots, \lambda_n P_{Y|X}(y|x_n)\}. \end{aligned} \quad (12)$$

Hence, we define the quantity  $\tilde{\tau}_{\max}(\lambda, P_{Y|X}) \triangleq R_\lambda^* - R_\lambda^*(P_{Y|X})$  as the *max-DeGroot distance*:

$$\begin{aligned} \tilde{\tau}_{\max}(\lambda, P_{Y|X}) &= \sum_{y \in \mathcal{Y}} \max\{\lambda_1 P_{Y|X}(y|x_1), \dots, \lambda_n P_{Y|X}(y|x_n)\} \\ &\quad - \max\{\lambda_1, \dots, \lambda_n\}. \end{aligned} \quad (13)$$

Observe that for the special case where  $n = 2$ , both min-DeGroot and max-DeGroot distances are equivalent and the same as classical DeGroot distance [1]. Indeed, we have

$$\begin{aligned} &\min\{\lambda, \bar{\lambda}\} - \sum_{y \in \mathcal{Y}} \min\{\lambda P_{Y|X}(y|x_1), \bar{\lambda} P_{Y|X}(y|x_2)\} \\ &= \sum_{y \in \mathcal{Y}} \max\{\lambda P_{Y|X}(y|x_1), \bar{\lambda} P_{Y|X}(y|x_2)\} - \max\{\lambda, \bar{\lambda}\} \\ &= \|\lambda P_{Y|X=x_1} - \bar{\lambda} P_{Y|X=x_2}\|_1 - \frac{1}{2}|1 - 2\lambda|, \end{aligned} \quad (14)$$

where  $\bar{\lambda} = 1 - \lambda$  for  $\lambda \in [0, 1]$ .

## B. Simultaneously Maximal Coupling

In this section, we will generalize Goldstein's simultaneous coupling result, cf. [2], [26], to the case where we have  $n$  distributions rather than just two. Given a finite collection of probability distributions  $P_{X_1, Y_1}, P_{X_2, Y_2}, \dots, P_{X_n, Y_n}$ , the next theorem constructs a joint coupling which is simultaneously maximal with respect to the joint distributions of  $(X_i, Y_i)$  for  $i \in [n]$  and the marginal distributions of  $X_1, \dots, X_n$ .

*Theorem 3 (Simultaneously Maximal Coupling)*: Given  $n$  probability distributions  $P_{X_1, Y_1}, P_{X_2, Y_2}, \dots, P_{X_n, Y_n}$ , over finite alphabets with  $X_i \in \mathcal{X}$  and  $Y_i \in \mathcal{Y}$  for  $i \in [n]$ , there exists a coupling  $P_{X_1, Y_1, X_2, Y_2, \dots, X_n, Y_n}$  that is simultaneously maximal in the following sense

$$\begin{aligned} \tau(P_{X_1, Y_1}, \dots, P_{X_n, Y_n}) &= \mathbb{P}(X_1 = \dots = X_n, Y_1 = \dots = Y_n), \\ \tau(P_{X_1}, \dots, P_{X_n}) &= \mathbb{P}(X_1 = \dots = X_n), \end{aligned}$$

where  $\mathbb{P}(\cdot)$  denotes the probability law under the coupling. The proof is provided in Appendix A-C. We will use Theorem 3 to establish the information contraction properties of Doeblin coefficients over Bayesian networks.

## IV. CONTRACTION OVER BAYESIAN NETWORKS

In this section, building on the ideas in [2], we derive the contraction properties of Doeblin coefficients over Bayesian networks. Specifically, given Doeblin coefficients for each of the channels comprising a Bayesian network, we obtain a bound on the Doeblin coefficient of the composite channel from a single source to any sink of nodes. To this end, consider a Bayesian network represented by a finite directed acyclic graph with vertex set  $\mathcal{V}$ . Every vertex  $U$  in  $\mathcal{V}$  is a random variable that takes values in a finite alphabet set. Let the source node be denoted by  $X$ , and suppose each vertex  $U$  (other than the source node) is associated with the conditional distribution  $P_{U|\text{pa}(U)}$ , where  $\text{pa}(U)$  denotes the set of parents of  $U$ . These conditional distributions collectively define the Bayesian network [27]. We assume that the vertices are topologically sorted, and for any node  $U$  and subset of nodes  $V \subseteq \mathcal{V}$ , we write  $U > V$  when there is no directed path from  $U$  to  $V$ . Moreover, in the spirit of [2], we define a *site percolation* process on  $\mathcal{V}$  so that each vertex  $U$  in  $\mathcal{V}$  is removed independently with probability  $\tau_U$ , where  $\tau_U = \tau(P_{U|\text{pa}(U)})$  denotes the Doeblin coefficient at  $U$ . Finally, let  $\text{perc}(V)$  be the probability (with respect to the percolation process) that there is an (open) path from  $X$  to a subset of nodes  $V \subseteq \mathcal{V}$ .

*Theorem 4 (Doeblin Coefficients in Bayesian Networks)*: For any node  $U$  in  $\mathcal{V}$  and any subset of nodes  $V \subseteq \mathcal{V}$  such that  $U > V$ , we have

$$\tau(P_{V \cup \{U\}|X}) \geq \tau_U \tau(P_{V|X}) + (1 - \tau_U) \tau(P_{V \cup \text{pa}(U)|X}).$$

Furthermore, under the aforementioned site percolation model, we have for every  $V \subseteq \mathcal{V}$ ,

$$\gamma(P_{V|X}) = 1 - \tau(P_{V|X}) \leq \text{perc}(V).$$

The proof is provided in Appendix A-D. Note that  $\tau(\cdot)$  is tending towards one over the network, which is intuitive

since a stochastic matrix typically transforms input probability distributions into “more uniform” distributions. Therefore, a Doeblin coefficient closer to one implies that the corresponding channel is *losing information* contained in source distributions. Moreover, the percolation bound in Theorem 4 is a strengthening of that in [2] as  $1 - \tau(P_{V|X}) \geq \eta_{TV}(P_{V|X})$ .

