# PMATH450 — Lebesgue Integration and Fourier Analysis

Classnotes for Spring 2019

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# List of Procedures



The pre-requisite to this course is Real Analysis. We will use a lot of the concepts introduced in Real Analysis, at times without explicitly stating it. Refer to notes on PMATH351.

This course is spiritually broken into 2 pieces:

- Lebesgue Integration; and
- Fourier Analysis,

which is as the name of the course.

In this set of notes, we use a special topic environment called **culture** to discuss interesting contents related to the course, but will not be throughly studied and not tested on exams.

### Lecture 1 May 07th 2019

Since many of our results work for both  $\mathbb C$  and  $\mathbb R$ , we shall use  $\mathbb K$  throughout this course to represent either  $\mathbb C$  or  $\mathbb R$ .

#### 1.1 Riemannian Integration

#### **■** Definition 1 (Norm and Semi-Norm)

Let V be a vector space over  $\mathbb{K}$ . We define a **semi-norm** on V as a function

$$\nu:V\to\mathbb{R}$$

that satisfies

- 1. (Positive Semi-Definite)  $v(x) \ge 0$  for all  $x \in V$ ;
- 2.  $\nu(\kappa x) = |\kappa| \nu(x)$  for any  $\kappa \in \mathbb{K}$  and  $x \in V$ ; and
- 3. (Triangle Inequality)  $v(x+y) \le v(x) + v(y)$  for all  $x, y \in V$ .

If  $v(x) = 0 \implies x = 0$ , then we say that v is a **norm**. In this case, we usually write  $\|\cdot\|$  to denote the norm, instead of v.

#### 66 Note 1.1.1

• We sometimes call a semi-norm a pseudo-length.

#### Remark 1.1.1

Notice that we wrote  $v(x) = 0 \implies x = 0$  instead of  $v(x) = 0 \iff x = 0$ . This is because if  $z = 0 \in V$ , then

$$v(z) = v(0z) = 0.$$

#### Exercise 1.1.1

Show that if v is a semi-norm on a vector space V, then  $\forall x, y \in V$ ,

$$|\nu(x) - \nu(y)| \le \nu(x - y).$$

#### Proof

Notice that by condition (2) and (3), we have

$$\nu(x-y) \le \nu(x) + \nu(-y) = \nu(x) - \nu(y),$$

and

$$\nu(x - y) = -\nu(y - x) \ge -(\nu(y) - \nu(x)) = \nu(x) - \nu(y).$$

It follows that indeed

$$|\nu(x) - \nu(y)| \le \nu(x - y).$$

#### Example 1.1.1

The absolute value  $|\cdot|$  is a **norm** on  $\mathbb{K}$ .

#### 7

#### Example 1.1.2 (*p*-norms)

Consider  $N \ge 1$  an integer. We define a family of norms on

$$\mathbb{K}^N = \underbrace{K \times K \times \ldots \times K}_{N \text{ times}}.$$

1-norm

$$\|(x_n)_{n=1}^N\|_1 := \sum_{n=1}^N |x_n|.$$

#### Infinity-norm, ∞-norm

$$\left\| (x_n)_{n=1}^N \right\|_{\infty} \coloneqq \max_{1 \le n \le N} |x_n|.$$

#### Euclidean-norm, 2-norm

$$\left\| (x_n)_{n=1}^N \right\|_2 := \left( \sum_{n=1}^N |x_n|^2 \right)^{\frac{1}{2}}$$

It is relatively easy to check that the above norms are indeed norms, except for the 2-form. In particular, the triangle inequality is not as easy to show 1.

<sup>1</sup> See Minkowski's Inequality.

Less obviously so, but true nonetheless, we can define the following *p*-norms on  $\mathbb{K}^N$ :

$$\|(x_n)_{n=1}^N\|_p := \left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}},$$

for  $1 \le p < \infty$ .



#### Culture

Consider  $V = \mathbb{M}_n(\mathbb{C})$ , where  $n \in \mathbb{N}$  is fixed. For  $T \in \mathbb{M}_n(\mathbb{C})$ , we define the singular numbers of T to be

$$s_1(T) \ge s_2(T) \ge \dots \ge s_n(T) \ge 0,$$

where  $\sigma(T^*T) = \{s_1(T)^2, s_2(T)^2, \dots, s_n(T)^2\}$ , including multiplicity. Then we can define

$$||T||_p := \left(\sum_{i=1}^n s_i(T)^p\right)^{\frac{1}{p}}$$

for  $1 \leq p < \infty$ , which is called the p-norm of T on  $\mathbb{M}_n(\mathbb{C})$ .

#### <sup>2</sup> Note that $\mathbb{M}_n(\mathbb{C})$ is the set of $n \times n$ matrices over C.

#### Example 1.1.3

Let

$$V = \mathcal{C}([0,1],\mathbb{K}) = \{f : [0,1] \to \mathbb{K} \mid f \text{ is continuous } \}.$$

Then

$$||f||_{\sup} := \sup\{|f(x)| \mid x \in [0,1]\}$$

<sup>3</sup> defines a norm on  $C([0,1], \mathbb{K})$ .

A sequence  $(f_n)_{n=1)^{\infty}}$  in V converges in this norm to some  $f \in V$ , i.e.

$$\lim_{n\to\infty}\|f_n-f\|_{\sup}=0,$$

which means that  $(f_n)_{n=1}^{\infty}$  converges uniformly to f on [0,1].

 $^3$  Some authors use  $\|f\|_{\infty}$ , but we will have the notation  $\|[f]\|_{\infty}$  later on, and so we shall use  $\|f\|_{\sup}$  for clarity.

#### **■** Definition 2 (Normed Linear Space)

A normed linear space (NLS) is a pair  $(V, \|\cdot\|)$  where V is a vector space over  $\mathbb{K}$  and  $\|\cdot\|$  is a norm on V.

#### **■** Definition 3 (Metric)

Given an NLS  $(V, \|\cdot\|)$ , we can define a metric d on V (called the metric induced by the norm) as follows:

$$d: V \times V \to \mathbb{R}$$
  $d(x,y) = ||x - y||$ ,

such that

- $d(x,y) \ge 0$  for all  $x,y \in V$  and  $d(x,y) = 0 \iff x = y$ ;
- d(x, y) = d(y, x); and
- $d(x,y) \leq d(x,z) + d(y,z)$ .

#### **66** Note 1.1.2

Norms are all metrics, and so any space that has a norm will induce a metric on the space.

#### **■** Definition 4 (Banach Space)

We say that an NLS  $(V, \|\cdot\|)$  is complete or is a **Banach Space** if the corresponding (V,d), where d is the metric induced by the norm, is complete 4.

<sup>4</sup> Completeness of a metric space is such that any of its Cauchy sequences converges in the space.

#### Example 1.1.4

$$(\mathcal{C}([0,1],\mathbb{K}),\left\|\cdot\right\|_{sup})$$
 is a Banach space.

#### Example 1.1.5

We can define a 1-norm  $\|\cdot\|_1$  on  $\mathcal{C}([0,1],\mathbb{K})$  via

$$||f||_1 \coloneqq \int_0^1 |f|.$$

Then  $(\mathcal{C}([0,1],\mathbb{K}),\|\cdot\|_1)$  is an NLS.

#### Exercise 1.1.2

Show that  $(C([0,1],\mathbb{K}),\|\cdot\|_1)$  is not complete, which will then give us an example of a normed linear space that is not Banach.

#### Proof

Consider the sequence  $(f_n)_{n=1}^{\infty}$  of continuous functions given by

$$f_n(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2} \\ n\left(x + \frac{1}{2}\right) & \frac{1}{2} \le x \le \frac{1}{2} + \frac{1}{n} \\ 1 & \text{otherwise} \end{cases}$$

Note that the sequence  $(f_n)_{n=1}^{\infty}$  is indeed Cauchy: let  $\varepsilon > 0$  and  $|n-m|<rac{\varepsilon}{|x-rac{1}{2}|}$ , and then we have

$$|f_n(x) - f_m(x)| = \left| n\left(x - \frac{1}{2}\right) - m\left(x - \frac{1}{2}\right) \right|$$
$$= \left| (n - m)\left(x - \frac{1}{2}\right) \right| = |n - m|\left|x - \frac{1}{2}\right| < \varepsilon.$$

However, it is clear that the sequence  $(f_n)_{n=1}^{\infty}$  converges to the

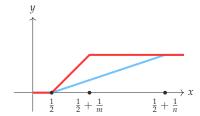


Figure 1.1: Sequence of functions  $(f_n)_{n=1}^{\infty}$ . We show for two indices n < m.

piecewise function (in particular, a non-continuous function)

$$f(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2} \\ 1 & x \ge \frac{1}{2} \end{cases}.$$

#### Example 1.1.6

If  $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$  and  $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$  are NLS's, and if  $T: \mathfrak{X} \to \mathfrak{Y}$  is a linear map, we define the **operator norm** of T to be

$$||T|| := \sup\{||T(x)||_{\mathfrak{Y}} \mid ||x||_{\mathfrak{X}} \le 1\}.$$

We set

$$B(\mathfrak{X},\mathfrak{Y}) := \{T : \mathfrak{X} \to \mathfrak{Y} \mid T \text{ is linear }, ||T|| < \infty\}.$$

Note that for any such linear map T,  $||T|| < \infty \iff T$  is continuous. Thus  $B(\mathfrak{X}, \mathfrak{Y})$  is the set of all continuous functions from  $\mathfrak{X}$  into  $\mathfrak{Y}$ .

Then 
$$(B(\mathfrak{X},\mathfrak{Y}),\|\cdot\|)$$
 is an NLS.

It is likely that we have seen this in Real Analysis.

#### Exercise 1.1.3

Show that  $(B(\mathfrak{X},\mathfrak{Y}),\|\cdot\|)$  is complete iff  $(\mathfrak{Y},\|\cdot\|_{\mathfrak{Y}})$  is complete.

#### 66 Note 1.1.3

One example of the last example is when  $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}}) = (\mathbb{K}, |\cdot|)$ . In this case,  $B(\mathfrak{X}, \mathbb{K})$  is known as the dual space of  $\mathfrak{X}$ , or simple the dual of  $\mathfrak{X}$ .

We are interested in integrating over Banach spaces.

#### **■** Definition 5 (Partition of a Set)

Let  $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$  be a Banach space and  $f : [a, b] \to \mathfrak{X}$  a function, where  $a < b \in \mathbb{R}$ . A partition P of [a, b] is a finite set

$$P = \{a = p_0 < p_1 < \dots < p_N = b\}$$

for some  $N \geq 1$ . The set of all partitions of [a,b] is denoted by  $\mathcal{P}[a,b]$ .

#### **■** Definition 6 (Test Values)

Let  $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$  be a Banach space and  $f: [a,b] \to \mathfrak{X}$  a function, where  $a < b \in \mathbb{R}$ . Let  $P \in \mathcal{P}[a,b]$ . A set

$$P^* := \{p_k^*\}_{k=1}^N$$

satisfying

$$p_{k-1} \le p_k^* \le p_k$$
, for  $1 \le k \le n$ 

is called a set of test values for P.

#### **■** Definition 7 (Riemann Sum)

Let  $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$  be a Banach space and  $f: [a,b] \to \mathfrak{X}$  a function, where  $a < b \in \mathbb{R}$ . Let  $P \in \mathcal{P}[a,b]$  and  $P^*$  its corresponding set of test values. We define the Riemann sum as

$$S(f, P, P^*) = \sum_{k=1}^{N} f(p_k^*)(p_k - p_{k-1}).$$

#### Remark 1.1.2

- 1. Note that because  $\square$  Definition 5,  $p_k p_{k-1} > 0$ .
- 2. When  $(\mathfrak{X},\|\cdot\|)\,=\,(\mathbb{R},|\cdot|),$  then this is the usual Riemann sum from first-year calculus.
- 3. In general, note that

$$\frac{1}{b-a}S(f, P, P^*) = \sum_{k=1}^{N} \lambda_k f(p_k^*),$$

where  $0 < \lambda_k = \frac{p_k - p_{k-1}}{b-a} < 1$  and 5

 $\sum_{k=1}^{N} \lambda_k = 1.$ 

 $^{5}$  via the fact that the  $\lambda_{k}$ 's form a telescoping sum

So  $\frac{1}{b-a}S(f,P,P^*)$  is an averaging of f over [a,b]. We call  $\frac{1}{b-a}S(f,P,P^*)$  the convex combination of the  $f(p_k^*)$ 's.

#### Example 1.1.7 (Silly example)

Let 
$$(\mathfrak{X} = \mathcal{C}([-\pi, \pi], \mathbb{K}), \|\cdot\|_{\sup})$$
. Let

$$f: [0,1] \to \mathfrak{X}$$
 such that  $x \mapsto e^{2\pi x} \sin 7\theta + \cos x \cos(12\theta)$ ,

where  $\theta \in [-\pi, \pi]$ . Now if we consider the partition

$$P = \left\{-\pi, \frac{1}{10}, \frac{1}{2}, \pi\right\}$$

and its corresponding test value

$$P^* = \left\{0, \frac{1}{3}, 2\right\},\,$$

then

$$\begin{split} S(f,P,P^*) &= f(0) \left(\frac{1}{10} + \pi\right) + f\left(\frac{1}{3}\right) \left(\frac{1}{2} - \frac{1}{10}\right) + f(2) \left(\pi - \frac{1}{2}\right) \\ &= \left(\sin 7\theta + \cos 12\theta\right) \left(\pi + \frac{1}{10}\right) \\ &+ \left(e^{\frac{2\pi}{3}} \sin 7\theta + \cos \frac{1}{3} \cos 12\theta\right) \left(\frac{2}{5}\right) \\ &+ \left(e^{4\pi} \sin 7\theta + \cos 2 \cos 12\theta\right) \left(\pi - \frac{1}{2}\right) \end{split}$$

#### **■** Definition 8 (Refinement of a Partition)

Let  $a < b \in \mathbb{R}$ , and  $P \in \mathcal{P}[a,b]$ . We say Q is a **refinement** of P is  $Q \in \mathcal{P}[a,b]$  and  $P \subseteq Q$ .

#### **66** Note 1.1.4

*In simpler words, Q is a "finer" partition that is based on P.* 

E Definition 9 (Riemann Integrable)

Let  $a < b \in \mathbb{R}$ ,  $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$  be a Banach space and  $f : [a, b] \to \mathfrak{X}$  be a function. We say that f is Riemann integrable over [a, b] if  $\exists x_0 \in \mathfrak{X}$ such that

$$\forall \varepsilon > 0 \quad \exists P \in \mathcal{P}[a, b],$$

such that if Q is any refinement of P, and  $Q^*$  is any set of test values of Q, then

$$||x_0 - S(f, Q, Q^*)||_{\mathfrak{X}} < \varepsilon.$$

In this case, we write

$$\int_a^b f = x_0.$$

#### ♦ Proposition 1 (Uniqueness of the Riemann Integral)

*If* f *is Riemann integrable over* [a,b]*, then the value of*  $\int_a^b f$  *is unique.* 

#### Proof

Suppose not, i.e.

$$\int_a^b f = x_0 \text{ and } \int_a^b f = y_0$$

for some  $x_0 \neq y_0$ . Then, let

$$\varepsilon = \frac{\|x_0 - y_0\|}{2},$$

which is > 0 since  $||x_0 - y_0|| > 0$ . Let  $P_{x_0}, P_{y_0} \in \mathcal{P}[a, b]$  be partitions corresponding to  $x_0$  and  $y_0$  as in the definition of Riemann integrability.

Then, let  $R = P_{x_0} \cup P_{y_0}$ , so that R is a common refinement of  $P_{x_0}$  and  $P_{y_0}$ . If Q is any refinement of R, then Q is also a common refinement of  $P_{x_0}$  and  $P_{y_0}$ . Then for any test values  $Q^*$  of Q, we have

$$2\varepsilon = \|x_0 - y_0\|$$

$$\leq \|x_0 - S(f, Q, Q^*)\| + \|S(f, Q, Q^*) - y_0\| < \varepsilon + \varepsilon = 2\varepsilon,$$

which is a contradiction.

Thus  $x_0 = y_0$  as required.

#### ■ Theorem 2 (Cauchy Criterion of Riemann Integrability)

Let  $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$  be a Banach space,  $a < b \in \mathbb{R}$  and  $f : [a, b] \to \mathfrak{X}$  be a function. TFAE:

- 1. f is Riemann integrable over [a, b];
- 2.  $\forall \varepsilon > 0$ ,  $R \in \mathcal{P}[a,b]$ , if P,Q is any refinement of R, and  $P^*$  (respectively  $Q^*$ ) is any test values of P (respectively Q), then

$$||S(f, P, P^*) - S(f, Q, Q^*)||_{\mathfrak{X}} < \varepsilon.$$

#### Proof

This is a rather straightforward proof. Suppose  $P,Q \in \mathcal{P}[a,b]$  is some refinement of the given partition  $R \in \mathcal{P}[a,b]$ , and  $P^*,Q^*$  any test values for P,Q, respectively. Then by assumption and  $P^*$  Proposition 1,  $\exists x_0 \in \mathfrak{X}$  such that

$$||x_0 - S(f, P, P^*)||_{\mathfrak{X}} < \frac{\varepsilon}{2} \text{ and } ||x_0 - S(f, Q, Q^*)||_{\mathfrak{X}} < \frac{\varepsilon}{2}.$$

It follows that

$$||S(f, P, P^*) - S(f, Q, Q^*)||_{\mathfrak{X}}$$

$$\leq ||x_0 - S(f, P, P^*)||_{\mathfrak{X}} + ||x_0 - S(f, Q, Q^*)||_{\mathfrak{X}}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

By hypothesis, wma  $\varepsilon = \frac{1}{n}$  for some  $n \ge 1$ , such that if P, Q are any refinements of the partition  $R_n \in \mathcal{P}[a,b]$ , and  $P^*, Q^*$  are the respective arbitrary test values, then

$$||S(f, P, P^*) - S(f, Q, Q^*)||_{\mathfrak{X}} < \frac{1}{n}$$

Now for each  $n \ge 1$ , define

$$W_n := \bigcup_{k=1}^n R_k \in \mathcal{P}[a,b],$$

so that  $W_n$  is a common refinement for  $R_1, R_2, \ldots, R_n$ . For each  $n \ge n$ 1, let  $W_n^*$  be an arbitrary set of test values for  $W_n$ . For simplicity, let us write

$$x_n = S(f, W_n, W_n^*)$$
, for each  $n \ge 1$ .

6

Claim:  $(x_n)_{n=1}^{\infty}$  is a Cauchy sequence If  $n_1 \ge n_2 > N \in \mathbb{N}$ , then

$$\|x_{n_1} - x_{n_2}\|_{\mathfrak{X}} = \|S(f, W_{n_1}, W_{n_1}^*) - S(f, W_{n_2}, W_{n_2}^*)\| < \frac{1}{N}$$

by our assumption, since  $W_{n_1}$ ,  $W_{n_2}$  are refinements of  $R_N$ . Then by picking  $N = \frac{1}{\varepsilon}$  for any  $\varepsilon > 0$ , we have that  $(x_n)_{n=1}^{\infty}$  is indeed a Cauchy sequence in  $\mathfrak{X}$ .

Since  $\mathfrak{X}$  is a Banach space, it is complete, and so  $\exists x_0 := \lim_{n \to \infty} x_n \in$  $\mathfrak{X}$ . It remains to show that, indeed,

$$x_0 = \int_a^b f.$$

Let  $\varepsilon > 0$ , and choose  $N \ge 1$  such that

- $\frac{1}{N} < \frac{\varepsilon}{2}$ ; and
- $k \ge N$  implies that  $||x_k x_0|| < \frac{\varepsilon}{2}$ .

Then suppose that V is any refinement of  $W_N$ , and  $V^*$  is an arbitrary set of test values of V. Then we have

$$||x_{0} - S(f, V, V^{*})||_{\mathfrak{X}} \leq ||x_{0} - x_{N}||_{\mathfrak{X}} + ||x_{N} - S(f, V, V^{*})||_{\mathfrak{X}}$$

$$< \frac{\varepsilon}{2} + ||S(f, W_{N}, W_{N}^{*}) - S(f, V, V^{*})||_{\mathfrak{X}}$$

$$< \frac{\varepsilon}{2} + \frac{1}{N} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

It follows that

$$\int_a^b f = x_0,$$

as desired.

<sup>6</sup> Note that it would be nice if for the finer and finer partitions that we have constructed, i.e. the  $W_n$ 's, give us a convergent sequence of Riemann sums, since it makes sense that this convergence will give us the final value that we want.

In first-year calculus, all continuous functions over  $\mathbb{R}$  are integrable. A similar result holds in Banach spaces as well. In the next lecture, we shall prove the following theorem.

#### **■** Theorem (Continuous Functions are Riemann Integrable)

Let  $(\mathfrak{X}, \|\cdot\|)$  be a Banach space and  $a < b \in \mathbb{R}$ . If  $f : [a,b] \to \mathfrak{X}$  is continuous, then f is Riemann integrable over [a,b].

### Lecture 2 May 9th 2019

#### 2.1 Riemannian Integration (Continued)

We shall now prove the last theorem stated in class.

#### Theorem 3 (Continuous Functions are Riemann Integrable)

Let  $(\mathfrak{X}, \|\cdot\|)$  be a Banach space and  $a < b \in \mathbb{R}$ . If  $f : [a, b] \to \mathfrak{X}$  is continuous, then f is Riemann integrable over [a, b].

#### **✓** Strategy

This is rather routine should one have gone through a few courses on analysis, and especially on introductory courses that involves Riemannian integration.

We shall show that if  $P_N \in \mathcal{P}[a,b]$  is a partition of [a,b] into  $2^N$  subintervals of equal length  $\frac{b-a}{2^N}$ , and if we use  $P_N^* = P_n \setminus \{a\}$  as the set of test values for  $P_N$ , which consists of the right-endpoints of each the subintervals in  $P_N$ , then the sequence  $(S(f,P_N,P_N^*))_{N=1}^{\infty}$  converges in  $\mathfrak{X}$  to  $\int_a^b f$ .

Note that this choice of partition is a valid move, since any of these  $P_N$ 's, for different N's, is a refinement of some other partition of [a,b], and if we choose a different set of test values, then we may as well consider an even finer partition.

First, note that since [a, b] is closed and bounded in  $\mathbb{R}$ , it is compact. Also, we have that X is a metric space (via the metric induced by the norm). This means that **any continuous function** f **on** [a, b] **is uniformly continuous on** [a, b]. In other words,

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, y \in [a, b]$$
$$|x - y| < \delta \implies ||f(x) - f(y)|| < \frac{\varepsilon}{2(b - a)}.$$

Claim:  $(S(f, P_N, P_N^*))_{N=1}^\infty$  is Cauchy Now by picking  $P_N \in \mathcal{P}[a, b]$  and set of test values  $P_N^*$  as described in the strategy above, we proceed by picking M > 0 such that  $\frac{b-a}{2^M} < \delta$ . Then for any  $K \ge L \ge M$ , since each of the subintervals have length  $\frac{b-a}{2^L}$  and  $\frac{b-a}{2^K}$  for  $P_L$  and  $P_K$  respectively, if we write

$$P_L = \{a = p_0 < p_1 < \ldots < p_{2^L} = b\}$$

and

$$P_K = \{a = q_0 \le q_1 < \ldots < q_{2^K} = b\},\$$

then  $p_j=q_j2^{K-L}$  1 for all  $0\leq j\leq 2^L$ . By uniform continuity, for  $1\leq j\leq 2^L$ , wma

$$||f(p_j^*) - f(q_s^*)|| < \frac{\varepsilon}{2(b-a)}, \text{ where } (j-1)2^{K-L} < s \le j2^{K-L}.$$

We can see that

$$||S(f, P_L, P_L^*) - S(f, P_K, P_K^*)||$$

$$= \left\| \sum_{j=1}^{2^L} \sum_{s=(j-1)2^{K-L}+1}^{j2^{K-L}} (f(p_j) - f(q_s))(q_s - q_{s-1}) \right\|$$

$$\leq \sum_{j=1}^{2^L} \sum_{s=(j-1)2^{K-L}+1}^{j2^{K-L}} ||f(p_j) - f(q_s)|| (q_s - q_{s-1})$$

$$\leq \sum_{j=1}^{2^L} \sum_{s=(j-1)2^{K-L}+1}^{j2^{K-L}} \frac{\varepsilon}{b-a} (q_s - q_{s-1})$$

$$= \frac{\varepsilon}{b-a} \sum_{s=1}^{2^K} (q_s - q_{s-1})$$

$$= \frac{\varepsilon}{2(b-a)} (b-a) = \frac{\varepsilon}{2}.$$

<sup>1</sup> This is not immediately clear on first read. Think of *a* as 0.

This proves our claim.

