VI. Proof of Theorem 1

Recall that since $\hat{\pi}$ is the stationary distribution of $\hat{S}$, we have

$$\hat{\pi}_i = \frac{\sum_{j:j\neq i} \hat{p}_{ji}\hat{\pi}_j}{\sum_{j:j\neq i} \hat{p}_{ij}}.$$

We introduce the auxiliary quantity $\bar{\pi}_i$ by replacing $\hat{\pi}_j$ with π_j in the above equation, i.e., for $i \in [n]$ we have

$$\bar{\pi}_i = \frac{\sum_{j:j\neq i} \hat{p}_{ji}\pi_j}{\sum_{j:j\neq i} \hat{p}_{ij}}.$$

We further define the error term δ_i as:

$$\delta_i = \frac{\hat{\pi}_i - \bar{\pi}_i}{\pi_i} = \frac{\sum_{j:j\neq i} \hat{p}_{ji}(\hat{\pi}_j - \pi_j)}{\pi_i \sum_{j:j\neq i} \hat{p}_{ij}}. \tag{12}$$

We introduce $\hat{S}^{(-m)}$ which is a leave-one-out version of $\hat{S}$ with mth row and columns replaced by their expected values. Thus for $i \neq j$, its (i,j) entry is defined as

$$(\hat{S}^{(-m)})_{ij} = \begin{cases} \hat{S}_{ij} & \text{if } i \neq m \text{ and } j \neq m \\ \frac{p_{ij}q_{ij}}{d} & \text{if } i = m \text{ or } j = m \end{cases}.$$

Also, we define $(\hat{S}^{(-m)})_{ii} = 1 - \sum_{l:l\neq i} \left(S^{(-m)}\right)_{il}$. Let $\hat{\pi}^{(-m)}$ be the stationary distribution of $\hat{S}^{(-m)}$. Similarly, we define $\bar{\pi}^{(-m)}$ be the leave-one-out version of $\bar{\pi}$ as

$$\bar{\pi}_i^{(-m)} = \frac{\sum_{j\in[n]\setminus\{i,m\}} \hat{p}_{ji}\pi_j + p_{mi}q_{mi}\pi_m}{\sum_{j\in[n]\setminus\{i,m\}} \hat{p}_{ij} + p_{im}q_{im}}.$$

Additionally, we define $\delta_i^{(-m)}$, as the leave-one-out version of δ_i as follows

$$\delta_i^{(-m)} = \frac{\hat{\pi}_i^{(-m)} - \bar{\pi}_i^{(-m)}}{\pi_i}.$$

We will decompose δ_i as

$$\delta_i = \underbrace{\frac{\sum_{j:j\neq i} \hat{p}_{ji}(\hat{\pi}_j - \hat{\pi}_j^{(-i)})}{\pi_i \sum_{j:j\neq i} \hat{p}_{ij}}}_{I_1} + \underbrace{\frac{\sum_{j:j\neq i} \hat{p}_{ji}\pi_j\delta_j^{(-i)}}{\pi_i \sum_{j:j\neq i} \hat{p}_{ij}}}_{I_2} + \underbrace{\frac{\sum_{j:j\neq i} \hat{p}_{ji}(\bar{\pi}_j^{(-i)} - \pi_j)}{\pi_i \sum_{j:j\neq i} \hat{p}_{ij}}}_{I_3}. \tag{13}$$

First, note that using Hoeffding's inequality, for all $i \in [n]$, the denominator term of each of I_1, I_2 and I_3 is lower bounded with probability at least $1 - O(1/n^5)$ as:

$$\pi_i \sum_{j:j\neq i} \hat{p}_{ij} \geq \pi_i \sum_{j:j\neq i} p_{ij} + \pi_i \sum_{j:j\neq i} (\hat{p}_{ij} - p_{ij}) \geq \frac{\pi_i}{h+1}\left(\sum_{j:j\neq i} A_{ij} - \sqrt{\frac{\sum_{j:j\neq i} A_{ij}^2}{k}\log n}\right). \tag{14}$$

Define $d_i = \sum_{j:j\neq i} A_{ij}$ be the degree of node i. Note that, with probability at least $1 - O(1/n^5)$, we have the following degree condition for all $i \in [n]$ and $q_{ij} \in [p, 1]$ with $p \geq \log n$:

$$\frac{1}{2}\sum_{j:j\neq i} q_{ij} \leq d_i \leq 2\sum_{j:j\neq i} q_{ij} \tag{15}$$