#### A. Samorodnitsky’s SDPI for Doeblin Coefficients

We next establish a generalization of Samorodnitsky’s SDPI for Doeblin coefficients in analogy with the results in [2], [4]. To this end, consider discrete random variables  $U, X_1, X_2, \dots, X_n$  and  $Y_1, Y_2, \dots, Y_n$  on finite alphabets. Suppose  $U \rightarrow X^n \rightarrow Y^n$  forms a Markov chain for a channel  $P_{X^n|U}$  and a *memoryless* channel  $P_{Y^n|X^n}$ , which is a tensor product of the channels  $P_{Y_j|X_j}$  for  $j \in [n]$ :

$$P_{Y^n|X^n} = \prod_{j=1}^n P_{Y_j|X_j}. \quad (15)$$

Define the Doeblin coefficient of each individual channel as  $\tau_j = \tau(P_{Y_j|X_j})$  for  $j \in [n]$ . Using the tensorization and sub-multiplicativity properties in Theorem 1, we obtain the following bound on the Doeblin coefficient of  $P_{Y^n|U}$ :

$$\tau(P_{Y^n|U}) \geq \tau(P_{Y^n|X^n}) + \tau(P_{X^n|U})(1 - \tau(P_{Y^n|X^n})) \quad (16)$$

$$\geq \prod_{j=1}^n \tau_j. \quad (17)$$

This bound parallels the bound on contraction coefficients for operator convex  $f$ -divergences in [4, Equation 61]. In [2, Section 6.2] and [4, Section 3.4], the authors argue that stronger bounds can be obtained using more refined knowledge of the distribution  $P_{U, X^n}$ . In this vein, the ensuing theorem presents tighter bounds on  $\tau(P_{Y^n|U})$  in terms of the single-letter Doeblin coefficients  $\{\tau_j : j \in [n]\}$ .

**Theorem 5 (Samorodnitsky’s SDPI for Doeblin Coefficients):** For the Markov chain  $U \rightarrow X^n \rightarrow Y^n$  with a given memoryless channel  $P_{Y^n|X^n} = \prod_{j=1}^n P_{Y_j|X_j}$ , we have

$$\tau(P_{Y^n|U}) \geq \sum_{T \subseteq [n]} P(T) \tau(P_{X_T|U}),$$

where  $P(T)$  is the probability of a subset  $T$  of  $[n]$  generated by independently drawing each element  $i \in [n]$  with probability  $1 - \tau_i$ , and  $X_T = \{X_i : i \in T\}$  for any subset  $T \subseteq [n]$ . Specifically, when  $\tau_j = \tau$  for all  $j \in [n]$ , the following bound holds:

$$\tau(P_{Y^n|U}) \geq \sum_{T \subseteq [n]} \tau^{n-|T|} (1 - \tau)^{|T|} \tau(P_{X_T|U}).$$

The proof is provided in Section A-E.

## V. APPLICATION TO PMF FUSION

In this section, we will apply the techniques developed above to the general problem of aggregation or *fusion* of PMFs [28]. Pooling or fusion of PMFs is often used in statistics and machine learning to combine multiple sources of information or data into a single, more accurate estimate of a probability

distribution. This can be useful in many applications, such as in resource-constrained Bayesian inference, where multiple sources of data or prior beliefs need to be combined to produce a single posterior distribution [29]. We next propose a new framework for PMF fusion through an optimization-based approach.

#### A. PMF Fusion: Optimization-based Approach

Suppose we have  $n$  agents, each with their own PMFs  $P_1, \dots, P_n \in \mathcal{P}_m$ , representing their beliefs about some phenomenon of interest. The goal is to generate a combined belief by aggregating their individual PMFs. One approach to achieving this is to use an optimization method to find the aggregated PMF that maximizes or minimizes some measure of discrepancy between the individual PMFs. This approach has been studied using various measures, such as weighted KL divergence and weighted  $\alpha$ -divergences (see [28] for a detailed survey).

Inspired by Doeblin coefficients, we propose a new approach that focuses on maximizing the probability of agreement among the agents under all possible coupling of their beliefs. More precisely, we find the joint distribution or coupling that maximizes the likelihood that the agents’ beliefs, represented as random variables  $X_1, \dots, X_n$ , agree with each other over all possible couplings of their beliefs, i.e.,

$$\max_{\substack{P_{X_1, \dots, X_n} : \\ P_{X_1} = P_1, \dots, P_{X_n} = P_n}} \mathbb{P}(X_1 = \dots = X_n), \quad (18)$$

where  $\mathbb{P}(\cdot)$  denotes the probability law corresponding to a coupling. Hence, we define the optimal fused PMF as the conditional PMF under the coupling conditioned on the agreement among the agents i.e.,  $P_* \in \mathcal{P}_m$  via  $P_*(x) \propto \mathbb{P}(X_1 = \dots = X_n = x)$  for the maximal coupling. Since Theorem 2 tells us that  $\mathbb{P}(X_1 = \dots = X_n = x) = \min\{P_1(x), \dots, P_n(x)\}$  for the maximal coupling, we can write the optimal fused PMF as:

$$\forall x \in \mathcal{Y}, P_*(x) = \frac{\min\{P_1(x), \dots, P_n(x)\}}{\sum_{x \in \mathcal{X}} \min\{P_1(x), \dots, P_n(x)\}}. \quad (19)$$

We refer to the above fusion method as the *min-rule*. Such a min-rule had been proposed in the literature, cf. [30], but lacked a formal probabilistic understanding. Indeed, the min-rule was used for distributed hypothesis testing in [30] based on the intuition that if there is a true state of the world  $x^* \in \mathcal{Y}$ , and if there is an agent that can distinguish  $x$  from  $x^*$  for every false state  $x \in \mathcal{Y} \setminus \{x^*\}$ , then the min-rule can drive the beliefs of distributed agents to zero on each of the false states. While [30] established that the min-rule enjoys faster asymptotic convergence than other belief averaging-based approaches, such as arithmetic pooling [31], [32] and geometric pooling [33]–[36], our development reveals the underlying probabilistic interpretation of the min-rule.

