

PMATH347 - Groups and Rings (Class Notes)

Johnson Ng

September 25, 2017

Contents

1	Lecture 1: Sept 8, 2017	6
1.1	Logistics	6
1.2	Group theory: Dihedral and Permutation groups	6
2	Lecture 2: Sept 11, 2017	9
2.1	Last time	9
2.2	Continuing on Dihedral groups	9
2.3	Groups	10
3	Lecture 3: Sept 13, 2017	12
3.1	Properties of Groups	12
3.2	Homomorphism	14
4	Lecture 4: Sep 15, 2017	16
4.1	Examples of Homomorphism	16
5	Lecture 5: Sep 18, 2017	20
5.1	Group Actions	20
6	Lecture 5: Sep 20, 2017	24
6.1	Logistics	24
6.2	More on Group Actions	24
7	Lecture 7: Sep 22, 2017	28
7.1	Cosets	28
8	Lecture 8: Sep 25, 2017	33
8.1	Continuing with Cosets	33
9	Lecture 9: Sep 27, 2017	37

List of Definitions

1.2.1 Symmetry	6
1.2.2 Dihedral Group	6
1.2.3 Permutation	7
2.3.1 Group	10
2.3.2 Subgroup	10
2.3.3 Abelian Group	11
3.2.1 Group Homomorphism	14
3.2.2 Kernel	15
4.1.1 Inverse of a Group Homomorphism	18
5.1.1 Group Action	20
5.1.2 Trivial Homomorphism	22
5.1.3 Kernel of the Action	22
6.2.1 Faithful	25
6.2.2 Image Set	25
6.2.3 Conjugation	26
6.2.4 Center of the Group	27
7.1.1 Left Cosets	28
7.1.2 Order	31
8.1.1 Set of Left Cosets	33
8.1.2 Normal Subgroup	34

<i>CONTENTS</i>	3
8.1.3	35

List of Theorems

Proposition 3.1.1	12
Proposition 3.1.2	Cancellation law	13
Proposition 3.2.1	14
Proposition 3.2.2	15
Proposition 4.1.1	18
Corollary 4.1.1	19
Lemma 5.1.1	σ_g as a bijection	20
Proposition 5.1.1	Permutation Representation	21
Proposition 6.2.1	25
Lemma 6.2.1	Lemma 9	25
Corollary 6.2.1	Cayley's Theorem	26
Proposition 7.1.1	Proposition 11	29
Corollary 7.1.1	29
Proposition 7.1.2	29
Proposition 7.1.3	30
Corollary 7.1.2	30
Corollary 7.1.3	12. Lagrange's Theorem	31
Lemma 8.1.1	Lemma 13	35
Lemma 8.1.2	Lemma 14	36

List of Symbols

S_n	symmetric group on n letters
D_{2n}	dihedral group of order 2n
$A \times B$	Cartesian product of A and B
$A \simeq B$	A is isomorphic to B
$\ker(\phi)$	kernel of ϕ
$I_m(\phi)$	image set of ϕ
gH, Hg	left coset, right coset of H with coset representative g
$[G : H]$	index of the subgroup H in G
$H \triangleleft G$	H is a normal subgroup of G
$\text{rem}_n(a)$	remainder of a when divided by n

Chapter 1

Lecture 1: Sept 8, 2017

1.1 Logistics

Textbook is relatively important. The level of the text is about the same as the class, so it works to read ahead. (Problem is, the syllabus is not listed in the course outline, so what should we read?)

1.2 Group theory: Dihedral and Permutation groups

Fix $n \geq 3$, regular n -gon on a plane. For e.g. $n=7$

Definition 1.2.1 (Symmetry)

Rigid motions in \mathbb{R}^3 , in which we can move it(?) around in \mathbb{R}^3 and put it back to get the same region.

Definition 1.2.2 (Dihedral Group)

D_{2n} - set of all such symmetries (which is a “group”).

Our interest: the end results

Two symmetries are the same if they have the same final position.

Fix a labelling of the vertices.

An element of D_{2n} determines and is determined by how it permutes the vertices (labels).

Definition 1.2.3 (Permutation)

A permutation of a set X is a bijection $\sigma : X \rightarrow X$. S_X is the set (or “group”) of all permutations of X . $S_n := S_{\{1, \dots, n\}}$

So $D_{2n} \subset S_n$, we view $\sigma \in D_{2n}$ as the permutation $\{1, \dots, n\} \rightarrow \{1, \dots, n\}$. So $i \mapsto$ the vertex that the symmetry σ takes i to.

Not all permutations are symmetries!

Example 1.2.1

$n = 4$, with labels 1, 2, 3, 4

Let $\sigma \in S_4$ with mapping $1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 3, 4 \mapsto 4$.

So $\sigma \notin D_8$

Claim 1.2.1

$$|S_n| = n!$$

Proof

There are n labels. Firstly, pick 1, then there will be $n - 1$ choices for 2. Then, pick 2, then there remains $n - 2$ choices. Continue this argument. This completes the proof.

Claim 1.2.2

$$|D_{2n}| = 2n$$

Proof

For each vertex, $i \in \{1, \dots, n\}$, we have the rigid motion that takes vertex 1 to vertex i . Then we have a choice of placing vertex 2 at vertex $i + 1$ (where $n + 1 = 1$) or $i - 1$ (where $1 - 1 = n$). Both choices are possible and give distinct symmetries. E.g. They take the pair $(1, 2)$ to distinct pairs. So we have $2n$ elements in D_{2n} so far. But a symmetry is determined by where it takes $(1, 2)$.

We can “multiply” the elements of D_{2n} – and also of S_n – by composition.

