

VI. PROOF OF THEOREM 1

Recall that since $\hat{\pi}$ is the stationary distribution of \hat{S} , we have

$$\hat{\pi}_i = \frac{\sum_{j:j \neq i} \hat{p}_{ji} \hat{\pi}_j}{\sum_{j:j \neq i} \hat{p}_{ij}}.$$

We introduce the auxiliary quantity $\bar{\pi}_i$ by replacing $\hat{\pi}_j$ with π_j in the above equation, i.e., for $i \in [n]$ we have

$$\bar{\pi}_i = \frac{\sum_{j:j \neq i} \hat{p}_{ji} \pi_j}{\sum_{j:j \neq i} \hat{p}_{ij}}.$$

We further define the error term δ_i as:

$$\delta_i = \frac{\hat{\pi}_i - \bar{\pi}_i}{\pi_i} = \frac{\sum_{j:j \neq i} \hat{p}_{ji} (\hat{\pi}_j - \pi_j)}{\pi_i \sum_{j:j \neq i} \hat{p}_{ij}}. \quad (12)$$

We introduce $\hat{S}^{(-m)}$ which is a leave-one-out version of \hat{S} with m th row and columns replaced by their expected values. Thus for $i \neq j$, its (i, j) entry is defined as

$$(\hat{S}^{(-m)})_{ij} = \begin{cases} \hat{S}_{ij} & \text{if } i \neq m \text{ and } j \neq m \\ \frac{p_{ij} q_{ij}}{d} & \text{if } i = m \text{ or } j = m \end{cases}.$$

Also, we define $(\hat{S}^{(-m)})_{ii} = 1 - \sum_{l:l \neq i} (\hat{S}^{(-m)})_{il}$. Let $\hat{\pi}^{(-m)}$ be the stationary distribution of $\hat{S}^{(-m)}$. Similarly, we define $\bar{\pi}^{(-m)}$ be the leave-one-out version of $\bar{\pi}$ as

$$\bar{\pi}_i^{(-m)} = \frac{\sum_{j \in [n] \setminus \{i, m\}} \hat{p}_{ji} \pi_j + p_{mi} q_{mi} \pi_m}{\sum_{j \in [n] \setminus \{i, m\}} \hat{p}_{ij} + p_{im} q_{im}}.$$

Additionally, we define $\delta_i^{(-m)}$, as the leave-one-out version of δ_i as follows

$$\delta_i^{(-m)} = \frac{\hat{\pi}_i^{(-m)} - \bar{\pi}_i^{(-m)}}{\pi_i}.$$

We will decompose δ_i as

$$\delta_i = \underbrace{\frac{\sum_{j:j \neq i} \hat{p}_{ji} (\hat{\pi}_j - \hat{\pi}_j^{(-i)})}{\pi_i \sum_{j:j \neq i} \hat{p}_{ij}}}_{I_1} + \underbrace{\frac{\sum_{j:j \neq i} \hat{p}_{ji} \pi_j \delta_j^{(-i)}}{\pi_i \sum_{j:j \neq i} \hat{p}_{ij}}}_{I_2} + \underbrace{\frac{\sum_{j:j \neq i} \hat{p}_{ji} (\bar{\pi}_j^{(-i)} - \pi_j)}{\pi_i \sum_{j:j \neq i} \hat{p}_{ij}}}_{I_3}. \quad (13)$$

First, note that using Hoeffding's inequality, for all $i \in [n]$, the denominator term of each of I_1, I_2 and I_3 is lower bounded with probability at least $1 - O(1/n^5)$ as:

$$\begin{aligned} \pi_i \sum_{j:j \neq i} \hat{p}_{ij} &\geq \pi_i \sum_{j:j \neq i} p_{ij} + \pi_i \sum_{j:j \neq i} (\hat{p}_{ij} - p_{ij}) \\ &\geq \frac{\pi_i}{h+1} \left(\sum_{j:j \neq i} A_{ij} - \sqrt{\frac{\sum_{j:j \neq i} A_{ij}^2}{k} \log n} \right). \end{aligned} \quad (14)$$

Define $d_i = \sum_{j:j \neq i} A_{ij}$ be the degree of node i . Note that, with probability at least $1 - O(1/n^5)$, we have the following degree condition for all $i \in [n]$ and $q_{ij} \in [p, 1]$ with $p \geq \log n$:

$$\frac{1}{2} \sum_{j:j \neq i} q_{ij} \leq d_i \leq 2 \sum_{j:j \neq i} q_{ij} \quad (15)$$

