ℓ^∞-Error Bounds for Spectral Ranking Under Semi-Random Adversaries

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***Abstract*—THIS PAPER IS ELIGIBLE FOR THE STUDENT PAPER AWARD. The Bradley-Terry-Luce (BTL) model is one of the most widely used frameworks for ranking and analyzing preference data. Entry-wise guarantees for maximum likelihood techniques for parameter estimation of the BTL model have been extensively studied under various observation graph settings, including deterministic graphs, random graphs, and semi-random adversaries. However, entry-wise estimation guarantees for spectral methods remain less explored. This paper addresses this gap by developing entry-wise guarantees for spectral methods for BTL parameter estimation under semi-random adversaries. We first establish that the *rank centrality algorithm* achieves similar ℓ^∞-error bounds as uniform sampling under the assumption that the semi-random adversary preserves the spectral gap. Next, to overcome such assumptions, we introduce a new *weighted rank centrality algorithm* and derive corresponding entry-wise error bounds for semi-random adversaries that are comparable to those for uniform sampling. Finally, we provide simulation results that support our theoretical findings.**

I. Introduction

The analysis of pairwise comparison data is central to several applications, such as sports rankings, consumer preferences, and search engine algorithms. Among the most widely used models for such tasks is the *Bradley-Terry-Luce* (BTL) model [1]–[12], which assumes an underlying latent skill parameter α_i for each item i and models the probability of item i being preferred over item j as $\frac{\alpha_i}{\alpha_i+\alpha_j}$. Two prominent methods for estimating the parameters of the BTL model include spectral algorithms, such as the rank centrality algorithm [8], and the maximum likelihood estimation (MLE) method [13]. Minimax error bounds for both the methods particularly in terms of ℓ^2-error bounds, have been well established under various graphs structures, such as Erdős-Rényi graphs [9] and deterministic graphs under certain regularity assumptions [8], [13].

However, real-world datasets often deviate from idealized graph models, exhibiting heterogeneity and adversarial structures. Several studies have shown that many established algorithms such as matrix completion [14], sparse recovery [15] can fail under non-uniform sampling patterns. This has motivated the study of more flexible and robust frameworks that can handle non-uniform or adversarial observation patterns. To model such phenomenon, we adopt the *semi-random comparison graph* model, where a semi-random adversary independently selects each internal edge (i, j) independently with probability $q_{ij} \geq p$, where $p \in (0, 1]$, but q_{ij} may be chosen adversarially for each edge. Clearly, this framework offers more flexibility compared to uniform sampling while retaining a degree of randomness.

Entry-wise error bounds for the MLE method in the BTL model have been well established for various graph structures, including Erdős-Rényi graphs [9], [16], general graphs [17], [18], and semi-random settings [19]. In particular, [19] showed that optimizing a weighted log-likelihood can can handle semi-random adversaries and provide estimation guarantees similar to those under uniform sampling. However, entry-wise error bounds for spectral methods remain relatively underexplored, particularly in scenarios involving deterministic graphs or semi-random adversaries. Spectral ranking methods continue to be popular due to their simplicity and their close connection to other well-known algorithms such as spectral clustering [20] and PageRank [21]. Importantly, entry-wise error bounds enable quantifying the number of observation needed for top-K ranking [9], [19], multiple testing [22], and applications to fair-allocation problems [23].

Existing theoretical results on the entry-wise error bounds of spectral methods are primarily limited to uniform sampling (Erdős-Rényi graphs) with edge-sampling probability p satisfying $np \geq O(\log n)$ or deterministic comparison graphs with individual node-degree scaling as $\Omega(\sqrt{n})$. Extending these work, we study the robustness of spectral methods under the semi-random adversary model and show that the original rank centrality algorithm achieves the same entry-wise ℓ^∞ error bounds as uniform sampling, provided $np \geq O(\log n)$ holds and the semi-random adversary satisfy a spectral gap condition. We also provide a examples of a semi-random stochastic block models that satisfy this spectral gap condition. Additionally, to overcome the spectral gap requirement, we introduce and analyze a weighted rank centrality algorithm under a general semi-random adversary, showing that incorporating weights can help relax the spectral gap assumptions while maintaining same ℓ^∞-error bounds as uniform sampling provided that $np \geq O(\sqrt{n})$ holds. Furthermore, we also present an efficient weight selection algorithm to select the weights and support our theoretical results with numerical simulations.

