PMATH352W18 Complex Analysis - Class Notes

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1 *Lecture 1 Jan 3rd 2018*

1.1 *Complex Numbers and Their Properties*

Definition 1.1.1 (Complex Number, Complex Plane)
*A **complex number** is a vector in $\mathbb{R}^2$. The **complex plane**, denoted by $\mathbb{C}$, is a set of complex numbers,*

$$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

In $\mathbb{C}$, we usually write

$$0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad 1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$i = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad x = \begin{pmatrix} x \\ 0 \end{pmatrix}$$
$$iy = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

where $x, y \in \mathbb{R}$. Consequently, we have that

$$x + iy = x + yi = \begin{pmatrix} x \\ y \end{pmatrix}$$

*If for $x, y \in \mathbb{R}$, $z = x + iy$, then x is called the **real part** of z and y is called the **imaginary part** of z, and we write*

$$\mathrm{Re}(z) = x \quad \mathrm{Im}(z) = y.$$

Note

- *It is easy to see how $\mathbb{R}$ is a subset of $\mathbb{C}$.*
- *Complex Numbers of the form $\left(\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right)$ where $y \in \mathbb{R}$ are called **purely imaginary numbers**.*

- *Certain authors may prefer to denote* $i = \binom{0}{1}$.

Definition 1.1.2 (Sum and Product)
We define the sum of two complex numbers to be the usual vector sum, i.e.

$$\begin{aligned}(a+ib)+(c+id) &= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \begin{pmatrix} a+c \\ b+d \end{pmatrix} \\ &= (a+c)+i(b+d)\end{aligned}$$

where $a, b, c, d \in \mathbb{R}$.

We define the product of two complex numbers by setting $i^2 = -1$, *and by requiring the product to be* ***commutative, associative, and distributive*** *over the sum. In this setup, we have that*

$$\begin{aligned}(a+ib)(c+id) &= ac + iad + ibc + i^2bd \\ &= (ac-bd)+i(ad+bc) \qquad (1.1)\end{aligned}$$

Note
It is interesting to note that ***any complex number times zero is zero***, *just like what we have with real numbers.*

$$\forall z = x + iy \in \mathbb{C}\ x, y \in \mathbb{R}\ \ 0 \in \mathbb{C}$$
$$z \cdot 0 = (x+iy)(0+i0) = 0 + i0 = 0$$

Example 1.1.1
Let $z = 2 + i, w = 1 + 3i$. *Find* $z + w$ *and* zw.

$$\begin{aligned}z + w &= (2+i)+(1+3i) \\ &= 3 + 4i\end{aligned}$$

$$\begin{aligned}zw &= (2+i)(1+3i) \\ &= (2-3)+i(6+1) \quad \textit{By Equation (1.1)} \\ &= -1 + 7i\end{aligned}$$

Example 1.1.2
Show that every non-zero complex number has a ***multiplicative inverse***, z^{-1}, *and find a formula for this inverse.*

Let $z = a + ib$ where $a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$. Then

$$\begin{aligned} & z(x+iy) = 1 \\ \iff & (ax - by) + i(ay + bx) = 1 \\ \iff & \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \iff & \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \iff & \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \iff & \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a \\ -b \end{pmatrix} \\ \iff & x + iy = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2} \end{aligned}$$

Therefore, we have that the formula for the inverse is

$$(a + ib)^{-1} = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2} \tag{1.2}$$

Notation

For $z, w \in \mathbb{C}$, we write

$$\begin{aligned} -z &= -1z & w - z &= w + (-z) \\ \tfrac{1}{z} &= z^{-1} & \tfrac{w}{z} &= wz^{-1} \end{aligned}$$

Example 1.1.3

Find $\frac{(4-i)-(1-2i)}{1+2i}$.

$$\begin{aligned} \frac{(4-i)-(1-2i)}{1+2i} &= \frac{3+i}{1+2i} \\ &= (3+i)(\frac{1}{5} - i\frac{2}{5}) \\ &= 1 - i \end{aligned}$$

Note

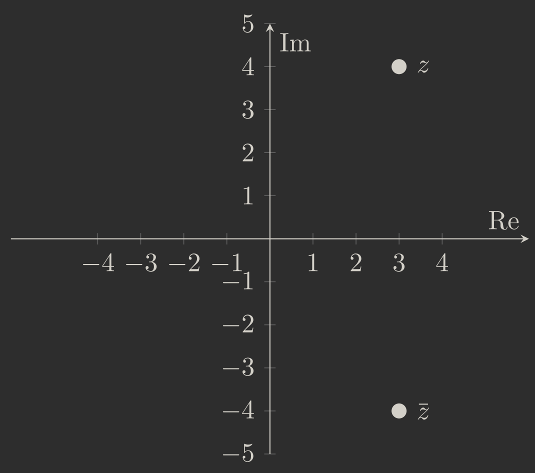
The set of complex numbers is a ***field*** *under the operations of addition and multiplication. This means that $\forall u, v, w \in \mathbb{C}$,*

$$\begin{aligned} u + v &= v + u & uv &= vu \\ (u + v) + w &= u + (v + w) & (uv)w &= u(vw) \\ 0 + u &= u & 1u &= u \\ u + (-u) &= 0 & uu^{-1} &= 1, \ u \neq 0 \\ u(v + w) &= uv + uw \end{aligned}$$

Since the distributive law holds for complex numbers, note that the ***binomial expansion works*** *for $(w+z)^n$ where $w, z \in \mathbb{C}$ and $n \in \mathbb{N}$. (I did not verify if this is still true for when $n \in \mathbb{R}$.)*

Definition 1.1.3 (Conjugate)
If $z = x + iy$ where $x, y \in \mathbb{R}$, then the ***conjugate of z*** *is given by $\bar{z} = x - iy$*

Example 1.1.4
Let $z = 3 + 4i$. Then the $\bar{z} = 3 - 4i$. Represented in the complex plane, we have the following:



We observe that on the complex plane, the conjugate of a complex number is simply its reflection on the real axis.

Definition 1.1.4 (Modulus)
We define the ***modulus*** *(length, magnitude) of $z = x+iy \in \mathbb{C}, x, y \in \mathbb{R}$, to be*

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}. \tag{1.3}$$

Note
Note that this definition is consistent with the notion of the absolute value in real numbers when z is a real number, since if $y = 0$, $|z| = |x + i0| = \sqrt{x^2} = \pm x$.

Note
For $z, w \in \mathbb{C}$ and $n \in \mathbb{N}$, we have

$$\begin{array}{ccc} \bar{\bar{z}} = z & z + \bar{z} = 2\,\mathrm{Re}(z) & z - \bar{z} = 2i\,\mathrm{Im}(z) \\ z\bar{z} = |z|^2 & |z| = |\bar{z}| & \overline{z \pm w} = \bar{z} \pm \overline{w} \\ \overline{zw} = \bar{z}\overline{w} & |zw| = |z|\,|w| & \bar{z}^n = \overline{z^n} \end{array}$$

but note that $|z + w| \neq |z| + |w|$.

Also, note that the last equation is a generalization of the ***highlighted equation****.*

Note

While inequalities such as $z_1 < z_2$*, where* $z_1, z_2 \in \mathbb{C}$*, are meaningless unless if both of them are real,* $|z_1| < |z_2|$ *means that the point* z_1 *in the complex plane is closer to the origin than the point* z_2*.*

Proposition 1.1.1 (Basic Inequalities)

1. $|\mathrm{Re}(z)| \leq |z|$
2. $|\mathrm{Im}(z)| \leq |z|$
3. $|z + w| \leq |z| + |w|$ *Triangle Inequality*
4. $|z + w| \geq \big|\, |z| - |w| \,\big|$ *Inverse Triangle Inequality*

Proof

Note that $|z|^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$ *and that we can express* $|x| = \sqrt{x^2}$ *for any* $x \in \mathbb{R}$*. 1 and 2 immediately follows from that.*

To prove 3, we have that

$$\begin{aligned} |z+w|^2 &= (z+w)(\bar{z} + \overline{w}) \\ &= |z|^2 + |w|^2 + (w\bar{z} + \overline{w}z) \\ &= |z|^2 + |w|^2 + 2\,\mathrm{Re}(w\bar{z}) \\ &\leq |z|^2 + |w|^2 + 2\,|w\bar{z}| \quad \textit{by 1} \\ &= |z|^2 + |w|^2 + 2\,|wz| \quad \textit{since } |w\bar{z}| = |w|\,|\bar{z}| \textit{ and } |z| = |\bar{z}| \\ &= (|z| + |w|)^2 \end{aligned}$$

To prove 4, note that

$$|z| = |z + w - w| \leq |z + w| + |w| \tag{1.4}$$

$$|w| = |w + z - z| \leq |z + w| + |z| \tag{1.5}$$

Observe that

$$\begin{aligned} &\textit{Equation } (1.4) \implies |z| - |w| \leq |z + w| \\ &\textit{Equation } (1.5) \implies |w| - |z| \leq |z + w| \end{aligned}$$

Thus, we have that

$$|z+w| \geq \big|\, |z| - |w| \,\big|$$

as required. □

Item 3 in Proposition 1.1.1 can be generalized by the means of mathematical induction to sums involving any finite number of terms, as:

$$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n| \tag{1.6}$$

where $n \in \mathbb{N} \setminus \{0, 1\}$.

To note the induction proof, when $n = 2$, Equation (1.6) is just Item 3. If Equation (1.6) is true for when $n = m$ where $m \in \mathbb{N} \setminus \{0, 1\}$, $n = m + 1$ is also true since by Item 3,

$$\begin{aligned} |(z_1 + z_2 + \ldots + z_m) + z_{m+1}| &\leq |z_1 + z_2 + \ldots + z_m| + |z_{m+1}| \\ &\leq (|z_1| + |z_2| + \ldots + |z_m|) + |z_{m+1}| \,. \end{aligned}$$

The distance between two points $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}, x_1, x_2, y_1, y_2 \in \mathbb{R}$ is $|z_1 - z_2|$, since $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 (y_1 - y_2)^2}$ is our usual notion of the Euclidean distance of two points on a plane.

Also, note that

$$z_1 - z_2 = z_1 + (-z_2)$$

and thus if we apply our knowledge of vector representation, $z_1 - z_2$ is the directed line segment from the point z_2 to z_1.

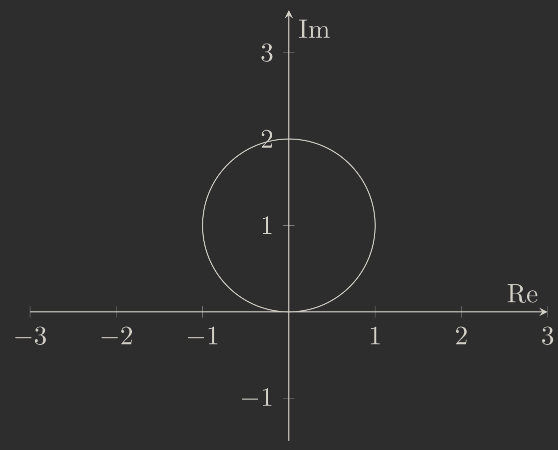
With the notion of a "distance" set on the complex plane, we can now explore upon points lying on a circle with a center z_0 and radius R, which satisfies the equation

$$|z - z_0| = R.$$

We may simply refer to this set of points as the circle $|z - z_0| = R$.

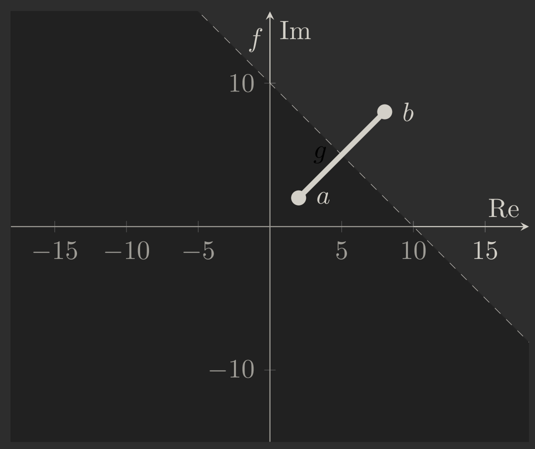
Example 1.1.5

We may describe a set $\{z \in \mathbb{C} : |z - i| = 1\}$ *as follows:*



Let $a, b \in \mathbb{C}$ *describe the set* $\{z \in \mathbb{C} : |z - a| < |z - b|\}$.

Suppose the following coordinates for a *and* b *are arbitrary,*

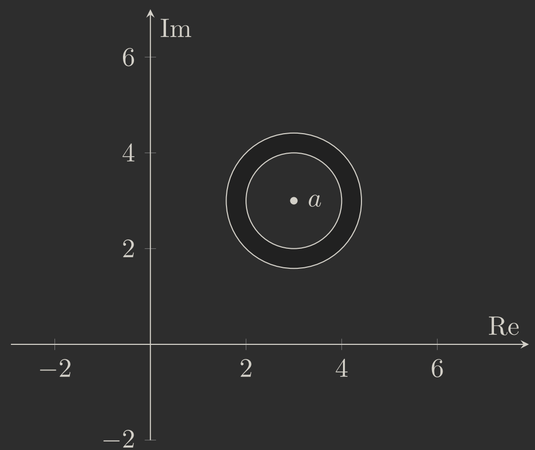


In the above, g *is the line segment that connects the points* a *and* b *on the complex plane, while* f *is the perpendicular bisector of the line segment* g. *The area described by the set* $\{z \in \mathbb{C} : |z - a| < |z - b|\}$ *is the shaded area which is below* f.

2 Lecture 2 Jan 5th 2018

2.1 Complex Numbers and Their Properties (Continued)

Example 2.1.1
Let $a \in \mathbb{C}$. Describe the set $\{z \in \mathbb{C} : 1 < |z - a| < 2\}$.



Example 2.1.2
Show that every non-zero complex number has exactly two complex square roots, and find a formula for the square roots.

Let $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$, and let $w = u + iv, u, v \in \mathbb{R}$. Then

$$\begin{aligned} w^2 = z &\iff (u + iv)^2 = x + iy \\ &\iff (u^2 - v^2) + i(2uv) = x + iy \\ &\iff x = u^2 + v^2 \quad \textit{and} \qquad (2.1) \\ &\quad\;\; y = 2uv \qquad (2.2) \end{aligned}$$

Square both sides of Equation (2.2), and thus we have $y^2 = 4u^2v^2$.

Multiply Equation (2.1) by $4u^2$, and we get

$$\begin{aligned} 4u^2x &= 4u^4 - 4u^2v^2 = 4u^4 - y^2 \\ \iff 0 &= 4u^4 - 4u^2x - y^2 \\ \iff u^2 &= \frac{4x \pm \sqrt{16x^2 + 16y^2}}{8} \\ &= \frac{x \pm \sqrt{x^2 + y^2}}{2} \end{aligned}$$

Suppose $y \neq 0$. Note that $x < \sqrt{x^2 + y^2}$. Thus $u^2 = \frac{x+\sqrt{x^2+y^2}}{2} \implies u = \left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}$.

Similarly, we can get

$$v = \pm \left(\frac{-x + \sqrt{x^2 + y^2}}{2} \right)^{\frac{1}{2}}$$

Note that all four choices of signs satisfy Equation (2.1). If $y > 0$, then u and v are either both positive or both negative by Equation (2.2).

Suppose $y = 0$. Then we have

$$w^2 = z = x$$

Therefore, we get

$$w = \begin{cases} \pm \left[\left(\frac{x+\sqrt{x^2+y^2}}{2} \right)^{\frac{1}{2}} + i \left(\frac{-x+\sqrt{x^2+y^2}}{2} \right)^{\frac{1}{2}} \right] & y > 0 \\ \pm \left[\left(\frac{x+\sqrt{x^2+y^2}}{2} \right)^{\frac{1}{2}} - i \left(\frac{-x+\sqrt{x^2+y^2}}{2} \right)^{\frac{1}{2}} \right] & y < 0 \\ \pm\sqrt{x} & y = 0, x > 0 \\ \pm i\sqrt{x} & y = 0, x < 0 \end{cases}$$

Remark
Let $z \in \mathbb{C}$. The notation $\sqrt{z}$ may represent either one of the square roots of z or both of the square roots, i.e. ***it is possible that $\sqrt{z}$ represents a set****.*

Exercise 2.1.1
Is it always okay for complex numbers such that $\sqrt{zw} = \sqrt{z}\sqrt{w}$, for $z, w \in \mathbb{C}$?

No. For example, consider $z = w = -1$. Then we have

$$\sqrt{zw} = \sqrt{1} = \pm 1$$

while

$$\sqrt{z}\sqrt{w} = i \cdot i = -1$$

and thus

$$\sqrt{zw} \neq \sqrt{z}\sqrt{w}.$$

Example 2.1.3

Find the values of $\sqrt{3-4i}$.

By Example 2.1.2,

$$\begin{aligned} \sqrt{3-4i} &= \pm\left(\sqrt{\frac{3+\sqrt{9+16}}{2}} - i\sqrt{\frac{-3+\sqrt{9+16}}{2}}\right) \\ &= \pm(2-i) \end{aligned}$$

Remark

The quadratic formula holds for complex polynomials, i.e.

$$\forall a, b, c \in \mathbb{C} \quad a \neq 0 \quad \forall z \in \mathbb{C}\ az^2 + bz + c = 0,$$

the solution for z is given by

$$z_{1,2} = \frac{-b + \sqrt{b^2 - 4ac}}{b} \tag{2.3}$$

The following is a short proof.

Proof

$$\begin{aligned} az^2 + bz + c = 0 &\iff z^2 + \frac{b}{a}z + \frac{c}{a} = 0 \\ &\iff z^2 + \frac{b}{a}z + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \\ &\iff \left(z + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2} \\ &\iff z = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

(Personal Note: where did the $-$ for the supposed $\pm$ go? Or should

it really be $\pm$?)

Example 2.1.4

Solve $iz^2 - (2+3i)z + 5(1+i) = 0$.

$$\begin{aligned} z &= \frac{2+3i+\sqrt{(2+3i)^2 - 4i[5(1+i)]}}{2i} \\ &= \frac{2+3i+\sqrt{-5+12i-20i+20}}{2i} \\ &= \frac{2+3i+\sqrt{15+8i}}{2i} \end{aligned}$$

Note that by Example 2.1.2,

$$\begin{aligned} \sqrt{15-8i} &= \pm\left[\sqrt{\frac{15+\sqrt{225+64}}{2}} - i\sqrt{\frac{-15+\sqrt{225+64}}{2}}\right] \\ &= \pm\left[\sqrt{\frac{15+17}{2}} - i\sqrt{\frac{-15+17}{2}}\right] \\ &= \pm(4-i) \end{aligned}$$

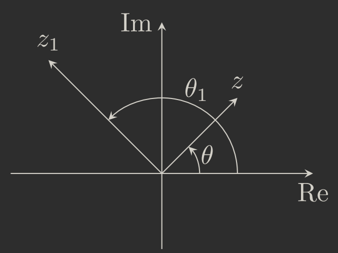
Thus we have

$$\begin{aligned} z &= \frac{2+3i+\sqrt{15+8i}}{2i} \\ &= \frac{2+3i\pm(4-i)}{2i} \\ &= (6+2i)\left(-\frac{1}{2}i\right) \text{ or } (-2+4i)\left(-\frac{1}{2}i\right) \quad \text{by Example 1.1.2} \\ &= (1-3i) \text{ or } (2+i) \end{aligned}$$

3 Lecture 3 Jan 8th 2018

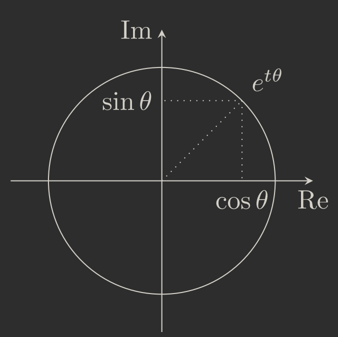
3.1 Complex Numbers and Their Properties (Continued 2)

Definition 3.1.1 (Argument of a Complex Number)
*Let $z \in \mathbb{C} \setminus \{0\}$. The **argument** (or the angle) of z, denoted by $\arg z$, $\operatorname{Arg} z$, or simply $\theta = \theta(z)$, is the angle modulo 2π (i.e. $0 \leq \theta < 2\pi$) between the vector defining z and the positive real axis (in the counterclockwise direction).*



Notation
*Let $e^{i\theta} := \cos\theta + i\sin\theta$. Note that this definition, called **Euler's formula**, can be derived by the extending the Taylor expansion of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for when $x \in \mathbb{C}$ (the sum of the real parts of the expansion is the Taylor expansion of cosine while the imaginary part for sine).*

Now $e^{i\theta}$ is on the unit circle.



Remark

If $z = 0$, the coordinate θ is undefined, and so it is implied that $z \neq 0$ whenever we use the polar form.

Example 3.1.1

Some examples of $\theta \in [0, 2\pi)$:

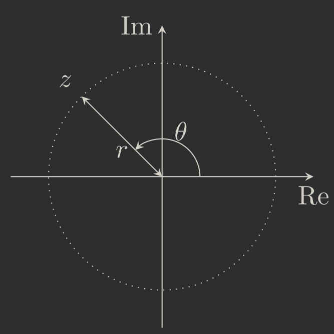
$$\begin{aligned} e^{i\frac{\pi}{4}} &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\frac{\pi}{2}} &= i \\ e^{i\frac{3\pi}{4}} &= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\pi} + 1 &= 0 \end{aligned}$$

Remark

$$\forall k \in \mathbb{Z} \;\; \forall \theta \in \mathbb{R} \;\; e^{i\theta} = e^{i(\theta + 2\pi k)}$$

Remark

The complex number $re^{i\theta}$, where $r > 0, \theta \in [0, 2\pi)$, represents the complex number with modulus r and argument θ.



Therefore, $\forall z \in \mathbb{C}$, we can express

$$z := |z| \, e^{i \operatorname{Arg} z}. \tag{3.1}$$

With that, we now have two representations of a complex number:

- **Cartesian representation**: $z = x + iy$ where $x = \operatorname{Re}(z)$ and $y = \operatorname{Im}(z)$
- **Polar representation**: $z = re^{i\theta}$ where $r = |z|$ and $\theta = \operatorname{Arg} z \in [0, 2\pi)$

To convert between the two representations, we have the following equations:

Polar $\to$ Cartesian:

$$x = r \cos\theta \quad y = r \sin\theta \tag{3.2}$$

Cartesian $\to$ Polar:

$$\begin{aligned} r &= |z| \\ x \neq 0 &\implies \tan\theta = \frac{y}{x} \\ x = 0 &\implies \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \end{aligned} \tag{3.3}$$

On another note,

$$z = re^{i\theta} \implies \bar{z} = re^{-i\theta}$$

and

$$z \neq 0 \implies \frac{1}{z} = \frac{1}{r}e^{-i\theta} \tag{3.4}$$

Remark

$$\begin{gathered} \forall r_1, r_2 \in \mathbb{R}\ \forall \theta_1, \theta_2 \in [0, 2\pi) \\ z_1 := r_1 e^{i\theta_1} \quad z_2 := r_2 e^{i\theta_2} \end{gathered}$$

Then

$$z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

Note that $e^{ix}e^{iy} = e^{i(x+y)}$ *is true for all* $x, y \in \mathbb{R}$ *since*

$$\begin{aligned} e^{ix}e^{iy} &= (\cos x + i \sin x)(\cos y + i \sin y) \\ &= (\cos x \cos y - \sin x \sin y) + i(\cos x \sin y + \cos y \sin x) \\ &= \cos(x+y) + i \sin(x+y) \\ &= e^{i(x+y)}. \end{aligned}$$

Generalizing the above, we get that

$$\forall n \in \mathbb{Z} \ \ z = (re^{in}) = r^n e^{in\theta} \tag{3.5}$$

which is commonly known as ***deMoivre's Law****. Note that by simply generalizing the above, all we have is that* $n \in \mathbb{Z}^+$*. But by Equation (3.4), we can have that for* $n \in \mathbb{Z}^-$*, let* $m = -n$*, and thus*

$$z^n = \left[\frac{1}{r}e^{i(-\theta)}\right]^m = \left(\frac{1}{r}\right)^m e^{im(-\theta)} = \left(\frac{1}{r}\right)^{-n} e^{i(-n)(-\theta)} = r^n e^{i\theta}$$

This proves that deMoivre's Law also holds for when $n \in \mathbb{Z}^-$.

Observe that if $r = 1$, Equation (3.5) becomes

$$(e^{i\theta})^n = e^{in\theta} \quad \textit{for all } n \in \mathbb{Z} \setminus \{0\} \tag{3.6}$$

When written in the form

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad (n \in \mathbb{Z} \setminus \{0\}) \tag{3.7}$$

*this is known as **deMoivre's formula**.*

Example 3.1.2
Equation (3.7) with $n = 2$ tells us that

$$(\cos\theta + i\sin\theta)^n = \cos 2\theta + i\sin 2\theta$$

or we can express the equation as

$$\cos^2\theta - \sin^2\theta + i2\sin\theta\cos\theta = \cos 2\theta + i\sin 2\theta$$

Equating real and imaginary parts, we have the familiar double angle trigonometric identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

3.1.1 *Roots of Complex Numbers*

Proposition 3.1.1 (nth Roots of a Complex Number)

$$\begin{gathered} \forall z = re^{i\theta} \in \mathbb{C} \;\; r = |z| \in \mathbb{R} \;\; \theta \in [0, 2\pi) \\ \exists w = se^{i\tau} \in \mathbb{C} \;\; s \in \mathbb{R} \;\; \tau \in [0, 2\pi) \\ \forall n \in \mathbb{Z} \\ w^n = \left(se^{i\tau}\right)^n = z = re^{i\theta} \end{gathered}$$

The nth roots of z is described by the set

$$\left\{ r^{\frac{1}{n}} e^{i\left(\frac{\theta + 2\pi k}{n}\right)} : k = 0, 1, ..., n-1 \right\} \tag{3.8}$$

Proof

$$\begin{aligned} s^n = r &\iff s = r^{\frac{1}{n}} \\ e^{in\theta} = e^{i\tau} &\iff \theta = \frac{\tau + 2\pi k}{n} \end{aligned}$$

Therefore, the set that describes the nth roots of z is

$$\left\{ w = r^{\frac{1}{n}} e^{i\left(\frac{\theta + 2\pi k}{n}\right)} : k = 0, 1, ..., n-1 \right\}$$

Remark (nth Roots of Unity)
*The **nth roots of unity** is a direct consequence of Proposition 3.1.1 where we solve for the equation $z^n = 1$ for any $z \in \mathbb{C}, n \in \mathbb{Z}$.*

The set that describes the nth roots of unity is

$$\left\{ e^{i\theta} : \theta = \frac{2\pi k}{n}, k = 0, 1, ..., n-1 \right\} \tag{3.9}$$

It is easy to see how the nth roots of unity **partitions the unit circle into n parts**.

Example 3.1.3
Find the cubic roots of $-2 + 2i$.

Let $z = -2 + 2i$. Note that $|z| = 2\sqrt{2}$ and $\operatorname{Arg} z = \frac{3\pi}{4}$.

Therefore, in polar form, $z = 2\sqrt{2}e^{i\frac{3\pi}{4}}$.

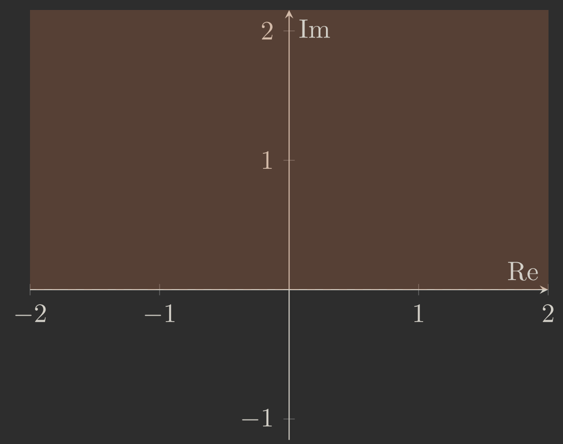
Let $w = re^{i\theta}$, where $\theta \in [0, 2\pi)$, and $w^3 = z$. Then

$$\begin{aligned} r &= (2\sqrt{2})^{\frac{1}{3}} \\ \theta &= \frac{\frac{3\pi}{4} + 2\pi k}{3}, \quad k = 0, 1, 2 \end{aligned}$$

The set that describes the cubic root of $-2 + 2i$ is thus

$$\left\{ (2\sqrt{2})^{\frac{1}{3}} e^{i\theta} : \theta = \frac{\frac{3\pi}{4} + 2\pi k}{3}, k = 0, 1, 2 \right\}$$

Example 3.1.4
Describe the set $\{z \in \mathbb{C} : \left|\operatorname{Arg} z - \frac{\pi}{2}\right| < \frac{\pi}{2}\}$. (Note: $\operatorname{Arg} z \in [0, 2\pi)$)



Exercise 3.1.1

Solve

1. $z^4 = -1$

$$\text{Let } z = re^{i\theta}$$

$$r = |-1| = 1 \quad \theta = \frac{\pi + 2\pi k}{4} = \frac{(2k+1)\pi}{4}, \quad k = 0, 1, 2, 3$$

2. $z^4 = -1 + \sqrt{3}i$

$$\text{Let } z = re^{i\theta}$$

$$r = \left| -1 + \sqrt{3}i \right| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\theta = \frac{\frac{2\pi}{3} + 2\pi k}{4} = \frac{(2k + \frac{2}{3})\pi}{4}, \quad k = 0, 1, 2, 3$$

4 Lecture 4 Jan 10th 2018

4.1 Examples for nth Roots of Unity

Recall that the nth roots of unity are given by $e^{i\frac{2\pi k}{n}}, k = 0, 1, ..., n-1$.

Exercise 4.1.1

Let z be any nth root of unity other than 1. Show that

$$z^{n-1} + z^{n-2} + \ldots + z + 1 = 0 \tag{4.1}$$

Proof

By the Sum of Finite Geometric Terms,

$$z^{n-1} + z^{n-2} + \ldots + z + 1 = \frac{1 - z^n}{1 - z}.$$

Since $z^n = 1$, RHS is thus zero, which in turn completes the proof.

As an aside, if we wish to remove the restriction that z can also be 1, we may consider that

$$z^n - 1 = (z - 1)(1 + z + \ldots + z^{n-1})$$

Since $z^n = 1$, LHS is zero. Then either $z = 1$ or $(1+z+\ldots+z^{n-1}) = 0$.

Exercise 4.1.2

Consider the $n - 1$ diagonals of a regular n-gon, inscribed in a circle of radius 1, obtained by connecting one vertex on the n-gon to all its other vertices.