Given $\sigma, \tau \in S_n$, denote by $\sigma\tau$ the permutation that first does τ then does σ , and then does σ , r times is expressed as $\sigma^r := \sigma\sigma \dots \sigma$.

Note

1. If $\sigma, \tau \in D_{2n}$, then $\sigma\tau \in D_{2n}$. We can also “invert” elements of D_{2n} . If $\sigma \in S_n, \sigma^{-1} \in S_n$.
2. $\sigma \in D_{2n} \implies \sigma^{-1} \in D_{2n}$.

$$3. (\sigma^r)^{-1} = (\sigma^{-1})^r =: \sigma^{-r}$$

In D_{2n} , we have a distinguished element called the **identity**, denoted by 1, which does nothing.

Convention: $\sigma \in S_n, \sigma^0 = 1$

Our proof that $|D_{2n}| = 2n$ actually showed us that:

Claim 1.2.3

Every element of D_{2n} can be written uniquely as $r^i s^k$ where $k = 0, 1$, $i = 0, \dots, n-1$, r is the rotation of the n -gon by $\frac{2\pi}{n}$ radians (by one vertex), and s = reflection across the line that passes through i and the origin.

Chapter 2

Lecture 2: Sept 11, 2017

2.1 Last time

$D_{2n} \subseteq S_n$ for $n \geq 3$.

Claim 2.1.1

Every element of D_{2n} can be written uniquely as $r^i s^k$ where $k = 0, 1$, $i = 0, \dots, n-1$, r is the rotation of the n -gon by $\frac{2\pi}{n}$ radians (by one vertex), and s = reflection across the line that passes through i and the origin.

2.2 Continuing on Dihedral groups

We can “compute” D_{2n} .

Example 2.2.1

Consider the element of D_{2n} given by $r_{-1} s r^2 s$ which is equals to $r^{-1} r^{-2} s s = r^{-3} s^2 = r^{-3} = r^{n-3}$.

Note (General identities in D_{2n})

1. $s r^i = r^{-i} s \quad i = 0, \dots, n-1$
2. $s^2 = 1$
3. $r^{-1} = r^{n-i}$

2.3 Groups

Definition 2.3.1 (Group)

A group is a non-empty set G equipped with a binary operation, i.e. a function

$$* : G \times G \rightarrow G \quad (2.1)$$

which is from ordered pairs of elements in G to G , satisfying the following three axioms:

1. Associativity: $\forall a, b, c \in G \ a * (b * c) = (a * b) * c$.
2. \exists identity element $e \in G$ with the property $\forall a \in G \ a * e = e * a = a$.
3. $\forall a \in G \ \exists$ an inverse $a^{-1} \in G \quad a * a^{-1} = a^{-1} * a = e$.

Example 2.3.1

S_n is a (finite) group with

1. $*$ = composition
2. $e = 1$
3. a^{-1} = inverse permutation

Example 2.3.2

D_{2n} is a group.

Definition 2.3.2 (Subgroup)

A subgroup of a group G is a non-empty subset $H \subseteq G$ that is closed under both $*$ and taking inverses, i.e.

- $a, b \in H \subseteq G \implies a * b \in H$
- $a \in H \subseteq G \implies a^{-1} \in H$

We denote H as a subgroup of G by $H \leq G$.

Remark

If $H \leq G$, then $*|_{H \times H} : H \times H \rightarrow H$ and this makes H into a group.

Proof

$*|_{H \times H}$ is the fact that H is closed under $*$.

Associativity of $*|_{H \times H}$ on H follows from associativity of $*$ on G .

For axiom (ii), take $a \in H$. So $a^{-1} \in H$. Then $a * a^{-1} \in H$. But since $a * a^{-1} = e \in G$ by axiom (iii). Thus $e \in H$.

Axiom (iii) is from the fact that H is closed under taking inverses.

Given a subgroup H of a group $(G, *)$, we call $(H, * \upharpoonright_{H \times H})$ the induced group structure on H .

Example 2.3.3

$$D_{2n} \leq S_n$$

Example 2.3.4

(a) S_n

(b) D_{2n}

(c) $\mathbb{R}^* = \text{non-zero real numbers}$, $*$ = usual multiplication, $a^{-1} = \frac{1}{a}$, $e = 1$.

(d) More generally, for $n \geq 1$, $GL_n(\mathbb{R}) = \text{set of } n \times n \text{ invertible matrices with real entries}$
 $(GL_n(\mathbb{R}) = \mathbb{R}^*)$, $*$ = matrix multiplication, $e = I$, inverse is M^{-1} if $M \in GL_n(\mathbb{R})$.

This works with \mathbb{R} replaced by $\mathbb{Q}, \mathbb{C}, \mathbb{Z}_p$.

Definition 2.3.3 (Abelian Group)

*A group $(G, *)$ is called abelian if $*$ is commutative, i.e. $a * b = b * a$ for all $a, b \in G$.*

Example 2.3.5

From our previous example, (a) and (b) are non-abelian (for $n \geq 3$). (c) is abelian. (d) is non-abelian for $n > 1$.

Continuing the numbering of the example,

(e) *The following are abelian groups: $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{C}, +)$, $(\mathbb{Z}_p, +)$*

where we have $$ = +, $e = 0$, $a^{-1} = -a$*

Note (Multiplicative Notation)

*We often write ab for $a * b$ and we tend to write 1 for e .*

Note (Additive Notation)

*If we are working with a group $(G, *)$ that we know is abelian, we often write $a + b$ instead of $a * b$. We write 0 instead of e and write $-a$ instead of a^{-1} .*

We never use Additive Notation if we are unsure about the commutativity of the group.

Chapter 3

Lecture 3: Sept 13, 2017

3.1 Properties of Groups

Proposition 3.1.1

Suppose G is a group.