The author ordering is alphabetical.

A. Related Work

The BTL model [1]–[3] is probably the most widely adopted framework for analyzing pairwise comparison data across various domains, such as sports [24], [25], psychology [26], analyzing team performance over time [27], investigating market and sports competitiveness [11], [12]. Several techniques have been developed for estimating the parameters of the BTL model, with the most prominent being maximum likelihood estimation [4], [5], spectral methods [8], [28], least-squares approaches [10], and non-parametric techniques [27], [29].

Several works have focused on deriving the error bounds for the BTL model under various conditions. Early work in this area has focused on establishing ℓ^2-error bounds for MLE [9], [13] and spectral methods [8], [28] under both deterministic and Erdős-Rényi random graphs. Further works have explored stronger error bounds, such as ℓ^∞-error bounds, under various settings. For example, [9] establishes ℓ^∞-error bounds for the MLE method in the context of Erdős-Rényi graphs. [18] estabilised entry-wise error bounds for the MLE method under general fixed graphs. Moreover, [19] established entry-wise bounds for the MLE method in the presence of semi-random adversaries, showing that by optimizing a weighted log-likelihood, one can achieve similar estimation guarantees to those of uniform sampling. For spectral methods, [9] also provides entry-wise error bound for Erdős-Rényi graphs. [17] studies spectral methods for deterministic graphs, providing entry-wise error bounds under conditions where each node degree scales as $\Omega(\sqrt{n})$. Additionally, [16] establishes both entry-wise error bounds and provides uncertainity uncertainty quantification technique for both spectral and MLE methods under Erdős-Rényi graphs.

Our work can also be viewed from the lens of robustness as we try to the study the robustness of error bounds to various assumptions. In this context, [30] examines the impact of Byzantine voters on spectral ranking algorithms, showing that these methods fail when even small fractions of data are adversarially corrupted and propose a more robust spectral ranking algorithm in presence of Byzantine voters. Similarly, [31], [32] explores hypothesis testing for the BTL model and provide estimation error bound when the data may be generated by a more general pairwise comparison framework rather than a BTL model.

B. Notation

We use the following notation throughout the paper. Let $[n] = \{1, 2, \ldots, n\}$ denote the set of integers from 1 to n. Let $\|\cdot\|_2$ represents the ℓ^2-norm for vectors and the spectral norm (operator norm) for matrices, while $\|\cdot\|_\infty$ denotes the ℓ^∞-norm for vectors. Let $\mathrm{diag}(x) \in \mathbb{R}^{n\times n}$ is the diagonal matrix with $x \in \mathbb{R}^n$ along its principal diagonal. The all-ones vector of length n is denoted by $\mathbf{1}_n$. For a matrix A, $\lambda_l(A)$ refers to the l-th largest eigenvalue of A in absolute value. We employ standard Bachmann-Landau asymptotic notation, e.g., $O(\cdot)$, where the parameter $n \to \infty$ or $k \to \infty$, with n representing the number of items and k the number of pairwise comparisons per pair of items. Throughout this paper, we use the term "high probability" to describe an event that occurs with probability at least $1 - 1/\mathrm{poly}(n)$, where $\mathrm{poly}(n)$ denotes a polynomial function of n.

II. Problem Formulation

We consider a set of n items/agents, denoted by $[n]$, which are involved in a series of pairwise comparisons. Each comparison corresponds to an observation between two distinct items i and j from $[n]$. These comparisons are represented by an undirected observation graph $\mathcal{G} = ([n], \mathcal{E})$, where the vertices correspond to the items, and the edges between the corresponding items indicate whether those items have been compared at least once. We assume an underlying semi-random adversary that selects each edge (i, j) with $j > i$ independently with probability q_{ij}, where $q_{ij} \in [p, 1]$ for $p \in (0, 1]$. The selection of edges is independent of the outcomes of any observations. For notational convenience, we set $q_{ij} = q_{ji}$. We emphasize that we assume $np \geq c_0 \log n$, for some constant constant $c_0 \geq 1$. This also ensures that the graph $\mathcal{G}$ is connected with high probability.