We now focus on the numerator term of I_1 . An application of Cauchy-Schwarz inequality gives

$$\begin{aligned} \sum_{j:j \neq i} \hat{p}_{ji} (\hat{\pi}_j - \hat{\pi}_j^{(-i)}) &\leq \sqrt{\sum_{j:j \neq i} \hat{p}_{ji}^2} \|\hat{\pi} - \hat{\pi}^{(-i)}\|_2 \\ &\leq \sqrt{\sum_{j:j \neq i} A_{ji}} \|\hat{\pi} - \hat{\pi}^{(-i)}\|_2. \end{aligned} \quad (16)$$

The final bound on I_1 is

$$|I_1| \lesssim \frac{\sqrt{d_i} (h+1)^2 \|\hat{\pi} - \hat{\pi}^{(-i)}\|_2}{\|\pi_i\|_\infty \left(d_i - \sqrt{\frac{d_i}{k} \log n} \right)}. \quad (17)$$

The following lemma proved in Section VI-A provides bounds on $\|\hat{\pi} - \hat{\pi}^{(-i)}\|_2$.

Lemma 1 (Error Bound for Leave-one-out). *Under the assumption of Theorem 1, there exists a constant $c > 1$ such that for $np > c\gamma^2 \log n$, the following bound holds with probability at least $1 - O(n^{-5})$*

$$\|\hat{\pi} - \hat{\pi}^{(-i)}\|_2 \leq \frac{c}{\gamma} \sqrt{\frac{\log n}{knp}} \|\pi\|_\infty + \|\hat{\pi} - \pi\|_\infty.$$

Now we focus on the numerator term of I_3 . Observe that

$$\begin{aligned} \sum_{j:j \neq i} \hat{p}_{ji} (\bar{\pi}_j^{(-i)} - \pi_j) &= \sum_{j:j \neq i} \hat{p}_{ji} \frac{\sum_{l:l \in [n] \setminus \{i, j\}} \hat{p}_{lj} \pi_l - \pi_j \hat{p}_{jl}}{\sum_{l:l \in [n] \setminus \{i, j\}} \hat{p}_{jl} + p_{ji} q_{ji}} \\ &= \sum_{j:j \neq i} \hat{p}_{ji} \max_{j \in [n] \setminus \{i\}} \left| \frac{\sum_{l:l \in [n] \setminus \{i, j\}} \hat{p}_{lj} \pi_l - \pi_j \hat{p}_{jl}}{\sum_{l:l \in [n] \setminus \{j\}} \hat{p}_{jl} + p_{ji} q_{ji}} \right|. \end{aligned}$$

Note that $\sum_{j:j \neq i} \hat{p}_{ji} \leq \sum_{j:j \neq i} A_{ji}$. Moreover, for the other term, note that by our BTL model, we have that $\pi_l p_{lj} = \pi_j p_{jl}$. Simplifying the term, we obtain

$$\begin{aligned} \left| \frac{\sum_{l:l \in [n] \setminus \{i, j\}} \hat{p}_{lj} \pi_l - \pi_j \hat{p}_{jl}}{\sum_{l:l \in [n] \setminus \{i, j\}} \hat{p}_{jl} + p_{ji} q_{ji}} \right| &\leq \left| \frac{\sum_{l:l \neq \{j, i\}} A_{lj} (\hat{p}_{lj} - p_{lj}) \pi_l}{\sum_{l:l \in [n] \setminus \{i, j\}} \hat{p}_{jl} + p_{ji} q_{ji}} \right| \\ &\quad + \left| \frac{\sum_{l:l \in [n] \setminus \{i, j\}} A_{lj} \pi_j (p_{jl} - \hat{p}_{jl})}{\sum_{l:l \in [n] \setminus \{i, j\}} \hat{p}_{jl} + p_{ji} q_{ji}} \right|. \end{aligned} \quad (18)$$

By the same reasoning as in Eq. (14), we have the following lower bound on the denominator term $\sum_{l:l \neq j} \hat{p}_{jl} + p_{ji} q_{ji}$ as

$$\begin{aligned} \sum_{l:l \in [n] \setminus \{i, j\}} \hat{p}_{jl} + p_{ji} q_{ji} &\geq \\ \frac{1}{(1+h)} \left(\sum_{l:l \in [n] \setminus \{i, j\}} A_{jl} - \sqrt{\frac{\sum_{l:l \in [n] \setminus \{i, j\}} A_{jl}^2}{k} \log n} \right). \end{aligned} \quad (19)$$