1. *The identity element is unique.*
2. *For each $a \in G$, a has a unique inverse.*
3. $(a^{-1})^{-1} = a$
4. $(ab)^{-1} = b^{-1}a^{-1}$
5. *Generalised associativity law: $\forall a_1, \dots, a_n \in G$, then $a_1a_2\dots a_n$ gives the same value regardless of how we associate the expressions.*
6. *In any group G , $1^{-1} = 1$*

Proof

1. *Suppose $e, e' \in G$ are both identity elements (WTP $e = e'$).*

$$\begin{aligned} e &= e'e && \text{since } e' \text{ is an identity} \\ &= e' && \text{since } e \text{ is an identity} \end{aligned}$$

2. Suppose $b, c \in G$ are both inverses of $a \in G$. So

$$\begin{aligned}
 ab &= 1 = ac \\
 \implies b(ab) &= b(ac) \\
 \implies (ba)b &= (ba)c \quad \text{associativity} \\
 \implies 1b &= 1c \quad (\text{since } ba = 1) \\
 \implies b &= c
 \end{aligned}$$

3. By defn of inverse,

$$\begin{aligned}
 aa^{-1} &= 1 \\
 a^{-1}a &= 1
 \end{aligned}$$

Thus a is the inverse of a^{-1} .

4. $\forall a, b \in G$

$$\begin{aligned}
 (b^{-1}a^{-1})(ab) &= b^{-1}(a^{-1}(ab)) \\
 &= b^{-1}((a^{-1}a)b) \\
 &= b^{-1}(1b) \\
 &= b^{-1}b = 1
 \end{aligned}$$

Similarly, $(ab)(b^{-1}a^{-1}) = 1$. Therefore $(ab)^{-1} = b^{-1}a^{-1}$.

5. Proof for this proposition is intuitive, and we can use induction on n . See Dummit Section 1.1.

6. $1 \cdot 1 = 1$

The fifth proposition allows us to drop the parenthesis without ambiguity.

Note (Notation)

$\forall a \in G \ n > 0$

$$a^n = a \cdot a \cdot \dots \cdot a, \quad a^0 = 1, \quad a^{-n} = (a^n)^{-1}$$

Remark

1. By Proposition (d), $a^{-n} = (a^n)^{-1}$

2. In general, $(ab)^n \neq a^n b^n$ (especially for non-abelian)

Proposition 3.1.2 (Cancellation law)

For any $a, b, u, v \in G$

1. $au = av \implies u = v$ (Left-cancellation)
2. $ub = vb \implies u = v$ (Right-cancellation)

Proof

1.

$$\begin{aligned}
 au &= av \\
 a^{-1}au &= a^{-1}av \\
 u &= v
 \end{aligned}$$

2. Similar to above.

Remark

Note that 0 is not in a group, since every element must have an inverse but 0 does not.

3.2 Homomorphism

Definition 3.2.1 (Group Homomorphism)

A group homomorphism $\phi : G \rightarrow H$ where G, H are groups, is a function from G to H with the property that for all $a, b \in G$

$$\phi(ab) = \phi(a)\phi(b)$$

(aka a morphism of groups)

A homomorphism $\phi : G \rightarrow H$ is called an isomorphism if it is bijective.

We say that $G \simeq H$, and say that G is isomorphic to H , if there exists an isomorphism $\phi : G \rightarrow H$.

Remark

ab is a multiplication in G .

$\phi(a)\phi(b)$ is a multiplication in H .

Proposition 3.2.1

If $\phi : G \rightarrow H$ is a group homomorphism, then

1. $\phi(1_G) = 1_H$
2. $\forall a \in G \quad \phi(a^{-1}) = \phi(a)^{-1}$

Proof

1. $1_H \phi(1_G) = \phi(1_G) = \phi(1_G 1_G) = \phi(1_G) \phi(1_G)$. Thus, by Cancellation Law, $1_H = \phi(1_G)$.

2. $\forall a \in G$

$$\begin{aligned}\phi(a^{-1})\phi(a) &= \phi(a^{-1}a) = \phi(1_G) = 1_H \\ \phi(a)\phi(a^{-1}) &= \phi(aa^{-1}) = \phi(1_G) = 1_H \\ \implies \phi(a^{-1}) &= \phi(a)^{-1}\end{aligned}$$

Definition 3.2.2 (Kernel)

If $\phi : G \rightarrow H$ is a group homomorphism, then the kernel of ϕ is

$$\ker(\phi) = \{a \in G : \phi(a) = 1\} \tag{3.1}$$

Proposition 3.2.2

$\ker(\phi) \leq G$

Proof

Suppose $a, b \in \ker(\phi)$.

$$\begin{aligned}\phi(ab) &= \phi(a)\phi(b) \\ &= 1_H 1_H = 1_H\end{aligned}$$

So $ab \in \ker(\phi)$. So $\ker(\phi)$ is closed under the group operation of G .

Suppose $a \in \ker(\phi)$.

$$\phi(a^{-1}) = \phi(a)^{-1} = 1_H^{-1} = 1_H$$

So $\ker(\phi)$ is closed under inverses.

Also, $\ker(\phi) \neq \emptyset$ since $\phi(1_G) = 1_H$ by proposition 3(a), so $1_G \in \ker(\phi)$.

Chapter 4

Lecture 4: Sep 15, 2017

4.1 Examples of Homomorphism

Example 4.1.1

Fix $n \geq 2$.

1. $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$

addition: $a \oplus b = \text{remainder of } a + b \text{ when divided by } n$

multiplication: $a \otimes b = \text{remainder of } ab \text{ when divided by } n$

(\mathbb{Z}_n, \oplus) is an abelian group with identity 0.

The fact that this is a (finite) group uses basic arithmetic of congruences.

