PMATH351 - Real Analysis

Classnotes for Fall 2018

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1 Lecture 1 Sep 06th

1.1 Course Logistics

No content is covered in today's lecture so this chapter will cover some of the important logistical highlights that were mentioned in class.

- Assignments are designed to help students understand the content.
- Due to shortage of manpower, not all assignment questions will be graded; however, students are encouraged to attempt all of the questions.
- To further motivate students to work on ungraded questions, the midterm and final exam will likely recycle some of the assignment questions.
- There are no required text, but the professor has prepared course notes for reading. The course note are self-contained.
- The approach of the class will be more interactive than most math courses.
- Due to the size of the class, students are encouraged to utilize Waterloo Learn for questions, so that similar questions by multiple students can be addressed at the same time.

1.2 Preview into the Introduction

How do we compare the size of two sets?

- If the sets are finite, this is a relatively easy task.
- If the sets are infinite, we will have to rely on functions.

- Injective functions tell us that the **domain is of size that is lesser than or equal to the codomain**.
- Surjective functions tell us that the **codomain is of size that is lesser than or equal to the domain**.
- So does a bijective function tell us that the domain and codomain have the same size? Yes, although this is not as intuitive as it looks, as it relies on **Cantor-Schröder-Bernstein Theorem**.

Now, given two arbitrary sets, are we guaranteed to always be able to compare their sizes? It would be very tempting to immediately say yes, but to do that, one would have to agree on the **Axiom of Choice**. Fortunately, within the realm of this course, the Axiom of Choice is taken for granted.

2 Lecture 2 Sep 10th

2.1 Basic Set Theory

We shall use the following notations for some of the common set of numbers that we are already familiar with:

- $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \ldots\}$;
- $\mathbb{Z}$ denotes the set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$;
- $\mathbb{Q}$ denotes the set of rational numbers $\left\{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N}\right\}$; and
- $\mathbb{R}$ denotes the set of real numbers.

We shall start with having certain basic properties of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$.

We will use the notation $A \subset B$ and $A \subseteq B$ interchangably to mean that A is a subset of B with the possibility that $A = B$. When we wish to explicitly emphasize this possibility, we shall use $A \subseteq B$. When we wish to explicitly state that A is a **proper subset** of B, we will either specify that $A \neq B$ or simply $A \subsetneq B$.

Definition 1 (Universal Set)

A universal set, which we shall generally give the label X, is a set that contains all the mathematical objects that we are interested in.

This is a hand-wavy definition, but it is not in the interest of this course to further explore on this topic.

With a universal set in place, we can have the following definitions:

Definition 2 (Union)

Let X be a set. If $\{A_\alpha\}_{\alpha \in I}$ such that $A_\alpha \subset X$, then the ***union*** *for all A_α is defined as*

$$\bigcup_{\alpha \in I} A_\alpha := \{x \in X \mid \exists \alpha \in I, x \in A_\alpha\}.$$

Definition 3 (Intersection)

Let X be a set. If $\{A_\alpha\}_{\alpha \in I}$ such that $A_\alpha \subset X$, then the ***intersection*** *for all A_α is defined as*

$$\bigcap_{\alpha \in I} A_\alpha := \{x \in X \mid \forall \alpha \in I, x \in A_\alpha\}.$$

Definition 4 (Set Difference)

Let X be a set and $A, B \subseteq X$. The ***set difference*** *of A from B is defined as*

$$A \setminus B := \{x \in X \mid x \in A, x \notin B\}.$$

On a similar notion:

Definition 5 (Symmetric Difference)

Let X be a set and $A, B \subseteq X$. The ***symmetric difference*** *of A and B is defined as*

$$A \Delta B := \{x \in X \mid (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)\}.$$

In words, for an element in the symmetric difference of two sets, the element is either in A or B but not both. We can also think of the symmetric difference as

$$(A \cup B) \setminus (A \cap B)$$

or

$$(A \setminus B) \cup (B \setminus A).$$

We can also talk about the non-members of a set:

Definition 6 (Set Complement)

Let X be a set and $A \subset X$. The set of all non-members of A is called the ***complement*** *of A, which we denote as*

$$A^c := \{x \in X \mid x \notin A\}.$$

Note

Note that

$$(A^c)^c = \{x \in X \mid x \notin A^c\} = \{x \in X \mid x \in A\} = A.$$

Now taking a step away from that, we define the following:

Definition 7 (Empty Set)

An ***empty set****, denoted by $\emptyset$, is a set that contains nothing.*

Note

The empty set is set to be a subset of all sets.

Definition 8 (Power Set)

Let X be a set. The power set of X is the set that contains all subsets of X, i.e.

$$\mathcal{P}(X) := \{A \mid A \subset X\}.$$

Note

A power set is always non-empty, since $\emptyset \in \mathcal{P}(\emptyset)$, and since $\emptyset \subset X$ for any set X, we have $\emptyset \in \mathcal{P}(X)$.

Example 2.1.1

Let $X = \{1, 2, ..., n\}$. There are several ways we can show that the size of $\mathcal{P}(X)$ is 2^n. One of the methods is by using a characteristic function that *maps from A to $\{0,1\}$, defined by*

$$\begin{gathered} \mathcal{X}_A : A \to \{0,1\} \\ \mathcal{X}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}. \end{gathered}$$

Using this function, each element in X have 2 states: one being in the subset, and the other being not in the subset, which are represented by 1 and 0 respectively. It is then clear that there are 2^n of such configurations.

Theorem 1 (De Morgan's Laws)

Let X be a set. Given $\{A_\alpha\}_{\alpha \in I} \subset \mathcal{P}(X)$, we have

1. $\left(\bigcup_{\alpha \in I} A_\alpha \right)^c = \bigcap_{\alpha \in I} A_\alpha^c$; *and*
2. $\left(\bigcap_{\alpha \in I} A_\alpha \right)^c = \bigcup_{\alpha \in I} A_\alpha^c$.