For example, if we are given $n = 6$, we obtain the following diagram.

Show that the product of the lengths of these diagonals is equal to n.

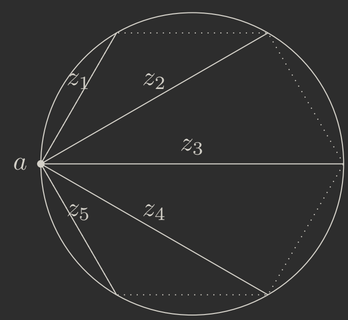


Figure 4.1: $n = 6$, where a is an arbitrary vertex on the hexagon

Proof

Note that Figure 4.1 can be translated into Figure 4.2.

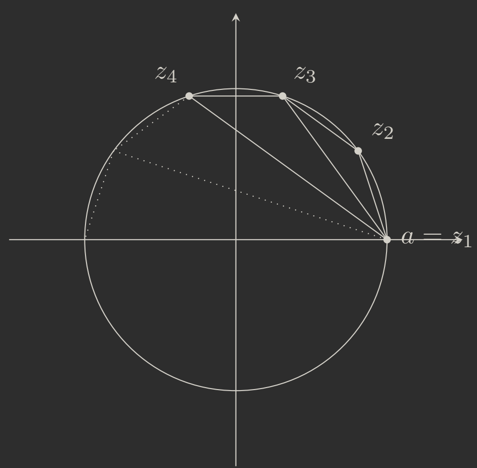


Figure 4.2: A regular n-gon with the roots of unity on its vertices

Thus the equation that we wish to prove becomes

$$|1 - z_2|\,|1 - z_3| \ldots |1 - z_n| = n \tag{4.2}$$

Note that $z_2, ..., z_n$ are the nth roots of unity other than 1.

Let z be a variable and consider the polynomial

$$P(z) := 1 + z + z^2 + \ldots + z^{n-1} \tag{4.3}$$

Since the roots of $P(z)$ are the nth roots of unity other than 1, we can factorize Equation (4.3) into

$$P(z) = (z - z_2)(z - z_3) \ldots (z - z_n)$$

Now let $z = 1$ and take the modulus of $P(z)$, and we get Equation (4.2).

Exercise 4.1.3

Let $n \in \mathbb{N}$. Show that $\sum_{j=0}^{n} \binom{3n}{3j} = \frac{2^{3n} + 2(-1)^n}{3}$.

Proof

Let $\alpha = e^{i\frac{2\pi}{3}}$*. Then* α *is a cubic root of unity, i.e.* $\alpha^3 = 1$*, and from Exercise 4.1.1,* $1 + \alpha + \alpha^2 = 0$.

Consider

$$\begin{aligned}(1+1)^{3n} = &\binom{3n}{0} + \binom{3n}{1} + \binom{3n}{2} + \binom{3n}{3} + \binom{3n}{4} \\ &+ \binom{3n}{5} + \binom{3n}{6} + \ldots + \binom{3n}{3n}\end{aligned} \tag{4.4}$$

$$\begin{aligned}(1+\alpha)^{3n} = &\binom{3n}{0} + \binom{3n}{1}\alpha + \binom{3n}{2}\alpha^2 + \binom{3n}{3} + \binom{3n}{4}\alpha \\ &+ \binom{3n}{5}\alpha^2 + \binom{3n}{6} + \ldots + \binom{3n}{3n}\end{aligned} \tag{4.5}$$

$$\begin{aligned}(1+\alpha^2)^{3n} = &\binom{3n}{0} + \binom{3n}{1}\alpha^2 + \binom{3n}{2}\alpha + \binom{3n}{3} + \binom{3n}{4}\alpha^2 \\ &+ \binom{3n}{5}\alpha + \binom{3n}{6} + \ldots + \binom{3n}{3n}\end{aligned} \tag{4.6}$$

Adding Equation (4.4), Equation (4.5) and Equation (4.6), we observe that the terms with coefficients $\binom{3n}{k}$ *where* k *is not a multiple of 3 sums to 0 as given by* $1 + \alpha + \alpha^2 = 0$*, and therefore we obtain*

$$\begin{aligned}
2^{3n} + (1+\alpha)^{3n} + (1+\alpha^2)^{3n} &= 3\sum_{j=0}^{n}\binom{3n}{3j} \\
\frac{1}{3}\left[2^{3n} + (1+\alpha)^{3n} + (1+\alpha^2)^{3n}\right] &= \sum_{j=0}^{n}\binom{3n}{3j} \\
\frac{1}{3}\left[2^{3n} + (-\alpha^2)^{3n} + (-\alpha)^{3n}\right] &= \sum_{j=0}^{n}\binom{3n}{3j} \quad \textit{since } 1+\alpha+\alpha^2 = 0 \\
\frac{1}{3}\left[2^{3n} + (-1)^n + (-1)^n\right] &= \sum_{j=0}^{n}\binom{3n}{3j} \quad \textit{since } \alpha^3 = 1 \\
\frac{2^{3n} + 2(-1)^n}{3} &= \sum_{j=0}^{n}\binom{3n}{3j}
\end{aligned}$$

as required.

Exercise 4.1.4

Note that we can define $\operatorname{Arg} z$ *in any interval of length* 2π*, i.e. it is not necessary that* $\operatorname{Arg} z \in [0, 2\pi)$.

For example, if we restrict $\operatorname{Arg} z \in [-\pi, \pi]$*, then we can write*

$$\operatorname{Arg}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -\frac{3\pi}{4}$$

Let z be on the unit circle and $\operatorname{Arg} z \in [-\pi, \pi]$. Suppose that $z \notin \mathbb{R}$, i.e. $z \neq 1, z \neq -1$. Show that

$$\operatorname{Arg}\left(\frac{z-1}{z+1}\right) = \begin{cases} \frac{\pi}{2} & \operatorname{Im} z > 0 \\ -\frac{\pi}{2} & \operatorname{Im} z < 0 \end{cases}$$

Proof

Note that $\forall w_1, w_2 \in \mathbb{C}$, where $\operatorname{Arg} w_1 = \tau_1, \operatorname{Arg} w_2 = \tau_2$ for τ_1, τ_2 in the same 2π-interval,

$$\operatorname{Arg} \frac{w_1}{w_2} = \frac{e^{i\tau_1}}{e^{i\tau_2}} \equiv e^{i(\tau_1 - \tau_2)} = \operatorname{Arg} w_1 - \operatorname{Arg} w_2 \tag{4.7}$$

in modulo 2π.

Suppose $\operatorname{Im} z > 0$. Let $\theta_1 = \operatorname{Arg}(z-1)$ and $\theta_2 = \operatorname{Arg}(z+1)$. Consider Figure 4.3. Note that since both $\theta_1, \theta_2 \in [0, \pi]$, we have that $\theta_1 - \theta_2 \in [-\pi, \pi]$, and thus Equation (4.7) holds true without the need of the condition of being in modulo 2π. We observe that

$$\begin{aligned} \frac{\pi}{2} &= \theta_2 + \pi - \theta_1 \\ \theta_1 - \theta_2 &= \frac{\pi}{2} \end{aligned}$$

as desired.

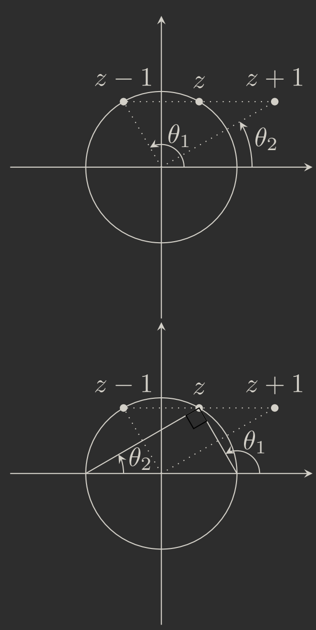


Figure 4.3: (Right) Depicted question, (Left) Translated Angles

Similarly, we can obtain $\theta_1 - \theta_2 = -\frac{\pi}{2}$ for when $\operatorname{Im} z < 0$. This completes the proof.

Exercise 4.1.5

Let $f(z) = e^z$ for $z \in \mathbb{C}$. Let $A = \{z = x + iy \in \mathbb{C} : x \leq 1, y \in [0, \pi]\}$. Describe the image of $f(A)$.

Solution

Firstly, note that

$$e^z = e^{x+iy}$$
$$e^x \in (0, e]$$
$$y \in [0, \pi]$$

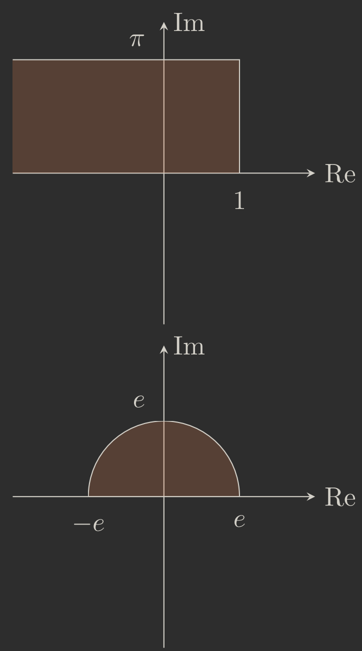


Figure 4.4: (Right) Domain of $f(A)$, (Left) Image of $f(A)$

It is clear that the image will be in on the positive side of the imaginary-axis. Also, since $e^x \in (0, e]$, we get the right graph represented in Figure 4.4. The image of $f(A)$ is described in the left image of Figure 4.4.

5 Lecture 5 Jan 12 2018

5.1 Complex Functions

5.1.1 Limits

Definition 5.1.1 (Convergence)

*A sequence of complex numbers $z_1, z_2, z_3, \ldots$ **converges** to $z \in \mathbb{C}$ if*

$$\lim_{n \to \infty} |z_n - z| = 0 \tag{5.1}$$

or we may say

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n > N \quad |z_n - z| < \varepsilon \tag{5.2}$$

Note

If $\{z_n\}_{n \in \mathbb{N}}$ converges to z, we may write $\lim_{n \to \infty} z_n = z$ or $z_n \to z$ (as $n \to \infty$).

Example 5.1.1

For $|z| > 1$, does $\{\frac{1}{z^n}\}_{n=1}^{\infty}$ converge? Explain.

Solution

We claim that the limit is 0. Since $|z| > 1$, we have that

$$\begin{aligned} \lim_{n \to \infty} \left| \frac{1}{z^n} - 0 \right| &= \lim_{n \to \infty} \left| \frac{1}{z} \right|^n \\ &= 0 \end{aligned}$$

Another way to prove this, since $|z| > 1 \implies 0 < \left|\frac{1}{z}\right| < 1$,

$$\forall \varepsilon = \left| \frac{1}{z} \right| > 0$$
$$\left| \frac{1}{z^n} - 0 \right| = \left| \frac{1}{z} \right|^n < \left| \frac{1}{z} \right| = \varepsilon$$

Definition 5.1.2 (Convergence for Complex Functions)
$\forall \Omega \subseteq \mathbb{C}$, let $f : \Omega \to \mathbb{C}$. We say that

$$\lim_{z \to z_0} f(z) = L \tag{5.3}$$

for some $L \in \mathbb{C}$ if for every sequence $\{z_n\}_n \subseteq \Omega$ (not including z_0 if it is in Ω), we have that

$$z_n \to z_0 \implies f(z_n) \to L \tag{5.4}$$

Note that L need not be in Ω.

Example 5.1.2
Let $f(z) = \frac{\bar{z}}{z}, z \in \mathbb{C} \setminus \{0\}$. Find $\lim_{z \to 0} f(z)$.

Solution
Suppose $z = x \in \mathbb{R} \setminus \{0\}$. Then $f(z) = f(x) = \frac{x}{x} = 1$.

Suppose $z = iy, y \in \mathbb{R} \setminus \{0\}$. Then $f(z) = f(iy) = \frac{-iy}{iy} = -1$.

Therefore, the limit $\lim_{z \to 0} f(z)$ does not exist.

Exercise 5.1.1
Show that $z_n \to z \iff \mathrm{Re}(z_n) \to \mathrm{Re}(z) \land \mathrm{Im}(z_n) \to \mathrm{Im}(z)$.
(Hint: $|\mathrm{Re}(z)|, |\mathrm{Im}(z)| \leq |z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$)

Solution
Suppose $z_n \to z$. Then $\forall \varepsilon_0 > 0 \ \exists N \in \mathbb{N} \ \forall n > N \ \ |z_n - z| < \varepsilon$. Note once and for all that

$$\begin{aligned} \mathrm{Re}(z_n - z) &= \mathrm{Re}(z_n) - \mathrm{Re}(z) \\ \mathrm{Im}(z_n - z) &= \mathrm{Im}(z_n) - \mathrm{Im}(z). \end{aligned}$$

Thus

$$\begin{aligned} |\mathrm{Re}(z_n) - \mathrm{Re}(z)| &= |\mathrm{Re}(z_n - z)| \\ &\leq |z_n - z| < \varepsilon \\ |\mathrm{Im}(z_n) - \mathrm{Im}(z)| &= |\mathrm{Im}(z_n - z)| \\ &\leq |z_n - z| < \varepsilon \end{aligned}$$

For the other direction,

$$\begin{aligned} &\forall \frac{\varepsilon}{2} > 0 \ \exists N_0 \in \mathbb{N} \ \forall n > N_0 \ \ |\mathrm{Re}(z_n) - \mathrm{Re}(z)| < \frac{\varepsilon}{2} \\ &\forall \frac{\varepsilon}{2} > 0 \ \exists N_1 \in \mathbb{N} \ \forall n > N_1 \ \ |\mathrm{Im}(z_n) - \mathrm{Im}(z)| < \frac{\varepsilon}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned}|z_n - z| &= |\mathrm{Re}(z_n) + \mathrm{Im}(z_n) - \mathrm{Re}(z) - \mathrm{Im}(z)| \\ &\leq |\mathrm{Re}(z_n) - \mathrm{Re}(z)| + |\mathrm{Im}(z_n) - \mathrm{Im}(z)| \\ &\leq \varepsilon\end{aligned}$$

□

5.1.2 *Continuity*

Definition 5.1.3 (Continuity)
$\forall \Omega \subseteq \mathbb{C}$, *let* $f : \Omega \to \mathbb{C}$. *We say that* f *is* ***continuous*** *at* $z_0 \in \Omega$ *if*

1. $\forall \{z_n\}_{n \in \mathbb{N}}$
   $z_n \to z_0 \implies f(z_n) \to f(z_0)$
2. $\forall \varepsilon > 0 \ \exists \delta > 0$
   $|z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon$

Remark

1. f *is continuous on* Ω *if it is continuous on every point in* Ω.
2. *We may* ***split*** f *into its feal and imaginary parts, i.e.*

$$f(z) = f(x, y) = u(x, y) + iv(x, y) \tag{5.5}$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$.

Example 5.1.3
Let $f : \mathbb{C} \to \mathbb{C}$ *and for* $z \in \mathbb{C}, f(z) = \frac{\bar{z}}{z}$. *To split* f *into real and imaginary parts:*

$$\begin{aligned}f(z) &= \frac{\bar{z}}{z} \\ &= (x + iy)\left(\frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}\right) \\ &= \frac{x^2 - y^2}{x^2 + y^2} + i\frac{(-2xy)}{x^2 + y^2}\end{aligned}$$

and we get

$$\begin{aligned}u(x, y) &= \frac{x^2 - y^2}{x^2 + y^2} \\ v(x, y) &= -\frac{2xy}{x^2 + y^2}\end{aligned}$$

6 *Lecture 6 Jan 15th 2018*

6.1 *Continuity (Continued)*

Exercise 6.1.1

Let $f : \Omega \to \mathbb{C}$. Prove that $f(z)$ is continuous at $z_0 = x_0 + iy_0 \in \mathbb{C} \iff$ functions $u, v : \mathbb{R}^2 \to \mathbb{R}$, such that $f(z) = u(x, y) + iv(x, y)$ are both continuous at (x_0, y_0).

Solution

We shall first prove the forward direction. Suppose that $f(z)$ is continuous at $z_0 = x_0 + iy_0 \in \mathbb{C}$. By Definition 5.1.3, $\forall \{z_n\}_{n \in \mathbb{N}} \subseteq \Omega$, $z_n \to z_0 \implies f(z_n) \to f(z_0)$. By Exercise 5.1.1,

$$\begin{aligned} z_n \to z_0 &\iff \operatorname{Re} z_n \to \operatorname{Re} z_0 \wedge \operatorname{Im} z_n \to \operatorname{Im} z_0 \\ &\iff x_n \to x_0 \wedge y_n \to y_0 \end{aligned} \tag{6.1}$$

where $z_n = x_n + iy_n$ for $x_n, y_n \in \mathbb{R}$.

Similarly so, and by Equation (5.5),

$$f(z_n) + f(z_0) \iff u(x_n, y_n) \to u(x_0, y_0) \wedge v(x_n, y_n) \to v(x_0, y_0) \tag{6.2}$$

Putting together Equation (6.1) and Equation (6.2), we get

$$(x_n, y_n) \to (x_0, y_0) \implies u(x_n, y_n) \to u(x_0, y_0) \wedge v(x_n, y_n) \to v(x_0, y_0)$$

as desired.

The proof of the other direction is simply a reversed process of the above. □

6.2 Differentiability

Definition 6.2.1 (Neighbourhood)
For $z_0 \in \mathbb{C}, r \in \mathbb{R}$, let

$$D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}. \tag{6.3}$$

On the complex plane, this is seen as a open disk centered around the point z_0 with radius r, as shown below. This open disk is called a

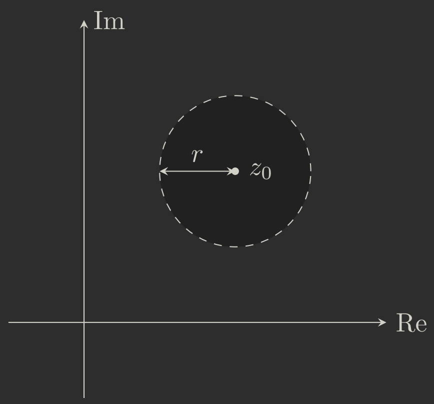


Figure 6.1: Open disk centered around z_0 with radius r

neighbourhood *of z_0.*

Definition 6.2.2 (Differentiable/Holomorphic)
Let $f(z)$ be defined in a neighbourhood of $z_0 \in \mathbb{C}$. We say f is ***differentiable/holomorphic*** *at z_0 if for some $h \in \mathbb{C}$,*

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{6.4}$$

exists. If such a limit exists, we denote the limit by $f'(z_0)$.

Remark
$h \in \mathbb{C}$: h need not necessarily be real. In this sense, h approaches 0 from ***any direction*** *around $0 \in \mathbb{C}$.*

Example 6.2.1
For $z \in \mathbb{C} \setminus \{0\}$, let $f(z) = \frac{1}{z}$. Let $z_0 \in \mathbb{C} \setminus \{0\}$. Note that

$$\lim_{h \to 0} \frac{\frac{1}{z_0 + h} - \frac{1}{z_0}}{h} = \lim_{h \to 0} \frac{1}{h} \left[\frac{-h}{(z_0 + h) z_0} \right] = -\frac{1}{z_0^2}$$

Thus f is holomorphic at any $z \in \mathbb{C} \setminus \{0\}$, and hence $f'(z) = -\frac{1}{z}$.

Example 6.2.2

For $z \in \mathbb{C}$, let $f(z) = \bar{z}$. Let $z_0 \in \mathbb{C}$. Notice that

$$\lim_{h \to 0} \frac{\overline{z_0 + h} - \bar{z}}{h} = \lim_{h \to 0} \frac{\bar{h}}{h}.$$

From Example 5.1.2, we know that such a limit does not exist. Thus f is not holomorphic on any $z \in \mathbb{C}$.

Exercise 6.2.1 (Holomorphic Functions Properties)

If f, g are holomorphic at $z \in \mathbb{C}$, prove that

1. *$f + g$ is holomorphic and $(f+g)' = f' + g'$.*

2. *fg is holomorphic and $(fg)' = f'g + fg'$.*

3. *if $g(z) \neq 0$, $\frac{f}{g}$ is holomorphic and $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$.*

Solution

1. *For $f + g$,*

$$\begin{aligned} &\lim_{h \to 0} \frac{f(z+h) + g(z+h) - f(z) - g(z)}{h} \\ =&\lim_{h \to 0} \left[\frac{f(z+h) - f(z)}{h} + \frac{g(z+h) - g(z)}{g} \right] \\ =&f'(z) + g'(z) \end{aligned}$$

Thus $(f+g)' = f' + g'$.

2. *For fg,*

$$\begin{aligned} &\lim_{h \to 0} \frac{f(z+h)g(z+h) - f(z)g(z)}{h} \\ =&\lim_{h \to 0} \frac{f(z+h)g(z+h) + f(z)g(z+h) - f(z)g(z+h) - f(z)g(z)}{h} \\ =&\lim_{h \to 0} \left[\frac{f(z+h) - f(z)}{h} g(z+h) + f(z) \frac{g(z+h) - g(z)}{h} \right] \\ =&f'(z)g(z) + f(z)g'(z) \end{aligned}$$

Therefore, $(fg)' = f'g + fg'$.

3. *When* $\forall z \in \mathbb{C}\ g(z) \neq 0$, *for* $\frac{f}{g}$,

$$\begin{aligned}
&\lim_{h\to 0} \frac{\frac{f(z+h)}{g(z+h)} - \frac{f(z)}{g(z)}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{f(z+h)g(z) - f(z)g(z+h)}{g(z+h)g(z)}\right] \\
&= \lim_{h\to 0} \frac{1}{g(z+h)g(z)}\left[\frac{f(z+h)g(z) + f(z)g(z) - f(z)g(z) - f(z)g(z+h)}{g}\right] \\
&= \lim_{h\to 0} \frac{1}{g(z+h)g(z)}\left[\frac{[f(z+h) - f(z)]g(z) - f(z)[g(z+h) - g(z)]}{h}\right] \\
&= \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}
\end{aligned}$$

Hence, $\frac{f}{g} = \frac{f'g - fg'}{g^2}$

Note

If we look at the example above from the perspective of f being treated as a real-valued function, i.e. $f(z) = u(x,y) + iv(x,y)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$ and $z = x + iy$, observe that $\forall (x,y) \in \mathbb{R}^2, (x,y) \mapsto (x,-y)$, which we see that u and v are partially differentiable in $\mathbb{R}^2$.

We will now look into this "discrepancy".

6.2.1 *Cauchy-Riemann Equations*

Consider the following function taken from Equation (6.4),

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{6.5}$$

While h may approach $0 \in \mathbb{C}$ from infinitely many sides on the complex plane, we will consider 2 cases.

Case 1: $h \to 0$ via the real axis

In this case, $h = x + i(0)$ and $x \to 0 \in \mathbb{R}$. Then Equation (6.5) gives

$$\begin{aligned}
f'(z_0) &= \lim_{x\to 0} \frac{u(x_0 + x, y_0) + iv(x_0 + x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{x} \\
&= \lim_{x\to 0} \left[\frac{u(x_0 + x, y_0) - u(x_0, y_0)}{x} + i\frac{v(x_0 + x, y_0) - v(x_0, y_0)}{x}\right] \\
&= \frac{\partial u}{\partial x}\bigg|_{(x_0,y_0)} + i\frac{\partial v}{\partial x}\bigg|_{(x_0,y_0)} \qquad (6.6)
\end{aligned}$$

Case 2: $h \to 0$ via the imaginary axis

In this case, $h = 0 + iy$ and $y \to 0 \in \mathbb{R}$. In a similar fashion, Equation (6.5) becomes

$$\begin{aligned} f'(z_0) &= \lim_{y \to 0} \left[\frac{u(x_0, y_0 + y) - u(x_0, y_0)}{iy} + \frac{v(x_0, y_0 + y) - v(x_0, y_0)}{y} \right] \\ &= \frac{1}{i} \cdot \frac{\partial u}{\partial y}\Big|_{(x_0, y_0)} + \frac{\partial v}{\partial y}\Big|_{(x_0, y_0)} \end{aligned} \tag{6.7}$$

Note that since $f'(z_0)$ exists, the real and imaginary part of Equation (6.6) and Equation (6.7) must equate. Also note that $\frac{1}{i} = -i$. With that, we obtain the following theorem.

Theorem 6.2.1 (Cauchy-Riemann Equations)
If $f(z)$ is holomorphic at $z_0 = x_0 + iy_0 \in \mathbb{C}$ where $x_0, y_0 \in \mathbb{R}$, then, at (x_0, y_0),

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{6.8}$$

7 Lecture 7 Jan 17th 2018

7.1 Differentiability (Continued)

7.1.1 Cauchy-Riemann Equations (Continued)

It is natural to wonder if the **converse** of Theorem 6.2.1 is true. We present the following example.

Example 7.1.1

Let

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Check if

1. *f is holomorphic at 0.*
2. *Theorem 6.2.1 holds at $(0,0)$.*

Proof

1. *Observe that by letting $h = x_h + iy_h$ where $x_h, y_h \in \mathbb{R}$,*

$$\lim_{h \to 0} \frac{\frac{\overline{0+h}^2}{0+h} - 0}{h} = \lim_{h \to 0} \frac{\bar{h}^2}{h} = \lim_{x_h + iy_h \to 0} \left(\frac{x_h - iy_h}{x_h + iy_h} \right)^2$$

Consider $y_h = kx_h$, for $k \in \mathbb{R} \setminus \{0\}$. Then

$$\lim_{x_h \to 0} \left(\frac{x_h - ikx_h}{x_h + ikx_h} \right)^2 = \left(\frac{1 - ik}{1 + ik} \right)^2,$$

where we see that the limit depends on the value of k. Therefore, the limit DNE. Hence f is not holomorphic at 0.

2. *Let $z = x + iy$ for $x, y \in \mathbb{R}$. Then*

$$\frac{\bar{z}^2}{z} = \frac{(x - iy)^2}{x + iy} = \frac{(x - iy)^3}{x^2 + y^2} = \frac{x^3 - 3xy^2}{x^2 + y^2} + i \frac{(-3x^2 y + y^3)}{x^2 + y^2}$$

Therefore, we obtain

$$u(x,y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$v(x,y) = \begin{cases} \frac{y^3 - 3x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Observe that

$$\begin{aligned} \frac{\partial u}{\partial x}\bigg\rvert_{(0,0)} &= \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = 1 \\ \frac{\partial v}{\partial y}\bigg\rvert_{(0,0)} &= \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = 1 \\ &\text{and} \\ \frac{\partial u}{\partial y}\bigg\rvert_{(0,0)} &= \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = 0 \\ \frac{\partial v}{\partial x}\bigg\rvert_{(0,0)} &= \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = 0 \end{aligned}$$

satisfies Equation (6.8).

This illustrates that the converse of Theorem 6.2.1 is not true. We will, however, show that the converse will be true given an extra condition.

Theorem 7.1.1 (Conditional Converse of CRE)
Let $z_0 = x_0 + iy_0 \in \Omega \subseteq \mathbb{C}, x_0, y_0 \in \mathbb{R}$, *and* $u, v : \mathbb{R}^2 \to \mathbb{R}, f = u + iv : \Omega \to \mathbb{C}$. *If*

1. *the partials of* u, v *exist in a neighbourhood of* (x_0, y_0),
2. *the partials of* u, v *are continuous at* (x_0, y_0), *and*
3. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ *and* $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ *at* (x_0, y_0),

then f *is holomorphic at* z_0.

A proof of the theorem is in page 36 of Newman and Bak (recommended text of PMATH352W18). I may include the proof whenever I am free.

7.1.2 *Power Series*

Definition 7.1.1 (Power Series)
A ***power series*** *in* $\mathbb{C}$ *is an infinite series of the form*

$$\sum_{n \in \mathbb{N}} c_n z^n, \tag{7.1}$$

where each $c_n \in \mathbb{C}$ *is the coefficient of* z *of the* n*-th power.*

In this subsection, we are interested to see if Equation (7.1) converges.

Recall the notion of convergence in series from $\mathbb{R}$. Equation (7.1) converges if the sequence of partial sums $\{S_N\}$ converges as $N \to \infty$, where

$$S_N := \sum_{n=0}^{N} c_n z^n$$

In other words, using the same definition of S_N,

$$\begin{gathered} \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \setminus \{0\} \quad \forall n > N \\ |S_n - L| < \varepsilon \end{gathered}$$

where $L \in \mathbb{C}$ is the limit that the sequence converges to.

We also know that Equation (7.1) converges absolutely if $\sum_{n=0}^{\infty} |c_n| \, |z|^n$ converges. This is a stronger statement (i.e. absolute convergence $\implies$ convergence)

$$\because \left| \sum_{n=0}^{N} c_n z^n \right| \leq \sum_{n=0}^{N} |c_n| \, |z|^n \quad \text{for each } N \in \mathbb{N}$$

Example 7.1.2
$\sum_{n=0}^{\infty} z^n$ *converges absolutely for* $|z| < 1$.

Note that the partial sum of a geometric series is

$$\sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r}$$

and so the limit as $N \to \infty$ *exists if* $|r| < 1$*, and hence we see that*

$$\sum_{n=0}^{N} r^n \to \frac{1}{1 - r}$$

if $|r| < 1$ *as* $N \to \infty$.

However, if $|z| = 1$*, the power series diverges.*

Another note that we shall point out is that if Equation (7.1) converges absolutely for some $z_0 \in \mathbb{C}$, then it converges absolutely for any z where $|z| < |z_0|$.

These notions, in turn, begs the question of **what is the largest possible $|z_0|$ for the series to converge absolutely**.

8 Lecture 8 Jan 19 2018

8.1 Power Series (Continued)

8.1.1 Radius of Convergence

Theorem 8.1.1 (Convergence in the Radius of Convergence)
For any power series $\sum_{n \in \mathbb{N}} c_n z^n$, $\exists 0 \leq R < \infty$, *such that*

1. $|z| < R \implies$ *series converges absolutely.*
2. $|z| > R \implies$ *series diverges.*

Moreover, R *is given by* ***Hadamard's Formula****:*

$$\frac{1}{R} := \limsup_{n \to \infty} |c_n|^{\frac{1}{n}} \tag{8.1}$$

Remark

1. *R is called the* ***radius of convergence*** *of the series.* $\{z \in \mathbb{C} : |z| < R\}$ *is called the disk of convergence of the series.*
2. *Recall the definition of the* ***limit supremum***

$$\limsup_{n \to \infty} a_n := \lim_{n \to \infty} \left(\sup_{m \geq n} a_m \right) \tag{8.2}$$

which we may colloquially say as the "highest peak 'reached' by a_n*'s as* $n \to \infty$*"*

Proposition 8.1.1 (A Property of limsup)

$$\forall \{a_n\}_{n \in \mathbb{N}} \quad L := \limsup_{n \to \infty} a_n \implies$$
$$\forall \varepsilon > 0 \quad \exists N > 0 \quad \forall n > N$$
$$L - \varepsilon < a_n < L + \varepsilon$$

(Proof to be included)

Proof (Theorem 8.1.1)
Let $L := \frac{1}{R} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$. *Clearly,* $L \geq 0$.

