# PMATH450 — Lesbesgue Integration and Fourier Analysis

Classnotes for Spring 2019

by

Johnson Ng

BMath (Hons), Pure Mathematics major, Actuarial Science Minor University of Waterloo

## Table of Contents

Index	22
Bibliography	21
1 Lecture 1 May 07th 2019 1.1 Riemannian Integration	<b>9</b> 9
Preface	7
List of Procedures	5
List of Theorems	4
List of Definitions	3
Table of Contents	2

## List of Definitions

1	Definition (Norm and Semi-Norm)	ç
2	■ Definition (Normed Linear Space)	12
3	■ Definition (Metric)	12
4	■ Definition (Banach Space)	13
5	<b>■</b> Definition (Partition of a Set)	14
6	■ Definition (Test Values)	15
7	■ Definition (Riemann Sum)	15
8	■ Definition (Refinement of a Partition)	16
9	■ Definition (Riemann Integrable)	17

## List of Theorems

1	♦ Proposition (Uniqueness of the Riemann Integral)	17
2	■ Theorem (Cauchy Criterion of Riemann Integrability)	18

## List of Procedures



The pre-requisite to this course is Real Analysis. We will use a lot of the concepts introduced in Real Analysis, at times without explicitly stating it. Refer to notes on PMATH351.

This course is spiritually broken into 2 pieces:

- Lesbesgue Integration; and
- Fourier Analysis,

which is as the name of the course.

In this set of notes, we use a special topic environment called **culture** to discuss interesting contents related to the course, but will not be throughly studied and not tested on exams.

## Lecture 1 May 07th 2019

Since many of our results work for both  $\mathbb C$  and  $\mathbb R$ , we shall use  $\mathbb K$  throughout this course to represent either  $\mathbb C$  or  $\mathbb R$ .

#### 1.1 Riemannian Integration

#### **■** Definition 1 (Norm and Semi-Norm)

Let V be a vector space over  $\mathbb{K}$ . We define a semi-norm on V as a function

$$\nu:V\to\mathbb{R}$$

that satisfies

- 1. (Positive Semi-Definite)  $v(x) \ge 0$  for all  $x \in V$ ;
- 2.  $\nu(\kappa x) = |\kappa| \nu(x)$  for any  $\kappa \in \mathbb{K}$  and  $x \in V$ ; and
- 3. (Triangle Inequality)  $\nu(x+y) \leq \nu(x) + \nu(y)$  for all  $x,y \in V$ .

If  $v(x) = 0 \implies x = 0$ , then we say that v is a norm. In this case, we usually write  $\|\cdot\|$  to denote the norm, instead of v.

#### 66 Note 1.1.1

• We sometimes call a semi-norm a pseudo-length.

#### Remark 1.1.1

Notice that we wrote  $v(x) = 0 \implies x = 0$  instead of  $v(x) = 0 \iff x = 0$ . This is because if  $z = 0 \in V$ , then

$$v(z) = v(0z) = 0.$$

#### Exercise 1.1.1

Show that if v is a semi-norm on a vector space V, then  $\forall x, y \in V$ ,

$$|\nu(x) - \nu(y)| \le \nu(x - y).$$

#### Proof

Notice that by condition (2) and (3), we have

$$\nu(x-y) \le \nu(x) + \nu(-y) = \nu(x) - \nu(y),$$

and

$$\nu(x - y) = -\nu(y - x) \ge -(\nu(y) - \nu(x)) = \nu(x) - \nu(y).$$

It follows that indeed

$$|\nu(x) - \nu(y)| \le \nu(x - y).$$

#### Example 1.1.1

The absolute value  $|\cdot|$  is a **norm** on  $\mathbb{K}$ .