Proof

1. *Note that*

$$\begin{aligned} x \in \left(\bigcup_{\alpha \in I} A_\alpha \right)^c &\iff \nexists \alpha \in I \;\; x \in A_\alpha \\ &\iff \forall \alpha \in I \;\; x \notin A_\alpha \\ &\iff \forall \alpha \in I \;\; x \in A_\alpha^c \text{ by set complementation} \\ &\iff x \in \bigcap_{\alpha \in I} A_\alpha^c . \end{aligned}$$

2. *Observe that, by part 1,*

$$\left(\bigcap_{\alpha \in I} A_\alpha \right)^c = \left(\left(\bigcup_{\alpha \in I} A_\alpha^c \right)^c \right)^c = \bigcup_{\alpha \in I} A_\alpha^c .$$

$\square$

Example 2.1.2

Suppose $I = \emptyset$. *Then what is* $\bigcup_{\alpha \in \emptyset} A_\alpha$? *It is sensible to think that all we are left with is simply a union of empty sets, and so*

$$\bigcup_{\alpha \in \emptyset} A_\alpha = \emptyset. \tag{2.1}$$

And what about $\bigcap_{\alpha \in \emptyset} A_\alpha$? *By Theorem 1, it is quite clear from Equation (2.1) that*

$$\bigcap_{\alpha \in \emptyset} A_\alpha = X.$$

2.2 Products of Sets

Definition 9 (Product of Sets)

Given 2 *sets* X *and* Y, *the* ***product*** *of* X *and* Y *is given by*

$$X \times Y := \{(x, y) \mid x \in X, y \in Y\}.$$

We often refer to elements of $X \times Y$ *as* ***tuples***.

Note

Now if

$$X = \{x_1, x_2, ..., x_n\},$$
$$Y = \{y_1, y_2, ..., y_m\},$$

then

$$X \times Y = \{(x_i, y_j) \mid i = 1, 2, ..., n, \; j = 1, 2, ..., m\}$$

and so the size of $X \times Y$ *is* mn.

Consequently, we can think of tuples as two elements being in some "relation".

Definition 10 (Relation)

A ***relation*** *on sets* X *and* Y *is a subset* R *of the product* $X \times Y$. *We write*

$$xRy \quad \text{if } (x, y) \in R \subset X \times Y.$$

We call

- $\{x \in X \mid \exists y \in Y, (x, y) \in R\}$ *as the* ***domain*** *of R; and*
- $\{y \in Y \mid \exists x \in X, (x, y) \in R\}$ *as the* ***range*** *of R.*

In relation to that, functions are, essentially, relations.

Definition 11 (Function)

A ***function*** *from X to Y is a relation R such that*

$$\forall x \in X \; \exists! y \in Y \; (x, y) \in R.$$

Suppose $X_1, X_2, \ldots, X_n$ are non-empty[1] sets. We can define

$$X_1 \times X_2 \times \ldots \times X_n = \prod_{i=1}^{n} X_i := \{(x_1, x_2, \ldots, x_n) \mid x_i \in X_i\}.$$

Now if $X_i = X_j = X$ for all $i, j = 1, 2, \ldots, n$, we write

$$\prod_{i=1}^{n} X_i = \prod_{i=1}^{n} X = X^n.$$

And now comes the problem: given a collection $\{X_\alpha\}_{\alpha \in I}$ of non-empty sets[2], what do we mean by

$$\prod_{\alpha \in I} X_\alpha ?$$

To motivate for what comes next, consider

$$\prod_{i=1}^{n} X_i = X_1 \times \ldots \times X_n = \{(x_1, \ldots, x_n) \mid x_i \in X_i\}.$$

Choose $(x_1, \ldots, x_n) \in \prod_{i=1}^{n} X_i$. This induces a function

$$f_{(x_1, \ldots, x_n)} : \{1, \ldots, n\} \to \bigcup_{i=1}^{n} X_i$$

[1] We are typically only interested in non-empty sets, since empty sets usually lead us to vacuous truths, which are not interesting.

[2] i.e. we now talk about arbitrary $\alpha \in I$.

with

$$\begin{aligned} f(1) &= x_1 \in X_1 \\ f(2) &= x_2 \in X_2 \\ &\vdots \\ f(n) &= x_n \in X_n \end{aligned}$$

Now assume for a more general f such that

$$f : \{1, ..., n\} \to \bigcup_{i=1}^{n} X_i$$

is defined by

$$f(i) \in X_i.$$

Then, we have

$$(f(1), f(2), ..., f(n)) \in \prod_{i=1}^{n} X_i,$$

which leads us to the following notion:

Definition 12 (Choice Function)

Given a collection $\{X_\alpha\}_{\alpha \in I}$ of non-empty sets, let

$$\prod_{\alpha \in I} X_\alpha = \left\{ f : I \to \bigcup_{\alpha \in I} X_\alpha \right\}$$

*such that $f(\alpha) \in X_\alpha$. Such an f is called a **choice function**.*

And so we may ask a similar question as before: if each X_α is non-empty, is $\prod_{\alpha \in I} X_\alpha$ non-empty? Turns out this is not as easy to show. In fact, it is essentially impossible to show, because this is exactly the **Axiom of Choice**.

3 Lecture 3 Sep 12th

3.1 Axiom of Choice

RECALL our final question of last lecture: If $\{X_\alpha\}_{\alpha \in I}$ is a non-empty collection of non-empty sets, is

$$\prod_{\alpha \in I} X_\alpha \neq \emptyset \; ?$$

Turns out this is widely known (in the world of mathematics) as the **Axiom of Choice**.

Axiom 2 (Zermelo's Axiom of Choice)

If $\{X_\alpha\}_{\alpha \in I}$ is a non-empty collection of non-empty sets, then

$$\prod_{\alpha \in I} X_\alpha \neq \emptyset.$$

An equivalent statement of the above axiom is:

Axiom 3 (Zermelo's Axiom of Choice v2)

$X \neq \emptyset \implies$

$$\exists f : \mathcal{P}(X) \setminus \{\emptyset\} \to X \; \forall A \in \mathcal{P}(X) \setminus \{\emptyset\} \; f(A) \in A$$

where f is the choice function.

