# Testing for the Bradley-Terry-Luce Model

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***Abstract*—The Bradley-Terry-Luce (BTL) model is one of the most widely used models for ranking a set of items given data about pairwise comparisons among them. While several studies in the literature have attempted to empirically test how accurately a BTL model can model some given pairwise comparison data, this work aims to develop a formal, computationally efficient hypothesis test to determine whether the BTL model accurately represents the data. Specifically, we first propose such a formal hypothesis test, establish an upper bound on the critical radius of the proposed test, and then provide a complementary lower bound on the critical radius. Our bounds prove the minimax optimality of the scaling of the critical radius with respect to the number of items (up to constant factors). Finally, we also take the first step towards characterizing the stability of rankings under the BTL model when there is a small model mismatch.**

## I. Introduction

Many applications, such as sports tournaments, consumer preference surveys, and political voting, generate data in the form of pairwise comparisons between a set of items or agents (e.g., choices, teams). These datasets are useful for performing various data analysis tasks, such as ranking the items, analyzing the skill level of a particular team over time, and examining market or sports competitiveness, cf. [1]–[15]. A popular modeling assumption to perform such learning and inference tasks is the *Bradley-Terry-Luce* (BTL) model [1]–[6]. The BTL model assigns a latent skill score $\alpha_i > 0$ to each item $i$, representing its relative merit compared to other items, and posits that the likelihood of $i$ being preferred over an item $j$ in a pairwise comparison is given by

$$\mathbb{P}(i \text{ is preferred over } j) = \frac{\alpha_i}{\alpha_i + \alpha_j}. \quad (1)$$

The BTL model is a natural consequence of the assumption of *independence of irrelevant alternatives* (IIA), which is widely used in economics and social choice theory [2]. Despite its widespread popularity, it is known that the IIA assumption does not hold well for various real-world datasets [16]–[18]. For example, the BTL model is oblivious to the "home-advantage effect" in sports, which refers to a home team's possible advantage when playing against a visiting team (see, e.g., cricket [19], soccer [20]). Hence, several other models of pairwise comparisons have been proposed in the literature, e.g., modifications of the BTL model to incorporate home-advantage effect [21, Chapter 10], Thurstonian models [3], other generalizations of the BTL model [7], models of rankings based on Borda scores [22], [23], and other non-parametric stochastically transitive models [24], [25].

Nevertheless, primarily because of its simplicity and interpretability, the BTL framework remains one of the most widely-used models. A large fraction of the associated results in the literature focuses on estimation of the skill score parameters of the BTL model. Some popular approaches include maximum likelihood estimation [6], [7], rank centrality (or Markov-chain-based) methods [10], [26], least-squares methods [13], and non-parametric methods [12], [24] (also see [8], [9] for Bayesian inference for BTL models). Once the parameters are estimated, they are then used for inference tasks, such as ranking items and learning skill distributions [14], [15]. An inherent assumption for such statistical analysis to be meaningful in real-world scenarios is that the BTL model accurately represents the given pairwise comparison data. Hence, it is important to understand precisely when the BTL assumption holds in a systematic manner.

### A. Main Contributions

In contrast to the above works, we focus on the problem of testing whether the BTL assumption accurately models data generated from an underlying pairwise comparison model. First, we devise a notion of "distance" that allows us to quantify the deviation of a general pairwise comparison model from BTL models. We then use this distance to formally construct a hypothesis test to determine whether a BTL model accurately models the underlying data. We establish an upper bound on the minimax critical radius for this test. Furthermore, we also prove an information theoretic lower bound on the critical radius for this problem, thereby demonstrating the minimax optimality of the critical radius. Finally, we utilize the notion of distance mentioned above to analyze the stability of BTL assumptions in the context of rankings. More specifically, we investigate the deviation from the BTL condition that is sufficient for the ranking produced under the BTL assumption to differ from the classical Borda count ranking [27].

### B. Related Literature

This work lies at the confluence of two fields of study: hypothesis testing and preference learning. The analysis of preference data, such as pairwise comparisons, has a rich history starting from the seminal works [1]–[6]. As mentioned earlier, the BTL model is one of the most well-studied models for pairwise comparison data [1]. It was first proposed by [6] as a method for estimating participants' skill levels in chess

The author ordering is alphabetical.

tournaments. Moreover, it is a special case of the Plackett-Luce model [2], [4], initially developed in mathematical psychology. We refer readers to [28], [29] for a comprehensive overview of different models of rankings. In the literature, many studies have focused on estimating parameters for the BTL model and characterizing the error bounds, cf. [7], [10], [11], [25], [26], [30]–[32] . For example, [11] presents non-asymptotic bounds for relative $\ell^\infty$ and $\ell^2$-norm estimation errors of normalized vectors of skill scores. We use some of these bounds in our arguments, although we need to make alterations since the proofs do not apply to general pairwise comparison models.

Hypothesis testing also has a rich history in statistics, ranging from Pearson's $\chi^2$-test [33] to non-parametric tests [34]. Yet, to the best of our knowledge, no study has developed rigorous hypothesis tests to determine the validity of the BTL assumption in the literature. The minimax perspective of hypothesis testing, which we specialize in our setting, was initially proposed by [35]. It is worth mentioning that recently, [36] analyzed two-sample testing on pairwise comparison data, and [37] derived lower bounds for testing the IIA assumption given general preference data. For the special case of pairwise comparisons, the lower bounds in [37] agree with ours in terms of the high-level scaling law of the critical radius. However, our hypothesis testing problem is formulated differently to [37] and [37] does not provide upper bounds.

Furthermore, investigating the stability of the BTL assumption is another interesting question in the literature. For example, [22] provided empirical evidence that the BTL assumption is not very robust to changes in the pairwise comparison matrix. So, in this work, we also take the first steps towards rigorously characterizing the stability of rankings under the BTL model.

## II. Formal Setup and Decision Rule

### A. Notational Preliminaries

We briefly collect some notation here that is used throughout this work. Let $\mathbf{1}_n \in \mathbb{R}^n$ be the column vector with all entries equal to 1, where we drop the subscript when it is clear from context, and $[n] \triangleq \{1, \dots, n\}$. Furthermore, for any vector $x \in \mathbb{R}^n$, $\operatorname{diag}(x) \in \mathbb{R}^{n\times n}$ is the diagonal matrix with $x$ along its principal diagonal. For a vector $x \in \mathbb{R}^n$, $\|x\|_2$ denotes its $\ell^2$-norm and $x^p$ denotes the entry-wise $p$th power of $x$, i.e., $x^p = [x_1^p, x_2^p, \cdots, x_n^p]^{\mathrm{T}}$. For any matrix $A \in \mathbb{R}^{n\times n}$, $\|A\|_2$ and $\|A\|_{\mathrm{F}}$ denote the operator norm and Frobenious norm of $A$, respectively. For a strictly positive vector $\pi \in \mathbb{R}^n$, we define a Hilbert space on $\mathbb{R}^n$ with inner product $\langle x, y\rangle_\pi = \sum_{i=1}^n \pi_i x_i y_i$ and the corresponding vector and matrix norms $\|x\|_\pi = \sqrt{\langle x, x\rangle_\pi}$ and $\|A\|_\pi = \sup_{\|x\|_\pi=1} \|Ax\|_\pi$ and $\|A\|_{\pi,\mathrm{F}} = \left(\sum_i \sum_j \pi_j A_{ij}^2\right)^{1/2}$.

### B. Formal Model and Goal

We begin by introducing general pairwise comparison models. Consider a set of $n$ agents, indexed by $[n]$ with $n \in \mathbb{N}\backslash\{1\}$, that engage in a tournament consisting of several pairwise comparisons. This scenario is ubiquitous in many real-world applications. For example, in a sports tournament, $[n]$ represents the teams or players that play pairwise games with each other, and in discrete choice models from economics, $[n]$ represents alternatives that an individual may choose from. Several probabilistic models exist in the literature to capture such pairwise comparison settings, e.g., BTL model [1], [2], [5], Thurstonian models [3], and non-parametric models [24], [25], and all of them turn out to be specializations of the following general pairwise comparison model.

*Definition 1 (Pairwise Comparison Model):* For any pair of agents $i, j \in [n]$, $i \neq j$, let $p_{ij} \in (0, 1)$ denote the probability that agent $j$ beats agent $i$ in a "i vs. j" pairwise comparison. We refer to the collection of $n(n-1)$ parameters $\{p_{ij} : i, j \in [n],\, i \neq j\}$ as a *pairwise comparison model*.

Specifically, we consider an asymmetric setting where an "*i vs. j*" comparison may have a different meaning to a "*j vs. i*" comparison. Such asymmetric settings are commonly observed in sports like cricket, football, etc. [19]. Hence, such a model can be aptly summarized by a matrix $P \in \mathbb{R}^{n\times n}$ with

$$P_{ij} = \begin{cases} p_{ij}, & i \neq j, \\ \frac{1}{2}, & i = j, \end{cases} \tag{2}$$

where we have set $P_{ii} = \frac{1}{2}$ for notational convenience.

In our analysis, we will find it convenient to assign a time-homogenous Markov chain (or row stochastic matrix) on the finite state space $[n]$ to any pairwise comparison model. This canonical assignment is defined next.

*Definition 2 (Canonical Markov Chain):* For any pairwise comparison model $\{p_{ij} \in (0, 1) : i, j \in [n], i \neq j\}$ with matrix $P \in \mathbb{R}^{n\times n}$, its *canonical Markov chain* is given by the row stochastic matrix $S \in \mathbb{R}^{n\times n}$, where

$$S_{ij} = \begin{cases} \dfrac{p_{ij}}{n}, & i \neq j, \\ 1 - \dfrac{1}{n} \displaystyle\sum_{k\in[n]\backslash\{i\}} p_{ik}, & i = j. \end{cases}$$

As noted earlier, the most well-known specialization of the pairwise comparison model in Definition 1 is the BTL model defined below [1], [2], [5].

