PMATH451 — Measure and Integration

Class notes for Fall 2019

by

Johnson Ng

BMath (Hons), Pure Mathematics major, Actuarial Science Minor

University of Waterloo

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Preface

Assignment problems is introduced in class as we go, so we have the special environment homework for these in this note.

I included a special chapter in the appendix (see Appendix A) that records and provides insights into what drove the direction(s) of certain proofs. This is an attempt to resolve the problem of proofs being overly obscure with its motivations. Contents presented in this appendix are typically like rough work, and so are typically much longer than the presented proof.

I also made an appendix for some of the common themes and tricks (see Appendix B) that are seen repeatedly in this topic I think it is invaluable that they are noted down, because the ideas that these commonalities carry forward.

1 Lecture 1 Sep 04th, 2019

1.1 Motivation for the Study of Measures

Recall Riemann integration.

Definition (Riemann Integration)

Let $f : [a, b] \to \mathbb{R}$ be a ***bounded*** *function. We call*

$$P = \{a = x_0 < x_1 < \ldots < x_n = b\} \subseteq [a, b]$$

a ***partition*** *of $[a, b]$, and*

$$\Delta x_i = x_i - x_{i-1}$$

as the ***length of the i^{th} interval*** *for $i = 1, \ldots, n$.*

Let

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

be the ***supremum of f on the i^{th} interval****, and*

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$

be the ***infimum of f on the i^{th} interval****. We define the* ***Riemann upper sum*** *as*

$$U(f, P) = \sum_i M_i \Delta x_i,$$

and the ***Riemann lower sum*** *as*

$$L(f, P) = \sum_i m_i \Delta x_i.$$

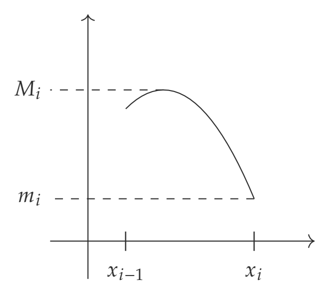

Figure 1.1: Idea of Riemann integration

We define the ***Riemann upper integral*** *as*

$$\overline{\int_a^b} f\,dx = \inf_P U(f,P)$$

and the ***Riemann lower integral*** *as*

$$\underline{\int_a^b} f\,dx = \sup_P L(f,P).$$

We say that f is ***Riemann integrable*** *if*

$$\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx,$$

and we write the integral of f as

$$\int_a^b f\,dx = \overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx.$$

As hyped up as one does earlier in university about Riemann integration, there are functions that are not Riemann integrable!

Example 1.1.1

Consider a function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}.$$

Then

$$\overline{\int_a^b} f\,dx = 1 \text{ and } \underline{\int_a^b} f\,dx = 0.$$

Thus f is not Riemann integrable.

Note 1.1.1 (Shortcomings of the Riemann integral)

1. *We cannot characterize functions that are Riemann integrable, i.e. we do not have a list of characteristics that we can check against to see if a function is Riemann integrable.*

This remained an open problem in the earlier 1920s.

2. *The Riemann integral behaves badly when it comes to pointwise limits of functions. The next example shall illustrate this.*

3. *The Riemann integral is awkward when f is unbounded. In particular, we used to hack our way around by looking at whether the Riemann integral converges to some value the function approaches the unbounded point, and then "conclude" that the integral is the limit of that convergence.*

4. *Recall that the* ***Fundamental Theorem of Calculus*** *states that*

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

We know that this works for Riemann integrals. By the first shortcoming, the problem here is that we do not fully know what are the functions that the Fundamental Theorem is true for.