1. *Suppose* $|z| < R$. $\exists \varepsilon > 0, r := |z|(L+\varepsilon)$ *such that* $0 < r < 1$. *By Proposition 8.1.1,* $\exists N \in \mathbb{N}, \forall n > N, |c_n|^{\frac{1}{n}} < L + \varepsilon$.

   Now since $L = \frac{1}{R}$,

   $$\sum_{n=N}^{\infty} |c_n| |z|^n = \sum_{n=N}^{\infty} (|c_n|^{\frac{1}{n}} |z|)^n < \sum_{n=N}^{\infty} r^n$$

   and since $0 < r < 1$, *the final summation converges (as it is a geometric sum). Thus by comparison test,* $\sum_{n=N}^{\infty} |c_n| |z|^n$ *converges.*

   We may also proceed with noticing that the partial sum of $\sum_{n=N}^{\infty} |c_n| |z|^n$ *is* ***bounded and monotonic****, which shows that the series converges.*

2. *Suppose* $|z| > R$. $\exists \varepsilon > 0, r := |z|(L-\varepsilon)$ *such that* $r > 1$. *By Proposition 8.1.1,* $\exists N \in \mathbb{N}, \forall n > N, |c_n|^{\frac{1}{n}} > L - \varepsilon$. *Then analogous to the proof above,*

   $$\sum_{n=N}^{\infty} |c_n| |z|^n = \sum_{n=N}^{\infty} (|c_n|^{\frac{1}{n}} |z|)^n > \sum_{n=N}^{\infty} r^n$$

   where the final summation diverges, and thus implying that $\sum_{n=N}^{\infty} |c_n| |z|^n$ *diverges.*

Theorem 8.1.2 (Power function, holomorphic function, region of convergence)
Suppose $f(z) = \sum_{n\in\mathbb{N}} c_n z^n$ *has a radius of convergence* $R \in \mathbb{R}$. *Then* $f'(z)$ *exists and equals*

$$\sum_{n=1}^{\infty} n c_n z^{n-1}$$

throughout $|z| < R$.

Moreover, f' *has the* ***same radius of convergence*** *as* f.

Proof
Note that f' *has the same radius of convergence as* f *since*

$$\limsup_{n\to\infty} |nc_n|^{\frac{1}{n}} = \limsup_{n\to\infty} |n|^{\frac{1}{n}} |c_n|^{\frac{1}{n}} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$$

where note that $\lim_{n\to\infty} |n|^{\frac{1}{n}} = 1$.

Let $|z_0| \leq r < R$ *and* $g(z_0) := \sum_{n=1}^{\infty} n c_n z_0^{n-1}$.

WTS

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = g(z_0)$$

OR

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \qquad (\dagger)$$
$$|h| < \delta \implies \left| \frac{f(z_0+h)-f(z_0)}{h} - g(z_0) \right| < \varepsilon$$

(Note that we would want $\delta > 0$ such that $|z_0 + h| \leq r < R$)

WTP $(\dagger)$. *$\forall \varepsilon > 0$, choose $\delta > 0$. Suppose $|z_0 + h| < |z_0 + \delta| \leq r < R$. Write*

$$f(z) = \sum_{n=0}^{N} c_n z^n + \sum_{n=N}^{\infty} c_n z^n$$

and let $S_N(z)$ and $E_N(z)$ be the first and second terms respectively. Then we have that

$$S_N'(z) = \sum_{n=1}^{N} n c_n z^{n-1}$$

Consider

$$\begin{aligned} &\left| \frac{S_N(z_0+h)-S_N(z_0)}{h} + \frac{E_N(z_0+h)-E_N(z_0)}{h} - g(z_0) + S_N'(z_0) - S_N'(z_0) \right| \\ &\leq \left| \frac{S_N(z_0+h)-S_N(z_0)}{h} - S_N'(z_0) \right| + \left| \frac{E_N(z_0+h)-E_N(z_0)}{h} \right| + |S_N'(z_0) - g(z_0)| \end{aligned} \tag{8.3}$$

For the second term, since $a^2 - b^2 = (a-b)(a^{n-1} + a^{n-2}b + \ldots ab^{n-2} + b^{n-1})$ and $|z_0|, |z_0 + h| < r \leq R$, we obtain $(z_0+h)^n - z_0^n \leq hnr^{n-1}$. Thus

$$\left| \frac{E_N(z_0+h)-E_N(z_0)}{h} \right| = \left| \frac{1}{h} \sum_{n=N+1}^{\infty} c_n [(z_0+h)^n - z_0^n] \right| \leq \sum_{n=N+1}^{\infty} n c_n r^{n-1}.$$

Note that

$$\sum_{n=1}^{\infty} n c_n r^{n-1} = g(r), \tag{8.4}$$

and since $r < R$, Equation (8.4) converges absolutely by Theorem 8.1.1, thus the tail $\sum_{n=N+1}^{\infty} n c_n r^{n-1}$ converges absolutely. Therefore, by comparison, we can pick $\frac{\varepsilon}{3} > 0$ so that $\exists N_1 \in \mathbb{N}, \forall n > N_1$

$$\left| \frac{E_N(z_0+h)-E_N(z_0)}{h} \right| < \frac{\varepsilon}{3}.$$

For the third term in Equation (8.3), we observe that by definition,

$S'_N(z_0) = \sum_{n=1}^{N} nc_n z^{n-1}$. Since

$$\lim_{N\to\infty} S'_N(z_0) = \lim_{N\to\infty} \sum_{n=1}^{N} nc_n z^{n-1} = \sum_{n=1}^{\infty} nc_n z^{n-1} = g(z_0)$$

we know that we can pick $\frac{\varepsilon}{3} > 0, \exists N_2 \in \mathbb{N}, \forall n > N_2$, we have

$$|S'_N(z_0) - g(z_0)| < \frac{\varepsilon}{3}$$

.

For the first term in Equation (8.3), note that $(S_N(z_0))' = S'_N(z_0)$. Let $\frac{\varepsilon}{3} > 0, \exists \delta > 0, \exists N > \max\{N_1, N_2\}, \forall n > N$, since $|h| < \delta$, we have that

$$\left| \frac{S_N(z_0 + h) - S_N(z_0)}{h} - S'_N(z_0) \right| < \frac{\varepsilon}{3}$$

This completes the proof. $\square$

9 Lecture 9 Jan 22nd 2018

9.1 Power Series (Continued 2)

9.1.1 Radius of Convergence (Continued)

Example 9.1.1

Let $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$. To find the radius of convergence, we use Hadamard's Formula:

$$\frac{1}{R} = \limsup_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = 1 \qquad \because \lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

Therefore $R = 1$. Thus, by Theorem 8.1.1, f converges absolutely when $|z| < 1$ and diverges when $|z| > 1$. As for the boundary, i.e. $|z| = 1$, consider the following two cases:

1. *If $z = 1$, then $f(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ is a **harmonic series**, and hence f diverges.*

2. *If $z = i$, then*

$$\begin{aligned} f(i) &= \sum_{n=1}^{\infty} \frac{i^n}{n} \\ &= i - \frac{1}{2} + \frac{-i}{3} + \frac{1}{4} + \frac{i}{5} - \frac{1}{6} \\ &= \left(-\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \ldots\right) + i\left(1 - \frac{1}{3} + \frac{1}{5} + \ldots\right). \end{aligned}$$

   *Observe that both the real and imaginary parts are alternating series where the absolute values of each term is decreasing, which, by the **alternating series test**, converge. Thus in this case, f converges.*

*Therefore, we observe that **both convergence and divergence may occur** on the boundary, depending on the value of z.*

Note

We may not always exchange the position of $\lim$ *and* $\sum_{a=1}^{b}$ *when we consider an infinite sum (i.e.* $b = \infty$*). Here's an example why this is true. Consider the function* $f(x) = \sum_{n=1}^{\infty}(x^n - x^{n-1})$ *for* $|x| < 1$*. Is*

$$\lim_{x\to 1}\sum_{n=1}^{\infty}(x^n - x^{n-1}) = \sum_{n=1}^{\infty}\lim_{x\to 1}(x^n - x^{n+1})$$

true?

Clearly, RHS is 0. For LHS, note that

$$\begin{aligned} f(x) &= \lim_{N\to\infty}\sum_{n=1}^{N}(x^n - x^{n+1}) \\ &= \lim_{N\to\infty}(x - x^2 + x^2 - x^3 + \ldots + x^N - x^{N+1}) \\ &= \lim_{N\to\infty}(x - x^{N+1}) = x. \end{aligned}$$

So,

$$LHS = \lim_{x\to 1} x = 1$$

And we see that RHS $\neq$ *LHS.*

Definition 9.1.1 (Entire Function)

A function f *is said to be* ***entire*** *if* f *is holomorphic in* ***the entire complex plane****.*

Exercise 9.1.1

Define $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$*. Show that*

1. *the radius of convergence of this series is* ∞*, and hence that* e^z *is an entire function. (Hint: Use* ***Stirling's formula****:* $n! \sim (\frac{n}{e})^n\sqrt{2\pi n}$*)*

2. $(e^z)' = e^z$

Solution

1. *Using Stirling's formula, note that we have*

$$e^z = \sum_{n=0}^{\infty}\frac{1}{\sqrt{2\pi n}}\left(\frac{ez}{n}\right)^n$$

To find R, we have

$$\begin{aligned}\frac{1}{R} &= \limsup_{n\to\infty} \left| \frac{1}{\sqrt{2\pi n}} \left(\frac{e}{n}\right)^n \right|^{\frac{1}{n}} \\ &= \limsup_{n\to\infty} \left|\frac{e}{n}\right| \limsup_{n\to\infty} \left| \frac{1}{\sqrt{2\pi n}} \right|^{\frac{1}{n}} \\ &= 0\end{aligned}$$

since $\limsup_{n\to\infty} \frac{e}{n} = 0$ *and* $\limsup_{n\to\infty} \left| \frac{1}{\sqrt{2\pi n}} \right|^{\frac{1}{n}} = 1$. *Thus* $R = \infty$. *By Theorem 8.1.1,* e^z *is an entire function.*

2. *Note that*

$$\begin{aligned}\lim_{h\to 0} \frac{e^{z+h} - e^z}{h} &= e^z \lim_{h\to 0} \frac{e^h - 1}{h} \\ &= e^z \lim_{h\to 0} \frac{1 + h + \frac{h^2}{2} + \frac{h^3}{3} + \ldots - 1}{h} \\ &= e^z\end{aligned}$$

Thus $(e^z)' = e^z$ *as required.*

10 Lecture 10 Jan 24th 2018

10.1 Power Series (Continued 3)

10.1.1 Radius of Convergence (Continued 2)

A power series is infinitely $\mathbb{C}$-differentiable in its radius of convergence. All its derivatives are also power series, obtained by term-wise differentiation.

E.g.

$$f(z) - \sum_{n=0}^{\infty} c_n z^n \quad \text{then} \quad f^{(2)}(z) = \sum_{n=0}^{\infty} n(n-1)c_n z^{n-2}$$

In general, we may have $\sum_{n=0}^{\infty} c_n (z - z_0)^n$, which is a power series centered at $z_0 \in \mathbb{C}$. Then, as before, the radius of convergence of this power series is given by

$$\frac{1}{R} = \limsup_{n \to \infty} |c_n|^{\frac{1}{n}}$$

So instead of having the disc of convergence centered around 0, we now have one that is centered around z_0.

Corollary 10.1.1 (Corollary of Theorem 8.1.2)
From Theorem 8.1.2, we have shown that

$f(z)$ has a power series expansion at z_0 (i.e. $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ in some neighbourhood of z_0) with radius of convergence $R > 0$	$\Longrightarrow$	*f is holomorphic at z_0*

The converse of the statement above is true, i.e.

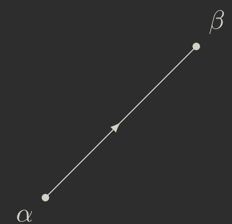
f is holomorphic at z_0	$\implies$	$f(z)$ has a power series expansion at z_0 (i.e. $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ in some neighbourhood of z_0) with radius of convergence $R > 0$

This converse, however, is not possible to be proven given the current tools on our belt. And so we now have to venture into integrals in $\mathbb{C}$.

10.2 *Integration in $\mathbb{C}$*

10.2.1 *Curves and Paths*

Before we begin with the definition of a curve in $\mathbb{C}$, let us consider how a straight line should be described as a vector-valued function in the complex plane. For instance, if we have two points $\alpha, \beta \in \mathbb{C}$, and we want to describe the straight line connecting the two.

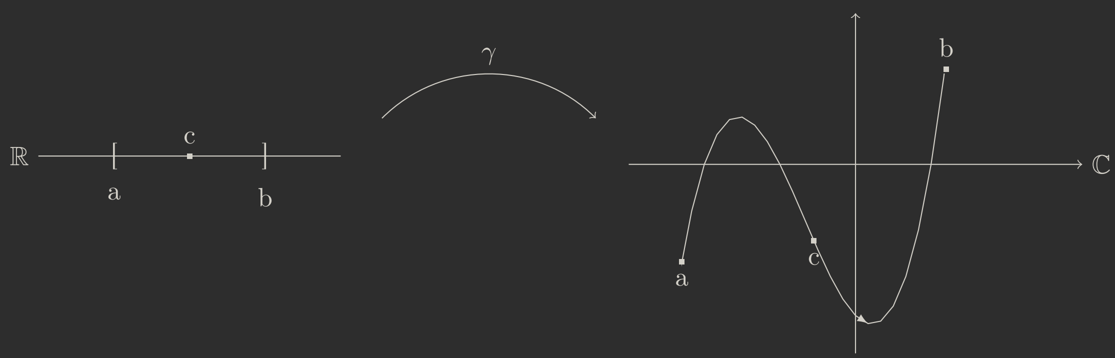


Let γ be the function that describes this line. We may then define $\gamma : [0,1] \to \mathbb{C}$ to be either

$$\gamma(t) = \alpha + (\beta - \alpha)t \quad \text{or} \quad \gamma = \alpha(1-t) + \beta t.$$

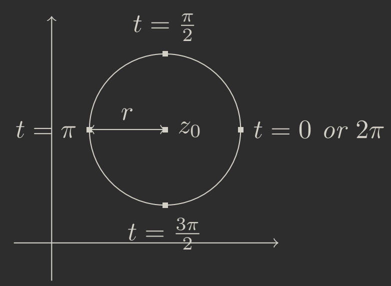
We would then have the following mapping:

Figure 10.1: Mapping from $\mathbb{R} \to \mathbb{C}$ with γ, which is called **the curve** γ



Definition 10.2.1 (Curves in $\mathbb{C}$)
A curve in $\mathbb{C}$ is a continuous function, $\gamma(t) : [a,b] \to \mathbb{C}$, where $a, b \in \mathbb{R}$. The image of γ in $\mathbb{C}$ is called γ^*.

Example 10.2.1
Let $z_0 \in \mathbb{C}, r > 0$.



1. *Let $\gamma : [0, 2\pi] \to \mathbb{C}$, such that $\gamma(t) = z_0 + re^{it}$.*
2. *Let $\gamma' : [0, 1] \to \mathbb{C}$, such that $\gamma'(t) = z_0 + re^{2\pi it}$.*

The two functions above describe a circle centered at z_0 with radius r, anticlockwise-oriented.

We say that γ and γ' are equivalent parameterizations for the same oriented path.

Definition 10.2.2 (Equivalent Parameterization)
*Let $\gamma_1 : [a, b] \to \mathbb{C}, \gamma_2 : [c, d] \to \mathbb{C}$ where $a, b, c, d \in \mathbb{C}$ describe the path γ^*. The two parameterizations are said to be **equivalent parameterizations** if $\exists h : [a, b] \to [c, d]$ that is a bijection and a continuous function such that*

$$\gamma_1(t) = \gamma_2(h(t))$$

where $t \in [a, b]$.

Note
We will not look at functions like the Weierstrass function in this course.

Definition 10.2.3 (Smooth Curve)
Let $\gamma : [a, b] \to \mathbb{C}, a, b, \in \mathbb{C}$. γ is said to be smooth if its derivative γ' exists and is continuous on $[a, b]$ and $\forall t \in [a, b], \gamma'(t) \neq 0$.

Definition 10.2.4 (Piecewise Smooth)
Let $\gamma : [a, b] \to \mathbb{C}$. γ is said to be piecewise smooth if it is smooth on $[a, b]$ except on finitely many points in $[a, b]$.

Remark
Piecewise smooth curves shall be called paths.

10.2.2 Integral

Definition 10.2.5 (Contour)
*Given a path $\gamma : [a, b] \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$, a function continuous on γ. We define the integral f along γ, called a **contour**, as*

$$\int_\gamma f(z)\, dz := \int_a^b f(\gamma(t))\gamma'(t)dt \tag{10.1}$$

where we let $z = \gamma(t)$ and hence $dz = \gamma'(t)dt$.

Remark

1. *Suppose g is a complex-valued function, then*

$$\int_a^b g(t)dt = \int_a^b \mathrm{Re}(g(t))dt + i\int_a^b \mathrm{Im}(g(t))dt$$

2. *The integral of f along γ can be shown to be independent of the chosen parameterization for γ^*.*

   Proof
   Let $a, b, c, d \in \mathbb{R}, \gamma_1 : [a, b] \to \mathbb{C}, \gamma_2 : [c, d] \to \mathbb{C}$ describe the same path γ^*. By Definition 10.2.2, define a bijection $h : [a, b] \to [c, d]$ that is a continuous function such that $t \mapsto \tau$, so that

$$\gamma_1(t) = \gamma_2(h(t)) = \gamma(\tau).$$

   Note that

$$\gamma_1'(t) = h'(t)\gamma_2'(h(t)) \text{ and}$$
$$h(t) = \tau \implies h'(t)dt = d\tau.$$

   Now since h is a bijection, we claim that $h(a) = c$ while $h(b) = d$.

   We know that h cannot be a constant function. Suppose h is an increasing function, then since $a \leq b$ and $c \leq d$, it is clear that $h(a) = c$ and $h(b) = d$. Similarly, if h is a decreasing function, then $h(a) = d$ and $h(b) = c$. But this is a contradiction to our supposition that γ_1 and γ_2 describe the same orientation. Thus h must be an increasing function, and hence we have $h(a) = c$ and $h(b) = d$.

   (This can be more rigorous but that is an easy proof, and we may use perhaps the Approximation Property of $\mathbb{R}$ to

that end, which is a fun exercise that shall not be included within these covers.)

Now

$$\begin{aligned}
\int_{\gamma_1} f(z)\,dz &= \int_a^b f\big(\gamma_1(t)\big)\gamma_1'(t)dt \\
&= \int_a^b f\Big(\gamma_2\big(h(t)\big)\Big)h'(t)\gamma_2'(h(t))dt \\
&= \int_c^d f\big(\gamma_2(\tau)\big)\gamma_2'(\tau)d\tau \\
&= \int_{\gamma_2} f(z)\,dz
\end{aligned}$$

This completes the proof. $\square$

11 Lecture 11 Jan 26th 2018

11.1 Integration in $\mathbb{C}$ (Continued)

11.1.1 Integral (Continued)

Note (Recall)
Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth curve. For a function f that is continuous on γ, we defined

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b f\big(\gamma(t)\big)\gamma'(t)dt \\ &= \int_a^b \operatorname{Re}\Big(f\big(\gamma(t)\big)\gamma'(t)\Big)dt + i\int_a^b \operatorname{Im}\Big(f\big(\gamma(t)\big)\gamma'(t)\Big)dt\end{aligned}$$

and have

$$\begin{aligned}&\gamma'(t) = u'(t) + iv'(t) \\ &\textit{if } \gamma(t) = u(t) + iv(t)\end{aligned}$$

Example 11.1.1
Let $f(z) = f(x + iy) = x^2 + y^2$ be continuous along $\gamma : [0, 1] \to \mathbb{C}$ $t \mapsto t + it$. Evaluate $\int_\gamma f(z)\,dz$.

Solution

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_0^1 f(t + it)(1 + i)dt \\ &= (1 + i)^2 \int_0^1 t^2 dt \\ &= (1 + i)^2 \cdot \frac{1}{3}t^3\Big|_0^1 \\ &= \frac{2i}{3}\end{aligned}$$

Example 11.1.2

$\forall n \in \mathbb{Z}$, evaluate $\int_\gamma z^n \, dz$ that is continue on the path γ that describes any circle centered at origin oriented anticlockwise.

Solution

Let $R \in \mathbb{R}$, and define

$$\gamma : [0,1] \to \mathbb{C} \;\; t \mapsto Re^{2\pi it}$$
$$\gamma'(t) = 2R\pi i e^{2\pi it} = 2\pi i \gamma(t)$$

Then

$$\begin{aligned}
\int_\gamma z^n \, dz &= \int_0^1 R^n e^{2\pi int} \cdot 2\pi i \cdot Re^{2\pi it} dt \\
&= 2\pi i R^{n+1} \int_0^1 e^{2\pi i(n+1)t} dt \\
&= \begin{cases} \frac{R^{n+1}}{n+1} e^{2\pi i(n+1)t} \Big|_0^1 & \text{if } n \in \mathbb{Z} \setminus \{-1\} \\ 2\pi it \Big|_0^1 & \text{if } n = -1 \end{cases} \\
&= \begin{cases} \frac{R^{n+1}}{n+1} \left(e^{2\pi i(n+1)} - 1 \right) & \text{if } n \in \mathbb{Z} \setminus \{-1\} \\ 2\pi i & \text{if } n = -1 \end{cases} \quad \because e^{2\pi ki} \equiv 1 \mod 2\pi \\
&= \begin{cases} 0 & \text{if } n \in \mathbb{Z} \setminus \{-1\} \\ 2\pi i & \text{if } n = -1 \end{cases}
\end{aligned}$$

Note that our final answer does not depend on R, the radius of the circle.

Proposition 11.1.1 (Properties of integrals in $\mathbb{C}$)

1. ***(Linearity)*** *Let $\alpha, \beta \in \mathbb{C}$. $\int_\gamma (\alpha f(z) + \beta g(z)) \, dz = \alpha \int_\gamma f(z) \, dz + \beta \int_\gamma g(z) \, dz$.*

2.(a) *For any complex-valued function g, and $b \geq a$,*

$$\left| \int_a^b g(t) dt \right| \leq \int_a^b |g(t)| \, dt$$

(b) *For any function $f(z)$ that is continuous on a path $\gamma : [a,b] \to \mathbb{C}$,*

$$\left| \int_\gamma f(z) \, dz \right| \leq \sup_{z \in \gamma} |f(z)| \cdot \underbrace{\int_a^b |\gamma'(t)| \, dt}_{\textit{length of the path}}$$

3. If γ^- is the path $\gamma : [a, b] \to \mathbb{C}$ with a reversed direction, then

$$\int_{\gamma^-} f(z)\,dz = -\int_{\gamma} f(z)\,dz$$

Proof

1.

$$\begin{aligned} LHS &= \int_\gamma \alpha f(z) + \beta g(z)\,dz \\ &= \int_\gamma \alpha\Big(\operatorname{Re}(f(z)) + i\operatorname{Im}(f(z))\Big) + \beta\Big(\operatorname{Re}(g(z)) + i\operatorname{Im}(g(z))\Big)\,dz \\ &= \begin{array}{l} \alpha \int_a^b \Big(\operatorname{Re}(f(z)) + i\operatorname{Im}(f(z))\Big) dt \\ \quad + \beta \int_a^b \Big(\operatorname{Re}(g(z)) + i\operatorname{Im}(g(z))\Big) dt \end{array} \qquad (\dagger) \\ &= \alpha \int_a^b f(z)\,dz + \beta \int_a^b g(z)\,dz \end{aligned}$$

where $(\dagger)$ is because of Item 1 from an earlier remark and Linearity of Integrals in $\mathbb{R}$. $\square$

2.(a) Let $R \in \mathbb{R}$. Suppose $\int_a^b g(t)dt = Re^{i\theta}$. Then

$$\begin{aligned} LHS = R &= \int_a^b \underbrace{e^{-i\theta} g(t)}_{u(t)+iv(t)}\,dt \\ &= \int_a^b u(t)dt + i\int_a^b v(t)dt \end{aligned}$$

Since $R \in \mathbb{R}$, we have $\int_a^b v(t)dt = 0$, so

$$\begin{aligned} R &= \int_a^b u(t)dt \\ &\leq \int_a^b |u(t)|\,dt \quad \because a \leq b \wedge u(t) \leq |u(t)| \in \mathbb{R} \\ &\leq \int_a^b |u(t) + iv(t)|\,dt \\ &= \int_a^b \left|e^{-i\theta} g(t)\right| dt \\ &= \int_a^b |g(t)|\,dt = RHS \end{aligned}$$

$\square$

(b)

$$\begin{aligned}
LHS &= \left| \int_\gamma f(z)\, dz \right| \\
&= \left| \int_a^b f\big(\gamma(t)\big)\gamma'(t) dt \right| \\
&\leq \int_a^b \left| f\big(\gamma(t)\big)\gamma'(t) \right| dt \quad \text{by Item 2a} \\
&\leq \int_a^b \sup_{z\in\gamma} |f(z)| \, |\gamma'(t)| \, dt \quad \text{since } |f(z)| \leq \sup_{z\in\gamma} |f(z)| \\
&= \sup_{z\in\gamma} |f(z)| \cdot \int_a^b |\gamma'(t)| \, dt = RHS
\end{aligned}$$

□

3. *Let $\gamma^- : [b, a] \to \mathbb{C}$ such that $\gamma^- = \gamma(b - t + a)$. Let $k = b - t + a$ so that $dk = -dt$, and $t = b \implies k = a \wedge t = a \implies k = b$. Note that $k \in [a, b]$ with this definition. Then*

$$\begin{aligned}
LHS &= \int_{\gamma^-} f(z)\, dz \\
&= \int_b^a f\big(\gamma^-(t)\big)\gamma^{-'}(t) dt \\
&= \int_b^a f\big(\gamma(b - t + a)\big)\gamma'(b - t + a) dt \\
&= -\int_a^b f\big(\gamma(k)\big)\gamma'(k) dk \\
&= -\int_\gamma f(z)\, dz = RHS
\end{aligned}$$

as required. □

We are now in a position to generalize the **Fundamental Theorem of Calclus** for $\mathbb{C}$.

12 Lecture 12 Jan 29th 2018

12.1 Integration in $\mathbb{C}$ (Continued 2)

12.1.1 Fundamental Theorem of Calculus

To simplify statements from hereon, we shall use the following notations.

Notation

Let $\Omega \subseteq \mathbb{C}$ be an open set in $\mathbb{C}$. We denote $f \in H(\Omega) \iff f$ is holomorphic on Ω.

Theorem 12.1.1 (Fundamental Theorem of Calculus)

Let $\gamma : [a, b] \to \mathbb{C}$ be a path inside an open set $\Omega \subseteq \mathbb{C}$. Suppose $f(z)$ is continuous on γ, and has an antiderivative $F \in \Omega$. Then

$$\int_\gamma f(z)\, dz = F\big(\gamma(b)\big) - F\big(\gamma(a)\big) \tag{12.1}$$

Proof

Let $G = F \circ \gamma$ and suppose γ is a smooth function. Since γ is smooth, γ' exists and is continuous on $[a, b]$ and $\gamma'(t) \neq 0$ for all $t \in [a, b]$, and since f is continuous on $[a, b]$, $G(t) = F'(\gamma(t))\gamma'(t)$ is continuous as well.

Now

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b f\big(\gamma(t)\big)\gamma'(t)dt \\ &= \int_a^b F'\big(\gamma(t)\big)\gamma'(t)dt \\ &= \int_a^b G'(t)dt \\ &= G(b) - G(a) \quad \text{by applying FTC in } \mathbb{R} \text{ to real and imaginary parts} \\ &= F(\gamma(b)) - F(\gamma(a))\end{aligned}$$

If γ is piecewise smooth, then we can simply apply the above to each of the smooth paths separately and sum up all of the integrals. □

Definition 12.1.1 (Closed Path)
A path $\gamma : [a, b] \to \mathbb{C}$ is said to be ***closed*** *if $\gamma(a) = \gamma(b)$.*

Corollary 12.1.1 (Corollary of FTC)
If $F \in H(\Omega), \Omega \subseteq \mathbb{C}$ (hence F' is continuous on Ω), then

$$\int_\gamma F'(z)\,dz = 0$$

on any closed path γ on Ω.

Proof
A closed path $\gamma : [a, b] \to \mathbb{C}$ has $\gamma(a) = \gamma(b)$. By Theorem 12.1.1, $\int_\gamma F'(z)\,dz = F(\gamma(b)) - F(\gamma(a)) = 0$ as required. □

Example 12.1.1
Take $f(z) = z^n$ where $n \in \mathbb{Z} \setminus \{-1\}$ as in Example 11.1.2. Then f is continuous on $\mathbb{C} \setminus \{0\}$ ***(not sure why this would be problematic when we've already excluded -1 for n)****. Then $f = F'$ for $F(z) = \frac{z^{n+1}}{n+1}$ and $F \in H(\mathbb{C} \setminus \{0\})$. Therefore by Corollary 12.1.1, $\int_\gamma z^n\,dz = 0$ for any closed path γ not passing through 0.*

If we do include -1 for n, note that F' would not be continuous on 0, and thus the corollary would not apply. We have also shown in the earlier example that $\int_\gamma \frac{1}{z}\,dz = 2\pi i$.

Note (Recall)
The ***interior*** *of a set Ω is defined as $\{z \in \Omega : \forall \varepsilon > 0 \; B(z, \varepsilon) \subseteq \Omega\}$, and denoted as Ω^0.*

Theorem 12.1.2 (Goursat's Theorem / Cauchy's Theorem for a triangle)
Let $\Omega \subseteq \mathbb{C}$ be an open set. Suppose $\Delta \subseteq \Omega$ is a closed triangle whose interior is also contained in Ω. Let $f \in H(\Omega)$. Then

$$\int_{\Delta} f(z)\, dz = 0$$

This theorem holds more meaning than the presented statement, as it implies that, essentially, given any two points connected by two different paths in an open set in $\mathbb{C}$, and a function that is holomorphic over the two paths, the **two path integrals of the function will yield the same result**!