*Definition 3 (BTL Model [1], [2], [5]):* A pairwise comparison model $\{p_{ij} \in (0, 1) : i, j \in [n], i \neq j\}$ is known as a BTL (or multinomial logit) model if there exist skill score parameters $\alpha_i > 0$ for every agent $i \in [n]$ such that:

$$\forall i, j \in [n],\, i \neq j,\;\; p_{ij} = \frac{\alpha_j}{\alpha_i + \alpha_j}.$$

Hence, we can describe a BTL model entirely using the collection of its $n$ skill score parameters $\{\alpha_i : i \in [n]\}$.

We next describe how a pairwise comparison model characterizes the likelihood of a tournament between $n$ agents. To this end, fix any pairwise comparison model $\{p_{ij} \in (0, 1) : i, j \in [n],\, i \neq j\}$. For any $i \neq j$, define the outcome of the $m$th *i vs j* pairwise comparison between them as the Bernoulli random variable

$$Z_{m_{ij}} \triangleq \begin{cases} 1, & \text{if } j \text{ beats } i \text{ (with probability } p_{ij}), \\ 0, & \text{if } i \text{ beats } j \text{ (with probability } 1 - p_{ij}), \end{cases} \tag{3}$$

for $m \in [k_{ij}]$, where $k_{ij}$ denotes the number of *i vs j* comparisons. We assume throughout that the observation random variables $\mathcal{Z} \triangleq \{Z_{m_{ij}} : i, j \in [n], i \neq j, m \in [k_{i,j}]\}$ are mutually independent. Let $Z_{ij} \triangleq \sum_{m=1}^{k_{ij}} Z_{m_{ij}}$. Clearly, it follows that for any $i \neq j$, $Z_{ij}$ is a binomial random variable, i.e., $Z_{ij} \sim \mathrm{Bin}(k_{ij}, p_{ij})$, and for simplicity, we set $Z_{ii} = 0$. We also make the following assumption on the pairwise comparison model.

*Assumption 1 (Dynamic Range):* We assume that there is a constant $\delta \in (0,1)$ such that for all $i, j \in [n]$,

$$\frac{\delta}{1+\delta} \leq p_{ij} \leq \frac{1}{1+\delta}. \tag{4}$$

**Goal.** Given the observations $\mathcal{Z}$ of a tournament as defined above, our objective is to determine whether the underlying pairwise comparison model is a BTL model. This corresponds to solving a *composite hypothesis testing* problem:

$$\begin{aligned} &H_0 : \mathcal{Z} \sim \text{BTL model for some } \alpha_1, \ldots, \alpha_n > 0, \\ &H_1 : \mathcal{Z} \sim \text{pairwise comparison model that is not BTL,} \end{aligned} \tag{5}$$

where the null hypothesis $H_0$ states that $\mathcal{Z}$ is distributed according to a BTL model, and the alternative hypothesis $H_1$ states that $\mathcal{Z}$ is distributed according to a general non-BTL pairwise comparison model. To pose this hypothesis testing problem more rigorously, we demonstrate an interesting relation between a BTL model and its canonical Markov chain.

Recall that a Markov chain on the state space $[n]$, defined by the row stochastic matrix $W \in \mathbb{R}^{n\times n}$, is said to be *reversible* if it satisfies the *detailed balance conditions* [38, Section 1.6]:

$$\forall i, j \in [n],\, i \neq j,\;\; \pi_i W_{ij} = \pi_j W_{ij}, \tag{6}$$

where $W_{ij}$ denotes the probability of transitioning from state $i$ to state $j$, and $\pi = (\pi(1), \ldots, \pi(n))$ denotes the invariant distribution of the Markov chain (which always exists). Equivalently, the Markov chain $W$ is reversible if and only if

$$\mathrm{diag}(\pi) W = W^{\mathrm{T}} \mathrm{diag}(\pi). \tag{7}$$

It turns out that there is a tight connection between reversible Markov chains and the BTL model. This is elucidated in the ensuing proposition, cf. [39, Lemma 6], [10].

*Proposition 1 (BTL Model and Reversibility):* A pairwise comparison model $\{p_{ij} \in (0,1) : i, j \in [n], i \neq j\}$ is a BTL model if and only if its canonical Markov chain $S \in \mathbb{R}^{n\times n}$ is reversible and $p_{ij} + p_{ji} = 1$ for all $i, j \in [n]$.

*Proof:* We provide a proof for completeness. If the pairwise comparison model is BTL, it implies that for some weight vector $\alpha \in \mathbb{R}_+^n$, the pairwise comparison matrix $P$ is given by

$$p_{ij} = \frac{\alpha_j}{\alpha_i + \alpha_j}$$

for $i \neq j$. It is easy verify that $\pi \triangleq \left(\sum_{i=1}^n \alpha_i\right)^{-1} [\alpha_1 \cdots \alpha_n]^{\mathrm{T}}$ is the stationary distribution of canonical Markov chain matrix $S$ corresponding to $P$. Moreover, $S$ is reversible as

$$\pi_i S_{ij} = \frac{\alpha_i}{\sum_{i=1}^n \alpha_i} \times \frac{\alpha_j}{n(\alpha_i + \alpha_j)} = \pi_j S_{ij}.$$

For the converse, since $p_{ij} > 0$ for all $i, j \in [n]$, $S$ is irreducible and has a unique stationary distribution $\pi$. By reversibility of $S$, we have for all $i \neq j$,

$$\pi_i S_{ij} = \pi_j S_{ij} \implies \pi_i p_{ij} = \pi_j p_{ji} \implies p_{ij} = \frac{\pi_j}{\pi_i + \pi_j},$$

where last step follows from the fact that $p_{ij} + p_{ji} = 1$. Thus, $P$ corresponds to a BTL model with weight vector $\pi$. $\square$

Let $\pi$ be the stationary distribution of the canonical Markov chain matrix $S$ corresponding to a valid pairwise comparison matrix $P$. By Proposition 1, any pairwise comparison matrix $P$ is BTL if and only if it satisfies the reversibility condition $\Pi P = P^{\mathrm{T}} \Pi$, where $\Pi = \mathrm{diag}(\pi)$, and translated skew-symmetry $P + P^{\mathrm{T}} = \mathbf{1}_n \mathbf{1}_n^{\mathrm{T}}$. It turns out that both conditions are elegantly captured by the matrix $\Pi P + P\Pi - \mathbf{1}_n \pi^{\mathrm{T}}$ as illustrated in the Proposition 2, and we will later use the norm of this matrix to quantify the deviation of a pairwise comparison matrix from being BTL.

*Proposition 2 (Orthogonal Decomposition):* For any pairwise comparison matrix $P \in \mathbb{R}^{n\times n}$ and vector $\pi \in \mathbb{R}^n$ with strictly positive entries, we have

$$\begin{aligned} \left\|\Pi P + P\Pi - \mathbf{1}_n \pi^{\mathrm{T}}\right\|_{\pi^{-1},\mathrm{F}}^2 &= \left\|\Pi P - P^{\mathrm{T}} \Pi\right\|_{\pi^{-1},\mathrm{F}}^2 \\ &\quad + \left\|P + P^{\mathrm{T}} - \mathbf{1}_n \mathbf{1}_n^{\mathrm{T}}\right\|_{\pi,\mathrm{F}}^2, \end{aligned}$$

where $\Pi = \mathrm{diag}(\pi)$.

The proof can be found in Appendix A-A. It is important to note here that Assumption 1 and the *Perron-Frobenius theorem* [40, Chapter 8] imply that $\pi_i > 0$ for all $i \in [n]$. Hence, the norm $\|\cdot\|_{\pi^{-1},\mathrm{F}}$ is always well-defined.

**Hypothesis testing problem.** For a given tolerance parameter $\epsilon > 0$, we can formulate the hypothesis testing problem in (5) as:

$$\begin{aligned} H_0 :&\quad \Pi P + P\Pi = \mathbf{1}_n \pi^{\mathrm{T}}, \\ H_1 :&\quad \frac{1}{n} \|\Pi P + P\Pi - \mathbf{1}_n \pi^{\mathrm{T}}\|_{\mathrm{F}} \geq \epsilon \|\pi\|_\infty, \end{aligned} \tag{8}$$

where $\Pi = \mathrm{diag}(\pi)$ and $\pi$ is the stationary distribution of the canonical Markov chain matrix $S$ corresponding to $P$. Proposition 3, which is proved in Appendix A-B, verifies that the null hypothesis indeed captures BTL models.

*Proposition 3 (BTL Model Characterization):* The pairwise comparison matrix $P$ defined in Section II-B corresponds to a BTL model if and only if the hypothesis $H_0$ in (8) is true.

### *C. Minimax Risk and Decision Rule*

Let $\phi$ denote a hypothesis test (or decision rule) that maps the consolidated observations $\{Z_{ij}, k_{ij}\}_{i,j\in[n]}$ to $\{0,1\}$, where $0$ represents the null hypothesis and $1$ represents the alternative hypothesis. Let $\mathbb{P}_{H_0}$ and $\mathbb{P}_{H_1}$ denote the probability distributions of the input variables under $H_0$ and $H_1$, respectively. Let $\mathcal{M}_0$ and $\mathcal{M}(\epsilon)$ denote the sets of matrices $P$ that satisfy the null and alternative hypotheses in (8), respectively. Now, define the *minimax risk* as

$$\mathcal{R}_{\mathsf{m}} \triangleq \inf_{\phi} \left\{ \sup_{P\in\mathcal{M}_0} \mathbb{P}_{H_0}(\phi = 1) + \sup_{P\in\mathcal{M}(\epsilon)} \mathbb{P}_{H_1}(\phi = 0) \right\}, \tag{9}$$

where the infimum is taken over all $\{0,1\}$-valued tests $\phi$. Finally, we can define the *critical threshold* of the hypothesis testing problem in (8) as the smallest value of $\epsilon$ for which the minimax risk is bounded by $\frac{1}{3}$:

$$\varepsilon_{\mathsf{c}} = \inf\left\{\epsilon > 0 : \mathcal{R}_{\mathsf{m}} \leq \frac{1}{3}\right\}. \tag{10}$$

The constant $\frac{1}{3}$ is arbitrary and can be replaced by any constant in $(0, 1)$.