Exercise 3.1.1

Prove that Axiom 2 and Axiom 3 are equivalent.

Proof

From Axiom 2 to Axiom 3:

Since $X \neq \emptyset$, we have that $\mathcal{P}(X) \setminus \{\emptyset\}$ is a non-empty collection of non-empty sets. Therefore,

$$\prod_{A \in \mathcal{P}(X) \setminus \{\emptyset\}} A \neq \emptyset.$$

So we know that

$$\exists (x_A)_{A \in \mathcal{P}(X) \setminus \{\emptyset\}} \in \prod_{A \in \mathcal{P}(X) \setminus \{\emptyset\}} A.$$

We then simply need to choose the choice function $f : \mathcal{P}(X) \setminus \{\emptyset\} \to X$ such that

$$f(A) = x_A \in A.$$

From Axiom 3 to Axiom 2:

Let $X_\alpha \in \mathcal{P}(X)$ for $\alpha \in I$, where I is some index set. We know that not all $X_\alpha = \emptyset$ since $X \neq \emptyset$. Choose $J \subseteq I$ such that $\{X_\alpha\}_{\alpha \in J}$ is a non-empty collection of non-empty sets. Let $f : \mathcal{P}(X) \setminus \{\emptyset\}$ be any choice function. By Axiom 3,

$$\forall X_\alpha \in \mathcal{P}(X) \setminus \{\emptyset\} \quad f(X_\alpha) \in X_\alpha.$$

Therefore,

$$(f(X_\alpha))_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha.$$

$\square$

3.2 Relations

Now, it is in our interest to start talking about comparisons or relations between the mathematical objects that we have defined.

Definition 13 (Relations)

A relation R on a set X is [1]

- *(**Reflexive**)* $\forall x \in X \;\; xRx$;
- *(**Symmetric**)* $\forall x, y \in X \;\; xRy \iff yRx$;
- *(**Anti-symmetric**)* $\forall x, y \in X \;\; xRy \wedge yRx \implies x = y$;
- *(**Transitive**)* $\forall x, y, z \in X \;\; xRy \wedge yRz \implies xRz$.

[1] We can look at this definition as $R \subseteq X \times X$. Under such a definition, we would have

- (**Reflexive**) $\forall x \in X \;\; (x, x) \in R$;
- (**Symmetric**) $\forall x, y \in X \;\; (x, y) \in R \iff (y, x) \in R$;
- (**Anti-symmetric**) $\forall x, y \in X \;\; (x, y), (y, x) \in R \implies x = y$;
- (**Transitive**) $\forall x, y, z \in X \;\; (x, y), (y, z) \in R \implies (x, z) \in R$.

Example 3.2.1

Let $X = \mathbb{R}$, and let $xRy \iff x \leq y$, where $\leq$ is the notion of "less than or equal to", which we shall assume that it has the meaning that we know. Observe that $\leq$ is:

- *reflexive:* $\forall x \in \mathbb{R} \;\; x \leq x$ *is true;*
- *anti-symmetric:* $\forall x, y \in \mathbb{R} \;\; x \leq y \wedge y \leq x \implies x = y$; *and*
- *transitive:* $\forall x, y, z \in \mathbb{R} \;\; x \leq y \wedge y \leq z \implies x \leq z$.

Example 3.2.2

Let $Y \neq \emptyset$, $X = \mathcal{P}(Y)$, with $ARB \iff A \subseteq B$. Observe that $\subseteq$ is:

- *reflexive:* $\forall A \in \mathcal{P}(Y) \;\; ARA \iff A \subseteq A$ *is true;*
- *anti-symmetric:* $\forall A, B \in \mathcal{P}(Y) \;\; ARB \wedge BRA \iff A \subseteq B \wedge B \subseteq A \implies A = B$;
- *transitive:* $\forall A, B, C \in \mathcal{P}(Y) \;\; ARB \wedge BRC \iff A \subseteq B \wedge B \subseteq C \implies A \subseteq C$.

Example 3.2.3

Let $Y \neq \emptyset$, $X = \mathcal{P}(Y)$, with $ARB \iff A \supseteq B$. Observe that $\supseteq$ is:

- *reflexive:* $\forall A \in \mathcal{P}(Y) \;\; ARA \iff A \subseteq A$;
- *anti-symmetric:* $\forall A, B \in \mathcal{P}(Y) \;\; ARB \wedge BRA \iff A \supseteq B \wedge B \supseteq A \implies A = B$;
- *transitive:* $\forall A, B, C \in \mathcal{P}(Y) \;\; ARB \wedge BRC \iff A \supseteq B \wedge B \supseteq C \implies A \supseteq C$.

All the above examples are also known as *partially ordered sets*.

Definition 14 (Partially Ordered Sets)

*The set X with the relation R on X is called a **partially ordered set** (or a **poset**) if R is*

- *reflexive;*
- *anti-symmetric; and*
- *transitive.*

We denote a poset by (X, R).

The "partial" in 'partially ordered" indicates that not every pair of elements need to be comparable, i.e. there may be pairs for which neither precedes the other (anti-symmetry).

Note

If (X, R) is a poset, then if $A \subseteq X$, and $R_1 = R \restriction_{A \times A}$, then (A, R_1) is also a poset.

Example 3.2.4

How many possible relations can we define on these sets to make them into posets?

1. $X = \emptyset$

   Solution
   We have that $R = \emptyset \times \emptyset$, and so the only relation we have is an empty relation. Then it is vacuously true that (X, R) a poset.

2. $X = \{x\}$

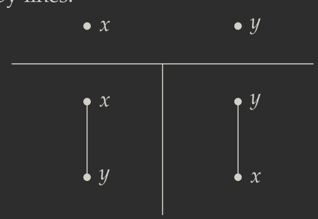
   Solution
   We have that $R = X \times X = \{(x, x)\}$. It it clear that (X, R) is a poset.