We now focus on the numerator term of I_1. An application of Cauchy-Schwarz inequality gives

$$\sum_{j:j\neq i} \hat{p}_{ji}(\hat{\pi}_j - \hat{\pi}_j^{(-i)}) \leq \sqrt{\sum_{j:j\neq i} \hat{p}_{ji}^2}\|\hat{\pi} - \hat{\pi}^{(-i)}\|_2 \leq \sqrt{\sum_{j:j\neq i} A_{ji}}\|\hat{\pi} - \hat{\pi}^{(-i)}\|_2. \tag{16}$$

The final bound on I_1 is

$$|I_1| \lesssim \frac{\sqrt{d_i}(h+1)^2\|\hat{\pi} - \hat{\pi}^{(-i)}\|_2}{\|\pi_i\|_\infty\left(d_i - \sqrt{\frac{d_i}{k}\log n}\right)}. \tag{17}$$

The following lemma proved in Section VI-A provides bounds on $\|\hat{\pi} - \hat{\pi}^{(-i)}\|_2$.

Lemma 1 (Error Bound for Leave-one-out)**.** *Under the assumption of Theorem 1, there exists a constant $c > 1$ such that for $np > c\gamma^2 \log n$, the following bound holds with probability at least $1 - O(n^{-5})$*

$$\|\hat{\pi} - \hat{\pi}^{(-i)}\|_2 \leq \frac{c}{\gamma}\sqrt{\frac{\log n}{knp}}\|\pi\|_\infty + \|\hat{\pi} - \pi\|_\infty.$$

Now we focus on the numerator term of I_3. Observe that

$$\sum_{j:j\neq i} \hat{p}_{ji}(\bar{\pi}_j^{(-i)} - \pi_j) = \sum_{j:j\neq i} \hat{p}_{ji} \frac{\sum_{l:l\in[n]\setminus\{i,j\}} \hat{p}_{lj}\pi_l - \pi_j\hat{p}_{jl}}{\sum_{l:l\in[n]\setminus\{i,j\}} \hat{p}_{jl} + p_{ji}q_{ji}} = \sum_{j:j\neq i} \hat{p}_{ji} \max_{j\in[n]\setminus\{i\}} \left|\frac{\sum_{l:l\in[n]\setminus\{i,j\}} \hat{p}_{lj}\pi_l - \pi_j\hat{p}_{jl}}{\sum_{l:l\in[n]\setminus\{j\}} \hat{p}_{jl} + p_{ji}q_{ji}}\right|.$$

Note that $\sum_{j:j\neq i} \hat{p}_{ji} \leq \sum_{j:j\neq i} A_{ji}$. Moreover, for the other term, note that by our BTL model, we have that $\pi_l p_{lj} = \pi_j p_{jl}$. Simplifying the term, we obtain

$$\left|\frac{\sum_{l:l\in[n]\setminus\{i,j\}} \hat{p}_{lj}\pi_l - \pi_j\hat{p}_{jl}}{\sum_{l:l\in[n]\setminus\{i,j\}} \hat{p}_{jl} + p_{ji}q_{ji}}\right| \leq \left|\frac{\sum_{l:l\neq\{j,i\}} A_{lj}(\hat{p}_{lj} - p_{lj})\pi_l}{\sum_{l:l\in[n]\setminus\{i,j\}} \hat{p}_{jl} + p_{ji}q_{ji}}\right| + \left|\frac{\sum_{l:l\in[n]\setminus\{i,j\}} A_{lj}\pi_j(p_{jl} - \hat{p}_{jl})}{\sum_{l:l\in[n]\setminus\{i,j\}} \hat{p}_{jl} + p_{ji}q_{ji}}\right|. \tag{18}$$

By the same reasoning as in Eq. (14), we have the following lower bound on the denominator term $\sum_{l:l\neq j} \hat{p}_{jl} + p_{ji}q_{ji}$ as

$$\sum_{l:l\in[n]\setminus\{i,j\}} \hat{p}_{jl} + p_{ji}q_{ji} \geq \frac{1}{(1+h)}\left(\sum_{l:l\in[n]\setminus\{i,j\}} A_{jl} - \sqrt{\frac{\sum_{l:l\in[n]\setminus\{i,j\}} A_{jl}}{k}\log n}\right). \tag{19}$$