Proof
Let $\Delta_1^{(1)}, \Delta_2^{(1)}, \Delta_3^{(1)}, \Delta_4^{(1)}$ be smaller triangles by bisecting each side of Δ. $\forall i \in \{1, 2, 3, 4\}$, orient $\Delta_i^{(1)}$ anticlockwise. Then we have

$$J := \int_{\Delta} f(z)\, dz = \sum_{i=1}^{4} \int_{\Delta_i^{(1)}} f(z)\, dz \tag{12.2}$$

Note that there must at least one of the $\Delta_i^{(1)}$ such that $\left| \int_{\Delta_i^{(1)}} \right| \geq \frac{|J|}{4}$, since $\forall i \in \{1, 2, 3, 4\}, \left| \int_{\Delta_i^{(1)}} \right| < \frac{|J|}{4}$ would contradict Equation (12.2). Without loss of generality, let $\Delta_1^{(1)}$ be the largest triangle of the four.

Now note that each of the perimeter of $\Delta_i^{(1)}$ is half of the perimeter of Δ. Let $\ell(x)$ be the perimeter of x. Continue with taking bisectors of $\Delta_1^{(1)}, \Delta_1^{(2)}, \ldots$ such that

$$\Delta \supseteq \Delta_1^{(1)} \supseteq \Delta_1^{(2)} \supseteq \ldots,$$

then we have that for each $j \in \mathbb{N} \setminus \{0\}$, $\Delta_i^{(j)}$ is such that

$$\left| \int_{\Delta_i^{(j)}} f(z)\, dz \right| \geq \frac{|J|}{4^j}$$

*and $\ell(\Delta_i^{(j)}) = \frac{1}{2^j} \ell(\Delta)$. By the **Nested Rectangle Theorem from Real Analysis**, $\exists z_0 \in \mathbb{C}$ such that $z_0 \in \Delta_i^{(j)}$ for all $j \in \mathbb{N} \setminus \{0\}$ that is a limit point. Since $z_0 \in \Omega \ \land \ f \in H(\Omega)$, we have that*

$$\forall z \in \Omega \ \ \forall \varepsilon > 0 \ \ \exists \delta > 0$$
$$0 < |z - z_0| < \delta \implies \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

Consider $D(z_0, \delta), \exists n \in \mathbb{N} \setminus \{0\}, \Delta_1^{(n)} \subseteq D(z_0, \delta)$. *Consider*

$$\int_{\Delta_1^{(n)}} f(z)\,dz = \int_{\Delta_1^{(n)}} \big(f(z) - f(z_0) - f'(z_0)(z - z_0)\big)\,dz,$$

where we note that $\int_{\Delta_1^{(n)}} f(z_0)\,dz = \int_{\Delta_1^{(n)}} f'(z_0)(z - z_0)\,dz = 0$ *by Corollary 12.1.1.*

By Item 2a in Proposition 11.1.1,

$$\begin{aligned} \left| \int_{\Delta_1^{(n)}} f(z)\,dz \right| &\leq \int_{\Delta_1^{(n)}} |f(z) - f(z_0) - f'(z_0)(z - z_0)|\,dz \\ &\leq \int_{\Delta_1^{(n)}} \varepsilon\,|z - z_0|\,dz \leq \varepsilon\ell(\Delta_1^{(n)}) \int_{\Delta_1^{(n)}} dz = \varepsilon\ell(\Delta_1^{(n)})^2 \end{aligned}$$

where we note that $|z - z_0| \leq \ell(\Delta_1^{(n)}$ *since* $z \in \Delta_1^{(n)}$. *This implies that*

$$\frac{|J|}{4^n} \leq \varepsilon\ell(\Delta_1^{(n)})^2 \leq \varepsilon\frac{\ell(\Delta)^2}{4^n} = |J| \leq \varepsilon$$

and therefore, $|J| = 0$*!*

Tutorial Jan 31 2018

Note

Consider the power series $\sum_{n\geq 0} a_n(z-z_0)^n$ *and let* $\frac{1}{R} := \limsup_{n\to\infty} \sup \sqrt[n]{|a_n|} \in [0,\infty)$.

- *If* $|z-z_0| < R$, $\sum_{n\geq 0} a_n(z-z_0)^n$ *converges absolutely.*
- *If* $|z-z_0| > R$, $\sum_{n\geq 0} a_n(z-z_0)^n$ *diverges.*
- *If* $0 < r < R$, *then* $\sum_{n\geq 0} a_0(z-z_0)^n$ *converges uniformly on* $\{z : |z-z_0| < r\}$.

12.2 *Practice Problems*

1. Parameterize the semicircle $|z-4-5i| = 3$ clockwise, starting from $z = 4+8i$ to $z = 4+2i$.

   Solution

   Let $\gamma : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{C}$ *such that* $\gamma(t) = 3e^{-it} + 4 + 5i$. *Note that* γ *parameterizes the given semicircle:*

$$\begin{aligned}\gamma\left(-\frac{\pi}{2}\right) &= 4+8i \\ \gamma(0) &= 7+5i \\ \gamma\left(\frac{\pi}{2}\right) &= 4+2i\end{aligned}$$

2. If the power series $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$ centered at z_0 has a non-zero radius of convergence, then show that

$$c_m = \frac{f^{(m)}(z_0)}{m!}$$

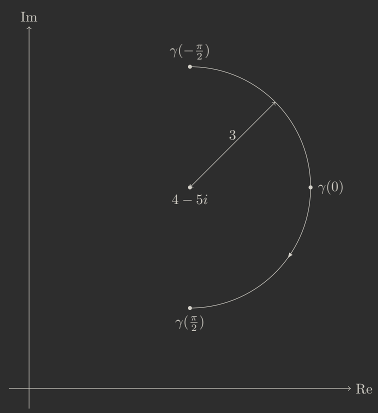


Figure 12.1: Semicircle $|z-4-5i| = 3$ oriented clockwise, parameterized by γ

for any $m \in \mathbb{Z}, m \geq 0$, where $f^{(m)}(z_0)$ denotes the mth derivative of f at z_0.

Solution

Since $f(z)$ is a power series and the radius of convergence $R \neq 0$, by Theorem 8.1.2, $f(z)$ is $\mathbb{C}$-differentiable and each derivative has the same radius of convergence. By induction, it can be shown that

$$f^{(m)}(z) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} c_n (z-z_0)^{n-m}$$

Evaluating $f^{(m)}$ at z_0, we have

$$\begin{aligned} f^{(m)}(z_0) &= \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} c_n (z_0-z_0)^{n-m} \\ &= m! c_m \end{aligned}$$

where all terms above m are 0. Then we obtain

$$c_m = \frac{f^{(m)}(z_0)}{m!}$$

as desired. □

3. Let γ be the arc of the unit circle centered at the origin in the first quadrant oriented clockwise (from i to 1). Evaluate the integral

$$\int_\gamma \bar{z}^2 \, dz$$

by parameterizing the curve.

Solution

Consider the parameterization $\gamma : [-\frac{\pi}{2}, 0] \to \mathbb{C}$ given by $\gamma(t) = e^{-it}$.

Note that $\overline{e^{-it}} = e^{it}$. *Then*

$$\begin{aligned} \int_\gamma \bar{z}^2 \, dz &= \int_{-\frac{\pi}{2}}^{0} e^{2it} \cdot \left(-ie^{-it}\right) dt \\ &= -i \int_{-\frac{\pi}{2}}^{0} e^{it} dt \\ &= -e^{it}\Big|_{\frac{\pi}{2}}^{0} \\ &= -1 - i \end{aligned}$$

□

4. Evaluate the above integral by finding an antiderivative. (Hint: Use $\left(\frac{z\bar{z}}{z}\right)^2$)

   Solution

   Note that $z\bar{z} = |z|^2$, *so on the circle, we have* $\bar{z} = \frac{1}{z}$. *Thus the integral is equivalent to*

$$\int_\gamma \frac{1}{z^2} \, dz$$

   Note that the antiderivative of $\frac{1}{z^2}$ *is* $-\frac{1}{z}$. *Thus by Theorem 12.1.1,*

$$\int_\gamma \bar{z}^2 \, dz = \int_\gamma \frac{1}{z^2} = F(\gamma(0)) - F\left(\gamma\left(-\frac{\pi}{2}\right)\right) = -\frac{1}{e^{-i(0)}} + \frac{1}{e^{-i(-\pi/2)}} = -1-i$$

5. Let $\{c_n\}_{n=0}^\infty$ be a sequence of positive real numbers such that

$$L = \lim_{n\to\infty} \frac{c_{n+1}}{c_n}$$

   exists. Then show that

$$\lim_{n\to\infty} c_n^{\frac{1}{n}} = L$$

   This shows that, when applicable, the **ratio test** can be used instead of the root test to calculate the radius of convergence of a power series.

   Solution

   Suppose that

$$L = \lim_{n\to\infty} \frac{c_{n+1}}{c_n}$$

   exists. By definition, we have

$$\begin{gathered} \forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n > N \\ \left| \frac{c_n}{c_{n-1}} - L \right| < \varepsilon \end{gathered}$$

Thus for $n \geq N$,

$$
\begin{aligned}
c_n^{\frac{1}{n}} &= \left(\frac{c_n}{c_{n-1}} \cdot \frac{c_{n-1}}{c_{n-2}} \dots \frac{c_N}{c_{N-1}} \cdot c_{N-1} \right)^{\frac{1}{n}} \\
&= \left(\frac{c_n}{c_{n-1}} \right)^{\frac{1}{n}} \left(\frac{c_{n-1}}{c_{n-2}} \right)^{\frac{1}{n}} \dots \left(\frac{c_N}{c_{N-1}} \right)^{\frac{1}{n}} c_{N-1}^{\frac{1}{n}}
\end{aligned}
$$

Now

$$
\begin{gathered}
(L-\varepsilon)^{\frac{1}{n}} (L-\varepsilon)^{\frac{1}{n}} \dots (L-\varepsilon)^{\frac{1}{n}} c_{N-1}^{\frac{1}{n}} \leq c_n^{\frac{1}{n}} \leq (L+\varepsilon)^{\frac{1}{n}} (L+\varepsilon)^{\frac{1}{n}} \dots (L+\varepsilon)^{\frac{1}{n}} c_{N-1}^{\frac{1}{n}} \\
(L-\varepsilon)^{\frac{n-N+1}{n}} c_{N-1}^{\frac{1}{n}} \leq c_n^{\frac{1}{n}} \leq (L+\varepsilon)^{\frac{n-N+1}{n}} c_{N-1}^{\frac{1}{n}}
\end{gathered}
$$

Note that

$$
\begin{aligned}
\lim_{n \to \infty} (L-\varepsilon)^{\frac{n-N+1}{n}} c_{N-1}^{\frac{1}{n}} = L - \varepsilon \\
\lim_{n \to \infty} (L+\varepsilon)^{\frac{n-N+1}{n}} c_{N-1}^{\frac{1}{n}} = L + \varepsilon
\end{aligned}
$$

Thus we have

$$
\begin{gathered}
L - \varepsilon \leq c_n^{\frac{1}{n}} \leq L + \varepsilon \\
\left| c_n^{\frac{1}{n}} - L \right| \leq \varepsilon
\end{gathered}
$$

as desired. □

6. Find the radius of convergence of

   (a) $\sum_{n=0}^{\infty} \frac{n^n z^n}{n!}$

   (b) $\sum_{n=0}^{\infty} z^{2^n}$

   Solution

   (a) By Stirling's Approximation, i.e. $n! \sim (\frac{n}{e})^n \sqrt{2\pi n}$, we have that Hadamard's formula is

$$
\begin{aligned}
\frac{1}{R} &= \limsup_{n \to \infty} \left| \frac{n^n}{n!} \right|^{\frac{1}{n}} \\
&= \limsup_{n \to \infty} \left| \frac{n^n}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} \right|^{\frac{1}{n}} \\
&= \limsup_{n \to \infty} \left| \frac{e^n}{\sqrt{2\pi n}} \right|^{\frac{1}{n}} \\
&= e \limsup_{n \to \infty} \left| \frac{1}{\sqrt{2\pi n}} \right|^{\frac{1}{n}} = e
\end{aligned}
$$

Therefore, $R = \frac{1}{e}$.

(b) ***no solution yet: current problem, not being able to express the sum as a power series, in turn failing to get*** c_n ***which is needed for*** $\frac{1}{R}$.

7. Show that for any path $\gamma : [a, b] \to \mathbb{C}$ and $f(z)$ continuous on γ, we have

$$\left| \int_\gamma f(z)\, dz \right| \leq \sup_{z \in \gamma} |f(z)| \int_a^b |\gamma'(t)|\, dt$$

Solution

$$\begin{aligned} LHS &= \left| \int_\gamma f(z)\, dz \right| \\ &= \left| \int_a^b f\big(\gamma(t)\big)\gamma'(t) dt \right| \quad \textit{by definition} \\ &\leq \int_a^b \left| f\big(\gamma(t)\big)\gamma'(t) \right| dt \quad \textit{by Item 2a of Proposition 11.1.1} \\ &\leq \int_a^b \sup_{z \in \gamma} |f(z)|\, |\gamma'(t)|\, dt \quad \textit{since } |f(z)| \leq \sup_{z \in \gamma} |f(z)| \\ &= \sup_{z \in \gamma} |f(z)| \cdot \int_a^b |\gamma'(t)|\, dt = RHS \end{aligned}$$

13 Lecture 13 Feb 9th 2018

13.1 Cauchy's Integral Formula

Definition 13.1.1 (Convex Set)
A set $S \subseteq \mathbb{C}$ is called a ***convex set*** *if the line segment joining any pair of points in S lies entirely in S.*

Theorem 13.1.1 (Cauchy's Theorem for Convex Set)
Let $\Omega \subseteq \mathbb{C}$ be a convex open set, and $f \in H(\Omega)$. Then

1. $f = F'$ *for some* $F \in H(\Omega)$.
2. $\int_\gamma f(z)\,dz = 0$ *for any closed path* $\gamma \in \Omega$.

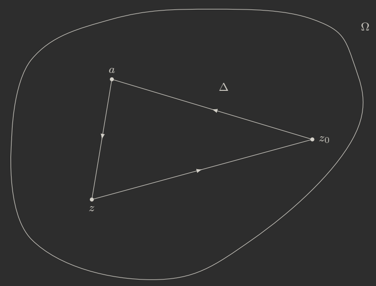
Proof
Note that it is sufficient to prove 1 since 1 $\implies$ 2 by Theorem 12.1.1.

Let $a \in \Omega$, and let $[a, z]$ denote the straight line from a to z. Since Ω is a convex set, $[a, z]$ is in Ω. Define $F(z)$[1] $= \int_{[a,z]} f(z)\,dz$[2].

WTS $F \in H(\Omega), F'(z_0) = f(z_0)$ *for any* $z_0 \in \Omega$.

Now by Theorem 12.1.2,

$$\begin{aligned}
0 &= \int_\Delta f(z)\,dz \\
&= \int_{[a,z]} f(z)\,dz + \int_{[z,z_0]} f(z)\,dz + \int_{[z_0,a]} f(z)\,dz \\
&= F(z) + \int_{[z,z_0]} f(z)\,dz + (-F(z_0))
\end{aligned}$$



This implies that

$$F(z) - F(z_0) = \int_{[z_0,z]} f(z)\,dz.$$

[1] It can be verified that F is continuous.
[2] This is a keys step: defining an "antiderivative" as how we would expect it to be.

Divide both sides by $z - z_0$, then

$$\begin{aligned}\frac{F(z)-F(z_0)}{z-z_0} - f(z_0) &= \frac{1}{z-z_0}\int_{[z_0,z]} f(z)\,dz - f(z_0) \\ &= \frac{1}{z-z_0}\int_{[z_0,z]} f(z) - f(z_0)\,dz \quad \text{since } \int_{[z_0,z]} dz = z - z_0\end{aligned}$$

Since $f \in H(\Omega)$ and is hence continuous, we have that

$$\begin{gathered}\forall \varepsilon > 0\ \exists \delta > 0 \\ |z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon\end{gathered}$$

which in turn implies that

$$\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| = \left|\frac{1}{z-z_0}\int_{[z_0,z]} [f(z)-f(z_0)]\,dz\right| \leq \frac{1}{|z-z_0|}\left|\int_{[z_0,z]} \varepsilon\ dz\right| = \varepsilon$$

Hence, by first principle, $F'(z_0) = f(z_0)$. □

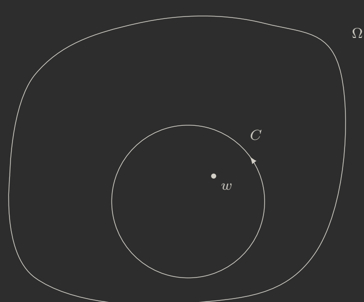
Theorem 13.1.2 (Cauchy's Integral Formula 1)
Let $\Omega \subseteq \mathbb{C}$ be a convex open set, and C be a closed circle path in Ω. If $w \in \Omega \setminus \partial C$, where ∂C is the ***boundary of C****, and $f \in H(\Omega)$, then*

$$f(w)\,\mathrm{Ind}_C\,(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dz$$

where

$$\mathrm{Ind}_C\,(w) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-w}$$

denotes the number of times the countour C winds around the point w.

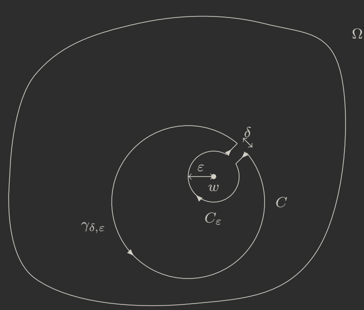


is called the ***index of w with respect to C****, or the* ***winding number*** *of C around w.*

Proof
Let $w \in \Omega \setminus \partial C$. Define

$$g(w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w \end{cases}$$

By the construction of g, g is continuous on Ω, and $g \in H(\Omega \setminus \{w\})$.



We need to *construct a convex set $\Omega' \subseteq \Omega$ that contains $\gamma_{\delta,\varepsilon}$ such that $g \in H(\Omega')$.*

We now follow a similar argument as in the proof for Theorem 13.1.1. Let $\varepsilon > 0$ such that $\exists \delta > 0$, so that we can define the "keyhole"

$\gamma_{\delta,\varepsilon}$ which omits w. Consider $D(w,\varepsilon)$, call the image of the border of $D(w,\varepsilon)$ as C_ε, let δ be the width of the "corridor", and the two paths that are the "sides of the corridor" be called $C_{\delta_1}, C_{\delta_2}$ respectively. Define $G(z) = \int_{[a,z]} g(z)\,dz$, where a and z are in the interior of C but not in the interior of C_ε. Then if we define a set Ω' such that it contains the interior of $\gamma_{\delta,\varepsilon}$, we have that Ω' is a convex open set, and $G \in H(\Omega')$. By Theorem 13.1.1, $G' = g$.

Also from Theorem 13.1.1, we have that $\int_{\gamma_{\delta,\varepsilon}} g(z)\,dz = 0$ for any $\varepsilon, \delta > 0$. As $\delta \to 0^+$, we have that the integrals over C_{δ_1} and C_{δ_2} cancel out. Hence, we are left with

$$\int_C g(z)\,dz + \int_{C_\varepsilon} g(z)\,dz = 0$$

Let's put our focus on the smaller circle, C_ε. Now as $\varepsilon \to 0^+$, $\frac{f(z)-f(w)}{z-w} \to 0$, and thus

$$\int_{C_\varepsilon} g(z)\,dz = \int_{C_\varepsilon} \frac{f(z)-f(w)}{z-w}\,dz \to 0$$

Therefore,

$$\int_C g(z)\,dz = 0$$

which implies, in the limit, that

$$\int_C \frac{f(z)}{z-w}\,dz = \int_C \frac{f(w)}{z-w}\,dz = f(w)\int_C \frac{dz}{z-w}$$

We now require $\int_C \frac{dz}{z-w} = 2\pi i$, but we shall prove for a more general case as a lemma.

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14.1 Cauchy's Integral Formula (Continued)

Lemma 14.1.1

(Lemma and proof from Newman & Bak on Complex Analysis, 3rd Ed.)

Suppose $a \in C_\rho^0$ *such that* $\exists \alpha \in C_\rho$ *that is the center of the circle* C_ρ, *where* ρ *is the radius of* C_ρ, *and hence* $|a - \alpha| < \rho$. *Then*

$$\int_{C_\rho} \frac{dz}{z-a} = 2\pi i$$

Proof

Let $z \equiv \alpha + \rho e^{i\theta}$, *then* $dz = i\rho e^{i\theta} d\theta$. *Thus*

$$\int_{C_\rho} \frac{dz}{z-\alpha} = \int_0^{2\pi} \frac{i\rho e^{i\theta}}{\rho e^{i\theta}} d\theta = 2\pi i$$

while

$$\int_{C_\rho} \frac{dz}{(z-\alpha)^{k+1}} = 0 \quad \text{for } k = 1, 2, 3, \dots \tag{14.1}$$

The Equation (14.1) follows not only from a direct evaluation of the integral

$$\int_{C_\rho} \frac{dz}{(z-\alpha)^{k+1}} = \int_0^{2\pi} \frac{i\rho e^{i\theta}}{(\rho e^{i\theta})^{k+1}} d\theta = \frac{i}{\rho^k} \int_0^{2\pi} e^{-ik\theta} d\theta = 0$$

but also the fact that $\frac{1}{(z-\alpha)^{k+1}}$ *is the derivative of* $-\frac{1}{k(z-\alpha)^k}$, *which can be verified to be holomorphic on* C_ρ, *which simply makes Equation (14.1) true by Theorem 12.1.1.*

To evaluate $\int_{C_\rho} \frac{dz}{z-a}$*, write*

$$\begin{aligned}\frac{1}{z-a} &= \frac{1}{(z-\alpha)-(a-\alpha)} = \frac{1}{(z-\alpha)[1-\frac{a-\alpha}{z-\alpha}]} \\ &= \frac{1}{z-\alpha} \cdot \frac{1}{1-\omega}\end{aligned}$$

where

$$\omega = \frac{a-\alpha}{z-\alpha} \text{ has fixed modulus } \frac{|a-\alpha|}{\rho} < 1 \text{ throughout } C_\rho \tag{14.2}$$

By Equation (14.2) and by the ***Infinite Geometric Sum*** *that* $\frac{1}{1-\omega} = 1 + \omega + \omega^2 + \ldots$*, we get*

$$\begin{aligned}\frac{1}{z-a} &= \frac{1}{z-\alpha}\left[1 + \frac{a-\alpha}{z-\alpha} + \frac{(a-\alpha)^2}{(z-\alpha)^2} + \ldots\right] \\ &= \frac{1}{z-\alpha} + \frac{a-\alpha}{(z-\alpha)^2} + \frac{(a-\alpha)^2}{(z-\alpha)^3} + \ldots.\end{aligned}$$

Since the convergence is uniform throughout C_ρ*,*

$$\int_{C_\rho} \frac{1}{z-a}\,dz = \int_{C_\rho} \frac{1}{z-\alpha}\,dz + \sum_{k=1}^{\infty} \int_{C_\rho} \frac{(a-\alpha)^k}{(z-\alpha)^{k+1}}\,dz = 2\pi i$$

□

We may now continue with completing the previous proof.

Proof (Continued - Theorem 13.1.2)
Lemma 14.1.1 completes the part where we required $\int_C \frac{dz}{z-w} = 2\pi i$*.*

We now have

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-w}\,dz$$

Now note that if we further generalize the number of times the contour C_ρ *made around* a*, where in this case* C_ρ *is a closed path instead of a simple circle in* Ω*, in Lemma 14.1.1, we would get* $\int_{C_\rho} \frac{dz}{z-a} = 2k\pi i$ *where* k *would represent that number.*

In this case, we would get

$$f(w)k = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-w}\,dz$$

where $k = \operatorname{Ind}_C(w) = \frac{1}{2\pi i} \int_C \frac{dz}{z-w}$ *which represents the number of times the contour* C *winds around* w*.* □

Remark

As noted, Theorem 13.1.2 holds for any closed path $\gamma \in \Omega$ instead of a simple circle C. If $w \in \Omega \setminus \gamma^*$, we get

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w} \, dz = f(w) \operatorname{Ind}_\gamma(w)$$

Proposition 14.1.1 (Holomorphic Functions can be expressed as Power series)

Let $\Omega \subseteq \mathbb{C}$ be an open set, $f \in H(\Omega)$. Then f can be expressed as a power series.

Proof

$\forall w \in \Omega, \exists C \subseteq \Omega$ that is a closed circle path with $w \in C^0$. By Theorem 13.1.2, and since C is a circle, i.e. the contour winds around w only once, we have

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-w} \, dz.$$

Let $w_0 \in \Omega$ be the center of C. Then $\forall z \in \partial C, 0 < |w - w_0| < |z - w_0|$[1]. This implies that

[1] This is the key step

$$\begin{aligned} & 0 < \frac{|w-w_0|}{|z-w_0|} < 1 \\ \implies & \sum_{n=0}^{\infty} \left(\frac{w-w_0}{z-w_0} \right)^n = \frac{1}{1 - \frac{w-w_0}{z-w_0}} = \frac{z-w_0}{z-w} \quad \text{by the Infinite Geometric Sum} \\ \implies & \frac{1}{2\pi i} \int_C \frac{f(z)}{z-w} \, dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-w_0} \frac{z-w_0}{z-w} \, dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-w_0} \sum_{n=0}^{\infty} \left(\frac{w-w_0}{z-w_0} \right)^n dz \end{aligned}$$

*Note that each of the terms in the integrand of the last expression are absolutely convergent, thus by **Fubini's Theorem**, we can interchange the summation and integral sign to get*

$$f(w) = \sum_{n=0}^{\infty} \underbrace{\left[\frac{1}{2\pi i} \int_C \frac{f(z)}{(z-w_0)^{n+1}} \, dz \right]}_{a_n} (w - w_0)^n$$

which is a power series centered at w_0 with coefficient a_n.

Note (Recall)

Consider the power series $f(w) = \sum_{n=0}^{\infty} a_n (w-w_0)^n$. Recall Item 2 from Section 12.2 that

$$a_n = \frac{f^{(n)}(w_0)}{n!}$$

Applying this to Proposition 14.1.1, we get

$$\frac{f^{(n)}(w_0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - w_0)^{n+1}} \, dz$$

which holds for any $w_0 \in \Omega$ by having $C \subseteq \Omega$ centered at w_0.

Theorem 14.1.1 (Cauchy's Integral Formula 2)
Let $\Omega \subseteq \mathbb{C}$ be open, $f \in H(\Omega)$. Then

1. *$\forall w \in \Omega$, f has a power series expansion at w.*

2. *f is differentiable infinitely many times in Ω.*

3. *$\forall C \subseteq \Omega$ that is a closed circle oriented anticlockwise, we have that $\forall w \in C^0$,*

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - w)^{n+1}} \, dz \tag{14.3}$$

Remark
Item 3 is the actual Cauchy's Integral Formula in the theorem.

Proof
We have shown 1 from Proposition 14.1.1 and 2 from Theorem 8.1.2. It remains to prove 3, which we shall prove by induction.

When $n = 0$, it is simply Theorem 13.1.2. Suppose f has up to $n-1$ complex derivatives and that

$$f^{(n-1)}(w) = \frac{(n-1)!}{2\pi i} \int_C \frac{f(z)}{(z - w)^n} \, dz.$$

Consider $h > 0$, the difference of the quotient for $f^{(n-1)}$ is

$$\frac{f^{(n-1)}(w - h) - f^{(n-1)}(w)}{h} = \frac{(n-1)!}{2\pi i} \int_C f(z) \frac{1}{h} \left[\frac{1}{z - w - h} - \frac{1}{z - w} \right] dz \tag{14.4}$$

Note that

$$A^n - B^n = (A - B)(A^{n-1} + A^{n-2}B + \ldots + AB^{n-2} + B^{n-1})$$

Let $A = \frac{1}{z-w-h}, B = \frac{1}{z-w}$[2], then the term in square brackets in Equation (14.4) becomes

[2] Key step

$$\frac{h}{(z - w - h)(z - w)} \left[A^{n-1} + A^{n-2}B + \ldots + AB^{n-2} + B^{n-1} \right]$$

Thus as $h \to 0$, we have

$$f^{(n)} = \frac{(n-1)!}{2\pi i} \int_C f(z) \left[\frac{1}{(z-w)^2} \right] \left[\frac{n}{(z-w)^{n-1}} \right] dz = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-w)^{n+1}}\, dz$$

which completes the induction proof and proves 3. □

Corollary 14.1.1 (Taylor Expansion of Entire Functions)
If f is an entire function, then $\forall z_0 \in \mathbb{C}$, we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots$$

which is a ***Taylor Expansion*** *of f around z_0.*

Proof
By Proposition 14.1.1, we have that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^{n+1}}\, dw \right] (z - z_0)^n \\ &= \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_0}\, dw + \left[\frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^2} \right] (z - z_0) \qquad (14.5) \\ &\quad + \left[\frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^3}\, dw \right] (z-z_0)^2 + \dots \\ &\quad + \left[\frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^{k+1}}\, dw \right] (z-z_0)^k + \dots \end{aligned}$$

Now by Theorem 14.1.1, we have

$$\begin{aligned} f(z_0) = f^{(0)}(z_0) &= \frac{0!}{2\pi i} \int_C \frac{f(w)}{w - z_0}\, dw = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_0}\, dw \\ f^{(1)}(z_0) &= \frac{1!}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^2}\, dw \\ f^{(2)}(z_0) &= \frac{2!}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^3}\, dw \\ &\vdots \\ f^{(k)}(z_0) &= \frac{k!}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^{k+1}}\, dw \\ &\vdots \end{aligned}$$

Thus Equation (14.5) becomes

$$f(z) = f(z_0) + f^{(1)}(z_0)(z - z_0) + \frac{f^{(2)}(z_0)}{2!}(z - z_0)^2 + \dots + \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k + \dots$$

as required. □

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15.1 Cauchy's Integral Formula (Continued 1)

At this point, it is important that we provide the following definition:

Definition 15.1.1 (Analytic Functions)
*We say that f is **analytic** in Ω if f has a power series expansion at every $z \in \Omega$.*

Remark

1. *We have proven, in the previous lecture, that Holomorphicity $\implies$ Analyticity*

2. *Should we have defined, in Theorem 14.1.1, that the closed circle orients clockwise, then we would have a negative equation for Equation (14.3).*

15.1.1 Applications of Cauchy's Integral Formula

Exercise 15.1.1

1. ***(Cauchy's Inequality)***[1] *Prove that $\forall z_0 \in \mathbb{C}\ \forall R > 0 \in \mathbb{R}\ \forall f \in H(C = D(z_0, R))$*

$$f^{(n)}(z_0) \leq \frac{n!}{R^n} \cdot \sup_{z \in \mathbb{C}} |f(z)|$$

Proof
From Equation (14.3), we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

[1] In a sense, this inequality implies that as we take higher derivatives, the value of the derivatives become smaller.