#### **\***

#### Example 1.1.2 (p-norms)

Consider  $N \ge 1$  an integer. We define a family of norms on

$$\mathbb{K}^N = \underbrace{K \times K \times \ldots \times K}_{N \text{ times}}.$$

1-norm

$$\|(x_n)_{n=1}^N\|_1 := \sum_{n=1}^N |x_n|.$$

Infinity-norm, ∞-norm

$$\left\| (x_n)_{n=1}^N \right\|_{\infty} \coloneqq \max_{1 \le n \le N} |x_n|.$$

Euclidean-norm, 2-norm

$$\left\| (x_n)_{n=1}^N \right\|_2 := \left( \sum_{n=1}^N |x_n|^2 \right)^{\frac{1}{2}}$$

It is relatively easy to check that the above norms are indeed norms, except for the 2-form. In particular, the triangle inequality is not as easy to show 1.

<sup>1</sup> See Minkowski's Inequality.

Less obviously so, but true nonetheless, we can define the following *p*-norms on  $\mathbb{K}^N$ :

$$\|(x_n)_{n=1}^N\|_p := \left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}},$$

for  $1 \le p < \infty$ .



#### Culture

Consider  $V = \mathbb{M}_n(\mathbb{C})$ , <sup>2</sup> where  $n \in \mathbb{N}$  is fixed. For  $T \in \mathbb{M}_n(\mathbb{C})$ , we define the singular numbers of T to be

<sup>2</sup> Note that  $\mathbb{M}_n(\mathbb{C})$  is the set of  $n \times n$ matrices over C.

$$s_1(T) \geq s_2(T) \geq \ldots \geq s_n(T) \geq 0$$
,

where  $\sigma(T^*T) = \{s_1(T)^2, s_2(T)^2, \dots, s_n(T)^2\}$ , including multiplicity. Then we can define

$$\|T\|_p \coloneqq \left(\sum_{i=1}^n s_i(T)^p\right)^{\frac{1}{p}}$$

for  $1 \le p < \infty$ , which is called the p-norm of T on  $\mathbb{M}_n(\mathbb{C})$ .

#### Example 1.1.3

Let

$$V = \mathcal{C}([0,1],\mathbb{K}) = \{f : [0,1] \to \mathbb{K} \mid f \text{ is continuous } \}.$$

Then

$$||f||_{\sup} := \sup\{|f(x)| \mid x \in [0,1]\}$$

<sup>3</sup> defines a norm on  $\mathcal{C}([0,1],\mathbb{K})$ .

A sequence  $(f_n)_{n=1)^\infty}$  in V converges in this norm to some  $f \in V$ , i.e.

$$\lim_{n\to\infty}\|f_n-f\|_{\sup}=0,$$

which means that  $(f_n)_{n=1}^{\infty}$  converges uniformly to f on [0,1].

 $^3$  Some authors use  $\|f\|_{\infty}$ , but we will have the notation  $\|[f]\|_{\infty}$  later on, and so we shall use  $\|f\|_{\sup}$  for clarity.

#### **■** Definition 2 (Normed Linear Space)

A normed linear space (NLS) is a pair  $(V, \|\cdot\|)$  where V is a vector space over  $\mathbb{K}$  and  $\|\cdot\|$  is a norm on V.

#### **■** Definition 3 (Metric)

Given an NLS  $(V, \|\cdot\|)$ , we can define a metric d on V (called the metric induced by the norm) as follows:

$$d: V \times V \to \mathbb{R}$$
  $d(x,y) = ||x - y||$ ,

such that

- $d(x,y) \ge 0$  for all  $x,y \in V$  and  $d(x,y) = 0 \iff x = y$ ;
- d(x, y) = d(y, x); and
- $d(x,y) \leq d(x,z) + d(y,z)$ .

#### 66 Note 1.1.2

Norms are all metrics, and so any space that has a norm will induce a metric on the space.

#### **■** Definition 4 (Banach Space)

We say that an NLS  $(V, \|\cdot\|)$  is **complete** or is a **Banach Space** if the corresponding (V,d), where d is the metric induced by the norm, is complete 4.