5. *In PMATH450, we saw that Fourier developed the Fourier series, which is an extremely useful tool in solving* ***Differential Equations*** *using sines and cosines. However, the convergence of the Fourier series remains largely unexplained by Fourier, and we have but developed some roundabout ways of showing some convergence.*

6. *Consider the set R if Riemann integrable functions on the interval $[a, b]$. The set R has a natural metric:*

$$d(f, g) = \int_a^b \left| f - g \right| dx.$$

However, the metric space (R, d) is ***not complete****. This means many of our favorite results in PMATH351 are not usable!*

7. *There are many functions that seem like they should have an integral, but turned out that they did not under Riemann integration.*

Example 1.1.2 (Pointwise Limits of Riemann Integrable Functions is not necessarily Riemann Integrable)

Let $\mathbb{Q} = \{x_n\}_{n \in \mathbb{N}}$. Then consider a sequence of functions

$$f_n(x) = \begin{cases} 1 & x \in \{x_1, \ldots, x_n\} \\ 0 & x \notin \{x_1, \ldots, x_n\} \end{cases}.$$

It is rather clear that

$$\overline{\int_a^b} f \, dx = \underline{\int_a^b} f \, dx = 0.$$

However, the pointwise limit of the f_n's, and that is

$$\lim_{n \to \infty} f_n(x) = f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases},$$

is, as mentioned in the last example, not Riemann integrable.

To address the shortcomings of the Riemann integral, Henri Lebesgue developed the **Lebesgue integral**, of which we have seen in PMATH450.

Instead of dividing the x-axis, Lebesgue decided to divide the y-axis first.

If the range of a function f is $[c, d]$, where c, d can be infinite, then we partition the interval such that

$$P = \{c = y_0 < y_1 < \ldots < y_n = d\},$$

and we define

$$E_i = \{x : f(x) \in [y_{i-1}, y_i]\}.$$

Then if A_i is the area of the "rectangle" for the i^{th} interval of $[c, d]$, we have

$$y_{i-1} \cdot \ell(E_i) \leq A_i \leq y_i \cdot \ell(E_i),$$

where $\ell(E_i)$ is the **Lebesgue measure** of the set E_i. Then if we let $\int_a^b f$ denote the Lebesgue integral of f, we would expect

$$\sum_{i=1}^{n} y_{i-1} \cdot \ell(E_i) \leq \int_a^b f \leq \sum_{i=1}^{n} y_i \cdot \ell(E_i).$$

However, to truly understand what this means, we need to understand what the Lebesgue measure is.

Furthermore, recall that in PMATH450, we saw that not all sets, in $\mathbb{R}$ for example, are measurable, and for 'good' reasons, there always exists non-measurable sets.

1.2 *Algebras and σ-Algebra of Sets*

For this course, we shall use the convention that

- the 'ambient' space X is always non-empty;
- $\mathcal{P}(X)$, the power set of X, has non-trivial elements; and
- we denote $A^C = \{x \in X : x \notin A\}$ for $A \subseteq X$.

Definition 1 (Algebra of Sets)

Given X, a non-empty collection of subsets of X, i.e. $\emptyset \neq \mathcal{A} \subseteq \mathcal{P}(X)$, is called an ***algebra of sets*** *of X provided that:*

1. $A_1, \ldots, A_n \in \mathcal{A} \implies \bigcup_{i=1}^n A_i \in \mathcal{A}$; *and*
2. $A \in \mathcal{A} \implies A^C \in \mathcal{A}$.

Proposition 1 (Properties of Algebra of Sets)

If $\mathcal{A}$ is an algebra of sets of X, then

3. $\emptyset, X \in \mathcal{A}$;
4. $A, B \in \mathcal{A} \implies A \setminus B = \{x \in X \mid x \in A \wedge x \notin B\} \in \mathcal{A}$; *and*
5. $A_1, \ldots, A_n \in \mathcal{A} \implies \bigcap_{i=1}^n A_i \in \mathcal{A}$.

Proof

3. $\mathcal{A} \neq \emptyset \implies \exists A \in \mathcal{A} \implies A^C \in \mathcal{A} \implies A \cup A^C = X \in \mathcal{A} \implies \emptyset = X^C \in \mathcal{A}$.
4. $A, B \in \mathcal{A} \implies A^C \in \mathcal{A} \implies A^C \cup B \in \mathcal{A} \implies A \setminus B = (A^C \cup B)^C \in \mathcal{A}$.
5. (**De Morgan's Law**) Notice that $(A_1 \cap A_2 \cap \ldots \cap A_n)^C = A_1^C \cup A_2^C \cup \ldots A_n^C \in \mathcal{A}$ since $A_i^C \in \mathcal{A}$. Thus the complement

$$A_1 \cap A_2 \cap \ldots \cap A_n \in \mathcal{A}. \qquad \square$$

Definition 2 (σ-Algebra of Sets)