Since  $\mathfrak{X}$  is a Banach space, and hence complete, we have that the sequence  $(S(f, P_N, P_N^*))_{N=1}^{\infty}$  has a limit  $x_0 \in \mathfrak{X}$ .

It remains to show that  $\int_a^b f = x_0$ . <sup>2</sup>

Let  $\varepsilon > 0$ , and choose  $T \ge 1$  such that  $\frac{b-a}{2^T} < \delta^3$ , so that we have

$$||x_0-S(f,P_T,P_T^*)||<\frac{\varepsilon}{2}.$$

Now let  $R = \{a = r_0 < r_1 < ... < r_I = b\} \in \mathcal{P}[a, b]$  such that  $P_T \subseteq R$ . Then there exists a sequence

$$0 = j_0 < j_1 < \dots < j_{2^T} = J$$

such that

$$r_{j_k} = p_k$$
, where  $0 \le k \le 2^T$ .

Let  $R^*$  be any set of test values of R. Note that for  $j_{k-1} \le s \le j_k$ , it is clear that

$$|p_k^* - r_s^*| \le |p_k - p_{k-1}| = \frac{b-a}{2^T} < \delta.$$

Thus

$$||S(f, P_T, P_T^*) - S(f, R, R^*)||$$

$$\leq \sum_{k=1}^{2^T} \sum_{s_{j_{k-1}+1}}^{j_k} ||f(p_k^*) - f(r_s^*)|| (r_s - r_{s-1})$$

$$< \frac{\varepsilon}{2(b-a)} \sum_{k=1}^{2^T} \sum_{s_{j_{k-1}+1}}^{j_k} (r_s - r_{s-1})$$

$$= \frac{\varepsilon}{2(b-a)} (b-a) = \frac{\varepsilon}{2}.$$

Putting everything together, we have

$$||x_{0} - S(f, R, R^{*})||$$

$$\leq ||x_{0} - S(f, P_{T}, P_{T}^{*})|| + ||S(f, P_{T}, P_{T}^{*}) - S(f, R, R^{*})||$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

<sup>2</sup> The rest of this proof is similar to the above proof.

 $^3$  Note that this is still the same  $\delta$  as in the first  $\delta$  in this entire proof.

We can also find another refinement of  $P_T$ , say Q, that works similarly as in the case of R. It follows from  $\square$  Theorem 2 that

$$x_0 = \int_a^b f,$$

i.e. that f is indeed Riemann integrable over [a, b].

The following is a corollary whose proof shall be left as an exercise.

#### Corollary 4 (Piecewise Functions are Riemann Integrable)

A piecewise continuous function is also Riemann integrable: if f:  $[a,b] \to \mathfrak{X}$  is piecewise continuous, then f is Riemann integrable.

#### Exercise 2.1.1

Prove Corollary 4.

Let us exhibit a function that is not Riemann integrable.

#### **■** Definition 10 (Characteristic Function)

Given a subset E of a set  $\mathbb{R}$ , we define the characteristic function of E as a function  $\chi_E : \mathbb{R} \to \mathbb{R}$  given by

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

#### Example 2.1.1

Consider the set  $E = \mathbb{Q} \cap [0,1] \subseteq \mathbb{R}$ . Let  $P \in \mathcal{P}[0,1]$  such that

$$P = \{0 = p_0 < p_1 < \ldots < p_N = 1\},$$

and let

$$P^* = \{p_k^*\}_{k=1}^N \text{ and } P^{**} = \{p_k^{**}\}_{k=1}^N$$

be 2 sets of test values for *P*, such that we have

$$p_k^* \in \mathbb{Q}$$
 and  $p_k^{**} \in \mathbb{R} \setminus \mathbb{Q}$ .

Then we have

$$S(\chi_E, P, P^*) = \sum_{k=1}^{N} \chi_E(p_k^*)(p_k - p_{k-1})$$
$$= \sum_{k=1}^{N} 1 \cdot (p_k - p_{k-1})$$
$$= p_N - p_0 = 1 - 0 = 1,$$

and

$$S(\chi_E, P, P^{**}) = \sum_{k=1}^{N} \chi_E(p_k^{**})(p_k - p_{k-1})$$
$$= \sum_{k=1}^{N} 0 \cdot (p_k - p_{k-1})$$
$$= 0.$$

It is clear that the Cauchy criterion fails for  $\chi_E$ . This shows that  $\chi_E$  is not Riemann integrable.

#### Remark 2.1.1

Let us once again consider  $E = \mathbb{Q} \cap [0,1]$ . Note that E is denumerable <sup>4</sup>. We may thus write

<sup>4</sup> This means that *E* is countably infinite.

$$E = \{q_n\}_{n=1}^{\infty}.$$

*Now, for*  $k \ge 1$ *, define* 

$$f_k(x) = \sum_{n=1}^k \chi_{\{q_n\}}(x).$$

In other words,  $f_k = \chi_{\{q_1,...,q_k\}}$ . Furthermore, we have that

$$f_1 \leq f_2 \leq f_3 \ldots \leq \chi_E$$
.

*Moreover, we have that*  $\forall x \in [0,1]$ *,* 

$$\chi_E(x) = \lim_{k \to \infty} f_k(x),$$

and

$$\int_0^1 f_k = 0 \text{ for all } k \ge 1.$$

And yet, we have that  $\int_0^1 \chi_E$  does not exist!

WE WANT TO develop a different integral that will 'cover' for this 'pathological' behavior of where the Riemann integral fails.

The rough idea is as follows.

In Riemann integration, when integrating over an interval [a, b], we partitioned [a, b] into subintervals. This happens on the x-axis.

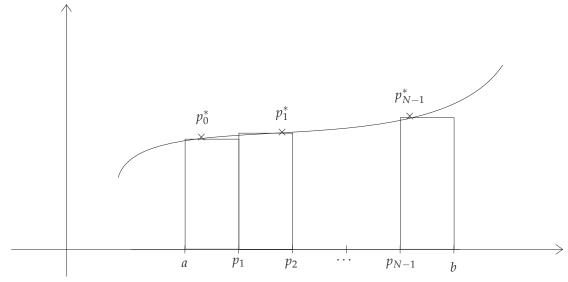


Figure 2.1: Rough illustration of how Riemann's integration works

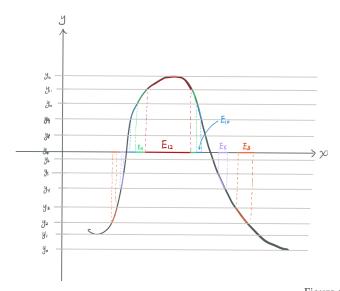
In each of the subintervals of the partition, we pick out a **test value**  $p_i^*$ , and basically draw a rectangle with base at  $[p_i, p_{i+1}]$  and height from 0 to  $p_i^*$ .

What we shall do now is that we **partition the range of** *f* **on the y-axis**, instead of the *x*-axis as we do in Riemannian integration.

In particular, given a function  $f : [a, b] \to \mathbb{R}$ , we first partition the

range of f into subintervals  $[y_{k-1}, y_k]$ , where  $1 \le k \le N$ . Then, we set

$$E_k = \{x \in [a, b] : f(x) \in [y_{k-1}, y_k]\} \text{ for } 1 \le k \le N.$$

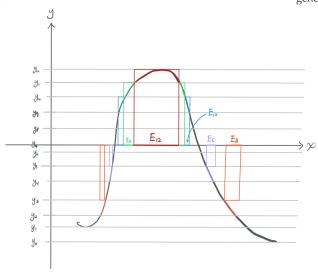


This will then allow us to estimate the integral of f over [a, b] by the expression

$$\sum_{k=1}^{N} y_k m E_k,$$

where each of the  $y_k m E_k$  are called **simple functions**. In the expression,  $mE_k$  denotes a "measure" <sup>5</sup> of  $E_k$ .

<sup>5</sup> Note that a measure is simply a generalization of the notion of 'length'.



We observe that  $E_k$  need not be a particularly well-behaved set.

Figure 2.3: Drawing out the rectangles of  $y_k m E_k$  from Figure 2.2.

Figure 2.2: A sketch of what's happening with the construction of the

However, note that we may rearrange the possibly scattered pieces of each  $E_k$  together, so as to form a 'continuous' base for the rectangle. We need our definition of a measure to be able to capture this.

The following is an analogy from Lebesgue himself on comparing Lebesgue integration and Riemann integration <sup>6</sup>:

I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral.

But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral.

The insight here is that one can freely arrange the values of te functions, all the while preserving the value of the integral.

- This requires us to have a better understanding of what a measure is.
- This process of rearrangement converts certain functions which are extremely difficult to deal with, or outright impossible, with the Riemann integral, into easily digestible pieces using Lebesgue integral.

#### 2.2 Lebesgue Outer Measure

Goals of the section

- 1. Define a "measure of length" on as many subsets of  $\mathbb R$  as possible.
- The definition should agree with our intuition of what a 'length' is.

#### Definition 11 (Length)

For  $a \leq b \in \mathbb{R}$ , we define the *length* of the interval (a,b) to be b-a, and

<sup>6</sup> Siegmund-Schultze, R. (2008). Henri Lesbesgue, in Timothy Gowers, June Barrow-Green, Imre Leader (eds.), Princeton Companion to Mathematics. Princeton University Press we write

$$\ell((a,b)) := b - a.$$

We also define

- $\ell(\emptyset) = 0$ ; and
- $\ell((a,\infty)) = \ell((-\infty,b)) = \ell((-\infty,\infty)) = \infty.$

#### **■** Definition 12 (Cover by Open Intervals)

Let  $E \subseteq \mathbb{R}$ . A countable collection  $\{I_n\}_{n=1}^{\infty}$  of open intervals is said to be a cover of E by open intervals if  $E \subseteq \bigcup_{n=1}^{\infty} I_n$ .

#### 66 Note 2.2.1

In this course, the only covers that we shall use are open intervals, and so we shall henceforth refer to the above simply as covers of E.

Before giving what immediately follows from the above, I shall present the following notion of an outer measure.

#### **■** Definition 13 (Outer Measure)

Let  $\emptyset \neq X$  be a set. An **outer measure**  $\mu$  on X is a function

$$\mu: \mathcal{P}(X) \to [0, \infty] := [0, \infty) \cup \{\infty\}$$

which satisfies

- 1.  $\mu \emptyset = 0$ ;
- 2. (monotone increment or monotonicity)  $E \subseteq F \subseteq X \implies \mu E \le$ μF; and
- 3. (countable subadditivity or  $\sigma$ -subadditivity)  $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu E_n.$$

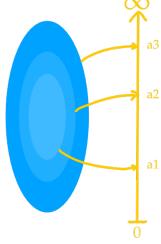


Figure 2.4: Idea of the outer measure

#### **66** Note 2.2.2

Note that by the monotonicity, the  $\sigma$ -subadditivity condition is equivalent to: given  $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$  and  $F \subseteq \bigcup_{n=1}^{\infty} E_n$ , we have that

$$\mu(F) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

#### **■** Definition 14 (Lebesgue Outer Measure)

We define the Lebesgue outer measure as a function  $m^*: \mathcal{P}(X) \to \mathbb{R}$  such that

$$m^*E := \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : E \subseteq \bigcup_{n=1}^{\infty} I_n \right\}.$$

We cheated a little bit by calling the above an outer measure, so let us now justify our cheating.

#### **♦** Proposition 5 (Validity of the Lebesgue Outer Measure)

*m*\* *is indeed an outer measure.* 

#### Proof

 $\mu\emptyset = 0$  We consider a sequence of sets  $\{I_n\}_{n=1}^{\infty}$  such that  $I_n = \emptyset$  for each  $n = 1, ..., \infty$ . It is clear that  $\emptyset \subseteq \bigcup_{n=1}^{\infty} I_n$ . Also, we have that  $\ell(I_n) = 0$  for all  $n = 1, ..., \infty$ . It follows that

$$0 \le m^*(\emptyset) \le \sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} 0 = 0,$$

where the inequality is simply by the definition of  $m^*$  being an infimum, **not to be confused with**  $\sigma$ **-subadditivity**. We thus have that

$$m^*(\emptyset) = 0.$$

Monotonicity Suppose  $E \subseteq F \subseteq \mathbb{R}$ , and  $\{I_n\}_{n=1}^{\infty}$  a cover of F. Then

$$E \subseteq F \subseteq \bigcup_{n=1}^{\infty} I_n$$
.

In particular, all covers of *F* are also covers of *E*, i.e.

$$\left\{\{J_m\}_{m=1}^{\infty}: E \subseteq \bigcup_{m=1}^{\infty} J_m\right\} \subseteq \left\{\{I_n\}_{n=1}^{\infty}: F \subseteq \bigcup_{n=1}^{\infty} I_n\right\}.$$

It follows that

$$m^*E < m^*F$$
.

 $\sigma$ -subaddivitity Consider  $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$  such that  $E \subseteq \bigcup_{n=1}^{\infty} E_n$ . **WTS** 

$$m^*E \leq \sum_{n=1}^{\infty} m^*E_n.$$

Now if the sum of the RHS is infinite, i.e. if any of the  $m^*E_n$  is infinite, then the inequality comes for free. Thus WMA  $\sum_{n=1}^{\infty} E_n <$  $\infty$ , and in particular that  $m^*E_n < \infty$  for all  $n = 1, ..., \infty$ .

To do this, let  $\varepsilon > 0$ . Since  $m^*E_n < \infty$  for all n, we can find covers  $\left\{I_k^{(n)}\right\}_{k=1}^{\infty}$  for each of the  $E_n$ 's such that

$$\sum_{k=1}^{\infty} \ell\left(I_k^{(n)}\right) < m^* E_n + \frac{\varepsilon}{2^n}.$$

Then, we have that

$$E \subseteq \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} I_k^{(n)}.$$

Then by  $m^*E$  being the infimum of the sum of lengths of the covering intervals, we have that

$$m^*E \le \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \ell\left(I_k^{(n)}\right)$$
$$\le \sum_{n=1}^{\infty} \left(m^*E_n + \frac{\varepsilon}{2^n}\right)$$
$$= \sum_{n=1}^{\infty} m^*E_n + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}$$
$$= \sum_{n=1}^{\infty} m^*E_n + \varepsilon.$$

Since  $\varepsilon$  was arbitrary, we have that

$$m^*E_n\leq \sum_{n=1}^\infty m^*E_n,$$

as desired.

## Corollary 6 (Lebesgue Outer Measure of Countable Sets is Zero)

*If*  $E \subseteq \mathbb{R}$  *is countable, then*  $m^*E = 0$ .

#### Proof

We shall prove for when E is denumerable, for the finite case follows a similar proof. Let us write  $E = \{x_n\}_{n=1}^{\infty}$ . Let  $\varepsilon > 0$  and

$$I_n = \left(x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}\right).$$

Then it is clear that  $\{I_n\}_{n=1}^{\infty}$  is a cover of E.

It follows that

$$0 \le m^* E \le \sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Thus as  $\varepsilon \to 0$ , we have that

$$m^*E = 0$$
,

as expected.

#### Corollary 7 (Lebesgue Outer Measure of Q is Zero)

We have that  $m^*\mathbb{Q} = 0$ .

IN THE PROOFS above that we have looked into, and based on the

intuitive notion of the length of an open interval, it is compelling to simply conclude that

$$m^*(a,b) = \ell(a,b) = b - a.$$

However, looking back at <a> Definition 14</a>, we know that that is not how  $m^*(a, b)$  is defined.

This leaves us with an interesting question:

how does our notion of measure  $m^*(a,b)$  of an interval compare with the notion of the length of an interval?

By taking  $I_1 = (a, b)$  and  $I_n = \emptyset$  for  $n \ge 2$ , it is rather clear that  ${I_n}_{n=1}^{\infty}$  is a cover of (a,b), and so we have

$$m^*(a,b) \le \ell(a,b) = b - a.$$
 (2.1)

However, the other side of the game is not as easy to confirm: we would have to consider all possible covers of (a, b), which is a lot.

Another question that we can ask ourselves seeing Equation (2.1) is why can't  $m^*(a, b)$  be something that is strictly less than the length to give us an even more 'precise' measurement?

To answer these questions, it is useful to first consider the outer measure of a closed and bounded interval, e.g. [a, b], since these intervals are compact under the Heine-Borel Theorem. This will give us a finite subcover for every infinite cover of the compact interval, which is easy to deal with.

We shall see that with the realization of the outer measure of a compact interval, we will also be able to find the outer measure of intervals that are neither open nor closed.

We shall prove the following proposition in the next lecture. Note that for the sake of presentation, I shall abbreviate the Lebesgue Outer Measure as LOM.

Suppose  $a < b \in \mathbb{R}$ . Then

1. 
$$m^*([a,b]) = b - a$$
; and therefore

2. 
$$m^*((a,b]) = m^*([a,b)) = m^*((a,b)) = b - a$$
.

# 3.1 Lebesgue Outer Measure Continued

# **♦** Proposition 8 (LOM of Arbitrary Intervals)

Suppose  $a < b \in \mathbb{R}$ . Then

- 1.  $m^*([a,b]) = b a$ ; and therefore
- 2.  $m^*((a,b)) = m^*([a,b)) = m^*((a,b)) = b a$ .

#### Proof

1. Consider  $a < b \in \mathbb{R}$ . Let  $\varepsilon > 0$ , and let

$$I_1 = \left(a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right)$$

and  $I_n = \emptyset$  for  $n \ge 2$ . Then  $\{I_n\}_{n=1}^{\infty}$  is a cover of [a,b]. This means that

$$m^*([a,b]) \le \sum_{n=1}^{\infty} \ell(I_n) = b - a + \varepsilon.$$

So for all  $\varepsilon \to 0$ , we have that

$$m^*([a,b]) \le b-a.$$

¹ Conversely, if [a,b] is covered by open intervals  $\{I_n\}_{n=1}^{\infty}$ , then by compactness of [a,b] (via the Heine-Borel Theorem), we know that we can cover [a,b] by finitely many of these intervals, and let us denote these as  $\{I_n\}_{n=1}^N$ , for some  $1 \le N < \infty$ .

¹ For the converse, we know that  $m^*([a,b]) = \inf \bigstar$ , where  $\bigstar$  is just a placeholder for you-know-what. So  $m^*([a,b])$  is one of the sums. So if we can show that for an arbitrary sum,  $\geq$  holds, our work is done.

WTS

$$\sum_{n=1}^{N} \ell(I_n) \ge b - a.$$

If LHS =  $\infty$ , then our work is done. Thus wlog, WMA each  $I_n = (a_n, b_n)$  is a finite interval. Note that we have

$$[a,b]\subseteq\bigcup_{n=1}^N(a_n,b_n).$$

In particular,  $a \in \bigcup_{n=1}^{N} I_n$ . Thus,  $\exists 1 \leq n_2 \leq N$  such that  $a \in I_{n_1}$ . Now if  $b_{n_1} > b$ , we shall stop this process for our work is done, since then  $[a,b] \subseteq I_{n_1}$ . Otherwise, if  $b_{n_1} \leq b$ , then  $b_{n_1} \in [a,b] \subseteq \bigcup_{n=1}^{N} I_n$ , which means that  $\exists 1 \leq n_2 \leq N$  such that  $b_{n_1} \in I_{n_2}$ . Notice that  $n_1 \neq n_2$ , since  $b_{n_1} \notin I_{n_1}$  but  $b_{n_1} \in I_{n_2}$ .



Figure 3.1: Our continual picking of  $I_{n_1}, I_{n_2}, \dots, I_{n_k}$ 

Now once again, if  $b_{n_2} > b$ , then we shall stop this process since our work is done. Otherwise, we have  $a < b_{n_2} \le b$ , and so  $\exists 1 \le n_3 \le N, n_3 \ne n_1, n_2$ , such that  $b_{n_2} \in I_3$ ...

We continue with the above process for as long as  $b_{n_k} \leq b$ . We can thus find, for each k,  $I_{n_{k+1}}$ , where  $n_{k+1} \in \{1, ..., N\} \setminus \{n_1, n_2, ..., n_k\}$ , such that  $b_{n_k} \in I_{n_{k+1}}$ .

However, since each of the  $I_{n_k}$ 's are different, and since we only have N such intervals, there must exists a  $K \leq N$  such that

$$b_{n_{K-1}} \leq b$$
 and  $b_{n_K} > b$ .

It now suffices for us to show that

$$\sum_{j=1}^K \ell(I_{n_j}) \ge b - a.$$

Observe that

$$\sum_{j=1}^{K} \ell(I_{n_j}) = (b_{n_K} - a_{n_K}) + (b_{n_{K-1}} - a_{n_{K-1}}) + \dots$$

$$+ (b_{n_2} - a_{n_2}) + (b_{n_1} - a_{n_1})$$

$$= b_{n_K} + (b_{n_{K-1}} - a_{n_K}) + (b_{n_{K-2}} - a_{n_{K-1}}) + \dots$$

$$\geq 0$$

$$+ (b_{n_1} - a_{n_2}) - a_{n_1}$$

$$\geq b_{n_K} - a_{n_1} \geq b - a.$$

Thus

$$\sum_{n=1}^{\infty} \ell(I_n) \geq \sum_{n=1}^{N} \ell(I_n) \geq \sum_{j=1}^{K} \ell(I_{n_j}) \geq b - a,$$

whence

$$m^*([a,b]) \ge b - a.$$

It follows that, indeed,

$$m^*([a,b]) = b - a.$$

#### 2. First, note that

$$m^*((a,b)) \le m^*([a,b]) \le b-a.$$

On the other hand, notice that  $\forall 0 < \varepsilon < \frac{b-a}{2}$ , we have that

$$[a+\varepsilon,b-\varepsilon]\subseteq (a,b),$$

and so by monotonicity,

$$(b-a)-2\varepsilon=m^*([a+\varepsilon,b-\varepsilon])\leq m^*((a,b)).$$

As  $\varepsilon \to 0$ , we have that

$$b - a \le m^*((a, b)) \le b - a.$$

So

$$m^*((a,b)) = b - a$$

as desired.

Finally, we have that

$$b-a=m^*((a,b)) \le m^*((a,b]) \le m^*([a,b]) = b-a,$$

and similarly

$$b-a=m^*((a,b)) \le m^*([a,b)) \le m^*([a,b]) = b-a.$$

Thus

$$m^*((a,b)) = m^*((a,b]) = m^*([a,b)) = b - a$$

as required.

# **♦** Proposition 9 (LOM of Infinite Intervals)

We have that  $\forall a, b \in \mathbb{R}$ ,

$$m^*((a,\infty)) = m^*([a,\infty))$$
$$= m^*((-\infty,b)) = m^*((-\infty,b])$$
$$= m^*\mathbb{R} = \infty.$$

## Proof

Observe that

$$(a, a + n) \subseteq (a, \infty)$$

for all  $n \ge 1$ . Thus

$$n = m^*((a, a + n)) \le m^*((a, \infty))$$

for all  $n \ge 1$ . Hence

$$m^*((a,\infty)) = \infty$$

by definition.

All other cases follow similarly.

# Corollary 10 (Uncountability of R)

 $\mathbb{R}$  is uncountable.

#### Proof

We have that

$$m^*\mathbb{R}=\infty\neq 0$$
,

and so it follows from ightharpoonup Corollary 6, we must have that  $\mathbb{R}$  is uncountable.

# **■** Definition 15 (Translation Invariant)

Let  $\mu$  be an outer measure on  $\mathbb{R}$ . We say that  $\mu$  is translation invariant if  $\forall E \subseteq \mathbb{R}$ ,

$$\mu(E) = \mu(E + \kappa)$$

for all  $\kappa \in \mathbb{R}$ , where

$$E + \kappa := \{x + \kappa : x \in E\}.$$

# ♦ Proposition 11 (Translation Invariance of the LOM)

The Lebesgue outer measure is translation invariant.

## Proof

Let  $E \subseteq \mathbb{R}$  and  $\kappa \in \mathbb{R}$ . Note that E is covered by open intervals  ${I_n}_{n=1}^{\infty}$  iff  $E + \kappa$  is covered by  ${I_n + \kappa}_{n=1}^{\infty}$ .

Claim:  $\forall n \geq 1$ ,  $\ell(I_n + \kappa) = \ell(I_n)$  Write

$$I_n = (a_n, b_n).$$

Then

$$I_n + \kappa = (a_n + \kappa, b_n + \kappa).$$

Observe that

$$\ell(I_n + \kappa) = b_n + \kappa - (a_n - \kappa) = b_n - a_n = \ell(I_n),$$

as claimed. ⊢

By the claim, it follows that

$$m^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : E \subseteq \bigcup_{n=1}^{\infty} \right\}$$
$$= \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n + \kappa) : E + \kappa \subseteq \bigcup_{n=1}^{\infty} (I_n + \kappa) \right\}$$
$$= m^*(E + \kappa).$$

#### Remark 3.1.1

Suppose  $E \subseteq \mathbb{R}$  and  $E = \bigcup_{n=1}^{\infty} E_n$ , where

$$E_i \cap E_j = \emptyset$$
 if  $i \neq j$ .