<sup>4</sup> Completeness of a metric space is such that any of its Cauchy sequences converges in the space.

#### Example 1.1.4

 $(\mathcal{C}([0,1],\mathbb{K}),\|\cdot\|_{\sup})$  is a Banach space.

#### Example 1.1.5

We can define a 1-norm  $\lVert \cdot \rVert_1$  on  $\mathcal{C}([0,1],\mathbb{K})$  via

$$||f||_1 := \int_0^1 |f|.$$

Then  $(\mathcal{C}([0,1],\mathbb{K}),\|\cdot\|_1)$  is an NLS.

#### Exercise 1.1.2

Show that  $(C([0,1], \mathbb{K}), \|\cdot\|_1)$  is not complete, which will then give us an example of a normed linear space that is not Banach.

#### Proof

Consider the sequence  $(f_n)_{n=1}^{\infty}$  of continuous functions given by

$$f_n(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2} \\ n\left(x + \frac{1}{2}\right) & \frac{1}{2} \le x \le \frac{1}{2} + \frac{1}{n} \\ 1 & \text{otherwise} \end{cases}$$

Note that the sequence  $(f_n)_{n=1}^{\infty}$  is indeed **Cauchy**: let  $\varepsilon > 0$  and  $|n-m|<rac{arepsilon}{|x-rac{1}{2}|}$ , and then we have

$$|f_n(x) - f_m(x)| = \left| n\left(x - \frac{1}{2}\right) - m\left(x - \frac{1}{2}\right) \right|$$
$$= \left| (n - m)\left(x - \frac{1}{2}\right) \right| = |n - m|\left|x - \frac{1}{2}\right| < \varepsilon.$$

However, it is clear that the sequence  $(f_n)_{n=1}^{\infty}$  converges to the

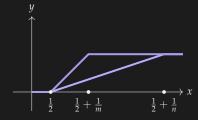


Figure 1.1: Sequence of functions  $(f_n)_{n=1}^{\infty}$ . We show for two indices n < m.

piecewise function (in particular, a non-continuous function)

$$f(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2} \\ 1 & x \ge \frac{1}{2} \end{cases}.$$

#### Example 1.1.6

If  $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$  and  $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$  are NLS's, and if  $T: \mathfrak{X} \to \mathfrak{Y}$  is a linear map, we define the **operator norm** of T to be

$$||T|| := \sup\{||T(x)||_{\mathfrak{Y}} \mid ||x||_{\mathfrak{X}} \le 1\}.$$

We set

$$B(\mathfrak{X},\mathfrak{Y}) := \{T : \mathfrak{X} \to \mathfrak{Y} \mid T \text{ is linear }, \|T\| < \infty\}.$$

Note that for any such linear map T,  $||T|| < \infty \iff T$  is continuous. Thus  $B(\mathfrak{X}, \mathfrak{Y})$  is the set of all continuous functions from  $\mathfrak{X}$  into  $\mathfrak{Y}$ .

Then 
$$(B(\mathfrak{X},\mathfrak{Y}),\|\cdot\|)$$
 is an NLS.

\*

It is likely that we have seen this in Real Analysis.

#### Exercise 1.1.3

Show that  $(B(\mathfrak{X},\mathfrak{Y}),\|\cdot\|)$  is complete iff  $(\mathfrak{Y},\|\cdot\|_{\mathfrak{Y}})$  is complete.

#### 66 Note 1.1.3

One example of the last example is when  $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}}) = (\mathbb{K}, |\cdot|)$ . In this case,  $B(\mathfrak{X}, \mathbb{K})$  is known as the dual space of  $\mathfrak{X}$ , or simple the dual of  $\mathfrak{X}$ .

We are interested in integrating over Banach spaces.

