PMATH351 - Real Analysis

CLASSNOTES FOR FALL 2018

bv

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1 Lecture 1 Sep 06th

1.1 Course Logistics

No content is covered in today's lecture so this chapter will cover some of the important logistical highlights that were mentioned in class.

- Assignments are designed to help students understand the content.
- Due to shortage of manpower, not all assignment questions will be graded; however, students are encouraged to attempt all of the questions.
- To further motivate students to work on ungraded questions, the midterm and final exam will likely recycle some of the assignment questions.
- There are no required text, but the professor has prepared course notes for reading. The course note are self-contained.
- The approach of the class will be more interactive than most math courses.
- Due to the size of the class, students are encouraged to utilize Waterloo Learn for questions, so that similar questions by multiple students can be addressed at the same time.

1.2 Preview into the Introduction

How do we compare the size of two sets?

• If the sets are finite, this is a relatively easy task.

- If the sets are infinite, we will have to rely on functions.
 - Injective functions tell us that the domain is of size that is lesser than or equal to the codomain.
 - Surjective functions tell us that the codomain is of size that is lesser than or equal to the domain.
 - So does a bijective function tell us that the domain and codomain have the same size? Yes, although this is not as intuitive as it looks, as it relies on Cantor-Schröder-Bernstein Theorem.

Now, given two arbitrary sets, are we guaranteed to always be able to compare their sizes? It would be very tempting to immediately say yes, but to do that, one would have to agree on the Axiom of Choice. Fortunately, within the realm of this course, the Axiom of Choice is taken for granted.

2 Lecture 2 Sep 10th

2.1 Basic Set Theory

We shall use the following notations for some of the common set of numbers that we are already familiar with:

- N denotes the set of natural numbers {1,2,3,...};
- \mathbb{Z} denotes the set of integers $\{..., -2, -1, 0, 1, 2, ...\}$;
- Q denotes the set of rational numbers $\left\{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N}\right\}$; and
- \mathbb{R} denotes the set of real numbers.

We shall start with having certain basic properties of \mathbb{N} , \mathbb{Z} , and \mathbb{Q} .

WE WILL USE the notation $A \subset B$ and $A \subseteq B$ interchangably to mean that A is a subset of B with the possibility that A = B. When we wish to explicitly emphasize this possibility, we shall use $A \subseteq B$. When we wish to explicitly state that A is a **proper subset** of B, we will either specify that $A \neq B$ or simply $A \subseteq B$.

Definition 1 (Universal Set)

A universal set, which we shall generally give the label X, is a set that contains all the mathematical objects that we are interested in.

With a universal set in place, we can have the following definitions:

Definition 2 (Union)

This is a hand-wavy definition, but it is not in the interest of this course to further explore on this topic.

Let X be a set. If $\{A_{\alpha}\}_{{\alpha}\in I}$ such that $A_{\alpha}\subset X$, then the union for all A_{α} is defined as

$$\bigcup_{\alpha\in I}A_{\alpha}:=\{x\in X\mid \exists \alpha\in I, x\in A_{\alpha}\}.$$

Definition 3 (Intersection)

Let X be a set. If $\{A_{\alpha}\}_{{\alpha}\in I}$ such that $A_{\alpha}\subset X$, then the **intersection** for all A_{α} is defined as

$$\bigcap_{\alpha\in I}A_{\alpha}:=\{x\in X\mid \forall \alpha\in I, x\in A_{\alpha}\}.$$

Definition 4 (Set Difference)

Let X be a set and A, $B \subseteq X$. The **set difference** of A from B is defined as

$$A \setminus B := \{ x \in X \mid x \in A, x \notin B \}.$$

On a similar notion:

Definition 5 (Symmetric Difference)

Let X be a set and A, B \subseteq X. The **symmetric difference** of A and B is defined as

$$A\Delta B := \{ x \in X \mid (x \in A \land x \notin B) \lor (x \notin A \land x \in B) \}.$$

We can also talk about the non-members of a set:

In words, for an element in the symmetric difference of two sets, the element is either in A or B but not both. We can also think of the symmetric difference

$$(A \cup B) \setminus (A \cap B)$$

or

 $(A \setminus B) \cup (B \setminus A).$

Definition 6 (Set Complement)

Let X be a set and $A \subset X$. The set of all non-members of A is called the **complement** of A, which we denote as

$$A^c := \{ x \in X \mid x \notin A \}.$$

66 Note

Note that

$$(A^c)^c = \{x \in X \mid x \notin A^c\} = \{x \in X \mid x \in A\} = A.$$

Now taking a step away from that, we define the following:

Definition 7 (Empty Set)

An *empty set*, denoted by \emptyset , is a set that contains nothing.

66 Note

The empty set is set to be a subset of all sets.

Definition 8 (Power Set)

Let X be a set. The power set of X is the set that contains all subsets of X,

$$\mathcal{P}(X) := \{ A \mid A \subset X \}.$$

66 Note

A power set is always non-empty, since $\emptyset \in \mathcal{P}(\emptyset)$ *, and since* $\emptyset \subset X$ *for* any set X, we have $\emptyset \in \mathcal{P}(X)$.

Example 2.1.1

Let $X = \{1, 2, ..., n\}$. There are several ways we can show that the size of $\mathcal{P}(X)$ is 2^n . One of the methods is by using a characteristic function that

maps from A to $\{0,1\}$, defined by

$$X_A: A \to \{0,1\}$$

$$X_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

Using this function, each element in X have 2 states: one being in the subset, and the other being not in the subset, which are represented by 1 and 0 respectively. It is then clear that there are 2^n of such configurations.

