Foreword

Usage

- Notes are presented in two columns: main notes on the left, and sidenotes on the right. Main notes will have a larger margin.
- The following is the color code for the notes:

Blue	Definitions
Red	Important points
Yellow	Points to watch out for / comment for incompletion
Green	External definitions, theorems, etc.
Light Blue	Regular highlighting
Brown	Secondary highlighting

- The following is the color code for boxes, that begin and end with a line of the same color:

Blue	Definitions
Red	Warning
Yellow	Notes, remarks, etc.
Brown	Proofs
Magenta	Theorems, Propositions, Lemmas, etc.

- Hyperlinks are underlined in magenta. If your PDF reader supports it, you can follow the links to either be redirected to an external website, or a theorem, definition, etc., in the same document. Note that this is only reliable if you have the full set of notes as a single document, which you can find on: `https://japorized.github.io/TeX_notes`

13 Lecture 13 May 30 2018

13.1 Isomorphism Theorems (Continued)

13.1.1 Quotient Groups (Continued)

Proposition 35

Let $K \triangleleft G$ and write $G/K = \{Ka : a\, in G\}$ for the set of cosets of K.

1. *G/K is a group under the operation $KaKb = Kab$.*
2. *The mapping $\phi : G \to G/K$ given by $\phi(a) = Ka$ is a surjective homomorphism.*[1]
3. *If $[G : K]$ is finite, then $\left|G/K\right| = [G : K]$. In particular, if $|G|$ is finite, then $\left|G/K\right| = \frac{|G|}{|K|}$.*

[1]

Exercise 13.1.1
Is ϕ injective?

Solution
We know that we cannot uniquely express a coset, since for $a, b \in Ka$ such that $a \neq b$, we have that $Ka = Kb$.

Proof

1. *By Lemma 34, the operation is well-defined, and G/K is closed under the operation. The identity of G/K is $K = K(1)$ since $\forall Ka \in G/K$,*

$$KaK(1) = Ka = K(1)Ka.$$

Also, since

$$KaKa^{-1} = K(1) = Ka^{-1}Ka,$$

the inverse of Ka is Ka^{-1}. Finally, by associativity of G, we have that

$$Ka(KbKc) = Kabc = (KaKb)Kc.$$

It follows that G/K is a group.

2. *Clearly, ϕ is surjective. For $a, b \in G$,*

$$\phi(ab) = Kab = KaKb = \phi(a)\phi(b).$$

Thus ϕ is a surjective homomorphism.

3. *If $[G : K]$ is finite, then by definition of the index $[G : K]$, we have that $[G : K] = \left| {}^{G}\!/_{K} \right|$. Also, if $|G|$ is finite, then by Theorem 23,*

$$\left| {}^{G}\!/_{K} \right| = [G : K] = \frac{|G|}{|K|}.$$

$\square$

Definition 26 (Quotient Group)
*Let $K \lhd G$. The group ${}^{G}\!/_{K}$ of all cosets of K in G is called the **quotient group** of G by K. Also, the mapping*

$$\phi : G \to {}^{G}\!/_{K} \text{ defined by } a \mapsto Ka$$

*is called the **coset** (pr **quotient**) **map**.*

13.1.2 Isomorphism Theorems

Definition 27 (Kernel and Image)
*Let $\alpha : G \to H$ be a group homomorphism. The **kernel** of α is defined by*

$$\ker \alpha := \{ g \in G \, : \, \alpha(g) = 1_H \} \subseteq G$$

and the image of α is defined by

$$\operatorname{im} \alpha := \alpha(G) = \{ \alpha(g) \, : \, g \in G \} \subseteq H.$$

Proposition 36
Let $\alpha : G \to H$ be a group homomorphism.