3. $X = \{x, y\}$

   Solution
   There are 3 possible relations:

   - *a relation where xRx and yRy;*
   - *a relation where xRy; or*
   - *a relation where yRx.*

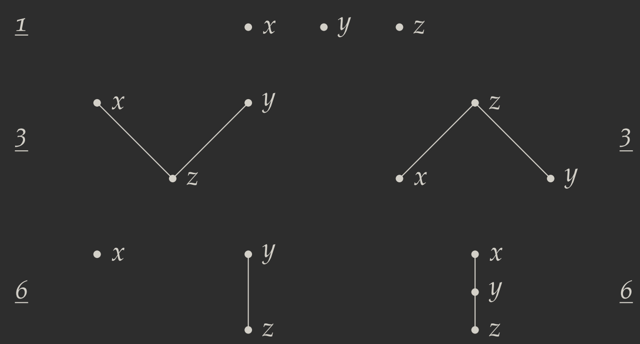
3 possibilities represented as graphs (known as Hasse diagram), separated by lines:



4. $X = \{x, y, z\}$

Solution

The following are all the possibilities represented by graphs, where the underlined numbers represent the number of ways we can rearrange the elements for unique relations:



Therefore, we see that there are a total of

$$1 + 3 + 3 + 6 + 6 = 19 \text{ relations}.$$

Exercise 3.2.1

How many possible relations can we define on a set of 6 elements to the set into a poset?

Solution

to be added

Definition 15 (Totally Ordered Sets / Chains)

The set X with the relation R on X is called a ***totally ordered set*** *(or a* ***chain****) if (X, R) is a poset with the exception that, for any $x, y \in X$, either xRy or yRx but not both.*

Definition 16 (Bounds)

Let $(X, \leq)$ be a poset. Let $A \subset X$. We say $x_0 \in X$ is an ***upper bound*** *for A if*

$$\forall a \in A \quad a \leq x_0.$$

If A has an upper bound, we say that A is ***bounded above****. If A is bounded above, then x_0 is the* ***least upper bound*** *(or* ***supremum****) of*

A is for any $x_1 \in X$ that is an upper bound of A, we have

$$x_0 \leq x_1.$$

We write $x_0 = \mathrm{lub}(A) = \sup(A)$. If $\sup(A) \in A$, then $\sup(A) = \max(A)$ is the ***maximum*** *of A.*

We can analogously define for:

$$\begin{aligned} \textit{upper bound } &\to \textit{ lower bound} \\ \textit{bounded above } &\to \textit{ bounded below} \\ \textit{least upper bound, } \mathrm{lub} &\to \textit{ greatest lower bound, } \mathrm{glb} \\ \textit{supremum, } \sup &\to \textit{ infimum, } \inf \\ \textit{maximum, } \max &\to \textit{ minimum, } \min \end{aligned}$$

Note

By ***anti-symmetry*** *of posets, we have that $\max, \sup, \min, \inf$ are all unique if they exists.*

Example 3.2.5 (Least Upper Bound Property of $\mathbb{R}$)

Let $X = \mathbb{R}$, and $\leq$ be the order that we have defined. Every bounded non-empty subset of X has a supremum.

Example 3.2.6

Let $Y \neq \emptyset$, and $X = \mathcal{P}(Y)$, and $\subseteq$ the ordering by inclusion. We know that Y is the maximum element of $(X, \subseteq)$. Then the collection $\{A_\alpha\}_{\alpha \in I} \subset \mathcal{P}(Y)$ is bounded above by Y, and we have that

$$\sup\left(\{A_\alpha\}_{\alpha \in I}\right) = \bigcup_{\alpha \in I} A_\alpha$$
$$\inf\left(\{A_\alpha\}_{\alpha \in I}\right) = \bigcap_{\alpha \in I} A_\alpha$$

Now if $Y = \emptyset$, we would end up having

$$\sup\left(\{A_\alpha\}_{\alpha \in I}\right) = \emptyset$$
$$\inf\left(\{A_\alpha\}_{\alpha \in I}\right) = X$$

But this makes sense, since the empty set would be the least of upper bounds, and since $X = \mathcal{P}(Y)$ would have to be the greatest of lower bounds.

4 Lecture 4 Sep 14th

4.1 Zorn's Lemma

Definition 17 (Maximal Element)

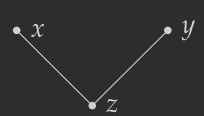
*Let $(X, \leq)$ be a poset. An element $x \in X$ is **maximal** if whenever $y \in X$ is such that $x \leq y$, we must have $y = x$.*

Example 4.1.1

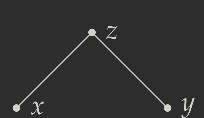
Looking back at Example 3.2.4, on the set $X = \{x, y, z\}$, we have that the maximal element in each possible poset is/are:

This shows to us that the maximal element does not have to be unique.

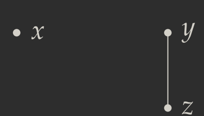
x, y, z are all maximal



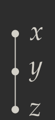
z is maximal



x, y are both maximal



x, z are both maximal



z is maximal

Example 4.1.2

- *Given $X \neq \emptyset$, the maximal element of the poset $(\mathcal{P}(X), \subseteq)$ is X.*
- *Given $X \neq \emptyset$, the maximal element of the poset $(\mathcal{P}(X), \supseteq)$ is $\emptyset$.*
- *The poset $(\mathbb{R}, \leq)$ has no maximal element.*

Axiom 4 (Zorn's Lemma)

If $(X, \leq)$ is a non-empty poset such that every chain $S \subset X$ has an upper bound, then $(X, \leq)$ has a maximal element.

Theorem 5 (★ Non-Zero Vector Spaces has a Basis)

Every non-zero vector space, V, has a basis.

The flow of this proof is a typical approach when Zorn's Lemma is involved.

Proof (★)

Let

$$\mathcal{L} := \{A \subset V \mid A \text{ is linearly independent } \}.$$

Note that $\mathcal{L} \neq \emptyset$ since $V \neq \{0\}$. Now order elements of $\mathcal{L}$ with $\subseteq$. It suffices to show that $(\mathcal{L}, \subseteq)$ has a maximal element, since this maximal element must be a basis. Otherwise, we would contradict the maximality of such an element.[1]

[1] This is the key to this proof.

