MATH 352: Complex Analysis
Assignment 1 (due: January 24, 2018)

1. A total ordering on $\mathbb{C}$ is a relation $\succ$ between complex numbers that satisfies *all* the following conditions:

   (C1) For any two complex numbers z, w, one and only one of the following is true: $z \succ w$, $w \succ z$ or $z = w$.

   (C2) For all $z_1, z_2, z_3 \in \mathbb{C}$, the relation $z_1 \succ z_2$ implies $z_1 + z_3 \succ z_2 + z_3$.

   (C3) For all $z_1, z_2, z_3 \in \mathbb{C}$, if $z_3 \succ 0$, then the relation $z_1 \succ z_2$ implies $z_1 z_3 \succ z_2 z_3$.

   Show that it is impossible to define a total ordering on $\mathbb{C}$. [Hint: Assume a relation between i and 0]

2. Let w be a complex number with $0 < |w| < 1$. Show that the set of all $z \in \mathbb{C}$ with $|z - w| < |1 - \bar{w}z|$ is the disc $\{z \in \mathbb{C} : |z| < 1\}$.

3. Let $P(z)$ be a polynomial with real coefficients. Show that the complex roots of P appear in conjugate pairs.

4. Suppose $f, g : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ are holomorphic at $z_0 \in \Omega$. Show that fg is holomorphic at z_0 and $(fg)' = f'g + fg'$ at z_0.

5. A function f is said to be entire if f is holomorphic in the entire complex plane. Consider a polynomial in z of degree $n \geq 1$:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0, \quad (a_i \in \mathbb{C}, a_n \neq 0).$$

   Show that

   (i) $P(z)$ is an entire function and $P'(z) = n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \ldots + a_1$.

   (ii) P cannot take only imaginary values.