We denote the probability of j being preferred over i in a given comparison by p_{ij}, where $p_{ij} \in (0, 1)$, and by our BTL model, we have:

$$p_{ji} = \frac{\alpha_j}{\alpha_i + \alpha_j}, \forall (i, j) \in \mathcal{E} \tag{1}$$

where $\alpha_i \in \mathbb{R}_+$ denotes the underlying skill parameters for item $i \in [n]$. Assume that the ratio of the skill scores are upper bounded by a constant h, such that:

$$\frac{\max_{j\in[n]} \alpha_j}{\min_{i\in[n]} \alpha_i} \leq h. \tag{2}$$

We can collect these probabilities to define the pairwise comparison matrix $P \in [0, 1]^{n\times n}$, where for each $(i, j) \in \mathcal{E}$, the entry $P_{ij} = p_{ij}$, and 0 otherwise. Next, we introduce the canonical Markov matrix $S \in [0, 1]^{n\times n}$, which is a row-stochastic matrix derived from the pairwise comparison matrix P. The entries of S are defined as follows:

$$S_{ij} \triangleq \begin{cases} \frac{p_{ij}}{d}, & \text{if } i \neq j \text{ and } (i, j) \in \mathcal{E}, \\ 1 - \frac{1}{d}\sum_{l:(i,l)\in\mathcal{E}} p_{il}, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

where $d \geq d_{\max}$, and $d_{\max}$ denotes the maximum degree of the graph. Under the assumption that the graph is connected and the Markov chain is aperiodic (since $S_{ii} > 0$), the matrix S has a unique stationary distribution π [33]. Moreover, this distribution is given by:

$$\pi = \frac{1}{\sum_{i\in[n]} \alpha_i}[\alpha_1, \ldots, \alpha_n]. \tag{4}$$

Let k denote the number of observed comparisons for each pair $(i, j) \in \mathcal{E}$. For simplicity, we assume that the number of comparisons is identical across different item pairs; however, our analysis can be extended to the case where the number of comparisons varies across pairs. Each observation corresponds to an independent comparison between items $(i, j) \in \mathcal{E}$ and

is modeled as a sequence of independent Bernoulli random variables $Z_{ij}^{(m)} \sim \text{Bernoulli}(p_{ij})$, where $Z_{ij}^{(m)} = 1$ if item j is preferred over item i in the m-th comparison, and $Z_{ij}^{(m)} = 0$ otherwise. Let Z_{ij} denote the total number of times item j is preferred over i and is given by $Z_{ij} = \sum_{m=1}^{k_{ij}} Z_{ij}^{(m)}$. Let $\hat{p}_{ij} = Z_{ij}/k$ denote the empirical probability of j being preferred over i. The empirical Markov matrix $\hat{S}$, computed from the observed pairwise comparison data, is given by

$$\hat{S}_{ij} \triangleq \begin{cases} \frac{\hat{p}_{ij}}{d}, & \text{if } i \neq j \text{ and } (i,j) \in \mathcal{E}, \\ 1 - \frac{1}{d}\sum_{l:(i,l)\in\mathcal{E}} \hat{p}_{il}, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

The rank centrality method [8] estimates the parameters of the BTL model by computing the stationary distribution $\hat{\pi}$ of the Markov chain induced by $\hat{S}$ (or selecting one arbitarily, if it is not unique).

III. Entry-wise Error Bounds for Rank Centrality

In this section, we establish entry-wise error bounds for the rank centrality algorithm applied to an observation graph generated by a semi-random adversary. Our results emphasize the impact of the spectral gap $1 - \lambda_2(S)$ on the algorithm's accuracy, where S is the canonincal Markov matrix corresponding to the observation graph generated by the semi-random model. Specifically, we show that the rank centrality algorithm, without modifications such as trimming or re-weighting, maintains its entry-wise error bounds when the graph generation process preserves a sufficient spectral gap. The following theorem highlights the impact of spectral gap on the accuracy of the rank centrality algorithm.