It is important to note that the min-rule in (19) is only suitable under certain settings, such as group decision problems [37], where different experts are given equal importance and each expert is trying to construct optimal estimates based on

their knowledge. Clearly, if some of the agents are either adversarial or have imperfect knowledge about the environment, then the min-rule may not be appropriate.

## VI. CONCLUSION

In this work, we have developed new insights into the structure and geometry of Doeblin coefficients. For instance, we have illustrated using extremal coupling characterizations how Doeblin coefficients generalize TV distance to measure “distances” between multiple probability distributions. We have also demonstrated the contraction properties of Doeblin coefficients over Bayesian networks, similar to other established contraction coefficients, such as those for KL divergence and TV distance. Furthermore, we have introduced a new approach for fusing PMFs based on Doeblin coefficients dubbed the *min-rule*, which serves as a means of aggregating the beliefs of agents in certain applications. Finally, some immediate directions for future research include identifying the axioms that uniquely characterize the min-rule and developing robust versions of it that take into account the reliability of agents.

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APPENDIX A  
PROOFS

A. Proof of Theorem 1

**Part 1:** Follows from the definition of  $\tau(W) = \sum_{x \in \mathcal{X}} \min\{P_1(x), \dots, P_n(x)\}$ . Clearly,

$$0 \leq \tau(W) \leq \sum_{x \in \mathcal{X}} P_i(x) = 1.$$

It is easy to see that  $\tau(W) = 1$  if  $P_1 = P_2 = \dots = P_n$ . For converse, without loss of generality assume that  $P_1, P_2$  differ at some  $x_0 \in \mathcal{X}$ , then we have

$$\tau(W) \leq \sum_{x \in \mathcal{X}} \min\{P_1(x), P_2(x)\} = 1 - \|P_1 - P_2\|_{\text{TV}} < 1$$

Similarly, It is easy to see that  $\tau(W) = 0$  if and only if at least one of  $P_1(y), P_2(y), \dots, P_n(y)$  is zero for all  $y \in \mathcal{Y}$ .

**Part 2:** The first two properties namely total symmetry and positive definiteness follows trivially. Define  $P_{\min}(x) \triangleq \min\{P_1(x), P_2(x), \dots, P_n(x)\}$ . To prove Polyhedron inequality we need to show

$$(n-1) \left( 1 - \sum_{x \in \mathcal{X}} \min\{P_1(x), \dots, P_n(x)\} \right) \leq \sum_{i=1}^n \left( 1 - \sum_{x \in \mathcal{X}} \min\{P_1(x), \dots, P_{i-1}(x), P_{i+1}(x), \dots, P_{n+1}(x)\} \right)$$

The above condition simplifies to

$$\sum_{x \in \mathcal{X}} \left( \sum_{i=1}^n \min\{P_1(x), \dots, P_{i-1}(x), P_{i+1}(x), \dots, P_{n+1}(x)\} - (n-1) \min\{P_1(x), \dots, P_n(x)\} \right) \leq 1 \quad (20)$$

Define event  $\mathcal{A} \triangleq \{x : P_{n+1}(x) \leq \min\{P_1(x), P_2(x), \dots, P_n(x)\}\}$ . On the set  $\mathcal{A}$ , the terms inside the parenthesis in (20) reduces to

$$nP_{n+1}(x) - (n-1) \min\{P_1(x), \dots, P_n(x)\} \leq P_{n+1}(x)$$

Now consider any  $x \in \mathcal{A}^C$ . Without loss of generality we assume  $P_1(x)$  is smallest among  $P_1(x), P_2(x), \dots, P_n(x)$ , the term on left of Eq. (20) simplifies to

$$(n-1)P_1(x) + \min\{P_2(x), \dots, P_{n+1}(x)\} - (n-1)P_1(x) \leq P_{n+1}(x) \quad (21)$$

Hence, on both  $\mathcal{A}$  and  $\mathcal{A}^C$ , the terms inside the parenthesis in Eq. (20) is upper bounded  $P_{n+1}(x)$  which sums to atmost 1. Thus proving the polyhedron Inequality.

**Part 3:** For some  $\lambda \in [0, 1]$ , let  $\bar{\lambda} \triangleq 1 - \lambda$

$$\begin{aligned} \tau(\lambda V + \bar{\lambda} W) &= \sum_{x \in \mathcal{X}} \min\{\lambda V_1(x) + \bar{\lambda} W_1(x), \dots, \lambda V_n(x) + \bar{\lambda} W_n(x)\} \\ &\geq \lambda \sum_{x \in \mathcal{X}} \min\{V_1(x), \dots, V_n(x)\} \\ &\quad + \bar{\lambda} \sum_{x \in \mathcal{X}} \min\{W_1(x), \dots, W_n(x)\} \\ &= \lambda \tau(V) + \bar{\lambda} \tau(W) \end{aligned}$$

