

Contents

1	Lecture 1 Jan 3 2018	5
1.1	Complex Numbers and Their Properties	5
2	Lecture 2 Jan 5th 2018	12
2.1	Complex Numbers and Their Properties (Continued)	12
3	Lecture 3 Jan 8th 2018	16
3.1	Complex Numbers and Their Properties (Continued 2)	16
3.1.1	Roots of Complex Numbers	19
4	Lecture 4 Jan 10th 2018	22
4.1	Examples for nth Roots of Unity	22
5	Lecture 5 Jan 12 2018	28
5.1	Complex Functions	28
5.1.1	Limits	28
5.1.2	Continuity	30
6	Lecture 6 Jan 15th 2018	32
6.1	Continuity (Continued)	32
6.2	Differentiability	33
6.2.1	Cauchy-Riemann Equations	35
7	Lecture 7 Jan 17 2018	37
7.1	Differentiability (Continued)	37
7.1.1	Cauchy-Riemann Equations (Continued)	37
7.1.2	Power Series	39
8	Lecture 8 Jan 19 2018	41
8.1	Power Series (Continued)	41
8.1.1	Radius of Convergence	41

List of Definitions

1.1.1	Complex Number, Complex Plane	5
1.1.2	Sum and Product	6
1.1.3	Conjugate	8
1.1.4	Modulus	8
3.1.1	Argument of a Complex Number	16
5.1.1	Convergence	28
5.1.2	Convergence for Complex Functions	29
5.1.3	Continuity	30
6.2.1	Neighbourhood	33
6.2.2	Differentiable/Holomorphic	33
7.1.1	Power Series	39
9.1.1	Entire Function	46
10.2.1	Curves in \mathbb{C}	50
10.2.2	Equivalent Parameterization	50
10.2.3	Smooth Curve	51
10.2.4	Piecewise Smooth	51
10.2.5	Integral of f over a path γ	51

List of Theorems

Proposition 1.1.1	Basic Inequalities	9
Proposition 3.1.1	n th Roots of a Complex Number	19
Theorem 6.2.1	Cauchy-Riemann Equations	36
Theorem 7.1.1	Conditional Converse of CRE	38
Theorem 8.1.1	Convergence in the Radius of Convergence	41
Proposition 8.1.1	A Property of \limsup	41
Theorem 8.1.2	Power function, holomorphic function, region of convergence .	42
Corollary 10.1.1	Corollary of Theorem 8.1.2	48
Proposition 10.2.1	Properties of integrals in \mathbb{C}	52

Chapter 1

Lecture 1 Jan 3 2018

1.1 Complex Numbers and Their Properties

Definition 1.1.1 (Complex Number, Complex Plane)

A **complex number** is a vector in \mathbb{R}^2 . The **complex plane**, denoted by \mathbb{C} , is a set of complex numbers,

$$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

In \mathbb{C} , we usually write

$$\begin{aligned} 0 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & 1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ i &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} & x &= \begin{pmatrix} x \\ 0 \end{pmatrix} \\ iy &= \begin{pmatrix} 0 \\ y \end{pmatrix} \end{aligned}$$

where $x, y \in \mathbb{R}$. Consequently, we have that

$$x + iy = x + yi = \begin{pmatrix} x \\ y \end{pmatrix}$$

If for $x, y \in \mathbb{R}$, $z = x + iy$, then x is called the **real part** of z and y is called the **imaginary part** of z , and we write

$$\operatorname{Re}(z) = x \quad \operatorname{Im}(z) = y.$$

Note

- It is easy to see how \mathbb{R} is a subset of \mathbb{C} .

- Complex Numbers of the form $\begin{pmatrix} 0 \\ y \end{pmatrix}$ where $y \in \mathbb{R}$ are called **purely imaginary numbers**.
- Certain authors may prefer to denote $i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Definition 1.1.2 (Sum and Product)

We define the sum of two complex numbers to be the usual vector sum, i.e.

$$\begin{aligned} (a + ib) + (c + id) &= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \begin{pmatrix} a + c \\ b + d \end{pmatrix} \\ &= (a + c) + i(b + d) \end{aligned}$$

where $a, b, c, d \in \mathbb{R}$.

We define the product of two complex numbers by setting $i^2 = -1$, and by requiring the product to be **commutative, associative, and distributive** over the sum. In this setup, we have that

$$\begin{aligned} (a + ib)(c + id) &= ac + iad + ibc + i^2bd \\ &= (ac - bd) + i(ad + bc) \end{aligned} \tag{1.1}$$

Note

It is interesting to note that **any complex number times zero is zero**, just like what we have with real numbers.

$$\begin{aligned} \forall z = x + iy \in \mathbb{C} \quad x, y \in \mathbb{R} \quad 0 \in \mathbb{C} \\ z \cdot 0 = (x + iy)(0 + i0) = 0 + i0 = 0 \end{aligned}$$

Example 1.1.1

Let $z = 2 + i, w = 1 + 3i$. Find $z + w$ and zw .

$$\begin{aligned} z + w &= (2 + i) + (1 + 3i) \\ &= 3 + 4i \end{aligned}$$

$$\begin{aligned} zw &= (2 + i)(1 + 3i) \\ &= (2 - 3) + i(6 + 1) \quad \text{By Equation (1.1)} \\ &= -1 + 7i \end{aligned}$$

Example 1.1.2

Show that every non-zero complex number has a **multiplicative inverse**, z^{-1} , and find a formula for this inverse.

Let $z = a + ib$ where $a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$. Then

$$\begin{aligned}
 z(x + iy) &= 1 \\
 \iff (ax - by) + i(ay + bx) &= 1 \\
 \iff \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \iff \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \iff \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \iff \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{a^2 + b^2} \begin{pmatrix} a \\ -b \end{pmatrix} \\
 \iff x + iy &= \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2}
 \end{aligned}$$

Therefore, we have that the formula for the inverse is

$$(a + ib)^{-1} = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2} \quad (1.2)$$

Notation

For $z, w \in \mathbb{C}$, we write

$$\begin{aligned}
 -z &= -1z & w - z &= w + (-z) \\
 \frac{1}{z} &= z^{-1} & \frac{w}{z} &= wz^{-1}
 \end{aligned}$$

Example 1.1.3

Find $\frac{(4-i)-(1-2i)}{1+2i}$.

$$\begin{aligned}
 \frac{(4-i)-(1-2i)}{1+2i} &= \frac{3+i}{1+2i} \\
 &= (3+i)\left(\frac{1}{5} - i\frac{2}{5}\right) \\
 &= 1 - i
 \end{aligned}$$

Note

The set of complex numbers is a **field** under the operations of addition and multiplication. This means that $\forall u, v, w \in \mathbb{C}$,

$$\begin{array}{ll}
u + v = v + u & uv = vu \\
(u + v) + w = u + (v + w) & (uv)w = u(vw) \\
0 + u = u & 1u = u \\
u + (-u) = 0 & uu^{-1} = 1, \quad u \neq 0 \\
u(v + w) = uv + uw &
\end{array}$$

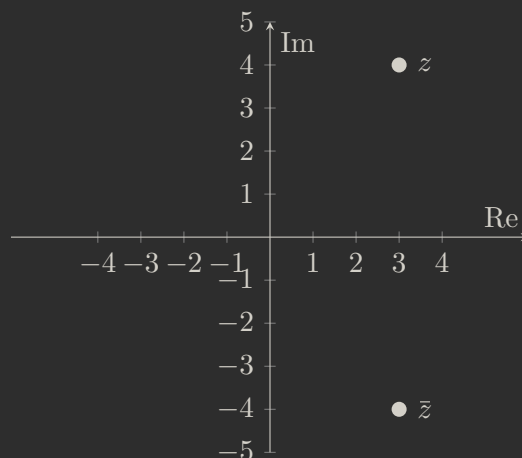
Since the distributive law holds for complex numbers, note that the **binomial expansion works** for $(w + z)^n$ where $w, z \in \mathbb{C}$ and $n \in \mathbb{N}$. (I did not verify if this is still true for when $n \in \mathbb{R}$.)