Now let $S = \{A_\alpha\}_{\alpha \in I}$ be a chain in $\mathcal{L}$. Let

$$A_0 = \bigcup_{\alpha \in I} A_\alpha.$$

Require clarification before proceeding...

Definition 18 (Well-Ordered)

*We say that a poset $(X, \leq)$ is **well-ordered** if every non-empty subset $A \subset X$ has a least/minimal element in A.*

Exercise 4.1.1
Prove that well-ordered sets are chains.

Example 4.1.3

$(\mathbb{N}, \leq)$ is well-ordered.

Axiom 6 (Well-Ordering Principle)

Every non-empty set can be well-ordered.

Theorem 7 (Axioms of Choice and Its Equivalents)

TFAE:

1. *Axiom of Choice, Axiom 2*
2. *Zorn's Lemma, Axiom 4*
3. *Well-Ordering Principle, Axiom 6.*

Exercise 4.1.2
Prove Theorem 7

Proof
(3) $\implies$ (1) is simple; let the choice function be such that we pick the minimal element from each set among a non-empty collection of non-empty sets. It is clear that the product of these sets will always have an element, in particular the tuple where each component is the minimal element of each set.

The rest will be added once I've worked it out

Example 4.1.4

Let $X = \mathbb{Q}$. Let $\phi : \mathbb{Q} \to \mathbb{N}$ be defined such that

$$\phi\left(\frac{m}{n}\right) = \begin{cases} 2^m 5^n & m > 0 \\ 1 & m = 1 \\ 3^{-m} 7^n & m < 0 \end{cases}$$

*By the **unique prime factorization of natural numbers** (or **Fundamental Theorem of Arithmetic**), we have that ϕ is injective. In fact,*

$$r \leq s \iff \phi(r) \leq \phi(s),$$

showing to us that we have a well-ordering on $\mathbb{Q}$.

4.2 Cardinality

4.2.1 Equivalence Relation

Definition 19 (Equivalence Relation)

*Let X be non-empty set. A relation $\sim$ on X is an **equivalence relation** if it is*

- *reflexive;*
- *symmetric; and*
- *transitive.*

Definition 20 (Equivalence Class)

*Let X be a non-empty set, and $x \in X$. An **equivalence class** of x under the equivalence relation $\sim$ is defined as*

$$[x] := \{y \in X \mid x \sim y\}.$$

Note

Note that we either have $[x] = [y]$ or $[x] \cap [y] = \emptyset$. This is sensible, since if $w \in [x]$, then $w \sim x$. If $w \in [y]$, then we are done. If $w \notin [y]$, suppose $\exists v \in [y]$ such that $w \sim v$, which then implies $w \in [y]$ which is a contradiction.

This results shows to us that

$$X = \bigcup_{x \in X} [x],$$

or in words, equivalence classes partition the set.

Definition 21 (Partition)

Let $X \neq \emptyset$. A ***partition*** *of X is a collection $\{A_\alpha\}_{\alpha \in I} \subset \mathcal{P}(X)$ such that*

1. $A_\alpha \neq \emptyset$;
2. $A_\alpha \cap A_\beta = \emptyset$ *if* $\alpha \neq \beta$ *in* I; *and*
3. $X = \bigcup_{\alpha \in I} A_\alpha$.

With this, we have ourselves another method to show that $\sim$ is an equivalence relation.

Proposition 8 (Characterization of An Equivalence Relation)

If $\{A_\alpha\}_{\alpha \in I}$ is a partition of X and $x \sim y \iff x, y \in A_\alpha$, then $\sim$ is an equivalence relation.

The proof of this statement has been stated above.

Similar to when we defined partial orders, we can ask ourselves the following question:

Example 4.2.1

How many equivalence relations are there on the set $X = \{1, 2, 3\}$?[2]

[2] By Proposition 8, this question is equivalent to asking for the number of partitions we can create from the set X. The study of counting partitions is what is covered by Bell's Number.

Solution

Note that we can partition X as

$$\{\{1\}, \{2\}, \{3\}\}, \{\{1, 2, 3\}\},$$

and

$$\{\{1, 2\}, \{3\}\},$$

which we can rearrange in 3 different ways. Therefore, there are 5 different equivalence relations that we can define on X.

Example 4.2.2

Let X be any set. Consider $\mathcal{P}(X)$. Define $\sim$ on $\mathcal{P}(X)$ by

$$A \sim B \iff \exists f : A \to B$$

such that f is surjective[3]*. It is easy to verify that $\sim$ is an equivalence relation.*

[3] $\sim$ partitions X into sets that have the same number of elements.

5 Lecture 5 Sep 17th

5.1 Cardinality (Continued)

Definition 22 (Finite Sets)

*A set X is **finite** if $X = \emptyset$ or if $X \sim \{1, 2, ..., n\}$ for some $n \in \mathbb{N}$, where $\sim$ is the equivalence relation defined in Example 4.2.2.*

Definition 23 (Cardinality)

*If $X \sim n$, we say X has **cardinality** n and write $|X| = n$. We also let $|\emptyset| = 0$.*

Now a good question here is: if $n \neq m$, is $\{1, 2, ..., n\} \sim \{1, 2, ..., m\}$?

Theorem 9 (Pigeonhole Principle)

The set $\{1, 2, ..., n\}$ is not equivalent to any of its proper subset.

Proof

We shall prove this by induction on n.

Base case: $\{1\} \nsim \emptyset$.

Assume that the statement holds for $\{1, ..., k\}$. Suppose we have an injec-

This is a **proof by contradiction**, using the fact that we cannot find an injective function from a "larger" set to a "smaller" set.

We can assume that the function f is not surjective, since if the larger set is indeed equivalent to the smaller set, then it should not matter if f is surjective or not. In particular, we only require that there be an injective function.

