

Foreword

Usage

- Notes are presented in two columns: main notes on the left, and sidenotes on the right. Main notes will have a larger margin.
- The following is the color code for the notes:

Blue	Definitions
Red	Important points
Yellow	Points to watch out for / comment for incompleteness
Green	External definitions, theorems, etc.
Light Blue	Regular highlighting
Brown	Secondary highlighting
- The following is the color code for boxes, that begin and end with a line of the same color:

Blue	Definitions
Red	Warning
Yellow	Notes, remarks, etc.
Brown	Proofs
Magenta	Theorems, Propositions, Lemmas, etc.
- Hyperlinks are underlined in magenta. If your PDF reader supports it, you can follow the links to either be redirected to an external website, or a theorem, definition, etc., in the same document. Note that this is only reliable if you have the full set of notes as a single document, which you can find on:
https://japorized.github.io/TeX_notes

11 Lecture 11 May 25th 2018

The following theorem is useful for A2. The proof is not provided in this lecture, but expect the corollary to be restated and proven in a later lecture.

Corollary

Let G be a finite group and $H, K \triangleleft G$, $H \cap K = \{1\}$ and $|H| |K| = |G|$. Then $G \cong H \times K$.

11.1 Normal Subgroup (Continued 2)

11.1.1 Normal Subgroup (Continued)

Note (Recall)

Recall the definition of a normal subgroup as in Definition 23. Let H be a subgroup of G . If $gH = Hg$ for all $g \in G$, then $H \triangleleft G$.

Proposition 27 (Normality Test)

Let H be a subgroup of a group G . The following are equivalent:

1. $H \triangleleft G$
2. $\forall g \in G \quad gHg^{-1} \subseteq H$
3. $\forall g \in G \quad gHg^{-1} = H$

Note

Note that item 3 is indeed a stronger statement than item 2. But since the statements are equivalent, while using the [Normality Test](#), if we can show that item 2 is true, item 3 is automatically true.

Proof(1) \implies (2):

$$\begin{aligned}
x \in gHg^{-1} &\implies \exists h \in H \ x = ghg^{-1} \\
&\implies \exists h_1 \in H \ gh = h_1g \quad \because gh \in gH = Hg \\
&\implies x = ghg^{-1} = h_1gg^{-1} = h_1 \in H \\
&\implies gHg^{-1} \subseteq H
\end{aligned}$$

(2) \implies (3):

$$\begin{aligned}
(2) &\implies \forall g \in G \ gHg^{-1} \subseteq H \\
&\implies \exists g^{-1} \in G \ g^{-1}Hg \subseteq H \\
&\implies H \subseteq gHg^{-1} \\
&\stackrel{(2)}{\implies} gHg^{-1} = H
\end{aligned}$$

(3) \implies (1):

$$\begin{aligned}
(3) &\implies \forall g \in G \ gHg^{-1} = H \\
&\implies \forall x \in gH \ xg^{-1} \in gHg^{-1} = H \\
&\implies x \in Hg \quad \because gg^{-1} = 1 \\
&\implies gH \subseteq Hg
\end{aligned}$$

Using a similar argument, we would have $Hg \subseteq gH$. And so $gH = Hg$ as required. \square

Example 11.1.1

Let $G = GL_n(\mathbb{R})$ and $H = SL_n(\mathbb{R})$.¹ For $A \in G$ and $B \in H$ we have

$$\det ABA^{-1} = \det A \det B \det A^{-1} = \det A(1) \frac{1}{\det A} = 1.$$

Thus $\forall A \in G, ABA^{-1} \in H$. By Proposition 27, $H \triangleleft G$, i.e. $SL_n(\mathbb{R}) \triangleleft GL_n(\mathbb{R})$.²

¹ Recall Definition 8 and Definition 11.

²

Note

The normality is true for any field, not just \mathbb{R} .

Proposition 28 (Subgroup of Index 2 is Normal)

$$\forall H \text{ subgroup of } G \wedge [G : H] = 2 \implies H \triangleleft G$$

Proof

Let $a \in G$.

$$a \in H \implies aH = H = Ha$$

$$a \notin H \implies G = H \cup Ha \implies Ha = G \setminus H \quad \because \text{Proposition 22}$$

$$a \notin H \implies G = H \cup aH \implies aH = G \setminus H \quad \because \text{Proposition 22}$$

That implies that $aH = Ha$ for any $a \in G$. Hence, by Proposition 27, $H \triangleleft G$. \square

Example 11.1.2

Let A_n be the **Alternating Group** contained by S_n .³ By Proposition 28, since $[S_n : A_n] = 2$ because $S_n = A_n \cup O_n$ and O_n is a coset of A_n , we have that

$$A_n \triangleleft S_n.$$

³ Recall the definition of alternating group from Theorem 11 and S_n from Definition 4

Example 11.1.3

Let

$$D_{2n} = \{1, a, a^2, \dots, a^{n-1}, b, ba, ba^2, \dots, ba^{n-1}\}$$

be the **Dihedral Group** of order $2n$. Since $[D_{2n} : \langle a \rangle] = 2$,⁴ we have that

$$\langle a \rangle \triangleleft D_{2n} \quad \because \text{Proposition 27.}$$

⁴ The coset of $\langle a \rangle$ is $b\langle a \rangle$.