Parameterize C with $\gamma : [0, 2\pi] \to \mathbb{C}$, where $t \mapsto z_0 + Re^{it}$. Then

$$\begin{aligned} f^{(n)}(z_0) &= \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{(Re^{it})^{n+1}} Rie^{it}\, dt \\ \left| f^{(n)}(z_0) \right| &\leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{\left| f(z_0 + Re^{it}) \right|}{R^n}\, dt \quad \because \left| Re^{it} \right| = R \\ &\leq \frac{n!}{2\pi R^n} \sup_{z \in C} |f(z)| \int_0^{2\pi} dt \\ &= \frac{n!}{R^n} \sup_{z \in C} |f(z)| \end{aligned}$$

This completes the proof. □

2. ***(Liouville's Theorem)*** *A bounded entire function $f : \mathbb{C} \to \mathbb{C}$ is a constant*[2] [3]*.*

   Proof

   Since f is entire, we may take R, in Item 1, to be any large value. Let M be the bound of f, i.e. $\exists M \in \mathbb{C}, \forall z_0 \in \mathbb{C}, \left| f^{(n)}(z_0) \right| \leq \frac{n!}{R^n} \sup_{z \in \mathbb{C}} |f(z)| = \frac{n!}{R^n} \sup_{z \in \mathbb{C}} M$. Let $n = 1$, then $|f'(z_0)| = \frac{M}{R}$. Thus we observe that $R \to \infty \implies f(z_0) \to 0$ for any $z_0 \in \mathbb{C}$. By A2Q5(a), f is a constant.

[2] The theorem is not true in $\mathbb{R}$, since $\sin x$ is a bounded function differentiable everywhere, but is not a constant.

[3] The theorem also implies that "trigonometry" in $\mathbb{C}$ is unbounded, whatever the definition of "trigonometry" may be.

3. ***(Parseval's Theorem)*** *Let $\Omega \subseteq \mathbb{C}$ be open, $f \in H(\Omega)$, $\overline{D(z_0, R)} \subseteq \Omega$. Then $\forall z \in \overline{D(z_0, R)}$, $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$, which in turn implies that* [4]

$$\forall z \in \overline{D(z_0, R)} \quad f(z_0 + re^{i\theta}) = \sum_{n=0}^{\infty} c_n (re^{i\theta})^n \tag{$\dagger$}$$

[4] This is why the L^2-norm is perserved, as seen in AMATH231.

Consider (the L^2 norm)

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + re^{i\theta}) \right|^2 d\theta \\ =&\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=0}^{\infty} c_n (re^{i\theta})^n \right|^2 d\theta \\ =&\frac{1}{2\pi} \int_0^{2\pi} \left[\sum_{n=0}^{\infty} c_n r^n e^{in\theta} \right] \left[\sum_{m=0}^{\infty} \overline{c_m} r^m e^{-in\theta} \right] d\theta \\ =&\frac{1}{2\pi} \int_0^{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n \overline{c_m} r^{n+m} e^{i(n-m)\theta}\, d\theta \end{aligned}$$

Since the series are absolutely convergent, use may use Fubini's

Theorem, and thus

$$\begin{aligned}
&= \frac{1}{2\pi} \sum_{n,m=0}^{\infty} c_n \overline{c_m} r^{n+m} \int_0^{2\pi} e^{i(n-m)\theta} \, d\theta \\
&= \begin{cases} \frac{1}{2\pi} \sum_{n,m=0}^{\infty} c_n \overline{c_m} r^{n+m} 2\pi & \text{if } n = m \\ \frac{1}{2\pi} \sum_{n,m=0}^{\infty} c_n \overline{c_m} r^{n+m} \frac{e^{i(n-m)\theta}}{i(n-m)} \Big|_0^{2\pi} = 0 & \text{if } n \neq m \end{cases} \\
&= \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \quad \text{if } n = m
\end{aligned}$$

Therefore, we have what is known as ***Parseval's Identity****:*

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + re^{i\theta}) \right|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \tag{15.1}$$

Parseval's Theorem states that:

L^2-norm of LHS in Equation (15.1) = L^2-norm of RHS of Equation (†)

Before going into the next application, please see Lemma 15.1.1.

4. ***(Maximum Modulus Principle)*** *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and $f \in H(\Omega)$. Then*

$$\sup_{z \in \Omega} |f(z)| = \max_{z \in \partial\Omega} |f(z)| .$$

This implies that f cannot attain its maximum value in Ω^0.

Proof

Suppose not, i.e. $\exists z_0 \in \Omega^0$, $\forall z \in \Omega$ such that $|f(z_0)| = \max_{z \in \Omega} |f(z)| \geq |f(z)|$

$$\begin{aligned}
&\implies \exists r > 0 \quad \overline{D(z_0, r)} \subseteq \Omega \\
&\implies \forall z \in \overline{D(z_0, r)} \quad f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n
\end{aligned}$$

Note that $c_0 = \frac{f^{(0)}(z_0)}{0!} = f(z_0)$. *By Item 3,*

$$\begin{aligned}
\sum_{n=0}^{\infty} \left| c_n \right|^2 r^{2n} &= \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + re^{i\theta}) \right|^2 d\theta \\
\implies f(z_0)^2 + \sum_{n=1}^{\infty} \left| c_n \right|^2 r^{2n} &= \frac{1}{2\pi} \left| f(z_0 + re^{i\theta}) \right|^2 d\theta \\
&\leq \frac{1}{2\pi} \left| f(z_0) \right|^2 (2\pi) \quad \because f(z_0) = \max_{z \in \Omega} f(z) \\
\implies f(z_0)^2 + \sum_{n=1}^{\infty} \left| c_n \right|^2 r^{2n} &\leq \left| f(z_0) \right|^2 \\
\implies \sum_{n=1}^{\infty} \left| c_n \right|^2 r^{2n} &\leq 0 \\
\implies c_1, c_2, \ldots &= 0 \\
&\implies f \text{ is a constant in } \overline{D(z_0, r)} \\
&\implies f \text{ is a constant in } \Omega \text{ by Lemma 15.1.1}
\end{aligned}$$

which is a contradiction. □

Lemma 15.1.1 (Principle of Analytic Continuation)
Let $\Omega \subseteq \mathbb{C}$ *be open and connected, and* $f \in H(\Omega)$. *Let* $Z(f) = \{a \in \Omega : f(a) = 0\}$. *Then either*

- $Z(f) = \Omega$, *i.e.* $\forall z \in \Omega$, $f(z) = 0$; *or*
- $Z(f)$ *has no limit point, i.e. points where* $f = 0$ *are isolated*

This is a powerful result, since if we can find a small region for where f is 0 in Ω, then f would be 0 in the entirety of Ω. If not, then f is only 0 at isolated points, i.e. points where $f = 0$ are all apart from each other.

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16.1 Cauchy's Integral Formula (Continued 3)

16.1.1 Applications of Cauchy's Integral Formula (Continued)

Exercise 15.1.1 (Continued)

We shall restate the Item 4 in the following manner.

4. **Maximum Modulus Principle (MMP)** Let $\Omega \subseteq \mathbb{C}$, $f \in H(\Omega)$, $D_{z_0} = \overline{D(z_0, r)} \subseteq \Omega$. Then $|f(z_0)| \leq \max_{z \in \partial D_{z_0}} |f(z)|$ with

$$|f(z_0)| = \max_{z \in \partial D_{z_0}} |f(z)| \iff f \text{ is a constant on } \Omega$$

Remark

(a) This implies that for a non-constant analytic function f, $\forall z \in \Omega^0$, $f(z) \neq \max_{w \in \Omega} f(w)$.

(b) Since a global maximum is also a local maximum, we observe that for any smaller region $\Omega_0 \subseteq \Omega$, f cannot attain its maximum value for any point in Ω_0^0. This is a stronger statement than the our previous statement about the MMP.

Proof

Suppose for $\lightning$ that f has a maximum in Ω^0, say at z_0. Hence $\exists r > 0, D_{z_0} = \overline{D(z_0, r)}$ where

$$|f(z_0)| \geq \max_{z \in D_{z_0}} |f(z)|$$

On D_{z_0}, we have

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \tag{16.1}$$

Note that $c_0 = f(z_0)$. *By Item 3, on* D_{z_0},

$$\begin{aligned}\sum_{n=0}^{\infty} \left|c_n\right|^2 r^{2n} &= \frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0 + re^{i\theta})\right|^2 d\theta \quad \text{by Equation (15.1)} \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0)\right|^2 d\theta \quad \text{by Equation (16.1)} \\ &= \left|f(z_0)\right|^2 .\end{aligned}$$

Then we have

$$\begin{aligned}\left|c_0\right|^2 + \sum_{n=1}^{\infty} \left|c_n\right|^2 r^{2n} &= \left|f(z_0)\right|^2 \\ \left|f(z_0)\right|^2 + \sum_{n=1}^{\infty} \left|c_n\right|^2 r^{2n} &= \left|f(z_0)\right|^2 \\ \sum_{n=1}^{\infty} \left|c_n\right|^2 r^{2n} &= 0\end{aligned}$$

which $\implies$ $c_1 = c_2 = \ldots = 0$. *Thus* $\forall z \in D_{z_0}$, $f(z) \equiv c_0 \mod 2\pi$. *Then by Lemma 15.1.1, since* $f(z_0) - c_0$, *as* $f(z) - c_0$) *contains* $\overline{D(z-0, r)}$, *we see that* $f(z) - c_0 \equiv 0$ *in* Ω, *which implies the equality of Item 4.* □

5. **Fundamental Theorem of Algebra** (FTA) Any polynomial $P(z) \in \mathbb{C}[z]$ of degree greater than 1 has precisely n roots in $\mathbb{C}$, given by $\alpha_1, \alpha_2, ..., \alpha_n$. $P(z)$ can be factored as $P(z) = A(z - \alpha_1)\ldots(z-\alpha_n)$ for some $A \in \mathbb{C}$.

Proof

We may write $P(z) = A(z^n + a_{n-1}z^{n-1} + \ldots a_1 z + a_0)$, *which then*

$$\frac{P(z)}{z^n} = A\left(1 + \frac{a_{n+1}}{z} + \ldots \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right)$$

which then, by the Reverse Triangle Inequality,

$$\implies \left|\frac{P(z)}{z^n}\right| \geq |A| \left[1 - \frac{|a_{n-1}|}{|z|} - \ldots - \frac{|a_1|}{|z^{n-1}|} - \frac{|a_0|}{|z^n|}\right] \tag{16.2}$$

So as $|z| \to \infty$, $\left|\frac{P(z)}{z^n}\right| \to |A|$, *from Equation (16.2). Since* $|z| \to \infty$, $\exists R > 0$, $\forall |z| > R$, *then* $\forall \theta \in [0, 2\pi]$,

$$\left|P(Re^{i\theta})\right| = |P(z)| \geq \frac{|A|}{2}|z|^n \geq \frac{|A|}{2}R^n$$

Taking R to be even larger if necessary, we can get

$$\left|P(Re^{i\theta})\right| \geq |P(0)| \tag{†}$$

Suppose, for contradiction, $P(z)$ has no root in $\mathbb{C}$. Then $g(z) = \frac{1}{P(z)}$ is an entire function. By Equation (†), we have that $\left|g(Re^{i\theta})\right| \leq |g(0)|$ for all $\theta \in [0, 2\pi]$. But this contradicts Item 4 unless if $g(z)$ is constant on $\mathbb{C}$, which in turn implies that P is a constant, but that contradicts that P has degree greater than 1.

$\therefore$ *$P(z)$ has to have a zero in $\mathbb{C}$, say α_1. This implies that*

$$P(z) = A(z - \alpha_1)P_1(z)$$

where $P_1(z) \in \mathbb{C}[z]$. By repeatedly taking the above steps, inductively so, for $P_1, P_2, \ldots$, the proof is completed. $\square$

17 Lecture 17 Feb 26th 2018

17.1 Analytic Continuity

We shall restate the important lemma that we have been using in the last two lectures, and proceed to prove this lemma.

Lemma 17.1.1 (Principle of Analytic Continuity)
Let $\Omega \subseteq \mathbb{C}$ be open and connected, and $f \in H(\Omega)$. Let $Z(f) = \{ain\Omega : f(a) = 0\}$. Then either

- *$Z(f) = \Omega$, i.e. $\forall z \in \Omega$, $f(z) = 0$; or*
- *$Z(f)$ has no limit point, i.e. points where $f = 0$ are isolated*

Proof
Let $z_0 \in Z(f)^*$.

Step 1: *Show that $z_0 \in Z(f)^0$, i.e. f is identically 0 on some $\overline{D(z_0, r)} \subseteq \Omega$ for $r > 0$.*

On $\overline{D(z_0, r)}$, $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$. Suppose f is not identically 0 on $\overline{D(z_0, r)}$. Then $\exists m \in \mathbb{N}$, $c_m \neq 0$, $\forall j < m$, $c_j = 0$, i.e. $f(z) = c_m (z - z_0)^m + c_{m+1}(z - z_0)^{m+1} + \ldots$.

Define, in Ω,

$$g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^m} & z \in \Omega \setminus \{z_0\} \\ c_m & z = z_0 \end{cases}$$

Clearly, $g \in H(\Omega \setminus \{z_0\})$. But on $\overline{D(z_0, r)}$,

$$g(z) = c_m + c_{m+1}(z - z_0) + c_{m+2}(z - z_0)^2 + \ldots$$

which implies $g \in H(\Omega)$. Now $g(z_0) = c_m \neq 0$, so there exists a neighbourhood U_{z_0} of z_0, such that $g \neq 0$ on U_{z_0}.

$\forall a \neq z_0 \in Z(f)$, we have that $g(a) = 0$ by defintion of $Z(f)$, which implies that $a \notin U_{z_0}$, which contradicts that $z_0 \in Z(f)^*$. This implies $f \equiv 0$ in $\overline{D(z_0, r)}$.

Step 2: *$Z(f)^0$ is both open and closed.*

Note that

$$Z(f)^0 := \left\{ a \in Z(f) : \exists r > 0, \overline{D(a,r)} \subseteq Z(f) \right\}$$

is open by definition.

WTP *$[Z(f)^0]^* \subseteq [Z(f)]^*$.*

*From **Step 1**, we know that $[Z(f)^0]^* \subseteq Z(f)^0$. Thus $Z(f)^0$ contains its limit points and is hence closed by definition.*

Step 3: *$Z(f) = \emptyset$ or Ω.*

$$\begin{aligned}
&\Omega \text{ is connected} \\
&\implies \Omega = Z(f)^0 \sqcup \left(Z(f)^0\right)^c \\
&\implies \left(Z(f)^0\right)^c \text{ is open and closed by } \textbf{Step 2}
\end{aligned}$$

A connected set cannot be expressed as a disjoint union of non-trivial open sets. Therefore, either $Z(f)^0 = \emptyset$ or $Z(f)^0 = \Omega$.

$$\begin{aligned}
Z(f)^0 = \emptyset &\implies Z(f)^* = \emptyset \text{ by } \textbf{Step 1} \implies Z(f) = \emptyset \\
Z(f)^0 = \Omega &\implies Z(f) = \Omega \text{ by } \textbf{Step 1}
\end{aligned}$$

□

Corollary 17.1.1 (Uniqueness of a Function)
Let $\Omega \subseteq \mathbb{C}$ be open and connected. $\forall f, g \in H(\Omega)$ with $f(z) = g(z)$ for $z \in \Omega_1 \subseteq \Omega$ where Ω_1 has limit points. Then $\forall z \in \Omega$, $f(z) = g(z)$.

Proof
Apply Lemma 15.1.1 to the function $f - g$.

Remark

1. *In $\mathbb{C}$, we cannot have two functions sharing a region of points in their images. (But this is possible in $\mathbb{R}$)*
2. *Suppose $f \in H(\Omega)$, $\Omega \in \mathbb{C}$ is open and connected, $F \in H(\Omega')$ with $\Omega \subseteq \Omega'$. If f, F agree on Ω, then F is called an analytic contin-*

uation of f in Ω' (i.e. F 'extends' f in Ω'). Lemma 15.1.1 states that F is uniquely determined by f, i.e. there is a unique way to analytically 'continue' f.

17.2 Morera's Theorem

Remark (Recall)

From Cauchy's Theorem, we know that $\forall f \in H(\Omega) \implies \forall \gamma \in \Omega \int f = 0$. We used Goursat's Theorem, i.e. $\forall \Delta \in \Omega \int_\Delta f = 0$ to proof this, and in the process we constructed an antiderivative. Now, our question is, is the converse of the said Cauchy's Theorem true?

Unfortunately for us, that is not true (**example needed**). But a "partial" converse exists.

Theorem 17.2.1 (Morera's Theorem)

Let f be continuous on $\Omega \subseteq \mathbb{C}$, which is an open set, and $\forall \Delta \in \Omega$, $\int_\Delta f = 0$, where Δ is a triangular path. Then $f \in H(\Omega)$.

Proof

Use the same construction as in Cauchy's Theorem for Convex Sets to get an antiderivative F for f, where $F \in H(\Omega)$, i.e.

$$F(z) := \int_{[a,z]} f(z)\, dz$$

Then $F'(z) = f(z)$, which in turn implies that $f \in H(\Omega)$ since F is $\mathbb{C}$-differentiable on Ω by Theorem 14.1.1.

18 Lecture 18 Feb 28th 2018

18.1 Winding Numbers

Recall Cauchy's Integral Formula. We claimed that

$$\mathrm{Ind}_C(w) = \begin{cases} 1 & w \in C^0 \\ 0 & w \notin C \end{cases}$$

We will now formally define this index.

Definition 18.1.1 (Winding Numbers)
Let $\gamma : [a, b] \to \mathbb{C}$ be a closed and oriented anti-clockwise, and γ^* be the image of γ in $\mathbb{C}$. Let $\Omega = \mathbb{C} \setminus \gamma^*$. $\forall w \in \Omega$, define the index of w with respect to γ as

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z - w}$$

in which shall be called the winding number of γ around w.

Theorem 18.1.1 (Winding Number Theorem)
We shall use notation as the definition above. $\mathrm{Ind}_\gamma(w)$ is

1. *always an integer;*
2. *constant on any connected component of Ω; and*
3. *zero on the unbounded component of Ω.*

Note
γ is compact in $\mathbb{C}$ (since it creates a ring from $[a, b]$ under γ). So for some disc D, $\gamma^* \subseteq D$. Let $\Omega \supset \mathbb{C} \setminus D$, where we note that the contained set is connected and unbounded. Then Ω contains one unbounded component, while other components of Ω are inside D. Therefore, we know that components in D are bounded.

Proof

1. *By definition,*

$$\begin{aligned}\mathrm{Ind}_\gamma(w) &= \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-w} \\ &= \frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)\,dt}{\gamma(t)-w}\end{aligned}$$

WTS $\mathrm{Ind}_\gamma(w) \in \mathbb{Z} \equiv \int_a^b \frac{\gamma'(t)\,dt}{\gamma(t)-w} \in 2\pi i\mathbb{Z}$.

Note that $z \in 2\pi i\mathbb{Z} \iff e^z = 1$. *Thus it suffices to show that*

$$e^{\int_a^b \frac{\gamma'(t)\,dt}{\gamma(t)-w}} = 1$$

Idea: *Think of* $\exp\left(\int_a^u \frac{\gamma'(t)\,dt}{\gamma(t)-w}\right)$ *as a function of* u, *call it* $\phi(u)$. *Then we just need to show that* $\phi(b) = 1$. *We know that* $\phi(a) = \exp\left(\int_a^a \ldots\right) = 1$. *This motivates us to find the derivative of* ϕ.

Define ϕ *accordingly, and then since* $(e^{f(u)})' = e^{f(u)} \cdot f'(u)$,

$$\begin{aligned}\phi'(u) &= \phi(u) \cdot \frac{d}{du}\int_a^u \frac{\gamma'(t)\,dt}{\gamma(t)-w} \\ \text{by FTC} \implies \frac{\phi'(u)}{\phi(u)} &= \frac{\gamma'(u)}{\gamma(u)-w} \\ &\implies \phi'(u)\left(\gamma(u)-w\right) - \gamma'(u)\phi(u) = 0 \\ &\implies \frac{d}{du}\left(\frac{\phi(u)}{\gamma(u)-w}\right) = 0 \quad \text{by quotient rule} \\ &\implies \frac{\phi(b)}{\gamma(b)-w} = \frac{\phi(a)}{\gamma(a)-w} \quad \text{since } \frac{\phi(u)}{\gamma(u)-w} \text{ is a constant function of } u \\ &\implies \phi(b) = \phi(a) = 1 \quad \because \gamma \text{ is closed.}\end{aligned}$$

We will prove that $\mathrm{Ind}_\gamma(w)$ *is continuous.*

$$\forall w \in \Omega\ \forall z \in \gamma^*\ \ \exists M > 0\ \ |w - z| > M$$

$$\forall \varepsilon > 0\ \exists \delta = \frac{M^2 \pi \varepsilon}{\int_\gamma dz} > 0\ \forall w_0 \in \Omega$$

$$|w - w_0| < \delta \wedge |w_0 - z| > \frac{M}{2}$$

then

$$\begin{aligned}
|\mathrm{Ind}_\gamma(w) - \mathrm{Ind}_\gamma(w_0)| &= \left| \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-w} - \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-w_0} \right| \\
&= \frac{1}{2\pi} \left| \int_\gamma \frac{w-w_0}{(z-w)(z-w_0)} \, dz \right| \\
&\leq \frac{1}{2\pi} \int_\gamma \left| \frac{w-w_0}{(z-w)(z-w_0)} \right| dz \\
&< \frac{1}{2\pi} \delta \int_\gamma \left| \frac{2}{M \cdot M} \right| dz \\
&= \frac{1}{M^2 \pi} \delta \int_\gamma dz = \varepsilon
\end{aligned}$$

2. *Also* $\mathrm{Ind}_\gamma(w)$ *takes only integer values, thus it must be constant on each open connected component*[1] *(**why?**).*

[1] We may invoke Lemma 15.1.1 but it is, to an extent, unnecessary for such a powerful statement.

3. *Note that*

$$|\mathrm{Ind}_\gamma(w)| = \frac{1}{2\pi} \left| \int_a^b \frac{\gamma'(t) \, dt}{\gamma(t) - w} \right|$$

Let w be in the unbounded component in the complement of γ such that $|w| \to \infty$. Then $\forall t \in [a, b]$, $\exists M > 0$ such that

$$\frac{1}{|\gamma(t) - w|} \leq \frac{1}{M}$$

which implies that

$$|\mathrm{Ind}_\gamma(w)| \leq \frac{1}{2\pi} \frac{1}{M} \cdot \underbrace{\int_a^b |\gamma'(t)| \, dt}_{\substack{\text{is a fixed constant} \\ \text{as } \gamma \text{ is a fixed path}}}$$

$$\implies (|w| \to \infty \implies M \to \infty \implies |\mathrm{Ind}_\gamma(w)| \to 0)$$

Then by parts 1 and 2, the proof is completed. □

Remark

Note that by 2, we have that $\forall w \in C^0$,

$$\frac{1}{2\pi i} \int_C \frac{dz}{z-w} = \frac{1}{2\pi i} \int_C \frac{dz}{z-z_0} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{Rie^{i\theta}}{Re^{i\theta}} \, d\theta = 1$$

where z_0 is the center of the circle path C.

19 Lecture 19 Mar 2nd 2018

19.1 Singularities

Exercise 19.1.1
Let $C : [0, 2\pi] \to \mathbb{C}$ such that $\forall t \in [0, 2\pi]$, $t \to e^{it}$. Suppose $f \in H(\Omega)$, then by Cauchy

$$\int_C f(z)\,dz = 0$$

Let $f(z) = \frac{1}{z}$, then $\int_C \frac{1}{z}\,dz = 2\pi i \operatorname{Ind}_C(0) = 2\pi i$ when it is "supposed" to be 0 by the argument above. Then in this case, $f \notin H(\Omega)$. In fact, f is undefined at 0.

The example above introduces us to the study of such exceptional points.

Definition 19.1.1 ((Isolated) Singularity)
$\forall a \in \mathbb{C}$, $\exists r > 0$, $\exists D = D(a, r)$.

$$f \in H(D \setminus \{a\}) \,\wedge\, f(a) \text{ is undefined} \iff$$

f has a(n) ***point/isolated singularity*** *at $z = a$.*

Example 19.1.1

1. *Given $f \in H(\mathbb{C} \setminus \{0\})$, define $f(z) = \frac{e^z - 1}{z}$. Clearly, z is a singularity. Consider the function $(e^z - 1) \in H(\mathbb{C})$. Then we have that the function has a power series expansion around $z = 0$. So $\forall z \in \mathbb{C}$,*

$$e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

And for $z \neq 0$, we have

$$\frac{e^z - 1}{z} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots \tag{19.1}$$

This motivates us to define

$$g(z) = \begin{cases} \frac{e^z - 1}{z} & z \in \mathbb{C} \setminus \{0\} \\ 1 & z = 0 \end{cases}$$

Clearly then $g \in H(\mathbb{C})$, where in $\mathbb{C} \setminus \{0\}$ its holomorphicity is given by f, and in a neighbourhood of 0, from Equation (19.1). Therefore, y assigning f the value of 1 at $z = 0$, we can make f "entire".

*We call such a point z as a **removable singularity** for f.*

2. *Given $f \in H(\mathbb{C} \setminus \{0\})$, define $f(z) = \frac{1}{z}$. Is the singularity at 0 removable?*

   Suppose $\exists g \in H(\mathbb{C})$ such that

$$\forall z \in \mathbb{C} \setminus \{0\} \quad g(z) = f(z) \tag{19.2}$$

   $\therefore \exists r > 0 \; \forall z \in D(0, r)$

$$g(z) = c_0 + c_1 z + c_2 z^2 + \ldots \tag{19.3}$$

   Consider the function $zg(z)$. By Equation (19.2),

$$\forall z \in \mathbb{C} \setminus \{0\} \quad zg(z) = 1$$

   *By Equation (19.3), $z = 0 \implies zg(z) = 0$. But this cannot happen since $zg(z) \in H(\mathbb{C})$ (**if we pick an open ball of, say, $\frac{1}{2}$ around 0, then there are no points in the entirety of $\mathbb{C}$ that is close to 0**). Therefore $z = 0$ is not a removable singularity for f.*

Definition 19.1.2 (Removable Singularity, Pole, Essential Singularity)
Let f have a singularity at $z_0 \in \mathbb{C}$.

1. $\exists r > 0 \; \forall z \in D = D(z_0, r) \; \exists g(z) \in H(D) \; \forall z \in D \setminus \{z_0\} \; g(z) = f(z)$
   $\implies f$ *has a **removable singularity** at z_0*[1].
2. $\exists r > 0 \; \forall z \in D = D * (z_0, r) \; \exists A, B \in H(D) \; A(z_0) \neq 0 \wedge B(z_0) = 0 \; f(z) = \frac{A(z)}{B(z)}$
   $\implies f$ *has a **pole** at z_0 (a non-removable singularity)*[2]
3. *f has a singularity at z_0 which is neither removeable nor a pole*
   $\implies f$ *has an **essential singularity** at z_0.*

[1] For the laymen, "the value of f at z_0 can be corrected or defined to make it holomorphic in its designated region."

[2] For the laymen, "the singularity of f comes from a zero of its denominator."

Example 19.1.2
To show an example of an essential singularity, consider the function

$f(z) = e^{\frac{1}{z}}$. If we attempt to do a "Taylor expansion" on the function (which is invalid at $z = 0$), we have

$$f(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots$$

The point 0 for f is said to be a "pole of infinite order" (this shall be defined later on)

While **removable singularities** are nice to have, they are not as interesting to us. On the other hand, we are more interested in their non-removable counterpart, the **poles**. This motivates the study of zeros of holomorphic functions.

Theorem 19.1.1 (Theorem 9)

Let $\Omega \subseteq \mathbb{C}$ be open and connected. Suppose that $f \in H(\Omega)$ with $f \not\equiv 0$ on Ω and that f has a zero at $z_0 \in \Omega$. Then

$$\begin{gathered} \exists r > 0 \;\; \forall z \in D = D(z_0, r) \;\; \exists g \in H(D) \;\; g(z_0) \neq 0 \;\; \exists! n \in \mathbb{N} \\ f(z) = (z - z_0)^n \cdot g(z) \end{gathered} \tag{19.4}$$

Proof

By Analytic Continuation, zeros of f are isolated since $f \not\equiv 0$. So $\exists r > 0$ such that $\exists D = D(z_0, r)$, in which $\forall z \in D \setminus \{z_0\}$, $f(z) \neq 0$.

Since $f \in H(\Omega)$, $\forall z \in D$,

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k$$

As $f \not\equiv 0$ in D, $\exists n \in \mathbb{N} \setminus \{0\}$ that is the smallest such that $c_n \neq 0$[3].

[3] $n \neq 0$ since we have $f(z_0) = 0$ which implies $c_0 = 0$.

$$\begin{aligned} \therefore f(z) &= c_n (z - z_0)^n + c_{n+1} (z - z_0)^{n+1} + \ldots \\ &= (z - z_0)^n \underbrace{\left[c_n + c_{n+1}(z - z_0) + \ldots \right]}_{\textit{call this } g(z)} \end{aligned}$$

Note that $g(z_0) \neq 0$ since $c_n \neq 0$. Thus $g(z) \in H(D)$ since it has the same radius of convergence as f.

To prove uniqueness, suppose that we may write

$$f(z) = \sum_{k=0}^{\infty} (z - z_0)^n \cdot g(z) = (z - z_0)^m \cdot h(z)$$

for some $m \in \mathbb{N}$ *and that* $h(z) \neq 0$*. If* $m > n$*, dividing both sides by* $(z - z_0)^n$*,*

$$g(z) = (z - z_0)^{m-n} h(z).$$

As $z \to z_0$*, we would have* $g(z_0) = 0$*, which is a contradiction. If* $m < n$*, we can perform a similar argument and have that as* $z \to z_0$*,* $h(z_0) = 0$*, also a contradiction. Therefore,* $m = n$ *and* $h = g$*.* □

We say that f has a **zero of order** n at z_0 if Equation (19.4) holds.

20 *Lecture 20 Mar 5th 2018*

20.1 *Singularity (Continued)*

Recall the definition of a **removable singularity** from Definition 19.1.2.