#### **■** Definition 5 (Partition of a Set)

Let  $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$  be a Banach space and  $f: [a,b] \to \mathfrak{X}$  a function, where  $a < b \in \mathbb{R}$ . A partition P of [a,b] is a finite set

$$P = \{a = p_0 < p_1 < \ldots < p_N = b\}$$

for some  $N \ge 1$ . The set of all partitions of [a,b] is denoted by  $\mathcal{P}[a,b]$ .

#### **■** Definition 6 (Test Values)

Let  $(\mathfrak{X},\|\cdot\|_{\mathfrak{X}})$  be a Banach space and  $f:[a,b] o \mathfrak{X}$  a function, where  $a < b \in \mathbb{R}$ . Let  $P \in \mathcal{P}[a,b]$ . A set

$$P^* := \{p_k^*\}_{k=1}^N$$

satisfying

$$p_{k-1} \le p_k^* \le p_k$$
, for  $1 \le k \le n$ 

is called a set of test values for P.

#### **■** Definition 7 (Riemann Sum)

Let  $(\mathfrak{X},\|\cdot\|_{\mathfrak{X}})$  be a Banach space and  $f:[a,b]\to\mathfrak{X}$  a function, where  $a < b \in \mathbb{R}$ . Let  $P \in \mathcal{P}[a,b]$  and  $P^*$  its corresponding set of test values. We define the Riemann sum as

$$S(f, P, P^*) = \sum_{k=1}^{N} f(p_k^*)(p_k - p_{k-1}).$$

#### Remark 1.1.2

- 1. Note that because  $\blacksquare$  Definition 5,  $p_k p_{k-1} > 0$ .
- 2. When  $(\mathfrak{X},\|\cdot\|)=(\mathbb{R},|\cdot|)$ , then this is the usual Riemann sum from first-year calculus.
- 3. In general, note that

$$\frac{1}{b-a}S(f, P, P^*) = \sum_{k=1}^{N} \lambda_k f(p_k^*),$$

where  $0 < \lambda_k = \frac{p_k - p_{k-1}}{b-a} < 1$  and 5

 $\sum_{k=1}^{N} \lambda_k = 1.$ 

 $^{5}$  via the fact that the  $\lambda_{k}$ 's form a telescoping sum

So  $\frac{1}{b-a}S(f,P,P^*)$  is an averaging of f over [a,b]. We call  $\frac{1}{b-a}S(f,P,P^*)$  the convex combination of the  $f(p_k^*)$ 's.

#### Example 1.1.7 (Silly example)

Let 
$$(\mathfrak{X} = \mathcal{C}([-\pi,\pi],\mathbb{K}),\|\cdot\|_{\sup})$$
. Let

$$f: [0,1] \to \mathfrak{X}$$
 such that  $x \mapsto e^{2\pi x} \sin 7\theta + \cos x \cos(12\theta)$ ,

where  $\theta \in [-\pi, \pi]$ . Now if we consider the partition

$$P = \left\{-\pi, \frac{1}{10}, \frac{1}{2}, \pi\right\}$$

and its corresponding test value

$$P^* = \left\{0, \frac{1}{3}, 2\right\},\,$$

then

$$\begin{split} S(f,P,P^*) &= f(0) \left(\frac{1}{10} + \pi\right) + f\left(\frac{1}{3}\right) \left(\frac{1}{2} - \frac{1}{10}\right) + f(2) \left(\pi - \frac{1}{2}\right) \\ &= \left(\sin 7\theta + \cos 12\theta\right) \left(\pi + \frac{1}{10}\right) \\ &+ \left(e^{\frac{2\pi}{3}} \sin 7\theta + \cos \frac{1}{3} \cos 12\theta\right) \left(\frac{2}{5}\right) \\ &+ \left(e^{4\pi} \sin 7\theta + \cos 2 \cos 12\theta\right) \left(\pi - \frac{1}{2}\right) \end{split}$$

#### **■** Definition 8 (Refinement of a Partition)

Let  $a < b \in \mathbb{R}$ , and  $P \in \mathcal{P}[a,b]$ . We say Q is a refinement of P is  $Q \in \mathcal{P}[a,b]$  and  $P \subseteq Q$ .