Let H and K be subgroups of a group G . Recall an earlier discussion: $H \cap K$ is the largest subgroup contained in both H and K .

What is the “smallest” subgroup that contains both H and K ? Since $H \cap K$ is the largest, it makes sense to think about $H \cup K$. However,

$$H \cup K \text{ is a subgroup of } G \iff H \subseteq K \vee K \subseteq H$$

While we know that $H \cup K$ can indeed be such a subgroup, the price of the restriction is too high, since it is overly restrictive.

A more “useful” construction turns out to be the **product** of the

subgroups.

Definition 24 (Product of Groups)

$$HK := \{hk : h \in H, k \in K\}$$

However, HK is not necessarily a subgroup. For example, for $h_1k_1, h_2k_2 \in HK$, it is not necessary that $h_1k_1h_2k_2 \in HK$, since k_1h_2 is not necessarily equal to h_2k_1 .

Lemma 29 (Product of Groups as a Subgroup)

Let H and K be subgroups of G . The following are equivalent:

1. HK is a subgroup of G
2. $HK = KH$ ⁵
3. KH is a subgroup of G

⁵ If one of H or K is normal, then the lemma immediately kicks in.

Proof

It suffices to prove (1) \iff (2), since (1) \iff (3) simply through exchanging H and K .

(1) \implies (2): Let $kh \in KH$ such that $k \in K$ and $h \in H$. Their inverses are $k^{-1} \in K$ and $h^{-1} \in H$, since K and H are groups. Note that

$$kh = (h^{-1}k^{-1})^{-1} \in HK \quad \because HK \text{ is a subgroup of } G.$$

Therefore $kh \in HK$, which implies $KH \subseteq HK$. By a similar argument, we can arrive at $HK \subseteq KH$ and so $HK = KH$.

(2) \implies (1): Note that $1 = 1 \cdot 1 \in HK$. For all $hk \in HK$, $(hk)^{-1} = k^{-1}h^{-1} \in KH = HK$. For $h_1k_1, h_2k_2 \in HK$, note that $k_1h_2 \in KH = HK$, so there exists $hk \in HK$ such that $k_1h_2 = hk$. Therefore,

$$h_1k_1h_2k_2 = h_1hkk_2 \in HK.$$

By the [Subgroup Test](#), HK is a subgroup of G . □

Proposition 30 (Product of Normal Subgroups is Normal)

Let H and K be subgroups of G .

1. $H \triangleleft G \vee K \triangleleft G \implies HK = KH$ is a subgroup of G
 2. $H, K \triangleleft G \implies HK = KH \triangleleft G$
-

Proof

1. Without loss of generality, suppose $H \triangleleft G$. Then

$$HK = \bigcup_{k \in K} Hk = \bigcup_{k \in K} kH = KH \quad (11.1)$$

By Lemma 29, $HK = KH$ is a subgroup of G .

2. Suppose $H, K \triangleleft G$. Then

$$\forall g \in G \ \forall hk \in HK \quad g^{-1}(hk)g = (g^{-1}hg)(g^{-1}kg) \in HK$$

Thus $gHKg^{-1} \subseteq HK$. Thus by Proposition 27, we have that $HK \triangleleft G$.

□

Note

Note that Equation (11.1) is a weaker statement than the regular normality that we have defined, since it only requires all elements of K to work instead of the entire G .

With that, we define the following notion:

Definition 25 (Normalizer)

Let H be a subgroup of G . The **normalizer of H** , denoted by $N_G(H)$, is defined to be

$$N_G(H) := \{g \in G : gH = Hg\}$$

Note

By the above definition, we immediately see that $H \triangleleft G \iff N_G(H) = G$ by Equation (11.1). Observe that since we only needed $kH = Hk$ in Equation (11.1) for all $k \in K$, we have that $k \in N_G(H)$.

Corollary 31

Let H and K be subgroups of a group G .

$$K \subseteq N_G(H) \vee H \subseteq N_G(K) \implies HK = KH \text{ is a subgroup of } G$$

The proof of Corollary 31 is embedded in the proof of Proposition 30 while using the definition of a **normalizer**.