*Formally, our objective is to characterize the scaling of the critical radius with respect to $n$.* To this end, we consider a hypothesis test which takes the consolidated observations $\{Z_{ij}, k_{ij}\}_{i,j\in[n]}$ as input and evaluates the following expression:

$$T = \sum_{i=1}^{n}\sum_{j=1}^{n}(\hat{\pi}_i + \hat{\pi}_j)^2\frac{Z_{ij}(Z_{ij}-1)}{k_{ij}(k_{ij}-1)} + \hat{\pi}_j^2 - 2\hat{\pi}_j(\hat{\pi}_i + \hat{\pi}_j)\frac{Z_{ij}}{k_{ij}} \tag{11}$$

where $\hat{\pi}$ denotes the stationary distribution (choosing one arbitrarily if there are several) of the empirical Markov chain matrix $\hat{S} \in \mathbb{R}^{n\times n}$ defined via

$$\hat{S}_{ij} \triangleq \begin{cases} \frac{Z_{ij}}{k_{ij}n}, & i \neq j, \\ 1 - \frac{1}{n}\sum_{u:u\neq i}\frac{Z_{iu}}{k_{iu}}, & i = j. \end{cases} \tag{12}$$

The alternative hypothesis is selected if $T > \gamma/n$ for some appropriately chosen constant $\gamma$ independent of $n$ (see (22) for the analytical expression). The test has constructed such that if $\hat{\pi} = \pi$ then $\mathbb{E}[T] = \|\Pi P + P\Pi - \mathbf{1}_n\pi^{\mathrm{T}}\|_{\mathrm{F}}^2$, i.e., we "plug-in" $\hat{\pi}$ in an unbiased estimator of $\|\Pi P + P\Pi - \mathbf{1}_n\pi^{\mathrm{T}}\|_{\mathrm{F}}^2$.

## III. Upper Bound on Critical Threshold

The ensuing theorem establishes an upper bound on the critical radius of the hypothesis testing problem for the BTL model. For simplicity of analysis, we will assume that $k_{ij} = k_{ji} = k$ for all $i, j \in [n]$ throughout the sequel.

*Theorem 1 (Upper Bound on $\varepsilon_{\mathsf{c}}$):* Consider the hypothesis testing problem in (8), and suppose the number of comparisons per pair of agents satisfies $k \geq \max\{2, \frac{36C^2\log(n)}{n\delta^4}\}$ for some constant $C > 0$. Then, there exists another constant $c > 0$ such that for any $\epsilon > 0$ with $\epsilon^2 \geq \frac{c}{n}$, we have $\mathcal{R}_{\mathsf{m}} \leq \frac{1}{3}$. Hence, we obtain the bound

$$\varepsilon_{\mathsf{c}}^2 \leq \frac{c}{n}.$$

The proof is provided in Appendix B.

## IV. Lower Bound on Critical Threshold

We now prove an information theoretic lower bound on the critical radius for the BTL hypothesis testing problem, thus proving the minimax optimality of the scaling provided in the upper bound (up to constant factors).

*Theorem 2 (Lower Bound on $\varepsilon_{\mathsf{c}}$):* Consider the hypothesis testing problem in (8). Then, there exists a constant $c > 0$ such that the critical radius $\varepsilon_{\mathsf{c}}$ is lower bounded as

$$\varepsilon_{\mathsf{c}}^2 \geq \frac{c}{kn}.$$

*Proof:* We will use the *Ingster-Suslina method* for constructing a lower bound on the critical radius [41]. The method is similar to the well-known *Le Cam's method*, but it establishes a minimax lower bound by considering a point and a mixture on the parameter space instead of just two points. (Although Le Cam's method could also be used for this proof in principle, the Ingster-Suslina method greatly simplifies the calculations to bound total variation distance in our setting.)

Under the null hypothesis, we assume that the pairwise comparison matrix $P$ is fixed to be an all $1/2$ matrix, i.e.,

$$H_0: \quad P = P_0 \triangleq \frac{1}{2}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}. \tag{13}$$

We will denote the distribution corresponding to the pairwise comparison matrix $P_0$ by $\mathbb{P}_0$. Moreover, note that under $H_0$, the stationary distribution of the canonical Markov chain matrix $S$ is uniform, i.e., $\pi = \frac{1}{n}\mathbf{1}_n$. Under the alternative hypothesis, we assume that the pairwise comparison matrix $P_\theta$ is generated by sampling the parameter $\theta$ uniformly from the set $\Theta$, i.e.,

$$H_1 : P = P_\theta \text{ and } \theta \sim \mathrm{Unif}(\Theta), \tag{14}$$

and for some $\theta \in \Theta$, $P_\theta$ is given by

$$P_\theta = \begin{bmatrix} \frac{1}{2}\mathbf{1}_{n/2}\mathbf{1}_{n/2}^{\mathrm{T}} & \frac{1}{2}\mathbf{1}_{n/2}\mathbf{1}_{n/2}^{\mathrm{T}} + \eta Q_\theta \\ \frac{1}{2}\mathbf{1}_{n/2}\mathbf{1}_{n/2}^{\mathrm{T}} - \eta Q_\theta & \frac{1}{2}\mathbf{1}_{n/2}\mathbf{1}_{n/2}^{\mathrm{T}} \end{bmatrix}, \tag{15}$$

where $\Theta$ is set of all permuation matrices, and $Q_\theta$ is the $n/2\times n/2$ permutation matrix corresponding to the permutation $\theta$. Let $\mathbb{P}_\Theta$ denote the distribution corresponding to the pairwise comparison matrix $P_\theta$. The construction of this mixture was inspired by [36]. However, there are two notable differences. Firstly, the problem in [36] is distinguishing whether two sets of data samples consisting of pairwise comparisons are coming from the same underlying distribution or two different distributions described by a pairwise comparison model. In contrast, our work tests whether or not a single dataset is sampled from a BTL model. Secondly, the manner in which a notion of distance is used to define the deviation of the given data from the null hypothesis is very different in the two works.

Let $S_\theta$ denote the canonical Markov chain matrix corresponding to $P_\theta$. It is straightforward to verify that the stationary distribution of $S_\theta$ is independent of the permutation $\theta$. Let $\pi$ denote the stationary distribution of $S_\theta$. By the symmetry of $S_\theta$, the set of first $n/2$ elements, and respectively, last $n/2$ elements, of $\pi$ are equal, i.e., $\pi_1 = \cdots = \pi_{n/2} \triangleq x$ and $\pi_{(n/2)+1} = \cdots = \pi_n \triangleq y$. Now $x$ and $y$ can be determined by solving the set of linear equations:

$$\pi^{\mathrm{T}} = \pi^{\mathrm{T}}S_\theta \text{ and } \sum_{i=1}^{n}\pi_i = 1\,.$$

Solving these equations gives

$$x = \frac{1}{n}\left(1 - \frac{4\eta}{n}\right) \text{ and } y = \frac{1}{n}\left(1 + \frac{4\eta}{n}\right).$$

It is also easy to verify that the deviation from BTL $\|\Pi P_\theta + P_\theta \Pi - \mathbf{1}_n \pi^{\mathrm{T}}\|_{\mathrm{F}}$ is independent of the permutation $\theta$ and is given by

$$\begin{aligned}
&\|\Pi P_\theta + P_\theta^{\mathrm{T}} \Pi - \mathbf{1}_n \pi^{\mathrm{T}}\|_{\mathrm{F}}^2 = \frac{n}{2}\left((x+y)\left(\frac{1}{2}+\eta\right) - y\right)^2 \\
&\quad + \frac{n}{2}\left((x+y)\left(\frac{1}{2}-\eta\right) - x\right)^2 \\
&\quad + \frac{n}{2}\left(\frac{n}{2}-1\right)\left(\frac{x+y}{2} - y\right)^2 + \frac{n}{2}\left(\frac{n}{2}-1\right)\left(\frac{x+y}{2} - x\right)^2 \\
&\quad = \frac{2\eta^2}{n}\left(1-\frac{2}{n}\right)^2 + \frac{2\eta^2}{n^2}\left(1-\frac{2}{n}\right).
\end{aligned} \tag{16}$$

Let $\epsilon = \|\Pi P_\theta + P_\theta \Pi - \mathbf{1}_n \pi^{\mathrm{T}}\|_{\mathrm{F}}/(n\|\pi\|_\infty)$ to ensure that $P_\theta$'s satisfy the condition of the alternative hypothesis in (8). Substituting the values of $\|\pi\|_\infty = y$ and $\|\Pi P_\theta + P_\theta^{\mathrm{T}} \Pi - \mathbf{1}_n \pi^{\mathrm{T}}\|_{\mathrm{F}}$ implies that $\epsilon \le C\eta/\sqrt{n}$ for some constant $C > 0$.

Now, the Ingster-Suslina method [41] states that

$$\mathcal{R}_{\mathsf{m}} \ge \frac{1}{2}\left(1 - \sqrt{\chi^2(\mathbb{P}_0 || \mathbb{P}_\Theta)}\right),$$

where $\chi^2(\cdot||\cdot)$ denotes the $\chi^2$-divergence. Similar to the analysis in [36, Theorem 3], for the distributions $\mathbb{P}_0$ and $\mathbb{P}_\Theta$, if we have $\eta^2 \le \frac{c}{k}$ for some constant $c$ independent of $n, k$, then we can upper bound the $\chi^2(\mathbb{P}_0||\mathbb{P}_\Theta)$ term as $\chi^2(\mathbb{P}_0||\mathbb{P}_\Theta) \le \frac{1}{9}$. Hence, we have shown that there exists a constant $c > 0$, such that if $n\epsilon^2 \le c/k$, then $\chi^2(\mathbb{P}_0||\mathbb{P}_\Theta) \le \frac{1}{9}$, which implies that the minimax risk $\mathcal{R}_{\mathsf{m}} \ge \frac{1}{3}$. Hence, $\varepsilon_{\mathsf{c}}^2 \ge c/(kn)$ as desired. □

## V. Stability of the BTL Assumption

In this section, we analyze the stability of the BTL assumption in the context of rankings. The BTL ranking orders agents based on the stationary distribution $\pi$ of the canonical Markov-chain matrix. Meanwhile, the Borda count ranking is more general, as it doesn't rely on the BTL assumption, and instead is based on Borda scores [27] (defined below). If the BTL assumption holds, then Borda ranking equals the BTL ranking. Our goal is to determine the size of the deviation from the BTL condition in a pairwise comparison model that is sufficient to produce a discrepancy between the BTL and Borda count rankings. For this section, we will consider the symmetric setting in which the pairwise comparison matrix $P$ satisfies, $p_{ij} + p_{ji} = 1$. Define the *Borda count* $\tau_i(P)$ of an agent $i \in [n]$ as the (scaled) probability that $i$ beats any other agent selected uniformly at random [27]:

$$\tau_i(P) \triangleq \sum_{j=1}^{n} (1 - p_{ij}). \tag{17}$$

Now we show that the stability of the BTL assumption decreases as $n$ grows meaning that even smaller deviations from the BTL condition can lead to a reversed ranking.