Requires clarification and confirmation of proof.

tive function

$$f : \{1, 2, ..., k, k+1\} \to \{1, 2, ..., k, k+1\}$$

that is not surjective.

***Case 1**: $k+1 \notin \mathrm{Range}(f)$, where $\mathrm{Range}(f)$ is the range of f. Then we have*

$$f \restriction_{\{1,...,k\}}: \{1, ..., k\} \to \{1, ..., k\} \setminus \{f(k+1)\}.$$

Note: $\restriction$ is the restriction sign.

But f is an injective function and clearly

$$\{1, ..., k\} \setminus \{f(k+1)\} \subsetneq \{1, ..., k\},$$

a contradition.

***Case 2**: $k+1 \in \mathrm{Range}(f)$. Then $\exists j_0 \in \{1, ..., k, k+1\}$ such that $f(j_0) = k+1$, and since f is not surjective, $\exists m \in \{1, ..., k\}$ such that $m \notin \mathrm{Range}(f)$. Then consider a new function $g : \{1, ..., k, k+1\} \to \{1, ..., k\}$ such that*

$$g(a) = \begin{cases} m & a = k+1 \\ f(k+1) & a = j_0 \\ f(a) & a \neq j_0, k+1 \end{cases}$$

Corollary 10 (Pigeonhole Principle (Finite Case))

If the set X is finite, then X is not equivalent to any proper subset.

Sketch of proof:

$$\begin{tikzcd} \{1, ..., n\} \arrow[r] \arrow[d, shift right, "f^{-1}"'] & \{1, ..., n\} \\ X \arrow[u, shift right, "f"'] \arrow[r, "1-1", "\text{onto}"'] & A \subsetneq X \arrow[u, "f"'] \end{tikzcd}$$

Exercise 5.1.1

Prove Corollary 10.

Definition 24 (Infinite Sets)

*X is **infinite** if it is not finite.*

Example 5.1.1

Observe that we can construct a function $f : N \to \{2, 3, ...\}$ by $f(n) = n+1$. It is clear that f is a bijective funciton, and so $\mathbb{N} \sim \{2, 3, ...\}$.

Proposition 11 ($\mathbb{N}$ is the Smallest Infinite Set)

Every infinite set contains a subset $A \sim \mathbb{N}$.

Proof

Suppose X is infinite. Let

$$f : \mathcal{P}(X) \setminus \{\emptyset\} \to X$$

such that for $S \subset X$ where $S \neq \emptyset$, $f(S) \in S$ [1]*. Let $x_1 = f(X)$. Let $x_2 = f(X \setminus \{x_1\})$. Recursively, define*

[1] Axiom 3 ahoy!

$$x_n = f(X \setminus \{x_1, ..., x_{n-1}\}).$$

This gives us a sequence

$$X \supset S = \{x_1, ..., x_n, ...\}$$

which is equivalent to $\mathbb{N}$ via the map $n \mapsto x_n$. □

Corollary 12 (Infinite Sets are Equivalent to Its Proper Subsets)

Every infinite set X is equivalent to a proper subset of X.

Proof

Given such an X, we construct a sequence $\{x_n\}$ as in the previous proof. Define $f : X \to X \setminus \{x_n\}$ by

$$f(x) = \begin{cases} x_{n+1} & x \in \{x_n\} \\ x & x \notin \{x_n\}. \end{cases}$$

Clearly so, f is injective. □

Definition 25 (Countable)

We say that a set is ***countable*** *(or* ***denumerable****) is either X is finite or if $X \sim \mathbb{N}$. If $X \sim \mathbb{N}$, we can say that X is* ***countably infinite*** *and write $|X| = |\mathbb{N}| = \aleph_0$.*

Definition 26 (Smaller Cardinality)

Given 2 sets X, Y, we write

$$|X| \leq |Y|$$

if $\exists f : X \to Y$ injective.

Proposition 13 (Injectivity is Surjectivity Reversed)

TFAE:

1. *$\exists f : X \to Y$ injective*
2. *$\exists g : Y \to X$ surjective*

Proof

$(1) \implies (2)$: *Define*

$$g(y) = \begin{cases} x & \exists x \in X \;\; f(x) = y \\ x_0 & \text{any } x_0 \in X \end{cases}$$

Clearly g is surjective.

$(2) \implies (1)$: *Since g is surjective, for each $x \in X$, we have that*[2]

$$g^{-1}(|x|) = \{y \in Y : g(y) = x\} \neq \emptyset.$$

By the Axiom of Choice, there exists a choice function $h : \mathcal{P}(Y) \setminus \{\emptyset\} \to Y$ such that for each $A \subset Y$, $h(A) \in A$. Then, let $f : X \to Y$ such that

$$f(x) = h(g^{-1}(\{x\})).$$

Clearly so, f is injective. $\square$

[2] The idea here is to collect the preimages into a set, and use the choice function as an injective map.

Note

Note that we have $|\mathbb{N}| \leq |\mathbb{Q}|$*, since we can define an injective function* $f : \mathbb{N} \to \mathbb{Q}$ *such that* $f(n) = \frac{n}{1}$.

We have also shown that $|\mathbb{Q}| \leq |\mathbb{N}|$ *using our injective function* $g : \mathbb{Q} \to \mathbb{N}$*, given by*

$$g\left(\frac{m}{n}\right) = \begin{cases} 2^m 3^n & m > 0 \\ 1 & m = 0 \\ 5^{-m} 7^n & m < 0 \end{cases}$$

QUESTION: Is $|\mathbb{N} = |\mathbb{Q}||$? In other words, given $|X| \leq |Y| \wedge |Y| \leq |X|$, is $|X| = |Y|$?

6 Lecture 6 Sep 19th

6.1 Cardinality (Continued 2)

Before delving into resolving our last question in the previous lecture, note the following:

Note

Suppose $f : X \to Y$ is bijective. Let $A \subseteq B$, then

$$f(B \setminus A) = f(B) \setminus f(A).$$

Prove this observation as an exercise:

Exercise 6.1.1
Prove the note on the left.