Now by  $\sigma$ -subadditivity of  $m^*$ , we have that

$$m^*E \leq \sum_{n=1}^{\infty} m^*E_n.$$

However, equality is not guaranteed. Consider the following case: if E = [0,1], we may have  $E_n = [0,1]$  for all n >= 1, in which case  $E = \bigcup_{n=1}^{\infty} E_n = [0,1]$ , but

$$m^*E = m^*[0,1] = 1 < \infty = \sum_{n=1}^{\infty} m^*E_n.$$

It would be desirable to have

$$m^*E=\sum_{n=1}^{\infty}m^*E_n,$$

when the  $E_i$ 's are pairwise disjoint, i.e.  $E = \bigcup_{n=1}^{\infty} E_n$ . In fact, this would agree with our intuition, that if the outer measure is going to be our 'length'. Consider the example  $A = [0,2] \cup [5,7]$ . Then we would expect  $m^*A = 2+2=4$ .

However, this is actually impossible for an arbitrary number of collections.

# Theorem 12 (Non-existence of a sensible Translation Invariant Outer Measure that is also $\sigma$ -additive)

There does not exist a translation-invariant outer measure  $\mu$  on  $\mathbb R$  that satisfies

- 1.  $\mu(\mathbb{R}) > 0$ ;
- 2.  $\mu[0,1] < \infty$ ; and
- 3.  $\mu$  is  $\sigma$ -additive; i.e. if  $\{E_n\}_{n=1}^{\infty}$  is a countable collection of disjoint subsets of  $\mathbb{R}$  that covers  $E \subseteq \mathbb{R}$ , then

$$\mu E = \sum_{n=1}^{\infty} \mu E_n.$$

Consequently, the Lebesgue outer measure  $m^*$  is **not**  $\sigma$ -additive.

#### Proof

Suppose to the contrary that such a  $\mu$  exists.

Step 1 Consider the relation  $\sim$  on  $\mathbb{R}$  such that  $x \sim y$  if  $x - y \in \mathbb{Q}$ .

Claim:  $\sim$  is an equivalence relation

- (reflexivity) We know that  $0 \in \mathbb{Q}$  and x x = 0. Thus  $x \sim x$ .
- (symmetry) Since Q is a field, it is closed under multiplication, and  $-1 \in \mathbb{Q}$ . Thus if  $x \sim y$ , then  $x - y \in \mathbb{Q}$ , and so (-1)(x  $y) = y - x \in \mathbb{Q}$ , which means  $y \sim x$ .
- (transitivity) Again, since Q is a field, it is closed under (this time) addition. Thus

$$x \sim y \land y \sim z \implies (x - y), (y - z) \in \mathbb{Q}$$
  
 $\implies (x - y) + (y - z) = x - z \in \mathbb{Q}.$ 

Thus  $x \sim z$ .

This proves the claim.  $\dashv$ 

Let

$$[x] := x + \mathbb{Q} := \{x + q : q \in \mathbb{Q}\}\$$

denote the equivalence class of x wrt  $\sim$ . Note that the set of equivalence classes, which we shall represent as

$$\mathcal{F} := \{ [x] : x \in \mathbb{R} \},$$

partitions  $\mathbb{R}$ , i.e.

- $[x] = [y] \iff x y \in \mathbb{Q}$ ; and
- $[x] \cap [y] = \emptyset$  otherwise.

Note that since  $\mathbb{Q}$  is **dense** in  $\mathbb{R}$ , we have that  $[x] = x + \mathbb{Q}$  is also dense in  $\mathbb{R}$ , for all  $x \in \mathbb{R}$ . Then for each  $^2 F \in \mathcal{F}$ ,  $\exists x_F \in F$  such that

<sup>2</sup> Notice that here, we have invoked the Axiom of Choice .

$$0 \le x_F \le 1$$
.

Now consider the set

$$\mathbb{V} := \{x_F : F \in \mathcal{F}\} \subseteq [0,1],$$

which is called Vitali's Set.

Step 2 Since  $\mathcal{F}$  partitions  $\mathbb{R}$ , we have that

$$\mathbb{R} = \bigcup_{F \in \mathcal{F}} F = \bigcup_{F \in \mathcal{F}} [x_F]$$

$$= \bigcup_{F \in \mathcal{F}} x_F + \mathbb{Q}$$

$$= \mathbb{V} + \mathbb{Q} := \{x + q : q \in \mathbb{Q}, x \in \mathbb{V}\}.$$

Step 3 Claim:  $p \neq q \in \mathbb{Q} \implies (\mathbb{V} + p) \cap (\mathbb{V} + q) = \emptyset$  Suppose not, and suppose  $\exists y \in (\mathbb{V} + p) \cap (\mathbb{V} + q)$ . Then  $\exists F_1, F_2 \in \mathcal{F}$  such that

$$y = x_{F_1} + p = x_{F_2} + q. (3.1)$$

Then we may rearrange the above equation to get

$$x_{F_1}-x_{F_2}=q-p\in\mathbb{Q}.$$

This implies that

$$[x_{F_1}] = [x_{F_2}] \implies F_1 = F_2$$

since V consists of one unique representative from each of the equivalence classes. However, this would mean that

$$x_{F_1} = x_{F_2}$$
.

Since  $p \neq q$ , we have that

$$x_{F_1} + p \neq x_{F_2} + q,$$

which contradicts Equation (3.1). Thus

$$(\mathbb{V}+p)\cap(\mathbb{V}+q)=\emptyset,$$

as claimed.  $\dashv$ 

This in turn means that the  $\mathbb{V} + q$ , for each  $q \in \mathbb{Q}$ , also partitions  $\mathbb{R}$ . In other words, if we write  $\mathbb{Q} = \{p_n\}_{n=1}^{\infty}$ , then

$$\mathbb{R} = \mathbb{V} + \mathbb{Q} = \bigcup_{n=1}^{\infty} \mathbb{V} + p_n.$$

Now, note that

$$0 \neq \mu \mathbb{R} \stackrel{(1)}{=} \sum_{n=1}^{\infty} \mu(\mathbb{V} + p_n) \stackrel{(2)}{=} \sum_{n=1}^{\infty} \mu(\mathbb{V}),$$

where (1) is by  $\mu$  being  $\sigma$ -additive and (2) is by  $\mu$  being translation invariant, both directly from our assumptions. This means that

$$\mu V > 0$$
.

Step 4 Now consider  $S = \mathbb{Q} \cap [0,1]$  such that S is denumerable. Write

$$S = \{s_n\}_{n=1}^{\infty}.$$

Note that for all  $n \ge 1$ ,

$$\mathbb{V} \subseteq [0,1] \implies \mathbb{V} + s_n \subseteq [0,2],$$

and as proven above

$$i \neq j \implies (\mathbb{V} + s_i) \cap (\mathbb{V} + s_i) = \emptyset.$$

Thus it follows that

$$\mu\left(\bigcup_{n=1}^{\infty} \mathbb{V} + s_n\right) = \sum_{n=1}^{\infty} \mu(\mathbb{V} + s_n) = \sum_{n=1}^{\infty} \mu(\mathbb{V}) = \infty.$$

Also,

$$\mu\left(\bigcup_{n=1}^{\infty} \mathbb{V} + s_n\right) = \sum_{n=1}^{\infty} \mu(\mathbb{V} + s_n)$$

$$\leq \mu([0,2]) = \mu([0,1] \cup ([0,1]+1))$$

$$\leq \mu[0,1] + \mu([0,1]+1)$$

$$= 2\mu([0,1]) = 2 < \infty,$$

contradicting what we have right above.

Therefore, no such  $\mu$  exists.

With the realization of Theorem 12, we find ourselves facing a losing dilemma: we may either

- 1. be happy with the Lebesgue outer measure  $m^*$  for all subsets  $E \subseteq \mathbb{R}$ , which would agree with our intuitive notion of length, at the price of  $\sigma$ -additivity; or
- 2. restrict the **domain** of our function  $m^*$  to some family of subsets of  $\mathbb{R}$ , where  $m^*$  would have  $\sigma$ -additivity.

We shall adopt the second approach. We shall call the collection of sets where  $m^*$  has  $\sigma$ -additivity as the collection of Lebesgue measurable sets.

# 3.2 *Lebesgue Measure*

We shall first introduce Carathéodory's definition of a Lebesgue measurable set.

# **■** Definition 16 (Lebesgue Measureable Set)

*A set*  $E \subseteq \mathbb{R}$  *is said to be Lebesgue measurable if,*  $\forall X \subseteq \mathbb{R}$ *,* 

$$m^*X = m^*(X \cap E) + m^*(X \setminus E).$$

We denote the collection of all Lebesgue measurable sets as  $\mathfrak{M}(\mathbb{R})$ .

#### Remark 3.2.1

Since we shall almost exclusively focus on the Lebesgue measure, we shall hereafter refer to "Lebesgue measurable sets" as simply "measurable sets".

# **66** Note 3.2.1

*I shall quote and paraphrase this remark from our course notes* <sup>3</sup>:

*Informally, we see that a set E*  $\mathbb{R}$  is measurable provided that it is a "universal slicer", that it "slices" every other set X into two disjoint sets, into where the Lebesgue outer measure is

Also, note that we get the following inequality for free, simply from  $\sigma$ -subadditivity of  $m^*$ :

$$m^*X \le m^*(X \cap E) + m^*(X \setminus E).$$

Thus, it suffices for us to check if the reverse inequality holds for all sets  $X \subseteq \mathbb{R}$ .

Before ploughing forward to getting out hands dirty with examples, let us first study a result on a structure of  $\mathfrak{M}(\mathbb{R})$  that is rather

<sup>3</sup> Marcoux, L. W. (2018). PMath 450 Introduction to Lebesgue Measure and Fourier Analysis. (n.p.)

interesting. 4

# **■** Definition 17 (Algebra of Sets)

A collection  $\Omega \subseteq \mathcal{P}(\mathbb{R})$  is said to be an algebra of sets if

- 1.  $\mathbb{R} \in \Omega$ ;
- 2. (closed under complementation)  $E \in \Omega \implies E^C \in \Omega$ ; and
- 3. (closed under finite union) given  $N \ge 1$  and  $\{E_n\}_{n=1}^N \subseteq \Omega$ , then

$$\bigcup_{n=1}^N E_n \in \Omega.$$

We say that  $\Omega$  is a  $\sigma$ -algebra of sets if

- 1.  $\Omega$  is an algebra of sets; and
- 2. (closed under countable union) if  $\{E_n\}_{n=1}^{\infty} \subseteq \Omega$ , then

$$\bigcup_{n=1}^{\infty} E_n \in \Omega.$$

## 66 Note 3.2.2

We often call a  $\sigma$ -algebra of sets as simply a  $\sigma$ -algebra.

### **P**Theorem 13 ( $\mathfrak{M}(\mathbb{R})$ is a $\sigma$ -algebra)

*The collection*  $\mathfrak{M}(\mathbb{R})$  *of Lebesgue measurable sets in*  $\mathbb{R}$  *is a*  $\sigma$ *-algebra.* 

Due to time constraints, we shall prove the first 2 requirements in this lecture and prove the last requirement next time (which is also really long).

Proof

<sup>4</sup> For those who has dirtied themselves in the world of probability and statistics, especially probability theory, get ready to get excited!  $\mathbb{R} \in \mathfrak{M}(\mathbb{R})$  Observe that  $\forall X \subseteq \mathbb{R}$ ,

$$m^*X = m^*X + 0 = m^*X + m^*\emptyset = m^*(X \cap \mathbb{R}) + m^*(X \setminus \mathbb{R})$$

 $E \in \mathfrak{M}(\mathbb{R}) \implies E^C \in \mathfrak{M}(\mathbb{R})$  Observe that  $\forall X \subseteq \mathbb{R}$ , since  $E \in \mathbb{R}$  $\mathfrak{M}(\mathbb{R})$ , we have

$$m^*X = m^*(X \cap E) + m^*(X \setminus E)$$

$$= m^*(X \cap (E^C)^C) + m^*(X \cap E^C)$$

$$= m^*(X \setminus E^C) + m^*(X \cap E^C)$$

$$= m^*(X \cap E^C) + m^*(X \setminus E^C)$$

$$= m^*(X \cap E^C) + m^*(X \setminus E^C)$$
rearrangement

Thus  $E^C \in \mathfrak{M}(\mathbb{R})$ .

# 4.1 Lebesgue Measure (Continued)

Recalling the last theorem we were in the middle of proving, it remains for us to prove that  $\mathfrak{M}(\mathbb{R})$  is closed under arbitrary unions of its elements.

But before we dive in, let's first have a little pep talk.

# **✓** Strategy

Since  $m^*$  is  $\sigma$ -subadditive, given  $\{E_n\}_{n=1}^{\infty}$ , we need only prove that  $\forall X \subseteq \mathbb{R}$ ,

$$m^*X \ge m^* \left(X \cap \bigcup_{n=1}^{\infty} E_n\right) + m^* \left(X \setminus \bigcup_{n=1}^{\infty} E_n\right).$$

Recall our discussion near the end of Section 3.1. We want  $\sigma$ -additivity, especially when we are given a set of disjoint intervals. However, our  $E_n$ 's are arbitrary, and so they are not necessarily disjoint.

It helps if one has seen how we can slice  $\mathbb{R}$  up into disjoint unions, and consequently we can do so for any of its subsets. We shall not take that for granted and immediately use it, but we shall work through this proof in the spirit of that. We shall see how we can slice  $\mathbb{R}$  up in A1.

Once we can, in some way, express  $\bigcup_{n=1}^{\infty} E_n$  as a disjoint union of intervals, we will then show that, indeed, we have  $\sigma$ -additivity instead of  $\sigma$ -subadditivity on this disjoint union.

 $\mathfrak{M}(\mathbb{R})$  is closed under arbitrary unions Suppose  $\{E_n\}_{n=1}^{\infty}\subseteq \mathfrak{M}(\mathbb{R})$ . To show that  $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{M}(\mathbb{R})$ , WTS

$$m^*X = m^* \left(X \cap \bigcup_{n=1}^{\infty} E_n\right) + m^* \left(X \setminus \bigcup_{n=1}^{\infty} E_n\right).$$

Since  $m^*$  is  $\sigma$ -subadditive, it suffices for us to show that

$$m^*X \ge m^* \left(X \cap \bigcup_{n=1}^{\infty} E_n\right) + m^* \left(X \setminus \bigcup_{n=1}^{\infty} E_n\right).$$
 (4.1)

Step 1 Consider

$$H_n = \bigcup_{i=1}^n E_i, \quad \forall n \geq 1.$$

Claim:  $H_n \in \mathfrak{M}(\mathbb{R})$ ,  $\forall n \geq 1$  We shall prove this by induction on n.

When n = 1, we have  $H_1 = E_1 \in \mathfrak{M}(\mathbb{R})$  by assumption, and so we are done. Suppose that  $H_k \in \mathfrak{M}(\mathbb{R})$  for some  $k \in \mathbb{N}$ . Consider n = k + 1.

Since we will need the piece  $X \cap H_{k+1}$ , first, notice that

$$X \cap H_{k+1} = X \cap (H_k \cup E_{k+1}) = (X \cap H_k) \cup ((X \setminus H_k) \cap E_{k+1}),$$

and in particular that

$$X \cap H_{k+1} = X \cap (H_k \cup E_{k+1}) \subseteq (X \cap H_k) \cup ((X \setminus H_k) \cap E_{k+1}).$$
 (4.2)

This may be (will be) useful later on, and we can guess that we will be using  $\sigma$ -subadditivity on this.

By the IH, since  $H_k \in \mathfrak{M}(\mathbb{R})$ , we have

$$m^*X = m^*(X \cap H_k) + m^*(X \setminus H_k).$$

Notice the similarity between the above equation and Equation (4.2), where we are just off by that  $\cap E_{k+1}$ .

Since  $E_{k+1} \in \mathfrak{M}(\mathbb{R})$ , we have

$$m^*(X \setminus H_k) = m^*(X \setminus H_k \cap E_{k+1}) + m^*(X \setminus H_k \setminus E_{k+1}).$$

To clean the above equation up a little bit, notice that by De Morgan's Law,

$$X \setminus H_k \setminus E_{k+1} = X \cap \bigcup_{i=1}^k E_i^C \cap E_{k+1}^C = X \setminus H_{k+1}.$$

So

$$m^*(X \setminus H_k) = m^*(X \setminus H_k \cap E_{k+1}) + m^*(X \setminus H_{k+1}).$$

Thus

$$m^*X = m^*(X \cap H_k) + m^*(X \setminus H_k \cap E_{k+1}) + m^*(X \setminus H_{k+1}).$$

Using Equation (4.2) and  $\sigma$ -subadditivity, we have that

$$m^*X \ge m^*(X \cap H_{k+1}) + m^*(X \setminus H_{k+1}),$$

which is what we need. Thus  $\forall k \geq 1$ ,  $H_k \in \mathfrak{M}(\mathbb{R})$ .  $\dashv$ 

Step 2 Consider  $F_1 = H_1 = E_1 \in \mathfrak{M}(\mathbb{R})$ , and for  $k \geq 2$ ,

$$F_k = H_k \setminus H_{k-1} = H_k \cap H_{k-1}^C.$$

<sup>1</sup> Claim:  $\forall k \geq 2$ ,  $F_k \in \mathfrak{M}(\mathbb{R})$  First, notice that

$$F_k^C = (H_k \cap H_{k+1}^C)^C = H_k^C \cup H_{k+1}.$$

By **step 1** <sup>2</sup>, we have that  $F_k^C \in \mathfrak{M}(\mathbb{R})$ , and thus by closure under complementation,  $F_k \in \mathfrak{M}(\mathbb{R})$ .

Also, note that the  $F_i$ 's are pairwise disjoint. Suppose not, i.e. that  $\exists x \in F_a \cap F_b$  for some  $a, b \ge 1$  and  $a \ne b$ . Wlog, wma a < b. Note that  $H_a \subseteq H_b$ , since

$$H_a = \bigcup_{i=1}^a E_i \subsetneq \bigcup_{i=1}^b E_i = H_b.$$

Since  $F_b = H_b \setminus H_{b-1}$ ,

$$x \in F_b \implies x \notin \bigcup_{i=1}^{b-1} E_i \supseteq \bigcup_{i=1}^a E_i,$$

- <sup>1</sup> Note that we cannot assume that  $\mathfrak{M}(\mathbb{R})$  is closed under finite intersections because that is part of what we want to prove.
- <sup>2</sup> I need to get this clarified.

and so  $x \notin E_i$  for  $1 \le i \le a \le b - 1$ . But we assumed that

$$x \in F_a = H_a \setminus H_{a-1}$$
,

i.e. it must be that  $x \in E_a$ , a contradiction.

Step 3 We now have

$$E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} H_i = \bigcup_{i=1}^{\infty} F_i.$$

Equation (4.1) becomes <sup>3</sup>

$$m^*X \ge m^* \left(X \cap \left(\bigcup_{i=1}^{\infty} F_i\right)\right) + m^* \left(X \setminus E\right).$$

Since the  $F_i$ 's are disjoint, we expect

$$m^*\left(X\cap\bigcup_{i=1}^{\infty}F_i\right)=\sum_{i=1}^{\infty}m^*(X\cap F_i).$$

i.e. for every n,

$$m^*\left(X\cap\bigcup_{i=1}^n F_i\right)=\sum_{i=1}^n m^*(X\cap F_i).$$

Let's prove this inductively. It is clear that case n=1 is trivially true. Suppose that this is true up to some  $k \in \mathbb{N}$ . Consider case n=k+1. Since  $F_{k+1} \in \mathfrak{M}(\mathbb{R})$ , we have that <sup>4</sup>

<sup>4</sup> This is quite a smart trick!

$$\begin{split} m^* \left( X \cap \bigcup_{i=1}^{k+1} F_i \right) \\ &= m^* \left( X \cap \bigcup_{i=1}^{k+1} F_i \cap F_{k+1} \right) + m^* \left( \left( X \setminus \bigcup_{i=1}^{k=1} F_i \right) \setminus F_{k+1} \right) \\ &= m^* \left( X \cap F_{k+1} \right) + m^* \left( X \cap \bigcup_{i=1}^{k} F_i \right) \\ &= m^* \left( X \cap F_{k+1} \right) + \sum_{i=1}^{k} m^* (X \cap F_i) \\ &= \sum_{i=1}^{k+1} m^* (X \cap F_i). \end{split}$$

Our claim is complete by induction.

<sup>3</sup> I refrained from changing the second term to the disjoint union. Retrospectively (i.e. once you're done with the proof), it makes sense to not consider this move, since there is no point looking at *X* take away a bunch of disjoint intervals.

Step 4 With Step 3, Equation (4.1) has become

$$m^*X \ge \sum_{i=1}^{\infty} m^*(X \cap F_i) + m^*(X \setminus E).$$

<sup>5</sup> Since  $H_k \in \mathfrak{M}(\mathbb{R})$  for each  $k \geq 1$ , we have

$$m^*X = m^*(X \cap H_k) + m^*(X \setminus H_k). \tag{*}$$

<sup>5</sup> This is a reward for the clear-minded, cause I certainly did not find it an obvious step to take.

Since

$$H_k = \bigcup_{i=1}^k E_i = \bigcup_{i=1}^\infty E_i = E,$$

we have that

$$X \setminus H_k \supseteq X \setminus E$$
,

for each  $k \ge 1$ . Thus by monotonicity, Equation (\*) becomes

$$m^*X \ge m^*(X \cap H_k) + m^*(X \setminus E)$$

$$= m^* \left( X \cap \left( \bigcup_{i=1}^{\infty} F_i \right) \right) + m^*(X \setminus E)$$

$$= \sum_{i=1}^{k} m^*(X \cap F_i) + m^*(X \setminus E),$$

for each  $k \ge 1$ .

By letting  $k \to \infty$ , we have that

$$m^*X \ge \sum_{i=1}^{\infty} m^*(X \cap F_i) + m^*(X \setminus E).$$

Note that

$$X \cap E = X \cap \bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} (X \cap F_i).$$

By  $\sigma$ -subadditivity, we have that

$$m^*(X \cap E) \le \sum_{i=1}^{\infty} m^*(X \cap F_i).$$

Therefore

$$m^*X \ge m^*(X \cap E) + m^*(X \setminus E),$$

which is what we want!

# 66 Note 4.1.1 (Post-mortem for proof of **PTheorem 13**)

In steps 1 - 3, we try to slice  $\bigcup_{n=1}^{\infty} E_n$  into disjoint **measurable** intervals  $F_i$ 's. Along the process of constructing them, it is the showing of them being measurable that takes up most of the proof, since we require induction.

# ♦ Proposition 14 (Some Lebesgue Measurable Sets)

- 1. If  $E \subseteq \mathbb{R}$  and  $m^*E = 0$ , then E is Lebesgue measurable.
- 2.  $\forall b \in \mathbb{R}, (-\infty, b) \in \mathfrak{M}(\mathbb{R}).$
- 3. Every open and every closed set is Lebesgue measurable.

# Proof

1. Let  $X \subseteq \mathbb{R}$ . Note that  $X \setminus E \subseteq X$ , and so  $\sigma$ -subadditivity gives

$$m^*X \ge m^*(X \setminus E). \tag{4.3}$$

On the other hand,  $X \cap E \subseteq E$ , and so

$$m^*(X \cap E) \le m^*E = 0 \implies m^*(X \cap E) = 0.$$

Thus, from Equation (4.3),

$$m^*X \ge ml * (X \setminus E) = m^*(X \cap E) + m^*(X \setminus E).$$

Hence  $E \in \mathfrak{M}(\mathbb{R})$  as required.

2. Let  $b \in \mathbb{R}$  and  $X \subseteq \mathbb{R}$  be arbitrary. WTS

$$m^*X > m^*(X \cap (-\infty, b)) + m^*(X \setminus (-\infty, b)).$$

<sup>6</sup> Let E = (-∞, b). Note that if  $m^*X = ∞$ , then there is nothing to show. Thus WMA  $m^*X < ∞$ . In this case, let ε > 0, and

<sup>6</sup> We will look at  $X \cap (\infty, b)$  and  $X \setminus (-\infty, b)$  more closely, and then realize that since we can cover X, we can "extend" this cover for these disjoint pieces by taking intersections and set removals on each of the covering sets.