*Proposition 4 (Stability of BTL Assumption):* There exists a pairwise comparison matrices $P \in \mathbb{R}^{n\times n}$ such that two rows $i, j \in [n]$ having constant $\Delta\tau \triangleq \tau_i(P) - \tau_j(P) > 0$, but has an opposite ranking BTL ranking, i.e., $\pi_i < \pi_j$. Let $\Pi = \mathrm{diag}(\pi)$. Moreover, the deviation of $P$ from BTL condition decays as

$$\left\|\Pi P + P\Pi - \mathbf{1}_n \pi^{\mathrm{T}}\right\|_{\mathrm{F}} \le \frac{c}{\sqrt{n}}$$

for some constant $c > 0$.

The proof is deferred to Appendix A-C. Proposition 4 highlights that the BTL assumption may potentially give a wrong ranking when the underlying pairwise matrix is $O(1/\sqrt{n})$ "distance" away from the BTL condition. Interestingly, this $O(1/\sqrt{n})$ deviation coincides with the critical threshold for the BTL testing problem (up to constant factors). It would be interesting to further explore the stability of the BTL assumption in the context of rankings.

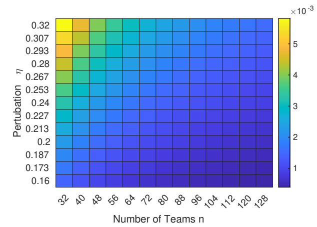


Fig. 1. Plot of $\mathbb{E}_{H_1}[T]$.

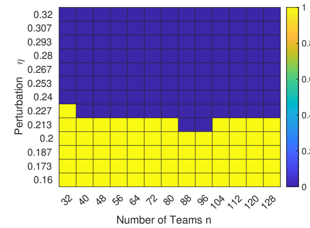


Fig. 2. Plot of $\mathbb{1}_{\mathcal{R}_{\mathsf{m}} > 4/5}$.

## VI. Numerical Simulations

In this section, we will empirically analyze the behavior of the minimax risk and the deviation $\|\Pi P + P\Pi - \mathbf{1}\pi^{\mathrm{T}}\|_{\mathrm{F}}$ via a synthetic experiment. We will use the same construction for the pairwise comparison matrix $P$ that we utilized in (2) under the null and alternate hypothesis which are presented in (13) and (14). We set the number of pairwise comparisons per pair of agents $k = 12$, the number of agents $n$ is linearly increased from 32 to 128, and the perturbation $\eta$ in (15) is increased from 0.16 to 0.32. Simulations are performed for each value of $\eta$ and $n$, and the corresponding value of expected values of test $T$ under hypothesis $H_1$ and *minimax risk* $\mathcal{R}_{\mathsf{m}}$ is estimated. The threshold used for the decision rule is set to $\eta^2/n$. Fig. 2 plots the $\mathbb{E}_{H_1}[T]$ and $\mathbb{1}_{\mathcal{R}_{\mathsf{m}} > 4/5}$ for different values of $\eta$ and $n$. Note that the behavior of $\mathbb{E}_{H_1}[T]$ is consistent with (16) and moreover for a fixed value of $\eta$ the behavior of $\mathcal{R}_{\mathsf{m}}$ is independent of $n$ and is consistent with **??** and the arguments therein.

## VII. Conclusion

In this work, we studied the problem of testing whether a BTL model accurately represents the data generated from an underlying pairwise comparison model. We developed a rigorous hypothesis test (8) for this purpose and established that the (miminax) critical threshold for this test is $\varepsilon_{\mathsf{c}} = \Theta(1/\sqrt{n})$. We also took the first steps towards rigorously characterizing the stability of the BTL assumption for rankings. Our results indicated that the BTL assumption may lead to different rankings if the pairwise comparison matrix deviates from the BTL condition by $O(1/\sqrt{n})$.

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## Appendix A
## Proofs of Propositions

We prove Propositions 2 to 4 in this section.

### A. Proof of Proposition 2

Observe that

$$\begin{aligned}
\|\Pi P + P\Pi - \mathbf{1}_n\pi^{\mathrm{T}}\|_{\pi^{-1},\mathrm{F}}^2 &= \|\Pi P - P^{\mathrm{T}}\Pi\|_{\pi^{-1},\mathrm{F}}^2 \\
&\quad + \|(P^{\mathrm{T}} + P - \mathbf{1}_n\mathbf{1}_n^{\mathrm{T}})\Pi\|_{\pi^{-1},\mathrm{F}}^2 \\
&\quad + 2\mathrm{Tr}\Big(\Pi^{-1/2}(\Pi P - P^{\mathrm{T}}\Pi)^{\mathrm{T}}(P^{\mathrm{T}} + P - \mathbf{1}_n\mathbf{1}_n^{\mathrm{T}})\Pi^{1/2}\Big) \\
&= \|\Pi P - P^{\mathrm{T}}\Pi\|_{\pi^{-1},\mathrm{F}}^2 + \|P^{\mathrm{T}} + P - \mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}\|_{\pi,\mathrm{F}}^2 \\
&\quad + 2\mathrm{Tr}\big((\Pi P - P\Pi)^{\mathrm{T}}(P^{\mathrm{T}} + P - \mathbf{1}_n\mathbf{1}_n^{\mathrm{T}})\big),
\end{aligned}$$

where $\mathrm{Tr}(\cdot)$ dentoes the trace operator. Now it remains to show that $\mathrm{Tr}\big((\Pi P - P^{\mathrm{T}}\Pi)^{\mathrm{T}}(P^{\mathrm{T}} + P - \mathbf{1}_n\mathbf{1}_n^{\mathrm{T}})\big) = 0$. This follows from the fact that $\mathrm{Tr}(A^{\mathrm{T}}B) = 0$ when $A$ is anti-symmetric, i.e., $A^{\mathrm{T}} = -A$, and $B$ is symmetric, i.e., $B = B^{\mathrm{T}}$:

$$\begin{aligned}
\mathrm{Tr}(A^{\mathrm{T}}B) &= -\mathrm{Tr}(AB) = -\mathrm{Tr}(AB^{\mathrm{T}}) \\
&= -\mathrm{Tr}(A^{\mathrm{T}}B) \implies \mathrm{Tr}(A^{\mathrm{T}}B) = 0.
\end{aligned}$$

This completes the proof. □

### B. Proof of Proposition 3

If $P$ corresponds to a BTL model, it implies that for some strictly positive weights $\alpha_i,\ i \in [n]$, we have $p_{ij} = \alpha_j/(\alpha_i + \alpha_j)$. It is easy to verify that the stationary distribution for the corresponding canonical Markov chain matrix is

$$\pi \triangleq \left[\frac{\alpha_1}{\sum_{i=1}^n \alpha_i}, \cdots, \frac{\alpha_n}{\sum_{i=1}^n \alpha_i}\right]^{\mathrm{T}},$$

and the stationary distribution satisfies $H_0$. Conversely, if $H_0$ is true, this implies that

$$p_{ij}(\pi_i + \pi_j) = \pi_j \implies p_{ij} = \frac{\pi_j}{\pi_i + \pi_j}.$$

This completes the proof. □

### C. Proof of Proposition 4

The construction of matrix $P \in \mathbb{R}^{n\times n}$ is as follows. For simplicity, we assume $n$ is even, otherwise we can replace $n/2$ in the construction with $\lceil n/2 \rceil$ to get the corresponding results. Below, we define the pairwise comparison matrix $P_n$ for $j \geq i$. The rest of the values can be obtained by skew-symmetric condition $p_{ij} + p_{ji} = 1$

$$p_{ij} = \begin{cases} 1/2 & 1 \leq i \leq j \leq \frac{n}{2} \\ 1/2 & \frac{n}{2} < i \leq j \leq n \\ 1/2 + 2\eta/n & i = 1, j > n/2 \\ 1/2 & i = 2, \frac{n}{2} < j \leq n - l \\ \frac{1}{2} + (\eta - \alpha)/l & i = 2, n - l < j \leq n \\ 1/2 & 3 \leq i \leq \frac{n}{2}, \frac{n}{2} < j \leq n \end{cases}$$

where $l$ is a integer less than $n/2$ and will be defined below. We are primarily interested in the first two rows of $P$. Their respective Borda scores are $\tau_1(P) = \frac{n}{2} - \eta$ and $\tau_2(P) = \frac{n}{2} - \eta + \alpha$ and hence $\Delta\tau = \tau_1 - \tau_2 = -\alpha$. Clearly, under Borda count ranking item 2 is ranked higher than 1. We will show that 1 is always ranked higher than 2 under BTL ranking. Moreover, we will show that the deviation of $P$ from BTL condition decays as

$$\left\|\Pi P + P\Pi - \mathbf{1}_n\pi^{\mathrm{T}}\right\|_{\mathrm{F}} \leq \frac{c}{\sqrt{n}},$$