**Part 4:** To prove sub-multiplicativity, we will utilize the following lemma from [14].

*Lemma 1 (Doeblin Minorization and Degradation [14, Lemma 5]):* Let  $\mathcal{A}$  and  $\mathcal{R}$  be arbitrary discrete and finite sets. Let  $\mathcal{B} = \mathcal{A} \cup \{\mathbf{e}\}$  and assume that  $P_{B|A} = E(\epsilon)$  is an erasure channel with erasure probability  $\epsilon$  (here  $\mathbf{e}$  is the erasure symbol). Consider an arbitrary  $P_{R|A}$ . There exists a conditional distribution  $P_{R|B}$  such that

$$P_{R|A} = E(\epsilon) P_{R|B},$$

if and only if

$$\tau(P_{R|A}) = \sum_{r \in \mathcal{R}} \min_a P_{R|A}(r|a) \geq \epsilon.$$

Let  $P_{R|A} = VW$  and index the elements of set  $\mathcal{A}$  as  $\{1, 2, \dots, n\}$ . From Lemma 1, the sub-multiplicativity property follows if we show a construction of  $P_{R|B} \in \mathbb{R}_{\text{sto}}^{n+1 \times m}$  such that

$$VW = E(1 - (1 - \tau(V))(1 - \tau(W))) P_{R|B} \quad (22)$$

For simplicity define shorthand notations  $\tau_v = \tau(V)$ ,  $\tau_w = \tau(W)$ , and row vectors  $W_{\min} \in \mathbb{R}^m$ ,  $V_{\min} \in \mathbb{R}^k$  such that  $W_{\min}(j) = \min_{i \in [k]} W_{ij}$  for  $j \in [m]$  and  $V_{\min}(j) = \min_{i \in [n]} V_{ij}$  for  $j \in [k]$ . Let  $V_j$  denote the  $j$ -th row of  $V$ . For a fixed  $b \in \mathcal{B}$ , the row vector  $P_{R|B=b}$  is constructed as follows

$$P_{R|B=b} = \begin{cases} \frac{(1-\tau_v)\tau_w}{\tau_v + \tau_w - \tau_v\tau_w} \frac{W_{\min}}{(V_j - V_{\min})} + \frac{\tau_v}{\tau_v + \tau_w - \tau_v\tau_w} \frac{V_{\min}W}{\tau_v} & \text{if } b = \mathbf{e} \\ \frac{\tau_w}{(1-\tau_v)} \frac{(W - \mathbf{1}W_{\min})}{(1-\tau_w)} & \text{if } b = j \end{cases}$$

It is easy verify  $P_{R|B}$  is a stochastic matrix and also satisfies (22) as

$$\begin{aligned} &(E(1 - (1 - \tau(V))(1 - \tau(W))) P_{R|B})_i = \\ &= (1 - \tau_v)(1 - \tau_w) P_{R|B=i} + (\tau_v + \tau_w - \tau_v\tau_w) P_{R|B=\mathbf{e}} \\ &= V_i W - W_{\min} - V_{\min} W + \tau_v W_{\min} \\ &\quad + (1 - \tau_v) W_{\min} + V_{\min} W \\ &= V_i W = (VW)_i \end{aligned}$$



**Part 5:** For  $V \in \mathbb{R}_{\text{sto}}^{n \times k}$  and  $W \in \mathbb{R}_{\text{sto}}^{k \times m}$

$$\begin{aligned}\tau(V \otimes W) &= \sum_{x \in [n], y \in [k]} \min_{i \in [k], j \in [m]} V_{ix} W_{jy} \\ &= \sum_{x, y} \min\{V_1(x), \dots, V_n(x)\} \times \min\{W_1(y), \dots, W_n(y)\} \\ &= \left( \sum_x \min\{V_1(x), \dots, V_n(x)\} \right) \times \\ &\quad \left( \sum_y \min\{W_1(y), \dots, W_n(y)\} \right) \\ &= \tau(V) \tau(W)\end{aligned}$$

**Part 6:** This upper bound for  $\tau(W)$  was proved in [1, Remark III.2].

**Part 7:** It is easy to see that  $\text{Tr}(PW) = \sum_{j=1}^m P_j W_{:,j} \geq \sum_{j=1}^m \min_{i \in [n]} W_{ij}$  where  $W_{:,j}$  is the  $j$ -th column of  $W$  in column vector form. The equality is attained when  $P_{ij} = 1$  when  $j = \arg \min_k W_{ki}$  and 0 otherwise.

**Part 8:** From the Bayesian decision theory, we know that for an observation model  $P_{Y|X} \in \mathbb{R}_{\text{sto}}^{n \times m}$ , and a prior PMF  $P_X \in \mathbb{P}_n$ , and a loss function  $l : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ , and any randomized estimator  $P_{\hat{X}|Y} \in \mathbb{R}_{\text{sto}}^{n \times m}$ , the risk function is defined as

$$\begin{aligned}R(P_{Y|X}, P_X, l, P_{\hat{X}|Y}) &\triangleq \mathbb{E}[l(X, \hat{X})] \\ &= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \sum_{\hat{x} \in \mathcal{X}} l(x, \hat{x}) P_X(x) P_{Y|X}(y|x) P_{\hat{X}|Y}(\hat{x}|y) \\ &= \text{Tr}(L^T \text{diag}(P_X) P_{Y|X} P_{\hat{X}|Y}),\end{aligned}$$

where  $L \in \mathbb{R}^{n \times n}$  is the matrix representation of the loss function  $l : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ . Specifically, the  $(i, j)$ th entry of the matrix  $L$  is  $l(x_i, \hat{x}_j)$  for  $i, j \in [n]$  and  $x_i, \hat{x}_j \in \mathcal{X}$ . By the choice of our loss function, we have  $L = I$  and  $\text{diag}(P_X) = \frac{1}{n} I$ . This gives the Bayes optimal risk  $R_\lambda^*(P_{Y|X})$  as

$$\begin{aligned}R_\lambda^*(P_{Y|X}) &= \min_{P_{\hat{X}|Y} \in \mathbb{R}_{\text{sto}}^{m \times n}} \mathbb{P}(\hat{X}(Y) = X) \\ &= \min_{P_{\hat{X}|Y} \in \mathbb{R}_{\text{sto}}^{m \times n}} \text{Tr}\left(I \frac{1}{n} P_{Y|X} P_{\hat{X}|Y}\right) \\ &= \frac{1}{n} \sum_x \min\{P_1(x), P_2(x), \dots, P_n(x)\} = \tau(W)/n.\end{aligned}$$