 $\{I_n\}_{n=1}^{\infty}$  a cover of *X* by open intervals, where we write

$$I_n = (a_n, b_n)$$

for each  $n \ge 1$ , so that <sup>7</sup>

$$\sum_{n=1}^{\infty} \ell(I_n) < m^* X + \varepsilon.$$

For each  $n \ge 1$ , consider the sets

$$J_n = I_n \cap E + I_n \cap (-\infty, b)$$

and

$$K_n = I_n \setminus E = I_n \setminus (\infty, b) = I_n \cap [b, \infty).$$

The following table captures all possible  $J_n$ 's and  $K_n$ 's:

$$\begin{array}{c|cccc} Case & 1 & 2 & 3 \\ \hline b & > b_n & \in I_n & < a_n \\ \hline J_n & I_n & (a_n,b) & \varnothing \\ K_n & \varnothing & [b,b_n) & I_n \\ \hline \end{array}$$

Notice that  $\{J_n\}_{n=1}^{\infty}$  is an open cover for  $X \cap E$ .  $\{K_n\}_{n=1}^{\infty}$  is also a cover of  $X \setminus E$  but it is not an open cover (the only covers of which we consider in this course). Thus, we consider a small extension  $L_n$  of  $K_n$  such that

- if  $K_n = \emptyset$ , then  $L_n = \emptyset$ ;
- if  $K_n = I_n$ , then  $L_n = I_n$ ; and
- if  $K_n = [b, b_n)$ , then  $L_n = (b \frac{\varepsilon}{2^n}, b_n)$ .

Then  $\{L_n\}_{n=1}^{\infty}$  is a cover of  $X \setminus E$ . By  $\sigma$ -subadditivity of  $m^*$ , we have that

$$m^*(X \cap E) \le \sum_{n=1}^{\infty} \ell(J_n)$$

and

$$m^*(X \setminus E) \leq \sum_{n=1}^{\infty} \ell(L_n).$$

Thus

$$m^*(X \cap E) + m^*(X \setminus E) \le \sum_{n=1}^{\infty} (\ell(J_n) + \ell(L_n)).$$

<sup>7</sup> Note that this is legitimate because  $m^*X$  is the infimum of such sums on the LHS, and we can definitely find such a cover as a result. Also, there is no harm in assuming that each of the  $I_n$ 's are non-empty, since we may simply remove all the empty  $I_n$ 's from the cover.

Table 4.1: Possible outcomes of  $J_n$  and  $K_n$ , for each  $n \ge 1$ 

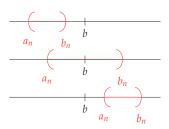


Figure 4.1: Three possible scenarios of where b stands for different  $I_n$ 's

Now, notice that in cases 1 and 3,

$$\ell(I_n) + \ell(L_n) = \ell(I_n).$$

In case 2, we have that

$$(\ell(I_n) + \ell(L_n)) - \ell(I_n) < \frac{\varepsilon}{2^n}$$

and so

$$\ell(J_n) + \ell(L_n) < \ell(I_n) + \frac{\varepsilon}{2^n}.$$

Therefore

$$m^{*}(X \cap E) + m^{*}(X \setminus E)$$

$$\leq \sum_{n=1}^{\infty} (\ell(J_{n}) + \ell(L_{n}))$$

$$\leq \sum_{n=1}^{\infty} (\ell(I_{n}) + \frac{\varepsilon}{2^{n}})$$

$$= \sum_{n=1}^{\infty} \ell(I_{n}) + \varepsilon$$

$$< (m^{*}X + \varepsilon) + \varepsilon$$

$$= m^{*}X + 2\varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, we have that

$$m^*X \ge m^*(X \cap E) + m^*(X \setminus E),$$

and since *X* is arbitrary, we have that  $E = (-\infty, b) \in \mathfrak{M}(\mathbb{R})$ .

3. Wlog, suppose  $a < b \in \mathbb{R}$ . By part 2, we have that

$$(-\infty,b)\in\mathfrak{M}(\mathbb{R}),$$

and similarly, for  $n \ge 1$ ,

$$\left(\infty, a + \frac{1}{n}\right) \in \mathfrak{M}(\mathbb{R}).$$

Since  $\mathfrak{M}(\mathbb{R})$  is a  $\sigma$ -algebra, we have that

$$\left[a+\frac{1}{n},\infty\right)=\left(-\infty,a+\frac{1}{n}\right)^{C}\in\mathfrak{M}(\mathbb{R}),$$

for each  $n \ge 1$ . Consequently,

$$(a,\infty) = \bigcup_{n=1}^{\infty} \left[ a + \frac{1}{n}, \infty \right) \in \mathfrak{M}(\mathbb{R}).$$

Therefore, we have that

$$(a,b) = (-\infty,b) \cap (a,\infty) \in \mathfrak{M}(\mathbb{R}).$$

<sup>8</sup> Since every open set  $G \subseteq \mathbb{R}$  is a countable disjoint union of open intervals in  $\mathbb{R}$ , it follows that  $G \in \mathfrak{M}(\mathbb{R})$  since  $\mathfrak{M}(\mathbb{R})$  is a  $\sigma$ -algebra. If  $F \subseteq \mathbb{R}$  is closed, notice that

<sup>8</sup> We shall prove this in A<sub>1</sub>.

$$F^C = G \in \mathfrak{M}(\mathbb{R})$$

since G is open, and so by closure under complementation of  $\sigma$ -algebras,  $F \in \mathfrak{M}(\mathbb{R})$ .

# **Definition 18 (Lebesgue Measure)**

Let  $m^*$  denote the Lebesgue outer measure on  $\mathbb{R}$ . We define the Lebesgue measure m to be

$$m=m^*\upharpoonright_{\mathfrak{M}(\mathbb{R})}$$
,

*i.e.*  $\forall E \in \mathfrak{M}(\mathbb{R})$ , we have that

$$mE = m^*E = \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) \mid E \subseteq \bigcup_{n=1}^{\infty} I_n \right\}.$$

In A2, we shall prove that

# **P**Theorem 15 ( $\sigma$ -additivity of the Lebesgue Measure on Lebesgue **Measurable Sets**)

The Lebesgue measure is  $\sigma$ -additive on  $\mathfrak{M}(\mathbb{R})$ , i.e. if  $\{E_n\}_{n=1}^{\infty}\subseteq \mathfrak{M}(\mathbb{R})$ with  $E_i \cap E_j = \emptyset$  for all  $i \neq j$ , then

$$m\bigcup_{n=1}^{\infty}E_n=\sum_{n=1}^{\infty}mE_n.$$

# Corollary 16 (Existence of Non-Measurable Sets)

There exists non-measurable sets.

#### Proof

Suppose not, i.e.  $\mathfrak{M}(\mathbb{R})=\mathcal{P}(\mathbb{R})$ . Then  $m=m^*$  is a translation invariant outer measure on  $\mathbb{R}$ , with  $m^*\mathbb{R}=\infty>0$ ,  $m^*[0,1]=1<\infty$ , and  $m^*$  is  $\sigma$ -additive, which contradicts  $\blacksquare$  Theorem 12. Thus  $\mathfrak{M}(\mathbb{R})\neq\mathcal{P}(\mathbb{R})$ .

The following proposition is left as an exercise.

# ♦ Proposition 17 (Non-measurability of the Vitali Set)

The Vitali set V, defined in PTheorem 12, is not measurable.

# $\blacksquare$ Definition 19 ( $\sigma$ -algebra of Borel Sets)

The  $\sigma$ -algebra of sets generated by the collection

$$\mathfrak{G} := \{G \subseteq \mathbb{R} : G \text{ is open } \}$$

is called the  $\sigma$ -algebra of Borel sets of  $\mathbb{R}$ , and is denoted by

$$\mathfrak{Bor}(\mathbb{R}).$$

## **66** Note 4.1.2

Since  $\mathfrak{Bor}(\mathbb{R})$  is generated by open sets in  $\mathbb{R}$  and all open subsets of  $\mathbb{R}$  are Lebesgue measurable (cf.  $\lozenge$  Proposition 14), we have that

$$\mathfrak{Bor}(\mathbb{R})\subseteq\mathfrak{M}(\mathbb{R}).$$

#### Exercise 4.1.1

Prove **O** Proposition 17.

#### Remark 4.1.1

Since  $\mathfrak{Bor}(\mathbb{R})$  is a  $\sigma$ -algebra, and it is, in particular, generated by open subsets of  $\mathbb{R}$ , it also contains all of the closed subsets of  $\mathbb{R}$ . Thus, we could have instead defined  $\mathfrak{Bor}(\mathbb{R})$  to be the  $\sigma$ -algebra of subsets of  $\mathbb{R}$  generated by the collection

$$\mathfrak{F} := \{ F \subseteq \mathbb{R} : F \text{ is closed } \},$$

and in turn conclude that  $\mathfrak{Bor}(\mathbb{R})$  contains  $\mathfrak{G}$ .

#### Remark 4.1.2

Let  $A \subseteq \mathcal{P}(\mathbb{R})$ , with  $\emptyset$ ,  $\mathbb{R} \in A$ . Let

$$\mathcal{A}_{\sigma} := \left\{ \bigcup_{n=1}^{\infty} A_n : A_n \in \mathcal{A}, n \ge 1 \right\}$$
$$\mathcal{A}_{\delta} := \left\{ \bigcap_{n=1}^{\infty} A_n : A_n \in \mathcal{A}, n \ge 1 \right\}.$$

We call the elements of  $A_{\sigma}$  as A-sigma sets, and elements of  $A_{\delta}$  as Adelta sets.

Recalling our definitions

$$\mathfrak{G} = \{ G \subseteq \mathbb{R} \mid G \text{ is open } \}$$
$$\mathfrak{F} = \{ F \subseteq \mathbb{R} \mid F \text{ is closed } \}$$

from above, notice that

$$\mathfrak{G}_{\delta} = \left\{ \bigcap_{n=1}^{\infty} G_n \mid G_n \in \mathfrak{G}, n \geq 1 \right\},$$

which is a countable intersection of open subsets of  $\mathbb{R}$ , and

$$\mathfrak{F}_{\sigma} = \left\{ \bigcup_{n=1}^{\infty} F_n \mid F_n \in \mathfrak{F}, n \geq 1 \right\},$$

which is a **countable union of closed subsets** of  $\mathbb{R}$ , are both subsets of  $\mathfrak{Bor}(\mathbb{R}).$ 

As MENTIONED BEFORE, the definition of which we provided for

a Lebesgue measurable set is from Carathéodory, which is not the most intuitive definition. We shall now show that it is equivalent to the original definition of which Lebesgue himself has provided.

# ■ Theorem 18 (Carathéodory's and Lebesgue's Definition of Measurability)

Let  $E \subseteq \mathbb{R}$ . TFAE:

- 1. E is Lebesgue measurable (Carathéodory).
- 2.  $\forall \varepsilon > 0$ , there exists an open  $G \supseteq E$  such that

$$m^*(G \setminus E) < \varepsilon$$
.

3. There exists a  $\mathfrak{G}_{\delta}$ -set H such that  $E \subseteq H$  and

$$m^*(H \setminus E) = 0.$$

## Proof

(1)  $\Longrightarrow$  (2) If we can find such a G that is open, then since E is Lebesgue measurable, we have

$$mG = m(G \cap E) + m(G \setminus E) = mE + m(G \setminus E),$$

and so

$$m(G \setminus E) = mG - mE. \tag{4.4}$$

So if we can construct such a G, that is particularly small enough (within  $\varepsilon$ -bigger) to contain E, our statement is good as done.

Case 1:  $mE < \infty$  In this case, we may consider a cover  $\{I_n\}_{n=1}^{\infty}$  of E such that

$$\sum_{n=1}^{\infty} \ell(I_n) < mE + \varepsilon.$$

Then we may simply let  $G = \bigcup_{n=1}^{\infty} I_n$ . Note that since  $\mathfrak{M}(\mathbb{R})$  is a

 $\sigma$ -algebra,  $G \in \mathfrak{M}(\mathbb{R})$ . Thus by monotonicity,

$$mG = m\left(\bigcup_{n=1}^{\infty} I_n\right) \le \sum_{n=1}^{\infty} mI_n = \sum_{n=1}^{\infty} \ell(I_n) < mE + \varepsilon.$$

With this, Equation (4.4) becomes

$$m(G \setminus E) < mE + \varepsilon - mE = \varepsilon$$
.

Case 2:  $\forall r \in \mathbb{R}, mE > r$  Consider

$$E_k = [-k, k] \cap E$$

measurable, and so for each  $k \ge 1$ ,  $E_k \in \mathfrak{M}(\mathbb{R})$ . Note that

$$E = \bigcup_{k>1} E_k.$$

<sup>10</sup> Note that  $E_k \subseteq [-k, k]$ , and so

$$mE_k \leq m[-k,k] = 2k < \infty.$$

Using a similar approach as in Case 1, we can construct an open set  $G_k$  such that  $G_k \supseteq E_k$ , and

$$m(G_k \setminus E_k) < \frac{\varepsilon}{2^k}$$

for each  $k \ge 1$ . Now let

$$G := \bigcup_{k>1} G_k \supseteq \bigcup_{k>1} E_k = E.$$

Note that if  $x \in G \setminus E$ , then  $x \notin E_k$  for all  $k \ge 1$ , and  $\exists N \ge 1$  such that  $x \in G_N$ . In particular, we have that

$$x \in G_N \setminus E_N$$

and so

$$G \setminus E \subseteq \bigcup_{k \ge 1} G_k \setminus E_k$$

11. Therefore

- <sup>9</sup> I should get clarification for my understanding of this approach. We picked closed intervals instead of open ones so that we deal with the possible quirkiness of E.
- 10 It would be a quick job if we take the union of the  $E_k$ 's but note that the  $E_k$ 's are not necessarily open!

<sup>11</sup> It is, however, true that equality holds, and it is not difficult to prove so.

$$m(G \setminus E) \leq \sum_{k>1} m(G_k \setminus E_k) \leq \sum_{k>1} \frac{\varepsilon}{2^k} = \varepsilon.$$

(2)  $\Longrightarrow$  (3) By (2), for each  $n \ge 1$ , let  $G_n \supseteq E$  such that

$$m(G_n \setminus E) < \frac{1}{n}.$$

Let  $H := \bigcap_{n \ge 1} G_n$ , which then  $H \in \mathfrak{G}_{\delta}$ . Also, since  $E \subseteq G_n$  for all  $n \ge 1$ , we have  $E \subseteq H$ . Also,  $H \subseteq G_n$  for each n. Thus

$$H \setminus E \subseteq G_n \setminus E$$
,

for each  $n \ge 1$ . By monotonicity,

$$m(H \setminus E) \le m(G_n \setminus E) < \frac{1}{n}$$

for each  $n \ge 1$ . Therefore

$$m(H \setminus E) = 0.$$

(3)  $\Longrightarrow$  (1) Notice that  $\mathfrak{G}_{\delta} \subseteq \mathfrak{Bor}(\mathbb{R}) \subseteq \mathfrak{M}(\mathbb{R})$ . Suppose  $G \in \mathfrak{G}_{\delta}$ , and  $E \subseteq H$  such that

$$m(H \setminus E) = 0.$$

By lacktriangle Proposition 14,  $H \setminus E \in \mathfrak{M}(\mathbb{R})$ . Since  $\mathfrak{M}(\mathbb{R})$  is a  $\sigma$ -algebra, notice that

$$E = H \setminus (H \setminus E) = H \cap (H \cap E^{C})^{C} = H \cap H^{C} \cup E \in \mathfrak{M}(\mathbb{R}). \quad \Box$$

# Lecture 5 May 21st 2019

# 5.1 Lebesgue Measure (Continued 2)

Recall from Corollary 6 that any countable subset  $E \subseteq \mathbb{R}$  has zero Lebesgue outer measure. From Proposition 14, we have that  $E \in \mathfrak{M}(\mathbb{R})$  and so  $mE = m^*E = 0$ . This shows that every countable set is Lebesgue measurable with Lebesgue measure zero.

But is the converse true? I.e., is every Lebesgue measurable set with Lebesgue measure zero countable?

We shall show that this is not true by giving a counterexample. We shall now construct an **uncountable set** *C* that has measure zero.

#### Example 5.1.1 (The Cantor Set)

Let  $C_0 = [0, 1]$ . Note that  $C_0$  is compact and

$$m^*C_0 = 1 < \infty$$
.



Figure 5.1: Cantor set showing up to n = 2, with the excluded interval in n = 3 shown.

Let

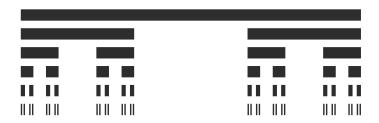
$$C_1 = C_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right).$$

Then  $C_1$  is closed <sup>1</sup> and  $C_0 \supseteq C_1$ .

 $^{\scriptscriptstyle 1}$   $C_1$  is an intersection of 2 closed sets.

Let

$$C_2 = C_1 \setminus \left( \left( \frac{1}{9}, \frac{2}{9} \right) \cup \left( \frac{7}{9}, \frac{8}{9} \right) \right).$$



Then  $C_2$  is closed and  $C_1 \supseteq C_2$ .

We continue this process indefinitely, and construct  $C_n$  for each  $n \ge 1$ , where

$$C_n = \frac{1}{3}C_{n-1} \cup \left(\frac{2}{3} + \frac{1}{3}C_{n-1}\right).$$

Then  $C_n$  will consist of  $2^n$  disjoint closed intervals. Thus each  $C_n$  is compact and measurable. Moreover,

$$m(C_n) = \left(\frac{2}{3}\right)^n,$$

for each  $n \ge 1$ .

Also, we have that

$$C_0 \supset C_1 \supset C_2 \supset \dots$$

is a **descending chain of measurable sets**. Note that the sequence  $\{C_n\}_{n=0}^{\infty}$  has the **finite intersection property**, and since  $\mathbb{R}$  is compact, the set

$$C := \bigcap_{n=1}^{\infty} C_n,$$

which we shall call it the Cantor Set, is non-empty <sup>2</sup>.

Now from A2, we have that

$$mC = \lim_{n \to \infty} mC_n = \lim_{n \to \infty} \left(\frac{2}{3}\right)^n = 0.$$

We shall now show that C is uncountable. To do this, we shall use the **ternary representation** for each  $x \in [0,1]$ . In particular, for each  $x \in [0,1]$ , we write

$$x = 0.x_1x_2x_3...$$

where each  $x_i \in \{0,1,2\}$  for all  $i \ge 1$ . Note that in base 10, we can

Figure 5.2: An illustration of the Cantor Set from https://mathforum.org/mathimages/index.php/Cantor\_Set.

<sup>2</sup> See FIP and Compactness from PMATH 351

express

$$x = \sum_{k=1}^{\infty} \frac{x_k}{10^k} = 0.x_1 + 0.0x_2 + 0.00x_3 + \dots$$

Thus, we can similarly express

$$x = \sum_{k=1}^{\infty} \frac{x_k}{3^k},$$

in ternary representation. However, just as

are indistinguishable, in ternary representation,

are indistinguishable. Fortunately, we can find out who exactly are the culprits that cannot be uniquely represented, which shall be left as an exercise.

#### Exercise 5.1.1

Show that the ternary expansion of  $x \in [0,1)$  is unique except when  $\exists N \geq$ 1 such that

$$x = \frac{r}{3^N},$$

for some  $0 < r < 3^N$ , where  $3 \nmid r$ .

In the cases where we have the above x, we have that  $^3$ 

$$x=0.x_1x_2x_3\ldots x_N,$$

where  $x_N \in \{1, 2\}$ .

- If  $x_N = 2$ , we shall keep this expression; otherwise
- if  $x_N = 1$ , then we write

$$x = 0.x_1x_2x_3...x_{N-2}x_{N-1}1000...$$
  
=  $0.x_1x_2x_3...x_{N}, x_{N-1}0222...$ ,

and we shall use the second expression.

I shall paraphrase the professor here because I like how the analogy brings good intuition, for me at least.

> Suppose there's this person that had only 3 fingers and is not aware of the existence of the base-10 system, and in turn invented the ternary system. Then, instead of having 10 regular intervals on [0,1], it had 3 regular intervals.

<sup>&</sup>lt;sup>3</sup> Note that the representation terminates somewhere, since it is a fraction, i.e. a rational number.

Also, we shall also use the convention that

$$1 = 0.22222...$$

With this, we have obtained a **unique** ternary expansion for each  $x \in [0,1]$ .

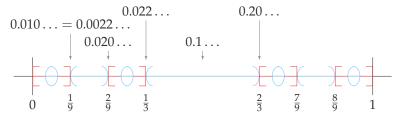


Figure 5.3: Some values on [0,1] in ternary representation

Now, observe that

$$C_1 = [0,1] \setminus \left(\frac{1}{3}, \frac{2}{3}\right)$$
  
=  $\{x \in [0,1] : x = 0.x_1x_2x_3..., x_1 \neq 1\},$ 

i.e. whichever  $x \in [0,1]$  with  $x_1 = 1$  sits in  $\left(\frac{1}{3}, \frac{2}{3}\right)$ . Similarly,

$$C_2 = \{x \in [0,1] : x = 0.x_1x_2x_3..., x_1 \neq 1, x_2 \neq 1\}.$$

In general, we have that

$$C_N = \{x \in [0,1] : x = 0.x_1x_2x_3..., x_i \neq 1, 1 \leq i \leq N\}.$$

Therefore,

$$C = \bigcap_{n=1}^{\infty} C_n$$

$$= \{ x \in [0,1] : x = 0.x_1 x_2 x_3 \dots, x_n \neq 1, n \geq 1 \}$$

$$= \{ x \in [0,1] : x = 0.x_1 x_2 x_3 \dots, x_n \in \{0,2\}, n \geq 1 \}$$

Now, consider the bijection

$$\varphi:C\to[0,1]$$

given by

$$x = 0.x_1x_2x_3... \mapsto y = 0.y_1y_2y_3...,$$

where  $x_n \in \{0,2\}$ , for  $n \ge 1$ , and x is the ternary expansion, while  $y_n = \frac{x_n}{2}$  for each  $n \ge 1$ , and so y is a binary expansion. Then  $\varphi$  is a bijection between C and [0,1], and therefore

$$|C| = |[0,1]| = |\mathbb{R}| = c = 2^{\aleph_0}.$$

# 66 Note 5.1.1

The lesson here is that the Lebesgue measure is not a measure on the cardinality of the set. Rather, it measures the distribution of points in the set.

# Lebesgue Measurable Functions

# 66 Note 5.2.1

We used

$$\mathfrak{M}(\mathbb{R}) = \{ E \subseteq \mathbb{R} \mid E \text{ is measurable } \}$$

to denote the set of measurable subsets of  $\mathbb{R}$ .

*In general, for*  $H \subseteq \mathbb{R}$ *, set shall denote by*  $\mathfrak{M}(H)$  *the collection of all* Lebesgue measurable subsets of H, i.e.

$$\mathfrak{M}(H) = \{ E \subseteq H \mid E \in \mathfrak{M}(\mathbb{R}) \}.$$

*In particular, for*  $E \in \mathfrak{M}(\mathbb{R})$ *, we also have* 

$$\mathfrak{M}(E) = \{ F \subseteq E \mid F \in \mathfrak{M}(\mathbb{R}) \}.$$

## Exercise 5.2.1

*Prove that the above*  $\mathfrak{M}(E)$  *is a*  $\sigma$ *-algebra of sets.* 

**■** Definition 20 (Lebesgue Measurable Function)

Let  $E \in \mathfrak{M}(E)$  and (X,d) a metric space. We say that a function

$$f: E \to X$$

is Lebesgue measurable (or simply measurable) if

$$f^{-1}(G) := \{x \in E : f(x) \in G\} \in \mathfrak{M}(E)$$

*for every open set*  $G \subseteq X$ *.* 

We write

$$\mathcal{L}(E, X) = \{ f : E \to X \mid f \text{ measurable } \}$$

for the set of measurable functions from E to X.

#### Exercise 5.2.2

Show that we can equivalently define that a function f is Lebesgue measurable if

$$f^{-1}(F) \in \mathfrak{M}(E)$$

*for all closed subsets*  $F \subseteq X$ .

## **66** Note 5.2.2

Note that we required that the domain of the function is a measurable set in **Definition 20**. Part of the reason is because we want constant functions to be measurable, and this happens iff the domain of the function is measurable <sup>4</sup>.

4 Why?

# ♦ Proposition 19 (Continuous Functions on a Measurable Set is Measurable)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and (X,d) a metric space. If  $f: E \to X$  is continuous, then  $f \in \mathcal{L}(E,X)$ .

#### Proof

Since f is continuous in a metric space, it implies that for all open  $G \subseteq X$ ,  $f^{-1}(G)$  is open in E 5. This means that  $f^{-1}(G) = U_G \cap E$ for some open  $U_G \subseteq \mathbb{R}$ . Since  $U_G$  is open, by  $\bigcirc$  Proposition 14,  $U_G \in \mathfrak{M}(\mathbb{R})$ . Since  $E \in \mathfrak{M}(\mathbb{R})$ , we have that

<sup>5</sup> We say that  $f^{-1}(G)$  is **relatively open** 

$$f^{-1}(G) = U_G \cap E \in \mathfrak{M}(E),$$

and so

$$f \in \mathcal{L}(E, X)$$
.