To show both these facts we need to calculate the stationary distribution $\pi$ of canonical Markov chain matrix corresponding to $P$. We set $l = \lceil 2\eta \rceil$ and $\eta = 4\alpha n$ and $\alpha = 0.01$. Let $\pi_1 = a$ and $\pi_2 = b$. Using the symmetric structure of $P$, it is easy to see that

$$\begin{aligned}
\pi_3 = \pi_4 = \ldots = \pi_{\frac{n}{2}} &= c \\
\pi_{n/2+1} = \ldots = \pi_{n-l} &= d \\
\pi_{n-l+1} = \cdots = \frac{\pi}{2} &= e
\end{aligned}$$

The above equations and the fact that $\pi$ is a probability vector gives

$$a + b + c\Big(\frac{n}{2} - 2\Big) + \Big(\frac{n}{2} - l\Big)d + le = 1$$

and by the stationarity of $\pi$, we have the following equations

$$\begin{aligned}
a\bigg(\frac{n-1}{2} + \eta\bigg) &= \frac{b}{2} + \frac{c}{2}\Big(\frac{n}{2} - 2\Big) + d\Big(\frac{n}{2} - l\Big)\bigg(\frac{1}{2} - \frac{2\eta}{n}\bigg) \\
&\quad + le\bigg(\frac{1}{2} - \frac{2\eta}{n}\bigg) \\
b\bigg(\frac{n-1}{2} + \eta - \alpha\bigg) &= \frac{a}{2} + \frac{c}{2}\Big(\frac{n}{2} - 2\Big) + \frac{d}{2}\Big(\frac{n}{2} - l\Big) \\
&\quad + le\bigg(\frac{1}{2} - \frac{\eta - \alpha}{l}\bigg) \\
c\bigg(\frac{n-1}{2}\bigg) &= \frac{a+b}{2} + \frac{c}{2}\Big(\frac{n}{2} - 3\Big) + \frac{d}{2}\Big(\frac{n}{2} - l\Big) + \frac{le}{2} \\
d\bigg(\frac{n-1}{2} - \frac{2\eta}{n}\bigg) &= a\bigg(\frac{1}{2} + \frac{2\eta}{n}\bigg) + \frac{b}{2} + \frac{c}{2}\Big(\frac{n}{2} - 2\Big) \\
&\quad + \frac{d}{2}\Big(\frac{n}{2} - l - 1\Big) + \frac{le}{2}
\end{aligned}$$

The system of equations was solved using Wolfram Mathematica, and the results are presented based on the dominant terms of the polynomials. The coefficients of the polynomials are accurate to four decimal places.

$$\begin{aligned}
a &= \frac{0.8518}{n}\frac{\big(n^9 - 1.4541n^8 + 0.6307n^7 + O(n^6)\big)}{g(n)} \\
b &= \frac{0.8518}{n}\frac{\big(n^9 - 1.5491n^8 + 0.6723n^7 + O(n^6)\big)}{g(n)} \\
c &= \frac{1}{n}, \quad d = \frac{1}{n}\frac{\big(n^9 - 1.1063n^8 + 0.2125n^7 + O(n^6)}{g(n)} \\
e &= \frac{1}{n}\frac{\big(n^9 + 0.7455n^8 - 0.5282n^7 + O(n^6)\big)}{g(n)}
\end{aligned}$$

where $g(n)$ is same across all terms and is given as $g(n) = n^9 - 1.4026n^8 + 0.5877n^7 + O(n^6)$. Clearly, we can see that $b < a$ finite although the difference between them decreases

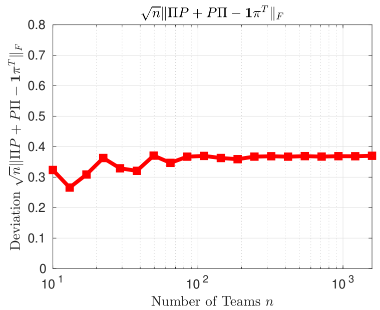


Fig. 3. Plot of $\sqrt{n}\|\Pi P + P\Pi - \mathbf{1}\pi^{\mathrm{T}}\|_{\mathrm{F}}$.

with $n$. Now it remains to show that $\|\Pi P + P\Pi - \mathbf{1}_n\pi^{\mathrm{T}}\|_{\mathrm{F}}$ decays as $O(1/\sqrt{n})$. To show this, we decompose $P$ as

$$P = \frac{1}{2}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}} + Q$$

where $Q$ contains the residual terms. Note that $Q$ has only $n + 2k$ non zero terms. Hence we can upper bound $\|\Pi P + P\Pi - \mathbf{1}\pi^{\mathrm{T}}\|_{\mathrm{F}}$ as

$$\begin{aligned}
&\|\Pi P + P\Pi - \mathbf{1}\pi^{\mathrm{T}}\|_{\mathrm{F}} \\
&= \|\frac{1}{2}(\pi\mathbf{1}_n^{\mathrm{T}} + \mathbf{1}_n\pi^{\mathrm{T}}) + \Pi Q + Q\Pi - \mathbf{1}_n\pi^{\mathrm{T}}\|_{\mathrm{F}} \\
&\leq \frac{1}{2}\|\pi\mathbf{1}_n^{\mathrm{T}} - \mathbf{1}_n\pi^{\mathrm{T}}\|_{\mathrm{F}} + \|\Pi Q + Q\Pi\|_{\mathrm{F}} \\
&\leq \|a\mathbf{1}_n^{\mathrm{T}} - \pi^{\mathrm{T}}\|_2 + \|b\mathbf{1}_n^{\mathrm{T}} - \pi^{\mathrm{T}}\|_2 \\
&\quad + \frac{1}{2}\|u\mathbf{1}_{n-2}^{\mathrm{T}} - \mathbf{1}_{n-2}u^{\mathrm{T}}\|_{\mathrm{F}} + \|\Pi Q + Q\Pi\|_{\mathrm{F}}
\end{aligned}$$

where $u \in \mathbb{R}^{n-2}$ is a vector such that and $u_i = \pi_{i+2}$ for $i \in [n-2]$. It is easy to see that $\|a\mathbf{1}_n^{\mathrm{T}} - \pi^{\mathrm{T}}\|_2 \leq O(1/\sqrt{n})$ as each term of the vector is bounded above by $2/n$. Moreover, observe that for any pair $x, y \in \{c, d, e\}$ the absolute difference $|x - y| \leq 2/n^2$ and hence, $\|u\mathbf{1}_{n-2}^{\mathrm{T}} - \mathbf{1}_{n-2}u^{\mathrm{T}}\|_{\mathrm{F}}$ is $O(1/n^2)$. Now it is easy to show that

$$\|\Pi Q + Q\Pi\|_{\mathrm{F}} \leq \frac{c}{\sqrt{n}}$$

as $\Pi Q + Q\Pi$ has only $O(n)$ non zero terms each of which is bounded above by $2/n$.

To verify our calculations, we also plot the numerically calculated values in Fig. 3 which confirms the theoretical calculation that $\sqrt{n}\|\Pi P + P\Pi - \mathbf{1}_n\pi^{\mathrm{T}}\|_{\mathrm{F}}$ converges to a constant. This completes the proof. □

## APPENDIX B
## PROOF OF THEOREM 1

This section is devoted to the proofs of various lemmata and existing results needed to prove our main result in Theorem 1. The main portion of the proof is presented in Appendix B-B. But first, we need to introduce some more notation and develop some key results. For any matrix $A \in \mathbb{R}^{n\times n}$, let $A_{:,i}$ denote the $i$th column of $A$ and $A_{j,:}$ denote the transpose of the $j$th row of $A$ (i.e., in column-vector form). Moreover, let $\mathcal{P}_n$ denote the $(n-1)$-dimensional probability simplex in $\mathbb{R}^n$.

### A. Preliminaries

In this section, we will prove key lemmata that will be used quite frequently to develop the proof of the main result in Theorem 1.

*1) Height of Perron-Frobenius Eigenvector:* We first present a modification of the result in [42, Theorem 3.1] that we will use quite frequently.

*Lemma 1 (Height of Perron–Frobenius Eigenvector):* Let $\pi$ be the stationary distribution of a canonical Markov chain matrix $S$, and suppose Assumption 1 holds. Then, we have

$$\min_{i\in[n]} \frac{\pi_i}{\|\pi\|_\infty} \geq \delta^2.$$

*Proof:* By Assumption 1 and the Perron-Frobenius theorem, we know that $\pi_i > 0$ for all $i \in [n]$. By stationarity of $\pi$, we have

$$\pi_i = \sum_{k=1}^{n}\pi_k S_{ki} = \pi_i\left(1 - \frac{1}{n}\sum_{k:k\neq i} p_{ik}\right) + \frac{1}{n}\sum_{k:k\neq i} p_{ki}\pi_k.$$

Rearranging the above equation gives

$$\pi_i = \frac{\sum_{k:k\neq i} p_{ki}\pi_k}{\sum_{k:k\neq i} p_{ik}}.$$

Hence, for any $i \neq j$ such that $\pi_i \leq \pi_j$, we have

$$\begin{aligned}
\frac{\pi_i}{\pi_j} &= \frac{\sum_{k:k\neq i} p_{ki}\pi_k}{\sum_{k:k\neq j} p_{kj}\pi_k}\frac{\sum_{k:k\neq j} p_{jk}}{\sum_{k:k\neq i} p_{ik}} \\
&\geq \delta\frac{\sum_{k:k\neq i} \pi_k}{\sum_{k:k\neq j} \pi_k}\frac{\sum_{k:k\neq j} p_{jk}}{\sum_{k:k\neq i} p_{ik}} \\
&\geq \delta^2\frac{1-\pi_i}{1-\pi_j} \geq \delta^2,
\end{aligned}$$

where last inequality holds because $\pi_i \leq \pi_j$ and the other inequalities use Assumption 1. The lemma follows by taking $j = \arg\max_k \pi_k$. □

*2) Auxiliary Lemmata for Estimation Error of Stationary Distribution:* The following lemma is similar to [11, Theorem 8] but also holds when $P$ is not reversible.