Moreover, by application of Hoeffding's inequality, we have the following high probability bound on the numerator term in Eq. (18), as

$$\sum_{l:l \neq \{j,i\}} A_{lj}(\hat{p}_{lj} - p_{lj})\pi_l \leq \sqrt{\sum_{l:l \neq \{j,i\}} A_{lj} \log n} \|\pi\|_\infty. \quad (20)$$

A similar bound can be obtained for the second term in Eq. (18). Therefore, we have the following bound on the numerator term of I_3 , with probability at least $1 - O(1/n^5)$,

$$\sum_{j:j \neq i} \hat{p}_{ji}(\bar{\pi}_j^{(-i)} - \pi_j) \lesssim d_i \max_{j \in [n] \setminus \{i\}} \sqrt{\frac{\log n}{d_j - 1}} \|\pi\|_\infty. \quad (21)$$

Now it remains to bound the numerator of I_2 . Let $\delta^{(-i)}$ be the vector of leave-one-out errors. Denote \mathcal{G}_{-i} as the graph with node i removed and \mathcal{Z}_i as the data corresponding to node i . Observe that since we have $\delta_j^{(-i)}$ independent of data corresponding to node i there we can split the numerator of I_2 as:

$$\begin{aligned} \sum_{j:j \neq i} \hat{p}_{ji} \pi_j \delta_j^{(-i)} &= \sum_{j:j \neq i} \mathbb{E}[\hat{p}_{ji} A_{ji} \mid \mathcal{G}_{-i}, \mathcal{Z}_i] \pi_j \delta_j^{(-i)} \\ &+ \sum_{j:j \neq i} (\hat{p}_{ji} A_{ji} - \mathbb{E}[\hat{p}_{ji} A_{ji} \mid \mathcal{G}_{-i}, \mathcal{Z}_i]) \pi_j \delta_j^{(-i)} \\ &= \sum_{j \neq i} p_{ij} q_{ij} \pi_j \delta_j^{(-i)} + \sum_{j \neq i} \hat{p}_{ji} (A_{ji} - q_{ij}) \pi_j \delta_j^{(-i)} \\ &\lesssim \sqrt{\sum_{j:j \neq i} q_{ij}^2} \|\hat{\pi}^{(-i)} - \bar{\pi}^{(-i)}\|_2 \\ &\quad + \|\hat{\pi}^{(-i)} - \bar{\pi}^{(-i)}\|_\infty \left(\sqrt{\sum_{j:j \neq i} q_{ij} (1 - q_{ij})} + \log n \right), \end{aligned}$$

where in ζ the first term follows from the Cauchy-Schwarz inequality while the second term follows with probability at least $1 - O(1/n^5)$ using the Hoeffding's inequality. Note that the ratio $\sqrt{\sum_{j:j \neq i} q_{ij}^2}/d_i$ and $(\sqrt{\sum_{j:j \neq i} q_{ij} (1 - q_{ij})} + \log n)/d_i$ is maximized when $q_{ij} = p$ for all j . Finally utilizing the bound on $\|\delta^{(-i)}\|_2$ from [37], we obtain the following bound on the numerator of I_2 with probability at least $1 - O(1/n^5)$

$$\sum_{j:j \neq i} \hat{p}_{ji} \pi_j \delta_j^{(-i)} \leq \frac{c}{\gamma} \sqrt{\frac{\log n}{k}} \|\pi\|_\infty \quad (22)$$

Combining the bounds (14), (16), (18), (19), (20), (21) and (22) we obtain the desired result. \square

A. Proof of Lemma 1

First, we will apply [9, Theorem 8], to obtain the following bound

$$\begin{aligned} \|\hat{\pi}^{(-m)} - \hat{\pi}\|_2 &\leq \frac{\|\hat{\pi}^{(-m)\text{T}}(\hat{S}^{(-m)} - \hat{S})\|_2}{1 - \max\{\lambda_2(S), -\lambda_n(S)\} - \|\hat{S} - S\|_\pi} \\ &\leq \frac{1}{\gamma} \sqrt{h} \|\hat{\pi}^{(-m)\text{T}}(\hat{S}^{(-m)} - \hat{S})\|_2 \end{aligned}$$