Theorem 20.1.1 (Theorem 10)
If $f \in H(\Omega \setminus \{z_0\})$ has an isolated singularity at z_0 and $\lim_{z \to z_0}(z - z_0)f(z) = 0$, then the singularity at z_0 is removable.

Proof
Since $f(z_0)$ is undefined, set

$$h(z) = \begin{cases} (z - z_0)^2 f(z) & \forall z \in \Omega \setminus \{z_0\} \\ 0 & z = z_0 \end{cases}$$

Clearly $h \in H(\Omega \setminus \{z_0\})$. At z_0,

$$\begin{aligned} \lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} &= \lim_{z \to z_0} \frac{(z - z_0)^2 f(z)}{z - z_0} \quad {}^{1} \\ &= 0 \text{ by assumption} \end{aligned}$$

[1] Goes to show that the definition of h is no foresight.

$\therefore h'(z_0)$ exists and equals 0. Clearly then that $h \in H(\Omega)$. So $\exists r > 0$ such that $\exists D = D(z_0, r)$, so that $\forall z \in D$,

$$h(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots$$

But $c_0 = h(z_0) = 0$ and $c_1 = h'(z_0) = 0$. Thus the power series can be written as

$$\begin{aligned} h(z) &= c_2(z - z_0)^2 + c_3(z - z_0)^3 + \ldots \\ &= (z - z_0)^2 \left[c_2 + c_3(z - z_0) + \ldots \right] \end{aligned}$$

Hence by the definition of h, $\forall z \in \Omega \setminus \{z_0\}$, $f(z) = c_2 + c_3(z - z_0) + \ldots$.

Therefore, by redefining $f(z_0) = c_2$, we see that the singularity at z_0 is removable.

We may also complete the proof by defining a function g as, $\forall z \in \Omega$,

$$g(z) = \begin{cases} f(z) & z \neq z_0 \\ c_2 & z = z_0 \end{cases}$$

$\square$

Recall Theorem 19.1.1

Let $\Omega \subseteq \mathbb{C}$ be open and connected, and $f \in H(\Omega)$ where $\forall z \in \Omega$, $f(z) \not\equiv 0$.

$f(z_0) = 0 \implies$

$$\exists r > 0 \;\; \exists D = D(z_0, r) \;\; \forall z \in D \;\; \exists! n \in \mathbb{N}$$
$$\exists! g \in H(D) \;\; g(z_0) \neq 0$$
$$f(z) = (z - z_0)^n g(z)$$

Definition 20.1.1 (Zero of Order n & Simple Zero)
*By the above setting, we say that f has a **zero of order n** at z_0.*[2]

*If $n = 1$, we say that z_0 is a **simple zero**.*

[2] In laymen terms, "Rate at which the function vanishes at z_0. The greater n is, the greater the rate."

Recall definition of a pole **from Definition 19.1.2**

Suppose f has an isolated singularity at z_0, and that there exists a neighbourhood D around z_0 where $A, B \in H(D)$, in which A and B are defined such that $\forall z \neq z_0 \in D$, $A(z_0) \neq 0 \wedge B(z_0) = 0$, so that we can let $f(z) = \frac{A(z)}{B(z)}$. Then f has a pole at z_0.

Theorem 20.1.2 (Theorem 9.1)
Stein & Shakarchi - Complex Analysis (pg. 74)

If f has a pole at $z_0 \in \Omega$, then in a neighbourhood of that point there exists a non-vanishing holomorphic function h and a unique positive integer n such that

$$f(z) = (z - z_0)^{-n} h(z)$$

Proof
By Theorem 19.1.1, we have $\frac{1}{f(z)} = (z - z_0)^n g(z)$, where g is holomorphic and non-vanishing in a neighbourhood of z_0, so the result follows

with $h(z) = \frac{1}{g(z)}$. □

Definition 20.1.2 (Pole of order n & Simple Pole)
*With the above setting, we say that f has a **pole of order n** at z_0 if the function B has a zero of order n*[3]

If $n = 1$, then z_0 is a simple pole.

[3] In laymen terms, "Rate at which f ‘grows’ near z_0."

Theorem 20.1.3 (Theorem 11)
Let f have a pole of order n at z_0. Then $\exists r > 0$, $\exists D = D(z_0, r)$, such that $\forall z \in D \setminus \{z_0\}$,

$$f(z) = \frac{c_{-n}}{(z-z_0)^n} + \frac{c_{-(n-1)}}{(z-z_0)^{n-1}} + \ldots + \frac{c_{-1}}{z-z_0} + G(z)$$

for some $G \in H(D)$.

Proof
By Theorem 20.1.2, write the holomorphic function h as $h(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$, then

$$f(z) = \frac{1}{(z-z_0)^n} \left[a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots \right].$$

The proof is complete by expanding the equation. □

Definition 20.1.3 (Principal Part)
*In Theorem 20.1.3, the sum $\sum_{j=1}^{n} = \frac{c_{-j}}{(z-z_0)^j}$ is called the **principal part** of f at the pole z_0.*

Definition 20.1.4 (Residue)
*In Theorem 20.1.3, the coefficient c_{-1} is called the **residue** of f at the pole z_0, denoted $\operatorname*{Res}_{z=z_0} f(z)$.*

The **residue** shall be more carefully studied later on.

21 Lecture 21 Mar 7th 2018

21.1 Singularity (Continued 2)

Theorem 21.1.1 (Casorati-Weierstrass)
Let $z_0 \in \Omega$ and $f \in H(\Omega \setminus \{z_0\})$. Suppose f has a singularity at z_0. Then one of the following occurs:

1. *f is a removable singularity at z_0;*
2. *$\exists m \in \mathbb{N}$, $\{c_j\}_{j=1}^m \subseteq \mathbb{C}$, $f(z) - \sum_{j=1}^m c_j (z - z_0)^- j$ has a removable singularity at z_0; or*
3. *$\forall r > 0$, $B(z_0, r) \subseteq \Omega$ such that $f(B^0(z_0, r))$ is dense in $\mathbb{C}$.*[1]

[1] $B^0(z_0, r)$ is the punctured ball.

Proof
Suppose 3. does not hold, i.e. $f(B^0(z_0, r))$ is not dense in $\mathbb{C}$ for some $r > 0$. Then $\exists w \in \mathbb{C}$, $\exists \delta > 0$, such that

$$\begin{aligned} &f\big(B^0(z_0, r)\big) \cap B(w, \delta) = \emptyset \\ \implies &\forall z \in B^0(z_0, r) \quad |f(z) - w| > \delta \end{aligned}$$

Consider $g(z) = \frac{1}{f(z) - w}$ for $z \in B^0(z_0, r)$, in which $g \in H(B^0(z_0, r))$. Then $|g(z)| \leq \frac{1}{\delta}$ for all $z \in B^0(z_0, r)$, which implies that

Plausible since $f(z) - w \neq 0$.

$$\lim_{z \to z_0} (z - z_0) g(z) = 0$$

Squeeze Theorem

By Theorem 20.1.1, g has a removable singularity at z_0, thus we can extend the function to a function $\tilde{g} \in H(B(z_0, r))$. From here, we try to construct a function that extends on f onto the singularity z_0, say, $\tilde{f}$. The construction of $\tilde{g}$ satisfies the equation $\frac{1}{\tilde{g}(z)} + w = f(z)$ except, possibly, at z_0.

Case 1: *Suppose $\tilde{g}(z_0) \neq 0$.*

We can simply define

$$\frac{1}{\tilde{g}(z)} = \begin{cases} f(z) - w & z \in B^0(z_0, r) \\ \frac{1}{\tilde{g}(z_0)} & z = z_0 \end{cases}$$

Clearly then $\frac{1}{\tilde{g}} \in H(B^0(z_0, r))$. *At* z_0,

$$\tilde{g}(z_0) \neq 0 \implies \exists r_1 > 0 \; D = D(z_0, r_1) \; \forall z \in D \; \tilde{g}(z) \neq 0 \implies \frac{1}{\tilde{g}} \in H(D)$$

Therefore, $\frac{1}{\tilde{g}} \in H(B(z_0, r))$[2]

[2] $\frac{1}{\tilde{g}}$ is the inverse of a non-zero holomorphic function.

Since $\forall z \in B^0(z_0, r)$ $f(z) = \frac{1}{\tilde{g}(z)} + w$ *by construction, we may define*

$$\tilde{f}(z) = \frac{1}{\tilde{g}(z)} + w$$

such that $f(z_0)$ *is defined as* $\frac{1}{\tilde{g}(z_0)} + w$. *By this construction, 1. holds.*

Case 2: $\tilde{g}(z_0) = 0$.

$\because \tilde{g} \in H(B^0(z_0, r)) \land \tilde{g}(z_0) = 0 \land (\forall z \in B^0(z_0, r) \; \tilde{g}(z_0) \neq 0)$

By Theorem 19.1.1,

$$\begin{aligned} &\exists! m \in \mathbb{N} \;\; \exists 0 < r_1 < r \;\; \exists g_1 \in H\big(B(z_0, r_1)\big) \\ &\quad \forall z \in B(z_0, r_1) \;\; \tilde{g}(z) = (z - z_0)^m g_1(z) \end{aligned} \tag{21.1}$$

$\because g_1 \in H\big(B(z_0, r_1)\big) \land g(z_0) \neq 0$, *we can repeat the argument as in* ***Case 1*** *for* g_1 *to get that* $\frac{1}{g_1} \in H\big(B(z_0, r_1)\big)$, *which implies*

$$\forall z \in B(z_0, r_1) \;\; \frac{1}{g_1(z)} = a_0 + a_1(z - z_0) + \ldots$$

By construction given by Equation (21.1), $\forall z \in B^0(z_0, r_1)$

$$\begin{aligned} f(z) - w &= \frac{1}{g_1(z)(z - z_0)^m} \\ &= \frac{a_0}{(z - z_0)^m} + \frac{a_1}{(z - z_0)^{m-1}} + \ldots + \frac{a_{m-1}}{z - z_0} + a_m + a_{m+1}(z - z_0) + \ldots \end{aligned}$$

$\therefore \forall z \in B^0(z_0, r_1)$,

$$f(z) - \frac{a_0}{(z - z_0)^m} - \ldots - \frac{a_{m-1}}{z - z_0} = w + a_m + a_{m+1}(z - z_0) \ldots$$

Thus we may define an "extended" function of f, $\tilde{f}$ *to be* $w + a_m$ *at the singularity* z_0. *Therefore,* $f(z) - \sum_{j=1}^{m} a_j (z - z_0)^{-j}$ *has a removable singularity at* z_0, *i.e. 2. holds.* $\square$

22 Lecture 22 Mar 9th 2018

22.1 Singularity (Continued 3)

Corollary 22.1.1

If f has an essential singularity at z_0 and is holomorphic in some $B^0(z_0, r)$ where $r > 0$, then $f(B^0(z_0, r))$ is dense in $\mathbb{C}$.

Proof

Suppose not, i.e. 3. of Theorem 21.1.1 does not hold. Then either 1., which implies that z_0 is removable, or 2., which implies that z_0 is a pole, is true. This contradicts the assumption that z_0 is an essential singularity. $\square$

Remark

*There are a lot more that are actually true from Theorem 21.1.1! **Picard** showed that in any such punctured ball $B^0(z_0, r)$ around the essential singularity z_0, f takes on every complex value (except possibly one value) infinitely often.*

22.2 The Residue Theorem

Note (Recall)

If f has a pole at z_0, $f \in H(\Omega \setminus \{z_0\})$, then in some open neighbourhood D of z_0, we can write $\forall z \in D \setminus \{z_0\}$

$$f(z) = \underbrace{\frac{c_{-k}}{(z - z_0)^k} + \ldots + \frac{c_{-1}}{(z - z_0)}}_{\textit{Principal Part}} + \underbrace{c_0 + c_1(z - z_0) + \ldots}_{G(z)} \tag{22.1}$$

with $G \in H(D)$.

Theorem 22.2.1 (Cauchy's Residue Theorem)

Let $\Omega \subseteq \mathbb{C}$ be open, $f \in H(\Omega \setminus \{z_0\})$ where $z_0 \in \Omega$ is a pole. If γ is a *closed path in* $\Omega \setminus \{z_0\}$ *such that* $\forall w \notin \Omega$, $\mathrm{Ind}_\gamma(w) = 0$. *Then*

$$\frac{1}{2\pi i}\int_\gamma f(z)\,d(z) = \left(\operatorname*{Res}_{z=z_0} f(z)\right)\mathrm{Ind}_\gamma(z_0)$$

where $\mathrm{Ind}_\gamma(z_0) := \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz$.

Proof

Using notation of Equation (21.1), define $g(z)$ *such that*

$$g(z) := \begin{cases} f(z) - \sum_{j=1}^{k} \frac{c_{-j}}{(z-z_0)^j} & z \in \Omega \setminus \{z_0\} \\ c_0 & z = z_0 \end{cases}$$

Clearly, $g \in H(\Omega \setminus \{z_0\})$, *since* $f(z)$ *minus finitely many polynomials with non-zero denominators is still a holomorphic function. At* z_0, *with a neighbourhood* D *around the point, we have, from Equation (21.1),*

$$g(z) = c_0 + c_1(z - z_0) + \ldots$$

which $g(z_0)$ *agrees with* c_0 *and for any point* $z \in D \setminus \{z_0\}$, *by definition of* g *using Equation (21.1). This implies that* $g \in H(D) \implies g \in H(\Omega)$.

Thus, by Cauchy's Theorem,

$$\int_\gamma g(z)\,dz = 0$$

Then $\forall z \in \gamma$ *and since* $z_0 \notin \gamma$, *we get*

$$\int_\gamma f(z)\,dz = \int_\gamma \sum_{j=1}^{k} \frac{c_{-j}}{(z-z_0)^j}\,dz$$

Consider each term of RHS in turn. Note that for $m \geq 2$, *since* $\frac{-1}{(m-1)(z-z_0)^{m-1}}$ *is the antiderivative of* $\frac{1}{(z-z_0)^m}$,

$$\begin{aligned}\int_\gamma \frac{1}{(z-z_0)^m}\,dz &= F\big(\gamma(b)\big) - F\big(\gamma(a)\big) \quad \textit{by FTC} \\ &= 0 \quad \textit{since } \gamma \textit{ is closed.}\end{aligned}$$

If $m = 1$, *then*

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz = \mathrm{Ind}_\gamma(z_0) \ \textit{ by definition}$$

$$\begin{aligned}\therefore \frac{1}{2\pi i}\int_\gamma f(z)\,dz &= c_{-1}\operatorname{Ind}_\gamma(z_0)\\ &= \left(\operatorname*{Res}_{z=z_0} f(z)\right)\operatorname{Ind}_\gamma(z_0)\end{aligned}$$

$\square$

Definition 22.2.1 (Meromorphic Functions)
A function f is said to be meromorphic on Ω if $\exists \mathscr{A} \subseteq \Omega$ such that

1. $\mathscr{A}^* = \emptyset$
2. $f \in H(\Omega \setminus \mathscr{A})$
3. $\forall z \in \mathscr{A} \quad f$ *has a pole of finite order on* z.

Remark
Holomorphicity $\subseteq$ *Meromorphicity (let* $\mathscr{A} = \emptyset$*)*

23 *Lecture 23 Mar 12th 2018*

23.1 *The Residue Theorem (Continued)*

We can generalize Theorem 22.2.1 for when there are more than one pole.

Theorem 23.1.1 (Cauchy's Residue Theorem - Generalized)
Let $\Omega \subseteq \mathbb{C}$ be open, f be meromorphic on Ω, $\mathscr{A}$ be a set of poles. If γ is a closed path in $\Omega \setminus \mathscr{A}$ such that $\forall w \notin \Omega \;\; \mathrm{Ind}_\gamma(w) = 0$, then

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \underbrace{\sum_{a\in\mathscr{A}}\left(\operatorname*{Res}_{z=a} f(z)\right)\mathrm{Ind}_\gamma(a)}_{\text{this is a finite sum}} \tag{23.1}$$

Proof
We will first need to prove that $\mathscr{A}$ has only finitely many points.

Since f is meromorphic, the set of poles, $\mathscr{A}$, must not contain its limit points by definition of meromorphisms, i.e. each of the poles are isolated singularities and $\mathscr{A}^* = \emptyset$. Let $\mathscr{A}' := \{a \in \mathscr{A} : \mathrm{Ind}_\gamma(a) \neq 0\}$. Suppose, for contradiction, that $\mathscr{A}'$ has infinitely many points. Since the union of γ and its interior is compact, by Bolzano-Weierstrass, any infinite subset (subsequence, to be exact, but we may simply index the points in the subset) must converge to a point in the union of γ and its interior, call it w. Since $\mathscr{A}'$ is a subset (subsequence) of the union of γ and its interior, $\mathscr{A}'$ converges to w. Now since $|f(z)| \to \infty$ and $z \to a$ for all $a \in \mathscr{A}'$, and since f is continuous since it is meromorphic and hence holomorphic, we have that $|f(z)| \to \infty$ as $z \to w$. Hence w is a pole in the interior of γ and is hence in $\mathscr{A}'$. In other words, $\mathscr{A}'$ contains its limit points, which contradicts the fact that $\mathscr{A}' \subseteq \mathscr{A}$ does not contain its limit points. Therefore, there are only finitely many points in $\mathscr{A}'$.

It remains to prove Equation (23.1).

Since $\mathscr{A}$ is finite, let $|\mathscr{A}| = n$ for some $n \in \mathbb{N}$. For any $a, b \in \mathbb{N}$, $\forall z_a, z_b \in \mathscr{A}$, $\exists r_a, r_b > 0$ such that $z_b \notin D(z_a, r_a) \wedge z_a \notin D(z_b, r_b)$. Therefore, $\forall l \in \mathbb{N}$, $\forall z_l \in \mathscr{A}$, we can write, $\forall z \in D(z_l, r_l)$, that

$$f(z) = \frac{c_{l,-k_l}}{(z-z_l)^{k_l}} + \ldots + \frac{c_{l,-1}}{z-z_l} + c_{l,0} + c_{l,1}(z-z_l) + \ldots, \tag{23.2}$$

where k_l is the order of the pole z_l. Since we are only concerned with the poles in the interior of γ, let $\mathscr{A}' := \{a \in \mathscr{A} : \mathrm{Ind}_\gamma(a) = 1\} = \{z_1, z_2, ..., z_n\}$.

Consider the function

$$g(z) = \begin{cases} f(z) - \sum_{l=1}^{n} \sum_{j=1}^{k_l} \frac{c_{l,-j}}{(z-z_l)^j} & \forall z \in \Omega \setminus \mathscr{A}' \\ c_{l,0} - \sum_{\substack{m=1 \\ m \neq l}}^{n} \sum_{j=1}^{k_m} \frac{c_{m,-j}}{(z-z_m)^j} & z \in \mathscr{A}' \end{cases}$$

At the neighbourhood of each $z_l \in \mathscr{A}'$ save z_l itself, that is in $D(z_l, r_l) \setminus \{z_l\}$ where $D(z_l, r_l)$ is as defined above, we have that

$$g(z) = \underbrace{f(z) - \sum_{j=1}^{k_l} \frac{c_{l,-j}}{(z-z_l)^j}}_{A_{z,l}} - \underbrace{\sum_{\substack{m=1 \\ m \neq l}}^{n} \sum_{j=1}^{k_m} \frac{c_{m,-j}}{(z-z_m)^j}}_{B_{z,l}} \tag{23.3}$$

in which we observe that $A_{z,l}$ is holomorphic by Casorati-Weierstrass, and $B_{z,l}$ is holomorphic in $D(z_l, r_l)$ since $\forall z \in D(z_l, r_l) \setminus \{z_l\}$, $z \notin \mathscr{A}'$ and hence the denominators cannot be zero. Thus $g \in H\big(D(z_l, r_l)\big)$. At z_l, $g(z_l)$ agrees with Equation (23.3) by Equation (23.2). Thus $g \in H\big(D(z_l, r_l)\big)$.

Then by Cauchy's Theorem,

$$\int_\gamma g(z)\, dz = 0.$$

Then, $\forall z \in \gamma^*$ (image of γ), since $\mathscr{A} \cap \gamma^* = \emptyset$ (since $\gamma* \in \Omega \setminus \mathscr{A}$), we get

$$\int_\gamma f(z)\, dz = \int_\gamma \sum_{l=1}^{n} \sum_{j=1}^{k_l} \frac{c_{l,-j}}{(z-z_0))^j}\, dz$$

For each of the terms in $\sum_{j=1}^{k_l} \frac{c_{l,-j}}{(z-z_l)^j}$, for $j \geq 2$, we have that the *antiderivative of* $\frac{1}{(z-z_l)^j}$ *is* $-\frac{1}{(j-1)(z-z_0)^{j-1}}$. *Thus by the Fundamental Theorem of Calculus, since we are integrating over the closed path* γ, *the integral of these terms are* 0. *When* $j = 1$, *since* $\frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_l} = \operatorname{Ind}_\gamma(z_l)$, *we have that*

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma f(z)\,dz &= \sum_{l=1}^{n} c_{l,-1}\operatorname{Ind}_\gamma(z_l) \\ &= \sum_{l=1}^{n}\left(\operatorname*{Res}_{z=z_l} f(z)\right)\operatorname{Ind}_\gamma(z_l)\end{aligned}$$

as required.

Another proof that we can be shown is as follows:

Using similar notations as above, on each of the neighbourhood of the points in $\mathscr{A}$, *we define a keyhole-like path around the points, with a circular path* γ_l, *oriented in the opposite direction of* γ, *of radius* $r_l \geq \varepsilon_l > 0$ *centered around each* z_l, *connected to the path* γ *through a corridor of width* δ_l. *This construction is similar to the proof of Cauchy's Integral Formula in our lectures, where we extend the path of* γ *inwards so as to avoid the poles. As each* $\delta_l \to 0$, *we have*

$$\int_\gamma f(z)\,dz = \sum_{l=1}^{n}\int_{\gamma_l} f(z)\,dz = \sum_{l=1}^{n}\int_{\gamma_l}\sum_{j=-k_l}^{\infty} c_{l,j}(z-z_l)^j\,dz$$

Now for $-k_l \leq j \leq 2$, *the term* $c_{l,j}(z-z_0)^j$ *have antiderivatives as described above, and are thus* 0 *by FTC, on the path* γ_l. *When* $j \geq 0$, $\sum_{j=0}^{\infty} c_{l,j}(z-z_l)^j$ *is a holomorphic function by construction, and hence their integrals are* 0 *by Cauchy's Theorem. We are, therefore, left with*

$$\int_\gamma f(z) = \sum_{l=1}^{n}\int_{\gamma_l}\frac{c_{l,-1}}{z-z_l}\,dz$$

which in turn implies

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_{l=1}^{n}\left(\operatorname*{Res}_{z=z_l} f(z)\right)\operatorname{Ind}_{\gamma_l}(z_l)$$

This implies the desired result (through a similar proof given in Theorem 22.2.1). □

Remark

We need Ω *to be connected with the interior of* γ *contained in* Ω, *i.e.* Ω *is simply connected.*

Now all the above begs the question: how exactly do we find the residue of a pole?

Suppose that f has a pole of order k at z_0. Then in some neighbourhood D of z_0, we have the Laurent expansion

$$f(z) = \frac{a_{-k}}{(z-z_0)^k} + \ldots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \ldots$$

which implies

$$f(z)(z-z_0)^k = a_{-k} + a_{-k+1}(z-z_0) + \ldots + a_{-1}(z-z_0)^{k-1} + \ldots$$

So a_{-1} is the $(k-1)^{\text{th}}$ coefficient for $f(z)(z-z_0)^k$, i.e. we can get

$$\operatorname*{Res}_{z=z_0} f(z) = a_{-1} = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} f(z)(z-z_0)^k$$

23.2 Applications of Cauchy's Residue Theorem

Exercise 23.2.1
Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx$.

The typical approach (from a complex analysis standpoint) is:

1. *Choose a complex function and integrate along some path / contour γ. By the Residue Theorem, we can get our answer in a straightforward way.*
2. *Break the contour into different parts*
   - *the needed real integral*
   - *use symmetry, decay of function, etc., in the limit (**we shall see more about this later on**)*

Let $f(z) = \frac{1}{1+z^4}$. The singularities are

$$z^4 = -1 \implies z = e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}$$

(Note: These are all simple poles)

Let $R > 0$, and let Γ_R be the semi-circular, anti-clockwise contour, centered at zero, sitting in the positive side of the imaginary axis on the complex plane. Theorem 23.1.1 gives that

$$\frac{1}{2\pi i} \int_\gamma f(z)\,dz = \operatorname*{Res}_{z=e^{i\frac{\pi}{4}}} f(z) + \operatorname*{Res}_{z=e^{i\frac{3\pi}{4}}} f(z)$$

24 Lecture 24 Mar 14 2018

24.1 Application of Cauchy's Residue Theorem (Continued)

We will continue with the previous example.

Exercise 24.1.1
Evaluate $I = \int_{-\infty}^{\infty} \frac{1}{1+x^4}\, dx$.

Consider the function $f(z) = \frac{1}{1+z^4}$. *Then* f *has simple poles at* $\alpha_1 = e^{i\frac{\pi}{4}}$, $\alpha_2 = e^{i\frac{3\pi}{4}}$, $\alpha_3 = e^{i\frac{5\pi}{4}}$, $\alpha_4 = e^{i\frac{7\pi}{4}}$. *Consider the contour* Γ_R, *where* R *is large, that consists of an anticlockwise semi-circle* C_R *going from* R *to* $-R$, *and a straight line from* $-R$ *to* R *on the real axis.*

By the Residue Theorem,

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{1+z^4}\, dz = \operatorname*{Res}_{z=\alpha_1} f(z) + \operatorname*{Res}_{z=\alpha_2} f(z) \tag{24.1}$$

Note that for Equation (24.1),

$$LHS = \frac{1}{2\pi i}\left[\int_{-R}^{R} \frac{1}{1+x^4}\, dx + \int_{C_R} \frac{1}{1+z^4}\, dz\right]$$

On C_R, *we have that* $|z| = R$, *so* $\left|1+z^4\right| \geq \left||1| - |z|^4\right| = R^4 - 1$, *and therefore*

$$\begin{aligned}
\left|\int_{C_R} \frac{1}{1+z^4}\, dz\right| &\leq \int_{C_R} \left|\frac{1}{1+z^4}\right| dz \\
&\leq \int_{C_R} \frac{1}{R^4-1}\, dz \\
&= \frac{1}{R^4-1}\int_{C_R} |dz| \\
&= \frac{1}{R^4-1}\cdot \pi R
\end{aligned}$$

As $R \to \infty$, *we have* $\int_{C_R} \frac{1}{1+z^4}\, dz \to 0$, *since it is bounded above by*

$\frac{\pi R}{R^4-1}$ *that goes to* 0.

Therefore, taking the limit of LHS (as well as RHS) as $R \to \infty$ *in Equation (24.1), we have*

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{1+x^4} = \operatorname*{Res}_{z=\alpha_1} f(z) + \operatorname*{Res}_{z=\alpha_2} f(z)$$

Next, we compute the residues:

$$\begin{aligned}
\operatorname*{Res}_{z=\alpha_1} f(z) &= \lim_{z\to\alpha_1} f(z)(z-\alpha_1) \\
&= \lim_{z\to\alpha_1} \frac{z-\alpha_1}{g(z)} \quad \textit{where } g(z) = 1+z^4 \\
&= \lim_{z\to\alpha_1} \frac{z-\alpha_1}{g(z)-g(\alpha_1)} \quad \because g(\alpha_1) = 0 \\
&= \frac{1}{g'(z)}\Big|_{z=\alpha_1} = \frac{1}{4z^3}\Big|_{\alpha_1} = \frac{1}{4\alpha_1^3} \\
\operatorname*{Res}_{z=\alpha_2} f(z) &= \frac{1}{4z^3}\Big|_{\alpha_2} = \frac{1}{4\alpha_2^3}
\end{aligned}$$

So RHS of Equation (24.1) is

$$RHS = \frac{1}{4}\left(\frac{1}{e^{3i\frac{\pi}{4}}} + \frac{1}{e^{9i\frac{\pi}{4}}}\right) = \frac{1}{4}\left(e^{-i\frac{3\pi}{4}} + e^{i\frac{\pi}{4}}\right) = \frac{i}{2}\sin\frac{\pi}{4} = -\frac{i}{2\sqrt{2}}$$

Therefore,

$$\int_{-\infty}^{\infty}\frac{1}{1+x^4}\,dx = 2\pi i\left(-\frac{i}{2\sqrt{2}}\right) = \frac{\pi}{2}$$

Exercise 24.1.2

Show that $\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx = \pi$

(Note: This integrand is not absolutely convergent)

If we try $f(z) = \frac{\sin z}{z}$ *on some semi-circle arc* C_R *with* $|f(z)| \leq \frac{1}{R}$, *then*

$$\left|\int_{C_R} f(z)\,dz\right| \leq \int_{C_R}\frac{1}{R}\,|dz| = \frac{\textit{length of } C_R}{R} \approx \pi$$

which means that the ***decay*** *of the* f *is insufficient to help us compute our desired result.*

Consider $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$. *We then need to show*

$$I = \int_{-\infty}^{\infty}\frac{e^{ix}-e^{-ix}}{2ix}\,dx = \pi$$

Let $f(z) = \frac{e^{iz}}{z} = \frac{1}{z}(1 + iz + \frac{(iz)^2}{2} + \ldots)$. *Thus* F *has a simple poles at* $z = 0$ *with residue* 1.