*Lemma 2 (Eigenvector Perturbation):* Let $\pi, \hat{\pi}$ be the stationary distributions of the row stochastic matrices $S, \hat{S}$, respectively. Then, if $\|\Pi^{1/2}S\Pi^{-1/2} - \sqrt{\pi}\sqrt{\pi}^{\mathrm{T}}\|_2 + \|\hat{S} - S\|_\pi < 1$, we have

$$\|\hat{\pi} - \pi\|_\pi \leq \frac{\|\pi^{\mathrm{T}}(S - \hat{S})\|_\pi}{1 - \|\Pi^{1/2}S\Pi^{-1/2} - \sqrt{\pi}\sqrt{\pi}^{\mathrm{T}}\|_2 - \|\hat{S} - S\|_\pi}.$$

*Proof:* By stationarity of $S$ and $\hat{S}$, we have

$$\begin{aligned}
&\hat{\pi}^{\mathrm{T}} - \pi^{\mathrm{T}} \\
&= \hat{\pi}^{\mathrm{T}}\hat{S} - \pi^{\mathrm{T}}S \\
&= \hat{\pi}^{\mathrm{T}}(\hat{S} - S) + (\hat{\pi} - \pi)^{\mathrm{T}}S - (\hat{\pi} - \pi)^{\mathrm{T}}\mathbf{1}_n\pi^{\mathrm{T}} \\
&= \pi^{\mathrm{T}}(\hat{S} - S) + (\hat{\pi} - \pi)^{\mathrm{T}}(\hat{S} - S) + (\hat{\pi} - \pi)^{\mathrm{T}}(S - \mathbf{1}_n\pi^{\mathrm{T}}).
\end{aligned}$$

Taking norm on both sides and using the Cauchy-Schwarz inequality gives

$$\begin{aligned}
\|\hat{\pi} - \pi\|_\pi \leq \|\pi^{\mathrm{T}}(\hat{S} - S)\|_\pi &+ \|\hat{\pi} - \pi\|_\pi\|\hat{S} - S\|_\pi \\
&+ \|\hat{\pi} - \pi\|_\pi\|S - \mathbf{1}_n\pi^{\mathrm{T}}\|_\pi.
\end{aligned}$$

It is straightforward to verify that

$$\|S - \mathbf{1}_n \pi^{\mathrm{T}}\|_\pi = \|\Pi^{1/2} S \Pi^{-1/2} - \sqrt{\pi}\sqrt{\pi}^{\mathrm{T}}\|_2.$$

Thus, rearranging the terms in the above inequality establishes the statement of Lemma 2. □

To use Lemma 2, we need to find upper bounds on the terms $\|\Pi^{1/2} S \Pi^{-1/2} - \sqrt{\pi}\sqrt{\pi}^{\mathrm{T}}\|_2$ and $\|\hat{S} - S\|_\pi$. When the Markov chain corresponding to $S$ is reversible, the matrix $\Pi^{1/2} S \Pi^{-1/2}$ is symmetric, and hence,

$$\|\Pi^{1/2} S \Pi^{-1/2} - \sqrt{\pi}\sqrt{\pi}^{\mathrm{T}}\|_2 = \max\{\lambda_2(S), -\lambda_n(S)\},$$

where $\lambda_l(S)$ is the $l$-th largest eigenvalue of $S$ for $l \in [n]$,. Moreover, by the Perron-Frobenius theorem, we have $\max\{\lambda_2(S), -\lambda_n(S)\} < 1$, and the corresponding upper bounds have been derived in [11], [26]. However, we are interested in the general case where $S$ need not be reversible. Hence, we bound each of these terms using the following two lemmata.

*Lemma 3 (Spectral Norm of Noise [26]):* For $\hat{S}$ as constructed in (12) and some constant $C \geq 8$, we have

$$\|\hat{S} - S\|_\pi \leq \sqrt{\frac{\|\pi\|_\infty}{\pi_{\min}}} \|\hat{S} - S\|_2 \leq \frac{C}{\delta}\sqrt{\frac{\log n}{nk}} \tag{18}$$

with probability at least $1 - 4n^{-C/8}$ and where $\pi_{\min} = \min_{i \in [n]} \pi_i$.

The quantity $\|\Pi^{1/2} S \Pi^{-1/2} - \sqrt{\pi}\sqrt{\pi}^{\mathrm{T}}\|_2$ is the second largest singular value of $\Pi^{1/2} S \Pi^{-1/2}$, which is known to be equal to the square root of the contraction coefficient for $\chi^2$-divergence $\eta_{\chi^2}(\pi, S)$ of the source-channel pair $(\pi, S)$ (see [43]–[45] for definitions and details).

*Lemma 4 (Spectral Norm Bound):*

$$\left\|\Pi^{1/2} S \Pi^{-1/2} - \sqrt{\pi}\sqrt{\pi}^{\mathrm{T}}\right\|_2 = \sqrt{\eta_{\chi^2}(\pi, S)} \leq \frac{1}{1 + \delta/3}.$$

*Proof:* We will find an upper bound on $\sqrt{\eta_{\chi^2}(\pi, S)}$ using [44, Proposition 2.5]:

$$\eta_{\chi^2}(\pi, S) \leq \eta_{\mathsf{TV}}(S) = \max_{i,j} \|S_{i,:} - S_{j,:}\|_{\mathsf{TV}},$$

where $\eta_{\mathsf{TV}}(S)$ is the Dobrushin contraction coefficient for total variation (TV) distance $\|\cdot\|_{\mathsf{TV}}$. Note that $S_{ii} \geq 1/n$ and $S_{ij} \leq 1/n$ for $i, j \in [n]$ with $i \neq j$. We can use Assumption 1 to bound $\eta_{\mathsf{TV}}(S)$ as

$$\begin{aligned}\|S_{i,:} - S_{j,:}\|_{\mathsf{TV}} &= 1 - \left(\sum_{k: k \neq i,j} \frac{\min\{p_{ik}, p_{jk}\}}{n} + \frac{p_{ij} + p_{ji}}{n}\right) \\ &\leq 1 - \frac{\delta}{1+\delta} = \frac{1}{1+\delta}.\end{aligned}$$

Hence, the lemma holds since $1/\sqrt{1+\delta} \leq 1/(1 + \delta/3)$. □

Lemma 3 and Lemma 4 give us the following corollary.

*Corollary 1 (Spectral Gap):* For $k \geq \max\{2, \frac{36C^2 \log n}{n\delta^4}\}$, the following bound holds with probability at least $1 - 4n^{-C/8}$:

$$1 - \|\Pi^{1/2} S \Pi^{-1/2} - \sqrt{\pi}\sqrt{\pi}^{\mathrm{T}}\|_2 - \|\hat{S} - S\|_\pi \geq \frac{\delta}{6}.$$

We will also require the following lemma from [11].

*Lemma 5 ($\mathcal{L}^2$-Error Bound [11, Theorem 9]):* Under the pairwise comparison model in Section II-B, for $k$ sufficiently large (i.e., larger than some constant), the following bound holds:

$$\|\pi - \hat{\pi}\|_2 \leq \frac{c_2 \|\pi\|_2}{\sqrt{nk}} \tag{19}$$

with probability at least $1 - c_2'/n^5$ for some constant $c_2, c_2'$ independent of $n, k$ .

### B. Upper Bound on Critical Threshold

*Proof of Theorem 1:* In this section, we will utilize the lemmata developed above to prove Theorem 1. Define $\hat{Y}_{ij} \triangleq \frac{Z_{ij}(Z_{ij}-1)}{k_{ij}(k_{ij}-1)}$ and $\hat{p}_{ij} \triangleq \frac{Z_{ij}}{k_{ij}}$. Since $Z_{ij} \sim \mathrm{Bin}(k_{ij}, p_{ij})$, it is easy to verify that

$$\begin{aligned}&\mathbb{E}[\hat{p}_{ij}] = p_{ij} \quad \text{and} \quad \mathrm{var}(\hat{p}_{ij}) = \frac{p_{ij}(1 - p_{ij})}{k_{ij}} \\ &\mathbb{E}[\hat{Y}_{ij}] = p_{ij}^2 \quad \text{and} \\ &\mathrm{var}(\hat{Y}_{ij}) = \frac{-2(2k_{ij} - 3)p_{ij}^4 + 4(k_{ij} - 2)p_{ij}^3 + 2p_{ij}^2}{k_{ij}(k_{ij} - 1)}\end{aligned} \tag{20}$$

Now, the test statistic $T$ in terms of $\hat{Y}_{ij}$ and $\hat{p}_{ij}$ is

$$T = \sum_i \sum_j (\hat{\pi}_i + \hat{\pi}_j)^2 \hat{Y}_{ij} + \hat{\pi}_j^2 - 2\hat{\pi}_j(\hat{\pi}_i + \hat{\pi}_j)\hat{p}_{ij}$$

We split $T$ as $T = T_1 + T_2 + T_3$ where

$$\begin{aligned}T_1 &\triangleq \sum_i \sum_j \left((\hat{\pi}_i + \hat{\pi}_j)^2 - (\pi_i + \pi_j)^2\right)\left(\hat{Y}_{ij} - p_{ij}^2\right) \\ &\quad - 2(\hat{\pi}_j(\hat{\pi}_i + \hat{\pi}_j) - \pi_j(\pi_i + \pi_{ij}))(\hat{p}_{ij} - p_{ij}) \\ T_2 &\triangleq \sum_i \sum_j \left((\hat{\pi}_i + \hat{\pi}_j)^2 - (\pi_i + \pi_j)^2\right)p_{ij}^2 + \hat{\pi}_j^2 - \pi_j^2 \\ &\quad - 2(\hat{\pi}_j(\hat{\pi}_i + \hat{\pi}_j) - \pi_j(\pi_i + \pi_j))p_{ij}. \\ T_3 &\triangleq \sum_i \sum_j \left((\pi_i + \pi_j)^2 \hat{Y}_{ij} + \pi_j^2 - 2\pi_j(\pi_i + \pi_j)\hat{p}_{ij}\right)\end{aligned}$$

The following bounds hold for $T_1$ and $T_2$.