## Example 5.2.1

Let  $E \in \mathfrak{M}(\mathbb{R})$  and  $H \subseteq E$ . Consider the characteristic function of H, which is

$$\chi_H : E \to \mathbb{R} \text{ given by } x \mapsto \begin{cases} 1 & x \in H \\ 0 & x \notin H \end{cases}.$$

Let  $G \subseteq \mathbb{R}$  be open. Then

$$\chi_{H}^{-1}(G) = \begin{cases} \emptyset & G \cap \{0,1\} = \emptyset \\ E & G \supseteq \{0,1\} \\ E \setminus H & G \cap \{0,1\} = \{0\} \end{cases},$$

$$H & G \cap \{0,1\} = \{1\}$$

in which case we observe that all the possible outcomes are measurable subsets of  $\mathbb{R}$ . Thus  $\chi_H$  is measurable iff  $H \in \mathfrak{M}(\mathbb{R})$ .

♦ Proposition 20 (Composition of a Continuous Function and a Measurable Function is Measurable)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and  $(X, d_X)$ ,  $(Y, d_Y)$  be metric spaces. Suppose that

 $f: E \to X$  is measurable and  $g: X \to Y$  is continuous.

Then

$$g \circ f : E \to Y$$
 is measurable.

The idea is simple:  $(gf)^{-1}(G) = f^{-1}g^{-1}(G)$  and continuity of G means that  $g^{-1}(G)$  is open in X.

## Proof

Let  $G \subseteq Y$  be open. Then since g is continuous, we have that

$$g^{-1}(G) \subseteq X$$
 is open.

Then since f measurable, we have that

$$(g \circ f)^{-1}(G) = f^{-1}(g^{-1}(G)) \in \mathfrak{M}(E).$$

Thus  $g \circ f \in \mathcal{L}(E, Y)$ .

## Example 5.2.2

Let  $E \in \mathfrak{M}(E)$  and  $f \in \mathcal{L}(E, \mathbb{K})$ . Let  $g : \mathbb{K} \to \mathbb{R}$  be given by  $z \mapsto |z|$ . Then g is continuous. By  $\Diamond$  Proposition 20, we have that

$$g \circ f = |f|$$
 is measurable.

#### Example 5.2.3

Note that the converse to the above is not true, i.e. that if we have that |f| is measurable, it is not necessary that f is not measurable.

Consider  $E = \mathbb{R} = \mathbb{K}$ . If we take  $H \subseteq \mathbb{R}$  that is not measurable, which we know exists, and then consider the function

$$f: E \to \mathbb{R}$$
 given by  $f(x) = \begin{cases} 1 & x \in H \\ -1 & x \notin H \end{cases}$ 

which is constructed by summing up two characteristic functions over H and then minus 1. Then |f|=1, but

$$f^{-1}(\{1\}) = H \notin \mathfrak{M}(\mathbb{R}).$$

Let  $E \in \mathfrak{M}(\mathbb{R})$  and  $f,g:E \to \mathbb{K}$ . Then TFAE:

- 1.  $f,g \in \mathcal{L}(E,\mathbb{K})$ ;
- 2.  $h: E \to \mathbb{K}^2$  given by  $x \mapsto (f(x), g(x))$  is measurable.



 $(2) \implies (1)^{6} \text{ Let}$ 

$$\pi_1: \mathbb{K}^2 \to \mathbb{K}$$
 given by  $(w, z) \mapsto w$   
 $\pi_2: \mathbb{K}^2 \to \mathbb{K}$  given by  $(w, z) \mapsto z$ 

so that  $\pi_1$ ,  $\pi_2$  are continuous. Then by  $\land$  Proposition 20, we have that

$$\pi_1 \circ h = f$$
 and  $\pi_2 \circ h = g$ 

are both measurable.

(1)  $\implies$  (2) Let  $G \subseteq \mathbb{K}^2$  be open. We can write G as a countable union of open sets 7, i.e.

$$G=\bigcup_{n=1}^{\infty}A_n\times B_n,$$

where  $A_n, B_n \subseteq \mathbb{K}$  are open. Then

$$h^{-1}(G) = h^{-1} \left( \bigcup_{n=1}^{\infty} A_n \times B_n \right)$$
$$= \bigcup_{n=1}^{\infty} \underbrace{f^{-1}(A_n)}_{\in \mathfrak{M}(\mathbb{K})} \cap \underbrace{g^{-1}(B_n)}_{\in \mathfrak{M}(\mathbb{K})} \in \mathfrak{M}(\mathbb{K})$$

Thus  $h \in \mathcal{L}(E, \mathbb{K}^2)$ .

<sup>6</sup> Awareness about projective maps is a plus here.

<sup>7</sup> If you are unsure about this, think

## • Proposition 22 ( $\mathcal{L}(E,\mathbb{K})$ is a Unital Algebra)

Let  $E \in \mathfrak{M}(\mathbb{R})$ . Then  $\mathcal{L}(E,\mathbb{K})$  is a unital algebra, i.e. if  $f,g \in$  $\mathcal{L}(E, \mathbb{K})$ , then

1.  $f + g \in \mathcal{L}(E, \mathbb{K})$ ;

2.  $fg \in \mathcal{L}(E, \mathbb{K})^8$ ;

<sup>8</sup> Here, it's multiplication of two functions, not compositions

3.  $g(x) \neq 0$ ,  $\forall x \in E \implies \frac{f}{g} \in \mathcal{L}(E, \mathbb{K})$ ; and

4. if  $h: E \to \mathbb{K}$  is constant, then  $h \in \mathcal{L}(E, \mathbb{K})$ .

#### Proof

We shall make use of this clever trick g. Let  $\mu: E \to \mathbb{K}^2$  given by  $x \mapsto (f(x), g(x))$ . Note that since  $f, g \in \mathcal{L}(E, \mathbb{K})$ , by  $\P$  Proposition 21,  $\mu \in \mathcal{L}(E, \mathbb{K}^2)$ .

<sup>9</sup> "Clever trick" = "Trick you should learn".

1. Consider the function

$$\sigma: \mathbb{K}^2 \to \mathbb{K}$$
 given by  $(w, z) \mapsto w + z$ .

It is clear that  $\sigma$  is continuous. Then

$$\sigma \circ \mu : x \mapsto f(x) + g(x)$$

is measurable by **\langle** Proposition 20.

2. Consider the function

$$\sigma: \mathbb{K}^2 \to \mathbb{K}$$
 given by  $(w, z) \mapsto wz$ .

Again, we see that  $\sigma$  is continuous. Then

$$\sigma \circ \mu : x \mapsto f(x)g(x)$$

is measurable by **\leftrightarrow** Proposition 20.

3. Consider the function

$$\sigma: \mathbb{K} \times (\mathbb{K} \setminus \{0\}) \to \mathbb{K}$$
 given by  $(w, z) \mapsto \frac{w}{z}$ .

Again,  $\sigma$  is continuous. Thus

$$\sigma \circ \mu : x \mapsto \frac{f(x)}{g(x)}$$

is measurable by **Operation** 20.

4. Suppose  $h: E \to \mathbb{K}$  is a constant, and we have  $h(x) = \alpha_0$  for all  $x \in E$ . Then for any  $G \subseteq \mathbb{K}$  that is open, we have that

$$h^{-1}(G) = \begin{cases} \emptyset & a_0 \notin G \\ E & a_0 \in G \end{cases},$$

both of which are measurable sets. Thus h is indeed measurable.

#### Remark 5.2.1

Note that  $(\mathbb{C},d)$ , where d(w,z) = |w-z|, is a metric space. Moreover, the тар

$$\gamma: \mathbb{C} \to \mathbb{R}^2$$
 given by  $x + iy \mapsto (x, y)$ ,

where  $x, y \in \mathbb{R}$  is a homeomorphism, which, in particular, is continuous. Then given a  $E \in \mathfrak{M}(\mathbb{R})$  with a measurable  $f \in E \to \mathbb{C}$ , then

$$\gamma \circ f : E \to \mathbb{R}^2 \in \mathcal{L}(E, \mathbb{R}^2).$$

Also, notice that

$$\gamma \circ f = (\Re f, \Im f).$$

By  $\land$  Proposition 21,  $\Re f$ ,  $\Im f \in \mathcal{L}(E,\mathbb{R})$ . This also means that

$$h: x \mapsto (\Re f(x), \Im f(x)) \in \mathcal{L}(E, \mathbb{R}^2).$$

Conversely, if  $\Re f$ ,  $\Im f \in \mathcal{L}(E,\mathbb{R})$ , then

$$f = \gamma^{-1} \circ h \in \mathcal{L}(E, \mathbb{C})$$

by **\langle** Proposition 21.

This means that a complex-valued function is measurable iff its real and imaginary parts are both measurable. Consequently, to study about complex-valued functions, it is sufficient for us to study about real-valued functions.

#### Absolute Part and a Scaling Part)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and suppose that  $f: E \to \mathbb{C}$  is measurable. Then there exists a measurable function  $\Theta: E \to \mathbb{T}$ , where

$$\mathbb{T} := \{ z \in \mathbb{C} \mid |z| = 1 \},$$

such that

$$f = \Theta \cdot |f|$$
.



Since  $\{0\} \subseteq \mathbb{C}$  is closed and f is measurable, we have that

$$K := f^{-1}(\{0\}) \in \mathfrak{M}(E).$$

Since  $\chi_K$  is a measurable function, we have that  $f + \chi_K$  is also measurable (cf.  $\wedge$  Proposition 22).

Claim:  $f + \chi_K \neq 0$  over E.

- If  $x \in E$  such that f(x) = 0, then  $x \in K$ , and so  $\chi_K(x) = 1$ .
- If  $x \in E$  such that  $\chi_K(x) = 0$ , then  $x \notin K$ , which means  $f(x) \neq 0$ .

Therefore, consider the function

$$\Theta = \frac{f + \chi_K}{|f + \chi_K|} : E \to \mathbb{T}.$$

By  $\Diamond$  Proposition 22,  $\Theta$  is measurable, and clearly

$$f = \Theta \cdot |f|$$
.

### Remark 5.2.2

As of now, given a set  $E \in \mathfrak{M}(\mathbb{R})$ , to verify that a function  $f \in \mathcal{L}(E, \mathbb{R})$ , we need to check that

$$\forall G \subseteq \mathbb{R} \text{ open }, f^{-1}(G) \in \mathfrak{M}(E).$$

Since there is an obscene amount of open (respectively closed) subsets of  $\mathbb{R}$ , we want to be able to reduce our workload. This shall be the first thing we do in the next lecture.

## 6.1 Lebesgue Measurable Functions (Continued)

## **♦** Proposition 24 (Function Measurability Check)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and  $f : E \to \mathbb{R}$  be a function. TFAE:

- 1. f is measurable, i.e.  $\forall G \subseteq \mathbb{R}$  that is open,  $f^{-1}(G) \in \mathfrak{M}(E)$ .
- 2.  $\forall a \in \mathbb{R}, f^{-1}((a, \infty)) \in \mathfrak{M}(E)$ .
- 3.  $\forall b \in \mathbb{R}, f^{-1}((-\infty, b]) \in \mathfrak{M}(E).$
- 4.  $\forall b \in \mathbb{R}, f^{-1}((-\infty, b)) \in \mathfrak{M}(E)$ .
- 5.  $\forall a \in \mathbb{R}, f^{-1}([a, \infty)) \in \mathfrak{M}(E)$ .

## Proof

- (1)  $\Longrightarrow$  (2) This is trivially true since  $\forall a \in \mathbb{R}$ ,  $(a, \infty)$  is open in  $\mathbb{R}$ , and so since f is measurable, we must have that  $f^{-1}((a, \infty)) \in \mathfrak{M}(E)$ .
- (2)  $\Longrightarrow$  (3) Notice that  $\forall b \in \mathbb{R}$ ,

$$f^{-1}((-\infty,b])=f^{-1}(\mathbb{R}\setminus (b,\infty))=E\setminus f^{-1}((b,\infty))$$

and  $f^{-1}((b,\infty))\in\mathfrak{M}(E)$  by assumption. Since  $\mathfrak{M}(E)$  is a  $\sigma$ -algebra,  $f^{-1}((-\infty,b])\in\mathfrak{M}(E)$ .

(3)  $\Longrightarrow$  (4) Notice that  $\forall b \in \mathbb{R}$ ,

$$f^{-1}((-\infty,b)) = \bigcup_{n=1}^{\infty} f^{-1}\left(\left(-\infty,b-\frac{1}{n}\right]\right),$$

and by assumption, for each  $n \ge 1$ ,  $f^{-1}\left(\left(-\infty, b - \frac{1}{n}\right]\right) \in \mathfrak{M}(E)$ . It follows that  $f^{-1}((-\infty, b)) \in \mathfrak{M}(E)$ .

(4)  $\Longrightarrow$  (5) Observe that  $\forall a \in \mathbb{R}$ , we have

$$f^{-1}([a,\infty)) = f^{-1}(\mathbb{R} \setminus (-\infty,a)) \in \mathfrak{M}(E)$$

by assumption.

(5)  $\Longrightarrow$  (1) <sup>1</sup> Notice that  $\forall a \in \mathbb{R}$ ,

$$f^{-1}((a,\infty)) = \bigcup_{n=1}^{\infty} f^{-1}\left(\left[a + \frac{1}{n},\infty\right)\right) \in \mathfrak{M}(E)$$

by assumption. Furthermore, we have that  $\forall b \in \mathbb{R}$ ,

$$f^{-1}((-\infty,b)) = E \setminus f^{-1}([b,\infty)) \in \mathfrak{M}(E),$$

also by assumption. Thus

$$f^{-1}((a,b)) = f^{-1}((a,\infty)) \cap f^{-1}((-\infty,b)) \in \mathfrak{M}(E),$$

for any  $a, b \in \mathbb{R}$ .

Since for any open  $G \subseteq \mathbb{R}$  can be written as a countable union of open intervals, i.e.

$$G=\bigcup_{n=1}^{\infty}I_n,$$

where each  $I_n$  is an open interval, we have that

$$f^{-1}(G) = \bigcup_{n=1}^{\infty} f^{-1}(I_n) \in \mathfrak{M}(E).$$

Thus f is measurable.

The proof of the following result is left to A2.

¹ This uses the same idea as in ♠ Proposition 14.

## Corollary 25 (Measurability Check on the Borel Set)

*If*  $E \in \mathfrak{M}(\mathbb{R})$  *and*  $f : E \to \mathbb{R}$  *is a function, then TFAE:* 

- 1. f is measurable.
- 2.  $\forall B \in \mathfrak{Bor}(\mathbb{R}), f^{-1}(B) \in \mathfrak{M}(E).$

#### Remark 6.1.1

Let  $E \in \mathfrak{M}(\mathbb{R})$  and  $f : E \to \mathbb{R}$ . Define

$$f^{+}(x) = \max\{f(x), 0\}, x \in E$$
$$f^{-}(x) = \max\{-f(x), 0\}, x \in E$$

Then  $f^+, f^- \geq 0$ , and

$$f = f^+ - f^-$$
 and  $|f| = f^+ + f^-$ .

Moreover.

$$f^{+} = \frac{|f| + f}{2}$$
 and  $f^{-} = \frac{|f| - f}{2}$ ,

and so both  $f^+$  and  $f^-$  are measurable.

By Remark 5.2.1, every complex-valued measurable function is a linear combination of 4 non-negative, real-valued measurable functions.

WE SHALL now examine a number of results dealing with pointwise limits of sequences of measurable, real-valued functions. We shall include the case where the limit of a given point is allowed to be an extended real number; i.e. the sequence diverges either to  $\infty$  or  $-\infty$ .

#### ■ Definition 21 (Extended Real Numbers)

We define the extended real numbers to be the set

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}.$$

*We also write*  $\overline{\mathbb{R}} = [-\infty, \infty]$ *.* 

By convention, we shall define

- $\infty + \infty = \infty$ ,  $-\infty \infty = -\infty$ ;
- $\forall \alpha \in \mathbb{R} \cup \{\infty\}, \alpha + \infty = \infty = \infty + \alpha;$
- $\forall \alpha \in \mathbb{R}, \alpha + (-\infty) = -\infty = -\infty + \alpha;$
- $\forall 0 < \alpha \in \overline{\mathbb{R}}, a \cdot \infty = \infty \cdot \alpha = (-\infty) \cdot (-\alpha) = (-\alpha) \cdot (-\infty) = \infty;$
- $\forall \alpha < 0 \in \overline{\mathbb{R}}$ ,  $a \cdot \infty = \infty \cdot \alpha = (-\infty) \cdot (-\alpha) = (-\alpha) \cdot (-\infty) = -\infty$ ; and
- $0 = 0 \cdot \infty = \infty \cdot 0 = 0 \cdot (-\infty) = (-\infty) \cdot 0.$

## \*Warning

*Notice that we do not define*  $\infty - \infty$  *and*  $-\infty + \infty$ .

#### 66 Note 6.1.1

While the space of extended real numbers is useful for treating measure-theoretic and analytic properties of sequences of functions, it has poor algebraic properties. In particular, it is no longer a vector space, since  $\infty$  and  $-\infty$  do not have their additive inverses.

#### **■** Definition 22 (Extended Real-Valued Function)

Given  $H \subseteq \mathbb{R}$ , the function  $f: H \to \overline{\mathbb{R}}$  is called an extended real-valued function.

## **■** Definition 23 (Measurable Extended Real-Valued Function)

If  $E \in \mathfrak{M}(\mathbb{R})$  and  $f : E \to \overline{\mathbb{R}}$  is an extended real-valued function, we say that f is Lebesgue measurable (or simply measurable) if

- 1.  $\forall G \subseteq \mathbb{R}$  open,  $f^{-1}(G) \in \mathfrak{M}(E)$ ; annd
- 2.  $f^{-1}(\{-\infty\}), f^{-1}(\{\infty\}) \in \mathfrak{M}(E)$ .

We denote the set of Lebesgue measurable extended real-valued functions on E by

$$\mathcal{L}(E,\overline{\mathbb{R}}) = \{f : E \to \overline{\mathbb{R}} : f \text{ is measurable } \}.$$

Since we shall often refer to only the non-negative elements of  $\mathcal{L}(E,\overline{\mathbb{R}})$ , we also define the notation

$$\mathcal{L}(E, [0, \infty]) = \{ f \in \mathcal{L}(E, \overline{\mathbb{R}}) : \forall x \in E, 0 \le f(x) \}.$$

#### 66 Note 6.1.2

Note that we can also replace the first condition of Lebesgue measurability of extended real-valued functions by

$$\forall F \subseteq \mathbb{R} \ closed \ , \ f^{-1}(F) \in \mathfrak{M}(E).$$

Just as in the case with regular real-valued measurable functions, we have the following shortcuts in testing whether an extended realvalued function is measurable.

#### **Notation**

We write

- $(a, \infty] = (a, \infty) \cup \{\infty\}$ ; and
- $[-\infty, b) = (-\infty, b) \cup \{-\infty\},$

for all  $a, b \in \mathbb{R}$ .

## ♦ Proposition 26 (Measurability Check for Extended Real-Valued Functions)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and suppose  $f : E \to \overline{\mathbb{R}}$  is a function. Then TFAE:

1. f is Lebesgue measurable.

- 2.  $\forall a \in \mathbb{R}, f^{-1}((a, \infty]) \in \mathfrak{M}(E)$ .
- 3.  $\forall b \in \mathbb{R}, f^{-1}([-\infty, b)) \in \mathfrak{M}(E)$ .

#### Exercise 6.1.1

Prove Proposition 26.

## ♦ Proposition 27 (Measurability of Limits and Extremas)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and suppose that  $(f_n)_{n=1}^{\infty}$  is a sequence in  $\mathcal{L}(E, \overline{\mathbb{R}})$ . Then the following extended real-valued functions are also measurable:

- 1.  $g_1 := \sup_{n \ge 1} f_n$ ;
- 2.  $g_2 := \inf_{n>1} f_n$ ;
- 3.  $g_3 := \limsup_{n \ge 1} f_n$ ; and
- 4.  $g_4 := \liminf_{n > 1} f_n$ .

#### Proof

1. Let  $a \in \mathbb{R}$ . Then

$$g_1^{-1}((a,\infty])=\bigcup_{n\geq 1}\underbrace{f_n^{-1}((a,\infty])}_{\in\mathfrak{M}(E)}\in\mathfrak{M}(E).$$

It follows from  $\land$  Proposition 26 that  $g_1 \in \mathcal{L}(E, \overline{\mathbb{R}})$ .

2. For any  $b \in \mathbb{R}$ , we have

$$g_2^{-1}([-\infty,b)) = \bigcap_{n>1} f_n^{-1}([-\infty,b)) \in \mathfrak{M}(E).$$

Thus by  $\land$  Proposition 26,  $g_2 \in \mathcal{L}(E, \overline{\mathbb{R}})$ .

3. Let  $h_n = \sup_{k \ge n} f_n$  for each  $n \ge 1$ . Then by part (1),  $h_n \in \mathcal{L}(E, \overline{\mathbb{R}})$  for each  $n \ge 1$ . Also, notice that  $h_1 \ge h_2 \ge h_3 \ge \ldots$ , i.e.  $\{h_n\}_{n=1}^{\infty}$  is an increasing sequence of functions. Then by part

<sup>&</sup>lt;sup>2</sup> Both notes and lecture notes used union, but should it not be intersection?

(2), 
$$g_3 = \lim_{n \to \infty} h_n = \inf_{n \ge 1} h_n \in \mathcal{L}(E, \overline{\mathbb{R}}).$$

4. Let  $h_n = \inf_{k \ge n} f_n$  for each  $n \ge 1$ . Then by part (2), each  $h_n \in$  $\mathcal{L}(E,\overline{\mathbb{R}})$ . Also,  $\{h_n\}_{n=1}^{\infty}$  is a decreasing sequence of functions. Then by part (1), we have that

$$g_4 = \lim_{n \to \infty} h_n = \sup_{n \ge 1} h_n \in \mathcal{L}(E, \overline{\mathbb{R}}).$$

## Corollary 28 (Extended Limit of Real-Valued Functions)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and suppose that  $(f_n)_{n=1}^{\infty}$  is a sequence of real-valued functions such that  $f(x) = \lim_{n\to\infty} f_n(x)$  exists as an extended realvalued number for all  $x \in E$ . Then

$$f \in \mathcal{L}(E, \overline{\mathbb{R}}).$$

## Proof

By A2, when the said limit exists, we have that

$$f = \limsup_{n \ge 1} f_n = \liminf_{n \ge 1} f_n,$$

and so  $f \in \mathcal{L}(E, \overline{\mathbb{R}})$  by  $\Diamond$  Proposition 27.

## **■** Definition 24 (Simple Functions)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and  $\varphi : E \to \overline{\mathbb{R}}$ . We say that  $\varphi$  is simple if range  $\varphi$  is finite. Furthermore, we denote the set of all simple, real-valued, measurable functions on E as

$$SIMP(E, \mathbb{R}).$$

## **■** Definition 25 (Standard Form)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and  $\varphi : E \to \overline{\mathbb{R}}$ . Suppose that

range 
$$\varphi = \{ \alpha_1 < \alpha_2 < ... < a_N \},$$

and set

$$E_n := \varphi^{-1}(\{\alpha_n\}), \text{ for } 1 \leq n \leq N.$$

We say that

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{E_n}$$

is the **standard form** of  $\varphi$ .

# **♦** Proposition 29 (Measurability of Simple Functions with Measurable Support)

Let  $E \in \mathfrak{M}(\mathbb{R})$ . Suppose  $\varphi : E \to \overline{\mathbb{R}}$  is simple with

range 
$$\varphi = \{\alpha_1 < \alpha_2 < \ldots < \alpha_N\}.$$

TFAE:

- 1.  $\varphi$  is measurable.
- 2. If  $\varphi = \sum_{n=1}^{N} \alpha_n \chi_{E_n}$  is the standard form of  $\varphi$ , then  $E_n \in \mathfrak{M}(E)$ , for all  $n \in \{1, ..., N\}$ .

#### Proof

 $(\Longrightarrow)$  Since  $\varphi$  is measurable, notice that for each  $n \in \{1, ..., N\}$ ,

• if  $\alpha_n \in \mathbb{R}$ , then  $\{\alpha_n\}$  is closed, and so

$$E = \varphi^{-1}(\{\alpha_n\}) \in \mathfrak{M}(E)$$
; and

• if  $\alpha_1 = -\infty$ , and similarly if  $\alpha_N = \infty$ , then by  $\blacksquare$  Definition 23,  $\varphi^{-1}(\{\alpha_1\}), \varphi^{-1}(\{\alpha_N\}) \in \mathfrak{M}(E)$ .

 $(\Leftarrow)$  By Example 5.2.1,  $\forall n \geq 1$ ,  $E_n \in \mathfrak{M}(E) \implies \forall n \geq 0 \chi_{E_n} \in$  $\mathfrak{M}(E)$ . Notice that  $\forall a \in \mathbb{R}$ ,

$$\varphi^{-1}((a,\infty])=\bigcup\{E_n:a<\alpha_n\},\,$$

and so  $\varphi^{-1}((a,\infty])$  is a finite (or empty) union of measurable sets, and is hence measurable. 