2. $(\mathbb{Z}, +)$ is an abelian group.

$$\text{rem} : \mathbb{Z} \rightarrow \mathbb{Z}_n$$

$\text{rem}(a) = \text{remainder of } a \text{ when divided by } n$.

This is a group homomorphism.

Proof

2. Need to show $\text{rem}(a + b) = \text{rem}(a) \oplus \text{rem}(b)$

We know

$$a \equiv \text{rem}(a) \pmod{n}$$

$$b \equiv \text{rem}(b) \pmod{n}$$

$$\implies a + b \equiv \text{rem}(a) + \text{rem}(b) \pmod{n}$$

and

$$a + b \equiv \text{rem}(a + b) \pmod{n}$$

$$\implies \text{rem}(a + b) \equiv (\text{rem}(a) + \text{rem}(b)) \pmod{n}$$

But $0 \leq \text{rem}(a + b) \leq n - 1$. Therefore,

$$\begin{aligned} \text{rem}(a + b) &= \text{remainder when } (\text{rem}(a) + \text{rem}(b)) \text{ is divided by } n \\ &= \text{rem}(a) \oplus \text{rem}(b) \end{aligned}$$

Note

$$\begin{aligned} \ker(\text{rem}) &= \{an : a \in \mathbb{Z}\} \\ &= \{b \in \mathbb{Z} : n|b\} = n\mathbb{Z} \end{aligned}$$

So $n\mathbb{Z} \leq \mathbb{Z}$, i.e. $n\mathbb{Z}$ is a subgroup of \mathbb{Z} .

Example 4.1.2

$G = \mathbb{R}^{>0}$ is a group under multiplication.

Note: $\mathbb{R}^{>0} \leq \mathbb{R}^X$

$$H = (\mathbb{R}, +)$$

$$\exp : \mathbb{R} \rightarrow \mathbb{R}^{>0} \quad \text{e.g. } r \mapsto e^r$$

Group homomorphism from $H \rightarrow G$: $\exp(a + b) = e^{a+b} = e^a e^b = \exp(a) \exp(b)$

$\exp(a + b)$ is a group operation on H .

$\exp(a) \exp(b)$ is a group operation on G .

So \exp is a group homomorphism from the additive group of reals to the multiplicative group of the positive reals. In fact, it is an isomorphism since it is bijective.

\exp *is injective*

$$\begin{aligned} e^a &= e^b \\ \implies \ln(e^a) &= \ln(e^b) \\ \implies a &= b \end{aligned}$$

\exp *is surjective*

$$\begin{aligned} r \in \mathbb{R}^{>0} \quad a &= \ln(r) \\ \implies e^a &= e^{\ln r} = r \\ \ln : (\mathbb{R}^{>0}, \times) &\rightarrow (\mathbb{R}, +) \text{ is also a group homomorphism} \\ \ln(ab) &= \ln(a) + \ln(b) \\ \exp \circ \ln : \mathbb{R}^{>0} &\rightarrow \mathbb{R}^{>0} \quad r \mapsto r \\ \exp \circ \ln &: id_G \\ \ln \circ \exp &: id_H \end{aligned}$$

Definition 4.1.1 (Inverse of a Group Homomorphism)

If $\phi : G \rightarrow H$ is a group homomorphism, then an inverse to ϕ is a group homomorphism

$$\psi : H \rightarrow G \tag{4.1}$$

such that

$$\begin{aligned} \psi \circ \phi &= id_G \\ \phi \circ \psi &= id_H \end{aligned}$$

Example 4.1.3 (Exercise)

A group homomorphism is an isomorphism iff it has an inverse group homomorphism.

Note

$$(\mathbb{R}, +) \simeq (\mathbb{R}^{>0}, \times)$$

Proposition 4.1.1

Suppose $\phi : G \rightarrow H$ is a surjective group homomorphism. If G is abelian, then so is H

Proof

$$\begin{aligned} a, b \in H \quad \exists r, s \in G \quad a &= \phi(r) \quad b = \phi(s) \\ ab &= \phi(r)\phi(s) = \phi(rs) = \phi(sr) = \phi(s)\phi(r) = ba \end{aligned}$$

Corollary 4.1.1

If $G \simeq H$ then G is abelian iff H is abelian.

Example 4.1.4

$$GL_1(\mathbb{C}) \not\simeq GL_2(\mathbb{C})$$

$GL_1(\mathbb{C})$ is abelian.

$GL_2(\mathbb{C})$ is not abelian.

Example 4.1.5

$$G = (\mathbb{Z}_4, \oplus)$$

$$H = (\mathbb{Z}_5^\times, \otimes)$$

$$\mathbb{Z}_5^\times = \{1, 2, 3, 4\} \text{ — identity} = 1$$

(Note \mathbb{Z}_6^\times is not a group under \otimes)

$$\phi : G \rightarrow H$$

$$\phi(0) = 1$$

$$\phi(1) = 2$$

$$\phi(2) = 3$$

$$\phi(3) = 4$$

But then $\phi(1 \oplus 1) = \phi(2) = 3$ and $\phi(1) \otimes \phi(1) = 2 \otimes 2 = 4$.

So $\phi(1 \oplus 1) \neq \phi(1) \otimes \phi(1)$.

But actually $G \simeq H$, since

$$\psi(0) = 1$$

$$\psi(1) = 2$$

$$\psi(2) = 4$$

$$\psi(3) = 3$$

is an isomorphism.

Chapter 5

Lecture 5: Sep 18, 2017

5.1 Group Actions

Definition 5.1.1 (Group Action)

A group action is a group G on a set A is a function

$$G \times A \rightarrow A \tag{5.1}$$

denoted by \cdot , i.e.

$$(g, a) \mapsto g \cdot a \in A \tag{5.2}$$

satisfying

1.

$$\begin{array}{c} \text{multiplication in } G \\ \downarrow \\ g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a \\ \uparrow \\ \text{group action} \end{array}$$

2. $1 \cdot a = a$

for all $g_1, g_2 \in G$, $a \in A$.