1. $\operatorname{im} \alpha$ *is a subgroup of H*

2. $\ker \alpha \triangleleft G$

Proof

1. *Note that* $1_H = \alpha(1_G) \in \alpha(G)$ *(i.e. the identity is in* $\operatorname{im} \alpha$*). Also, for* $h_1 = \alpha(g_1)$ *and* $h_2 = \alpha(g_2)$ *in* $\alpha(G)$ *and* $h_1, h_2 \in H$*, we have*

$$h_1 h_2 = \alpha(g_1)\alpha(g_2) = \alpha(g_1 g_2) \in \alpha(G).$$

(i.e. $\operatorname{im} \alpha$ *i closed under its operation). By Proposition 20,* $\alpha(g)^{-1} = \alpha(g^{-1}) \in \alpha(G)$ *(i.e. the inverse of an element is also in* $\operatorname{im} \alpha$*). Thus by the Subgroup Test, we have that* $\operatorname{im} \alpha$ *is a subgroup of* H*.*

2. *For* $\ker \alpha$, $\alpha(1_G) = 1_H$*. For* $k_1, k_2 \in \ker \alpha$*, we have*

$$\alpha(k_1 k_2) = \alpha(k_1)\alpha(k_2) = 1 \cdot 1 = 1.$$

Also,

$$\alpha(k_1^{-1}) = \alpha(k_1)^{-1} = 1^{-1} = 1.$$

By the Subgroup Test, $\ker \alpha$ *is a subgroup of* G*.*

If $g \in G$ *and* $k \in \ker \alpha$*, then*

$$\alpha(gkg^{-1}) = \alpha(g)\alpha(k)\alpha(g^{-1}) = \alpha(g)\alpha(g^{-1}) = 1.$$

Thus by Proposition 27, $\ker \alpha \triangleleft G$*.*

$\square$

Example 13.1.1
Consider the determinant map

$$\det : GL_n(\mathbb{R}) \to \mathbb{R}^* \text{ defined by } A \mapsto \det A.$$

Then $\ker \det = SL_n(\mathbb{R})$*. Then* $SL_n(\mathbb{R}) \triangleleft GL_n(\mathbb{R})$*, as proven before.*

Example 13.1.2
Define the ***sign of a permutation*** $\sigma \in S_n$ *by*

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even;} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Then the sign mapping, $\operatorname{sgn} : S_n \to \{\pm 1\}$ *defined by* $\sigma \mapsto \operatorname{sgn}(\sigma)$ *is a* *homomorphism.*[2] *Also,* $\ker \operatorname{sgn} = A_n$*. Thus, we have* $A_n \triangleleft S_n$*, as proven before.*

[2] Think about why. It's quite straightforward using the defintion.

Proposition 37 (Normal Subgroup as the Kernel)
If $K \triangleleft G$*, then* $K = \ker \phi$ *where* $\phi : G \to G/K$ *is the coset map.*

Proof
Recall that $\phi : G \to G/K$ *is defined by* $g \mapsto Kg$*,* $\forall g \in G$*, and is a group homomorphism. By Proposition 22, we have*

$$Kg = K = K1 \iff g \in K.$$

Thus $K = \ker \phi$*.* $\square$

Theorem 38 (First Isomorphism Theorem)
Let $\alpha : G \to H$ *be a group homomorphism. We have*

$$G/\ker \alpha \cong \operatorname{im} \alpha$$

Proof
Let $K = \ker \alpha$*. Since* $K \triangleleft G$ *(by Proposition 36),* G/K *is a group. Let*[3]

$$\bar{\alpha} : G/K \to \operatorname{im} \alpha \text{ be defined by } Kg \mapsto \alpha(g)$$

[3] We must check that the function is well-defined, since cosets are not uniquely represented and so it is likely that a constructed mapping is not well-defined.

Note that

$$Kg = Kg_1 \iff gg_1^{-1} \in K \iff \alpha(gg_1^{-1}) = 1 \iff \alpha(g) = \alpha(g_1).$$

Thus $\bar{\alpha}$ *is well-defined and injective. Clearly,* $\bar{\alpha}$ *is surjective. It remains to show that* $\bar{\alpha}$ *is a group homomorphism.* $\forall g, h \in G$*, we have*

$$\bar{\alpha}(KgKh) = \bar{\alpha}(Kgh) = \alpha(gh) = \alpha(g)\alpha(h) = \bar{\alpha}(Kg)\bar{\alpha}(Kh).$$

Therefore, we have that $\bar{\alpha}$ *is an isomorphism and hence* $G/\ker \alpha \cong \operatorname{im} \alpha$ *as desired.* $\square$