Consider the countour Γ_R:

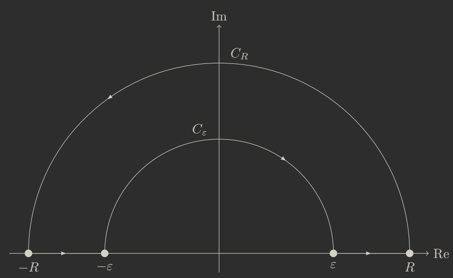


Figure 24.1: Contour Γ_R

The Residue Theorem gives

$$\frac{1}{2\pi i}\int_{\Gamma_R}\frac{e^{iz}}{z}\,dz = 0$$
$$\implies 0 = \int_{-R}^{-\varepsilon}\frac{e^{ix}}{x}\,dx + \int_{C_\varepsilon}\frac{e^{iz}}{z}\,dz + \int_{\varepsilon}^{R}\frac{e^{ix}}{x}\,dx + \int_{C_R}\frac{e^{iz}}{z}\,dz$$

Note that

$$\begin{aligned}\left|\int_{C_R}\frac{e^{iz}}{z}\,dz\right| &= \left|\int_0^\pi \frac{e^{iR(\cos\theta + i\sin\theta)}}{Re^{i\theta}}Rie^{i\theta}\,d\theta\right| \\ &\leq \int_0^\pi e^{-R\sin\theta}\,d\theta \\ &= \int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\,d\theta + \int_{\frac{\pi}{2}}^{\pi} e^{-R\sin\theta}\,d\theta \\ &= 2\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\,d\theta \quad \textit{by symmetry}\end{aligned}$$

By a similar argument as in A3Q1(c), on $[0, \frac{\pi}{2}]$,

$$\sin\theta \geq \frac{2}{\pi}\theta$$
$$e^{-\sin\theta} \leq e^{-\frac{2}{\pi}\theta}$$

Thus

$$\begin{aligned}\left|\int_{C_R}\frac{e^{iz}}{z}\,dz\right| &\leq 2\int_0^{\frac{\pi}{2}} e^{-\frac{2R\theta}{\pi}}\,d\theta \\ &= 2\left(-\frac{\pi}{2R}\right)e^{-\frac{2R\theta}{\pi}}\Big|_0^{\frac{\pi}{2}} = -\frac{\pi}{R}e^{-R-1}\end{aligned}$$

Thus we observe that as $R \to \infty$, *RHS* $\to 0$.

By A4Q4, on C_ε, *as* $\varepsilon \to 0$, *we pick up half of the residue at* $z = 0$, *i.e.*

$$\frac{1}{2\pi i}\int_{C_\varepsilon}\frac{e^{iz}}{z}\,dz = -\frac{1}{2},$$

in which we note that the value is negative since C_ε is clockwise.

Therefore,

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \left[\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\, dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\, dx \right] = \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx = \pi i$$

Using a similar argument, it can be shown that

$$\int_{-\infty}^{\infty} \frac{e^{-ix}}{x}\, dx = -\pi i$$

Work out the similar case as an exercise

Exercise 24.1.3
Show that

$$\int_{-\infty}^{\infty} \frac{e^{-ix}}{x}\, dx = -\pi i$$

And with that, we obtain the final solution of

$$\frac{\pi i - (-\pi i)}{2i} = \pi$$

as required.

Refer to Stein & Shakarchi, Section 2.1, for **more examples**.

Interesting Examples from Stein & Shakarchi

Example 24.1.1
Prove that for $0 < a < 1$,

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx = \frac{\pi}{\sin \pi a}$$

Choose $f(z) = \frac{e^{az}}{1+e^z}$. Note that f has one pole at $z = i\pi$. Consider a rectangle lying on the upper-half side of the plane with its base lying on the real axis and its top the line 2π.

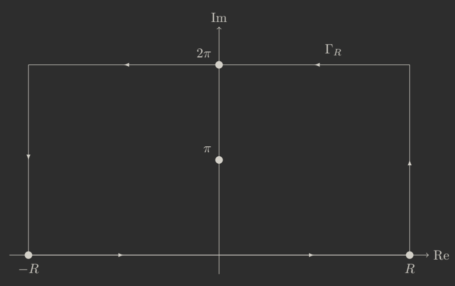


Figure 24.2: Contour Γ_R

By the Residue Theorem, we have that

$$\frac{1}{2\pi i} \int_{\Gamma_R} f(z)\, dz = \operatorname*{Res}_{z=i\pi} f(z)$$

To evaluate the residue, note that

$$(z - i\pi) \cdot \frac{e^{az}}{1+e^z} = e^{az} \cdot \frac{z - i\pi}{e^z - e^{i\pi}}$$

and $\lim_{z \to i\pi} \frac{z-i\pi}{e^z - e^{i\pi}}$ *is the inverse of the difference quotient (of first principles), and is hence* $e^{i\pi} = -1$*. Therefore*

$$\operatorname*{Res}_{z=i\pi} f(z) = \lim_{z\to\infty} (z - i\pi) \cdot \frac{e^{az}}{1+e^z} = -\lim_{z \to i\pi} e^{az} = -e^{ai\pi}.$$

Let I *be the desired integral and*

$$I_R := \int_{-R}^{R} f(z)\, dz$$

so that we may have $I_R \to I$ *as* $R \to \infty$*. This will describe the bottom part of the rectangle. For the top part of the integral, notice that it is simply*

$$\begin{aligned}
\int_{R}^{-R} \frac{e^{az}}{1+e^z}\, dz &= -\int_{-R}^{R} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}\, dx \\
&= -\int_{-R}^{R} \frac{e^{ax}e^{2\pi i a}}{1+e^x e^{2\pi i}}\, dx \\
&= -e^{2\pi i a} \int_{-R}^{R} \frac{e^{ax}}{1+e^x}\, dx \quad \because e^{2\pi i} = 1 \\
&= -e^{2\pi i a} I_R.
\end{aligned}$$

Let γ_r *parameterize the vertical path on the right side of the rectangle, i.e.* $\gamma_r : [0, 2\pi] \to \mathbb{C}$ *such that* $\gamma_r(t) = R + it$*. Then*

$$\begin{aligned}
\left| \int_{\gamma_r} \frac{e^{az}}{1+e^z}\, dz \right| &= \left| \int_0^{2\pi} \frac{e^{a(R+ti)}}{1+e^{R+ti}}\, dt \right| \\
&\leq \frac{e^{aR}}{1+e^R} \int_0^{2\pi} dt
\end{aligned}$$

which goes to 0 *as* $R \to \infty$ *since* $aR < R$ *for* $0 < a < 1$*. Thus the integral of the vertical path evaluates to* 0*. We can perform a similar procedure for the path on the left and end up with the same result.*

With that, we have that

$$I_R - e^{2\pi i a} I_R = -2\pi i e^{\pi i a}$$

and so

$$I_R = \frac{-2\pi i e^{\pi i a}}{1 - e^{2\pi i a}} = \frac{2\pi i}{e^{\pi i a} - e^{-\pi i a}} = \frac{\pi}{\sin \pi a}$$

and hence as $R \to \infty$*, we have*

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} \, dx = \frac{\pi}{\sin \pi a}$$

as required.

25 Lecture 25 Mar 16 2018

We are now in a position to look into how we can define "logarithms" for $\mathbb{C}$.

25.1 The Argument Principle

Since we may express $z = Re^{i(\theta + 2k\pi)}$ for some $k \in \mathbb{Z}$, we would expect a logarithm to be of the form

$$\log z = \log R + i(\theta + 2k\pi)$$

So in general,

$$\log f(z) = \log |f(z)| + i \arg f(z)$$

The derivative of $\log z$ is $\frac{f'(z)}{f(z)}$, should we expect the same idea extending from the reals, which is single-valued. Then the integral

$$\int_\gamma \frac{f'(z)}{f(z)} \, dz$$

can be interpreted as the change in the argument of f as z traverses the curve γ. Moreover, assuming that γ is a closed path, this change of argument is determined entirely by the zeros and poles of f in γ.

Note (Stein & Shakarchi, pg. 89)
The addivitity formula for log,

$$\log(f_1 f_2) = \log f_1 + \log f_2$$

fails in general.

Theorem 25.1.1 (Argument Principle)
Suppose f is meromorphic on a region (open & connected) $\Omega \subseteq \mathbb{C}$, γ a closed path such that $\gamma^* \in \Omega \setminus (\mathscr{A} \cup Z(f))$ such that

- $\forall w \notin \Omega \;\; \mathrm{Ind}_{\gamma}(w) = 0$
- $\forall w \in \Omega \setminus \gamma^* \;\; \mathrm{Ind}_{\gamma}(w) = 0$ *or* 1

Then

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}\, dz = \left|Z(f) \cap \gamma^0\right| - \left|\mathscr{A} \cap \gamma^0\right|$$

where the zeros and poles are counted by multiplicity.

Proof

(***include proof here: use Theorem 23.1.1 to CTP***).

Example 25.1.1

What are the poles of $\frac{f'}{f}$*?*

Suppose that f *has a zero of order* k *at* z_0*. Then* $\exists r > 0, \forall z \in D(z_0, r)$*,* $f(z) = (z - z_0)^k g(z)$ *where* $g \in H\big(D(z_0, r)\big)$ *and* $g \not\equiv 0$ *on* $D(z_0, r)$*. So*

$$\begin{aligned} f'(z) = k(z - z_0)^{k-1} g(z) + (z - z_0)^k g'(z) \implies \\ \frac{f'(z)}{f(z)} = \frac{k}{z - z_0} + \frac{g'(z)}{g(z)} \implies \end{aligned}$$

$\frac{f'}{f}$ *has a simple pole at* z_0 *with residue* k*.*

Suppose f *has a pole of order* k*. Then* $\exists r > 0, \forall z \in D(z_0, r), \exists h \in H\big(D(z_0, r)\big)$ $h \not\equiv 0$*,* $f(z) = (z - z_0)^{-k} h(z)$*. Then*

$$\begin{aligned} f'(z) = -k(z - z_0)^{-k-1} h(z) + (z - z_0)^{-k} h'(z) \implies \\ \frac{f'(z)}{f(z)} = \frac{-k}{z - z_0} + \frac{h'(z)}{h(z)} \implies \end{aligned}$$

$\frac{f'}{f}$ *has a simple pole at* z_0 *with residue* $-k$*.*

$\therefore$ f *is meromorphic on* $\Omega \implies \frac{f'}{f}$ *has simple zeros and poles at exactly the zeros and poles of* f *with residue equals to the order of zeros of* f *and negative of the order of poles of* f*, respectively.*

Theorem 25.1.2 (Rouché's Theorem)

Let $\Omega \subseteq \mathbb{C}$ *be a region,* $f, g \in H(\Omega)$*,* γ *a closed path on* Ω *with*

- $\forall w \notin \Omega \;\; \mathrm{Ind}_{\gamma}(w) = 0$*,*
- $\forall w \in \Omega \setminus \gamma^* \;\; \mathrm{Ind}_{\gamma}(w) = 0$ *or* 1*.*

If f, g *satisfy*

$$\forall z \in \gamma^* \;\; |f(z) - g(z)| < |f(z)|,$$

then f and g have the same number of zeros on γ^* (counted with multiplicity).

Proof (Idea)
WTS $\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \frac{1}{2\pi i}\int_\gamma \frac{g'}{g}$, *since the result would follow by Theorem 25.1.1. This is equivalent to* $\frac{1}{2\pi i} = \int_\gamma \left[\frac{f'}{f} - \frac{g'}{g}\right] = 0$. *By our small argument about how a* $\log$ *function should behave in* $\mathbb{C}$, *observe that we thus have*

$$(\log f)' - (\log g)' = \left(\log \frac{f}{g}\right)'$$

So we wish to get either $\frac{g}{f}$ *or* $\frac{f}{g}$.

Proof
Given that $\forall z \in \gamma^*$, $|f - g| < |f|$. *Note that* $f \not\equiv 0$; *otherwise we would have the impossible case of* $|0 - g| < |0|$.

Divide both sides by $|f|$, *then*

$$\forall z \in \gamma^* \quad \left|1 - \frac{g}{f}\right| < 1$$

Therefore, $\forall z \in \gamma^*$, $F = \frac{g}{f} \in B(1,1)$. *Let* $\gamma : [a,b] \to \mathbb{C}$ *be a parameterization of* γ^*. *Consider the function* $F \circ \gamma$, *which is a closed path that is contained in* $B(1,1)$. $\because z = 0$ *lies outside of* $F \circ \gamma$, *we have* $\mathrm{Ind}_{F\circ\gamma}(0) = 0$. *Then*

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{F\circ}^{\gamma} \frac{1}{z}\,dz = 0 \\
&\implies \frac{1}{2\pi i}\int_a^b \frac{F'\big(\gamma(t)\big)\gamma(t)\,dt}{F\big(\gamma(t)\big)} = 0 \\
&\implies \frac{1}{2\pi i}\int_\gamma \frac{F'(z)}{F(z)}\,dz = 0 \ \text{ by letting} \gamma(t) = z \\
&\implies \frac{1}{2\pi i}\int_\gamma \left(\frac{g'}{g} - \frac{f'}{f}\right) dz = 0 \qquad (\dagger)
\end{aligned}$$

where for Equation $(\dagger)$, *note that*

$$\begin{aligned}
F' &= \frac{g'f - gf'}{f^2} \ \text{ by Quotient Rule} \\
\frac{F'}{F} &= \frac{g'f - gf'}{f^2} \cdot \frac{f}{g} = \frac{g'}{g} - \frac{f'}{f}
\end{aligned}$$

The proof is complete by Theorem 25.1.1.

26 *Lecture 26 Mar 19 2018*

26.1 *The Argument Principle (Continued)*

Note (Notation)

Let f be a function meromorphic on a region $\Omega \subseteq \mathbb{C}$. We write

$$\begin{aligned} N_f &:= \#\text{zeros of } f \text{ inside } \gamma^* - \#\text{poles of } f \text{ inside } \gamma^* \\ &= \left|Z(f) \cap \gamma^0\right| - \left|\mathscr{A} \cap \gamma^0\right| \end{aligned}$$

Remark

If all conditions of Rouché's Theorem hold except that, instead, f & g are meromorphic on Ω, then if γ^* contains no poles of f & g then we can conclude that $N_f = N_g$

Exercise 26.1.1

Find the number of roots of $P(z) = z^8 - 5z^3 + z - 2$ lying in $|z| \leq 1$.

Solution

Let γ be the circle $|z| = 1$, oriented anticlockwise. Let $g(z) = P(z)$, $f(z) = -5z^3$[1]. Then $|f(z)| = \left|5z^3\right| = 5$, and

$$\begin{aligned} |f(z) - g(z)| &= \left|z^8 + z - 2\right| \\ &\leq 1 + 1 + 2 \quad \text{by Triangle Inequality, and on } \gamma \\ &= 4 < 5 = |f(z)| \end{aligned}$$

So the inequality in Rouché's Theorem holds. Hence by Rouché, $P(z) = g(z)$ has 3 roots (at $z = 0$, counted thrice since it has order 3) in $|z| < 1$.

To get the zeros for $|z| \leq 1$, change γ to be on $|z| = 1 + \varepsilon$ for some $\varepsilon > 0$ and proceed from there.

[1] We pick the dominant term in P for f

You should try more of these problems from the recommended texts.

26.1.1 *Alternative Proof for FTA*

Before proceeding with providing with alternative proof, note the following two definitions about polynomials.

Definition 26.1.1 (Monic Polynomial)
A monic polynomial is a polynomial with a leading coefficient of 1.

Definition 26.1.2 (Monomial)
A monomial is a polynomial with only one term.

Recall the statement of the Fundamental Theorem of Algebra (FTA)

$\forall P \in C[z]$ with $\deg P = n$ for some $n \in \mathbb{N}$, P has n roots in $\mathbb{C}$.

Proof
Without loss of generality, assume that the polynomial is monic (divide the polynomial by the leading coefficient if necessary). Take

$$g(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$$

with $a_1 \in \mathbb{C}$ *for* $i \in [1, n-1] \subset \mathbb{N}$*. Let* γ *be the circle* $|z| = R > \max\left\{\sum_{j=0}^{n-1} |a_j|, 1\right\}^2$*, oriented anticlockwise. Let* $f(z) = z^n$*. Then* $|f(z)| = R^n$ *on* γ*. We also have*

$$\begin{aligned} |g(z) - f(z)| &= \left|a_{n-1}z^{n-1} + \ldots + a_1 z + a_0\right| \\ &\leq |a_{n-1}| R^{n-1} + \ldots + |a_1| R + |a_0| \\ &\leq (|a_{n-1}| + \ldots + |a_1| + |a_0|) R^{n-1} \\ &< R^n \end{aligned}$$

Hence, the inequality for Rouché's Theorem holds. Hence by Rouché, $N_f = N_g$ *and* $N_f = n$*.*

Exercise: *Show that these are the only zeros of* $g(z)$*, using factorization of polynomials in the ring* $\mathbb{C}[z]$*.*

Suppose not, i.e. say g *has* $m \neq n$ *zeros. If* $m > n$*, then that would imply that* $\deg g = m$*, which* $\lightning$ *assumption. If* $m < n$*, then we can write*

$$g(z) = (z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_m) P_1(z)$$

[2] This is chosen from the later part of the proof

where each $\alpha_j \in \mathbb{C}$ *is a root of* g *and* $P_1 \in \mathbb{C}[z]$ *has* $\deg P_1 = n - m$ *and that* P_1 *has no roots (otherwise we would have* $m+1$ *roots). Then* P_1 *must be a constant polynomial, but that would imply that* $\deg g = m \neq n$*, which is yet another* $\lightning$*.* □

The above proof leads to the following result:

Corollary 26.1.1
All the zeros of a monic polynomial lie inside the disc $|z| \leq R$ *with* $R = \max \left\{ \sum_{j=0}^{n-1} |a_j|, a \right\}$ *where* $\{a_j\}_{j=0}^{n-1} \subset \mathbb{C}$ *are the coefficients of the monic polynomial.*

26.1.2 *Open Mapping Theorem*

Theorem 26.1.1 (Open Mapping Theorem)
If f *is holomorphic and non-constant in a region in* $\mathbb{C}$*, then* f *maps open sets to open sets.*

Proof
Let $w_0 = f(z_0)$ *for some* $z_0 \in \Omega \subseteq \mathbb{C}$*. Let* $d > 0$*.*

WTS $w_0 \in f\big(B(z_0, \delta)\big)^0$.

Let $\gamma = \partial B(z_0, \delta)$ *(i.e.* $|z - z_0| = \delta$*), oriented anticlockwise.* $\forall z \in B(z_0, \delta)$*, let* $F(z) := f(z) - w_0$*. Then* F *has at least one zero inside* γ *(in particular,* z_0*). Let* $G(z) := f(z) - w$ *for some* $w \in f\big(B(z_0, \delta)\big)$*.*

Want to have $G(z)$ *having a zero inside* γ *for* w *"close enough" to* w_0*.*

Our setup satisfies Rouché's inequality:

$$\begin{aligned} &\forall z \in \gamma^* \quad |F(z) - G(z)| < |F(z)| \\ &\text{or } \; |w - w_0| < |f(z) - w_0| \; \text{ on } \gamma^* \end{aligned}$$

We want $f(z) \neq w_0$ *on* γ*. Now we can choose a* $\delta > 0$ *such that* $B(z_0, \delta) \subseteq \Omega$ *and* $\forall z \in \partial B(z_0, \delta)$*,* $f(z) \neq w_0$*.*

Let $\varepsilon = \max_{z \in \gamma^*} |f(z) - w_0| > 0$*. Observe that*

$$\begin{aligned} |w - w_0| < \varepsilon &\implies Z(G) \cap \gamma^0 \neq \emptyset \\ &\implies w \in f\big(B(z_0, \delta)\big) \\ &\implies w_0 \in f\big(B(z_0, \delta)\big)^0 \end{aligned}$$

This completes the proof. □

27 Lecture 27 Mar 21 2018

27.1 Introductory Passage to Log Functions in $\mathbb{C}$

We have dealt with integrals of real numbers using our approach from complex analysis. But what would we do if we come across a problem of the form

$$\int_{-\infty}^{\infty} f(x)x^a \, dx \quad \text{for some } a \in \mathbb{R}?$$

If we try to apply residue integrals to the problem, we would need to consider $f(z)z^a$. But what is z^a, since $a \in \mathbb{R}$ and not simply $\in \mathbb{Z}$!?

When $a \in \mathbb{N}$, we know that $z^a = \underbrace{z \ldots z}_{a \text{ times}}$. When $a \in \mathbb{R}$, we want to be able to interpret z^a as $e^{a \log z}$ just as we can do so in $\mathbb{R}$. This leads to the study of log functions as complex variables.

We shall try to approach the problem via **analytic continuation**.

Exercise 27.1.1 (A simple problem in analytic continuation) *Let $f(z) = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$. We want to* ***analytically continue*** *f onto $\mathbb{C}$ if possible.*

For $|z| < 1$, we know that $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$. So let $g(z) = \frac{1}{1-z}$. Then we have that $g \in H(\mathbb{C} \setminus \{1\})$, where $z = 1$ is a simple pole, and g agrees with f on $|z| < 1$.

$\therefore$ g is an analytic continuation of f to $\mathbb{C} \setminus \{1\}$, or we say that f can be analytically continued except at $z = 1$, which is a simple pole.

In real analysis, log is the inverse of e[1]. But on $\mathbb{C}$, the exponential function is not 1-1, e.g.

[1] e in $\mathbb{R}$ is 1-1, and goes from $\mathbb{R} \to \mathbb{R}^+$

$$e^z = 1 \iff z \in 2\pi i\mathbb{Z}$$

As such, we would like to restrict the domain (**why?**) for the exponential function. That begs the question: what is the natural domain on which $\log z$ lives for $z \in \mathbb{C}$?

- Globally, we would require the notion of **Riemann Surfaces**
- Locally, we would require the notion of **Simply Connected Domains**

  (**What does local and global mean here?**)

27.2 Simply Connected Domains

Definition 27.2.1 (Homotopy (Poincaré))
Let X be a topological space[2]*. Recall that a curve in X is a continuous map $\gamma : I \to X$ where $I = [0, 1]$, and γ is said to be closed if $\gamma(0) = \gamma(1)$.*

Two closed curves γ_0 and γ_1 are said to be ***homotopic*** *if $\exists H : I \times I \to X$ with*

$$H(s, 0) = \gamma_0(s) \;\; H(s, 1) = \gamma_1(s)$$

and $H(s, t)$ be continuous with respect to s and t.

Alternative Definition from Stein-Shakarchi - Complex Analysis[3]

Let γ_0 and γ_1 be two curves in an open set Ω with common endpoints. So if γ_0 and γ_1 are two parameterizations on $[a, b]$, then

$$\gamma_0(a) = \gamma_1(a) = \alpha \;\text{ and }\; \gamma_0(b) = \gamma_1(b) = \beta$$

where $\alpha, \beta \in \Omega$. The two curves are said to be ***homotopic*** *in Ω if for each $0 \leq s \leq 1$, $\exists \gamma_s \subset \Omega$ parameterized by $\gamma_s(t)$ defined on $[a, b]$, such that $\forall s$,*

$$\gamma_s(a) = \alpha \;\text{ and }\; \gamma_s(b) = \beta,$$

and $\forall t \in [a, b]$,

$$\gamma_s(t)\Big|_{s=0} = \gamma_0(t) \;\text{ and }\; \gamma_s(t)\Big|_{s=1} = \gamma_1(t).$$

Moreover, $\gamma_s(t)$ should be jointly continuous in $s \in [0, 1]$ and $t \in [a, b]$.

Loosely speaking, γ_0, γ_1 are homotopic if we can **continuously**

[2] which we did not define

[3] I preferred this definition cause it's easier to read, but I shall be using the definition from the lecture for the class itself unless stated otherwise

deform γ_0 to γ_1 (wlog) without any obstruction in X.

Definition 27.2.2 (Simply Connected Domain)
*Let $\Omega \subseteq \mathbb{C}$ be open. We say Ω is **simply connected** if Ω is connected, and $\forall\gamma$ that is closed in Ω is homotopic to a point (i.e. a constant map $\gamma : I \to X$).*

Exercise 27.2.1

1. *$\mathbb{C}$ is simply connected.*
2. *$\mathbb{C} \setminus \{z = x + iy : x \leq 0, y = 0\}$ is simply connected.*
3. *$\mathbb{C} \setminus \{0\}$ is not simply connected.*

Note
I will temporarily use $\sim$ to represent homotopy, since it is an equivalence relation.

Here's a quick proof of that:

1. *(Reflexive) Define $H : I \times I \to X$, where $I = [a, b] \subseteq \mathbb{R}$, with $H(s, t) = \gamma_t(s)$, where, in this case, $t = 0$. This shows reflexivity.*
2. *(Symmetric) Suppose $\gamma_0 \sim \gamma_1$. Then $\exists H$ as above such that, this time, $t \in [0, 1]$. Choose $G : I \times I \to X$ with $G(s, t) = \gamma_{-t}(s)$ with $t \in [0, 1]$. Then $\gamma_1 \sim \gamma_0$.*
3. *(Transitive) Suppose $\gamma_0 \sim \gamma_1$ and $\gamma_1 \sim \gamma_2$. Then $\exists H_1, H, 2 : I \times I \to X$, I as above, with*

$$\begin{aligned} H_1(s,t) &= \gamma_t(s) \\ H_2(s,q) &= \gamma_q(s) \end{aligned}$$

   with $t \in [0, 1]$ and $q \in [1, 2]$. Then we can simply create $G : I \times I \to X$, now with the 2nd argument, say, $p \in [0, 2]$. such that

$$G(s,p) = \begin{cases} H_1(s,p) = \gamma_p(s) & p \in [0,1] \\ H_2(s,p) = \gamma_p(s) & p \in (1,2] \end{cases}$$

   Then $\gamma_0 \sim \gamma_2$.

One of the key facts about simply connected domains is that, if $f \in H(\Omega)$, then whenever $\gamma_0 \sim \gamma_1$ in Ω

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

i.e. in a simply connected domain, the ***integral does not depend on the chosen path***[4].

[4] This should remind you of Goursat's Theorem and Cauchy's Theorem

28 Lecture 28 Mar 23 2018

28.1 Constructing Logarithm

Theorem 28.1.1 (Theorem 17)
Suppose Ω is simply connected with $0 \notin \Omega$. Then in Ω, we can define a function, call it $\operatorname{Log} z$[1], such that

[1] This is called a branch of the logarithm

1. $\operatorname{Log} z \in H(\Omega) \wedge (\operatorname{Log} z)' = \frac{1}{z}$
2. $e^{\operatorname{Log} z} = z$ *for all* $z \in \Omega$
3. $\forall r \in \mathbb{R}^+ \; [1, r] \subseteq \Omega \implies \operatorname{Log} r - \operatorname{Log} 1 = \log r$ *where* $\log$ *denotes the usual natural logarithm on* $\mathbb{R}^+$.

Proof

1. *The proof can be completed using the method used in proving Cauchy's Theorem for Convex Sets if we define* Log *as follows:*

$$\forall z \in \Omega \;\; \exists w_0 \in \mathbb{C} \;\; e^{w_0} = z_0$$

(If we let $z_0 = Re^{i\theta}$, then we choose $w_0 = \log R + i\theta$) Define

$$\operatorname{Log} z = w_0 + \int_{z_0}^{z} \frac{1}{w} \, dw \tag{$\dagger$}$$

where the integral is over any path between the points z_0 and z in Ω. From here, use the proof provided in Cauchy's Theorem for Convex Sets to complete the proof.

2. *Let $G(z) = e^{-\operatorname{Log} z} \cdot z$. WTS $G(z) = 1$.*

Note that by part 1,

$$\forall z \in \Omega \;\; G'(z) = e^{-\operatorname{Log} z} - z \cdot \frac{1}{z} \cdot e^{-\operatorname{Log} z} = 0$$

$\therefore G' \equiv 0$ *in* Ω. $\because G \in H(\Omega)$, *we may write G as a power series,* *and since $G' \equiv 0$ on Ω, we have that $G(z) = G(z_0)$ in a neighbourhood of a chosen center $z_0 \in \Omega$, say with radius $r_0 > 0$. Therefore*

$$\exists c \in \mathbb{C}\, \forall z \in B(z_0, r_0) \;\; G(z) = c$$

Thus by Analytic Continuation, since Ω is connected, we have that $\forall z \in \Omega$, $G(z) = c$.

It is therefore sufficient to show that $G(z_0) = 1$, and this is true by the following:

$$\begin{aligned} G(z_0) &= e^{-\operatorname{Log} z_0} \cdot c_0 \\ &= e^{-w_0} \cdot z_0 \quad \because \textit{Equation } (\dagger) \\ &= \frac{z_0}{e^{w_0}} = 1 \quad \because e^{w_0} = z_0 \end{aligned}$$

Thus we have $G \equiv 1$ on Ω and hence $\forall z \in \Omega$, $e^{\operatorname{Log} z} = z$.

3. *Suppose $r \in \mathbb{R}^+$ and $[1, r] \subseteq \Omega$. By Equation $(\dagger)$,*

$$\begin{aligned} \operatorname{Log} r &= w_0 + \frac{z-0}{r} \frac{1}{w}\, dw \\ &= w_0 + \int_{z_0}^{1} \frac{1}{w}\, dw + \int_{1}^{r} \frac{1}{w}\, dw \\ &= \underbrace{w_0 + \int_{z_0}^{1} \frac{1}{w}\, dw}_{\operatorname{Log} 1 \text{ by } \textit{Equation } (\dagger)} + \underbrace{\int_{1}^{r} \frac{1}{t}\, dt}_{\log r - \log 1 = \log r} \end{aligned}$$

where we choose the straight line $[1, r]$ as the path for the 3rd term in the last line. Therefore we have

$$\operatorname{Log} r - \operatorname{Log} 1 = \log r$$

as required. □

Note
If we choose $z_0 = 1$ and $w_0 = 0$, then $\operatorname{Log} 1 = 0$, and hence $\operatorname{Log} r = \log r$ for any $r \in \mathbb{R}^+$ with $[1, r] \subseteq \Omega$.

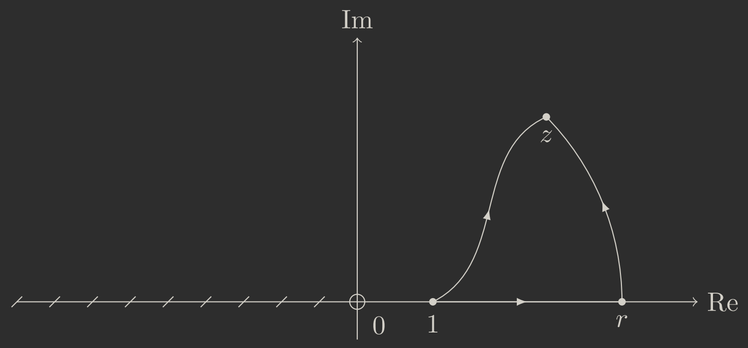
28.2 *Branches of the Logarithm*

1. (**Principal Branch**) Let $\Omega_1 = \mathbb{C} \setminus (-\infty, 0]$. We will write $z \in \Omega_1$ as $z = re^{i\theta}$ with $r > 0$ and $\theta \in (-\pi, \pi)$. Pick $z_0 = 1 \wedge w_0 = 0$> By

Equation (†), $\operatorname{Log} z = \int_1^z \frac{1}{w}\, dw$. Then in this case,

$$\operatorname{Log} z = \log r + i\theta \quad \text{when } z = re^{i\theta} \text{ with } \theta \in (-\pi, \pi)$$

To see this, pick the straight line path from 1 to r, and then any path from r to $z = re^{i\theta}$



Then

$$\begin{aligned} \int_1^z \frac{1}{w}\, dw &= \int_1^r \frac{1}{t}\, dt + \int_0^\pi \frac{ire^{it}}{re^{it}}\, dt \\ &= \log r + i\theta \end{aligned}$$

Exercise 28.2.1

Let $z_1 = e^{\frac{2\pi i}{3}}$*, then, using the Principal Branch,* $\operatorname{Log} z_1 = i\frac{2\pi}{3}$*. But note that* $\operatorname{Log}(z_1^2) \neq i\frac{4\pi}{3}$*. Instead, since*

$$z_1^2 = e^{\frac{4\pi i}{3}} = e^{-\frac{2\pi i}{3}}$$

($\because$ *the region in consideration is* $(-\pi, \pi)$*), we have that*

$$\operatorname{Log}(z_1^2) = -i\frac{2\pi}{3}$$

2. (a different branch) Let $\Omega_2 = \mathbb{C} \setminus [0, \infty)$. Write $z \in \Omega_2$ as $z = re^{i\theta}$ with $r > 0 \wedge \theta \in (0, 2\pi)$. Now we can pick **some function** so that

$$\operatorname{Log} z = \log r + i\theta \quad \text{with } z = re^{i\theta} \wedge \theta \in (0, 2\pi)$$

   In this case, we have that $\operatorname{Log}(z_1^2) = 2 \operatorname{Log} z_1$ does hold.