where $\gamma = 1 - \max\{\lambda_2(S), -\lambda_n(S)\} - \|\hat{S} - S\|_\pi$. We will bound the term $\|\hat{S} - S\|_\pi$ separately, first we will focus on the numerator term $\|\hat{\pi}^{(-m)\text{T}}(\hat{S}^{(-m)} - \hat{S})\|_2$ in the following steps. To bound this term, we introduce another Markov matrix $\hat{S}^{(-m, \mathcal{E})}$, which is also a leave-one-out version of \hat{S} defined conditional on edge-set \mathcal{E} . Similar to $\hat{S}^{(-m)}$, the matrix $\hat{S}^{(-m, \mathcal{E})}$ replaces all transition probabilities involving the m -th item with their expected values conditioned on \mathcal{E} . In particular, for $i \neq j$,

$$\hat{S}_{ij}^{(-m, \mathcal{E})} = \begin{cases} \hat{S}_{ij}, & i \neq m \text{ and } j \neq m \\ \frac{p_{ij}}{d} A_{ij}, & i = m \text{ or } j = m \end{cases}.$$

Similarly, we define the diagonal entries for each $1 \leq i \leq n$ as

$$\hat{S}_{ii}^{(-m, \mathcal{E})} = 1 - \sum_{j:j \neq i} \hat{S}_{ij}^{(-m, \mathcal{E})},$$

to make it a valid probability transition matrix. Using the triangle inequality, we obtain

$$\begin{aligned} \|\hat{\pi}^{(-m)\text{T}}(\hat{S}^{(-m)} - \hat{S})\|_2 &\leq \underbrace{\|\hat{\pi}^{(-m)\text{T}}(\hat{S} - \hat{S}^{(-m, \mathcal{E})})\|_2}_{J_1} \\ &\quad + \underbrace{\|\hat{\pi}^{(-m)\text{T}}(\hat{S}^{(-m)} - \hat{S}^{(-m, \mathcal{E})})\|_2}_{J_2}. \end{aligned}$$

Now we will bound each of the term J_1 and J_2 . The term J_1 is similar to [9] and yields the following bound with probability at least $1 - O(n^{-5})$

$$J_1 \leq c \sqrt{\frac{\log n}{kd}} \|\hat{\pi}^{(-m)}\|_\infty. \quad (23)$$

Now, we will focus on the term J_2 . Utilizing the identity $\hat{\pi}^{\text{T}}(\hat{S}^{(-m)} - \hat{S}^{(-m, \mathcal{E})}) = 0$, we obtain

$$\hat{\pi}^{(-m)\text{T}}(\hat{S}^{(-m)} - \hat{S}^{(-m, \mathcal{E})}) = (\hat{\pi}^{(-m)} - \pi)^{\text{T}}(\hat{S}^{(-m)} - \hat{S}^{(-m, \mathcal{E})}).$$

For $j \neq m$, it follows that

$$\begin{aligned} &[(\hat{\pi}^{(-m)} - \pi)^{\text{T}}(\hat{S}^{(-m)} - \hat{S}^{(-m, \mathcal{E})})]_j \\ &= \sum_i (\hat{\pi}_i^{(-m)} - \pi_i) (\hat{S}_{ij}^{(-m)} - \hat{S}_{ij}^{(-m, \mathcal{E})}) \\ &= -(\hat{\pi}_j^{(-m)} - \pi_j) (\hat{S}_{j,m}^{(-m)} - \hat{S}_{j,m}^{(-m, \mathcal{E})}) \\ &\quad + (\hat{\pi}_m^{(-m)} - \pi_m) (\hat{S}_{m,j}^{(-m)} - \hat{S}_{m,j}^{(-m, \mathcal{E})}). \end{aligned}$$

Since for (j, m) in \mathcal{E} , we have $|\hat{S}_{j,m}^{(-m)} - \hat{S}_{j,m}^{(-m, \mathcal{E})}| \leq 1/d$ and for $(j, m) \notin \mathcal{E}$, we have $|\hat{S}_{j,m}^{(-m)} - \hat{S}_{j,m}^{(-m, \mathcal{E})}| \leq \frac{q_{jm}}{d}$. Therefore, we have that

$$\begin{aligned} &\|[(\hat{\pi}^{(-m)} - \hat{\pi})^{\text{T}}(\hat{S}^{(-m)} - \hat{S}^{(-m, \mathcal{E})})]_j\| \\ &\leq \begin{cases} \frac{2}{d} \|\hat{\pi}^{(-m)} - \pi\|_\infty, & \text{if } (j, m) \in \mathcal{E} \\ \frac{2q_{jm}}{d} \|\hat{\pi}^{(-m)} - \pi\|_\infty, & \text{otherwise} \end{cases} \end{aligned}$$