Definition 1.1.3 (Conjugate)

If $z = x + iy$ where $x, y \in \mathbb{R}$, then the **conjugate of z** is given by $\bar{z} = x - iy$

Example 1.1.4

Let $z = 3 + 4i$. Then the $\bar{z} = 3 - 4i$. Represented in the complex plane, we have the following:



We observe that on the complex plane, the conjugate of a complex number is simply its reflection on the real axis.

Definition 1.1.4 (Modulus)

We define the **modulus** (length, magnitude) of $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$, to be

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}. \quad (1.3)$$

Note

Note that this definition is consistent with the notion of the absolute value in real numbers when z is a real number, since if $y = 0$, $|z| = |x + i0| = \sqrt{x^2} = \pm x$.

Note

For $z, w \in \mathbb{C}$ and $n \in \mathbb{N}$, we have

$$\begin{array}{lll} \bar{\bar{z}} = z & z + \bar{z} = 2 \operatorname{Re}(z) & z - \bar{z} = 2i \operatorname{Im}(z) \\ z\bar{z} = |z|^2 & |z| = |\bar{z}| & \overline{z \pm w} = \bar{z} \pm \bar{w} \\ \overline{zw} = \bar{z}\bar{w} & |zw| = |z| |w| & \bar{z}^n = \overline{z^n} \end{array}$$

but note that $|z + w| \neq |z| + |w|$.

Also, note that the last equation is a generalization of the **highlighted equation**.

Note

While inequalities such as $z_1 < z_2$, where $z_1, z_2 \in \mathbb{C}$, are meaningless unless if both of them are real, $|z_1| < |z_2|$ means that the point z_1 in the complex plane is closer to the origin than the point z_2 .

Proposition 1.1.1 (Basic Inequalities)

1. $|\operatorname{Re}(z)| \leq |z|$
2. $|\operatorname{Im}(z)| \leq |z|$
3. $|z + w| \leq |z| + |w|$ *Triangle Inequality*
4. $|z + w| \geq ||z| - |w||$ *Inverse Triangle Inequality*

Proof

Note that $|z|^2 = \operatorname{Re}(z)^2 + \operatorname{Im}(z)^2$ and that we can express $|x| = \sqrt{x^2}$ for any $x \in \mathbb{R}$. 1 and 2 immediately follows from that.

To prove 3, we have that

$$\begin{aligned} |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) \\ &= |z|^2 + |w|^2 + (w\bar{z} + \bar{w}z) \\ &= |z|^2 + |w|^2 + 2 \operatorname{Re}(w\bar{z}) \\ &\leq |z|^2 + |w|^2 + 2 |w\bar{z}| \quad \text{by 1} \\ &= |z|^2 + |w|^2 + 2 |wz| \quad \text{since } |w\bar{z}| = |w| |\bar{z}| \text{ and } |z| = |\bar{z}| \\ &= (|z| + |w|)^2 \end{aligned}$$

To prove 4, note that

$$|z| = |z + w - w| \leq |z + w| + |w| \quad (1.4)$$

$$|w| = |w + z - z| \leq |z + w| + |z| \quad (1.5)$$

Observe that

$$\text{Equation (1.4)} \implies |z| - |w| \leq |z + w|$$

$$\text{Equation (1.5)} \implies |w| - |z| \leq |z + w|$$

Thus, we have that

$$|z + w| \geq ||z| - |w||$$

as required. \square

Item 3 in Proposition 1.1.1 can be generalized by the means of mathematical induction to sums involving any finite number of terms, as:

$$|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n| \quad (1.6)$$

where $n \in \mathbb{N} \setminus \{0, 1\}$.

To note the induction proof, when $n = 2$, Equation (1.6) is just Item 3. If Equation (1.6) is true for when $n = m$ where $m \in \mathbb{N} \setminus \{0, 1\}$, $n = m + 1$ is also true since by Item 3,

$$\begin{aligned} |(z_1 + z_2 + \dots + z_m) + z_{m+1}| &\leq |z_1 + z_2 + \dots + z_m| + |z_{m+1}| \\ &\leq (|z_1| + |z_2| + \dots + |z_m|) + |z_{m+1}|. \end{aligned}$$

The distance between two points $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}, x_1, x_2, y_1, y_2 \in \mathbb{R}$ is $|z_1 - z_2|$, since $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ is our usual notion of the Euclidean distance of two points on a plane.

Also, note that

$$z_1 - z_2 = z_1 + (-z_2)$$

and thus if we apply our knowledge of vector representation, $z_1 - z_2$ is the directed line segment from the point z_2 to z_1 .

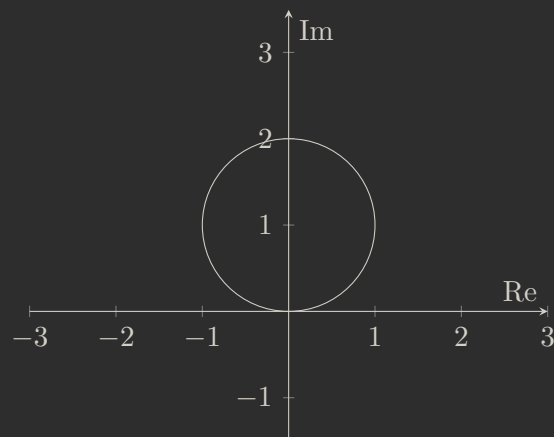
With the notion of a “distance” set on the complex plane, we can now explore upon points lying on a circle with a center z_0 and radius R , which satisfies the equation

$$|z - z_0| = R.$$

We may simply refer to this set of points as the circle $|z - z_0| = R$.

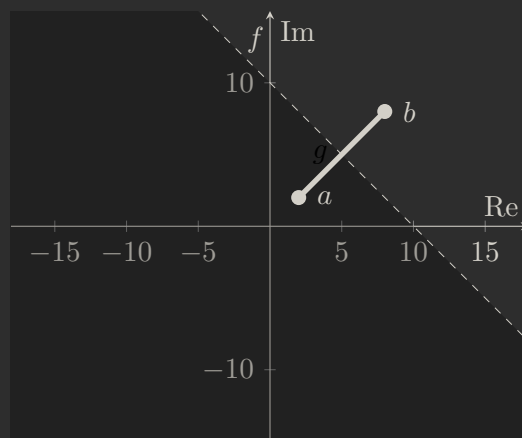
Example 1.1.5

We may describe a set $\{z \in \mathbb{C} : |z - i| = 1\}$ as follows:



Let $a, b \in \mathbb{C}$ describe the set $\{z \in \mathbb{C} : |z - a| < |z - b|\}$.

Suppose the following coordinates for a and b are arbitrary,



In the above, g is the line segment that connects the points a and b on the complex plane, while f is the perpendicular bisector of the line segment g . The area described by the set $\{z \in \mathbb{C} : |z - a| < |z - b|\}$ is the shaded area which is below f .

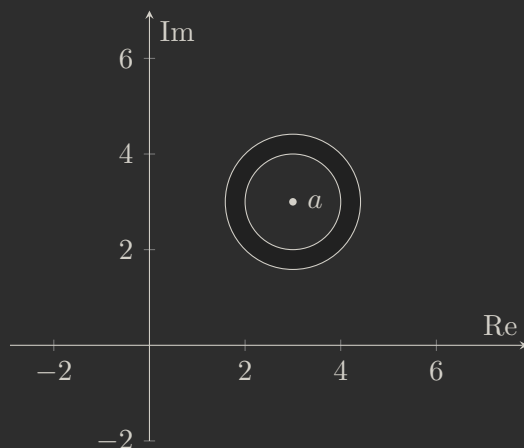
Chapter 2

Lecture 2 Jan 5th 2018

2.1 Complex Numbers and Their Properties (Continued)

Example 2.1.1

Let $a \in \mathbb{C}$. Describe the set $\{z \in \mathbb{C} : 1 < |z - a| < 2\}$.