If G acts on A , for each $g \in G$ we get $\sigma_g : A \rightarrow A$, i.e. $a \mapsto g \cdot a$

Lemma 5.1.1 (σ_g as a bijection)

G acts on A , $g \in G$. Then $\sigma_g : A \rightarrow A$ is a bijection.

Proof

For injection,

$$\begin{aligned}
 \forall a, b \in A \quad \sigma_g(a) &= \sigma_g(b) \\
 \implies g \cdot a &= g \cdot b \\
 \implies g^{-1} \cdot (g \cdot a) &= g^{-1} \cdot (g \cdot b) \\
 \implies (g^{-1}g) \cdot a &= (g^{-1}g) \cdot b \\
 \implies 1 \cdot a &= 1 \cdot b \\
 \implies a &= b \quad \text{by property 2}
 \end{aligned}$$

For surjection,

$$\begin{aligned}
 \forall b \in A \quad \text{Let } a &= g^{-1} \cdot b \\
 \sigma_g(a) &= g \cdot a = g \cdot (g^{-1} \cdot b) = (gg^{-1}) \cdot b = b
 \end{aligned}$$

Note (Warning)

Do not confuse the action of G on A and group operation on G , especially as we often write ga instead of $g \cdot a$ for the group action.

Hopefully, the difference is clear by context.

Note (Recall)

For any set A

$$S_A = \text{group of bijections of } \sigma : A \rightarrow A \text{ under composition} \quad (5.3)$$

If G acts on A , we have just defined a function

$$\begin{aligned}
 G &\rightarrow S_A \quad \sigma_g \text{ as a bijection 5.1.1} \\
 g &\mapsto \sigma_g
 \end{aligned}$$

Proposition 5.1.1 (Permutation Representation)

The function $G \rightarrow S_A$ given by $g \mapsto \sigma_g$ is a group homomorphism.

Proof

All we have to check is that for any $g, h \in G$

$$\sigma_{gh} = \sigma_g \circ \sigma_h \quad (5.4)$$

Both sides are permutations of A .

Let $a \in A$ be arbitrary.

$$\begin{aligned}\sigma_{gh}(a) &= (gh) \cdot a \\ &= g \cdot (h \cdot a) \\ &= g \cdot (\sigma_h(a)) \\ &= \sigma_g(\sigma_h(a))\end{aligned}$$

Exercise 5.1.1

Prove the converse of *Permutation Representation 5.1.1*: Suppose G is a group, A is a set, and $\phi : G \rightarrow S_A$ is a group homomorphism. Then we get an action of G on A by

$$g \cdot a := \phi(g)(a) \in A \quad (5.5)$$

Moreover, the associated $G \rightarrow S_A$ which $g \mapsto \sigma_g$ is just ϕ .

Definition 5.1.2 (Trivial Homomorphism)

For G, H groups, the trivial homomorphism $\phi : G \rightarrow H$ is $\phi(g) = 1_H$ for all $g \in G$, i.e. $\ker(\phi) = G$.

Example 5.1.1

1. Trivial action: Every group G acts on every set A by $g \cdot a = a$.

$$\bullet \quad g_1 \cdot (g_2 \cdot a) = g_1 \cdot a = a = (g_1 g_2) \cdot a$$

The associated permutation representation $G \rightarrow S_A$ is the trivial homomorphism, i.e. $\sigma_g = \text{id}_A : A \rightarrow A$ which $a \mapsto a$ for all $g \in G$. Everything is in the kernel of the action.

Definition 5.1.3 (Kernel of the Action)

Suppose G acts on A and $\phi : G \rightarrow S_A$ which $g \mapsto \sigma_g$ is the corresponding homomorphism. We call $\ker(\phi) \leq G$ the kernel of the action of G on A . It is the set of elements in G that acts trivially on A .

Example 5.1.2

2. Grenary set A, S_A acts on A by

$$\sigma \cdot a := \sigma(a) \quad (5.6)$$

The corresponding homomorphism

$$S_A \rightarrow S_A \quad (5.7)$$

is the identity homomorphism, i.e.

$$\sigma_\tau = \tau \quad \text{any } \tau \in S_A \quad (5.8)$$

3. V is an \mathbb{R} -vector space. Then scalar multiplication

$$\begin{aligned}\mathbb{R}^\times \times V &\rightarrow V \\ (r, v) &\mapsto rv \\ r(sv) &= (rs)v \quad \text{is a vector space axiom}\end{aligned}$$

Note: \mathbb{R}^\times is the non-zero reals

So $(\mathbb{R}^\times, \times)$ acts on the vector space V .

The associated homomorphism from $\mathbb{R}^\times \rightarrow S_V$ is injection if V is nontrivial (exercise).

Note (Bad notation last lecture)

$$\mathbb{Z}_5^\times = \{1, 2, 3, 4\} \quad \text{with } \otimes \tag{5.9}$$

but

$$\mathbb{Z}_6 \setminus \{0\} \tag{5.10}$$

is not a group.

(\mathbb{Z}_6^\times is something else, see homework)

Chapter 6

Lecture 5: Sep 20, 2017

6.1 Logistics

Note (Homework 1)

Q5 (\mathbb{Z}_n, \oplus)

Note (Midterm Confusion)

- check email

Note

symmetric group (S_A) = permutation groups

6.2 More on Group Actions

We continue with two important group actions:

Example 6.2.1

4. *Groups acting on themselves by left multiplication:*

Let G be a group. G acts on itself by

$$g \in G, a \in G \quad g \cdot a = ga \tag{6.1}$$

associativity of group operation

$$\iff g \cdot (h \cdot a) = (gh) \cdot a \tag{6.2}$$

group axiom about identity

$$\implies 1 \cdot a = a \tag{6.3}$$

Exercise 6.2.1

Multiplication on the right is not a group action.