Since we are given that $\hat{S}_{m,j}^{(-m)} - \hat{S}_{m,j}^{(-m, \mathcal{E})} = \frac{p_{mj}}{d} (q_{mj} - \mathbb{1}_{(m,j) \in \mathcal{E}})$, we obtain

$$J_2 = \sum_{j:j \neq m} (\hat{\pi}_m^{(-m)} - \hat{\pi}_m) \frac{p_{mj}}{d} (q_{mj} - \mathbb{1}_{(m,j) \in \mathcal{E}}). \quad (24)$$

Applying Hoeffdings inequality, we obtain

$$|J_2| \lesssim \frac{\sqrt{np \log n} + \log n}{d} \|\hat{\pi}^{(-m)} - \hat{\pi}\|_\infty$$

with high probability. Combining the above bounds, we derive

$$J_2 \lesssim \left(\frac{\sqrt{np \log n} + \log n}{d} + \frac{p\sqrt{n}}{d} + \frac{\sqrt{d}}{d} \right) \|\hat{\pi}^{(-m)} - \pi\|_\infty$$

By combining all components Eq. (23) and Eq. (24), we establish that

$$\|\hat{\pi}^{(-m)} - \hat{\pi}\|_2 \leq \frac{\sqrt{\kappa}}{\gamma} (J_1 + J_2).$$

Simplifying the above expression, we obtain

$$\begin{aligned} &\leq \frac{\sqrt{h}}{\gamma} \left(8\sqrt{\frac{\log n}{kd}} \|\hat{\pi}^{(-m)}\|_\infty \right. \\ &\quad \left. + C \left(\frac{\sqrt{np \log n} + \log n}{d} \right) \|\hat{\pi}^{(-m)} - \pi\|_\infty \right) \\ &\leq \frac{\sqrt{h}}{\gamma} \left(8\sqrt{\frac{\log n}{kd}} \|\hat{\pi}\|_\infty \right. \\ &\quad \left. + C \left(8\sqrt{\frac{\log n}{kd}} + \frac{\sqrt{np \log n} + \log n}{d} \right) \|\hat{\pi}^{(-m)} - \pi\|_\infty \right) \\ &\leq \frac{\sqrt{h}}{\gamma} \sqrt{\frac{\log n}{kd}} \|\hat{\pi}\|_\infty + \frac{1}{2} \|\hat{\pi}^{(-m)} - \pi\|_\infty, \end{aligned}$$

where ζ holds as long as $np\gamma^2 \geq ch \log n$ for sufficiently large constant c . Using the triangle inequality

$$\|\hat{\pi}^{(-m)} - \hat{\pi}\|_\infty \leq \|\hat{\pi}^{(-m)} - \hat{\pi}\|_2 + \|\hat{\pi} - \pi\|_\infty,$$

we obtain the following bound

$$\|\hat{\pi}^{(-m)} - \hat{\pi}\|_2 \leq \frac{16\sqrt{\kappa}}{\gamma} \sqrt{\frac{\log n}{kd}} \|\pi\|_\infty + \|\hat{\pi} - \pi\|_\infty,$$

which completes the proof. \square

VII. PROOF OF PROPOSITION 1

We will utilize the following lemma to prove the spectral gap condition of the NSSBM model.

Lemma 2 (Comparison theorem). *Let S, π and $\tilde{S}, \tilde{\pi}$ be reversible Markov chains on a finite set $[n]$ representing random walks on a graph, i.e. $S_{ij} = 0$ and $\tilde{S}_{ij} = 0$ if $(i, j) \notin \mathcal{E}$. Define $\alpha = \min_{(i,j) \in \mathcal{E}} \tilde{\pi}_i \tilde{S}_{ij} / \pi_i S_{ij}$ and $\beta = \max_i \tilde{\pi}_i / \pi_i$,*

$$\frac{1 - \lambda_2(S)}{1 - \lambda_2(\tilde{S})} \geq \frac{\alpha}{\beta}.$$

In order to apply Lemma 2, we will introduce the following empirical Markov matrix \tilde{P} with stationary distribution $\tilde{\pi} = 1/n$, and is defined as

$$\tilde{S}_{ij} = \begin{cases} \frac{1}{2d} A_{ij}, & \text{if } (i, j) \in \mathcal{E} \\ 0, & \text{otherwise} \end{cases} \quad (25)$$

where $d \geq d_{\max}$ the maximum degree of the graph. Applying Lemma 2, we obtain

$$\frac{1 - \lambda_2(S)}{1 - \lambda_2(\tilde{S})} \geq \frac{1}{2h^3}. \quad (26)$$