*Lemma 6 (Bounds on $T_1$ and $T_2$):* The following bounds on $T_1, T_2$ hold with probability at least 0.99 for some constants $c_0, c_2 > 0$:

$$\begin{aligned}|T_1| &\leq c_0 \frac{n\|\pi\|_\infty^2}{k} \\ |T_2| &\leq \frac{c_1 \sqrt{n}\|\pi\|_\infty}{\sqrt{k}} \|\Pi P + P\Pi + \mathbf{1}_n \pi^{\mathrm{T}}\|_{\mathrm{F}} + c_2 \frac{n\|\pi\|_\infty^2}{k}\end{aligned} \tag{21}$$

The proof is provided in Appendix B-C. The following lemma charaterizes the mean and the variance of $T_3$.

*Lemma 7 (Mean and Variance of $T_3$):* The following statements hold:

1) $\mathbb{E}[T_3] = \|\Pi P + P\Pi - \mathbf{1}_n \pi^{\mathrm{T}}\|_{\mathrm{F}}^2$.
2) $\mathrm{var}(T_3) \leq \frac{16\|\pi\|_\infty^2}{k} \|\Pi P + P\Pi - \mathbf{1}_n \pi^{\mathrm{T}}\|_{\mathrm{F}}^2 + \frac{32n^2}{k}\|\pi\|_\infty^4$.

The proof is provided in Appendix B-D.

Now under hypothesis $H_0$, by Lemma 7, $\mathbb{E}_{H_0}[T_3]=0$ and $\text{var}_{H_0}(T_3) \le \frac{32n^2\|\pi\|_\infty^4}{k}$. Moreover

$$\begin{aligned}\{T\ge t\} &= (\{T\ge t\}\cap\{T\le T'\})\cup(\{T\ge t\}\cap\{T>T'\})\\ &= (\{t\le T\le T'\})\cup(\{T\ge t\}\cap\{T\ge T'\})\end{aligned}$$

Let $T'=(c_0+c_2)\frac{\|\pi\|_\infty^2 n}{k}+T_3$

$$\begin{aligned}\mathbb{P}_{H_0}[T\ge t] &\le \mathbb{P}_{H_0}[T'\ge t]+\mathbb{P}_{H_0}[T>T']\\ &\le \mathbb{P}_{H_0}\big[T_3\ge t-(c_0+c_2)\|\pi\|_\infty^2/k\big]+0.01\\ &\overset{\zeta}{\le} \frac{\text{var}_{H_0}(T_3)}{\text{var}_{H_0}(T_3)+(t-(c_0+c_2)\|\pi\|_\infty^2/k)^2}+0.01\\ &\le \frac{32n^2\|\pi\|_\infty^4/k}{32n^2\|\pi\|_\infty^4/(k)+256n^2\|\pi\|_\infty^4/k}+0.01\le\frac{1}{6}\end{aligned}$$

where $\zeta$ follows from one-sided Chebyshev's inequality and we have substituted

$$t=16n\|\pi\|_\infty^2/\sqrt{k}+(c_0+c_2)\frac{n\|\pi\|_\infty^2}{k}. \tag{22}$$

Under hypothesis $H_1$, for some random variable $T'$,

$$\begin{aligned}\{T\le t\} &= (\{T\le t\}\cap\{T\le T'\})\cup(\{T\le t\}\cap\{T\ge T'\})\\ &= (\{T\le t\}\cap\{T\le T'\})\cup(\{T'\le T\le t\})\end{aligned}$$

In this case, define $T'=T_3-\Delta_t$, where

$$\Delta_t=\frac{c_1\sqrt{n}\|\pi\|_\infty}{\sqrt{k}}\|\Pi P+P\Pi-\mathbf{1}_n\pi^{\mathrm{T}}\|_{\mathrm{F}}+(c_0+c_2)\frac{n\|\pi\|_\infty^2}{k}$$

$$\begin{aligned}\mathbb{P}_{H_1}[T\le t] &= \mathbb{P}_{H_1}[T'\le t]+\mathbb{P}_{H_1}[T'\le T]\\ &\le \mathbb{P}_{H_1}[T_3\le t+\Delta_t]+0.01\\ &\le \frac{\text{var}_{H_1}(T_3)}{\text{var}_{H_1}(T_3)+(E_{H_1}(T_3)-t-\Delta_t)^2}+0.01\\ &\overset{\zeta}{\le} 1/6\end{aligned}$$

where $\zeta$ is true if

$$25\text{var}_{H_1}(T_3)\le 4(E_{H_1}(T_3)-t-\Delta_t)^2$$

Let $D=\|\Pi P+P\Pi-\mathbf{1}_n\pi^{\mathrm{T}}\|_{\mathrm{F}}$. The above equation is true if

$$5\left(\frac{4\|\pi\|_\infty}{\sqrt{k}}D+\frac{6n\|\pi\|_\infty^2}{\sqrt{k}}\right)\le 2\left(D^2-\frac{c_1D\sqrt{n}\|\pi\|_\infty}{\sqrt{k}}-\frac{16n\|\pi\|_\infty^2}{\sqrt{k}}-2(c_0+c_2)\frac{n\|\pi\|_\infty^2}{k}\right)$$

Substituting $D=\epsilon n\|\pi\|_\infty$

$$\frac{20n\epsilon}{\sqrt{k}}+\frac{30n}{\sqrt{k}}\le 2\epsilon^2n^2-2\frac{c_1\epsilon n^{1.5}}{\sqrt{k}}-\frac{32n}{\sqrt{k}}-4(c_0+c_2)\frac{n}{k} \tag{23}$$

Substituting $\epsilon=a/\sqrt{nk}$

$$\frac{20a}{k\sqrt{n}}+\frac{30}{\sqrt{k}}\le 2\frac{a^2}{k}-2\frac{c_1a}{k}-\frac{32}{\sqrt{k}}-4(c_0+c_2)\frac{1}{k} \tag{24}$$

Hence (24) is a polynomial in a which is positive for a suffuently large constant $a_0$

$$n\epsilon^2\geqslant a_0 \implies \epsilon \ \ \ge \frac{c_0}{\sqrt{n}}$$

This completes the proof. $\square$

## C. Proof of Lemma 6

For bounding $T_1$ we split $T_1$ as $T_1=T_{1a}+T_{1b}$, where

$$\begin{aligned}T_{1a} &\triangleq \sum_i\sum_j\Big((\hat\pi_i+\hat\pi_j)^2-(\pi_i+\pi_j)^2\Big)\Big(\hat Y_{ij}-p_{ij}^2\Big)\\ T_{1b} &\triangleq \sum_i\sum_j -2(\hat\pi_j(\hat\pi_i+\hat\pi_j)-\pi_j(\pi_i+\pi_{ij}))(\hat p_{ij}-p_{ij})\end{aligned}$$

Define a matrix $Q$ as follows

$$Q_{ij}=\begin{cases}\hat Y_{ij}-p_{ij}^2 & \text{if } i\ne j\\ 0 & \text{if } i=j\end{cases}$$

By definition of $\hat Y_{ij}$, $\mathbb{E}[Q_{ij}]=0$. Moreover, since $\hat Y_{ij}$ are bounded, each entry of $Q$ is sub-gaussian, and the variance of each entry is upper bounded by $4/k$. Hence, by [46, Theorem 4.4.5], the spectral norm $\|Q\|_2\le 2c_q(2\sqrt{n}+t)/\sqrt{k}$ for some constant $c_q$ with probability at least $1-2e^{-t^2}$. Substituting $t=3$, we get the following bound with probability at least 99.9%.

$$\|Q\|_2\le 12c_q\sqrt{n}/\sqrt{k} \tag{25}$$

Now we re-write $T_{1a}$ as

$$\begin{aligned}T_{1a} &= \sum_i\sum_j(\hat\pi_i-\pi_i+\hat\pi_j-\pi_j)Q_{ij}(\hat\pi_i+\pi_i+\hat\pi_j+\pi_j)\\ &= (\hat\pi-\pi)^{\mathrm{T}}Q(\hat\pi+\pi)+(\hat\pi-\pi)^{\mathrm{T}}Q^{\mathrm{T}}(\hat\pi+\pi)\\ &\quad+\big(\hat\pi^2-\pi^2\big)^{\mathrm{T}}Q\mathbf{1}_n+\mathbf{1}_n^{\mathrm{T}}Q\big(\hat\pi^2-\pi^2\big)\\ &\overset{\zeta_1}{\le} 2c_2\frac{\|\pi\|_2}{\sqrt{nk}}\left(12c_q\sqrt{\frac{n}{k}}\right)2\|\pi\|_2+\frac{\|\hat\pi^2-\pi^2\|_2(12c_qn)}{\sqrt{k}}\\ &\overset{\zeta_2}{\le} \frac{48n\|\pi\|_\infty^2}{k}+\frac{24c_qc_2n\|\pi\|_\infty^2}{\sqrt{k}\times\sqrt{k}}=O\left(\frac{n\|\pi\|_\infty^2}{k}\right)\end{aligned} \tag{26}$$

where $\zeta_1$ follows from the fact $x^{\mathrm{T}}Ay\le\|x\|_2\|y\|_2\|A\|_2$ and from (25) and Lemma 5. In $\zeta_2$, we utilize the fact that $\|\pi\|_2\le\sqrt{n}\|\pi\|_\infty$ and bound $\|\hat\pi^2-\pi^2\|_2\le 2\|\pi\|_\infty\|\hat\pi-\pi\|_2\le 2c_2\|\pi\|_\infty^2/\sqrt{k}$ by Lemma 5. Now we bound $T_{1b}$ as