We have a corresponding permutation representation

$$G \rightarrow S_G \tag{6.4}$$

Proposition 6.2.1

$G \rightarrow S_G$ is injective.

Proof

$$\forall g, h \in G \ \sigma_g = \sigma_h$$

In particular, $\sigma_g(1) = \sigma_h(1)$.

$$\begin{aligned} \sigma_g(1) &= g \cdot 1 = g1 = g \\ \sigma_h(1) &= h \cdot 1 = h1 = h \end{aligned}$$

Therefore, $g = h$.

In particular, the kernel of this action is trivial, i.e., the subgroup $\{1\} \leq G$.

Definition 6.2.1 (Faithful)

A group action G on A is said to be faithful if the kernel of the action is trivial, i.e. the only group element fixing all of A pointwise is the identity element 1_G .

Definition 6.2.2 (Image Set)

Suppose $\phi : G \rightarrow H$ is a group homomorphism. Then $I_m(\phi) = \{h \in H : h = \phi(g) \text{ for some } g \in G\}$

Lemma 6.2.1 (Lemma 9)

Suppose $\phi : G \rightarrow H$ is a group homomorphism.

1. $I_m(\phi) \leq H$
2. *If ϕ is injective then it induces an isomorphism*

$$G \xrightarrow{\sim} I_m(\phi) \tag{6.5}$$

Proof

1. Suppose $h_1, h_2 \in I_m(\phi) \implies \exists g_1, g_2 \in G \phi(g_1) = h_1 \phi(g_2) = h_2$.

$$\begin{aligned} h_1 h_2 &= \phi(g_1) \phi(g_2) \\ &= \phi(g_1 g_2) \end{aligned}$$

Since $g_1 g_2 \in G \implies h_1 h_2 \in I_m(\phi)$

Also $\phi(1_G) = 1_H$ so

$$\begin{aligned} 1_G \in I_m(\phi) &\implies I_m(\phi) \neq \emptyset \\ &\implies I_m(\phi) \leq H \end{aligned}$$

2. $I_m(\phi)$ is a group and

$$\phi : G \rightarrow I_m(\phi) \tag{6.6}$$

is a bijection group homomorphism. Hence

$$G \simeq I_m(\phi) \tag{6.7}$$

Corollary 6.2.1 (Cayley's Theorem)

Every group is isomorphic to a subgroup of some permutation. Moreover, if a group G is finite, i.e. $|G| = n$ for some n , then G is isomorphic to a subgroup S_n .

Proof

Consider the action of G on itself by left multiplication. By [Permutation Representation 6.2.1](#), this gives us an injective group homomorphism $\phi : G \rightarrow S_G$. By [Lemma 9 6.2.1](#), $G \simeq I_m(\phi) \leq S_G$. Moreover, if $|G| = n$, then $S_G \simeq S_n := S_{\{1,2,\dots,n\}}$.

Example 6.2.2

5. Groups acting on themselves by conjugation

Definition 6.2.3 (Conjugation)

For a group G , $\forall g, h \in G$. Then the conjugate of h by g is the element ghg^{-1} .

Remark

If G is abelian, then $ghg^{-1} = hgg^{-1} = h$, i.e. a conjugation does nothing in abelian groups. Thus the notion of a conjugation is only interesting for non-abelian groups.

Example 6.2.3

Conjugation as an action of G on itself, i.e. given $g \in G$, $a \in G$, $g \cdot a := gag^{-1}$

Given $g, h \in G$, $g \cdot (h \cdot a) = g \cdot (hah^{-1}) = g(hah^{-1})g^{-1} = (gh)a(h^{-1}g^{-1}) = (gh)a(gh)^{-1} = (gh) \cdot a$.

$$1 \cdot a = 1a1^{-1} = a$$

Example 6.2.4

We get another permutation representation

$$\psi : G \rightarrow S_G \tag{6.8}$$

coming from G acting on itself by conjugation.

$$\ker(\psi) = \{g \in G : ga = ag \ \forall a \in G\}$$

$$\text{Note that } gag^{-1} = 1 \iff ga = ag$$

If G is abelian, this is the trivial action.

Definition 6.2.4 (Center of the Group)

For any group G ,

$$Z(G) = \{g \in G : \forall h \in G \ gh = hg\} \tag{6.9}$$

is called the **center of G** .

Remark

1. If G is abelian, then $Z(G) = G$.
2. $Z(G) \leq G$ since $Z(G) = \ker(\psi)$ which is the kernel of the action of G on itself by conjugation.

Chapter 7

Lecture 7: Sep 22, 2017

7.1 Cosets

Definition 7.1.1 (Left Cosets)

Let G is a group and $H \leq G$.