Theorem 14 (★★★ Cantor-Schröder-Bernstein Theorem (CSB))

Let $A_2 \subset A_1 \subset A_0 = A$. Assume that $A_2 \sim A_0$. Then $A_0 \sim A_1$.

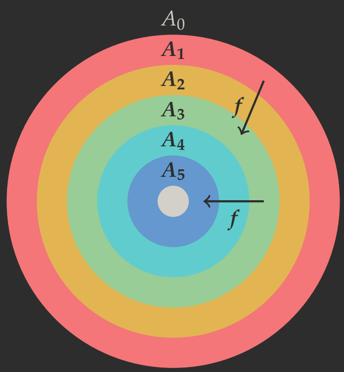


Figure 6.1: The core idea of the proof for Cantor-Schröder-Bernstein Theorem

Proof

Let $\phi : A_0 \to A_2$ be bijective, by assumption. Since $A_1 \subset A_1$, let $A_3 = \phi(A_1) \subset A_2$, and since $A_2 \subset A_0$, let $A_4 = \phi(A_2) \subset A_3$. Recursively so, let

$$A_{n+2} = \phi(A_n)$$

Notice that

$$A_0 = (A_0 \setminus A_1) \cup (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup (A_3 \setminus A_4) \cup \ldots \bigcap_{n=0}^{\infty} A_n$$

$$A_1 = (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup (A_3 \setminus A_4) \cup (A_4 \setminus A_5) \cup \ldots \bigcap_{n=1}^{\infty} A_n$$

Observe that

$$\bigcap_{n=0}^{\infty} A_n = \bigcap_{n=1}^{\infty} A_n.$$

[1]*Define* $f : A \to A_1$ *by*

$$f(x) = \begin{cases} x & x \in \bigcap_{n=0}^{\infty} A_n \\ x & x \in A_{2k+1} \setminus A_{2k+2},\ k = 0, 1, 2, \ldots \\ \phi(x) & x \in A_{2k} \setminus A_{2k+1},\ k = 0, 1, 2, \ldots \end{cases}$$

[1] Here, we employ the idea from Figure 6.1.

Corollary 15 (Cantor-Schröder-Bernstein Theorem - Restated)

If $A_1 \subset A \wedge B_1 \subset B \wedge A \sim B_1 \wedge B \sim A_1$, *then* $A \sim B$.[2]

[2] This is equivalent to the statement

$$|A| \leq |B| \wedge |B| \leq |A| \implies |A| = |B|.$$

Proof

By assumption, let $f : A \to B_1$ *be bijective, and let* $g : B \to A_1$ *be bijective. Let* $A_2 = g(B_1) \subseteq A_1 \subset A$ *Let* $A_2 = g(B_1) \subseteq A_1 \subset A$. *Then the composite function* $g \circ f : A \to A_2$ *is bijective, and so* $A \sim A_2$. *By* *Theorem 14*, *we have*

$$A \sim A_2 \sim A_1 \sim B.$$

$\square$

Example 6.1.1

Our question from last lecture now has an answer: by *Theorem 14*, *we have that* $|\mathbb{Q}| = |\mathbb{N}|$.[3]

[3] Now that we know that they have the ssme cardinality:

Exercise 6.1.2
Find a bijection between $\mathbb{Q}$ *and* $\mathbb{N}$.

Proposition 16 (Denumerability Check)

If X *is infinite, then*

$$|X| = |\mathbb{N}| = \aleph_0 \iff \exists f : X \to \mathbb{N} \text{ bijective.}$$

Proof

$(\implies)$ *is immediate. For* $(\impliedby)$, *suppose* $f : X \to \mathbb{N}$, *which implies that* $|X| \leq |\mathbb{N}|$. *By Proposition 11*, $|\mathbb{N}| \leq |X|$. *Therefore,* $|X| = |bbN| = \aleph_0$. $\square$

Example 6.1.2

$\mathbb{N} \times \mathbb{N}$ *is countable. The function*

$$f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \text{ given by } f(m,n) = 2^n 3^m$$

is injective.

Definition 27 (Uncountable)

*A set X is **uncountable** if it is not countable.*

Theorem 17 (Cantor's Diagonal Argument)

$(0,1)$ *is uncountable.*

Proof

Suppose, for contradiction, that $(0,1)$ *is countable. Then we can write*

$$\begin{aligned} a_1 &= .a_{11}a_{12}a_{13}\ldots \\ a_2 &= .a_{21}a_{22}a_{23}\ldots \\ &\vdots \\ a_n &= .a_{n1}a_{n2}a_{n3}\ldots \end{aligned}$$

in $(0,1)$. *This representation is unique if we do not allow the representation to end in a string of 9's. Let* $b \in (0,1)$, *expressed as* $b = .b_1b_2b_3\ldots$ *such that*

$$b_i = \begin{cases} 5 & a_i \neq 5 \\ 2 & a_i = 5 \end{cases}$$

Then $b \notin (0,1)$, *a contradiction*[4]. $\square$

[4] **Require more explanation.**

Corollary 18 (Uncountability of $\mathbb{R}$)

$\mathbb{R}$ is uncountable.

Proof

Let $f : (0, 1) \to \mathbb{R}$ be given by

$$f(x) = \tan\left(\pi x - \frac{\pi}{2}\right).$$

Clearly so, $(0, 1)$ is bijective. $\square$

Note

We denote $|\mathbb{R}| = c$.

Question: Given sets X, Y, is it always true that either[5]

1. $|X| = |Y|$;
2. $|X| < |Y|$; or
3. $|Y| < |X|$.

[5] As compare to $\leq$, $<$ implies that there is no surjection from the set on the LHS to the RHS.