$$\begin{aligned}T_{1b} &= -2\sum_i\sum_j\big(\hat\pi_j^2-\pi_j^2+\hat\pi_i\hat\pi_j-\pi_i\pi_j\big)(\hat p_{ij}-p_{ij})\\ &= \sum_j -2\big(\hat\pi_j^2-\pi_j^2\big)\left(\sum_i\hat p_{ij}-p_{ij}\right)\\ &\quad-2\sum_i\sum_j(\hat\pi_i-\pi_i)(\hat\pi_j-\pi_j)(\hat p_{ij}-p_{ij})\\ &\quad-2\sum_i\sum_j\pi_i(\hat p_{ij}-p_{ij})(\hat\pi_j-\pi_i)+\pi_j(\hat p_{ij}-p_{ij})(\hat\pi_i-\pi_i)\\ &= -2\mathbf{1}_n^{\mathrm{T}}(\hat P-P)\big(\hat\pi^2-\pi^2\big)-2(\hat\pi-\pi)^{\mathrm{T}}(\hat P-P)(\hat\pi-\pi)\\ &\quad-2\pi^{\mathrm{T}}(\hat P-P)(\hat\pi-\pi)-2(\hat\pi-\pi)\Big(\hat P^{\mathrm{T}}-P^{\mathrm{T}}\Big)\pi\\ &\le 2\sqrt{n}\|\hat P-P\|_2\big\|\hat\pi^2-\pi^2\big\|_2+2\|\hat\pi-\pi\|_2^2\|\hat P-P\|_2\\ &\quad+4\|\pi\|_2\|\hat P-P\|_2\|\hat\pi-\pi\|_2\\ &\overset{\zeta_1}{\le} \frac{12c_pc_2^2n\|\pi\|_\infty^2}{\sqrt{k}\times\sqrt{k}}+12c_pc_2^2\frac{\|\pi\|_2^2}{nk}\sqrt{n/k}\\ &\quad+\frac{24c_pc_2(\sqrt{n}\|\pi\|_\infty)\|\pi\|_2}{k}=O\left(\frac{n\|\pi\|_\infty^2}{k}\right)\end{aligned}$$

where $\zeta_1$ holds since $\|\hat{P}-P\|_2 \leq 6c_p\sqrt{n}/\sqrt{k}$ with probability at least 0.999 for some constant $c_p$ by the same argument as (25) and since $\text{var}(\hat{p}_{ij}) \leq 1/\sqrt{k}$ and the rest follows similar to (26). Now we will bound $T_2$ by simplifying it as

$$
\begin{aligned}
T_2 &= \sum_i \sum_j \big(\hat{\pi}_i^2 - \pi_i^2\big)p_{ij}^2 + \big(\hat{\pi}_j^2 - \pi_j^2\big)\big(p_{ij}^2 + 1 - 2p_{ij}\big) \\
&\qquad + 2(\hat{\pi}_i\hat{\pi}_j - \pi_i\pi_j)\big(p_{ij}^2 - p_{ij}\big) \\
&= \sum_i \sum_j (\hat{\pi}_i - \pi_i)(\hat{\pi}_i + \pi_i)\Big(p_{ij}^2 + (1-p_{ij})^2\Big) \\
&\quad + 2(\hat{\pi}_i - \pi_i)(\hat{\pi}_j - \pi_j)\big(p_{ij}^2 - p_{ij}\big) \\
&\quad + \pi_i(\hat{\pi}_j - \pi_j)\big(p_{ij}^2 - p_{ij}\big) + \pi_j(\hat{\pi}_i - \pi_i)\big(p_{ij}^2 - p_{ij}\big) \\
&= \sum_i \sum_j (\hat{\pi}_i - \pi_i)^2(p_{ij}^2 + (1-p_{ij})^2) + 2(\hat{\pi}_i - \pi_i)\times \\
&\Big(\pi_i(p_{ij}^2 + (1-p_{ij})^2) + \pi_j\big(p_{ji}^2 - p_{ji}\big) + \pi_j\big(p_{ij}^2 - p_{ij}\big)\Big) \\
&\quad + 2(\hat{\pi}_i - \pi_i)(\hat{\pi}_j - \pi_j)\big(p_{ij}^2 - p_{ij}\big) \\
&= \sum_i \sum_j 2(\hat{\pi}_i - \pi_i)p_{ij}(\pi_i p_{ij} + \pi_j p_{ij} - \pi_j) \\
&\quad + 2(\hat{\pi}_i - \pi_i)(1-p_{ji})(\pi_i - \pi_i p_{ji} - \pi_j p_{ji}) \\
&\quad + 2(\hat{\pi} - \pi)^{\mathrm{T}}(P_2 - P)(\hat{\pi} - \pi) + (\hat{\pi} - \pi)^{2^{\mathrm{T}}} P_3 \mathbf{1}_n \\
&\overset{\zeta_1}{\leq} 4\sqrt{n}\|\hat{\pi} - \pi\|_2 \|\Pi P + P\Pi - \mathbf{1}_n\pi^{\mathrm{T}}\|_{\mathrm{F}} \\
&\quad + \|\hat{\pi} - \pi\|_2^2 \|P_2 - P\|_2 + \|\hat{\pi} - \pi\|_2^2 \|P_3\mathbf{1}_n\|_\infty \\
&\overset{\zeta_2}{\leq} \frac{4c_2\|\pi\|_2}{\sqrt{k}} \|\Pi P + P\Pi - \mathbf{1}_n\pi^{\mathrm{T}}\|_{\mathrm{F}} + 2\frac{c_2^2\|\pi\|_2^2}{k} \\
&\leq \frac{4c_2\sqrt{n}\|\pi\|_\infty}{\sqrt{k}} \|\Pi P + P\Pi - \mathbf{1}_n\pi^{\mathrm{T}}\|_{\mathrm{F}} + \frac{2nc_2^2\|\pi\|_\infty^2}{k}
\end{aligned}
$$

where

$$
P_2 \triangleq \begin{cases} p_{ij}^2 & i \neq j \\ 0 & i = j \end{cases} \qquad P_3 \triangleq \begin{cases} (1-p_{ij})^2 + p_{ij}^2 & i \neq j \\ 1/2 & i = j \end{cases}
$$

and $\zeta_1$ follows from Cauchy-Schwarz inequality and $\zeta_2$ follows from Lemma 5 and the fact that $\|P_2 - P\|_2 \leq n$ and $\|P_3\|_2 \leq n$. This holds because the absolute value of each entry of $P_2 - P$ and $P_3$ is upper bounded by 1. Similarly, we can obtain the corresponding lower bounds on $T_1, T_2$ in the same fashion as above and thus completing the proof. □

### D. Proof of Lemma 7

Part 1: Since $Z_{ij} \sim \text{Bin}(k_{ij}, p_{ij})$, we have

$$
\begin{aligned}
\mathbb{E}[T_3] = \sum_{ij} (\pi_i + \pi_j)^2 &\frac{\mathbb{E}[Z_{ij}^2] - \mathbb{E}[Z_{ij}]}{k_{ij}(k_{ij}-1)} + \pi_j^2 \\
&- 2(\pi_i + \pi_j)\pi_j \frac{\mathbb{E}[Z_{ij}]}{k_{ij}} \\
= \sum_{ij} &(\pi_i + \pi_j)^2 p_{ij}^2 + \pi_j^2 - 2(\pi_i + \pi_j)\pi_j p_{ij} \\
= \|&\Pi P - P\Pi - \mathbf{1}_n\pi^{\mathrm{T}}\|_{\mathrm{F}}^2.
\end{aligned}
$$

Part 2: Recall that

$$
T_3 = \sum_i \sum_j (\pi_i + \pi_j)^2 \hat{Y}_{ij} + \pi_j^2 - 2\pi_j(\pi_i + \pi_j)\hat{p}_{ij}.
$$

Hence, we have

$$
\begin{aligned}
\text{var}(T_3) &= \sum_{i,j} (\pi_i + \pi_j)^4 \text{var}(\hat{Y}_{ij}) + 4\pi_j^2(\pi_i + \pi_j)^2 \text{var}(\hat{p}_{ij}) \\
&\qquad - 4(\pi_i + \pi_j)^3\pi_j\Big(\mathbb{E}[\hat{Y}_{ij}\hat{p}_{ij}] - \mathbb{E}[\hat{Y}_{ij}]\mathbb{E}[\hat{p}_{ij}]\Big) \\
&= \sum_{i,j} (\pi_i + \pi_j)^4 \left(\frac{2p_{ij}^2 + 4(k-2)p_{ij}^3 + (6-4k)p_{ij}^4}{k(k-1)}\right) \\
&\qquad + \frac{4\pi_j^2(\pi_i + \pi_j)^2 p_{ij}(1-p_{ij})}{k} \\
&\qquad - 4\pi_j(\pi_i + \pi_j)^3\left(\frac{2p_{ij}^2 - 2p_{ij}^3}{k}\right) \\
&\leq \sum_{i,j} \frac{2(\pi_i + \pi_j)^2}{k(k-1)}\Big((\pi_i + \pi_j)^2 p_{ij}^2 + 2(k-1)\Big((\pi_i + \pi_j)^2 p_{ij}^3 \\
&\quad + \pi_j^2 p_{ij} - 2(\pi_i + \pi_j)\pi_j p_{ij}^2\Big) + 4(k-1)(\pi_i + \pi_j)\pi_j p_{ij}^3\Big) \\
&\leq \sum_{i,j} \frac{8\|\pi\|_\infty^2}{k(k-1)}\Big(2(\pi_i p_{ij} + \pi_j p_{ij} - \pi_j)^2(k-1) \\
&\qquad + (\pi_i + \pi_j)^2 p_{ij}^2 + 4(k-1)(\pi_i + \pi_j)\pi_j p_{ij}^2\Big) \\
&\leq \frac{16\|\pi\|_\infty^2}{k}\|\Pi P + P\Pi - \mathbf{1}_n\pi^{\mathrm{T}}\|_2^2 + \frac{32n\|\pi\|_\infty^2\|\pi\|_2^2}{k} \\
&\leq \frac{16\|\pi\|_\infty^2}{k}\|\Pi P + P\Pi - \mathbf{1}_n\pi^{\mathrm{T}}\|_{\mathrm{F}}^2 + \frac{32}{k}n^2\|\pi\|_\infty^4.
\end{aligned}
$$

This completes the proof. □