A **left coset** of H is a set of the form $aH = \{a \cdot h : h \in H\}$ for some $a \in G$

Example 7.1.1

$$\begin{aligned} G &= S_3 \text{em} \quad (= D_6) \\ &= \langle r, s \mid r^3 = s^2 = 1, r^2s = sr \rangle \\ &= \{1, r, r^2, s, rs, r^2s\} \end{aligned}$$

$$H = \{1, s\} = \begin{array}{l} \text{subgroup generated by } s \\ \text{smallest subgroup of } G \text{ containing } s \\ (1 \in H, ss=1 \in H) \end{array}$$

The left cosets of H are:

$$\begin{aligned} 1H &= H & &= \{1, s\} \\ rH &= \{r, rs\} \\ r^2H &= \{r^2, r^2s\} \\ sH &= \{s, s^2\} & &= \{1, s\} \\ rsH &= \{rs, rs^2\} & &= \{r, rs\} \\ r^2sH &= \{r^2s, r^2s^2\} & &= \{r^2, r^2s\} \end{aligned}$$

Observe that

1. $1H = sH = \{1, s\}$
 $rH = rsH = \{r, rs\}$
 $r^2H = r^2sH = \{r^2, r^2s\}$
2. $aH \neq bH \implies aH \cap bH = \emptyset$
3. $\bigcup_{a \in G} aH = G$
4. All of the left cosets have the same size.

Proposition 7.1.1 (Proposition 11)

Let G be a group and $H \leq G$, and $a, b \in G$. Then $aH = bH \iff a \in bH$.

Proof

Suppose $aH = bH$, then $a = a1$ and since $1 \in H$, $a1 \in aH = bH$.

Suppose $a \in bH$. Then $a = bh$ for some $h \in H$. For any $h' \in H$, $ah' = bhh'$ and since $hh' \in H$, $ah' \in bH$. This implies that $aH \subset bH$.

For any $h'' \in H$, since $a = bh$ thus we have that $bh'' = ah^{-1}h''$ and since $h^{-1}h'' \in aH$. Thus $bH \subset aH$.

Therefore, $aH = bH$.

Corollary 7.1.1

1. $aH = bH \iff b^{-1}a \in H$
 $aH = H \iff a \in H$
2. $aH \cap bH \neq \emptyset \implies aH = bH$

Proof

1.

$$\begin{aligned}
 aH = bH &\iff a = bh \text{ for some } h \in H \\
 &\iff b^{-1}a = h \text{ for some } h \in H \\
 &\iff b^{-1}a \in H
 \end{aligned}$$

2. Suppose $c \in aH \cap bH$. Then $c \in aH \implies cH = aH$ and $c \in bH \implies cH = bH$.
Therefore $aH = bH$.

Proposition 7.1.2

$$\bigcup_{a \in G} aH = G \quad (7.1)$$

Proof

$$\bigcup_{a \in G} aH \subset G$$

For any $g \in G$, $g \in gH \subset \bigcup_{a \in G} aH \implies G \subset \bigcup_{a \in G} aH$

Proposition 7.1.3

Let G be a group, $H \leq G$, $a \in G$. Then the map

$$\begin{aligned} \sigma_a : H &\rightarrow aH \\ h &\mapsto ah \end{aligned}$$

is a bijection of sets.

Proof

By definition of left cosets, aH , σ_a is surjective, since $ah = \sigma_a(h)$.

If $\sigma_a(h_1) = \sigma_a(h_2)$ for $h_1, h_2 \in H$, then $ah_1 = ah_2$, which then $h_1 = h_2$. Thus σ_a is injective.

Corollary 7.1.2

If H is finite, then $|H| = |aH|$ for any $a \in G$. This means that all left cosets have the same size.

Remark

We now know that

- All left cosets have the same size.
- G is the union of all left cosets, and furthermore, it can be partitioned into all the distinct left cosets. Thus if G is finite, then

$$|G| = |H|(\text{number of distinct left cosets}) \quad (7.2)$$

We call the number of distinct left cosets of H the index of H in G , denoted by $[G : H]$.

Example 7.1.2

$$\begin{aligned}
G &= S_3 \\
H &= \{1, s\} \\
rH &= \{r, rs\} \\
r^2H &= \{r^2, r^2s\} \\
G &= \{1, s\} \cup \{r, rs\} \cup \{r^2, r^2s\} \\
|G| &= 2 + 2 + 2 \\
[G : H] &= 3
\end{aligned}$$

Corollary 7.1.3 (12. Lagrange's Theorem)

If G is a finite group and $H \leq G$, then

$$|H| \mid |G| \quad (7.3)$$

Proof

$$|G| = |H|[G : H] \text{ and } [G : H] \in \mathbb{Z}$$

Definition 7.1.2 (Order)

The order of a group G is the cardinality of G .

Example 7.1.3

In our previous example, note that S_3 cannot have a subgroup of order 4.

Example 7.1.4 (Subgroups of S_3)

$$S_3 = \{1, r, r^2, s, rs, r^2s\}$$

$$\begin{aligned}
H \leq S_3 &\implies |H| = 1 \text{ or } 2 \text{ or } 3 \text{ or } 6 \\
|H| = 1 &\implies H = \{1\} \quad |H| = 6 \implies H = S_3 \\
|H| = 2 &\implies H = \{1, s\} \text{ or } \{1, rs\} \text{ or } \{1, r^2s\} \\
|H| = 3 &\implies
\end{aligned}$$

Suppose $|H| = 2$. Then $H = \{1, a\}$ for some $a \in G$.

$$a^2 \in H \implies a^2 = 1 \text{ or } a^2 = a$$

but $a^2 = a \implies a = 1$. So $\{1, s\} \leq G \iff a^2 = 1$ since $a^2 = 1 \implies a^{-1} = a \in H$.

Suppose $|H| = 3$. Can H contain S ? No, since if $s \in H$, then $\{1, s\} \subset H$ and $\{1, s\}$, but by 12. Lagrange's Theorem 7.1.3 $2 \nmid 3$. Likewise, $rs, r^2s \notin H$. Thus

$$\begin{aligned} H &= \{1, r, r^2\} \\ &= \text{smallest subgroup containing } r \\ &= \text{subgroup generated by } r \end{aligned}$$

Converse of 12. Lagrange's Theorem 7.1.3 is false.

There exists a finite group G and a positive integer $m \mid |G|$ such that G does not have a subgroup of order m .