Moreover, by application of Hoeffding's inequality, we have the following high probability bound on the numerator term in Eq. (18), as

$$\sum_{l:l\neq\{j,i\}} A_{lj}(\hat{p}_{lj}-p_{lj})\pi_l \leq \sqrt{\sum_{l:l\neq j} A_{lj}\log n}\|\pi\|_\infty. \tag{20}$$

A similar bound can be obtained for the second term in Eq. (18). Therefore, we have the following bound on the numerator term of I_3, with probability at least $1-O(1/n^5)$,

$$\sum_{j:j\neq i} \hat{p}_{ji}(\bar{\pi}_j^{(-i)}-\pi_j) \lesssim d_i \max_{j\in[n]\setminus\{i\}} \sqrt{\frac{\log n}{d_j-1}}\|\pi\|_\infty. \tag{21}$$

Now it remains to bound the numerator of I_2. Let $\delta^{(-i)}$ be the vector of leave-one-out errors. Denote $\mathcal{G}_{-i}$ as the graph with node i removed and $\mathcal{Z}_i$ as the data corresponding to node i. Observe that since we have $\delta_j^{(-i)}$ independent of data corresponding to node i there we can split the numerator of I_2 as:

$$\begin{aligned}
&\sum_{j:j\neq i}\hat{p}_{ji}\pi_j\delta_j^{(-i)} = \sum_{j:j\neq i}\mathbb{E}[\hat{p}_{ij}A_{ji}\mid \mathcal{G}_{-i},\mathcal{Z}_i]\pi_j\delta_j^{(-i)}\\
&+\sum_{j:j\neq i}(\hat{p}_{ji}A_{ji}-\mathbb{E}[\hat{p}_{ji}A_{ji}\mid \mathcal{G}_{-i},\mathcal{Z}_i])\pi_j\delta_j^{(-i)}\\
&=\sum_{j\neq i}p_{ij}q_{ij}\pi_j\delta_j^{(-i)}+\sum_{j\neq i}\hat{p}_{ji}(A_{ji}-q_{ij})\pi_j\delta_j^{(-i)}\\
&\overset{\zeta}{\lesssim}\sqrt{\sum_{j:j\neq i}q_{ij}^2}\|\hat{\pi}^{(-i)}-\bar{\pi}^{(-i)}\|_2\\
&\qquad+\|\hat{\pi}^{(-i)}-\bar{\pi}^{(-i)}\|_\infty\left(\sqrt{\sum_{j:j\neq i}q_{ij}(1-q_{ij})}+\log n\right),
\end{aligned}$$

where in ζ the first term follows from the Cauchy-Schwarz inequality while the second term follows with probability at least $1-O(1/n^5)$ using the Hoeffding's inequality. Note that the ratio $\sqrt{\sum_{j:j\neq i}q_{ij}^2}/d_i$ and $(\sqrt{\sum_{j:j\neq i}q_{ij}(1-q_{ij})}+\log n)/d_i$ is maximized when $q_{ij}=p$ for all j. Finally utilizing the bound on $\|\delta^{(-i)}\|_2$ from [37], we obtain the following bound on the numerator of I_2 with probability at least $1-O(1/n^5)$

$$\sum_{j:j\neq i}\hat{p}_{ji}\pi_j\delta_j^{(-i)} \leq \frac{c}{\gamma}\sqrt{\frac{\log n}{k}}\|\pi\|_\infty \tag{22}$$

Combining the bounds (14), (16), (18), (19), (20), (21) and (22) we obtain the desired result.

□

A. *Proof of Lemma 1*

First, we will apply [9, Theorem 8], to obtain the following bound

$$\begin{aligned}
\|\hat{\pi}^{(-m)}-\hat{\pi}\|_2 &\leq \frac{\|\hat{\pi}^{(-m)\mathrm{T}}(\hat{S}^{(-m)}-\hat{S})\|_2}{1-\max\{\lambda_2(S),-\lambda_n(S)\}-\|\hat{S}-S\|_\pi}\\
&\leq \frac{1}{\gamma}\sqrt{h}\|\hat{\pi}^{(-m)\mathrm{T}}(\hat{S}^{(-m)}-\hat{S})\|_2
\end{aligned}$$