Bounding the spectral gap of \tilde{S} using [35, Lemma 18] yields the lemma. \square

VIII. PROOF OF THEOREM 2

We begin by defining the following quantities

$$L_\pi^w \triangleq \sum_{j>i} \frac{w_{ij} \pi_i \pi_j}{\pi_i + \pi_j} (e_i - e_j)(e_i - e_j)^T, \quad (27)$$

$$d_{\min}^\pi \triangleq \min_{i \in [n]} \sum_{j: (i,j) \in \mathcal{E}} \frac{\pi_i \pi_j}{\pi_i + \pi_j} w_{ij}. \quad (28)$$

We also define the matrix \hat{L}_π^w as

$$(\hat{L}_\pi^w)_{ij} \triangleq \begin{cases} -\hat{p}_{ji} \pi_j w_{ji}, & \text{if } (i, j) \in \mathcal{E} \\ \sum_{j: (i,j) \in \mathcal{E}} \hat{p}_{ji} \pi_j w_{ji}, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}. \quad (29)$$

Recall the definition of δ_i from Eq. (12), we begin by observing that

$$\begin{aligned} \frac{\hat{\pi}_i - \pi_i}{\pi_i} &= \frac{\bar{\pi}_i - \pi_i}{\pi_i} + \delta_i \\ &= \frac{\sum_{j:j \neq i} w_{ij} (\hat{p}_{ji} \pi_j - \hat{p}_{ij} \pi_i)}{\pi_i \sum_{j:j \neq i} w_{ij} \hat{p}_{ij}} + \delta_i \end{aligned} \quad (30)$$

Bounding the numerator of the first term in Eq. (30) using the Hoeffding inequality yields the following bound with high probability

$$\begin{aligned} \left| \sum_{j:j \neq i} w_{ij} (\hat{p}_{ji} \pi_j - \hat{p}_{ij} \pi_i) \right| &\leq \|\pi\|_\infty \sqrt{\frac{\log n}{k}} \sqrt{\sum_{j:j \neq i} w_{ij}^2} \\ &\leq \|\pi\|_\infty \sqrt{\frac{\log n}{k}} \sqrt{d_{\max}} \sqrt{w_{\max}}. \end{aligned}$$

We bound the denominator of the first-term in Eq. (30) using the following lemma

Lemma 3. *Let $\mathcal{G} = ([n], \mathcal{E}, \{w_{ij}\}_{(i,j) \in \mathcal{E}})$ be a weighted graph. We have*

$$\lambda_{n-1}(L_\pi^w) \leq 2d_{\min}^\pi.$$

Utilizing the lemma, we have the denominator term is lower bounded by $\lambda_{n-1}(L_\pi^w)/4$, if $\lambda_{n-1}(L_\pi^w)^2 k \geq d_{\min}^\pi \|\pi\|_\infty^2 w_{\max} \log n$ as shown below

$$\begin{aligned} \pi_i \sum_{j:j \neq i} w_{ij} \hat{p}_{ij} &= d_{\min}^\pi + \pi_i \sum_{j:j \neq i} w_{ij} (\hat{p}_{ij} - p_{ij}) \\ &\geq d_{\min}^\pi - \|\pi\|_\infty \sqrt{w_{\max}} \sqrt{\frac{\sum_{j:j \neq i} w_{ij}}{k}} \\ &\geq \frac{\lambda_{n-1}(L_\pi^w)}{4}. \end{aligned}$$

Now, we will focus on the second term δ_i in Eq. (30). Recall that

$$\begin{aligned} \delta_i &= \frac{\hat{\pi}_i - \bar{\pi}_i}{\pi_i} = \frac{\sum_{j:j \neq i} w_{ij} \hat{p}_{ji} (\hat{\pi}_j - \pi_j)}{\pi_i \sum_{j:j \neq i} \hat{p}_{ij} w_{ij}} \\ &= \frac{\sum_{j:j \neq i} w_{ij} \hat{p}_{ji} \pi_j \delta_j}{\pi_i \sum_{j:j \neq i} \hat{p}_{ij} w_{ij}} + \frac{\sum_{j:j \neq i} w_{ij} \hat{p}_{ji} (\bar{\pi}_j - \pi_j^*)}{\pi_i \sum_{j:j \neq i} w_{ij} \hat{p}_{ij}}. \end{aligned} \quad (31)$$