Example 7.1.5

$$\begin{aligned} G &= A_4 \\ &= \text{group of symmetries of a regular tetrahedron, order 12} \\ &= \{1, (1, 2)(3, 4), (1, 4)(2, 3), (1, 3)(2, 4), (2, 3, 4), (4, 3, 2), (1, 3, 4), (1, 2, 4), (4, 2, 1), (1, 2, 3), (3, 2, 1)\} \end{aligned}$$

A_4 has no subgroup of order 6 (exercise)

Chapter 8

Lecture 8: Sep 25, 2017

8.1 Continuing with Cosets

We can similarly define right cosets as

$$\forall a \in G \ H \leq G \quad H_a := \{ha : h \in H\} \quad (8.1)$$

Definition 8.1.1 (Set of Left Cosets)

Let $H \leq G$

$$G/H := \text{set of all left cosets of } H \text{ in } G \quad (8.2)$$

$$:= \{aH : a \in G\} \quad (8.3)$$

which we call as $G \bmod H$.

Note

We have a natural action of G on G/H given by: $\forall g \in G$

$$g \cdot aH := (ga)H \quad (8.4)$$

$$\text{The kernel of this action} = \{g \in G : gaH = aH \ \forall a \in G\} \quad (8.5)$$

$$= \{g \in G : a^{-1}ga \in H \ \forall a \in G\} \quad (8.6)$$

Example 8.1.1

$G = (\mathbb{Z}, +)$ $d > 0$ $H = d\mathbb{Z} \leq \mathbb{Z}$

$$\mathbb{Z}/d\mathbb{Z} = \{a + d\mathbb{Z} : a \in \mathbb{Z}\} \quad (8.7)$$

$$= \{[a]_d : a \in \mathbb{Z}\} \quad (8.8)$$

where $[a]_d = \{n \in \mathbb{Z} : n \equiv a \pmod{d}\} = a + d\mathbb{Z}$.

So the congruence class of $a \pmod{d}$ is just the left coset of $d\mathbb{Z}$ by a .

This has a natural group structure: $[a]_d + [b]_d = [a + b]_d$.

So left cosets generalises congruences classes to arbitrary groups.

We now try to put a natural group structure on G/H :

$$(aH)(bH) := abH \quad (8.9)$$

But in general, given any

$$\begin{aligned} X, Y &\subseteq G \\ XY &:= \{xy : x \in X, y \in Y\} \subseteq G \end{aligned} \quad (8.10)$$

Note that Equation 8.10 is a **natural definition**.

If

$$\begin{aligned} X = aH \quad Y = bH &\implies XY = \{ah_1bh_2 : h_1, h_2 \in H\} \\ abH &= \{abh : h \in H\} \\ abH &\subseteq XY \\ XY &\not\subseteq abH \quad \text{in general} \end{aligned}$$

If G is abelian, then $XY = abH$ and Equation 8.9 is a good definition.

G abelian: $XY \subseteq abH$

Proof

$ah_1bh_2 = abh_1h_2$ then we can just take $h = h_1h_2$.

All we need for $abH = (aH)(bH)$ is that for all $h_1 \in H$, for some $h' \in H$, $h_1b = bh'$. Then

$$ah_1bh_2 = abh'h_2 = abh$$

by $h = h'h_2 \in H$.

Definition 8.1.2 (Normal Subgroup)

A subgroup $H \leq G$ is normal if $\forall b \in H \quad Hb = bH$, i.e.

$$\forall h \in H \quad \exists h' \in H \quad hb = bh' \quad (8.11)$$

We denote a normal subgroup by $H \triangleleft G$.

Lemma 8.1.1 (Lemma 13) $H \leq G$. TFAE:

1. $H \triangleleft G$
2. $\forall b \in G \ b^{-1}Hb \subseteq H$
3. $b^{-1}Hb = H$

Proof1 \iff 3 Suppose $H \triangleleft G$ $b \in G$

$$Hb = \{hb : h \in H\}$$

$$bH = \{bh : h \in H\}$$

$$b^{-1}Hb = \{b^{-1}hb : h \in H\}$$

$$Hb = bH$$

$$\implies b^{-1}Hb = H$$

3 \implies 2 is straightforward2 \implies 3 Apply 2 to b^{-1} , so that

$$(b^{-1})^{-1}H(b^{-1}) \subseteq H$$

$$bHb^{-1} \subseteq H$$

$$Hb^{-1} \subseteq b^{-1}H$$

$$H \subseteq b^{-1}Hb$$

Therefore, $\forall b \in G \ H = b^{-1}Hb$ For 1 \implies 3, we needed the following**Definition 8.1.3** $X \subseteq G$

$$bX := \{bx : x \in X\}$$

$$Xb := \{xb : x \in X\}$$

Exercise 8.1.1

- $X = Y \implies bX = bY$
- $a(bX) = (ab)X$

Lemma 8.1.2 (Lemma 14)

If $H \triangleleft G$ and $a, b \in G$, then

$$\underbrace{(aH)(bH)}_{\text{set product}} = (ab)H \quad (8.12)$$

Proof

is an exercise (yay) - it is what motivated the definition of a normal subgroup

Remark

Suppose $H \triangleleft G$. Let G act on G/H . The kernel of the action is H .

Proof

$$\begin{aligned} h \in H \quad a \in G \quad \exists h' \in H \\ haH = ah'H \quad (\text{since } H \triangleleft G) = aH \end{aligned}$$

So h is in the kernel of the action.

Suppose $g \in G$ is in the kernel of the action. So

$$\forall a \in G \quad gaH = aH$$

in part, take $a = 1$. So $gH = H \implies g \in H$.

Chapter 9

Lecture 9: Sep 27, 2017