With that established, we may now use $z^a = e^{a \operatorname{Log} z}$ if we fix a branch (and a simply connected domain) and stick with it till the end of the problem.

Remark

For the Principal Branch of the logarithm, the following Taylor expan-

sion holds:

For $|z| < 1$*,*

$$\begin{aligned}\log(1+z) &= z - \frac{z^2}{2} + \frac{z^3}{3} - \dots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \frac{z^n}{n}\end{aligned}$$

(To remember this: note $-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$*) Let*

$$\begin{aligned}F(z) &= -\log(1-z) \\ G(z) &= \sum_{n=1}^{\infty} \frac{z^n}{n}.\end{aligned}$$

It can be shown that for $|z| < 1$*,*

$$F'(z) = \frac{1}{1-z} = G'(z)$$

which hence implies that

$$\begin{aligned}&(F-G)' \equiv 0 \\ \implies &F - C = c \quad \textit{for some } c \in \mathbb{C}\end{aligned}$$

Plug in $z = 0$ *and we will get* $c = 0$*.*

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29.1 Examples for Analytic Continuation

Gamma Function

For $s \in \mathbb{R}^+$, we define

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}$$

where

$$\begin{aligned} \int_0^\infty &: \text{ the integral over a locally compact topological group } \mathbb{R}^+ \\ e^{-t} &: \text{ additive character of } \mathbb{R}^+ \text{ (homomorphism from } (\mathbb{R}^+, +) \text{ to } \mathbb{R}) \\ t^s &: \text{ multiplicative character of } \mathbb{R}^+ \text{ (homomorphism from } (\mathbb{R}^+, \cdot) \text{ to } \mathbb{R}) \\ \frac{dt}{t} &: \textbf{ Haar measure} \text{ for } \mathbb{R}^+ \text{ (invariant under multiplication)} \end{aligned}$$

Exercise 29.1.1
The integral $\int_0^\infty e^{-t} t^s \frac{dt}{t}$ converges for $s > 0$. Prove this.

Note (Euler)
Euler observed that

$$\begin{aligned} \Gamma(n+1) &= \int_0^\infty e^{-t} t^{n+1} \frac{dt}{t} \\ &= \int_0^\infty e^{-t} t^n \, dt \\ &= -t^n e^{-t} \Big|_0^\infty + n \int_0^\infty e^{-t} t^{n-1} \, dt \quad \textit{by IBP} \\ &= n\Gamma(n) \\ &\vdots \\ &= n(n-1) \ldots 2 \cdot 1 \cdot \Gamma(1) \end{aligned}$$

and since $\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$*, we have that*

$$\Gamma(n+1) = n!$$

Remark
Euler observed that $\Gamma(s)$ *is a continuous and differentiable function of* s *that interpolates the factorials.*

We can extend $\Gamma(s)$ to complex numbers s as follows:

$$\forall s \in \mathbb{C} \quad \operatorname{Re} s > 0 \quad \Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}$$

Note

1. $\Gamma(s)$ *is holomorphic for* $\operatorname{Re} s > 0$
   - *It can be shown that* $\int_0^\infty e^{-t} t^s \frac{dt}{t}$ *converges for* $\operatorname{Re} s > 0$
   - *It can also show that this is* $\mathbb{C}$*-differentiable*
2. Γ *is a* ***Functional Equation****: We can repeat Euler's calculation to show that*

   $$\forall s \in \mathbb{C} \quad \operatorname{Re} s > 0 \quad \Gamma(s+1) = s\Gamma(s)$$

   which implies that, if $s \neq 0$,

   $$\underbrace{\Gamma(s)}_{\text{defined for } \operatorname{Re} s > 0} = \underbrace{\frac{\Gamma(s+1)}{s}}_{\text{defined for } \operatorname{Re} s > -1}$$

   because RHS makes sense for $\operatorname{Re} s > -1$*, in which we may do*

   $$-1 < \operatorname{Re} s < 0 \implies 0 < \operatorname{Re}(s+1) < 1.$$

   Thus, we can define, for $-1 < \operatorname{Re} s < 0$*, that, if* $s \neq 0$,

   $$\Gamma(s) = \frac{\Gamma(s+1)}{s}$$

   It is noteworthy that this definition agrees with our original definition of Γ *due to Equation* ($\dagger$).

Q: What happens at $s = 0$?
Consider Equation ($\dagger$), with $s \to 0^+$. Then

$$\lim_{s \to 0^+} [s\Gamma(s)] = \Gamma(1) = 0! = 1$$

$\therefore$ $\Gamma(s)$ behaves like $\frac{1}{s}$ near $s = 0$, i.e. Γ has a simple poles at $s = 0$.

Q: Can we continue the procedure above and go beyond $\operatorname{Re} s > -1$?

Yes. Equation (†) holes for $\Gamma(s+2)$ as well, which then we have, for $\operatorname{Re} s > 0$,

$$\Gamma(s+2) = (s+1)\Gamma(s+1) = (s+1)(s)\Gamma(s)$$

And thus for $\operatorname{Re} s > -2$ and $s \neq 0, -1$,

$$\Gamma(s) = \frac{\Gamma(s+2)}{s(s+1)}$$

We can proceed with this procedure inductively so and analytically continue Γ to $\mathbb{C} \setminus \{0, -1, -2, ...\}$.

29.2 *Characterizing Logarithms*

Theorem 29.2.1 (Theorem 18)
Any entire function $f(z)$ without any zeros has the form $Ae^{g(z)}$ where g is some entire function and $A \in \mathbb{C}$ is some constant.

This is a characterization of the function f that has no zeros or poles.

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30.1 Characterizing Logarithms

Theorem 30.1.1 (Theorem 18)
$\forall f$ that is entire with $Z(f) = \emptyset$,

$$f(z) = Ae^{g(z)}$$

where g is some entire function and $A \in \mathbb{C}$ is a constant.

Proof
Note

$$(f \in H(\mathbb{C}) \implies f' \in H(\mathbb{C})) \land Z(f) = \emptyset \implies \frac{f'}{f} \in H(\mathbb{C})$$

Choose

$$g'(z) = \frac{f'(z)}{f(z)} = c_0 + c_1 z + c_2 z^2 + \dots$$

where $\{c_j\}_{j \in \mathbb{Z}_{\geq 0}} \subseteq \mathbb{C}$[1].

[1] g can be obtained by term-wise integration of the Taylor series for $\frac{f'}{f}$

Consider $F(z) = f(z)e^{-g(z)}$. *Then* $\forall z \in \mathbb{C}$,

$$\begin{aligned} F'(z) &= f'(z)e^{-g(z)} - f(z)g'(z)e^{-g(z)} \\ &= f'(z)e^{-g(z)} - f(z)\frac{f'(z)}{f(z)}e^{-g(z)} \\ &= 0 \end{aligned}$$

$\therefore \forall z \in \mathbb{C}\ \ F'(z) \equiv 0$. *Now because of that and* $F \in H(\mathbb{C})$, $\exists A \in \mathbb{C}\ \forall z \in \mathbb{C}\ \ F(z) \equiv A$[2].

[2] By considering the Taylor series for F

$\therefore \forall z \in \mathbb{C}\ \ f(z) = Ae^{g(z)}$. $\square$

This characterizes any function f that has $Z(f) = \emptyset$. Suppose $f \in H(\mathbb{C})$ with $\mathscr{A}_f = \{a_1, a_2, a_3, ...\}$ for some $\{a_j\}_{j\in\mathbb{N}} \subseteq \mathbb{C}$. Construct some function $h \in H(\mathbb{C})$ with zeros at exactly every point in $\mathscr{A}_f$. For example,

$$h(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right)$$

is an entire function and has zeros at exactly $\mathscr{A}_f$. Then $\frac{f}{h} \in H(\mathbb{C})$ with $Z(\frac{f}{h}) = \emptyset$ on $\mathbb{C}$. Then by Theorem 30.1.1, $\exists g$, some entire function, and $A \in \mathbb{C}$ some constant, such that,

$$\frac{f}{h} = Ae^g \implies f(z) = Ah(z)e^{g(z)}$$

The construction of h motivates us to study our next topic: **infinite products**.

30.2 Infinite Products

Definition 30.2.1 (Infinite Products)
Let $u_1, u_2, ...$ be a sequence in $\mathbb{C}$. Let

$$P_N = \prod_{j=1}^{N} (1 + u_j)$$

*be the N^{th} **partial product**. If $\lim_{N\to\infty} P_N$ exists, then we say that the **infinite product**, $\prod_{j=1}^{\infty} (1 + u_j)$, converges, and write*

$$\lim_{N\to\infty} P_N = \prod_{j=1}^{\infty} (1 + u_j)$$

Before proceeding with an important result about infinite products, consider the following lemma.

Lemma 30.2.1 (Bounds of the Partial Product)
With $\{u_j\}_{j=1}^{\infty}$ being a sequence in $\mathbb{C}$, let

$$P_N^* = \prod_{j=1}^{N} (1 + |u_j|) .$$

Then

1. $P_N^* \leq \exp\left(\sum_{j=1}^{N} |u_j|\right)$

2. $\forall N \in \mathbb{N} \;\; |P_N - 1| \leq P_N^* - 1$[3]

[3] Note that $P_N^* \geq 1$

Proof

1. *Note that* $\forall x > 0 \;\; 1 + x \leq e^x$. *Thus*

$$\begin{aligned} P_N^* &= \prod_{j=1}^{N} (1 + |u_j|) \\ &\leq \prod_{j=1}^{N} \exp(|u_j|) \\ &= \exp\left(\sum_{j=1}^{N} |u_j| \right) \end{aligned}$$

2. *Using induction, note that* $N = 1$ *holds since,*

$$\begin{aligned} P_N - 1 &= 1 + u_1 - 1 = u_1 \\ P_N^* - 1 &= 1 + |u_1| - 1 = |u_1|. \end{aligned}$$

For $N + 1$, *we have*

$$\begin{aligned} |P_{N+1} - 1| &= |P_N(1 + u_{N+1}) - 1| \\ &= |P_N(1 + u_{N+1}) - (1 + u_{N+1}) + (1 + u_{N+1}) - 1| \\ &= |(P_N - 1)(1 + u_{N+1}) + u_{N+1}| \\ &\leq |P_N - 1| \, |1 + u_{N+1}| + |u_{N+1}| \quad \text{By Triangle Inequality} \\ &\leq (P_N^* - 1)(1 + |u_{N+1}|) + |u_{N+1}| \quad \text{By IH} \\ &= P_N^*(1 + |u_{N+1}|) - (1 + |u_{N+1}|) + |u_{N+1}| \\ &= P_{N+1}^* - 1 \end{aligned}$$

and the induction is complete. □

Remark

Lemma 30.2.1 continues to hold if P_N, P_N^* *are replaced by*

$$\begin{aligned} P_{N,M} &= \prod_{j=M+1}^{N} (1 + u_j) \\ P_{N,M}^* &= \prod_{j=M+1}^{N} (1 + |u_j|) \end{aligned}$$

respectively, where $N \geq M + 1$.

Theorem 30.2.1 (Theorem 19)
Suppose that $\sum_{n=1}^{\infty} |u_n|$ *converges. Then* $\prod_{n=1}^{\infty}(1+u_n)$ *converges. Moreover,* $\prod_{n=1}^{\infty}(1+u_n)$ *converges to zero iff* $u_{n_0} = -1$ *for some* $n_0 \in \mathbb{N}$.

Proof
Let $P_N = \prod_{n=1}^{N}(1+u_n)$.

WTS $\{P_N\}$ *is convergent. It suffices to show that* $\{P_N\}$ *is Cauchy, i.e.*[4]

$$\forall \varepsilon > 0 \;\; \exists N_0 > 0 \;\; \forall M, N > N_0 \;\; |P_N - P_M| < \varepsilon$$

[4] Cauchy convergence from real analysis

$$\begin{aligned}
|P_M - P_N| &= \left| P_N \left(\prod_{j=N+1}^{M} (1+u_j) - 1 \right) \right| \\
&= |P_N| \left| \prod_{j=N+1}^{M} (1+u_j) - 1 \right| \\
&\leq |P_N| \left| \prod_{j=N+1}^{M} (1+|u_j|) - 1 \right| \quad \textit{By 2 of Lemma } 30.2.1 \\
&\leq |P_N| \left| \exp\left(\sum_{j=N+1}^{M} |u_j| \right) - 1 \right| \quad \textit{By 1 of Lemma } 30.2.1 \\
&\leq P_N^* \left(\exp\left(\sum_{j=N+1}^{M} |u_j| \right) - 1 \right).
\end{aligned}$$

Thus $\forall \varepsilon > 0$, $\because \sum_{j=1}^{\infty} |u_j| < \infty$ *by assumption, we can choose an appropriate* $N_0(\varepsilon) > 0$ *such that* $\forall M > N \geq N_0$,

$$\sum_{j=N+1}^{M} |u_j| < \varepsilon^{5}$$

So by choosing the appropriate N_0 *we can have*

[5] Since it is convergent and hence Cauchy.

$$|P_M - P_N| \leq P_N^*(e^{\varepsilon} - 1)$$

Note that

$$\begin{aligned}
P_N &\leq \exp\left(\sum_{j=1}^{N} |u_j| \right) \quad \textit{By 1 of Lemma } 30.2.1 \\
&\leq \exp c \quad \textit{For some } c \in \mathbb{C} \\
&= c_0 \quad \textit{For some } c_0 \in \mathbb{C}
\end{aligned}$$

Therefore, for $M > N \geq N_0$, we have $|P_M - P_N| \leq c_0\varepsilon$ and P_N is hence Cauchy.

(Proof to be continued in next lecture)

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31.1 Infinite Products (Continued)

Proof (Continued)

Note that in the earlier part of the proof, we showed that

$$|P_M - P_N| \leq |P_N| \left(\exp\left(\sum_{j=N+1}^{M} u_j \right) - 1 \right) \tag{31.1}$$

Notice that by the Reverse Triangle Inequality,

$$|P_M| = |P_M - P_N + P_N| \geq ||P_M - P_N| - |P_N||.$$

So for large enough M, N,

$$|P_N - P_M| \leq |P_N| (e^{\varepsilon} - 1) \text{ by Equation (31.1) and the earlier part.}$$

Thus

$$\begin{aligned} |P_N| - P_M - P_N &\geq |P_N| (1 - (e^{\varepsilon} - 1)) \\ &= |P_N| (2 - e^{\varepsilon}). \end{aligned}$$

Therefore, for sufficiently large M, N,

$$|P_M| \geq ||P_N| - |P_M - P_N|| \geq |P_N| (2 - e^{\varepsilon}) \tag{31.2}$$

Now to prove the iff statement: Suppose that the infinite product converges to 0. Let $M \to \infty$ and fix N_0 from above to be sufficiently large. Then for Equation (31.2), $LHS \to 0$ as $M \to \infty$. Thus $RHS \to 0$ as well, and we thus have that, in the limit, $|P_{N_0}| (2 - e^{\varepsilon}) = 0$ and hence $|P_{N_0}| = 0$. But since P_{N_0} is a finite product, there must $\exists n_0 \in \mathbb{N}$ such that $u_{n_0} = -1$.

The converse is trivially true: suppose that $\exists n_0 \in \mathbb{N}$ *such that* $u_{n_0} = -1$. *Then we have that* $(1 - u_{n_0}) = 0$ *and hence the product is* 0.
$\square$

Remark
To apply Theorem 30.2.1 to a sequence of functions $\{u_n(z)\}$ *in some region* $\Omega \subseteq \mathbb{C}$, *we need* $\sum u_n(z)$ *to converge absolutely and uniformly*[1].

[1] No dependence on z, which is part of the definition of **uniform convergence**.

31.1.1 *Application to Riemann Zeta Function*

We define $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\operatorname{Re}(s) > 1$. This function is the well-known **Riemann Zeta Function**

Remark
1. *The series* $\sum_{n=1}^{\infty} \frac{1}{n^s}$ *is absolutely convergent for* $\operatorname{Re}(s) > 1$.

2. *By the construction of the function, it is holomorphic/analytic for* $\operatorname{Re}(s) > 1$[2]

[2] Requires the Weierstrass' M-test.

(History) Euler looked at the series with real numbers first. It was not until Riemann extended the function to become a function with complex variables that the series became well-known, and hence Riemann's name is prepended to the function instead of Euler.

The series can be analytically continued to the entire complex plane (using the functional equation[3]), except for a simple pole at $s = 1$, i.e.

$$\lim_{s \to 1^+} (1 - s)\zeta(s) = 1.^4$$

[3] This is similar to what we did for the Gamma function.

[4] Cauchy's Residue Theorem

Euler showed that for $\operatorname{Re}(s) > 1$, $\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{prime}} (1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots)$. Observe that RHS converges absolutely for $\operatorname{Re}(s) > 1$. This identity is known as **Euler's Identity** and it is simply a statement about the unique factorization of integers into primes[5].

[5] This is the **Fundamental Theorem of Arithmetic**

Note that for $\mathrm{Re}\,(s) > 1$, we can write

$$\begin{aligned}
\zeta(s) &= \prod_{p \text{prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right) \\
&= \prod_{p \text{prime}} \left(\frac{1}{1 - \frac{1}{p^s}}\right) \quad \text{(Infinite Geometric Sum)} \\
&= \prod_{p \text{prime}} \left(1 - \frac{1}{p^s}\right)^{-1}
\end{aligned}$$

This will be useful for the next statement.

Corollary 31.1.1 (Corollary for Theorem 19)
$\zeta(s) \neq 0$ *for* $\mathrm{Re}\,(s) > 1$.

Proof
Fix s *with* $\mathrm{Re}\,(s) > 1$. *Then* $\zeta(s) = \prod_{n=1}^{\infty}(1 + u_n)$ *with*

$$u_n = \begin{cases} 0 & \text{if } n \neq p \text{ prime} \\ \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots & \text{if } n = p \text{ prime} \end{cases}$$

For each s, *we have that each of the sums* $\frac{1}{p^s} + \frac{1}{p^{2s}} + \dots$ *converges absolutely for* $\mathrm{Re}\,(s) > 1$. *Also,* $\sum_{n=1}^{\infty} u_n$ *converges absolutely and uniformly for* $\mathrm{Re}\,(s) > 1$.[6]

[6] These two statements are not too hard to make reliable heuristics to make sense that they are true.

Basically, we can apply Theorem 30.2.1. So

$$\begin{aligned}
\forall s \in \mathbb{C}\ \mathrm{Re}\,(s) > 1\ \zeta(s) = 0 &\iff \exists n \in \mathbb{N}\ u_n = -1 \\
&\iff 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots \text{ for } p \text{ prime} \\
&\iff \frac{p^s}{p^s - 1} \text{ by the Infinite Geometric Sum} \\
&\iff p^s = 0 \iff e^{s \log p} = 0 \\
&\lightning\ \forall x \in \mathbb{R} e^x \neq 0.
\end{aligned}$$

This completes the proof. $\square$

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32.1 Infinite Products (Continued 2)

32.1.1 Weierstrass Products

PROBLEM: Given a sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$, where $\forall n \in \mathbb{N}$, $a_n \neq 0$, construct an entire function f with zeros precisely at each of the a_n's.

It is tempting to consider an infinite product such as $\prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right)$. But this may not converge all the time except for some specific sequence $\{a_n\}$.

The idea to approach this problem is to take the product of exponential factors so that it takes care of the convergence problem.

DEFINE: $\forall n \in \mathbb{N}$

$$P_n(z) := z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots + \frac{z^n}{n}$$
$$E_n(z) := (1 - z)\exp\left(P_n(z)\right)$$

and for $n = 0$, define

$$E_0(z) := 1 - z$$

OBSERVE that

- E_n is entire and has a simple zero at exactly $z = 1$.
- $E_n(0) = 1$ and $E_n(1) = 0$
- $P_n'(z) = 1 + z + z^2 + \ldots + z^{n-1} = \frac{z^n - 1}{z - 1}$[1]

[1] Finite geometric sum

Theorem 32.1.1 (Theorem 20)
For $|z| \leq 1$, $|1 - E_n(z)| \leq |z|^{n+1}$.

Proof
Note that

$$\begin{aligned} E_n'(z) &= (1-z)\exp(P_n(z)) \cdot \left(\frac{z^n - 1}{z-1} \right) - \exp(P_n(z)) \\ &= -z^n \exp(P_n(z)) \end{aligned}$$

and note that by the Taylor expansion of the exponent,

$$\exp(P_n(z)) = 1 + P_n(z) + \frac{(P_n(z))^2}{2!} + \dots.$$

Since the power/Taylor expansion of $P_n(z)$ *has non-negative and real coefficients, so does* $\exp(P_n(z))$*. Therefore, we have*

$$\begin{aligned} -E_n'(z) &= z^n \exp(P_n(z)) \\ &= z^n \cdot \sum_{m=0}^{\infty} a_m z^m \quad a_0 \neq 0^2,\ \forall m \in \mathbb{N}\ a_m \geq 0 \\ &= \sum_{m=n}^{\infty} c_m z^m \quad c_0 \neq 0,\ \forall m > n\ c_m \geq 0, \end{aligned} \tag{$\star$}$$

which holds $\forall z \in \mathbb{C}$ *since* E_n' *is entire*[3].

[2] This is because the exponential function is non-zero.
[3] This is true by Theorem 8.1.1, since E_n is entire.

Now note that

$$\begin{aligned} \int_0^z -E_n'(w)\,dw &= -E_n(z) + E_n(0) \quad \textit{by FTC} \\ &= 1 - E_n(z). \end{aligned}$$

Also, by Equation ($\star$),

$$\int_0^z -E_n'(w)\,dw = \sum_{m=n}^{\infty} \frac{c_m}{m+1} z^{m+1}.$$

Therefore,

$$1 - E_n(z) = \sum_{m=n}^{\infty} \frac{c_m}{m+1} z^{m+1} = \frac{c_n}{n+1} z^{n+1} + \dots,$$

which implies that

$$\frac{1 - E_n(z)}{z^{n+1}} = \sum_{m=n}^{\infty} \frac{c_m}{m+1} z^{m-n} \qquad \because c_n \neq 0$$

Let $\phi_n(z) := \frac{1-E_n(z)}{z^{n+1}} = \sum_{k=0}^{\infty} b_k z^k$*, where as defined, we have that* $b_0 \neq 0$ *and* $\forall k \in \mathbb{N}\ b_k \geq 0$*. By the Triangle Inequality, and since* $|z| \leq 1$*, we have that*

$$\begin{aligned}\left|\frac{1-E_n(z)}{z^{n+1}}\right| &= \left|\sum_{k=0}^{\infty} b_k z^k\right| \leq \sum_{k=0}^{\infty} \left|b_k z^k\right| \leq \sum_{k=0}^{\infty} (|b_k| \cdot 1) \\ &= \sum_{k=0}^{\infty} b_k = \phi_n(1) = \frac{1-E_n(1)}{1} = 1.\end{aligned}$$

Hence for $|z| \leq 1$*, we have the desired inequality,*

$$|1 - E_n(z)| \leq |z|^{n+1}$$

$\square$

Theorem 32.1.2 (Theorem 21)
Let $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ *with* $a_n \to \infty$ *as* $n \to \infty$*, and* $\forall n \in \mathbb{N}$*,* $a_n \neq 0$*. Then the* ***Weierstrass Product***

$$f(z) = \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right) \tag{32.1}$$

is entire and has zeros at exactly $\{a_n\}$ *and nowhere else.*

Proof
Let $R > 0$ *and* $\Omega = D(0, R) \subseteq \mathbb{C}$*. Let* $z \in \Omega$*.* $\because |a_n| \to \infty$*,* $\exists N_0 > 0\ \forall n \geq N_0\ \ |a_n| \geq 2R$*. Note*

$$\prod_{n=N_0}^{\infty} E_n\left(\frac{z}{a_n}\right) = \prod_{n=N_0}^{\infty}\left(1 + \underbrace{\left(E_n\left(\frac{z}{a_n}\right) - 1\right)}_{\text{call this } u_n(z)}\right)$$

On Ω*, since* $|z| \leq R \wedge \forall n \geq N_0\ \ |a_n| \geq 2R$*, by Theorem 32.1.1 we have that* $\forall n \geq N_0 \wedge \forall z \in \Omega$*,*

$$\left|\frac{z}{a_n}\right| \leq 1 \implies \left|E_n\left(\frac{z}{a_n}\right) - 1\right| \leq \left|\frac{z}{a_n}\right|^{n+1} \leq \frac{1}{2^{n+1}}$$

Then by Theorem 30.2.1 and its remark, since $\sum_{n=1}^{\infty} u_n(z)$ *converges absolutely and uniformly on* Ω*, the infinite product* $\prod_{n=N_0}^{\infty}(1 + u_n(z))$ *converges.*

Exercise *To prove that the infinite product converges uniformly on* Ω*, it suffices to show that it satisfies the Cauchy criterion: Define*

$f_n := \sum_{j=n}^{\infty}(1+u_j(z))$. Note that for $m > n \geq N_0$,

$$\begin{aligned}|f_m - f_n| &= \left|\sum_{j=n}^{m-1}\big(1+u_j(z)\big)\right| \\ &\leq \left|\sum_{j=n}^{m-1}\big(1+|u_j(z)|\big)\right| \quad \textit{by 2 of Lemma 30.2.1} \\ &\leq \exp\sum_{j=n}^{m-1}|u_j(z)| \quad \textit{by 1 of Lemma 30.2.1}\end{aligned}$$

So $\forall \varepsilon > 0$, since $\sum_{j=1}^{\infty}|u_n(z)|$ converges, $\exists N_0 > 0$[4] $\forall m > n \geq N_0$, such that

$$\sum_{j=n}^{m-1}|u_n(z)| \leq \varepsilon$$

and hence

$$|f_m - f_n| \leq e^{\varepsilon},$$

thus satisfying the Cauchy criterion and hence showing that the infinite product is uniformly convergent.

[4] choose an N_0 different from the earlier N_0 if necessary

Exercise *We now need to show that the infinite product is a holomorphic function on Ω. Note that by construction of E_n, it has no poles, and is entire. Note that we have that the $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathbb{C}$ that is uniformly convergent to the function $f(z) = \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right)$ in Ω. Let D be any disc whose closure[5] is contained in Ω, and T any triangle in D. Then by Goursat's Theorem,*

A more complete statement and proof of this statement is available in Stein & Shakarchi's Complex Analysis (Chapter 2, Section 5.2, Theorem 5.2).

[5] $\overline{D} := \{K \in \mathbb{C} : K \text{ closed } \wedge D \subseteq K\}$

$$\int_T f_n = 0.$$

Since $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f, f is continuous[6] and

$$\int_T f_n \to \int_T f.$$

Thus $\int_T f = 0$. Then by Morera's Theorem, we have that $f \in H(D)$ and since D is arbitrary, $f \in H(\Omega)$ as required.

[6] This is a relatively easy proof that can be done using techniques from Real Analysis.

Exercise 32.1.1
Prove that f is continuous under the supposition that $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent.

Now since R is arbitrary, we have that $f \in H(\mathbb{C})$, i.e. f is entire.

To find the zeros of f, let $z \in \Omega$. By Theorem 30.2.1 and in particular its remark, we have that

$$\prod_{j=N_0}^{\infty} E_j\left(\frac{z}{a_j}\right) = 0 \iff E_n\left(\frac{z}{a_n}\right) = 0 \text{ for some } n \geq N_0,$$

Therefore,

$$\prod_{j=1}^{\infty} E_N\left(\frac{z}{a_j}\right) = 0 \iff E_n\left(\frac{z}{a_n}\right) = 0 \iff z = a_n$$

for some $n \in \mathbb{N}$. *This shows that*

$$Z(f) = \{a_n\} \cap \Omega,$$

and since $R > 0$ *is arbitrary, we have that* $Z(f) = \{a_n : n \in \mathbb{N}\}$. *This completes the proof.* □

Example 32.1.1

Construct an entire function with simple zeros at $z = 0, \pm 1, \pm 2, ...,$ *i.e.* $z \in \mathbb{Z}$.

SOLUTION 1: *Define* $f_1(z) = \prod_{n=1}^{\infty} E_n\left(\frac{z}{n}\right)$, *which is entire and has zeros at exactly* $\mathbb{N}$ *by Theorem 32.1.2, and* $f_2(z) = f_1(-z)$, *which would have zeros at* $-\mathbb{N}$. *Then simply choose the function*

$$f(z) = z \cdot f_1(z) \cdot f_2(z)$$

so that we include the root $z = 0$.

SOLUTION 2: *We can also choose*

$$f(z) = z \cdot \prod_{n=1}^{\infty}\left(1 - \left(\frac{z}{n}\right)^2\right)$$

Exercise 32.1.2
Check that f *indeed has its zeros at all integers.*

Note that $\sin(\pi z)$ *is entire and has simple zeros at exactly* $z \in \mathbb{Z}$. *Then*

$$\frac{\sin(\pi z)}{f(z)} \text{ has no zeros or poles } \forall z,$$

and therefore by Theorem 30.1.1, $\exists g \in H(\mathbb{C})$, *such that*

$$\frac{\sin(\pi z)}{f(z)} = e^{g(z)}.$$

By the ***Product Formula of Sine***[7,8], *we have that*

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

and hence

$$f(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) = \frac{\sin(\pi z)}{\pi}.$$

[7] First proven by Euler, although incomplete.

[8] It turns out $g(z) = \log \pi$.

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