Define a vector $r \in \mathbb{R}^n$, such that $r_i = \sum_{j:j \neq i} w_{ji} \hat{p}_{ji} (\bar{\pi}_j - \pi_j^*)$ and also define $\hat{L}^w \in \mathbb{R}^{n \times n}$ such that

$$\hat{L}_{ij}^w = \begin{cases} -\hat{p}_{ji} \pi_j w_{ji} & \text{for } i \neq j, (i, j) \in \mathcal{E} \\ \sum_{j:j \neq i} \hat{p}_{ij} \pi_i w_{ij} & \text{if } i = j \end{cases}.$$

Also, note that $\mathbb{E}[\hat{L}^w] = L^w$. Then the above equation can be written as

$$\hat{L}_\pi^w \delta = r. \quad (32)$$

Taking ℓ^2 -norm on both sides and an application of triangle inequality yields

$$\begin{aligned} \|r\|_2 &= \|\hat{L}_\pi^w \delta\|_2 \geq \|L^w \delta\|_2 - \|L^w - \hat{L}_\pi^w\|_2 \|\delta\|_2 \\ &\geq \lambda_{n-1}(L_\pi^w) \left(\|\delta\|_2 - \frac{|\delta^T \mathbf{1}|}{\sqrt{n}} \right) - \|L_\pi^w - \hat{L}_\pi^w\|_2 \|\delta\|_2 \end{aligned}$$

This gives a bound on $\|\delta\|_2$ as

$$\|\delta\|_2 \leq \frac{\|r\|_2}{\lambda_{n-1}(L_\pi^w)} + \frac{|\delta^T \mathbf{1}|}{\sqrt{n}}, \quad (33)$$

provided $\|L_\pi^w - \hat{L}_\pi^w\|_2 \leq \frac{\lambda_{n-1}(L_\pi^w)}{2}$. Now we will bound the term as $\|r\|_2 \leq \sqrt{n} \|r\|_\infty$. Below, we bound the term $\|r\|_\infty$.

Bounding $\|r\|_\infty$: Now we focus on bounding $\|r\|_\infty$

$$\begin{aligned} r_i &= \sum_{j:j \neq i} w_{ij} \hat{p}_{ij} (\bar{\pi}_j - \pi_j) = \underbrace{\sum_{j:j \neq i} w_{ij} \hat{p}_{ij} (\bar{\pi}_j - \bar{\pi}_j^{(-i)})}_{J_1} \\ &\quad + \underbrace{\sum_{j:j \neq i} w_{ij} \hat{p}_{ij} (\bar{\pi}_j^{(-i)} - \pi_j)}_{J_2}, \end{aligned} \quad (34)$$

where we define $\bar{\pi}_j^{(-i)}$ as the leave-one-out version of $\bar{\pi}_j$.

$$\bar{\pi}_j^{(-i)} = \frac{\sum_{l \in [n] \setminus \{i, j\}} w_{lj} \hat{p}_{lj} \pi_l + w_{ij} \hat{p}_{ij} \pi_i}{\sum_{l \in [n] \setminus \{i, j\}} w_{jl} \hat{p}_{jl} + w_{ji} \hat{p}_{ji}}.$$