## B. Proof of Theorem 2

First, we will derive an upper bound on  $\mathbb{P}(X_1 = X_2 = \dots = X_n)$  as follows

$$\begin{aligned}\mathbb{P}(X_1 = \dots = X_n) &= \sum_{x \in \mathcal{Y}} \mathbb{P}(X_1 = \dots = X_n = x) \\ &\leq \sum_{x \in \mathcal{Y}} \min\{P_1(x), \mathbb{P}(X_2 = \dots = X_n = x)\} \\ &\leq \sum_{x \in \mathcal{Y}} \min\{P_1(x), P_2(x), \dots, P_n(x)\},\end{aligned}$$

where the first inequality follows from the fact that  $\mathbb{P}(A \cap B) \leq \min\{\mathbb{P}(A), \mathbb{P}(B)\}$  and the next inequality follows by applying

the same procedure recursively. Now we show a construction of a coupling that achieves the above upper bound. Let  $c \triangleq \sum_{x \in \mathcal{X}} \min\{P_1(x), P_2(x), \dots, P_n(x)\}$

Define distributions

$$\begin{aligned}Q(x) &= \frac{\min\{P_1(x), P_2(x), \dots, P_n(x)\}}{c} \\ Q_i(x) &= \frac{P_i(x) - \min\{P_1(x), P_2(x), \dots, P_n(x)\}}{1 - c}\end{aligned}$$

Let  $Z \sim \text{Ber}(c)$  and independent of  $X_1, X_2, \dots, X_n$ . We will now construct joint distribution  $P_{X_1 X_2 \dots X_n Z}$ . First define the conditional distribution

$$\begin{aligned}P_{X_1 X_2 \dots X_n | Z}(x_1, x_2, \dots, x_n | Z = 1) &= Q(x_1) \mathbb{1}_{x_1 = x_2 = \dots = x_n} \\ P_{X_1 X_2 \dots X_n | Z}(x_1, x_2, \dots, x_n | Z = 0) &= \prod_{i=1}^n Q_i(x_i)\end{aligned}$$

where  $\mathbb{1}_A$  denotes the indicator variable of set  $A$ . The above equations give us the joint distribution  $P_{X_1 X_2 \dots X_n Z}$ . Define  $x^n \triangleq (x_1, x_2, \dots, x_n)$ . Marginalizing with respect to  $Z$  we get  $P_{X_1 \dots X_n}$  as follows

$$\begin{aligned}P_{X_1 \dots X_n}(x^n) &= P(Z = 1) P_{X_1 \dots X_n | Z}(x^n | Z = 1) + \\ &\quad P(Z = 0) P_{X_1 \dots X_n | Z}(x^n | Z = 0) \\ &= c Q(x_1) \mathbb{1}_{x_1 = x_2 = \dots = x_n} + (1 - c) \prod_{i=1}^n Q_i(x_i)\end{aligned}$$

Define  $x^{(i)} \triangleq (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  and  $P_{\min}(x) = \min\{P_1(x), P_2(x), \dots, P_n(x)\}$ . Observe that for all  $i \in [n]$  and all  $x \in \mathcal{X}$

$$\begin{aligned}P_{X_i}(x) &= c \sum_{x^{(i)}} Q(x) \mathbb{1}_{x_1 = \dots = x_n = x} \\ &\quad + (1 - c) Q_i(x) \sum_{x^{(i)}} \prod_{j \neq i} Q_j(x_j) \\ &= P_{\min}(x) + P_i(x) - P_{\min}(x) \\ &= P_i(x)\end{aligned}$$

Hence  $P_{X_1 \dots X_n}$  has the right marginals. Since  $\{Z = 1\} = \{X_1 = X_2 = \dots = X_n\}$  under our coupling. Hence  $\mathbb{P}(X_1 = X_2 = \dots = X_n) = c$ .  $\square$

## C. Proof of Theorem 3

First, similar to Theorem 2 we will derive an upper bound on  $\mathbb{P}(X_1 = \dots = X_n, Y_1 = \dots = Y_n)$  and  $\mathbb{P}(X_1 = \dots = X_n)$

$$\begin{aligned}\mathbb{P}(X_1 = \dots = X_n, Y_1 = \dots = Y_n) &= \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \mathbb{P}(X_1 = \dots = X_n = x, Y_1 = \dots = Y_n = y) \\ &\leq \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \min\{P_{X_1, Y_1}(x, y), \\ &\quad \mathbb{P}(X_2 = \dots = X_n = x, Y_2 = \dots = Y_n = y)\} \\ &\leq \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \min\{P_{X_1, Y_1}(x, y), \dots, P_{X_n, Y_n}(x, y)\},\end{aligned}$$

and similarly we have

$$\mathbb{P}(X_1 = \dots = X_n) \leq \sum_{x \in \mathcal{X}} \min\{P_{X_1}(x), \dots, P_{X_n}(x)\}$$

Now the construction of the coupling achieving these bounds is given as follows:

Define  $c_{XY}, c_X$  as

$$c_{XY} \triangleq \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \min\{P_{X_1 Y_1}(x, y), \dots, P_{X_n Y_n}(x, y)\}$$

$$c_X \triangleq \sum_{x \in \mathcal{X}} \min\{P_{X_1}(x), \dots, P_{X_n}(x)\}.$$