THE STANDARD FORM is not a unique way of expressing a simple function as a finite linear combination of characteristic functions.

#### Example 6.1.1

Consider the function  $\varphi : \mathbb{R} \to \mathbb{R}$  given by

$$\varphi = \chi_{\mathbb{Q}} + 9\chi_{[2,6]}.$$

Then range  $\varphi = \{0, 1, 9, 10\}$ ; we see that

$$x \mapsto \begin{cases} 0 & x \in \mathbb{Q}^{C} \cap [2, 6]^{C} \\ 1 & x \in \mathbb{Q} \cap [2, 6]^{C} \\ 9 & x \in \mathbb{Q}^{C} \cap [2, 6] \\ 10 & x \in \mathbb{Q} \cap [2, 6] \end{cases}.$$

Then we may write  $\varphi$  as

$$\varphi = 0\chi_{\mathbb{Q}^C \cap [2,6]^C} + 1\chi_{\mathbb{Q} \cap [2,6]^C} + 9\chi_{\mathbb{Q}^C \cap [2,6]} + 10\chi_{\mathbb{Q} \cap [2,6]}.$$

#### **■** Definition 26 (Real Cone)

Let V be a vector space over  $\mathbb{K}$ . A subset  $C \subseteq V$  is said to be a (real) cone is

1. 
$$C \cap -C = \{0\}$$
, where  $-C = \{-w : w \in C\}$ ; and

2.  $y,z \in C$  and  $0 \le \kappa \in \mathbb{R}$  imply that

$$\kappa y + z \in C$$
.

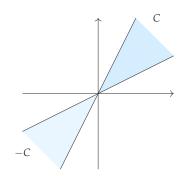


Figure 6.1: Typical figure of a cone

#### Example 6.1.2

1. Let  $\mathcal{V} = \mathbb{R}^3$  and

$$C = \{(x, y, z) \in \mathbb{R}^3 : 0 \le x, y, z\}.$$

Then *C* is a (real) cone.

2. Let  $\mathcal{V} = \mathbb{C}$  and

$$C = \left\{ w \in \mathbb{C} : w = re^{i\theta}, \frac{\pi}{6} \le \theta \le \frac{2\pi}{6}, 0 \le r < \infty \right\}.$$

The C is a cone in  $\mathbb{C}$ . Note that in both the above examples, C is not closed.

3. Let  $\mathcal{V} = \mathcal{C}([0,1],\mathbb{C})$ , and

$$C = \{ f \in \mathcal{V} : 0 \le f(x), \forall x \in [0, 1] \},$$

where we note that the condition means that we only look at those functions that return real positive values. Then C is a (real) cone in V.

#### Exercise 6.1.2

Show that  $SIMP(E, \mathbb{R})$  is an algebra, and hence a vector space over  $\mathbb{R}$ .

#### Remark 6.1.2

1. Note that

$$\mathrm{SIMP}(E,\overline{\mathbb{R}}) = \{f: E \to \overline{\mathbb{R}}: f \text{ is simple and measurable } \}.$$

is not a vector space. In fact, it is neither a field nor a ring.

2. We shall adopt the following notation:

$$SIMP(E, [0, \infty)) := \{ \varphi \in SIMP(E, \mathbb{R}) : 0 \le \varphi(x) \text{ for all } x \in E \}.$$

*Observe that this is a real cone in*  $SIMP(E, \mathbb{R})$ *.* 

In A<sub>3</sub>, we will show the following proposition.

# **♦** Proposition 30 (Increasing Sequence of Simple Functions that **Converges to a Measurable Function)**

Let  $E \in \mathfrak{M}(E)$  and  $f \in \mathcal{L}(E,[0,\infty])$ . Then there exists an increasing sequence

$$\varphi_1 \le \varphi_2 \le \varphi_3 \le \ldots \le f$$

of simple, real-valued functions  $\varphi_n$  such that

$$f(x) = \lim_{n \to \infty} \varphi_n(x)$$

for all  $x \in E$ .

## 7.1 Lebesgue Integration

We shall first begin by defining integration over simple, non-negative, extended real-valued functions. We shall then use this definition to define the integral of  $f \in \mathcal{L}(E,[0,\infty])$ , and derive several consequences of our definition. Furthermore, we shall also build the Lebesgue integral such that it is linear, which will require us to impose certain conditions to the range of functions which will retain this desirable property.

## **■** Definition 27 (Integration of Simple Functions)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and  $\varphi \in SIMP(E, [0, \infty])$ , such that its standard form is denoted as

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{E_n}.$$

We define

$$\int_{E} \varphi := \sum_{n=1}^{N} \alpha_{n} m E_{n} \in [0, \infty].$$

*If*  $F \subseteq E$  *is measurable, we define* 

$$\int_{F} \varphi = \int_{E} \varphi \cdot \chi_{F} = \sum_{n=1}^{N} \alpha_{n} m(F \cap E_{n}).$$

#### 66 Note 7.1.1

Note that since  $\varphi$  is measurable, so is each  $E_n$  for  $1 \le n \le N$ .

#### Example 7.1.1

1. Let  $\varphi = 0\chi_{[4,\infty)} + 17\chi_{\mathbb{Q}\cap[0,4)} + 29\chi_{[2,4)\setminus\mathbb{Q}}$ . Then

$$\int_{[0,\infty)} \varphi = 0m[4,\infty) + 17m(\mathbb{Q} \cap [0,4)) + 29m([2,4) \setminus \mathbb{Q})$$
$$= 0 + 17 \cdot 0 + 29(2) = 58.$$

2. Let  $C\subseteq [0,1]$  be the Cantor set from Example 5.1.1 and  $\varphi=1\chi_C+2\chi_{[5,9]}$ . Then

$$\int_{[0,6]} \varphi = 1m(C \cap [0,6]) + 2m([5,9] \cap [0,6])$$

$$= 1 \cdot 0 + 2m([5,6])$$

$$= 2.$$

Since our definition is fairly limited since it requires that our simple function be in standard form, let us try to relax that condition.

## **E** Definition 28 (Disjoint Representation)

Let  $E \in \mathfrak{M}(E)$  and  $\varphi \in SIMP(E, [0, \infty])$ . Suppose

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{H_n},$$

where  $H_n \subseteq E$  is measurable and  $\alpha_n \in \overline{\mathbb{R}}$  for each  $1 \leq n \leq N$ . We shall say that the above decomposition of  $\varphi$  is a disjoint representation of  $\varphi$  if

$$H_i \cap H_i = \emptyset$$
, for  $1 \le i \ne j \le N$ .

<sup>1</sup> Note that we did not require that the  $\alpha_n$ 's be distinct, nor do we require that they be written in any particular order, nor do we require that  $E = \bigcup_{n=1}^N H_n$ , unlike in the definition of simple functions.

♣ Lemma 31 (Common Disjoint Representation of Simple Functions over a Common Domain)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and suppose that  $\varphi, \psi \in \mathcal{L}(E, \mathbb{R})$ . Then there exists

1.  $N \in \mathbb{N}$ ;

- 2.  $H_1, H_2, \ldots, H_n \in \mathfrak{M}(E)$  with  $H_i \cap H_j = \emptyset$  for all  $i \neq j$ ; and
- 3.  $\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N$  such that

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{H_n}$$
 and  $\psi = \sum_{n=1}^{N} \beta_N \chi_{H_n}$ 

are disjoint representations of  $\varphi$  and  $\psi$ .

## Proof

Since  $\varphi$  and  $\psi$  are simple, from  $\square$  Definition 25, if we write

$$\varphi = \sum_{m=1}^{M_1} a_m \chi_{E_m}$$
 and  $\psi = \sum_{m=1}^{M_2} b_m \chi_{F_m}$ 

in their standard forms, we have that the  $E_m$ 's and  $F_m$ 's are respectively pairwise disjoint 2. Then

$${E_i \cap F_j : 1 \le i \le M_1, 1 \le j \le M_2}$$

is also a pairwise disjoint family of measurable sets, which we shall relabel them as

$$\{H_n\}_{n=1}^N$$
, where  $N = M_1 M_2$ .

Then

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{H_n},$$

where  $\alpha_n = a_i$  if  $H_n = E_i \cap F_j$  for some  $1 \le j \le M_2$ , and

$$\psi = \sum_{n=1}^{N} \beta_N \chi_{H_n},$$

where  $\beta_n = b_j$  if  $H_n = E_i \cap F_j$  for some  $1 \le i \le M_1$ .

<sup>2</sup> It is important to note here that the  $E_m$ 's and  $F_m$ 's are pairwise disjoint on E, which is why the next step is a sensible and correct one.

Lemma 32 (Integral of a Simple Funciton Using Its Disjoint Representation)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and suppose  $\varphi \in SIMP(E, [0, \infty])$ . If

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{H_n}$$

is any disjoint representation, then

$$\int_{E} \varphi = \sum_{n=1}^{n} \alpha_{n} m H_{n}.$$

## Proof

<sup>3</sup> If  $\bigcup_{n=1}^{N} H_n \neq E$ , then we set

$$H_{N+1} = E \setminus \bigcup_{n=1}^{N} H_n$$
 and  $\alpha_{N+1} = 0$ .

Then

$$\sum_{n=1}^{N} \alpha_n m H_n = \sum_{n=1}^{N+1} \alpha_n m H_n.$$

Thus, wlog, wma

$$\bigcup_{n=1}^{N} H_n = E.$$

Now since the  $H_n$ 's are mutually disjoint, wma

range 
$$\varphi = \{\alpha_1, \ldots, \alpha_N\},\$$

where we note that the above set may contain repeated elements, i.e. some  $\alpha_i = \alpha_j$ . We may thus rewrite this set such that

$$\{\alpha_1, \dots, \alpha_N\} = \{\beta_1 < \beta_2 < \dots < \beta_M\}$$

and set

$$E_i = \bigcup \{H_j : \alpha_j = \beta_i\}.$$

Note that since  $H_i \cap H_j = \emptyset$  for  $1 \le i \ne j \le N$ , for  $1 \le k \le M$ , we have

$$mE_k = \sum_{\alpha_j = \beta_k} m(H_j).$$

<sup>3</sup> One of the problems here is that the disjoint  $H_n$ 's may not cover the entire domain  $\varphi$ , but we can fill it up with zeros.

Then by definition,

$$\int_{E} \varphi = \sum_{k=1}^{M} \beta_{k} \xi_{E_{k}}$$

$$= \sum_{i=1}^{M} \beta_{i} \sum_{\alpha_{j} = \beta_{i}} mH_{j}$$

$$= \sum_{n=1}^{N} \alpha_{j} mH_{j},$$

as desired.

♦ Proposition 33 (Linearity and Monotonicity of the Integral of **Simple Functions**)

Let  $E \in \mathfrak{M}(\mathbb{R})$ . If  $\varphi, \psi \in SIMP(E, [0, \infty])$  and  $\kappa \in [0, \infty)$ , then

- 1.  $\int_E \kappa \varphi + \psi = \kappa \int_E \varphi + \int_E \psi$ ; and
- 2.  $\varphi \leq \psi$  on E implies

$$\int_{E} \varphi \leq \int_{E} \psi.$$

Proof

1. By Lemma 31, we can find a common disjoint representation of  $\varphi$  and  $\psi$ , say

$$\varphi = \sum_{n=1}^{N} a_n \chi_{H_n}$$
 and  $\psi = \sum_{n=1}^{N} b_n \chi_{H_n}$ ,

where the  $H_n$ 's are pairwise disjoint. Then

$$\kappa \varphi + \psi = \sum_{n=1}^{N} (\kappa a_n + b_n) \chi_{H_n}.$$

Thus by Lemma 32,

$$\int_{E} (\kappa \varphi + \psi) = \sum_{n=1}^{N} (\kappa a_n + b_n) m H_n$$
$$= \kappa \sum_{n=1}^{N} a_n m H_n * \sum_{n=1}^{N} b_n m H_n$$

$$=\kappa\int_{E}\varphi+\int_{E}\psi.$$

2. Using the disjoint representation, if  $\varphi \leq \psi$ , then  $a_n \leq b_n$  for all  $1 \leq n \leq N$ , and so by Lemma 32,

$$\int_{E} \varphi = \sum_{n=1}^{N} a_n m H_n \le \sum_{n=1}^{N} b_n m H_n = \psi.$$

We are now ready to define the Lebesgue integral for arbitrary measurable functions.

## **■** Definition 29 (Lebesgue Integral)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and  $f \in \mathcal{L}(E, [0, \infty])$ . We define the **Lebesgue integral** of f as

$$\int_{E}^{NEW} f = \sup \left\{ \int_{E} \varphi : \varphi \in \mathrm{SIMP}(e,[0,\infty)), \, 0 \leq \varphi \leq f \right\}.$$

#### 66 Note 7.1.2

- We can actually allow  $\varphi \in SIMP(E, [0, \infty])$ .
- We put "NEW" in the above integral because we now have "two" definitions for the integral of  $\varphi \in SIMP(E, [0, \infty])$ . Writing  $\varphi = \sum_{n=1}^{N} \alpha_n \chi_{H_n}$  in its standard form, by  $\square$  Definition 27,

$$\int_{E} \varphi = \sum_{n=1}^{N} \alpha_{n} m H_{n},$$

while 🖪 Definition 29 gives us

$$\int_{E}^{NEW} \varphi = \sup \left\{ \int_{E} \psi : \psi \in SIMP(E, [0, \infty)), \, 0 \le \psi \le \varphi \right\}.$$

#### Remark 7.1.1

Let us try reconciling these two definitions, which will allow us to drop the

dumb-looking "NEW" notation. First, note that

$$\varphi \in \{ \psi \in \text{SIMP}(E, [0, \infty]) : 0 \le \psi \le \varphi \},$$

and so by <a>E</a> Definition 29, then

$$\int_{E} \varphi \leq \int_{E}^{NEW} \varphi.$$

On the other hand, by  $\triangleleft$  *Proposition* 33, if  $\varphi \in SIMP(E, [0, \infty])$  and  $0 \le \psi \le \varphi$ , we have that

$$\int_{E} \psi \leq \int_{E} \varphi,$$

and so

$$\int_{E}^{NEW} \varphi = \sup \left\{ \int_{E} \psi : \psi \in SIMP(E, 0, \infty]), \, \psi \leq \varphi \right\} \leq \int_{E} \varphi.$$

Thus

$$\int_{E}^{NEW} \varphi = \int_{E} \varphi.$$

With that we shall drop the "NEW" notation from here on.

## **■** Definition 30 (Almost Everywhere (a.e.))

Let  $E \in \mathfrak{M}(\mathbb{R})$ . We say that a property (P) holds almost everywhere (a.e.) on E if the set

$$B := \{x \in E : (P) \text{ does not hold } \}$$

has Lebesgue measure zero.

#### Example 7.1.2

Let  $E \in \mathfrak{M}(\mathbb{R})$ . Given  $f, g \in \mathcal{L}(E, \overline{\mathbb{R}})$ , we say that f = g a.e. on E if

$$B := \{ x \in E : f(x) \neq g(x) \}$$

has measure zero, i.e. mB = 0.

An example of this is

$$\chi_{\mathbb{O}} = 0 = \chi_{\mathbb{C}}$$

a.e. on  $\mathbb{R}$ , where C is the Cantor set.



# ♣ Lemma 34 (Monotonicity of the Lebesgue Integral and Other Lemmas)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and let  $f, g, h : E \to [0, \infty]$  be functions. Suppose that g and h are measurable.

1. Suppose further that  $E = X \cup Y$ , where  $X, Y \in \mathfrak{M}(E)$ . Set  $f_1 := f \upharpoonright_X$  and  $f_2 := f \upharpoonright_Y$ . Then  $f \in \mathcal{L}(E, [0, \infty])$  iff  $f_1$  and  $f_2$  are measurable. When this is the case, then

$$\int_E f = \int_X f_1 + \int_Y f_2.$$

2. If  $g \leq h$ , then

$$\int_{E} g \le \int_{E} h.$$

3. If  $H \in \mathfrak{M}(E)$ , then

$$\int_{H} g = \int_{E} g \cdot \chi_{H} \le \int_{E} g.$$

#### Exercise 7.1.1

Prove Lemma 34.



1. f is measurable  $\iff$   $f_1$  and  $f_2$  are measurable  $(\implies)$  Note that

$$f_1 = f \cdot \chi_X$$
 and  $f_2 = f \cdot \chi_Y$ ,

and since X, Y are measurable, by  $\bigcirc$  Proposition 20, we have that  $f_1$  and  $f_2$  are measurable.

( $\iff$ ) Suppose  $f_1$  and  $f_2$  are measurable and  $X \cup Y$ . We have that

$$f = f_1 + f_2.$$

I will spare the details, but it is not difficult to see that  $\forall a \in \mathbb{R}$ , breaking  $(a, \infty]$  into disjoint pieces if necessary,  $f^{-1}((a, \infty])$  is measurable, and hence f is indeed measurable.

The integral 4 By Definition 27 and Proposition 33, we <sup>4</sup> This proof is iffy. have

$$\begin{split} \int_{E} f &= \sup \left\{ \int_{E} \varphi : \varphi \in \operatorname{SIMP}(E, [0, \infty]), \ \varphi \leq f \right\} \\ &= \sup \left\{ \int_{E} \varphi \cdot \chi_{X} + \varphi \cdot \chi_{Y} : \varphi \in \operatorname{SIMP}(E, [0, \infty]), \ \varphi \leq f \right\} \\ &= \sup \left\{ \int_{X} \varphi + \int_{Y} \varphi : \varphi \in \operatorname{SIMP}(E, [0, \infty]), \ \varphi \leq f \right\} \\ &\leq \sup \left\{ \int_{X} \varphi : \varphi \in \operatorname{SIMP}(X, [0, \infty]), \ \varphi \leq f_{1} \right\} \\ &+ \sup \left\{ \int_{Y} \psi : \psi \in \operatorname{SIMP}(Y, [0, \infty]), \ \psi \leq f_{2} \right\} \\ &= \int_{X} f_{1} + \int_{Y} f_{2}. \end{split}$$

On the other hand, since  $f_1 = f$  on X and  $f_2 = f$  on Y, and Xand Y are disjoint,

$$\begin{split} &\int_{X} f_{1} + \int_{Y} f_{2} \\ &= \sup \left\{ \int_{X} \varphi : \varphi \in \operatorname{SIMP}(X, [0, \infty]), \ \varphi \leq f_{1} = f \upharpoonright_{X} \right\} \\ &+ \sup \left\{ \int_{Y} \psi : \psi \in \operatorname{SIMP}(Y, [0, \infty]), \ \psi \leq f_{2} = f \upharpoonright_{Y} \right\} \\ &= \sup \left\{ \int_{X} \varphi + \int_{Y} \psi : \varphi \in \operatorname{SIMP}(X, [0, \infty]), \\ &\psi \in \operatorname{SIMP}(Y, [0, \infty]), \ \varphi \leq f \upharpoonright_{X}, \ \psi \leq f \upharpoonright_{Y} \right\} \\ &= \sup \left\{ \int_{E} \varphi \cdot \chi_{X} + \psi \cdot \chi_{Y} : \varphi, \psi \in \operatorname{SIMP}(E, [0, \infty]), \\ &\varphi + \psi \leq f \upharpoonright_{X} + f \upharpoonright_{Y} = f \right\} \\ &= \int_{E} f. \end{split}$$

2. By  $\Diamond$  Proposition 30, there exists sequences  $\{\varphi_n\}_n$  and  $\{\psi_n\}_n$ 

such that

$$\lim_{n\to\infty}\varphi_n=g\leq h=\lim_{n\to\infty}\psi_n.$$

In particular,

$$\sup_{n>1} \varphi_n = g \le h = \sup_{n>1} \psi_n.$$

Since the leftmost and rightmost terms are simple functions, by

♦ Proposition 33,

$$\begin{split} \int_{E} g &= \sup \left\{ \int_{E} \varphi : \varphi \in \text{SIMP}(E, [0, \infty]), \ \varphi \leq g \right\} \\ &\leq \sup \left\{ \int_{E} \psi : \psi \in \text{SIMP}(E, [0, \infty]), \ \psi \leq h \right\} \\ &= \int_{E} h. \end{split}$$

3. <sup>5</sup> For the first equality, by Definition 27, we have that

<sup>5</sup> This is also iffy.

$$\int_{H} g = \sup \left\{ \int_{H} \varphi : \varphi \in SIMP(H, [0, \infty]), \varphi \leq g \right\}$$

$$= \sup \left\{ \int_{E} \varphi \cdot \chi_{H} : \varphi \in SIMP(E, [0, \infty]), \varphi \leq g \right\}$$

$$= \int_{E} g \cdot \chi_{H}.$$

Note that we have  $g \cdot \chi_H \leq g$ , and so by part (2),

$$\int_{E} g \cdot \chi_{H} \le \int_{E} g.$$

**♦** Proposition 35 (Integration over Domains of Measure Zero and Integration of Functions Agreeing Almost Everywhere)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and  $f, g \in \mathcal{L}(E, [0, \infty])$ .

- 1. If mE = 0, then  $\int_{E} f = 0$ .
- 2. If f = g a.e. on E, then  $\int_E f = \int_E g$ .

1.  $\forall \varphi \in \text{SIMP}(E, [0, \infty])$  written in its standard form

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{E_n},$$

by monotonicity,

$$0 \leq \int_{E} \varphi = \sum_{n=1}^{N} \alpha_{n} m E_{n} \leq \sum_{n=1}^{N} \alpha_{n} m E = 0,$$

and so

$$\int_{\mathbb{F}} \varphi = 0.$$

Thus

$$\int_{E} f = \sup \left\{ \int_{E} \varphi : \varphi \in SIMP(E, [0, \infty]), \ \varphi \leq f \right\} = \sup \{0\} = 0.$$

2. Let  $B := \{x \in E : f(x) \neq g(x)\}$  so that mB = 0. Then by Lemma 34 and part (1), we have

$$\int_{E} f = \int_{E \setminus B} f + \int_{B}$$

$$= \int_{E \setminus B} f + 0$$

$$= \int_{E \setminus B} g + \int_{B} g$$

$$= \int_{F} g.$$

We are now in a position to prove the following important theorem, which we shall do so next lecture.

#### **■**Theorem (The Monotone Convergence Theorem)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and  $(f_n)_n$  be a sequence in  $\mathcal{L}(E,[0,\infty])$  such that  $f_n \leq$  $f_{n+1}$  a.e. on E. Suppose further that

$$f: E \to [0, \infty]$$

is a function such that  $f(x) = \lim_{n \to \infty} f_n(x)$  a.e. on E. Then  $f \in$ 

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$$\mathcal{L}(E,[0,\infty])$$
 and 
$$\int_E f = \lim_{n \to \infty} \int_E f_n.$$

# Lecture 8 May 30th 2019

## 8.1 Lebesgue Integration (Continued)

# **■**Theorem 36 (★ The Monotone Convergence Theorem)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and  $(f_n)_n$  be a sequence in  $\mathcal{L}(E,[0,\infty])$  such that  $f_n \leq f_{n+1}$  a.e. on E. Suppose further that

$$f: E \to [0, \infty]$$

is a function such that  $f(x)=\lim_{n\to\infty}f_n(x)$  a.e. on E. Then  $f\in\mathcal{L}(E,[0,\infty])$  and

$$\int_{E} f = \lim_{n \to \infty} \int_{E} f_{n}.$$

# **⊿** Strategy

- 1. Argue why we can proof for the case where we do not have the "a.e." assumption. There are 2 places here where have an "a.e." assumption:
  - (a)  $f_n \leq f_{n+1}$  on E; and
  - (b)  $f(x) = \lim_{n\to\infty} f_n(x)$  a.e. on E.
- 2. Look at where good things happen and bad things happen, and we'll be able to show that f is measurable.
- 3. Having gotten rid of the place where nasty things happen and showing that f is measurable. We will find that we need to show that

$$\int_{H} f = \lim_{n \to \infty} \int_{H} f_n,$$

where H is where our hopes and dreams live in.

4. One direction is easy, since  $f_n < f$  for all n, on H. For the other direction, we look at a simple function  $\varphi \leq f$ , which is then arbitrary. Then since  $\lim_{n\to\infty} f_n = f$  (pointwise), we want to be able to show something along the lines of

$$\int_{H} f_n - \int_{H} \varphi \ge 0.$$

Instead of trying to do this over the entire H, we can look at where this happens on H for each n. Since the  $f_n$ 's are increasing, and  $\varphi$  arbitrarily fixed,  $f_n - \varphi$  should give us more and more places where they are positive on H.

Proof

Step 1 Let

$$Z = \left\{ x \in E : f(x) \neq \lim_{n \to \infty} f_n(x) \right\}.$$

By hypothesis, mZ = 0 and  $Z \in \mathfrak{M}(E)$ .