Now we will focus on bounding the term J_1 in Eq. (34). We have

$$\begin{aligned} & \left| \sum_{j:j \neq i} w_{ji} \hat{p}_{ji} (\bar{\pi}_j - \bar{\pi}_j^{(-i)}) \right| \leq \\ & \left| \sum_{j:j \neq i} w_{ji} \hat{p}_{ji} \left(\frac{\sum_{l \in [n] \setminus \{i, j\}} \hat{p}_{lj} w_{lj} \pi_l + \hat{p}_{ij} w_{ij} \pi_i}{\sum_{l \in [n] \setminus \{i, j\}} \hat{p}_{jl} + \hat{p}_{ji} w_{ji}} - \frac{\sum_{l \in [n] \setminus \{i, j\}} \hat{p}_{lj} w_{lj} \pi_l + p_{ij} w_{ij} \pi_i}{\sum_{l \in [n] \setminus \{i, j\}} \hat{p}_{jl} + p_{ji} w_{ji}} \right) \right| \\ & \lesssim \left| \sum_{j:j \neq i} \frac{w_{ji}^3 \hat{p}_{ji} (\hat{p}_{ji} p_{ij} - \hat{p}_{ij} p_{ji}) \pi_i}{(\sum_{l \in [n] \setminus \{j\}} w_{jl} \hat{p}_{jl}) (\sum_{l \in [n] \setminus \{i, j\}} w_{jl} \hat{p}_{jl} + p_{ji} w_{ji})} \right| \\ & \quad + \left| \sum_{j:j \neq i} \frac{w_{ji}^2 \hat{p}_{ji} (\sum_{l \in [n] \setminus \{i, j\}} w_{lj} (\hat{p}_{lj} \pi_l - \pi_i \hat{p}_{jl}))}{(\sum_{l \in [n] \setminus \{i\}} w_{jl} \hat{p}_{jl}) (\sum_{l \in [n] \setminus \{i, j\}} w_{jl} \hat{p}_{jl} + p_{ji} w_{ji})} \right| \\ & \lesssim \frac{\sum_{j:j \neq i} w_{ij}^3 \|\pi\|_\infty^3}{\lambda^2(L_\pi^w)} \sqrt{\frac{\log n}{k}} + \\ & \quad \frac{\sum_{j:j \neq i} w_{ij}^2 \sqrt{\frac{\log n}{k}} \|\pi\|_\infty^3 \sum_{l \in [n] \setminus \{i, j\}} w_{lj}}{\lambda^2(L_\pi^w)}. \end{aligned} \quad (35)$$

The other term J_2 in Eq. (34) can be bounded as

$$\begin{aligned} & \sum_{j:j \neq i} w_{ji} \hat{p}_{ji} (\pi_j^{(-i)} - \pi_j) \\ &= \sum_{j:j \neq i} \frac{w_{ji} \hat{p}_{ji} \left(\sum_{l \in [n] \setminus \{i, j\}} w_{ji} (\hat{p}_{lj} \pi_l - \pi_j \hat{p}_{jl}) \right)}{\sum_{l \in [n] \setminus \{i, j\}} w_{lj} \hat{p}_{lj}} \\ &\lesssim \frac{(w_{\max})^2 d_{\max}}{\lambda_{n-1}(L_\pi^w)} \sqrt{\frac{\log n}{k}} \|\pi\|_\infty^2. \end{aligned}$$

Combining the bound on J_1 and J_2 gives the following bound on $\|r\|_\infty$

$$\|r\|_\infty \lesssim \frac{w_{\max}^2 d_{\max} \|\pi\|_\infty^3}{\lambda_{n-1}(L_\pi^w)^2} \sqrt{\frac{\log(n)}{k}} + \frac{(w_{\max} \|\pi\|_\infty)^2}{\lambda_{n-1}(L_\pi^w)} \sqrt{\frac{d_{\max}}{k}}. \quad (36)$$

Substituting the bound on $\|r\|_\infty$ in Eq. (33) bounds, we obtain (assuming $w_{\max} \leq n$)

$$\|\delta\|_2 \leq \frac{\sqrt{n} w_{\max}^2 \|\pi\|_\infty^3 d_{\max}}{\lambda_{n-1}(L^w)^3} \sqrt{\frac{\log n}{k}} + \frac{(w_{\max} \|\pi\|_\infty)^2}{\lambda_{n-1}(L^w)^2} \sqrt{\frac{d_{\max}}{k}}.$$

Utilizing Eq. (31) and recognizing that $\lambda_{n-1}(L_\pi^w) \gtrsim c \|\pi\|_\infty \lambda_{n-1}(L^w)$ and $d_{\max}^\pi \leq d_{\max} \|\pi\|_\infty$, we have the following bound on $\|\delta\|_\infty$ as

$$\|\delta\|_\infty \lesssim \frac{\sqrt{n} w_{\max}^{5/2} d_{\max}^{5/2}}{\lambda_{n-1}(L^w)^4} \sqrt{\frac{\log n}{k}} + \frac{w_{\max}^{5/2} (d_{\max})^{3/2}}{\lambda_{n-1}(L^w)^3} \sqrt{\frac{\log n}{k}}$$

Finally, the proof follows by Eq. (30). \square