Given X and $\emptyset \neq \mathcal{A} \subseteq \mathcal{P}(X)$, we say that $\mathcal{A}$ is a ***σ-algebra of sets*** *of X if it is an algebra of sets and*

$$\forall A_n \in \mathcal{A}, n \in \mathbb{N}, \quad \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}.$$

Example 1.2.1

1. $\mathcal{P}(X)$ is a σ-algebra.
2. Consider X as an infinite set. We say that a set A is **cofinite** if A^C is finite. Let

$$\mathcal{A} := \{A \in \mathcal{P}(X) \mid A \text{ is finite or cofinite }\}.$$

Then $\mathcal{A}$ is an algebra of sets:

- finite union of finite sets remains finite;
- finite union of finite and cofinite sets remains cofinite; and
- complement of finite sets are the cofinite sets and vice versa.

However, $\mathcal{A}$ is **not** a σ-algebra: consider $A_n = \{2^n\} \subseteq X = \mathbb{N}$, which we then realize that

$$\bigcup_{n \in \mathbb{N}} A_n = \text{ set of all even numbers },$$

but the set of all even numbers is clearly not finite, and its complement, which is the set of all odd numbers, is not finite.

3. Consider X as an uncountable set. We say that a set A is **co-countable** if A^C is countable. [1] The set

$$\mathcal{A} := \{A \subseteq X \mid A \text{ is countable or co-countable }\}$$

is a σ-algebra:

- countable union of countable sets is countable;

[1] Recall that a set A is said to be countable if there is a one-to-one correspondence between elements of A and the natural numbers.

- countable union of countable and co-countable sets is co-countable; and
- complement of countable sets are co-countable and vice versa.

2 Lecture 2 Sep 06th 2019

2.1 Algebra and σ-algebra of Sets (Continued)

We've seen some examples of σ-algebras. Let's now look at some other important properties of σ-algebras.

Proposition 2 (Closure of σ-algebras under Countable Intersection)

Let X be a set, $\mathcal{A}$ a σ-algebra on X. If $A_n \in \mathcal{A}$ for each $n \in \mathbb{N}$, then $\bigcap_n A_n \in \mathcal{A}$.

This follows rather similarly to Proposition 1 where we used **De Morgan's Law**.

Proof

We observe that

$$\begin{aligned} A_n \in \mathcal{A} &\implies A_n^C \in \mathcal{A} \\ &\implies \bigcup_n A_n^C \in \mathcal{A} \\ &\implies \bigcap_n A_n = \left(\bigcup_n A_n^C \right)^C \in \mathcal{A}. \end{aligned}$$

□

Let $\mathcal{A}_\alpha \subseteq \mathcal{P}(X)$, where α is from some index set. We denote

$$\bigcap_\alpha \mathcal{A}_\alpha = \{A \subseteq X : A \in \mathcal{A}_\alpha, \ \forall \alpha\}.$$

Proposition 3 (Existence of the 'Smallest' σ-algebra on a Set)

Let X be a set and $\{\mathcal{A}_\alpha\}_\alpha$ as a collection of σ-algebras on X. Then $\bigcap_\alpha A_\alpha$ is a σ-algebra.

Proof

$$\begin{aligned} A \in \bigcap_\alpha \mathcal{A}_\alpha &\implies \forall \alpha, A \in \mathcal{A}_\alpha \\ &\implies \forall \alpha, A^C \in \mathcal{A}_\alpha \\ &\implies A^C \in \bigcap_\alpha \mathcal{A}_\alpha \end{aligned}$$

and

$$\begin{aligned} \forall n \in \mathbb{N}, A_n \in \bigcap_\alpha \mathcal{A}_\alpha &\implies \forall n \in \mathbb{N}, \forall \alpha, A_n \in \mathcal{A}_\alpha \\ &\implies \forall \alpha, \bigcup_n A_n \in \mathcal{A}_\alpha \\ &\implies \bigcup_n A_n \in \bigcap_\alpha \mathcal{A}_\alpha. \end{aligned}$$

□

Due to the above proposition, the following definition is well-defined.