7 Lecture 7 Sep 21st

7.1 Cardinality (Continued 3)

Theorem 19 (★ Comparability of Cardinals)

If X and Y are non-empty, then either

$$|X| \leq |Y| \vee |Y| \leq |X|.$$

Proof

Let

$$S = \{(A, B, f) \mid A \subseteq X, B \subseteq Y, f : A \to B \text{ bijective }\}.$$

Note that $S \neq \emptyset$, since X and Y are non-empty, and so we can have $f(a) = b$ for $A = \{a\} \subset X$ and $B = |b| \subset Y$.[1] *We order S as follows: we say*

$$(A_1, B_1, f_1) \leq (A_2, B_2, f_2)$$

if

$$A_1 \subseteq A_2, \;\; B_1 \subseteq B_2, \;\; f_1 = f_2 \restriction_{A_1}.$$

Let $C = \{(A_\alpha, B_\alpha, f_\alpha)\}_{\alpha \in I}$ be a chain in $(S, \leq)$. Let $A_0 = \bigcup_{\alpha \in I} A_\alpha$, $B_0 = \bigcup_{\alpha \in I} B_\alpha$, and define $f_0 : A_0 \to B_0$ by

$$f_0(x) = f_{\alpha_0}(x) \text{ if } x \in A_{\alpha_0}.$$

Now if $x \in A_{\alpha_1}$, $x \in A_{\alpha_2}$ and

$$(A_{\alpha_1}, B_{\alpha_1}, f_{\alpha_1}) \leq (A_{\alpha_2}, B_{\alpha_2}, f_{\alpha_2}),$$

[1] We want to use the maximal element to obtain our result. To that end, we need Zorn's Lemma. So we need S to build this up.

we have that

$$f_{\alpha_1}(x) = f_{\alpha_2} \restriction_{A_{\alpha_1}} (x) = f_{\alpha_2}(x),$$

i.e. f_0 is well-defined.

Claim 1: $f_0 : A_0 \to B_0$ is injective.

Let $x_1, x_2 \in A_0$ such that $x_1 \neq x_2$.

$\implies \exists \alpha_1, \alpha_2 \in I \;\; x_1 \in A_{\alpha_1} \wedge x_2 \in A_{\alpha_2} \wedge A_{\alpha_1} \subseteq A_{\alpha_2}$ *(wlog)*

$\implies x_1 . x_2 \in A_{\alpha_2}$

$\implies (\because f_{\alpha_2}$ *injective* $\implies f_{\alpha_2}(x_1) \neq f_{\alpha_2}(x))$

$\implies f_0(x_1) \neq f_0(x_2) \implies f_0$ *injective.*

Claim 2: $f_0 : A_0 \to B_0$ is surjective.

Let $y_0 \in B_0$

$\implies \exists \alpha_0 \in I \;\; y_0 \in B_{\alpha_0}$

$\implies \exists x_0 \in A_{\alpha_0} \;\; f_{\alpha_0}(x_0) = y_0$ *($\because f_{\alpha_0}$ surjective)*

$\implies f_0(x_0) = y_0$

$\therefore (A_0, B_0, f_0)$ is an upper bound for C. Then by Zorn's Lemma, $(S, \leq)$ has a maximal element (A, B, f).

Case 1: If $A = X$, then injectivity of f implies $|X| \leq |Y|$.

Case 2: If $B = Y$, then surjectivity of f implies $|Y| \leq |A| \leq |X|$.

Case 3: If $A \neq X \wedge B \neq Y$, then $X \setminus A \neq \emptyset \wedge Y \setminus B \neq \emptyset$. Let $x_0 \in X \setminus A$, $y_0 \in Y \setminus B$. Let $A^* = A \cup \{x_0\}$, $B^* = B \cup \{y_0\}$, and $f^* : A^* \to B^*$ such that

$$f^*(x) = \begin{cases} f(x) & x \in A \\ y_0 & x = x_0 \end{cases}$$

Then $(A, B, f) \leq (A^*, B^*, f^*)$, contradicting maximality. □

7.1.1 Cardinal Arithmetic

Sum of Cardinals Observe that if $X = \{x_1, ..., x_n\}$, $Y = \{y_1, ..., y_m\}$, and $X \cap Y = \emptyset$, then $|X| = n$, $|Y| = m$ and $|X \cup Y| = n + m$. This motivates us to provide the following definition:

Definition 28 (Sum of Cardinals)

Assume that X and Y are such that $X \cap Y = \emptyset$. We define

$$|X| + |Y| = |X \cup Y|.$$

QUESTION: So what about $\aleph_0 + \aleph_0$?

A thought that motivates us to give the following answer lies in the observation that: if X is the set of even natural numbers and Y the odd natural numbers, then $X \cup Y$ is the set of all natural numbers. All three sets are countably infinite, i.e. they have cardinality $\aleph_0$.

QUESTION: What about $c + c$?

A similar motivation comes from the observation that: given $X = (0, 1)$ and $Y = (1, 2)$, we have

$$c = |X| \leq |X| + |Y| \leq |R| = c,$$

and so $|X| = |Y| = c \implies |X \cup Y| = c$.

Theorem 20 (Sums of Cardinals)

Given sets X and Y, if X is infinite, then

1. $|X| + |X| = |X|$
2. $|X| + |Y| = \max(|X|, |Y|)$

Exercise 7.1.1
Prove Theorem 20 as an exercise.

Multiplication of Cardinals Given

$$X = \{x_1, x_2, ..., x_n\}$$
$$Y = \{y_1, y_2, ..., y_m\}$$

we have that

$$X \times Y = \{(x_i, y_j) \mid i = 1, 2, ..., n, \ j = 1, 2, ..., m\}$$

and so

$$|X \times Y| = nm.$$

Definition 29 (Multiplication of Cardinals)

Given sets X and Y, define

$$|X|\,|Y| = |X \times Y|\,.$$

Example 7.1.1

We have $|\mathbb{N} \times \mathbb{N}| = \aleph_0$ since the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by

$$f(n, m) = 2^n 3^m$$

is injective.

Question: What about $c \cdot c$?

Theorem 21 (Multiplication of Cardinals)

If X is infinite and $Y \neq \emptyset$, then

- $|X \times X| = |X| \implies |X|\,|X| = |X|$;
- $|X \times Y| = \max(|X|, |Y|)$.

Exercise 7.1.2
Prove Theorem 21 as an exercise.

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