Now by Lemma 34,  $f_n \in \mathcal{L}(E, [0, \infty])$  and so  $f_n \upharpoonright_{E \setminus Z} \in \mathcal{L}(E \setminus Z, [0, \infty])$ . Since by hypothesis we have  $\forall x \in E \setminus Z$ ,

$$f(x) = \lim_{n \to \infty} f_n(x),$$

 $f \upharpoonright_{E \setminus Z} \in \mathcal{L}(E \setminus Z, [0, \infty])$  by Corollary 28.

Step 2 For each  $n \ge 1$ , let

$$Y_n := \{x \in E : f_n(x) > f_{n+1}(x)\}.$$

Then by hypothesis,  $mY_n = 0$  and  $Y_n \in \mathfrak{M}(E)$ . Let

$$Y = \bigcup_{n=1}^{\infty} Y_n.$$

Then since  $\mathfrak{M}(E)$  is a  $\sigma$ -algebra,  $Y \in \mathfrak{M}(E)$  and

$$0 \le mY \le \sum_{n=1}^{\infty} mY_n = 0 \implies mY = 0.$$

<sup>1</sup> Up till here, we have showed that we can, instead, turn our focus on wherever nice things happen, and that *f* is measurable as desired.

At this point, by Lemma 34,

$$\int_{E} f = \int_{E \setminus (Y \cup Z)} f + \int_{Y \cup Z} f = \int_{E \setminus (Y \cup Z)} f$$

and for each  $n \ge 1$ ,

$$\int_{E} f_{n} = \int_{E \setminus (Y \cup Z)} f_{n} + \int_{Y \cup Z} f_{n} = \int_{E \setminus (Y \cup Z)} f_{n}.$$

Thus, it remains for us to show that

$$\int_{E\setminus (Y\cup Z)} f = \lim_{n\to\infty} \int_{E\setminus (Y\cup Z)} f_n.$$

Step 3 Let  $X = Y \cup Z$ , which then  $X \in \mathfrak{M}(E)$  and

$$0 < mX < mY + mZ = 0 \implies mX = 0.$$

Let  $H = E \setminus X$ . Note that we then have  $H \in \mathfrak{M}(E)$  and  $\forall x \in H$ ,

$$\forall n \ge 1 \quad f_n(x) \le f_{n+1}(x) \tag{8.1}$$

and

$$f(x) = \lim_{n \to \infty} f_n(x). \tag{8.2}$$

For notational convenience, let

$$g_n = f_n \upharpoonright_H$$

and

$$g = f \upharpoonright_H$$
.

By Equation (8.1) and Equation (8.2), we have that

$$g_1 \leq g_2 \leq \ldots \leq g_n \leq g_{n+1} \leq \ldots \leq g$$
.

By Lemma 34,  $\forall x \in H$ 

$$\lim_{n\to\infty}g_n(x)=\sup_{n\geq 1}g_n(x)\leq g(x),$$

and so

$$\lim_{n\to\infty}\int_H g_n = \sup_{n>1}\int_H g_n \le \int_H g.$$

It remains to show that

$$\int_{H} g \le \lim_{n \to \infty} \int_{H} g_n.$$

If we can show that for any  $\varphi \in SIMP(H, [0, \infty])$ , we have

$$\lim_{n\to\infty}\int_H g_n \geq \int_H \varphi,$$

then our proof is done, since it would mean that

$$\int_{H} f = \int_{H} g = \lim_{n \to \infty} \int_{H} g_{n} = \lim_{n \to \infty} \int_{H} f_{n}.$$

Step 4 <sup>2</sup> Let  $\varphi \in \text{SIMP}(H, [0, \infty])$  such that  $\varphi \leq g$ . <sup>3</sup> Let 0 < r < 1, so that either

- $r\varphi = 0 \le g$ ; 4 or
- $r\varphi < g = \lim_{n \to \infty} g_n$ .

Then, consider

$$H_k = (g_k - r\varphi)^{-1}[0, \infty].$$

Notice that since  $g_{kk}$  is a sequence of increasing functions, we have

$$H_1 \subseteq H_2 \subseteq H_3 \subseteq \dots$$

Also, note that

$$H = \bigcup_{k=1}^{\infty} H_k.$$

- <sup>2</sup> Here, we do something like a race-check. We know that the  $g_n$ 's grow to be arbitrarily close to g, and the set  $\{\varphi \in SIMP(H, [0, \infty]) : \varphi \leq g\}$  also has elements arbitrarily close to g. It would suffice to show that for every  $\varphi$ , the limit of the integral of the  $g_n$ 's is greater than the integral of  $\varphi$ .
- $^3$  Note that we require this scaling factor, because we cannot allow  $\varphi=g$ , for otherwise our increasing sequence of  $g_n$ 's will never be able to 'catch up' to  $\varphi$ , which is what we want.  $^4$  In the case where g=0, we have that  $r\varphi=0$  and not something bigger.
- <sup>5</sup> The increasing-ness of the  $g_k$ 's guarantees that if  $(g_k r\varphi)(x) \ge 0$ , then  $(g_{k+1} r\varphi)(x) \ge 0$ . This is sort of like a rising water level scenario.

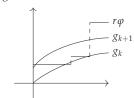


Figure 8.1: Increasing levels of  $g_k$  'covers' more and more parts of  $r\varphi$ 

<sup>6</sup> WTS

$$\int_{H} \varphi = \lim_{k \to \infty} \int_{H_k} \varphi.$$

Since  $\varphi \in SIMP(H, [0, \infty])$ , let us write

$$\varphi = \sum_{k=1}^{N} \alpha_k \chi_{J_k}$$

in its standard form, where  $J_k \in \mathfrak{M}(H)$ . Then

$$\int_{H} \varphi = \sum_{k=1}^{N} \alpha_{k} m J_{k},$$

while

$$\int_{H_n} \varphi = \sum_{k=1}^N \alpha_k m(J_k \cap H_n)$$

for each  $n \ge 1$ .

By the continuity of the Lebesgue measure (A1), notice that

$$\lim_{n\to\infty} m(J_k\cap H_n) = m\left(J_k\cap \left(\bigcup_{n=1}^{\infty} H_n\right)\right) = m(J_k\cap H) = m(J_k)$$

Thus

$$\lim_{n\to\infty}\int_{H_n}\varphi=\sum_{k=1}^N\alpha_km(J_n)=\int_H\varphi,$$

as claimed.

Then in particular, we have that

$$\int_{H} r \varphi = \lim_{k \to \infty} \int_{H_k} \varphi \leq \lim_{k \to \infty} \int_{H_k} g_k \leq \lim_{k \to \infty} \int_{H} g_k,$$

where the last inequality follows from Lemma 34.

This is exactly the final piece that we have set out to prove, and so we have completed the proof. 

#### Example 8.1.1

Recall our "pathological" sequence of Riemann integral functions earlier on, where  $E = \mathbb{Q} \cap [0,1] = \{q_n\}_{n=1}^{\infty}$ , and sequence of functions

$$f_n = \chi_{\{q_1,...,q_n\}}$$
, for  $n \ge 1$ ,

<sup>6</sup> By this construction, we have that  $r\varphi \leq g_k$  in  $H_k$  for each k. So we already

$$\lim_{k\to\infty}\int_{H_k}r\varphi\leq\lim_{k\to\infty}\int_{H_k}g_k$$

in our bag. Notice that since  $\varphi$  is a simple function, by <a> Definition 27</a>, we have

$$\int_{H_k} \varphi = \int_H \varphi \cdot \chi_{H_k}.$$

Since the  $H_k$ 's is an 'increasing sequence' of sets, and especially since  $H = \bigcup_{k=1}^{\infty} H_k$ , we expect

$$\lim_{k \to \infty} \int_{H_k} \varphi = \int_H \varphi.$$

and their limit

$$f = c_{\mathbb{Q} \cap [0,1]}.$$

We have that

$$0 \le f_1 \le f_2 \le \ldots \le f$$
,

and each 7

$$f_n \in \mathcal{L}([0,1],[0,\infty)).$$

By the Monotone Convergence Theorem (MCT), f is measurable and

$$\int_{[0,1]} f = \lim_{n \to \infty} \int_{[0,1]} f_n = \lim_{n \to \infty} 0 = 0.$$

This agrees with what we saw much earlier on, i.e.

$$0 \le \int_{[0,1]} f = \int_{[0,1]} \chi_E = mE \le m\mathbb{Q} = 0.$$

Note that  $f_n$  is Riemann integrable, but f is not, but it is Lebesgue integrable. In other words, this function f is an example of a Lebesgue integrable function that is not Riemann integrable.

Corollary 37 (Linearity of the Lebesgue Integral and Other Results)

Let  $E \in \mathfrak{M}(\mathbb{R})$ .

1. If  $f,g \in \mathcal{L}(E,[0,\infty])$  and  $\kappa \geq 0$ , then

$$\int_{E} \kappa f + g = \kappa \int_{E} f + \int_{E} g.$$

2. If  $(h_n)_{n=1}^{\infty}$  is a sequence in  $\mathcal{L}(E,[0,\infty])$  and

$$h(x) := \lim_{N \to \infty} \sum_{n=1}^{N} h_n(x), \quad \forall x \in E,$$

then  $h \in \mathcal{L}(E, [0, \infty])$  and

$$\int_E h = \sum_{n=1}^{\infty} \int_E h_n.$$

3. Let  $f \in \mathcal{L}(E, [0, \infty])$ . If  $(H_n)_{n=1}^{\infty}$  is a sequence  $\mathfrak{M}(E)$  with  $H_i \cap H_j =$ 

Exercise 8.1.1 Show that  $f_n \in \mathcal{L}([0,1],[0,\infty))$ .

 $\emptyset$  when  $1 \leq i \neq j \leq \infty$  and  $H = \bigcup_{n=1}^{\infty} H_n$ , then

$$\int_{H} f = \sum_{n=1}^{\infty} \int_{H_n} f.$$



1. By A3, there exists a sequence of simple, measurable functions  $(\varphi_n)_n$ ,  $(\psi_n)_n$  in  $\mathcal{L}(E,[0,\infty])$  such that

$$0 \le \varphi_1 \le \varphi_2 \le \dots \le f$$
$$0 \le \psi_1 \le \psi_2 \le \dots \le g$$

such that  $\forall x \in E$ ,

$$\lim_{n \to \infty} \varphi_n(x) = f(x)$$
$$\lim_{n \to \infty} \psi_n(x) = g(x)$$

By  $\land$  Proposition 33, we have that for each n, for any  $\kappa \in E$ , we have

$$\int_{E} \kappa \varphi_{n} + \psi_{n} = \kappa \int_{E} \varphi_{n} + \int_{E} \psi_{n}.$$

Furthermore, note that

$$\lim_{n\to\infty} (\kappa \varphi + \psi)(x) = (\kappa f + g)(x),$$

and  $(\kappa \varphi_n + \psi_n)_n$  is an increasing <sup>8</sup> sequence of non-negative, simple, measurable functions converging pointwise to the function  $\kappa f + g$ .

<sup>8</sup> If you are second-guessing yourself like I did, notice that that *n* is fixed for both of them, not just one of them.

Thus, by the MCT, we see that

$$\int_{E} (\kappa f + g) = \lim_{N \to \infty} (\kappa \varphi_N + \psi_N)$$
$$= \lim_{N \to \infty} \kappa \int_{E} \varphi_N + \int_{E} \psi_N = \kappa \int_{E} f + \int_{E} g.$$

2. 9 Let

$$g_N = \sum_{n=1}^N h_n$$

for each  $N \ge 1$ .

<sup>9</sup> Since range  $h_n \subseteq [0, \infty]$ , the partial sums form an increasing sequence of functions. Then, we can make use of the MCT.

Showing that  $g_N \in \mathcal{L}(E, [0, \infty])$  Let  $(a, \infty]$ , for any  $\alpha \in [0, \infty)$ .

Then since  $g_N$  is a finite sum of functions, we have that

$$g_N((a,\infty]) = h_1((a,\infty]) \cup h_2((a,\infty]) \cup \ldots \cup h_N((a,\infty]),$$

which is a countable union of measurable sets, and is hence measurable.

Then

$$0 \le g_1 \le g_2 \le \ldots \le h$$

and  $\forall x \in E$ 

$$\lim_{N\to\infty}g_N(x)=h(x),$$

both of which are from our assumptions.

By the MCT and part (1), we have that

$$\int_{E} h = \lim_{N \to \infty} \int_{E} g_{N}$$

$$= \lim_{N \to \infty} \int_{E} \sum_{n=1}^{N} h_{n}$$

$$= \lim_{N \to \infty} \sum_{n=1}^{N} \int_{E} h_{n}$$

$$= \sum_{n=1}^{\infty} \int_{E} h_{n}$$

as required.

3. 10 Let

$$h_n = f \cdot \chi_{H_n}$$

for each  $n \geq 1$ . Since each  $H_n \in \mathfrak{M}(E)$ , each  $\chi_{H_n}$ , and f being measurable implies that each  $h_n$  is measurable. Since  $H_i \cap H_j = \emptyset$  for all  $1 \leq i \neq j \leq \infty$ , we have that

$$f = \sum_{n=1}^{\infty} h_n.$$

By part (2), we have that

$$\int_{E} f = \sum_{n=1}^{\infty} \int_{E} h_n = \sum_{n=1}^{\infty} \int_{E} f \cdot \chi_{H_n} = \sum_{n=1}^{\infty} \int_{H_n} f.$$

<sup>10</sup> Since the RHS of the goal integrates over  $H_n$  ⊆ H, and the  $H_i$ 's are disjoint, we can break f down by where  $H_n$  is defined.

# **■** Definition 31 (Lebesgue Integrable)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and  $f \in \mathcal{L}(E,\overline{\mathbb{R}})$ . We say that f is Lebesgue integrable on E if 11

11 Recall Remark 6.1.1.

$$\int_E f^+ < \infty$$
 and  $\int_E f^- < \infty$ ,

in which case we set

$$\int_E f := \int_E f^+ - \int_E f^-.$$

We denote by  $\mathcal{L}_1(E,\overline{\mathbb{R}})$  the set of all Lebesgue integrable functions from E to  $\overline{\mathbb{R}}$ , and  $\mathcal{L}_1(E,\mathbb{R})$  all Lebesgue integrable functions from E to  $\mathbb{R}$ .

#### Remark 8.1.1

Let  $E \in \mathfrak{M}(\mathbb{R})$ .

- 1. By definition, every Lebesgue integrable function on E is Lebesgue measurable.
- 2. A measurable function f is Lebesgue integrable iff |f| is Lebesgue integrable. Notice that

$$\int_{E} f = \int_{E} f^{+} - \int_{E} f^{-}$$

while

$$\int_{E} |f| = \int_{E} f^{+} + f^{-} = \int_{E} f^{+} + \int_{E} f^{-},$$

and so if either of these are integrable, then

$$\int_E f^+ < \infty$$
 and  $\int_E f^- < \infty$ ,

which then the other must also be integrable.

It is important to note that this is a distinguishing feature of Lebesgue integration, in comparison to Riemann integration. For instance, if we consider the function

$$f(x) = \frac{\sin x}{x}$$
, for  $x \ge 1$ ,

improper Riemann integration gives us that  $\int_1^\infty f(x) dx = \frac{\pi}{2}$ . But from the POV of Lebesgue integration, notice that

$$\int_{[\pi,(N+1)\pi]} \left| \frac{(\sin x)^{+}}{x} \right|$$

$$= \sum_{k=1}^{N} \int_{[\pi k,\pi(k+1)]} \left| \frac{(\sin x)^{+}}{x} \right|$$

$$= \sum_{k=1}^{N} \int_{[0,\pi]} \frac{|\sin(t+k\pi)|}{t+k\pi}$$

$$= \sum_{k=1}^{N} \int_{[0,\pi]} \frac{|\sin t|}{t+k\pi}$$

$$\geq \sum_{k=1}^{N} \frac{1}{(k+1)\pi} \int_{[0,\pi]} \sin t.$$

Assuming we know some of the upcoming results, in particular, assuming that we know that for bounded functions the Lebesgue integral is the same as the Riemann integral, we see that the above is

$$= \frac{2}{\pi} \sum_{k=1}^{N} \frac{1}{k+1},$$

which is a harmonic series and hence divergent.

3. If  $f \in \mathcal{L}_1(E, \overline{\mathbb{R}})$ , then

$$mf^{-1}(\{-\infty\}) = 0 = mf^{-1}(\{\infty\}).$$

#### Exercise 8.1.2

Prove that the above is indeed the case.

4. Following the above, if we set

$$H = f^{-1}(\{-\infty, \infty\}),$$

then  $H \in \mathfrak{M}(E)$  and mH = 0. Letting

$$g = f \cdot \chi_{E \setminus H}$$
.

Then

$$g = f$$
 a.e. and  $g \in \mathcal{L}_1(E, \mathbb{R})$ .

This will prove itself more useful than it seems, especially since  $\mathcal{L}_1(E, \overline{\mathbb{R}})$ is that it is not a vector space!!!

5. Suppose that  $g: E \to \mathbb{C}$  is measurable. Let us write

$$g = (g_1 - g_2) + i(g_3 - g_4),$$

where  $g_1 = (\Re g)^+$ ,  $g_2 = (\Re g)^-$ ,  $g_3 = (\Im g)^+$ ) and  $g_4 = (\Im g)^-$ . Then we say that g is Lebesgue integrable, and write

$$g \in \mathcal{L}_1(E,\mathbb{C})$$
,

if

$$\int_{F} g_k < \infty \quad \forall 1 \le k \le 4,$$

and we write

$$\int_{E} g = \left( \int_{E} g_{1} - \int_{E} g_{2} \right) + i \left( \int_{E} g_{3} + \int_{E} g_{4} \right).$$

♦ Proposition 38 (Linearity of Lebesgue Integral for Lebesgue **Integrable Functions)** 

let  $E \in \mathfrak{M}(\mathbb{R})$ . Suppose that  $f, g \in \mathcal{L}_1(E, \mathbb{R})$  and  $\kappa \in \mathbb{R}$ .

- 1.  $\kappa f \in \mathcal{L}_1(E, \mathbb{R})$  and  $\int_E \kappa f = \kappa \int_E f$ .
- 2.  $f+g \in \mathcal{L}_1(E,\mathbb{R})$  and  $\int_E (f+g) = \int_E f + \int_E g$ .
- 3. Finally,

$$\left| \int_{E} f \right| \le \int_{E} |f| \, .$$

#### Proof

Note that ightharpoonup Corollary 37 covers for the cases where  $f,g \in \mathcal{L}(E,[0,\infty])$ and  $\kappa \geq 0$  for (1) and (2). This is, unfortunately, insufficient for the entire proposition

1. Let 
$$f = f^+ - f^-$$
.

Case 1:  $\kappa = 0$  We have that

$$\int_{E} \kappa f = \int_{E} 0 = 0 = \kappa \int_{E} f.$$

Case 2: k > 0 We have

$$\kappa f = (\kappa f)^+ - (\kappa f)^-.$$

Note

$$(\kappa f)^+ = \kappa f^+$$
 and  $(\kappa f)^- = \kappa f^-$ .

So, since  $f^+, -f^- \in \mathcal{L}(E, [0, \infty])$ , by Corollary 37,

$$\int_{E} \kappa f = \int_{E} \kappa f^{+} - \int_{E} \kappa f^{-}$$

$$= \kappa \int_{E} f^{+} - \kappa \int_{E} f^{-}$$

$$= \kappa \left( \int_{E} f^{+} - f^{-} \right)$$

$$= \kappa \int_{E} f.$$

Case 3:  $\kappa$  < 0 Similar to the above, we first observe that

$$(\kappa f)^+ = -\kappa f^-$$
 and  $(\kappa f)^- = -\kappa f^+$ .

Then by the same reason as in the last case, we have

$$\int_{E} \kappa f = \int_{E} -\kappa f^{-} - \int_{E} -\kappa f^{+}$$

$$= -\kappa \left( \int_{E} f^{-} - \int_{E} f^{+} \right)$$

$$= -\kappa \left( - \int_{E} f \right)$$

$$= \kappa \int_{E} f.$$

2.  $f + g \in \mathcal{L}_1(E, \mathbb{R})$  For convenience, let

$$h = f + g = f^{+} - f^{-} + g^{+} - g^{-}$$
.

Notice that

$$h^+, h^- \le |h| = |f + g| \le |f| + |g| = f^+ + f^- + g^+ + g^-.$$

Thus by Corollary 37,

$$\begin{split} \int_{E} h^{+} &\leq \int_{E} f^{+} + f^{-} + g^{+} + g^{-} \\ &= \int_{E} f^{+} + \int_{E} f^{-} + \int_{E} g^{+} + \int_{E} g^{-} < \infty. \end{split}$$

Similarly,  $\int_E h^-$  < ∞.

 $\int_{E} (f+g) = \int_{E} f + \int_{E} g$  Notice that

$$h^+ - h^- = h = f + g = f^+ - f^- + g^+ - g^-,$$

and so

$$h^+ + f^- + g^- = h^- + f^+ + g^+.$$

Then by Corollary 37,

$$\int_{E} h^{+} + \int_{E} f^{-} + \int_{E} g^{-} = \int_{E} h^{-} + \int_{E} f^{+} + \int_{E} g^{+},$$

and so

$$\int_{E} (f+g) = \int_{E} h = \int_{E} h^{+} - \int_{E} h^{-}$$

$$= \int_{E} f^{+} - \int_{E} f^{-} + \int_{E} g^{+} - \int_{E} g^{-}$$

$$= \int_{E} f + \int_{E} g.$$

3. First, notice that  $f \in \mathcal{L}_1(E,\mathbb{R})$ , by our previous remark,  $|f| \in$  $\mathcal{L}_1(E,\mathbb{R})$ . Now since  $|f|=f^++f^-$ , and  $\left|\int_E f^+\right|$ ,  $\left|\int_E f^-\right|\geq 0$ ,

$$\left| \int_{E} f \right| = \left| \int_{E} f^{+} - \int_{E} f^{-} \right|$$

$$\leq \left| \int_{E} f^{+} \right| + \left| \int_{E} f^{-} \right|$$

$$= \int_{E} f^{+} + \int_{E} f^{-}$$

$$= \int_{E} |f|.$$

# 9.1 Lebesgue Integration (Continued 2)

Thus far, we've only integrated simple functions, and never even did so for, say, f(x) = x. Trying to do that will lead to intense swearing, rising of blood pressure, heavy signs of nausea and mental pain. Why? Well just try doing it.

Exercise 9.1.1 (How a slime became one heck of a monster to deal with)

Calculate  $\int_{[0,1]} x$ .

We hate pain, and now we want to crawl back to Riemann integration and ask for forgiveness. Fortunately, the nice world of Riemann integration is kind enough to give us a bridge. We shall now study this bridge. In particular, we shall see that for **bounded** functions on **closed**, **bounded** intervals, Riemann integrability implies Lebesgue integrability, and, in fact, they coincide on these functions. In particular, this opens up the **Fundamental Theorem of Calculus** (for Riemann integration) to us.

# Lemma 39 (Riemann Integrability and Lebesgue Integrability of Step Functions)

Let  $a < b \in \mathbb{R}$  and  $\varphi : [a,b] \to \mathbb{R}$  be a step function. Then  $\varphi$  is both Riemann integrable and Lebesgue integrable, and

$$\int_{[a,b]} \varphi = \int_a^b \varphi.$$

## Proof

Let  $P = \{a = p_0 < p_1 < p_2 < \dots p_N = b\} \in \mathcal{P}([a,b])$ , where the  $p_n$ 's are chosen such that  $[p_{n-1}, p_n)$  do not contain a 'jump'. Since  $\varphi$  is a step function, let

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{[p_{n-1}, p_n)},$$

where  $\alpha_n = \varphi(x)$  for all  $x \in [p_{n-1}, p_n)$ ,  $1 \le n \le N$ .

Then

$$\int_{[a,b]} \varphi = \sum_{n=1}^{N} \alpha_n m[p_{n-1}, p_n)$$

$$= \sum_{n=1}^{N} \alpha_n (p_n - p_{n-1})$$

$$= \sum_{n=1}^{N} \int_{p_{n-1}}^{p_n} \alpha_n$$

$$= \int_a^b \sum_{n=1}^{N} \alpha_n \chi[p_{n-1}, p_n)$$

$$= \int_a^b \varphi.$$

# ■ Theorem 40 (Bounded Riemann-Integrable Functions are Lebesgue Integrable)

Let  $a < b \in \mathbb{R}$  and  $f : [a,b] \to \mathbb{R}$  be a bounded, Riemann-integrable function. Then  $f \in \mathcal{L}_1([a,b],\mathbb{R})$  and

$$\int_{[a,b]} f = \int_a^b f,$$

i.e. the Lebesgue and Riemann integrals of f over [a, b] coincide.