Definition 3 (Generator of a σ-algebra)

Let X be a set, and $\xi \subseteq \mathcal{P}(X)$ has some non-trivial set(s). Consider all σ-algebras $\mathcal{A}_\alpha$ with the property that $\xi \subseteq \mathcal{A}_\alpha$. Then we say that $\bigcap_\alpha \mathcal{A}_\alpha$

is the σ-algebra generated by ξ, and we denote this generated σ-algebra as

$$\mathfrak{M}(\xi) = \bigcap_{\alpha} \mathcal{A}_\alpha.$$

Remark 2.1.1

1. *It is clear from the definition that if $\mathcal{A}$ is a σ-algebra on X and $\xi \subseteq \mathcal{A}$, then $\mathfrak{M}(\xi) \subseteq \mathcal{A}$.*
2. *We often say that $\mathfrak{M}(\xi)$ is the "smallest σ-algebra containing ξ".*

The following is an example of such a σ-algebra.

Definition 4 (Borel σ-algebra)

Let X be a metric space (or topological space). The σ-algebra generated by the ***open subsets*** *of X is called the* ***Borel σ-algebra****, of which we denote by $\mathfrak{B}(X)$.*

Remark 2.1.2 (Some sets in $\mathfrak{B}(X)$)

Given an arbitrary metric space (or topological space) X. It is often hard to firmly grasp what kind of sets are in the Borel σ-algebra $\mathfrak{B}(X)$. The following are some examples that are in $\mathfrak{B}(X)$.

1. *Let $\{\mathcal{O}_n\}_{n\in\mathbb{N}}$ denote a countable collection of open sets. By Proposition 2, $\bigcap_n \mathcal{O}_n \in \mathfrak{B}(X)$. We call these countable union of open sets as G_δ sets.*
2. *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ denote a countable collection of closed sets. By Proposition 2, $\bigcup_n \mathcal{F}_n \in \mathfrak{B}(X)$. We call these countable intersection of closed sets as F_σ sets.*
3. *Let $\{H_n\}$ be a countable collection of G_δ sets. Then $\bigcup_n H_n \in \mathfrak{B}(X)$. These are called the $G_{\delta\sigma}$ sets.*
4. *Let $\{K_n\}$ be a countable collection of F_σ sets. Then $\bigcap_n K_n \in \mathfrak{B}(X)$. These are called the $F_{\sigma\delta}$ sets.*

We can continue constructing the $G_{\delta\sigma\dots}$ and $F_{\sigma\delta\dots}$ similarly, and all these sets belong to the Borel σ-algebra $\mathfrak{B}(X)$.

Proposition 4 (Other Formulations of the Borel σ-algebra (aka Proposition 1.2))

The following collection of sets are all equal:

1. $\mathfrak{B}_1 = \mathfrak{B}(\mathbb{R})$;
2. $\mathfrak{B}_2 =$ σ-algebra generated by open intervals (e.g. (a, b));
3. $\mathfrak{B}_3 =$ σ-algebra generated by closed intervals (e.g. $[a, b]$);
4. $\mathfrak{B}_4 =$ σ-algebra generated by half-open intervals (e.g. $(a, b]$);
5. $\mathfrak{B}_5 =$ σ-algebra generated by $(-\infty, a)$ and (b, ∞); and
6. $\mathfrak{B}_6 =$ σ-algebra generated by $(-\infty, a]$ and $[b, \infty)$.

As commented before, it is often hard knowing that is in a Borel σ-algebra, and what is not, despite knowing what its generator is. However, when talking about containments, this is a fairly straightforward discussion thanks to its closure under countable unions and Proposition 2. We simply need to talk about the generators.

Proof

$\mathfrak{B}_2 \subseteq \mathfrak{B}_1$ Given an arbitrary generator (a, b) in $\mathfrak{B}_2$, we know that (a, b) is an open set, and clearly $(a, b) \subseteq \mathbb{R}$. Thus $(a, b) \in \mathfrak{B}_1$, so $\mathfrak{B}_2 \subseteq \mathfrak{B}_1$.