Example 2.1.2

Show that every non-zero complex number has exactly two complex square roots, and find a formula for the square roots.

Let $z = x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$, and let $w = u + iv$, $u, v \in \mathbb{R}$. Then

Remark

Let $z \in \mathbb{C}$. The notation \sqrt{z} may represent either one of the square roots of z or both of the square roots, i.e. **it is possible that \sqrt{z} represents a set.**

Exercise 2.1.1

Is it always okay for complex numbers such that $\sqrt{zw} = \sqrt{z}\sqrt{w}$, for $z, w \in \mathbb{C}$?

No. For example, consider $z = w = -1$. Then we have

$$\sqrt{zw} = \sqrt{1} = \pm 1$$

while

$$\sqrt{z}\sqrt{w} = i \cdot i = -1$$

and thus

$$\sqrt{zw} \neq \sqrt{z}\sqrt{w}.$$

Example 2.1.3

Find the values of $\sqrt{3 - 4i}$.

By Example 2.1.2,

$$\begin{aligned} \sqrt{3 - 4i} &= \pm \left(\sqrt{\frac{3 + \sqrt{9 + 16}}{2}} - i \sqrt{\frac{-3 + \sqrt{9 + 16}}{2}} \right) \\ &= \pm(2 - i) \end{aligned}$$

Remark

The quadratic formula holds for complex polynomials, i.e.

$$\forall a, b, c \in \mathbb{C} \quad a \neq 0 \quad \forall z \in \mathbb{C} \quad az^2 + bz + c = 0,$$

the solution for z is given by

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (2.3)$$

The following is a short proof.

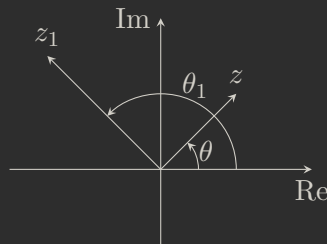
Chapter 3

Lecture 3 Jan 8th 2018

3.1 Complex Numbers and Their Properties (Continued 2)

Definition 3.1.1 (Argument of a Complex Number)

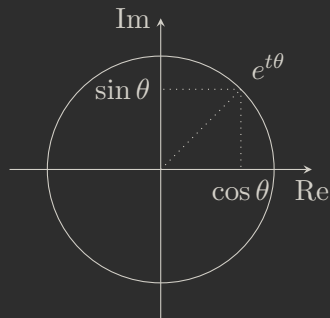
Let $z \in \mathbb{C} \setminus \{0\}$. The **argument** (or the angle) of z , denoted by $\arg z$, $\text{Arg } z$, or simply $\theta = \theta(z)$, is the angle modulo 2π (i.e. $0 \leq \theta < 2\pi$) between the vector defining z and the positive real axis (in the counterclockwise direction).



Notation

Let $e^{i\theta} := \cos \theta + i \sin \theta$. Note that this definition, called **Euler's formula**, can be derived by extending the Taylor expansion of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for when $x \in \mathbb{C}$ (the sum of the real parts of the expansion is the Taylor expansion of cosine while the imaginary part for sine).

Now $e^{i\theta}$ is on the unit circle.

**Remark**

If $z = 0$, the coordinate θ is undefined, and so it is implied that $z \neq 0$ whenever we use the polar form.

Example 3.1.1

Some examples of $\theta \in [0, 2\pi)$:

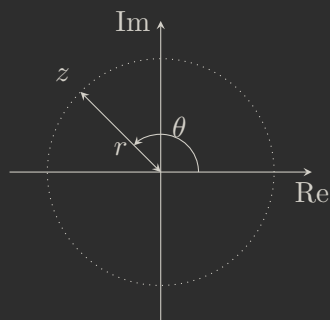
$$\begin{aligned} e^{i\frac{\pi}{4}} &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\frac{\pi}{2}} &= i \\ e^{i\frac{3\pi}{4}} &= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\pi} + 1 &= 0 \end{aligned}$$

Remark

$$\forall k \in \mathbb{Z} \quad \forall \theta \in \mathbb{R} \quad e^{i\theta} = e^{i(\theta+2\pi k)}$$

Remark

The complex number $re^{i\theta}$, where $r > 0, \theta \in [0, 2\pi)$, represents the complex number with modulus r and argument θ .



Therefore, $\forall z \in \mathbb{C}$, we can express

$$z := |z| e^{i \operatorname{Arg} z}. \quad (3.1)$$

The n th roots of z is described by the set

$$\left\{ r^{\frac{1}{n}} e^{i\left(\frac{\theta+2\pi k}{n}\right)} : k = 0, 1, \dots, n-1 \right\} \quad (3.8)$$

Proof

$$\begin{aligned} s^n = r &\iff s = r^{\frac{1}{n}} \\ e^{in\theta} = e^{i\tau} &\iff \theta = \frac{\tau + 2\pi k}{n} \end{aligned}$$

Therefore, the set that describes the n th roots of z is

$$\left\{ w = r^{\frac{1}{n}} e^{i\left(\frac{\theta+2\pi k}{n}\right)} : k = 0, 1, \dots, n-1 \right\}$$

Remark (nth Roots of Unity)

The ***nth roots of unity*** is a direct consequence of *Proposition 3.1.1* where we solve for the equation $z^n = 1$ for any $z \in \mathbb{C}, n \in \mathbb{Z}$.

The set that describes the n th roots of unity is

$$\left\{ e^{i\theta} : \theta = \frac{2\pi k}{n}, k = 0, 1, \dots, n-1 \right\} \quad (3.9)$$

It is easy to see how the n th roots of unity **partitions the unit circle into n parts**.

Example 3.1.3

Find the cubic roots of $-2 + 2i$.

Let $z = -2 + 2i$. Note that $|z| = 2\sqrt{2}$ and $\text{Arg } z = \frac{3\pi}{4}$.

Therefore, in polar form, $z = 2\sqrt{2}e^{i\frac{3\pi}{4}}$.

Let $w = re^{i\theta}$, where $\theta \in [0, 2\pi)$, and $w^3 = z$. Then

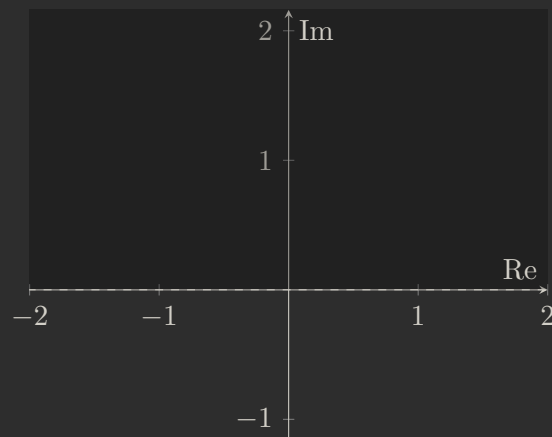
$$\begin{aligned} r &= (2\sqrt{2})^{\frac{1}{3}} \\ \theta &= \frac{\frac{3\pi}{4} + 2\pi k}{3}, \quad k = 0, 1, 2 \end{aligned}$$

The set that describes the cubic root of $-2 + 2i$ is thus

$$\left\{ (2\sqrt{2})^{\frac{1}{3}} e^{i\theta} : \theta = \frac{\frac{3\pi}{4} + 2\pi k}{3}, k = 0, 1, 2 \right\}$$

Example 3.1.4

Describe the set $\{z \in \mathbb{C} : |\operatorname{Arg} z - \frac{\pi}{2}| < \frac{\pi}{2}\}$. (Note: $\operatorname{Arg} z \in [0, 2\pi)$)

**Exercise 3.1.1**

Solve

1. $z^4 = -1$

$$\text{Let } z = re^{i\theta}$$

$$r = |-1| = 1 \quad \theta = \frac{\pi + 2\pi k}{4} = \frac{(2k+1)\pi}{4}, \quad k = 0, 1, 2, 3$$

2. $z^4 = -1 + \sqrt{3}i$

$$\text{Let } z = re^{i\theta}$$

$$r = \left| -1 + \sqrt{3}i \right| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\theta = \frac{\frac{2\pi}{3} + 2\pi k}{4} = \frac{(2k + \frac{2}{3})\pi}{4}, \quad k = 0, 1, 2, 3$$

Chapter 4

Lecture 4 Jan 10th 2018

4.1 Examples for n th Roots of Unity

Recall that the n th roots of unity are given by $e^{i\frac{2\pi k}{n}}, k = 0, 1, \dots, n-1$.