where $\gamma = 1-\max\{\lambda_2(S),-\lambda_n(S)\}-\|\hat{S}-S\|_\pi$. We will bound the term $\|\hat{S}-S\|_\pi$ separately, first we will focus on the numerator term $\|\hat{\pi}^{(-m)\mathrm{T}}(\hat{S}^{(-m)}-\hat{S})\|_2$ in the following steps. To bound this term, we introduce another Markov matrix $\hat{S}^{(-m,\mathcal{E})}$, which is also a leave-one-out version of $\hat{S}$ defined conditional on edge-set $\mathcal{E}$. Similar to $\hat{S}^{(-m)}$, the matrix $\hat{S}^{(-m,\mathcal{E})}$ replaces all transition probabilities involving the m-th item with their expected values conditioned on $\mathcal{E}$. In particular, for $i\neq j$,

$$\hat{S}_{ij}^{(-m,\mathcal{E})} = \begin{cases}\hat{S}_{i,j}, & i\neq m \text{ and } j\neq m\\ \frac{p_{ij}}{d}A_{ij}, & i=m \text{ or } j=m\end{cases}.$$

Similarly, we define the diagonal entries for each $1\leq i\leq n$ as

$$\hat{S}_{ii}^{(-m,\mathcal{E})} = 1-\sum_{j:j\neq i}\hat{S}_{ij}^{(-m,\mathcal{E})},$$

to make it a valid probability transition matrix. Using the triangle inequality, we obtain

$$\begin{aligned}
\|\hat{\pi}^{(-m)\mathrm{T}}(\hat{S}^{(-m)}-\hat{S})\|_2 \leq &\underbrace{\|\hat{\pi}^{(-m)\mathrm{T}}(\hat{S}-\hat{S}^{(-m,\mathcal{E})})\|_2}_{J_1}\\
&+\underbrace{\|\hat{\pi}^{(-m)\mathrm{T}}(\hat{S}^{(-m)}-\hat{S}^{(-m,\mathcal{E})})\|_2}_{J_2}.
\end{aligned}$$

Now we will bound each of the term J_1 and J_2. The term J_1 is similar to [9] and yields the following bound with probability at least $1-O(n^{-5})$

$$J_1 \leq c\sqrt{\frac{\log n}{kd}}\|\hat{\pi}^{(-m)}\|_\infty. \tag{23}$$

Now, we will focus on the term J_2. Utilizing the identity $\hat{\pi}^{\mathrm{T}}(\hat{S}^{(-m)}-\hat{S}^{(-m,\mathcal{E})})=0$, we obtain

$$\hat{\pi}^{(-m)\mathrm{T}}(\hat{S}^{(-m)}-\hat{S}^{(-m,\mathcal{E})}) = (\hat{\pi}^{(-m)}-\pi)^{\mathrm{T}}(\hat{S}^{(-m)}-\hat{S}^{(-m,\mathcal{E})}).$$

For $j\neq m$, it follows that

$$\begin{aligned}
&[(\hat{\pi}^{(-m)}-\pi)^{\mathrm{T}}(\hat{S}^{(-m)}-\hat{S}^{(-m,\mathcal{E})})]_j\\
&=\sum_i(\hat{\pi}_i^{(-m)}-\pi_i)(\hat{S}_{i,j}^{(-m)}-\hat{S}_{i,j}^{(-m,\mathcal{E})})\\
&=-(\hat{\pi}_j^{(-m)}-\pi_j)(\hat{S}_{j,m}^{(-m)}-\hat{S}_{j,m}^{(-m,\mathcal{E})})\\
&\qquad+(\hat{\pi}_m^{(-m)}-\pi_m)(\hat{S}_{m,j}^{(-m)}-\hat{S}_{m,j}^{(-m,\mathcal{E})}).
\end{aligned}$$

Since for (j,m) in $\mathcal{E}$, we have $|\hat{S}_{j,m}^{(-m)}-\hat{S}_{j,m}^{(-m,\mathcal{E})}|\leq 1/d$ and for $(j,m)\notin\mathcal{E}$, we have $|\hat{S}_{j,m}^{(-m)}-\hat{S}_{j,m}^{(-m,\mathcal{E})}|\leq\frac{q_{jm}}{d}$. Therfore, we have that

$$\begin{aligned}
&|[(\hat{\pi}^{(-m)}-\hat{\pi})^{\mathrm{T}}(\hat{S}^{(-m)}-\hat{S}^{(-m,\mathcal{E})})]_j|\\
&\qquad\leq\begin{cases}\frac{2}{d}\|\hat{\pi}^{(-m)}-\pi\|_\infty, & \text{if } (j,m)\in\mathcal{E}\\ \frac{2q_{jm}}{d}\|\hat{\pi}^{(-m)}-\pi\|_\infty, & \text{otherwise}\end{cases}
\end{aligned}$$