Theorem 1 (Spectral Gap and Entry-wise Error Bound)**.** *Let S be the canonical Markov matrix corresponding to the observation graph generated by a semi-random model with edge probabilities $q_{ij} \in [p, 1]$. Assume that the following spectral gap condition holds:*

$$\mathbb{P}(1 - \lambda_2(S) \leq \gamma) \leq \frac{1}{n^5}$$

for some $\gamma > 0$. Then, there exist constants $c_0 > 1, c_1, c_2 > 0$ such that for all p satisfying $np\gamma^2 \geq c_0 \log n$, the estimated parameters $\hat{\pi}$ returned by the rank centrality algorithm satisfy

$$\frac{\|\hat{\pi} - \pi\|_\infty}{\|\pi\|_\infty} \leq \frac{c_1}{\gamma}\sqrt{\frac{\log n}{npk}} + c_2\sqrt{\frac{\log n}{npk}}$$

with probability at least $1 - O(1/n^5)$.

The proof is provided in [34]. In particular, the result highlights that when the spectral gap is lower bounded by a constant (with high probability), the resulting error bounds are consistent with those obtained in the case of uniform sampling in [9]. Since it is well known that an Erdős–Rényi graph preserves the spectral gap with high probability, the following corollary recovers the results of [9] under uniform sampling.

Corollary 1 (Error Bound under Erdős–Rényi Graphs)**.** *Consider the setting in Section II, where the comparison graph follows an Erdős–Rényi model with edge-sampling probability p. Then there exist constants $c_0 > 1, c_1 > 0$ such that for $np \geq c_0 \log n$, with probability at least $1 - O(1/n^5)$, the estimate $\hat{\pi}$ returned by the rank-centrality algorithm satisfies the following error bound:*

$$\frac{\|\hat{\pi} - \pi\|_\infty}{\|\pi\|_\infty} \leq c_1\sqrt{\frac{\log n}{npk}}.$$

Building on this result, we now analyze the entry-wise error bound under the nonhomogeneous symmetric stochastic block model (NSSBM), as studied in [35], in the context of spectral clustering under semi-random adversaries. The NSSBM is defined as follows:

Definition 1 (Nonhomogeneous Symmetric Stochastic Block Model [35])**.** *Let n be an positive integer, and let $\{V_1, V_2\}$ be a partitioning of $[2n]$ into two equally-sized subsets. Given parameters satisfying $q < p \leq \bar{p}$, assume that the observation graph is generated such that for every $(i,j) \in V_1 \times V_1$ and $(i,j) \in V_2 \times V_2$, the edge (i,j) appears independently with probability $q_{ij} \in [p, \bar{p}]$, and for every $(i,j) \in V_1 \times V_2$, the edge (i,j) appears independently with probability q. The family of all such models is denoted as NSSBM$(n, p, \bar{p}, q)$.*

To better visualize the NSSBM model, we can examine the expected adjacency matrix of an NSSBM model:

$$\begin{bmatrix} p\cdot\mathbf{1}_n & q\cdot\mathbf{1}_n \\ q\cdot\mathbf{1}_n & p\cdot\mathbf{1}_n \end{bmatrix} \leq \begin{bmatrix} Q_{V_1} & q\cdot\mathbf{1}_n \\ q\cdot\mathbf{1}_n & Q_{V_2} \end{bmatrix} \leq \begin{bmatrix} \bar{p}\cdot\mathbf{1}_n & q\cdot\mathbf{1}_n \\ q\cdot\mathbf{1}_n & \bar{p}\cdot\mathbf{1}_n \end{bmatrix}, \tag{6}$$

where Q_{V_1} and Q_{V_2} denote the edge probability matrices for edges internal to V_1 and V_2. The following proposition, proved in [34], establishes that under a spectral gap condition, which is based on the parameters of the NSSBM, the rank centrality algorithm achieves the same error bound as in the case of uniform sampling.

Proposition 1 (Bound under NSSBM model)**.** *Under the setting of Theorem 1, if the observation graph follows an NSSBM model with parameters $(n, p, \bar{p}, q)$. Then, there exists constants $c, C > 0$ such that for the choice of parameters satisfying the condition:*

$$n(p - q) \geq C\Big(\sqrt{n\bar{p}\log n} + \log n\Big), \tag{7}$$

the rank centrality algorithm returns estimate $\hat{\pi}$ that satisfies the error bound:

$$\frac{\|\hat{\pi} - \pi\|_\infty}{\|\pi\|_\infty} \leq \frac{c}{\sqrt{npk}},$$

with probability at least $1 - O(1/n^5)$.