Exercise 4.1.1

Let z be any n th root of unity other than 1. Show that

$$z^{n-1} + z^{n-2} + \dots + z + 1 = 0 \quad (4.1)$$

Proof

By the Sum of Finite Geometric Terms,

$$z^{n-1} + z^{n-2} + \dots + z + 1 = \frac{1 - z^n}{1 - z}.$$

Since $z^n = 1$, RHS is thus zero, which in turn completes the proof.

As an aside, if we wish to remove the restriction that z can also be 1, we may consider that

$$z^n - 1 = (z - 1)(1 + z + \dots + z^{n-1})$$

Since $z^n = 1$, LHS is zero. Then either $z = 1$ or $(1 + z + \dots + z^{n-1}) = 0$.

Exercise 4.1.2

Consider the $n-1$ diagonals of a regular n -gon, inscribed in a circle of radius 1, obtained by connecting one vertex on the n -gon to all its other vertices.

For example, if we are given $n = 6$, we obtain the following diagram.

therefore we obtain

$$\begin{aligned}
 2^{3n} + (1 + \alpha)^{3n} + (1 + \alpha^2)^{3n} &= 3 \sum_{j=0}^n \binom{3n}{3j} \\
 \frac{1}{3} [2^{3n} + (1 + \alpha)^{3n} + (1 + \alpha^2)^{3n}] &= \sum_{j=0}^n \binom{3n}{3j} \\
 \frac{1}{3} [2^{3n} + (-\alpha^2)^{3n} + (-\alpha)^{3n}] &= \sum_{j=0}^n \binom{3n}{3j} \quad \text{since } 1 + \alpha + \alpha^2 = 0 \\
 \frac{1}{3} [2^{3n} + (-1)^n + (-1)^n] &= \sum_{j=0}^n \binom{3n}{3j} \quad \text{since } \alpha^3 = 1 \\
 \frac{2^{3n} + 2(-1)^n}{3} &= \sum_{j=0}^n \binom{3n}{3j}
 \end{aligned}$$

as required.

Exercise 4.1.4

Note that we can define $\text{Arg } z$ in any interval of length 2π , i.e. it is not necessary that $\text{Arg } z \in [0, 2\pi)$.

For example, if we restrict $\text{Arg } z \in [-\pi, \pi]$, then we can write

$$\text{Arg} \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) = -\frac{3\pi}{4}$$

Let z be on the unit circle and $\text{Arg } z \in [-\pi, \pi]$. Suppose that $z \notin \mathbb{R}$, i.e. $z \neq 1, z \neq -1$. Show that

$$\text{Arg} \left(\frac{z-1}{z+1} \right) = \begin{cases} \frac{\pi}{2} & \text{Im } z > 0 \\ -\frac{\pi}{2} & \text{Im } z < 0 \end{cases}$$

Proof

Note that $\forall w_1, w_2 \in \mathbb{C}$, where $\text{Arg } w_1 = \tau_1, \text{Arg } w_2 = \tau_2$ for τ_1, τ_2 in the same 2π -interval,

$$\text{Arg} \frac{w_1}{w_2} = \frac{e^{i\tau_1}}{e^{i\tau_2}} \equiv e^{i(\tau_1 - \tau_2)} = \text{Arg } w_1 - \text{Arg } w_2 \quad (4.7)$$

in modulo 2π .

Suppose $\text{Im } z > 0$. Let $\theta_1 = \text{Arg}(z-1)$ and $\theta_2 = \text{Arg}(z+1)$. Consider Figure 4.3. Note that since both $\theta_1, \theta_2 \in [0, \pi]$, we have that $\theta_1 - \theta_2 \in [-\pi, \pi]$, and thus Equation (4.7) holds

true without the need of the condition of being in modulo 2π . We observe that

$$\begin{aligned}\frac{\pi}{2} &= \theta_2 + \pi - \theta_1 \\ \theta_1 - \theta_2 &= \frac{\pi}{2}\end{aligned}$$

as desired.

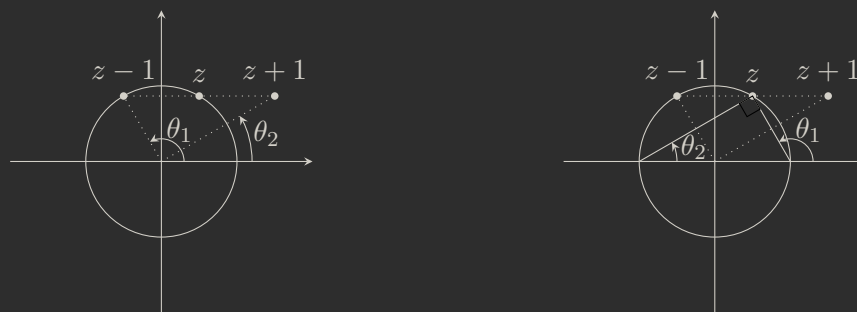


Figure 4.3: (Right) Depicted question, (Left) Translated Angles

Similarly, we can obtain $\theta_1 - \theta_2 = -\frac{\pi}{2}$ for when $\text{Im } z < 0$. This completes the proof.

Exercise 4.1.5

Let $f(z) = e^z$ for $z \in \mathbb{C}$. Let $A = \{z = x + iy \in \mathbb{C} : x \leq 1, y \in [0, \pi]\}$. Describe the image of $f(A)$.

Solution

Firstly, note that

$$\begin{aligned}e^z &= e^{x+iy} \\ e^x &\in (0, e] \\ y &\in [0, \pi]\end{aligned}$$

Chapter 5

Lecture 5 Jan 12 2018

5.1 Complex Functions

5.1.1 Limits

Definition 5.1.1 (Convergence)

A sequence of complex numbers z_1, z_2, z_3, \dots **converges** to $z \in \mathbb{C}$ if

$$\lim_{n \rightarrow \infty} |z_n - z| = 0 \quad (5.1)$$

or we may say

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \quad |z_n - z| < \epsilon \quad (5.2)$$

Note

If $\{z_n\}_{n \in \mathbb{N}}$ converges to z , we may write $\lim_{n \rightarrow \infty} z_n = z$ or $z_n \rightarrow z$ (as $n \rightarrow \infty$).

Example 5.1.1

For $|z| > 1$, does $\{\frac{1}{z^n}\}_{n=1}^{\infty}$ converge? Explain.

Solution

We claim that the limit is 0. Since $|z| > 1$, we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{1}{z^n} - 0 \right| &= \lim_{n \rightarrow \infty} \left| \frac{1}{z} \right|^n \\ &= 0 \end{aligned}$$

Another way to prove this, since $|z| > 1 \implies 0 < \left|\frac{1}{z}\right| < 1$,

$$\forall \epsilon = \left|\frac{1}{z}\right| > 0$$

$$\left|\frac{1}{z^n} - 0\right| = \left|\frac{1}{z}\right|^n < \left|\frac{1}{z}\right| = \epsilon$$

Definition 5.1.2 (Convergence for Complex Functions)

$\forall \Omega \subseteq \mathbb{C}$, let $f : \Omega \rightarrow \mathbb{C}$. We say that

$$\lim_{z \rightarrow z_0} f(z) = L \quad (5.3)$$

for some $L \in \mathbb{C}$ if for every sequence $\{z_n\}_n \subseteq \Omega$ (not including z_0 if it is in Ω), we have that

$$z_n \rightarrow z_0 \implies f(z_n) \rightarrow L \quad (5.4)$$

Note that L need not be in Ω .