It is easy to see that  $c_X \geq c_{XY}$ . If  $c_X = c_{XY}$ , then we have

$$\sum_{x \in \mathcal{X}} \min\{P_{X_1}(x), \dots, P_{X_n}(x)\} =$$

$$\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \min\{P_{X_1 Y_1}(x, y), \dots, P_{X_n Y_n}(x, y)\}$$

which gives

$$\sum_{x \in \mathcal{X}} \left( \min \left\{ \sum_{y \in \mathcal{Y}} P_{X_1 Y_1}(x, y) - P_{\min}(x, y), \dots, \right. \right.$$

$$\left. \left. \sum_{y \in \mathcal{Y}} P_{X_n Y_n}(x, y) - P_{\min}(x, y) \right\} \right) = 0$$

where  $P_{\min}(x, y) = \min\{P_{X_1 Y_1}(x, y), \dots, P_{X_n Y_n}(x, y)\}$ . Note that  $P_{X_i Y_i}(x, y) \geq P_{\min}(x, y)$  for all  $x \in \mathcal{X}, y \in \mathcal{Y}$ . Hence, for above equation to hold, there should exist an  $i_0$  such that

$$P_{X_{i_0} Y_{i_0}}(x, y) = P_{\min}(x, y) \text{ for all } x \in \mathcal{X}, y \in \mathcal{Y}$$

which is only possible if

$$P_{X_{i_0} Y_{i_0}}(x, y) = P_{XY}(x, y) = \dots = P_{X_n Y_n}(x, y)$$

In this case, the theorem holds trivially. Now for the general case let  $x^n = (x_1, x_2, \dots, x_n)$  and  $y^n = (y_1, y_2, \dots, y_n)$  and define distributions over the product space  $\mathcal{X}^n \times \mathcal{Y}^n$

$$R_1(x, y) = \frac{\min\{P_{X_1 Y_1}(x, y), \dots, P_{X_n Y_n}(x, y)\}}{c_{XY}}$$

$$R_2(x, y^n) = \frac{\min\{P_{X_1}(x), \dots, P_{X_n}(x)\} - \sum_{y \in \mathcal{Y}} P_{\min}(x, y)}{c_X - c_{XY}}$$

$$\times \prod_{i=1}^n \frac{P_{X_i Y_i}(x, y_i) - \min\{P_{X_1 Y_1}(x, y_i), \dots, P_{X_n Y_n}(x, y_i)\}}{P_{X_i}(x) - \sum_{y \in \mathcal{Y}} \min\{P_{X_1 Y_1}(x, y), \dots, P_{X_n Y_n}(x, y)\}}$$

$$R_3(x^n, y^n) = \prod_{i=1}^n \frac{P_{X_i Y_i}(x_i, y_i) - P_{\min}(x_i, y_i)}{P_{X_i}(x_i) - \sum_{y \in \mathcal{Y}} P_{\min}(x_i, y)}$$

$$\times \prod_{i=1}^n \frac{P_{X_i}(x_i) - \min\{P_{X_1}(x_i), \dots, P_{X_n}(x_i)\}}{1 - c_X}$$

Define a random variable  $Z$  independent of  $X_1, Y_1, X_2, Y_2, \dots, X_n, Y_n$  as

$$Z = \begin{cases} 0 & \text{with prob } c_{XY} \\ 1 & \text{with prob } c_X - c_{XY} \\ 2 & \text{with prob } 1 - c_X \end{cases}$$

We will now construct joint distribution  $P_{X_1 Y_1 X_2 Y_2 \dots X_n Y_n Z}$ . First define the conditional distribution

$$P_{X_1 Y_1 \dots X_n Y_n | Z}(x_1, \dots, x_n | Z = 0) = R_1(x, y) \mathbb{1}_{x^n=x, y^n=y}$$

$$P_{X_1 Y_1 \dots X_n Y_n | Z}(x_1, \dots, x_n | Z = 1) = R_2(x, y^n) \mathbb{1}_{x^n=x}$$

$$P_{X_1 Y_1 \dots X_n Y_n | Z}(x_1, \dots, x_n | Z = 2) = R_3(x^n, y^n)$$

The above equations give us the joint distribution  $P_{X_1 X_2 \dots X_n Z}$ . Marginalizing with respect to  $Z$  we get  $P_{X_1 \dots X_n}$  as follows

$$P_{X_1 \dots X_n}(x^n) = \sum_{i=0}^2 P(Z = i) P_{X_1, \dots, X_n | Z}(x^n | Z = i)$$

Now we need to show that the above joint distribution has the right marginals jointly with respect to  $X$  and  $Y$ . Define  $x^{(i)} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  and  $y^{(i)} = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$ . Define  $P_{X_{\min}}(x) = \{P_{X_1}(x), \dots, P_{X_n}(x)\}$ . Observe that for all  $x, y$  such that  $x_i = x$  and  $y_i = y$

$$\sum_{x^{(i)}, y^{(i)}} P_{X_1 Y_1 \dots X_n Y_n}(x^n, y^n)$$

$$= \sum_{x^{(i)}, y^{(i)}} c_{XY} R_1(x, y) \mathbb{1}_{x^n=x, y^n=y}$$

$$+ \sum_{x^{(i)}} \sum_{y^{(i)}} (c_X - c_{XY}) R_2(x, y^n) \mathbb{1}_{x^n=x}$$

$$+ \sum_{x^{(i)}} \sum_{y^{(i)}} (1 - c_X) R_3(x^n, y^n)$$

$$= P_{\min}(x, y) + \left( P_{X_{\min}}(x) - \sum_{y \in \mathcal{Y}} P_{\min}(x, y) \right) \times$$

$$\left( \frac{P_{X_i Y_i}(x, y) - P_{\min}(x, y)}{P_{X_i}(x) - \sum_{y \in \mathcal{Y}} P_{\min}(x, y)} \right) \times$$

$$\sum_{y^{(i)}} \prod_{j \neq i} \frac{P_{X_j Y_j}(x, y_j) - P_{\min}(x, y_j)}{P_{X_j}(x) - \sum_{y \in \mathcal{Y}} P_{\min}(x, y)}$$

$$+ \frac{(P_{X_i Y_i}(x, y) - P_{\min}(x, y))(P_{X_i}(x) - P_{X_{\min}}(x))}{P_{X_i}(x) - \sum_{y \in \mathcal{Y}} P_{\min}(x, y)} \times$$