We remarked that the condition in Eq. (7) ensures that the spectral gap is lower bounded by a constant and therefore by Theorem 1 the rank-centrality algorithm achieves similar error bounds as uniform sampling. We obtain bound on spectral gap by utilizing the comparison theorems for reversible Markov chains [36]. Interestingly, setting $\bar{p} = 1$ and $nq = o(\sqrt{n})$, we have that $np = O(\sqrt{n \log n})$, which matches degree

requirements of $\Omega(\sqrt{n})$ needed in the analysis of [37] for deterministic graphs. Notably, our result in Proposition 1 parallels that of [35, Theorem 1], which established the robustness of unnormalized spectral clustering algorithm for the NSSBM model under the same condition.

We note that Theorem 1 emphasizes the significance of the spectral gap condition. For the rank centrality algorithm, this condition also appears in the ℓ^2-error bounds for deterministic graphs [8] and also a related spectral assumption is needed in [37], to establish entry-wise error bounds. Interestingly, in the context of the maximum likelihood method, the corresponding error bounds depend on the second-largest eigenvalue of the Laplacian corresponding to the observation graph, both in ℓ^2 [13] and ℓ^∞-error bounds [17]. Recent work by [19] optimized a weighted log-likelihood function where the weights under a semi-random adversary are chosen such that the spectral properties of the Laplacian resemble those of random graphs. Notably, our simulation results also demonstrate that a general semi-random adversary model might not preserve the spectral gap condition (see Fig. 1). This motivates the exploration of a weighted approach in the context of spectral algorithms. In the following section, we introduce a corresponding weighted rank centrality algorithm and analyze its entry-wise error bounds under the semi-random adversary setting.

IV. Weighted Rank Centrality

In this section, we introduce a weighted variant of the rank centrality algorithm, where non-negative weights w_{ij}, potentially dependent on the observation graph, are assigned to comparison probabilities p_{ij}. These weights aim to reweight the semi-random graph and potentially improving its spectral properties. We begin by defining some key quantities central to our analysis. Let $\mathcal{E}$ denote the edge set selected by the semi-random adversary, and let w_{ij} be the non-negative weight assigned to the edge $(i,j) \in \mathcal{E}$. We assume that the weights are symmetric, i.e., $w_{ij} = w_{ji}$ for all $(i,j) \in \mathcal{E}$, and $w_{ij} = 0$ for edges not present in the observation graph.

Using these weights, we define the weighted Markov matrix S, where the transition probabilities are adjusted to incorporate the given weights. Specifically, the matrix S is defined as:

$$S_{ij} \triangleq \begin{cases} \frac{w_{ij}p_{ij}}{d}, & \text{if } i \neq j \text{ and } (i,j) \in \mathcal{E}, \\ 1 - \frac{1}{d}\sum_{l:(i,l)\in\mathcal{E}} w_{il}p_{il}, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

where d is a normalization factor satisfying $d \geq d_{\max} \triangleq \max_{i\in[n]} \sum_{j:(i,j)\in\mathcal{E}} w_{ij}$, ensuring that S is a valid row-stochastic matrix. Note that this definition of S extends the unweighted version from the previous section. Let π be the stationary distribution of S. Observe that as long as the weights are symmetric, the stationary distribution π of S remains unchanged and is given by Eq. (4). We also define a related quantity L^w, the weighted Laplacian matrix as:

$$L^w \triangleq \sum_{(i,j)\in\mathcal{E}:j>i} w_{ij}(e_i - e_j)(e_i - e_j)^{\mathrm{T}}, \tag{9}$$

where e_i is the standard basis vector. Additionally, $w_{\max} \triangleq \max_{(i,j)\in\mathcal{E}} w_{ij}$ represents the maximum edge weight in the graph and let $d_{\min} \triangleq \min_{i\in[n]} \sum_{j:(i,j)\in\mathcal{E}} w_{ij}$.

Similar to S, we define the weighted empirical Markov matrix $\hat{S}$, computed from the observed data as:

$$\hat{S}_{ij} \triangleq \begin{cases} \frac{w_{ij}\hat{p}_{ij}}{d}, & \text{if } i \neq j \text{ and } (i,j) \in \mathcal{E}, \\ 1 - \frac{1}{d}\sum_{l:(i,l)\in\mathcal{E}} w_{il}\hat{p}_{il}, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

The weighted rank centrality algorithm estimates the parameters by computing the stationary distribution $\hat{\pi}$ of the weighted empirical Markov matrix $\hat{S}$.