Example 5.1.2

Let $f(z) = \frac{z}{z}, z \in \mathbb{C} \setminus \{0\}$. Find $\lim_{z \rightarrow 0} f(z)$.

Solution

Suppose $z = x \in \mathbb{R} \setminus \{0\}$. Then $f(z) = f(x) = \frac{x}{x} = 1$.

Suppose $z = iy, y \in \mathbb{R} \setminus \{0\}$. Then $f(z) = f(iy) = \frac{-iy}{iy} = -1$.

Therefore, the limit $\lim_{z \rightarrow 0} f(z)$ does not exist.

Exercise 5.1.1

Show that $z_n \rightarrow z \iff \text{Re}(z_n) \rightarrow \text{Re}(z) \wedge \text{Im}(z_n) \rightarrow \text{Im}(z)$.

(Hint: $|\text{Re}(z)|, |\text{Im}(z)| \leq |z| \leq |\text{Re}(z)| + |\text{Im}(z)|$)

Solution

Suppose $z_n \rightarrow z$. Then $\forall \epsilon_0 > 0 \exists N \in \mathbb{N} \forall n > N |z_n - z| < \epsilon$. Note once and for all that

$$\text{Re}(z_n - z) = \text{Re}(z_n) - \text{Re}(z)$$

$$\text{Im}(z_n - z) = \text{Im}(z_n) - \text{Im}(z).$$

Thus

$$|\text{Re}(z_n) - \text{Re}(z)| = |\text{Re}(z_n - z)|$$

$$\leq |z_n - z| < \epsilon$$

$$|\text{Im}(z_n) - \text{Im}(z)| = |\text{Im}(z_n - z)|$$

$$\leq |z_n - z| < \epsilon$$

For the other direction,

$$\begin{aligned}\forall \frac{\epsilon}{2} > 0 \quad \exists N_0 \in \mathbb{N} \quad \forall n > N_0 \quad |\operatorname{Re}(z_n) - \operatorname{Re}(z)| < \frac{\epsilon}{2} \\ \forall \frac{\epsilon}{2} > 0 \quad \exists N_1 \in \mathbb{N} \quad \forall n > N_1 \quad |\operatorname{Im}(z_n) - \operatorname{Im}(z)| < \frac{\epsilon}{2}.\end{aligned}$$

Therefore,

$$\begin{aligned}|z_n - z| &= |\operatorname{Re}(z_n) + \operatorname{Im}(z_n) - \operatorname{Re}(z) - \operatorname{Im}(z)| \\ &\leq |\operatorname{Re}(z_n) - \operatorname{Re}(z)| + |\operatorname{Im}(z_n) - \operatorname{Im}(z)| \\ &\leq \epsilon\end{aligned}$$

□

5.1.2 Continuity

Definition 5.1.3 (Continuity)

$\forall \Omega \subseteq \mathbb{C}$, let $f : \Omega \rightarrow \mathbb{C}$. We say that f is **continuous** at $z_0 \in \Omega$ if

1. $\forall \{z_n\}_{n \in \mathbb{N}} \quad z_n \rightarrow z_0 \implies f(z_n) \rightarrow f(z_0)$
2. $\forall \epsilon > 0 \quad \exists \delta > 0 \quad |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$

Remark

1. f is continuous on Ω if it is continuous on every point in Ω .
2. We may **split** f into its real and imaginary parts, i.e.

$$f(z) = f(x, y) = u(x, y) + iv(x, y) \tag{5.5}$$

where $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Example 5.1.3

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ and for $z \in \mathbb{C}$, $f(z) = \frac{\bar{z}}{z}$. To split f into real and imaginary parts:

$$\begin{aligned}f(z) &= \frac{\bar{z}}{z} \\ &= (x + iy) \left(\frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2} \right) \\ &= \frac{x^2 - y^2}{x^2 + y^2} + i \frac{(-2xy)}{x^2 + y^2}\end{aligned}$$

Chapter 6

Lecture 6 Jan 15th 2018

6.1 Continuity (Continued)

Exercise 6.1.1

Let $f : \Omega \rightarrow \mathbb{C}$. Prove that $f(z)$ is continuous at $z_0 = x_0 + iy_0 \in \mathbb{C} \iff$ functions $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$, such that $f(z) = u(x, y) + iv(x, y)$ are both continuous at (x_0, y_0) .

Solution

We shall first prove the forward direction. Suppose that $f(z)$ is continuous at $z_0 = x_0 + iy_0 \in \mathbb{C}$. By Definition 5.1.3, $\forall \{z_n\}_{n \in \mathbb{N}} \subseteq \Omega$, $z_n \rightarrow z_0 \implies f(z_n) \rightarrow f(z_0)$. By Exercise 5.1.1,

$$\begin{aligned} z_n \rightarrow z_0 &\iff \operatorname{Re} z_n \rightarrow \operatorname{Re} z_0 \wedge \operatorname{Im} z_n \rightarrow \operatorname{Im} z_0 \\ &\iff x_n \rightarrow x_0 \wedge y_n \rightarrow y_0 \end{aligned} \tag{6.1}$$

where $z_n = x_n + iy_n$ for $x_n, y_n \in \mathbb{R}$.

Similarly so, and by Equation (5.5),

$$f(z_n) \rightarrow f(z_0) \iff u(x_n, y_n) \rightarrow u(x_0, y_0) \wedge v(x_n, y_n) \rightarrow v(x_0, y_0) \tag{6.2}$$

Putting together Equation (6.1) and Equation (6.2), we get

$$(x_n, y_n) \rightarrow (x_0, y_0) \implies u(x_n, y_n) \rightarrow u(x_0, y_0) \wedge v(x_n, y_n) \rightarrow v(x_0, y_0)$$

as desired.

The proof of the other direction is simply a reversed process of the above. □

6.2 Differentiability

Definition 6.2.1 (Neighbourhood)

For $z_0 \in \mathbb{C}$, $r \in \mathbb{R}$, let

$$D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}. \quad (6.3)$$

On the complex plane, this is seen as a open disk centered around the point z_0 with radius r , as shown below.

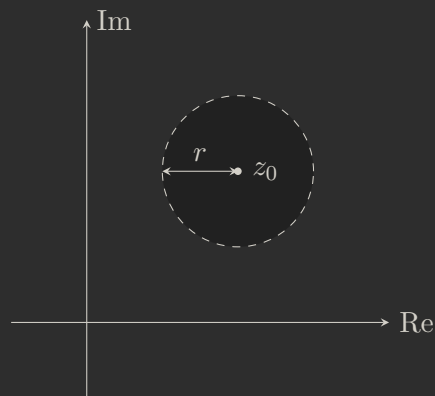


Figure 6.1: Open disk centered around z_0 with radius r

This open disk is called a **neighbourhood** of z_0 .

Definition 6.2.2 (Differentiable/Holomorphic)

Let $f(z)$ be defined in a neighbourhood of $z_0 \in \mathbb{C}$. We say f is **differentiable/holomorphic** at z_0 if for some $h \in \mathbb{C}$,

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad (6.4)$$

exists. If such a limit exists, we denote the limit by $f'(z_0)$.

Remark

$h \in \mathbb{C}$: h need not necessarily be real. In this sense, h approaches 0 from **any direction** around 0 $\in \mathbb{C}$.

Example 6.2.1

For $z \in \mathbb{C} \setminus \{0\}$, let $f(z) = \frac{1}{z}$. Let $z_0 \in \mathbb{C} \setminus \{0\}$. Note that

$$\lim_{h \rightarrow 0} \frac{\frac{1}{z_0+h} - \frac{1}{z_0}}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-h}{(z_0 + h)z_0} \right] = -\frac{1}{z_0^2}$$

Thus f is holomorphic at any $z \in \mathbb{C} \setminus \{0\}$, and hence $f'(z) = -\frac{1}{z}$.