$$\sum_{x^{(i)}, y^{(i)}} \prod_{j \neq i} \frac{P_{X_j Y_j}(x_j, y_j) - P_{\min}(x_j, y_j)}{P_{X_j}(x_j) - \sum_{y \in \mathcal{Y}} P_{\min}(x_j, y)} \times$$

$$\left( \frac{P_{X_j}(x_j) - P_{\min}(x_j)}{1 - c_X} \right)$$

$$= P_{\min}(x, y) + \left( P_{X_{\min}}(x) - \sum_{y \in \mathcal{Y}} P_{\min}(x, y) \right) \times$$

$$\begin{aligned}
& \frac{P_{X_i Y_i}(x, y) - P_{\min}(x, y)}{P_{X_i}(x) - \sum_{y \in \mathcal{Y}} P_{\min}(x, y)} \\
& + \frac{(P_{X_i Y_i}(x, y) - P_{\min}(x, y))(P_{X_i}(x) - P_{X_{\min}}(x))}{P_{X_i}(x) - \sum_{y \in \mathcal{Y}} P_{\min}(x, y)} \\
& = P_{\min}(x, y) + \frac{(P_{X_i Y_i}(x, y) - P_{\min}(x, y))}{P_{X_i}(x) - \sum_{y \in \mathcal{Y}} P_{\min}(x, y)} \\
& \quad \times \left( P_{X_i}(x) - \sum_{y \in \mathcal{Y}} P_{\min}(x, y) \right) \\
& = P_{\min}(x, y) + P_{X_i Y_i}(x, y) - P_{\min}(x, y) \\
& = P_{X_i Y_i}(x, y)
\end{aligned}$$

Now, it remains to show that the above coupling satisfies the following two conditions

$$\begin{aligned}
\mathbb{P}(X_1 = \dots = X_n, Y_1 = \dots = Y_n) &= \tau(P_{X_1 Y_1}, \dots, P_{X_n Y_n}) \\
\mathbb{P}(X_1 = X_2 = \dots = X_n) &= \tau(P_{X_1}, \dots, P_{X_n})
\end{aligned}$$

Define events  $A = \{X_1 = X_2 = \dots = X_n\}$ ,  $B = \{Y_1 = Y_2 = \dots = Y_n\}$ . Observe that under both  $R_2$  and  $R_3$  the measure of event  $A \cap B$  is zero. This is because for any  $y \in \mathcal{Y}$  and  $x \in \mathcal{X}$  if  $\{X_1 = X_2 = \dots = X_n = x\}$  and  $\{Y_1 = Y_2 = \dots = Y_n = y\}$  then the following term is zero for all  $x \in \mathcal{X}$

$$\prod_{i=1}^n (P_{X_i Y_i}(x, y) - \min\{P_{X_1 Y_1}(x, y), \dots, P_{X_n Y_n}(x, y)\}) = 0$$

Hence,

$$\begin{aligned}
P(A \cap B) &= \mathbb{P}(Z = 0) = c_{XY} \\
&= \tau(P_{X_1 Y_1}, \dots, P_{X_n Y_n})
\end{aligned}$$

To show  $\mathbb{P}(A) = \tau(P_{X_1}, \dots, P_{X_n})$ . Now the measure of set  $A$  under  $R_3(\cdot)$  is 0. Hence,

$$\begin{aligned}
& \mathbb{P}(X_1 = X_2 = \dots = X_n = x) \\
&= C_{XY} \sum_{y^n} R_1(x, y) + (c_X - c_{XY}) \sum_{y^n} R_2(x, y) \\
&= \sum_y P_{\min}(x, y) + \left( P_{X_{\min}}(x) - \sum_y P_{\min}(x, y) \right) \\
& \quad \times \left( \sum_{y^n} \prod_{i=1}^n \frac{P_{X_i Y_i}(x, y_i) - P_{\min}(x, y_i)}{P_{X_i}(x) - \sum_{y \in \mathcal{Y}} P_{\min}(x, y_i)} \right) \\
&= \sum_y P_{\min}(x, y) + P_{X_{\min}}(x) - \sum_y P_{\min}(x, y) \\
&= P_{X_{\min}}(x)
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{P}(A) &= \sum_{x \in \mathcal{X}} \mathbb{P}(X_1 = X_2 = \dots = X_n = x) \\
&= \sum_{x \in \mathcal{X}} P_{X_{\min}}(x) = \tau(P_{X_1}, P_{X_2}, \dots, P_{X_n})
\end{aligned}$$

This completes the proof.

#### D. Proof of Theorem 4

Denote  $Z = \text{pa}(U)$  and assume that it takes values in set  $\mathcal{Z}$ . Let  $Q_i$  denote the conditional distribution  $V, Z$  given  $X = x_i$ , i.e.,  $Q_i = P_{V, Z|X=x_i}$  for  $i \in [n]$ . Define random variables  $(V_i, Z_i) \sim Q_i$  for  $i \in [n]$ . By Theorem 3, there exists a coupling  $P_{V_1, Z_1, \dots, V_n, Z_n}$  of  $Q_1, \dots, Q_n$  denoted by  $\mathbb{P}(\cdot)$  such that

$$\begin{aligned}
\mathbb{P}(V_1 = \dots = V_n, Z_1 = \dots = Z_n) &= \tau(Q_1, Q_2, \dots, Q_n) \\
\mathbb{P}(V_1 = \dots = V_n) &= \tau(Q_{V|X=x_1}, \dots, Q_{V|X=x_n})
\end{aligned}$$