Theorem 1 (De Morgan's Laws)

Let X be a set. Given $\{A_{\alpha}\}_{{\alpha}\in I}\subset \mathcal{P}(X)$, we have

1.
$$\left(\bigcup_{\alpha\in I}A_{\alpha}\right)^{c}=\bigcap_{\alpha\in I}A_{\alpha}^{c}$$
; and

2.
$$\left(\bigcap_{lpha\in I}A_lpha\right)^c=igcup_{lpha\in I}A_lpha^c$$
.

Proof

1. Note that

$$x \in \left(\bigcup_{\alpha \in I} A_{\alpha}\right)^{c} \iff \nexists \alpha \in I \ x \in A_{\alpha}$$

$$\iff \forall \alpha \in I \ x \notin A_{\alpha}$$

$$\iff \forall \alpha \in I \ x \in A_{\alpha}^{c} \text{ by set complementation}$$

$$\iff x \in \bigcap_{\alpha \in I} A_{\alpha}^{c}.$$

2. Observe that, by part 1,

$$\left(\bigcap_{\alpha\in I}A_{\alpha}\right)^{c}=\left(\left(\bigcup_{\alpha\in I}A_{\alpha}^{c}\right)^{c}\right)^{c}=\bigcup_{\alpha\in I}A_{\alpha}^{c}.$$

Suppose $I = \emptyset$. Then what is $\bigcup_{\alpha \in \emptyset} A_{\alpha}$? It is sensible to think that all we are left with is simply a union of empty sets, and so

$$\bigcup_{\alpha \in \emptyset} A_{\alpha} = \emptyset. \tag{2.1}$$

And what about $\bigcap_{\alpha \in \emptyset} A_{\alpha}$? By \blacksquare Theorem 1, it is quite clear from Equation (2.1) that

$$\bigcap_{\alpha\in\emptyset}A_{\alpha}=X.$$

2.2 Products of Sets

Definition 9 (Product of Sets)

Given 2 sets X and Y, the **product** of X and Y is given by

$$X \times Y := \{(x,y) \mid x \in X, y \in Y\}.$$

We often refer to elements of $X \times Y$ as **tuples**.

66 Note

Now if

$$X = \{x_1, x_2, ..., x_n\},\$$

 $Y = \{y_1, y_2, ..., y_m\},\$

then

$$X \times Y = \{(x_i, y_j) \mid i = 1, 2, ..., n, j = 1, 2, ..., m\}$$

and so the size of $X \times Y$ is mn.

Consequently, we can think of tuples as two elements being in some "relation".

Definition 10 (Relation)

A **relation** on sets X and Y is a subset R of the product $X \times Y$. We write

$$xRy$$
 if $(x,y) \in R \subset X \times Y$.

We call

- $\{x \in X \mid \exists y \in Y, (x,y) \in R\}$ as the domain of R; and
- $\{y \in Y \mid \exists x \in X, (x,y) \in R\}$ as the range of R.

In relation to that, functions are, essentially, relations.

Definition 11 (Function)

A function from X to Y is a relation R such that

$$\forall x \in X \; \exists ! y \in Y \; (x,y) \in R.$$

Suppose $X_1, X_2, ..., X_n$ are non-empty¹ sets. We can define

$$X_1 \times X_2 \times ... \times X_n = \prod_{i=1}^n X_i := \{(x_1, x_2, ..., x_n) \mid x_i \in X_i\}.$$

Now if $X_i = X_j = X$ for all i, j = 1, 2, ..., n, we write

$$\prod_{i=1}^{n} X_{i} = \prod_{i=1}^{n} X = X^{n}.$$

And now comes the problem: given a collection $\{X_{\alpha}\}_{\alpha\in I}$ of non-empty sets², what do we mean by

$$\prod_{\alpha \in I} X_{\alpha}?$$

To motivate for what comes next, consider

$$\prod_{i=1}^{n} X_{i} = X_{1} \times \ldots \times X_{n} = \{(x_{1}, ..., x_{n}) \mid x_{i} \in X_{i}\}.$$

¹ We are typically only interested in non-empty sets, since empty sets usually lead us to vacuous truths, which are not interesting.

 2 i.e. we now talk about arbitrary $\alpha \in I$.

Choose $(x_1,...,x_n) \in \prod_{i=1}^n X_i$. This induces a function

$$f_{(x_1,...,x_n)}: \{1,...,n\} \to \bigcup_{i=1}^n X_i$$

with

$$f(1) = x_1 \in X_1$$

$$f(2) = x_2 \in X_2$$

$$\vdots f(n) = x_n \in X_n$$

Now assume for a more general *f* such that

$$f: \{1, ..., n\} \to \bigcup_{i=1}^{n} X_i$$

is defined by

$$f(i) \in X_i$$
.

Then, we have

$$(f(1), f(2), ..., f(n)) \in \prod_{i=1}^{n} X_{i},$$

which leads us to the following notion:

Definition 12 (Choice Function)

Given a collection $\{X_{\alpha}\}_{{\alpha}\in I}$ of non-empty sets, let

$$\prod_{\alpha \in I} X_{\alpha} = \left\{ f : I \to \bigcup_{\alpha \in I} X_{\alpha} \right\}$$

such that $f(\alpha) \in X_{\alpha}$. Such an f is called a choice function.

And so we may ask a similar question as before: if each X_{α} is nonempty, is $\prod_{\alpha \in I} X_{\alpha}$ non-empty? Turns out this is not as easy to show. In $\alpha \in I$ fact, it is essentially impossible to show, because this is exactly the Axiom of Choice.

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