Since we are given that $\hat{S}_{m,j}^{(-m)}-\hat{S}_{m,j}^{-m,\mathcal{E}}=\frac{p_{mj}}{d}(q_{mj}-\mathbb{1}_{(m,j)\in\mathcal{E}})$, we obtain

$$J_2 = \sum_{j:j\neq m}(\hat{\pi}_m^{(-m)}-\hat{\pi}_m)\frac{p_{mj}}{d}(q_{mj}-\mathbb{1}_{(m,j)\in\mathcal{E}}). \tag{24}$$

Applying Hoeffdings inequality, we obtain

$$|J_2| \lesssim \frac{\sqrt{np\log n}+\log n}{d}\|\hat{\pi}^{(-m)}-\hat{\pi}\|_\infty$$

with high probability. Combining the above bounds, we derive

$$J_2 \lesssim (\frac{\sqrt{np\log n}+\log n}{d}+\frac{p\sqrt{n}}{d}+\frac{\sqrt{d}}{d})\|\hat{\pi}^{(-m)}-\pi\|_\infty$$

By combining all components Eq. (23) and Eq. (24), we establish that

$$\|\hat{\pi}^{(-m)}-\hat{\pi}\|_2 \leq \frac{\sqrt{\kappa}}{\gamma}(J_1+J_2).$$

Simplifying the above expression, we obtain

$$\begin{aligned}
&\leq \frac{\sqrt{h}}{\gamma}(8\sqrt{\frac{\log n}{kd}}\|\hat{\pi}^{(-m)}\|_\infty \\
&\quad + C(\frac{\sqrt{np\log n}+\log n}{d})\|\hat{\pi}^{(-m)}-\pi\|_\infty) \\
&\leq \frac{\sqrt{h}}{\gamma}(8\sqrt{\frac{\log n}{kd}}\|\hat{\pi}\|_\infty \\
&+ C(8\sqrt{\frac{\log n}{kd}}+\frac{\sqrt{np\log n}+\log n}{d})\|\hat{\pi}^{(-m)}-\pi\|_\infty) \\
&\overset{\zeta}{\leq} \frac{\sqrt{h}}{\gamma}\sqrt{\frac{\log n}{kd}}\|\hat{\pi}\|_\infty+\frac{1}{2}\|\hat{\pi}^{(-m)}-\pi\|_\infty,
\end{aligned}$$

where ζ holds as long as $np\gamma^2 \geq ch\log n$ for sufficiently large constant c. Using the triangle inequality

$$\|\hat{\pi}^{(-m)}-\hat{\pi}\|_\infty \leq \|\hat{\pi}^{(-m)}-\hat{\pi}\|_2+\|\hat{\pi}-\pi\|_\infty,$$

we obtain the following bound

$$\|\hat{\pi}^{(-m)}-\hat{\pi}\|_2 \leq \frac{16\sqrt{\kappa}}{\gamma}\sqrt{\frac{\log n}{kd}}\|\pi\|_\infty+\|\hat{\pi}-\pi\|_\infty,$$

which completes the proof. □

VII. Proof of Proposition 1

We will utilize the following lemma to prove the spectral gap condition of the NSSBM model.

Lemma 2 (Comparison theorem). *Let S,π and $\tilde{S},\tilde{\pi}$ be reversible Markov chains on a finite set $[n]$ representing random walks on a graph, i.e. $S_{ij}=0$ and $\tilde{S}_{ij}=0$ if $(i,j)\notin\mathcal{E}$. Define $\alpha=\min_{(i,j)\in\mathcal{E}}\tilde{\pi}_i\tilde{S}_{ij}/\pi_i S_{ij}$ and $\beta=\max_i\tilde{\pi}_i/\pi_i$,*

$$\frac{1-\lambda_2(S)}{1-\lambda_2(\tilde{S})} \geq \frac{\alpha}{\beta}.$$

In order to apply Lemma 2, we will introduce the following empirical Markov matrix $\tilde{P}$ with stationary distribution $\tilde{\pi}=\mathbf{1}/n$, and is defined as

$$\tilde{S}_{ij}=\begin{cases}\frac{1}{2d}A_{ij}, & \text{if } (i,j)\in\mathcal{E}\\ 0, & \text{otherwise}\end{cases} \tag{25}$$

where $d\geq d_{\max}$ the maximum degree of the graph. Applying Lemma 2, we obtain

$$\frac{1-\lambda_2(S)}{1-\lambda_2(\tilde{S})} \geq \frac{1}{2h^3}. \tag{26}$$

Bounding the spectral gap of $\tilde{S}$ using [35, Lemma 18] yields the lemma. □