Given  $z_1, \dots, z_n \in \mathcal{Z}$ , define random variables  $U_i \sim P_{U|Z=z_i}$ . Construct a maximal coupling of  $P_{U|Z=z_1}, \dots, P_{U|Z=z_n}$ , and let  $(U_1, U_2, \dots, U_n)$  be distributed according to this coupling. The simultaneous maximal couplings of  $V_1, Z_1, V_2, Z_2, \dots, V_n, Z_n$  and maximal couplings of  $U_1, U_2, \dots, U_n$  defines a joint distribution  $P_{V_1, U_1, Z_1, \dots, V_n, U_n, Z_n}$  and generates a Markov chain

$$(V_1, V_2, \dots, V_n) \longrightarrow (Z_1, Z_2, \dots, Z_n) \longrightarrow (U_1, U_2, \dots, U_n)$$

Therefore, we have

$$\begin{aligned}
1 - \mathbb{P}(U_1 = U_2 = \dots = U_n \mid Z_1 = z_1, \dots, Z_n = z_n) \\
&= 1 - \tau(P_{U|Z=z_1}, \dots, P_{U|Z=z_n}) \\
&\leq (1 - \tau(P_{U|Z})) \mathbb{1}_{\{Z_1 = Z_2 = \dots = Z_n\}}^c
\end{aligned}$$

The bounds holds since the state space of  $Z$  may be larger than  $\{z_1, \dots, z_n\}$ . This gives

$$\begin{aligned}
1 - \mathbb{E}[\mathbb{P}(U_1 = \dots = U_n \mid Z_1 = z_1, \dots, Z_n = z_n) \mid V_1 = \dots = V_n] \\
\leq (1 - \tau(P_{U|Z})) \mathbb{P}(\{Z_1 = Z_2 = \dots = Z_n\}^c \mid V_1 = \dots = V_n)
\end{aligned}$$

Multiplying both sides by  $\mathbb{P}(V_1 = V_2 = \dots = V_n)$  gives

$$\begin{aligned}
\mathbb{P}(V_1 = \dots = V_n) - \mathbb{P}(U_1 = U_2 = \dots = U_n, V_1 = \dots = V_n) \\
\leq (1 - \tau(P_{U|Z})) (\mathbb{P}(V_1 = V_2 = \dots = V_n) \\
- \mathbb{P}(Z_1 = \dots = Z_n, V_1 = \dots = V_n))
\end{aligned}$$

By construction of simultaneous coupling, we have

$$\begin{aligned}
\tau(P_{V|X}) - (1 - \tau(P_{U|Z})) (\tau(P_{V|X}) - \tau(P_{ZV|X})) \leq \\
\mathbb{P}(U_1 = \dots = U_n, V_1 = \dots = V_n) \leq \tau(P_{VU|X})
\end{aligned}$$

which on rearranging proves the theorem

$$\tau(P_{VU|X}) \geq \tau(P_{V|X}) \tau(P_{U|Z}) + (1 - \tau(P_{U|Z})) \tau(P_{ZV|X}) \quad (23)$$

We will now prove the percolation bound. Using the steps from [2, Theorem 5], the set-function  $\text{perc}$  satisfies the following recursion

$$\begin{aligned}
\text{perc}(V \cup \{U\}) &\triangleq \mathbb{P}[\exists \pi : X \rightarrow V \cup \{U\}] \\
&= \tau_U \text{perc}(V) + (1 - \tau_U) \text{perc}(V \cup \text{pa}(U)).
\end{aligned}$$

The recursion holds trivially for  $V = \{X\}$ , since both sides are equal to 1. Moreover, we have the following recursive relation from (23)

$$\Box \quad 1 - \tau(P_{VU|X}) \leq \tau_U (1 - \tau(P_{V|X})) + (1 - \tau_U) (1 - \tau(P_{ZV|X}))$$

Then, by induction on the maximal element of  $V$  and applying (23), we get the following for all  $V$ .

$$1 - \tau(P_{V|X}) \leq \text{perc}(V).$$

This yields the desired result.  $\square$

#### E. Proof of Theorem 5

First we show a generalization of Lemma 1 for the channel  $P_{Y^n|X^n} = \prod_{j=1}^n P_{Y_j|X_j}$ . We will show that if  $X^n$  is first passed through an erasure channel  $P_{E^n|X^n}$  given by the tensor product  $P_{E^n|X^n} = \prod_{i=1}^n E_{E_i|X_i}(\tau_i)$ , where  $E_{E_i|X_i}(\tau_i)$  is an erasure channel with erasure probability  $\tau_i$ , then there exist a conditional distribution  $P_{Y^n|E^n}$  such that

$$P_{Y^n|X^n} = P_{E^n|X^n} P_{Y^n|E^n} \quad (24)$$

From Lemma 1, we know that for each  $P_{Y_j|X_j}$  there exist a conditional distribution  $P_{Y_j|E_j}$  such that for all  $j \in [n]$

$$P_{Y_j|X_j} = E_{E_j|X_j}(\tau_j) P_{Y_j|E_j}$$

Hence, taking tensor product of all the  $n$  equations on both sides, we get

$$\begin{aligned} P_{Y^n|X^n} &= \prod_{j=1}^n P_{Y_j|E_j} = \prod_{j=1}^n (E_{E_j|X_j}(\tau_j) P_{Y_j|E_j}) \\ &= \left( \prod_{j=1}^n E_{E_j|X_j}(\tau_j) \right) \left( \prod_{j=1}^n P_{Y_j|E_j} \right), \end{aligned} \quad (25)$$

where the last property follows from the mixed product property. Now, the Markov chain  $U \rightarrow X^n \rightarrow Y^n$  can be equivalently written as  $U \rightarrow X^n \rightarrow E^n \rightarrow Y^n$ . By submultiplicativity of Doeblin coefficient

$$\begin{aligned} \tau(P_{Y^n|U}) &\geq \tau(P_{E^n|U}) + \tau(P_{Y^n|E^n})(1 - \tau(P_{E^n|U})) \\ &\geq \tau(P_{E^n|U}) \\ &= \sum_{T \subset [n]} P(T) \tau(P_{X_T|U}), \end{aligned}$$

which completes the proof of Theorem 5.  $\square$