A. Error Bounds for Weighted Rank Centrality

In this section, we present the ℓ^∞-error bounds for the weighted rank centrality algorithm when applied to graphs generated by a semi-random adversary. We begin by analyzing the performance of the algorithm under a given set of weights and then discuss a method for efficiently estimating these weights to optimize the algorithm's performance.

Unlike the standard rank centrality algorithm, the analysis here cannot rely on a "row-concentration" based property [38], which was crucial in the proof of the unweighted case. Instead, we derive the following error bound by assuming the observation graph to be a deterministic graph:

Theorem 2 (Error Bounds for Weighted Rank Centrality)**.** *Suppose that the weighted graph $\mathcal{G} = ([n], \mathcal{E}, \{w_{ij}\}_{(i,j)\in\mathcal{E}})$ is connected. Assume that $w_{\max} \leq n$ and $d_{\min} \geq 1$. Further suppose that for a constant c, we have $\lambda_{n-1}(L^w)^2 k \geq c d_{\max} w_{\max} \log n$, where $\lambda_{n-1}(L^w)$ is the second smallest eigenvalue of L^w. Then, there exists a constant C such that we have the following error bound*

$$\begin{aligned} \frac{\|\hat{\pi} - \pi\|_\infty}{\|\pi\|_\infty} \leq C\Bigg(& \frac{1}{\lambda_{n-1}(L^w)}\sqrt{\frac{w_{\max} d_{\max} \log n}{k}} \\ & + \sqrt{\frac{n d_{\max}^5 w_{\max}^3 \log n}{k}} \frac{1}{\lambda_{n-1}(L^w)^4} \\ & + \sqrt{\frac{n d_{\max}^3 w_{\max}^3 \log n}{k}} \frac{1}{\lambda_{n-1}(L^w)^3} \Bigg), \end{aligned}$$

with probability at least $1 - O(1/n^5)$.

The above theorem highlights the dependence of the bounds on the spectral properties of the weighted graph, such as its maximum degree, and the second smallest eigenvalue of the weighted graph Laplacian. To ensure these spectral properties are well-behaved, we impose the following constraints on the weights:

$$w_{\max} \leq 1, \tag{11a}$$

$$d_{\max} \leq 2np, \tag{11b}$$

$$\lambda_{n-1}(L^w) \geq cnp. \tag{11c}$$

These constraints ensure that the weights are bounded and that the spectral properties of the weighted graph resemble

those of a random graph. Notably, if the observation graph is sampled from an Erdős–Rényi random graph model, uniform weights $w_{ij} = 1$ satisfy all the three constraints with high probability. Interestingly, these constraints also arise while analyzing the weighted non-negative likelihood, as discussed in [19]. To compute the weights for a given observation graph, we leverage an algorithm based on matrix multiplicative weights updates (MMWU) from [19, Algorithm 2]. The goal of the algorithm is to maximize the spectral gap $\lambda_{n-1}(L^w)$ of the weighted Laplacian L^w while ensuring that the constraints Eq. (11a) and Eq. (11b) are satisfied. The algorithm computes weights $\{w_{ij}\}_{(i,j)\in\mathcal{E}}$ by formulating this problem as a saddle-point semidefinite program (SDP). The MMWU algorithm solves the SDP approximately solved in nearly-linear time in the size of the graph. By combining the output of the MMWU algorithm with Theorem 2, we obtain the following guarantee for the weighted rank centrality method under the semi-random adversary model.

Theorem 3 (Semi-Random Adversary ℓ^∞-Error Bound)**.** *Under the semi-random adversary model described in Section II, there exists constant c_0 and c_1 such that for $np \geq c_0\sqrt{n}$, the weighted rank centrality algorithm, using weights selected by the MMWU algorithm, returns an estimate $\hat{\pi}$ such that the following bound holds*

$$\frac{\|\hat{\pi} - \pi\|_\infty}{\|\pi\|_\infty} \leq c_1\left(\sqrt{\frac{\log n}{npk}} + \sqrt{\frac{n\log n}{(np)^3 k}}\right),$$

with probability at least $1 - O(1/n^5)$.