#### 66 Note 1.1.4

*In simpler words, Q is a "finer" partition that is based on P.* 

#### **■** Definition 9 (Riemann Integrable)

Let  $a < b \in \mathbb{R}$ ,  $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$  be a Banach space and  $f: [a,b] \to \mathfrak{X}$  be a function. We say that f is Riemann integrable over [a, b] if  $\exists x_0 \in \mathfrak{X}$ such that

$$\forall \varepsilon > 0 \quad \exists P \in \mathcal{P}[a,b],$$

such that if Q is any refinement of P, and  $Q^*$  is any set of test values of Q, then

$$\|x_0 - S(f, Q, Q^*)\|_{\mathfrak{X}} < \varepsilon.$$

*In this case, we write* 

$$\int_a^b f = x_0.$$

#### ♦ Proposition 1 (Uniqueness of the Riemann Integral)

If f is Riemann integrable over [a, b], then the value of  $\int_a^b f$  is unique.

#### Proof

Suppose not, i.e.

$$\int_a^b f = x_0 \text{ and } \int_a^b f = y_0$$

for some  $x_0 \neq y_0$ . Then, let

$$\varepsilon=\frac{\|x_0-y_0\|}{2},$$

which is > 0 since  $||x_0 - y_0|| > 0$ . Let  $P_{x_0}, P_{y_0} \in \mathcal{P}[a, b]$  be partitions corresponding to  $x_0$  and  $y_0$  as in the definition of Riemann integrability.

Then, let  $R = P_{x_0} \cup P_{y_0}$ , so that R is a **common refinement** of  $P_{x_0}$  and  $P_{y_0}$ . If Q is any refinement of R, then Q is also a common refinement of  $P_{x_0}$  and  $P_{y_0}$ . Then for any test values  $Q^*$  of Q, we have

$$2\varepsilon = \|x_0 - y_0\|$$

$$\leq \|x_0 - S(f, Q, Q^*)\| + \|S(f, Q, Q^*) - y_0\| < \varepsilon + \varepsilon = 2\varepsilon,$$

which is a contradiction.

Thus  $x_0 = y_0$  as required.

#### ■Theorem 2 (Cauchy Criterion of Riemann Integrability)

Let  $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$  be a Banach space,  $a < b \in \mathbb{R}$  and  $f : [a, b] \to \mathfrak{X}$  be a function. TFAE:

- 1. f is Riemann integrable over [a, b];
- 2.  $\forall \varepsilon > 0$ ,  $R \in \mathcal{P}[a,b]$ , if P,Q is any refinement of R, and  $P^*$  (respectively  $Q^*$ ) is any test values of P (respectively Q), then

$$||S(f, P, P^*) - S(f, Q, Q^*)||_{\mathfrak{X}} < \varepsilon.$$

#### Proof

This is a rather straightforward proof. Suppose  $P,Q \in \mathcal{P}[a,b]$  is some refinement of the given partition  $R \in \mathcal{P}[a,b]$ , and  $P^*,Q^*$  any test values for P,Q, respectively. Then by assumption and  $\P$  Proposition 1,  $\exists x_0 \in \mathfrak{X}$  such that

$$\|x_0 - S(f, P, P^*)\|_{\mathfrak{X}} < \frac{\varepsilon}{2} \text{ and } \|x_0 - S(f, Q, Q^*)\|_{\mathfrak{X}} < \frac{\varepsilon}{2}.$$

It follows that

$$||S(f, P, P^*) - S(f, Q, Q^*)||_{\mathfrak{X}}$$

$$\leq ||x_0 - S(f, P, P^*)||_{\mathfrak{X}} + ||x_0 - S(f, Q, Q^*)||_{\mathfrak{X}}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