Example 6.2.2

For $z \in \mathbb{C}$, let $f(z) = \bar{z}$. Let $z_0 \in \mathbb{C}$. Notice that

$$\lim_{h \rightarrow 0} \frac{\overline{z_0 + h} - \bar{z}}{h} = \lim_{h \rightarrow 0} \frac{\bar{h}}{h}.$$

From [Example 5.1.2](#), we know that such a limit does not exist. Thus f is not holomorphic on any $z \in \mathbb{C}$.

Exercise 6.2.1 (Holomorphic Functions Properties)

If f, g are holomorphic at $z \in \mathbb{C}$, prove that

1. $f + g$ is holomorphic and $(f + g)' = f' + g'$.
2. fg is holomorphic and $(fg)' = f'g + fg'$.
3. if $g(z) \neq 0$, $\frac{f}{g}$ is holomorphic and $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$.

Solution

1. For $f + g$,

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(z+h) + g(z+h) - f(z) - g(z)}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{f(z+h) - f(z)}{h} + \frac{g(z+h) - g(z)}{h} \right] \\ &= f'(z) + g'(z) \end{aligned}$$

Thus $(f + g)' = f' + g'$.

2. For fg ,

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(z+h)g(z+h) - f(z)g(z)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(z+h)g(z+h) + f(z)g(z+h) - f(z)g(z+h) - f(z)g(z)}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{f(z+h) - f(z)}{h} g(z+h) + f(z) \frac{g(z+h) - g(z)}{h} \right] \\ &= f'(z)g(z) + f(z)g'(z) \end{aligned}$$

Therefore, $(fg)' = f'g + fg'$.

Case 2: $h \rightarrow 0$ via the imaginary axis

In this case, $h = 0 + iy$ and $y \rightarrow 0 \in \mathbb{R}$. In a similar fashion, Equation (6.5) becomes

$$\begin{aligned} f'(z_0) &= \lim_{y \rightarrow 0} \left[\frac{u(x_0, y_0 + y) - u(x_0, y_0)}{iy} + \frac{v(x_0, y_0 + y) - v(x_0, y_0)}{y} \right] \\ &= \frac{1}{i} \cdot \frac{\partial u}{\partial y} \Big|_{(x_0, y_0)} + \frac{\partial v}{\partial y} \Big|_{(x_0, y_0)} \end{aligned} \quad (6.7)$$

Note that since $f'(z_0)$ exists, the real and imaginary part of Equation (6.6) and Equation (6.7) must equate. Also note that $\frac{1}{i} = -i$. With that, we obtain the following theorem.

Theorem 6.2.1 (Cauchy-Riemann Equations)

If $f(z)$ is holomorphic at $z_0 = x_0 + iy_0 \in \mathbb{C}$ where $x_0, y_0 \in \mathbb{R}$, then, at (x_0, y_0) ,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \quad (6.8)$$

Chapter 7

Lecture 7 Jan 17 2018

7.1 Differentiability (Continued)

7.1.1 Cauchy-Riemann Equations (Continued)

It is natural to wonder if the **converse** of Theorem 6.2.1 is true. We present the following example.

Example 7.1.1

Let

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Check if

1. f is holomorphic at 0.
2. Theorem 6.2.1 holds at $(0,0)$.

Proof

1. Observe that by letting $h = x_h + iy_h$ where $x_h, y_h \in \mathbb{R}$,

$$\lim_{h \rightarrow 0} \frac{\overline{0+h}^2 - 0}{0+h} = \lim_{h \rightarrow 0} \frac{\bar{h}^2}{h} = \lim_{x_h + iy_h \rightarrow 0} \left(\frac{x_h - iy_h}{x_h + iy_h} \right)^2$$

Consider $y_h = kx_h$, for $k \in \mathbb{R} \setminus \{0\}$. Then

$$\lim_{x_h \rightarrow 0} \left(\frac{x_h - ikx_h}{x_h + ikx_h} \right)^2 = \left(\frac{1 - ik}{1 + ik} \right)^2,$$

where we see that the limit depends on the value of k . Therefore, the limit DNE. Hence f is not holomorphic at 0.

2. Let $z = x + iy$ for $x, y \in \mathbb{R}$. Then

$$\frac{\bar{z}^2}{z} = \frac{(x - iy)^2}{x + iy} = \frac{(x - iy)^3}{x^2 + y^2} = \frac{x^3 - 3xy^2}{x^2 + y^2} + i \frac{(-3x^2y + y^3)}{x^2 + y^2}$$

Therefore, we obtain

$$u(x, y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$v(x, y) = \begin{cases} \frac{y^3 - 3x^2y}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Observe that

$$\left. \frac{\partial u}{\partial x} \right|_{(0,0)} = \lim_{x \rightarrow 0} \frac{u(x, 0) - u(0, 0)}{x} = 1$$

$$\left. \frac{\partial v}{\partial y} \right|_{(0,0)} = \lim_{y \rightarrow 0} \frac{v(0, y) - v(0, 0)}{y} = 1$$

and

$$\left. \frac{\partial u}{\partial y} \right|_{(0,0)} = \lim_{y \rightarrow 0} \frac{u(0, y) - u(0, 0)}{y} = 0$$

$$\left. \frac{\partial v}{\partial x} \right|_{(0,0)} = \lim_{x \rightarrow 0} \frac{v(x, 0) - v(0, 0)}{x} = 0$$

satisfies Equation (6.8).

This illustrates that the converse of Theorem 6.2.1 is not true. We will, however, show that the converse will be true given an extra condition.

Theorem 7.1.1 (Conditional Converse of CRE)

Let $z_0 = x_0 + iy_0 \in \Omega \subseteq \mathbb{C}$, $x_0, y_0 \in \mathbb{R}$, and $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f = u + iv : \Omega \rightarrow \mathbb{C}$. If

1. the partials of u, v exist in a neighbourhood of (x_0, y_0) ,
2. the partials of u, v are continuous at (x_0, y_0) , and
3. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ at (x_0, y_0) ,

then f is holomorphic at z_0 .

A proof of the theorem is in page 36 of Newman and Bak (recommended text of PMATH352W18). I may include the proof whenever I am free.

7.1.2 Power Series

Definition 7.1.1 (Power Series)

A **power series** in \mathbb{C} is an infinite series of the form

$$\sum_{n \in \mathbb{N}} c_n z^n, \quad (7.1)$$

where each $c_n \in \mathbb{C}$ is the coefficient of z of the n -th power.

In this subsection, we are interested to see if Equation (7.1) converges.

Recall the notion of convergence in series from \mathbb{R} . Equation (7.1) converges if the sequence of partial sums $\{S_N\}$ converges as $N \rightarrow \infty$, where

$$S_N := \sum_{n=0}^N c_n z^n$$

In other words, using the same definition of S_N ,

$$\begin{aligned} \forall \epsilon > 0 \quad \exists N \in \mathbb{N} \setminus \{0\} \quad \forall n > N \\ |S_n - L| < \epsilon \end{aligned}$$

where $L \in \mathbb{C}$ is the limit that the sequence converges to.

We also know that Equation (7.1) converges absolutely if $\sum_{n=0}^{\infty} |c_n| |z|^n$ converges. This is a stronger statement (i.e. absolute convergence \implies convergence)

$$\because \left| \sum_{n=0}^N c_n z^n \right| \leq \sum_{n=0}^N |c_n| |z|^n \quad \text{for each } N \in \mathbb{N}$$

Example 7.1.2

$\sum_{n=0}^{\infty} z^n$ converges absolutely for $|z| < 1$.

Note that the partial sum of a geometric series is

$$\sum_{n=0}^N r^n = \frac{1 - r^{N+1}}{1 - r}$$

and so the limit as $N \rightarrow \infty$ exists if $|r| < 1$, and hence we see that

$$\sum_{n=0}^{\infty} r^n \rightarrow \frac{1}{1 - r}$$

if $|r| < 1$ as $N \rightarrow \infty$.

