PMATH352W18 Complex Analysis - Class Notes

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Chapter 1

Lecture 1 - Jan 3, 2018

1.1 Complex Numbers and Their Properties

Definition 1.1.1 (Complex Number, Complex Plane)
*A **complex number** is a vector in $\mathbb{R}^2$. The **complex plane**, denoted by $\mathbb{C}$, is a set of complex numbers,*

$$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

In $\mathbb{C}$, we usually write

$$\begin{aligned} 0 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & 1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ i &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} & x &= \begin{pmatrix} x \\ 0 \end{pmatrix} \\ iy &= \begin{pmatrix} 0 \\ y \end{pmatrix} \end{aligned}$$

where $x, y \in \mathbb{R}$. Consequently, we have that

$$x + iy = x + yi = \begin{pmatrix} x \\ y \end{pmatrix}$$

If for $x, y \in \mathbb{R}$, $z = x + iy$, then x is aclled the real part of z and y is called the imaginary part of z, and we write

$$\mathrm{Re}(z) = x \quad \mathrm{Im}(z) = y.$$

Definition 1.1.2 (Sum and Product)
We define the sum of two complex numbers to be the usual vector sum, i.e.

$$\begin{aligned}(a+ib)+(c+id) &= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \begin{pmatrix} a+c \\ b+d \end{pmatrix} \\ &= (a+c)+i(b+d)\end{aligned}$$

where $a, b, c, d \in \mathbb{R}$.

We define the product of two complex numbers by setting $i^2 = -1$, *and by requiring the product to be commutative, associative, and distributive over the sum. In this setup, we have that*

$$\begin{aligned}(a+ib)(c+id) &= ac + iad + ibc + i^2 bd \\ &= (ac-bd) + i(ad+bc) \qquad (1.1)\end{aligned}$$

Example 1.1.1
Let $z = 2+i, w = 1+3i$. *Find* $z+w$ *and* zw.

$$\begin{aligned}z+w &= (2+i)+(1+3i) \\ &= 3+4i\end{aligned}$$

$$\begin{aligned}zw &= (2+i)(1+3i) \\ &= (2-3)+i(6+1) \quad \textit{By Equation (1.1)} \\ &= -1+7i\end{aligned}$$

Example 1.1.2
Show that every non-zero complex number has a multiplicative inverse, z^{-1}, *and find a formula for this inverse.*

Let $z = a + ib$ where $a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$. Then

$$
\begin{aligned}
& z(x+iy) = 1 \\
\iff & (ax - by) + i(ay + bx) = 1 \\
\iff & \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\iff & \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\iff & \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\iff & \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a \\ -b \end{pmatrix} \\
\iff & x + iy = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}
\end{aligned}
$$

Therefore, we have that the formula for the inverse is

$$
(a + ib)^{-1} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2} \tag{1.2}
$$

Notation
For $z, w \in \mathbb{C}$, we write

$$
\begin{array}{ll}
-z = -1z & w - z = w + (-z) \\
\frac{1}{z} = z^{-1} & \frac{w}{z} = wz^{-1}
\end{array}
$$

Example 1.1.3
Find $\frac{(4-i)-(1-2i)}{1+2i}$.

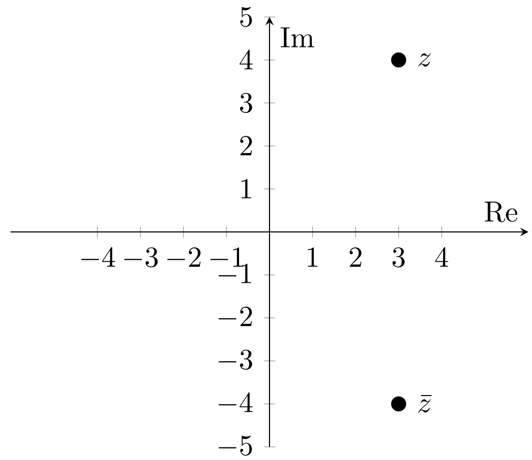
$$
\begin{aligned}
\frac{(4-i)-(1-2i)}{1+2i} &= \frac{3+i}{1+2i} \\
&= (3+i)(\frac{1}{5} - i\frac{2}{5}) \\
&= 1 - i
\end{aligned}
$$