By hypothesis, wma  $\varepsilon = \frac{1}{n}$  for some  $n \ge 1$ , such that if P, Q are any refinements of the partition  $R_n \in \mathcal{P}[a,b]$ , and  $P^*, Q^*$  are the respective arbitrary test values, then

$$\|S(f, P, P^*) - S(f, Q, Q^*)\|_{\mathfrak{X}} < \frac{1}{n}$$

Now for each  $n \ge 1$ , define

$$W_n := \bigcup_{k=1}^n R_k \in \mathcal{P}[a,b],$$

so that  $W_n$  is a common refinement for  $R_1, R_2, \ldots, R_n$ . For each  $n \ge n$ 1, let  $W_n^*$  be an arbitrary set of test values for  $W_n$ . For simplicity, let us write

$$x_n = S(f, W_n, W_n^*)$$
, for each  $n \ge 1$ .

6

Claim:  $(x_n)_{n=1}^{\infty}$  is a Cauchy sequence If  $n_1 \ge n_2 > N \in \mathbb{N}$ , then

$$\|x_{n_1} - x_{n_2}\|_{\mathfrak{X}} = \|S(f, W_{n_1}, W_{n_1}^*) - S(f, W_{n_2}, W_{n_2}^*)\| < \frac{1}{N}$$

by our assumption, since  $W_{n_1}$ ,  $W_{n_2}$  are refinements of  $R_N$ . Then by picking  $N = \frac{1}{\varepsilon}$  for any  $\varepsilon > 0$ , we have that  $(x_n)_{n=1}^{\infty}$  is indeed a Cauchy sequence in  $\mathfrak{X}$ .

Since  $\mathfrak{X}$  is a Banach space, it is complete, and so  $\exists x_0 := \lim_{n \to \infty} x_n \in$  $\mathfrak{X}$ . It remains to show that, indeed,

$$x_0 = \int_a^b f.$$

Let  $\varepsilon > 0$ , and choose  $N \ge 1$  such that

- $\frac{1}{N} < \frac{\varepsilon}{2}$ ; and
- $k \ge N$  implies that  $||x_k x_0|| < \frac{\varepsilon}{2}$ .

Then suppose that V is any refinement of  $W_N$ , and  $V^*$  is an arbitrary set of test values of V. Then we have

$$\begin{aligned} \|x_{0} - S(f, V, V^{*})\|_{\mathcal{X}} &\leq \|x_{0} - x_{N}\|_{\mathcal{X}} + \|x_{N} - S(f, V, V^{*})\|_{\mathcal{X}} \\ &< \frac{\varepsilon}{2} + \|S(f, W_{N}, W_{N}^{*}) - S(f, V, V^{*})\|_{\mathcal{X}} \\ &< \frac{\varepsilon}{2} + \frac{1}{N} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

It follows that

$$\int_a^b f = x_0,$$

as desired.

<sup>6</sup> Note that it would be nice if for the finer and finer partitions that we have constructed, i.e. the  $W_n$ 's, give us a convergent sequence of Riemann sums, since it makes sense that this convergence will give us the final value that we want.

In first-year calculus, all continuous functions over  $\mathbb{R}$  are integrable. A similar result holds in Banach spaces as well. In the next lecture, we shall prove the following theorem.

**Theorem (Continuous Functions are Riemann Integrable)** Let  $(\mathfrak{X}, \|\cdot\|)$  be a Banach space and  $a < b \in \mathbb{R}$ . If  $f : [a, b] \to \mathfrak{X}$  is continuous, then f is Riemann integrable over [a, b].





### Banach Space, 13

Cauchy Criterion of Riemann Integrability, 18 common refinement, 17 complete, 13 convex combination, 16

Metric, 12 metric, 12

Norm, 9 Normed Linear Space, 12

operator norm, 14

Partition, 14 pseudo-length, 9

Refinement, 16 Riemann Integrable, 17 Riemann Sum, 15

Semi-Norm, 9

Test Values, 15