VIII. Proof of Theorem 2

We begin by defining the following quantities

$$L_\pi^w \triangleq \sum_{j>i}\frac{w_{ij}\pi_i\pi_j}{\pi_i+\pi_j}(e_i-e_j)(e_i-e_j)^{\mathrm{T}}, \tag{27}$$

$$d_{\min}^\pi \triangleq \min_{i\in[n]}\sum_{j:(i,j)\in\mathcal{E}}\frac{\pi_i\pi_j}{\pi_i+\pi_j}w_{ij}. \tag{28}$$

We also define the matrix $\hat{L}_\pi^w$ as

$$(\hat{L}_\pi^w)_{ij} \triangleq \begin{cases}-\hat{p}_{ji}\pi_j w_{ji}, & \text{if } (i,j)\in\mathcal{E}\\ \sum_{j:(i,j)\in\mathcal{E}}\hat{p}_{ji}\pi_j w_{ji}, & \text{if } i=j\\ 0, & \text{otherwise}\end{cases}. \tag{29}$$

Recall the definition of δ_i from Eq. (12), we begin by observing that

$$\begin{aligned}
\frac{\hat{\pi}_i-\pi_i}{\pi_i} &= \frac{\bar{\pi}_i-\pi_i}{\pi_i}+\delta_i \\
&= \frac{\sum_{j:j\neq i}w_{ij}(\hat{p}_{ji}\pi_j-\hat{p}_{ij}\pi_i)}{\pi_i\sum_{j:j\neq i}w_{ij}\hat{p}_{ij}}+\delta_i
\end{aligned} \tag{30}$$

Bounding the numerator of the first term in Eq. (30) using the Hoeffding inequality yields the following bound with high probability

$$\begin{aligned}
\left|\sum_{j:j\neq i}w_{ij}(\hat{p}_{ji}\pi_j-\hat{p}_{ij}\pi_i)\right| &\leq \|\pi\|_\infty\sqrt{\frac{\log n}{k}}\sqrt{\sum_{j:j\neq i}w_{ij}^2} \\
&\leq \|\pi\|_\infty\sqrt{\frac{\log n}{k}}\sqrt{d_{\max}}\sqrt{w_{\max}}.
\end{aligned}$$

We bound the the denominator of the first-term in Eq. (30) using the following lemma

Lemma 3. *Let $\mathcal{G}=([n],\mathcal{E},\{w_{ij}\}_{(i,j)\in\mathcal{E}}\}$ be a weighted graph. We have*

$$\lambda_{n-1}(L_\pi^w) \leq 2d_{\min}^\pi.$$

Utilizing the lemma, we have the denominator term is lower bounded by $\lambda_{n-1}(L_\pi^w)/4$, if $\lambda_{n-1}(L_\pi^w)^2k \geq d_{\min}^\pi\|\pi\|_\infty^2 w_{\max}\log n$ as shown below

$$\begin{aligned}
\pi_i\sum_{j:j\neq i}w_{ij}\hat{p}_{ij} &= d_{\min}^\pi+\pi_i\sum_{j:j\neq i}w_{ij}(\hat{p}_{ij}-p_{ij}) \\
&\geq d_{\min}^\pi-\|\pi\|_\infty\sqrt{w_{\max}}\sqrt{\frac{\sum_{j:j\neq i}w_{ij}}{k}} \\
&\geq \frac{\lambda_{n-1}(L_\pi^w)}{4}.
\end{aligned}$$

Now, we will focus on the second term δ_i in Eq. (30). Recall that

$$\begin{aligned}
\delta_i &= \frac{\hat{\pi}_i-\bar{\pi}_i}{\pi_i} = \frac{\sum_{j:j\neq i}w_{ji}\hat{p}_{ji}(\hat{\pi}_j-\pi_j)}{\pi_i\sum_{j:j\neq i}\hat{p}_{ij}w_{ij}} \\
&= \frac{\sum_{j;j\neq i}w_{ji}\hat{p}_{ji}\pi_j\delta_j}{\pi_i\sum_{j:j\neq i}\hat{p}_{ij}w_{ij}}+\frac{\sum_{j:j\neq i}w_{ji}\hat{p}_{ji}(\bar{\pi}_j-\pi_j^*)}{\pi_i\sum_{j:j\neq i}w_{ij}\hat{p}_{ij}}.
\end{aligned} \tag{31}$$