$\mathfrak{B}_3 \subseteq \mathfrak{B}_2$ Given an arbitrary generator $[a, b]$ of $\mathfrak{B}_2$, we have

$$[a, b] = \bigcap_n \left(a - \frac{1}{n}, \, b + \frac{1}{n} \right) \in \mathfrak{B}_2.$$

Thus $\mathfrak{B}_3 \subseteq \mathfrak{B}_2$.

$\mathfrak{B}_4 \subseteq \mathfrak{B}_3$ Given an arbitrary generator $(a, b]$ of $\mathfrak{B}_4$,

$$(a, b] = \bigcup_n \left[a + \frac{1}{n},\, b\right] \in \mathfrak{B}_3.$$

Thus $\mathfrak{B}_4 \subseteq \mathfrak{B}_3$.

$\mathfrak{B}_5 \subseteq \mathfrak{B}_4$ Given an arbitrary generator $(-\infty, a)$ for $\mathfrak{B}_5$,

$$(-\infty, a) = \bigcup_n \left(-\infty,\, a - \frac{1}{n}\right) \in \mathfrak{B}_4.$$

On the other hand, for (b, ∞) in $\mathfrak{B}_5$,

$$(b, \infty) = \bigcup_n (b, n) \in \mathfrak{B}_4.$$

$\mathfrak{B}_6 \subseteq \mathfrak{B}_5$ We have that

$$(-\infty, a] = \bigcap_n \left(-\infty,\, a + \frac{1}{n}\right) \in \mathfrak{B}_5$$

and

$$[b, \infty) = \bigcap_n \left(b - \frac{1}{n}, \infty\right) \in \mathfrak{B}_5.$$

$\mathfrak{B}_1 \subseteq \mathfrak{B}_6$ Let $c < d \in \mathbb{R}$. Notice that

$$(-\infty, d] \cap [c, \infty) = [c, d] \in \mathfrak{B}_6.$$

Furthermore,

$$(c, d) = \bigcup_n \left[c + \frac{1}{n},\, d - \frac{1}{n}\right] \in \mathfrak{B}_6.$$

Recall that given an open set $\mathcal{O} \subseteq \mathbb{R}$, we have

$$\mathcal{O} = \bigcup \{(c, d) \subseteq \mathcal{O} : c, d \in \mathbb{Q}\},$$

which shows that $\mathcal{O}$ is a countable union of open sets (with rational endpoints). It follows that $\mathcal{O} \in \mathfrak{B}_6$ and so $\mathfrak{B}_1 \subseteq \mathfrak{B}_6$. $\square$

Exercise 2.1.1

Show that $\mathfrak{B}(\mathbb{R}^2)$ *is generated by open rectangles* $(a, b) \times (c, d)$.

Definition 5 (Infinitely Often)

Given $E_n \subseteq X$ for $n \in \mathbb{N}$, we say that $x \in E_n$ ***infinitely often (i.o.)*** *if*

$$\{n : x \in E_n\}$$

is an ***infinite set****. We typically let*

$$A := \{x \in X : x \in E_n \text{ i.o. }\}$$

be the set of x's that are in the E_n's infinitely often.

Definition 6 (Almost always)

Given $E_n \subseteq X$ for $n \in \mathbb{N}$, we say that $x \in E_n$ ***almost always (a.a.)*** *if*

$$\{n : x \notin E_n\}$$

is a ***finite set****. We typically let*

$$B := \{x \in X : x \in E_n \text{ a.a. }\}$$

be the set of x's that are in the E_n's almost always.

Homework (Homework 1)

Let X be a set, $\mathcal{A}$ a σ-algebra on X, and $E_n \in \mathcal{A}$ for $n \in \mathbb{N}$. Prove that

$$A := \{x \in X : x \in E_n \text{ i.o. }\}$$

and

$$B := \{x \in X : x \in E_n \text{ a.a. }\}$$

are both in $\mathcal{A}$.