Clearly, the bounds in the above theorem become vacuous when $np = o(\sqrt{n})$, as the terms can grow unbounded in this regime. This highlights the necessity of the condition $np \geq c_0\sqrt{n}$ for the bound to be meaningful. Notably, even when the average degrees are much greater than $\sqrt{n}$, an adversary can construct observation graph structures where the spectral gap decays towards 0. In the following section, we provide an example where $np \geq \sqrt{n}$, yet the spectral gap decays. This demonstrates that even when $np \geq \sqrt{n}$, the semi-random adversary may not preserve the spectral gap, as assumed in Theorem 1. This validates the necessity of our theoretical results and shows the advantage of the weighted rank centrality algorithm over the unweighted rank centrality algorithm. A promising direction for future work can be to improve the requirement of $np \geq c_0\sqrt{n}$ needed in our analysis and to establish lower bounds under semi-random or deterministic graph models.

V. Experiments

In this section, we conduct two experiments to support our theoretical results. In the first experiment, we consider a set of $3n$ nodes divided into three communities of equal size. The observation graph is constructed such that community 1 and community 3 are fully connected, while community 2 and all inter-community edges are connected with probability $p = 1/(3n)^{0.45}$, where n takes values in $\{9, 18, 36, 72, 144, 288, 576\}$. We compute the spectral gap of the Markov matrix S when all skill scores are equal (i.e., $p_{ij} = 1/2$). Figure 1 illustrates the decay of the spectral gap, confirming that even for $p \geq O(1/\sqrt{n})$, the semi-random adversary may not preserve the spectral gap as assumed in Theorem 1. This demonstrates the usefulness of the weighted spectral ranking in the same regime.

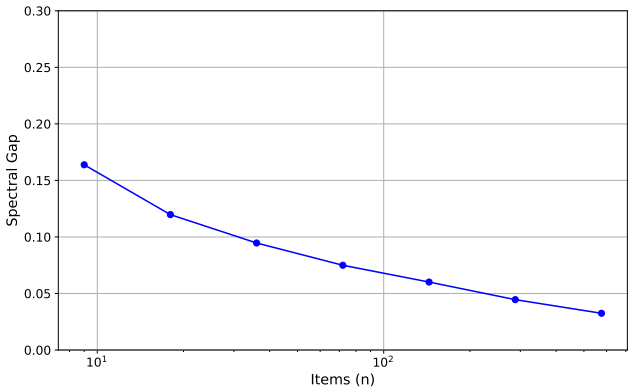

Fig. 1: Spectral gap $1 - \lambda_2(S)$ as a function of n.

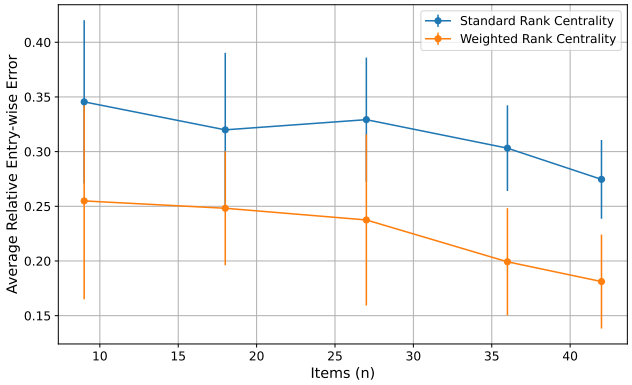

Fig. 2: Average relative ℓ^∞-error for weighted and standard rank centrality algorithm.

In the second experiment, we compare the performance of weighted rank centrality algorithm against the standard (unweighted) version using synthetic data generated from the BTL model. For the graph sampled from the same semi-random model as in the previous experiment, we consider $n \in \{9, 18, 27, 36, 42\}$. We optimize edge weights w_{ij} using the MMWU algorithm and generate synthetic comparisons with $k = 10$ samples per pair in the observation graph. We compute the relative ℓ^∞ error $\|\hat{\pi} - \pi\|_\infty / \|\pi\|_\infty$ between the estimated and true BTL parameters for both the methods. The results, averaged over 40 iterations, are shown in Fig. 2. Clearly, the figure shows that the weighted rank centrality method performs better than the standard method, achieving lower average error across all values of n.

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