# **✓** Strategy

Here is my understanding of the idea that motivates this proof.

- 1. It is important that the function is bounded both on its domain and its range. A bound on the domain allows us to do finite sums, and a bound on the range puts a cap on how high our rectangles can be.
- 2. We need to reduce the problem to deal only with step functions, using step functions as close to f as possible, and then use our earlier results and intuition to forge forward.

## Proof

First, since f is bounded, wma  $|f| < M \in \mathbb{R}$ . Let  $g = M\chi_{[a,b]}$ , which is a step-function and is hence integrable by Lemma 39. Then, notice that f + g is Riemann integrable. Furthermore, observe that

$$\int_{a}^{b} (f+g) = \int_{a}^{b} f + M(b-a).$$

So 
$$f + g \in \mathcal{L}_1([a,b],\mathbb{R})$$
 iff  $f \in \mathcal{L}_1([a,b],\mathbb{R})$ .

Now, by Pheorem 2, for each  $n \ge 1$ ,  $\exists R_n \in \mathcal{P}[a,b]$  partition such that  $\forall X, Y \supseteq R_n$  refinements,  $\forall X^*, Y^*$  test values of X and Yrespectively, we have

$$|S(f, X, X^*) - S(f, Y, Y^*)| < \frac{1}{N}.$$

<sup>1</sup> Now, let  $Q_N = \bigcup_{n=1}^N R_n$ , so that it is a **common refinement** of  $R_1, R_2, \ldots, R_N$ . Write

$$Q_N = \{ a = q_{0,N} < q_{1,N} < \ldots < q_{m_N,N} \}.$$

<sup>2</sup> Let

$$H_{k,N} = [q_{k,N}, q_{k+1,N}] \text{ for } 1 \le k \le m_N - 1,$$

and

$$H_{m_N,N} = [q_{m_N-1,N}, q_{m_N,N}].$$

<sup>3</sup> Define for each  $1 \le k \le m_N$ ,

$$\alpha_{k,N} := \inf\{f(t) : t \in H_{k,N}\} \le -M$$
$$\beta_{k,N} := \sup\{f(t) : t \in H_{k,N}\} \le M.$$

<sup>2</sup> Look at each subinterval of each refinement.

<sup>3</sup> Get the sup and inf of each interval under f.

<sup>4</sup> For each  $N \ge 1$ , let

$$egin{aligned} arphi_N &\coloneqq \sum_{k=1}^{m_N} lpha_{k,N} \chi_{H_{k,N}} \ \psi_N &= \sum_{k=1}^{m_N} eta_{k,N} \chi_{H_{k,N}}. \end{aligned}$$

Since each  $\varphi_N$ ,  $\psi_N$  is simple, they are all measurable and Lebesgue integrable (cf. Lemma 39).

Now, notice that

$$Q_1 \subseteq Q_2 \subseteq \ldots \subseteq Q_N \subseteq Q_{N+1} \subseteq \ldots$$

since it is a sequence of finer and finer refinements, we have

$$\varphi_1 \le \varphi_2 \le \varphi_3 \le \ldots \le f \le \ldots \le \psi_3 \le \psi_2 \le \psi_1.$$
 (9.1)

Thus, by Lemma 39 and Lemma 34, we have

$$\int_{[a,b]} \varphi_N = \int_a^b \varphi_N \le \int_a^b f \le \int_a^b \psi_N = \int_{[a,b]} \psi_N$$

for each N. Since  $Q_N$  is a refinement of  $R_N$ , we have that

$$|S(f,Q_N,Q_N^*) - S(f,Q_N,Q_N^{**})| < \frac{1}{N}$$

which implies

$$\left| \int_{[a,b]} \varphi_N - \int_{[a,b]} \psi_N \right| < \frac{1}{N},$$

for  $N \ge 1$ .

Due to Equation (9.1), let

$$\varphi \coloneqq \lim_{N \ge 1} \varphi_N$$
 and  $\psi \coloneqq \lim_{N \ge 1} \psi_N$ .

Then by the MCT, we have that

$$\int_{[a,b]} \varphi = \lim_{N \to \infty} \int_{[a,b]} \varphi_N = \lim_{N \to \infty} \int_a^b \varphi_N$$

$$= \int_a^b f$$

$$= \lim_{N \to \infty} \int_a^b \psi_N = \lim_{N \to \infty} \psi_N = \int_{[a,b]} \psi.$$

<sup>4</sup> Use the above  $\alpha$ 's and  $\beta$ 's to construct simple functions, which are step-like functions.

Then  $\int_{[a,b]} \varphi - \psi = 0$ . Since  $\varphi \leq \psi$ , we must thus have  $\varphi = \psi$  a.e. on [a, b]. Since  $\varphi \leq f \leq \psi$ , we have that  $\varphi = f = \psi$  a.e. on [a, b]. Since  $\varphi$ ,  $\psi$  are measurable, so is f, and thus

$$\int_{[a,b]} f = \int_{[a,b]} \varphi = \int_a^b f < \infty.$$

Our earlier demon-level slime has been reduced back to being a, well, slime-level monster.

#### Example 9.1.1

Let f(x) = x and  $x \in [0,1]$ . Then by the Fundamental Theorem of Calculus,

$$\int_{[0,1]} f = \int_0^1 f = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

#### Example 9.1.2

Let  $f(x) = \frac{1}{x^2}$  where  $x \in E := [1, \infty)$ . We want to calculate  $\int_{[1,\infty)} f$ . For each  $n \ge 1$ , set  $f_n := f \cdot \chi_{[1,n]}$ . Then f is measurable, since it is continuous except at one point on E, and

$$0 \le f_1 \le f_2 \le \dots$$

with

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \forall x \ge 1.$$

By  $\blacksquare$  Theorem 40, for all  $n \ge 1$ ,

$$\int_{[1,n]} f_n = \int_1^n f_n = \int_1^n \frac{1}{x^n} = -\frac{1}{x} \Big|_1^n = 1 - \frac{1}{n}.$$

By the MVT,

$$\int_{[1,\infty)} f = \lim_{n \to \infty} \int_{[1,n]} f_n$$

$$= \lim_{n \to \infty} \left( 1 - \frac{1}{n} \right) = 1.$$

#### **66** Note 9.1.1

*In the above example, the Lebesgue integral of f returns the value of the* 

improper Riemann integral of f over  $[1, \infty)$ , which is not what happened in another function that we looked at earlier. There are 2 things to note here:

- it is possible for an improper Riemann integral of a measurable function  $f:[1,\infty)\to\mathbb{R}$  to exist, even though the Lebesgue integral  $\int_{[1,\infty)} f$  does not exist!
- There is no notion of an "improper" Lebesgue integral. The domain of f,  $[1, \infty)$ , is just another measurable set.

IN THE Monotone Convergence Theorem, if the "increasing" assumption is dropped, then the result may not hold.

# Example 9.1.3 (The MCT needs an increasing/decreasing sequence of functions)

Consider the sequence  $(f_n)_n$  given by

$$f_n:[1,\infty)\to\mathbb{R}$$

where

$$x \mapsto \begin{cases} \frac{1}{nx} & 1 \le x \le e^n \\ 0 & x > e^n \end{cases}.$$

Then  $(f_n)_n$  converges uniformly to f = 0 on  $[1, \infty)$ . Note that for all  $n \ge 1$ ,  $f_n$  is Riemann integrable, and bounded on  $[1, e^n]$ , and so

$$\int_{[1,\infty)} f_n = \int_{[1,e^n]} \frac{1}{nx}$$

$$= \int_1^{e^n} \frac{1}{nx}$$

$$= \frac{1}{n} \ln x \Big|_1^{e^n}$$

$$= \frac{1}{n} (n-0) = 1,$$

for each  $n \ge 1$ . However,

$$\int_{[1,\infty)} f = \int_{[1,\infty)} f = 0.$$

## 10.1 Lebesgue Integration (Continued 3)

## **■**Theorem 41 (Fatou's Lemma)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and  $f_n \in \mathcal{L}(E,[0,\infty])$ , for  $n \geq 1$ . Then

$$\int_{E} \liminf_{n\geq 1} f_n \leq \liminf_{n\geq 1} \int_{E} f_n.$$

# Proof

For each  $N \ge 1$ , set  $g_N = \inf\{f_n : n \ge N\}$ . Then by  $\bigcirc$  Proposition 27, each  $g_N$  is measurable, and clearly

$$g_1 \leq g_2 \leq g_3 \leq \dots$$

Then by the MCT, we have

$$\int_{E} \liminf_{n\geq 1} f_n = \int_{E} \lim_{N\to\infty} g_N = \lim_{N\to\infty} \int_{E} g_N.$$

Since  $g_N \le f_n$  for all  $n \ge N$  (by construction), we have

$$\int_{E} g_{N} \leq \int_{E} f_{n}$$

for all  $n \ge N$ , whence

$$\int_E g_N \leq \liminf_{n\geq 1} \int_E f_n.$$

Since this holds for all  $N \ge 1$ , we have that

$$\int_{E} \liminf_{n \ge 1} f_n = \lim_{N \to \infty} \int_{E} g_N \le \liminf_{n \ge 1} \int_{E} f_n.$$

An example where the inequality in Fatou's Lemma is strict is the following.

#### Example 10.1.1

Consider a sequence of functions  $f_n = n\chi_{\left(0,\frac{1}{n}\right]}$ ,  $n \ge 1$ . It's clear that for any  $x \in [0,1]$ ,  $\lim_{n\to\infty} f_n(x) = 0$ . Thus

$$\int_{[0,1]} \liminf_{n \ge 1} f_n = \int_{[0,1]} 0 = 0.$$

On the other hand

$$\int_{[0,1]} f_n = nm\left(\left(0, \frac{1}{n}\right]\right) = 1$$

for all  $n \ge 1$ , and so  $\liminf_{n \ge 1} \int_{[0,1]} f_n = 1$ .

#### Example 10.1.2

Suppose  $E \in \mathfrak{M}(\mathbb{R})$ ,  $f \in \mathcal{L}(E, \overline{\mathbb{R}})$ . Recall that  $f \in \mathcal{L}_1(E, \overline{\mathbb{R}}) \iff |f| \in \mathcal{L}_1(E, \overline{\mathbb{R}})$ .

Suppose  $g \in \mathcal{L}_1(E,\overline{\mathbb{R}})$ ,  $f \in \mathcal{L}(E,\overline{\mathbb{R}})$  and suppose  $0 \leq |f| \leq g$  a.e. on E, and that  $\int_E g < \infty$ . Then  $\int_E |f| \leq \int_E g < \infty$ , which thus  $f \in \mathcal{L}_1(E,\overline{\mathbb{R}})$ .

#### Theorem 42 (Lebesgue Dominated Convergence Theorem)

Let  $E \in \mathfrak{M}(\mathbb{R})$  and  $(f_n)_n$  in  $\mathcal{L}_1(E,\overline{\mathbb{R}})$ . Suppose that there exists  $g \in \mathcal{L}_1(E.\overline{\mathbb{R}})$  such that  $|f_n| \leq g$  a.e. on E, for  $n \geq 1$ . Suppose furthermore that  $f: E \to \overline{\mathbb{R}}$  is a function, and that

$$f(x) = \lim_{n \to \infty} f_n(x)$$
, a.e. on E.

Then  $f \in \mathcal{L}_1(E, \overline{\mathbb{R}})$  and

$$\int_{E} f = \lim_{n \to \infty} \int_{E} f_{n}.$$

#### Proof

Isolating "bad" points Consider, for each  $n \ge 1$ , the set

$$Y_n := \{ x \in E : |f_n(x)| > g(x) \}.$$

By assumption,  $mY_n = 0$  for each  $n \ge 1$ . Letting

$$Y := \bigcup_{n=1}^{\infty} Y_n = \{ x \in E : |f_n(x)| > g(x), n \ge 1 \},$$

we have that

$$0 \le mY \le \sum_{n=1}^{\infty} mY_n = 0,$$

and so mY = 0.

Furthermore, consider

$$Z := \{ x \in E : f(x) \neq \lim_{n \to \infty} f_n(x) \}.$$

By assumption, mZ = 0.

Let

$$B := Y \cup Z$$
.

Then  $\forall x \in B$ , we have

$$f(x) \neq \lim_{n \to \infty} f_n(x)$$
 and  $|f_n(x)| > g(x)$  for each  $n \geq 1$ .

Most importantly, we have that

$$0 \le mB \le mY + mZ = 0,$$

and so mB = 0.

Let  $H = E \setminus B$ . Then  $\forall x \in H$ ,

$$f(x) = \lim_{n \to \infty} f_n(x)$$
 and  $|f_n(x)| \le g(x)$  for each  $n \ge 1$ .

It follows that if we can prove the statement for

$$f_n \upharpoonright_H$$
 and  $f \upharpoonright_H$ ,

then we obtain the result that we desire. Thus, wlog, we may replace E with H.

Proving the statement Since  $f(x) = \lim_{n\to\infty} f_n(x)$ , by A2, we have that

$$\limsup_{n\geq 1} f_n(x) = \liminf_{n\geq 1} f_n(x) = \lim_{n\to\infty} f_n(x) = f(x),$$

and so in particular we have

$$\int_{E} f = \int_{E} \liminf_{n \ge 1} f_n(x).$$

From Fatou's Lemma and A2, we have that

$$\int_{E} f \leq \liminf_{n \geq 1} \int_{E} f_{n} \leq \limsup_{n \geq 1} \int_{E} f_{n}.$$

Now, notice that  $g - f_n \ge 0$  <sup>1</sup>, and we have

<sup>1</sup> This is required to invoke Fatou's Lemma

$$\int_{E} (g - f) = \int_{E} (g - \limsup_{n \ge 1} f_n)$$

$$= \int_{E} \liminf_{n \ge 1} (g - f_n)$$

$$\leq \liminf_{n \ge 1} \int_{E} (g - f_n) \quad \because Fatou's$$

$$= \int_{E} g - \limsup_{n \ge 1} \int_{E} f_n.$$

Thus

$$\int_{E} f \geq \limsup_{n \geq 1} \int_{E} f_{n}.$$

Therefore

$$\limsup_{n\geq 1} \int_{E} f_n \leq \int_{E} f \leq \liminf_{n\geq 1} \int_{E} f_n \leq \limsup_{n\geq 1} \int_{E} f_n.$$

By the **Squeeze Theorem**, we obtain

$$\int_{E} f = \lim_{n \to \infty} \int_{E} f_{n}.$$

# 10.2 L<sub>p</sub> Spaces

Functional analysis is the study of normed linear spaces and the continuous linear maps between them. Amongst the most impor-

tant examples are the so-called  $L_p$ -spaces, and we will now turn our attention towards them. .

You may wish to refresh your memory on the definition of a semi-norm.

#### Example 10.2.1

Let  $E \in \mathfrak{M}(\mathbb{K})$  and mE > 0. Recall that

$$\mathcal{L}_1(E, \mathbb{K}) = \left\{ f \in \mathcal{L}(E, \mathbb{K}) : \int_E |f| < \infty \right\}.$$

Define the map

$$\nu_1: \mathcal{L}_1(E, \mathbb{K}) \to \mathbb{K}$$

$$f \mapsto \int_E |f|.$$

Observe that

- $\nu_1(f) \geq 0$  for all  $f \in \mathcal{L}_1(E, \mathbb{K})$ ;
- $\nu_1(0) = \int_F |0| = 0;$
- $\kappa \in \mathbb{K} \implies$

$$v_1(\kappa f) = \int_E |\kappa f| = |\kappa| \int_E |f| = |\kappa| v_1(f);$$

and

•  $\forall f, g \in \mathcal{L}_1(E, \mathbb{K})$ 

$$v_1(f+g) = \int_E |f+g| \le \int_E |f| + \int_E |g| = v_1(f) + v_1(g).$$

However, it is important to notice that for any  $x_0 \in E$ ,

$$\nu_1(\chi_{\{x_0\}}) = \int_{\{x_0\}} 1 = 0.$$

Thus  $\nu_1$  is **not a norm** since  $\chi_{\{x_0\}} \neq \emptyset$ .

# ♦ Proposition 43 (Kernel of a Vector Space is a Linear Manifold)

Let W be a vector space over the field  $\mathbb{K}$ , and suppose that v is a seminorm on W. Let

$$\mathcal{N} := \{ w \in W : \nu(w) = 0 \}.$$

Then N is a linear manifold  $^2$  in W and so W/N is a vector space over  $\mathbb{K}$ , whose elements we denote by

$$[x] := x + \mathcal{N}.$$

Furthermore, the map

$$\|\cdot\|: \mathcal{W}/\mathcal{N} \to \mathbb{K}$$

$$[x] \mapsto \nu(x)$$

is well-defined, and defines a norm on W/N.

<sup>2</sup> A subspace *M* of a Hilbert space, which is a vector space endowed with an inner product such that with an appropriate norm, turns it into a complete metric space, is called a linear manifold if it is closed under addition and scalar multiplication. (Source: Stover (nd))

Here, we can safely talk about Hilbert spaces because  $\mathbb{K}$  is endowed with an inner product. Furthermore, the check is to simply show that M is a subspace of the original space.



 $\mathcal{N}$  is a linear manifold Firstly, note that  $\nu(0) = 0 \implies 0 \in \mathcal{N}$ . Thus  $\mathcal{N} \neq \emptyset$ . Let  $x, y \in \mathcal{N}$  and  $\kappa \in \mathbb{K}$ . Then

$$0 \le \nu(\kappa x + y) \le |\kappa| \, \nu(x) + \nu(y) = 0,$$

which implies

$$\nu(\kappa x + y) = 0.$$

Thus  $\kappa x + y \in \mathcal{N}$ .

 $\mathcal{W}/\mathcal{N}$  is a vector space over  $\mathbb{K}$  This is a result from elementary linear algebra theory, but let's do it for revision. It is clear that  $\mathcal{N} \in \mathcal{W}/\mathcal{N}$ , so  $\mathcal{W}/\mathcal{N} \neq \emptyset$ . Notice that for any  $[x], [y] \in \mathcal{W}/\mathcal{N}$  and  $\kappa \in \mathbb{K}$ , we define the operations

$$[\kappa x + y] = \kappa[x] + [y].$$

By the commutativity of addition,

$$[x + y] = x + y + \mathcal{N} = y + x + \mathcal{N} = [y + x].$$

The additive identity is  $[0] = 0 + \mathcal{N}$ , multiplicative identity is  $[1] = 1 + \mathcal{N}$ , and additive inverse of [x] is [-x].

We note that W/N is normally referred to as the **quotient space** of W by N.

 $\|\cdot\|$  is well-defined Let  $[x_1] = [x_2] \in \mathcal{W}/\mathcal{N}$ . Then  $[x_1 - x_2] = [0]$ and so  $x_1 - x_2 \in \mathcal{N}$ . Then

$$\nu(x_1-x_2)=0,$$

which then since

$$0 \le |\nu(x_1) - \nu(x_2)| \le \nu(x_1 - x_2) = 0,$$

we have that  $v(x_1) = v(x_2)$ . Hence

$$||[x_1]|| = ||[x_2]||$$
,

and so  $\nu(\cdot)$  is well-defined.

 $\|\cdot\|$  is a norm Let  $[x], [y] \in \mathcal{W}/\mathcal{N}$  and  $\kappa \in \mathbb{K}$ . Then

- $||[x]|| = \nu(x) \ge 0$ ;
- $\|\kappa[x]\| = \|[\kappa x]\| = \nu(\kappa x) = |\kappa| \nu(x) = |\kappa| \|[x]\|;$
- $||[x] + [y]|| = ||[x + y]|| = v(x + y) \le v(x) + v(y) = ||[x]|| + ||[y]||$ ; and
- $\|[x]\| = 0 \implies \nu(x) = 0 \implies x \in \mathcal{N} \iff [x] = [0] \in \mathcal{W}/\mathcal{N}.$

Thus  $\|\cdot\|$  is indeed a norm.

Hence, W/N is a normed linear space.

## Example 10.2.2

In our last example, we determined that  $v_1(\cdot)$  is a seminorm on  $\mathcal{L}_1(E,\mathbb{K})$ . Suppose

$$g \in \mathcal{N}_1(E, \mathbb{K}) := \{ f \in \mathcal{L}_1(E, \mathbb{K}) : \nu_1(f) = 0 \}.$$

Then  $\int_{E} |g| = 0$ . Since mE > 0, this happens if and only if g = 0 a.e. on E.

Since g = 0 a.e. on E iff  $\int_{E} |g| = 0$ , we can also define

$$\mathcal{N}_1(E,\mathbb{K}) = \{g \in \mathcal{L}_1(E,\mathbb{K}) : g = 0 \text{ a.e. on } E\}.$$

Setting

$$L_1(E, \mathbb{K}) = \mathcal{L}_1(E, \mathbb{K}) / \mathcal{N}_1(E, \mathbb{K}),$$

we have that [f] = [g] iff  $f - g \in \mathcal{N}_1(E, \mathbb{K})$ , i.e. f = g a.e. on E.

# $\blacksquare$ Definition 32 ( $L_1$ -space)

Let  $E \in \mathfrak{M}(\mathbb{K})$  with mE > 0. We define the  $L_1$ -space as

$$L_1(E, \mathbb{K}) := \mathcal{L}_1(E, \mathbb{K}) / \mathcal{N}_1(E, \mathbb{K}),$$

with the norm

$$\|\cdot\|: L_1(E, \mathbb{K}) \to \mathbb{R}$$
 
$$\|[f]\| := \int_E f.$$

# $\blacksquare$ Definition 33 ( $\mathcal{L}_p(E,\mathbb{K})$ )

Let  $E \in \mathfrak{M}(\mathbb{K})$  with mE > 0. If  $1 in <math>\mathbb{R}$ , we define

$$\mathcal{L}_{p}(E, \mathbb{K}) := \{ f \in \mathcal{L}(E, \mathbb{K}) : \int_{E} |f|^{p} < \infty \}$$
$$= \{ f \in \mathcal{L}(E, \mathbb{K}) : |f|^{p} \in \mathcal{L}_{1}(E, \mathbb{K}) \}.$$

We need to show that  $\mathcal{L}_p(E, \mathbb{K})$  is a vector space for all 1 , and that

$$\nu_p(f) \coloneqq \left(\int_E |f|^p\right)^{\frac{1}{p}}$$

defines a semi-norm on  $\mathcal{L}_p(E,\mathbb{K})$ . If we can establish these results, we can then appeal to Proposition 43 and take the quotient space wrt to a similar kernel.

However, the proof of the triangle inequality of  $\nu_p$  is a non-trivial exercise.

# **■** Definition 34 (Lebesgue Conjugate)

Let  $1 \le p \le \infty$ . We associate to p the number  $1 \le q \le \infty$  as follows:

- if p = 1, then  $q = \infty$ ;
- if  $p = \infty$ , then q = 1; and finally
- 1

$$q = \left(1 - \frac{1}{p}\right)^{-1}.$$

We say that q is the Lebesgue conjugate of p. With the convention that  $\frac{1}{\infty} := 0$ , and we see that in all cases,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

#### 66 Note 10.2.1

When 1 , we see that the above equation is equivalent to each ofthe equations:

- p(q-1) = q and
- (p-1)q = p.

# Lemma 44 (Young's Inequality)

If  $1 and q is the Lebesgue conjugate of p, then for <math>0 < a, b \in \mathbb{R}$ ,

- 1.  $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ ; and
- 2. equality in the above holds iff  $a^p = b^q$ .

## Proof

Let  $g:(0,\infty)\to\mathbb{R}$  be such that

$$x \mapsto \frac{1}{p} x p^+ \frac{1}{q} - x.$$

There's another proof that I prefer over this construction here that feels like we just lucked out. See PMATH351.

Notice that g is differentiable on  $(0, \infty)$ , and we have

$$g'(x) = x^{p-1} - 1.$$

Furthermore,

- g'(x) < 0 for  $x \in (0,1)$ ;
- g'(1) = 0; and
- g'(x) > 0 for  $x \in (1, \infty)$ .

Also, note that  $g(1)=\frac{1}{p}+\frac{1}{q}-1=0$ . Thus by the above observation, we know that g attains its minimum at 1. Let  $x_0=\frac{a}{b^{q-1}}>0$ . Then we have

$$0 \le g(x_0) = \frac{1}{p} \left( \frac{a^p}{b^{(q-1)p}} \right) + \frac{1}{q} - \frac{a}{b^{q-1}}$$
$$= \frac{1}{p} \frac{a^p}{b^q} + \frac{1}{q} - \frac{a}{b^{q-1}}.$$

Thus

$$\frac{a}{b^{q-1}} \le \frac{1}{p} \frac{a^p}{b^q} + \frac{1}{q}.$$

Multiplying both sides by  $b^q$ , we get

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Furthermore, we notice that

$$g(x_0) = 0 \iff x_0 = 1 \iff a = b^{q-1} \iff a^p = b^{p(q-1)} = b^q$$
.



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