However, if $|z| = 1$, the power series diverges.

Another note that we shall point out is that if Equation (7.1) converges absolutely for some $z_0 \in \mathbb{C}$, then it converges absolutely for any z where $|z| < |z_0|$.

These notions, in turn, begs the question of **what is the largest possible $|z_0|$ for the series to converge absolutely.**

Chapter 8

Lecture 8 Jan 19 2018

8.1 Power Series (Continued)

8.1.1 Radius of Convergence

Theorem 8.1.1 (Convergence in the Radius of Convergence)

For any power series $\sum_{n \in \mathbb{N}} c_n z^n$, $\exists 0 \leq R < \infty$, such that

1. $|z| < R \implies$ series converges absolutely.
2. $|z| > R \implies$ series diverges.

Moreover, R is given by **Hadamard's Formula**:

$$\frac{1}{R} := \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}} \quad (8.1)$$

Remark

1. R is called the **radius of convergence** of the series. $\{z \in \mathbb{C} : |z| < R\}$ is called the disk of convergence of the series.
2. Recall the definition of the **limit supremum**

$$\limsup_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} \left(\sup_{m \geq n} a_m \right) \quad (8.2)$$

which we may colloquially say as the “highest peak ‘reached’ by a_n ’s as $n \rightarrow \infty$ ”

Proposition 8.1.1 (A Property of limsup)

$$\begin{aligned} \forall \{a_n\}_{n \in \mathbb{N}} \quad L := \limsup_{n \rightarrow \infty} a_n &\implies \\ \forall \epsilon > 0 \quad \exists N > 0 \quad \forall n > N & \\ L - \epsilon < a_n < L + \epsilon & \end{aligned}$$

(Proof to be included)

Proof (Theorem 8.1.1)

Let $L := \frac{1}{R} = \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}$. Clearly, $L \geq 0$.

1. Suppose $|z| < R$. $\exists \epsilon > 0, r := |z|(L + \epsilon)$ such that $0 < r < 1$. By Proposition 8.1.1, $\exists N \in \mathbb{N}, \forall n > N, |c_n|^{\frac{1}{n}} < L + \epsilon$.

Now since $L = \frac{1}{R}$,

$$\sum_{n=N}^{\infty} |c_n| |z|^n = \sum_{n=N}^{\infty} (|c_n|^{\frac{1}{n}} |z|)^n < \sum_{n=N}^{\infty} r^n$$

and since $0 < r < 1$, the final summation converges (as it is a geometric sum). Thus by comparison test, $\sum_{n=N}^{\infty} |c_n| |z|^n$ converges.

We may also proceed with noticing that the partial sum of $\sum_{n=N}^{\infty} |c_n| |z|^n$ is **bounded and monotonic**, which shows that the series converges.

2. Suppose $|z| > R$. $\exists \epsilon > 0, r := |z|(L - \epsilon)$ such that $r > 1$. By Proposition 8.1.1, $\exists N \in \mathbb{N}, \forall n > N, |c_n|^{\frac{1}{n}} > L - \epsilon$. Then analogous to the proof above,

$$\sum_{n=N}^{\infty} |c_n| |z|^n = \sum_{n=N}^{\infty} (|c_n|^{\frac{1}{n}} |z|)^n > \sum_{n=N}^{\infty} r^n$$

where the final summation diverges, and thus implying that $\sum_{n=N}^{\infty} |c_n| |z|^n$ diverges.

Theorem 8.1.2 (Power function, holomorphic function, region of convergence)

Suppose $f(z) = \sum_{n \in \mathbb{N}} c_n z^n$ has a radius of convergence $R \in \mathbb{R}$. Then $f'(z)$ exists and equals

$$\sum_{n=1}^{\infty} n c_n z^{n-1}$$

throughout $|z| < R$.

Moreover, f' has the **same radius of convergence** as f .

Chapter 9

Lecture 9 Jan 22 2018

9.1 Power Series (Continued 2)

9.1.1 Radius of Convergence (Continued)

Example 9.1.1

Let $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$. To find the radius of convergence, we use Hadamard's Formula:

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} \left(\frac{1}{n} \right)^{\frac{1}{n}} = 1 \quad \because \lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$$

Therefore $R = 1$. Thus, by [Theorem 8.1.1](#), f converges absolutely when $|z| < 1$ and diverges when $|z| > 1$. As for the boundary, i.e. $|z| = 1$, consider the following two cases:

1. If $z = 1$, then $f(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ is a **harmonic series**, and hence f diverges.
2. If $z = i$, then

$$\begin{aligned} f(i) &= \sum_{n=1}^{\infty} \frac{i^n}{n} \\ &= i - \frac{1}{2} + \frac{-i}{3} + \frac{1}{4} + \frac{i}{5} - \frac{1}{6} \\ &= \left(-\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \dots \right) + i \left(1 - \frac{1}{3} + \frac{1}{5} + \dots \right). \end{aligned}$$

Observe that both the real and imaginary parts are alternating series where the absolute values of each term is decreasing, which, by the **alternating series test**, converge. Thus in this case, f converges.

Therefore, we observe that **both convergence and divergence may occur** on the boundary, depending on the value of z .

Note

We may not always exchange the position of \lim and $\sum_{a=1}^b$ when we consider an infinite sum (i.e. $b = \infty$). Here's an example why this is true. Consider the function $f(x) = \sum_{n=1}^{\infty} (x^n - x^{n-1})$ for $|x| < 1$. Is

$$\lim_{x \rightarrow 1} \sum_{n=1}^{\infty} (x^n - x^{n-1}) = \sum_{n=1}^{\infty} \lim_{x \rightarrow 1} (x^n - x^{n-1})$$

true?

Clearly, RHS is 0. For LHS, note that

$$\begin{aligned} f(x) &= \lim_{N \rightarrow \infty} \sum_{n=1}^N (x^n - x^{n-1}) \\ &= \lim_{N \rightarrow \infty} (x - x^2 + x^2 - x^3 + \dots + x^N - x^{N+1}) \\ &= \lim_{N \rightarrow \infty} (x - x^{N+1}) = x. \end{aligned}$$

So,

$$LHS = \lim_{x \rightarrow 1} x = 1$$

And we see that $RHS \neq LHS$.

Definition 9.1.1 (Entire Function)

A function f is said to be **entire** if f is holomorphic in **the entire complex plane**.

Exercise 9.1.1

Define $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Show that

1. the radius of convergence of this series is ∞ , and hence that e^z is an entire function.
(Hint: Use **Stirling's formula**: $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$)
2. $(e^z)' = e^z$

Solution

1. Using Stirling's formula, note that we have

$$e^z = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2\pi n}} \left(\frac{ez}{n}\right)^n$$

Chapter 10

Lecture 10 Jan 24 2018

10.1 Power Series (Continued 3)

10.1.1 Radius of Convergence (Continued 2)

A power series is infinitely \mathbb{C} -differentiable in its radius of convergence. All its derivatives are also power series, obtained by term-wise differentiation.

E.g.

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \quad \text{then} \quad f^{(2)}(z) = \sum_{n=0}^{\infty} n(n-1)c_n z^{n-2}$$

In general, we may have $\sum_{n=0}^{\infty} c_n (z - z_0)^n$, which is a power series centered at $z_0 \in \mathbb{C}$. Then, as before, the radius of convergence of this power series is given by

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}$$

So instead of having the disc of convergence centered around 0, we now have one that is centered around z_0 .

Corollary 10.1.1 (Corollary of Theorem 8.1.2)

From Theorem 8.1.2, we have shown that

$f(z)$ has a power series expansion at z_0
 (i.e. $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ in some
 neighbourhood of z_0) with radius of
 convergence $R > 0$
 \implies
 f is holomorphic at z_0

The converse of the statement above is true, i.e.