Note
The set of complex numbers is a ***field*** *under the operations of additiona and multiplication. This means that $\forall u, v, w \in \mathbb{C}$,*

$$\begin{array}{cc}
u + v = v + u & uv = vu \\
(u + v) + w = u + (v + w) & (uv)w = u(vw) \\
0 + u = u & 1u = u \\
u + (-u) = 0 & uu^{-1} = 1,\ u \neq 0 \\
u(v + w) = uv + uw &
\end{array}$$

Definition 1.1.3 (Conjugate)
If $z = x + iy$ where $x, y \in \mathbb{R}$, then the ***conjugate of z*** *is given by $\bar{z} = x - iy$*

Example 1.1.4
Let $z = 3 + 4i$. Then the $\bar{z} = 3 - 4i$. Represented in the complex plane, we have the following:



Definition 1.1.4 (Modulus)
We define the ***modulus*** *(length, magnitude) of $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$, to be*

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}. \tag{1.3}$$

Note
For $z, w \in \mathbb{R}$, we have

$$\begin{array}{ccc}
\bar{\bar{z}} = z & z + \bar{z} = 2\,\mathrm{Re}(z) & z - \bar{z} = 2i\,\mathrm{Im}(z) \\
z\bar{z} = |z|^2 & |z| = |\bar{z}| & \overline{z \pm w} = \bar{z} \pm \bar{w} \\
\overline{zw} = \bar{z} - \bar{w} & |zw| = |z|\,|w| &
\end{array}$$

but note that $|z + w| \neq |z| + |w|$.

Proposition 1.1.1 (Basic Inequalities)

1. $|\mathrm{Re}(z)| \leq |z|$

2. $|\mathrm{Im}(z)| \leq |z|$

3. $|z+w| \leq |z| + |w|$ *Triangle Inequality*

4. $|z+w| \geq |\,|z| - |w|\,|$ *Inverse Triangle Inequality*

Proof
Note that $|z|^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$ *and that we can express* $|x| = \sqrt{x^2}$ *for any* $x \in \mathbb{R}$. *1 and 2 immediately follows from that.*

To prove 3, we have that

$$\begin{aligned}
|z+w|^2 &= (z+w)(\bar{z}+\bar{w}) \\
&= |z|^2 + |w|^2 + (w\bar{z} + \bar{w}z) \\
&= |z|^2 + |w|^2 + 2\,\mathrm{Re}(w\bar{z}) \\
&\leq |z|^2 + |w|^2 + 2\,|w\bar{z}| \quad \textit{by 1} \\
&= |z|^2 + |w|^2 + 2\,|wz| \quad \textit{since } |w\bar{z}| = |w|\,|\bar{z}| \textit{ and } |z| = |\bar{z}| \\
&= (|z| + |w|)^2
\end{aligned}$$

To prove 4, note that

$$|z| = |z+w-w| \leq |z+w| + |w| \tag{1.4}$$
$$|w| = |w+z-z| \leq |z+w| + |z| \tag{1.5}$$

Observe that

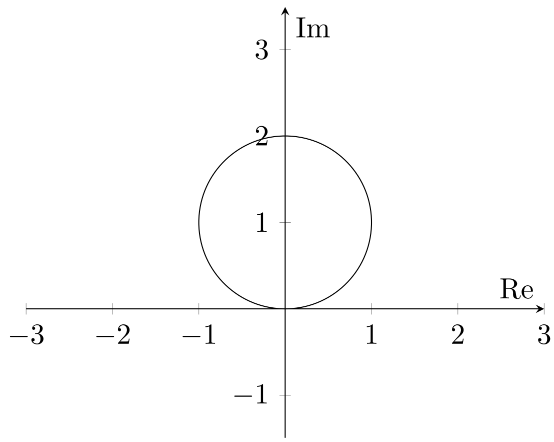
$$\begin{aligned}
\textit{Equation (1.4)} &\implies |z| - |w| \leq |z+w| \\
\textit{Equation (1.5)} &\implies |w| - |z| \leq |z+w|
\end{aligned}$$

Thus, we have that

$$|z+w| \geq |\,|z| - |w|\,|$$

as required. □

Example 1.1.5
We may describe a set $\{z \in \mathbb{C} : |z-i| = 1\}$ *as follows:*



Let $a, b \in \mathbb{C}$ *describe the set* $\{z \in \mathbb{C} : |z - a| < |z - b|\}$.