Define a vector $r \in \mathbb{R}^n$, such that $r_i = \sum_{j:j\neq i} w_{ji}\hat{p}_{ji}(\bar{\pi}_j - \pi_j^*)$ and also define $\hat{L}^w \in \mathbb{R}^{n\times n}$ such that

$$\hat{L}_{ij}^w = \begin{cases} -\hat{p}_{ji}\pi_j w_{ji} & \text{for } i\neq j, (i,j)\in\mathcal{E} \\ \sum_{j:j\neq i}\hat{p}_{ij}\pi_i w_{ij} & \text{if } i=j \end{cases}.$$

Also, note that $\mathbb{E}[\hat{L}^w] = L^w$. Then the above equation can be written as

$$\hat{L}_\pi^w \delta = r. \tag{32}$$

Taking ℓ^2-norm on both sides and an application of triangle inequality yields

$$\begin{aligned}\|r\|_2 &= \|\hat{L}_\pi^w\delta\|_2 \geq \|L^w\delta\|_2 - \|L^w - \hat{L}_\pi^w\|_2\|\delta\|_2 \\ &\geq \lambda_{n-1}(L_\pi^w)\left(\|\delta\|_2 - \frac{|\delta^{\mathrm{T}}\mathbf{1}|}{\sqrt{n}}\right) - \|L_\pi^w - \hat{L}_\pi^w\|_2\|\delta\|_2\end{aligned}$$

This gives a bound on $\|\delta\|_2$ as

$$\|\delta\|_2 \leq \frac{\|r\|_2}{\lambda_{n-1}(L_\pi^w)} + \frac{|\delta^{\mathrm{T}}1|}{\sqrt{n}}, \tag{33}$$

provided $\|L_\pi^w - \hat{L}_\pi^w\|_2 \leq \frac{\lambda_{n-1}(L^w)}{2}$. Now we will bound the term as $\|r\|_2 \leq \sqrt{n}\|r\|_\infty$. Below, we bound the term $\|r\|_\infty$.

Bounding $\|r\|_\infty$: Now we focus on bounding $\|r\|_\infty$

$$r_i = \sum_{j:j\neq i} w_{ij}\hat{p}_{ij}(\bar{\pi}_j - \pi_j) = \underbrace{\sum_{j:j\neq i} w_{ji}\hat{p}_{ji}(\bar{\pi}_j - \bar{\pi}_j^{(-i)})}_{J_1} + \underbrace{\sum_{j:j\neq i} w_{ji}\hat{p}_{ji}(\bar{\pi}_j^{(-i)} - \pi_j)}_{J_2}, \tag{34}$$

where we define $\bar{\pi}_j^{(-i)}$ as the leave-one-out version of $\bar{\pi}_j$.

$$\bar{\pi}_j^{(-i)} = \frac{\sum_{l\in[n]\setminus\{i,j\}} w_{lj}\hat{p}_{lj}\pi_l + w_{ij}p_{ij}\pi_i}{\sum_{l\in[n]\setminus\{i,j\}} w_{jl}\hat{p}_{jl} + w_{ji}p_{ji}}.$$