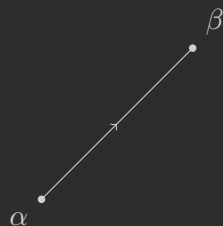
f is holomorphic at z_0
 \implies
 $f(z)$ has a power series expansion at z_0
 (i.e. $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ in some
 neighbourhood of z_0) with radius of
 convergence $R > 0$

This converse, however, is not possible to be proven given the current tools on our belt. And so we now have to venture into integrals in \mathbb{C} .

10.2 Integration in \mathbb{C}

10.2.1 Curves and Paths

Before we begin with the definition of a curve in \mathbb{C} , let us consider how a straight line should be described as a vector-valued function in the complex plane. For instance, if we have two points $\alpha, \beta \in \mathbb{C}$, and we want to describe the straight line connecting the two.



Let γ be the function that describes this line. We may then define $\gamma : [0, 1] \rightarrow \mathbb{C}$ to be either

$$\gamma(t) = \alpha + (\beta - \alpha)t \quad \text{or} \quad \gamma = \alpha(1 - t) + \beta t.$$

We would then have the following mapping:

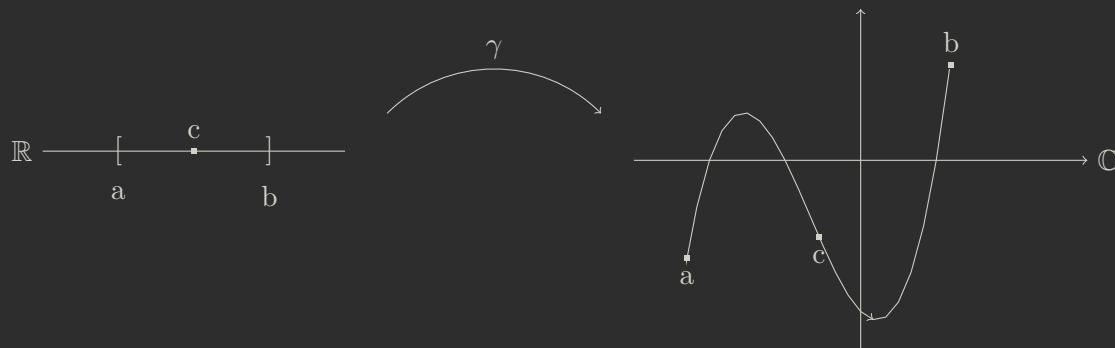


Figure 10.1: Mapping from $\mathbb{R} \rightarrow \mathbb{C}$ with γ , which is called **the curve γ**

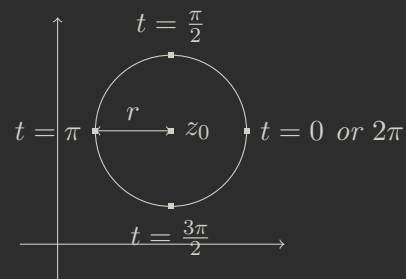
Definition 10.2.1 (Curves in \mathbb{C})

A curve in \mathbb{C} is a continuous function, $\gamma(t) : [a, b] \rightarrow \mathbb{C}$, where $a, b \in \mathbb{R}$. The image of γ in \mathbb{C} is called γ^* .

Example 10.2.1

Let $z_0 \in \mathbb{C}, r > 0$.

1. Let $\gamma : [0, 2\pi] \rightarrow \mathbb{C}$, such that $\gamma(t) = z_0 + re^{it}$.
 2. Let $\gamma' : [0, 1] \rightarrow \mathbb{C}$, such that $\gamma'(t) = z_0 + re^{2\pi it}$.
- The two functions above describe a circle centered at z_0 with radius r , anticlockwise-oriented.



We say that γ and γ' are equivalent parameterizations for the same oriented path.

Definition 10.2.2 (Equivalent Parameterization)

Let $\gamma_1 : [a, b] \rightarrow \mathbb{C}, \gamma_2 : [c, d] \rightarrow \mathbb{C}$ where $a, b, c, d \in \mathbb{R}$ describe the path γ^* . The two **parameterizations are said to be equivalent** if $\exists h : [a, b] \rightarrow [c, d]$ that is a bijection and a continuous function such that

$$\gamma_1(t) = \gamma_2(h(t))$$

where $t \in [a, b]$.

Note

We will not look at functions like the Weierstrass function in this course.

Definition 10.2.3 (Smooth Curve)

Let $\gamma : [a, b] \rightarrow \mathbb{C}$, $a, b \in \mathbb{C}$. γ is said to be smooth if its derivative γ' exists and is continuous on $[a, b]$ and $\forall t \in [a, b], \gamma'(t) \neq 0$.

Definition 10.2.4 (Piecewise Smooth)

Let $\gamma : [a, b] \rightarrow \mathbb{C}$. γ is said to be piecewise smooth if it is smooth on $[a, b]$ except on finitely many points in $[a, b]$.

Remark

Piecewise smooth curves shall be called paths.

10.2.2 Integral**Definition 10.2.5 (Integral of f over a path γ)**

Given a path $\gamma : [a, b] \rightarrow \mathbb{C}$ and $f : \mathbb{C} \rightarrow \mathbb{C}$, a function continuous on γ . We define the integral f along γ as

$$\int_{\gamma} f(z) dz := \int_a^b f(\gamma(t)) \gamma'(t) dt \quad (10.1)$$

where we let $z = \gamma(t)$ and hence $dz = \gamma'(t) dt$.

Remark

1. Suppose g is a complex-valued function, then

$$\int_a^b g(t) dt = \int_a^b \operatorname{Re}(g(t)) dt + i \int_a^b \operatorname{Im}(g(t)) dt$$

2. The integral of f along γ can be shown to be independent of the chosen parameterization for γ^* .

Proof

Let $a, b, c, d \in \mathbb{R}$, $\gamma_1 : [a, b] \rightarrow \mathbb{C}$, $\gamma_2 : [c, d] \rightarrow \mathbb{C}$ describe the same path γ^* . By *Definition 10.2.2*, define a bijection $h : [a, b] \rightarrow [c, d]$ that is a continuous function such that $t \mapsto \tau$, so that

$$\gamma_1(t) = \gamma_2(h(t)) = \gamma(\tau).$$

Note that

$$\begin{aligned} \gamma_1'(t) &= h'(t) \gamma_2'(h(t)) \text{ and} \\ h(t) = \tau &\implies h'(t) dt = d\tau. \end{aligned}$$

Now since h is a bijection, we claim that $h(a) = c$ while $h(b) = d$.

We know that h cannot be a constant function. Suppose h is an increasing function, then since $a \leq b$ and $c \leq d$, it is clear that $h(a) = c$ and $h(b) = d$. Similarly, if h is a decreasing function, then $h(a) = d$ and $h(b) = c$. But this is a contradiction to our supposition that γ_1 and γ_2 describe the same orientation. Thus h must be an increasing function, and hence we have $h(a) = c$ and $h(b) = d$.

(This can be more rigorous but that is an easy proof, and we may use perhaps the Approximation Property of \mathbb{R} to that end, which is a fun exercise that shall not be included within these covers.)

Now

$$\begin{aligned} \int_{\gamma_1} f(z)dz &= \int_a^b f(\gamma_1(t))\gamma_1'(t)dt \\ &= \int_a^b f(\gamma_2(h(t)))h'(t)\gamma_2'(h(t))dt \\ &= \int_c^d f(\gamma_2(\tau))\gamma_2'(\tau)d\tau \\ &= \int_{\gamma_2} f(z)dz \end{aligned}$$

This completes the proof.

Proposition 10.2.1 (Properties of integrals in \mathbb{C})

Let $\alpha, \beta \in \mathbb{C}$.

1. $\int_{\gamma} (\alpha f(z) + \beta g(z)) dz = \alpha \int_{\gamma} f(z)dz + \beta \int_{\gamma} g(z)dz.$