Now we will focus on bounding the term J_1 in Eq. (34). We have

$$\begin{aligned}&\Big|\sum_{j:j\neq i} w_{ji}\hat{p}_{ji}(\bar{\pi}_j - \bar{\pi}_j^{(-i)})\Big| \leq \\ &\Bigg|\sum_{j:j\neq i} w_{ji}\hat{p}_{ji}\Bigg(\frac{\sum_{l\in[n]\setminus\{i,j} \hat{p}_{lj}w_{lj}\pi_l + \hat{p}_{ij}w_{ij}\pi_i}{\sum_{l\in[n]\setminus\{i,j\}\}}\hat{p}_{jl} + \hat{p}_{ji}w_{ji}} - \\ &\qquad \frac{\sum_{l\in[n]\setminus\{i,j} \hat{p}_{lj}w_{lj}\pi_l + p_{ij}w_{ij}\pi_i}{\sum_{l\in[n]\setminus\{i,j\}\}}\hat{p}_{jl} + p_{ji}w_{ji}}\Bigg)\Bigg| \\ &\lesssim \Bigg|\sum_{j:j\neq i}\frac{w_{ji}^3\hat{p}_{ji}(\hat{p}_{ji}p_{ij} - \hat{p}_{ij}p_{ji})\pi_i}{(\sum_{l\in[n]\setminus\{j\}} w_{jl}\hat{p}_{jl})(\sum_{l\in[n]\setminus\{i,j\}} w_{jl}\hat{p}_{jl} + p_{ji}w_{ji})}\Bigg| \\ &+ \Bigg|\sum_{j:j\neq i}\frac{w_{ji}^2\hat{p}_{ji}(\sum_{l\in[n]\setminus\{i,j\}} w_{lj}(\hat{p}_{lj}\pi_l - \pi_i\hat{p}_{jl})}{(\sum_{l\in[n]\setminus\{i\}} w_{jl}\hat{p}_{jl})(\sum_{l\in[n]\setminus\{i,j\}} w_{jl}\hat{p}_{jl} + p_{ji}w_{ji})}\Bigg| \\ &\lesssim \frac{\sum_{j:j\neq i} w_{ij}^3\|\pi\|_\infty^3}{\lambda^2(L_\pi^w)}\sqrt{\frac{\log n}{k}} + \\ &\qquad \frac{\sum_{j:j\neq i} w_{ij}^2\sqrt{\frac{\log n}{k}}\|\pi\|_\infty^3\sum_{l\in[n]\setminus\{i,j\}} w_{lj}}{\lambda^2(L_\pi^w)}.\end{aligned} \tag{35}$$

The other term J_2 in Eq. (34) can be bounded as

$$\begin{aligned}&\sum_{j:j\neq i} w_{ji}\hat{p}_{ji}\left(\pi_j^{(-i)} - \pi_j\right) \\ &= \sum_{j:j\neq i}\frac{w_{ji}\hat{p}_{ji}\left(\sum_{l\in[n]\setminus\{i,j\}} w_{ji}(\hat{p}_{lj}\pi_l - \pi_j\hat{p}_{jl})\right)}{\sum_{l\in[n]\setminus\{i,j\}} w_{lj}\hat{p}_{lj}} \\ &\lesssim \frac{(w_{\max})^2 d_{\max}}{\lambda_{n-1}(L_\pi^w)}\sqrt{\frac{\log n}{k}}\|\pi\|_\infty^2.\end{aligned}$$

Combining the bound on J_1 and J_2 gives the following bound on $\|r\|_\infty$

$$\|r\|_\infty \lesssim \frac{w_{\max}^2 d_{\max}\|\pi\|_\infty^3}{\lambda_{n-1}(L_\pi^w)^2}\sqrt{\frac{\log(n)}{k}} + \frac{(w_{\max}\|\pi\|_\infty)^2}{\lambda_{n-1}(L_\pi^w)}\sqrt{\frac{d_{\max}}{k}}. \tag{36}$$

Substituting the bound on $\|r\|_\infty$ in Eq. (33) bounds, we obtain (assuming $w_{\max} \leq n$)

$$\|\delta\|_2 \leq \frac{\sqrt{n}w_{\max}^2\,\|\pi\|_\infty^3 d_{\max}}{\lambda_{n-1}(L^w)^3}\sqrt{\frac{\log n}{k}} + \frac{(w_{\max}\|\pi\|_\infty)^2}{\lambda_{n-1}(L_\pi^w)^2}\sqrt{\frac{d_{\max}}{k}}.$$

Utilizing Eq. (31) and recognizing that $\lambda_{n-1}(L_\pi^w) \gtrsim c\|\pi\|_\infty\lambda_{n-1}(L^w)$ and $d_{\max}^\pi \leq d_{\max}\|\pi\|_\infty$, we have the following bound on $\|\delta\|_\infty$ as

$$\|\delta\|_\infty \lesssim \frac{\sqrt{n}w_{\max}^{5/2}d_{\max}^{5/2}}{\lambda_{n-1}(L^w)^4}\sqrt{\frac{\log n}{k}} + \frac{w_{\max}^{5/2}(d_{\max})^{3/2}}{\lambda_{n-1}(L^w)^3}\sqrt{\frac{\log n}{k}}$$

Finally, the proof follows by Eq. (30). $\square$