Definition 7 (Characteristic Function)

Let $E \subseteq X$. We call the function

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

the ***characteristic function*** *of E.*

Homework (Homework 2 – A review on limsup and liminf)

Let $E_n \subseteq X$ for $n \in \mathbb{N}$, and

$$A := \{x \in X : x \in E_n \text{ i.o. } \}$$
$$B := \{x \in X : x \in E_n \text{ a.a. } \}.$$

Show that

$$\chi_A(x) = \limsup_n \chi_{E_n}(x)$$
$$\chi_B(x) = \liminf_n \chi_{E_n}(x).$$

Remark 2.1.3

Due to the above result, some people write

$$A = \limsup E_n$$
$$B = \liminf E_n.$$

2.2 Measures

Definition 8 (Measure)

Let X be a set and $\mathcal{A}$ a σ-algebra of subsets of X. A function $\mu : \mathcal{A} \to [0, \infty]$ is called a ***measure*** *on $\mathcal{A}$ provided that:*

1. $\mu(\emptyset) = 0$; *and*

2. *if* $E_n \in \mathcal{A}$ *for each* $n \in \mathbb{N}$*, and* $\{E_n\}$ *is disjoint, we have*

$$\mu\left(\bigcup_n E_n\right) = \sum_n \mu(E_n).$$

3 Lecture 3 Sep 9th 2019

3.1 Measures (Continued)

Definition 9 (Measure Space)

*Let X be a set, $\mathfrak{M}$ a σ-algebra of subsets of X and $\mu : \mathfrak{M} \to [0, \infty]$. We call the 3-tuple $(X, \mathfrak{M}, \mu)$ a **measure space**.*

Remark 3.1.1

*If $\mu(X) = 1$, we also call $(X, \mathfrak{M}, \mu)$ a **probability space**, and μ is called a **probability measure**.*

Example 3.1.1

1. (**Counting Measure**) Let X be a set and $\mathfrak{M} = \mathcal{P}(X)$. For $E \in \mathfrak{M}$, define
$$\mu(E) = \begin{cases} |E| & E \text{ is finite} \\ \infty & \text{otherwise} \end{cases}.$$
We verify that μ is indeed a measure:

   (a) We have that $\mu(\emptyset) = |\emptyset| = 0$.

   (b) Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}$ be a pairwise disjoint set. Notice that if any of the sets are infinite, say E_{N_0} is infinite, then
$$\mu(E_{N_0}) = \infty = \left|E_{N_0}\right|.$$

Since $\bigcup_{n=1}^{\infty} E_n$ is infinite in this case, we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \infty = \left|\bigcup_{n=1}^{\infty} E_n\right|.$$

On the other hand, if all the sets are finite, then since the E_n's are disjoint, we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \left|\bigcup_{n=1}^{\infty} E_n\right| = \sum_{n=1}^{\infty} |E_n| = \sum_{n=1}^{\infty} \mu(E_n).$$

We call μ a counting measure.

2. Let X be an uncountable set. Recall that in Example 1.2.1, we showed that

$$\mathfrak{M} := \{A \subseteq X \mid A \text{ is countable or co-countable }\}$$

is a σ-algebra. There are many measures that we can define on this σ-algebra. For instance,

$$\nu(E) = \begin{cases} 0 & E \text{ is countable} \\ 1 & E \text{ is uncountable} \end{cases},$$

and

$$\delta(E) = \begin{cases} 0 & E \text{ is countable} \\ \infty & E \text{ is uncountable} \end{cases}.$$

Verifying that both ν and δ are indeed measures shall be left to the reader as a straightforward exercise.

3. Let's make a non-example. Let X be an infinite set, and $\mathfrak{M} = \mathcal{P}(X)$. Define

$$\mu(E) = \begin{cases} 0 & E \text{ is finite} \\ \infty & E \text{ is infinite} \end{cases}.$$

Consider $X = \mathbb{N}$ and a sequence of sets with singletons,

$$E_n = \{2n+1\}, \quad \text{for } n \in \mathbb{N}.$$

Clearly,

$$\bigcup_{n=1}^{\infty} E_n = \text{ set of all odd numbers },$$

and clearly

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \infty.$$

However, notice that

$$\mu(E_n) = 0 \text{ for each } n \in \mathbb{N}.$$

Since each of the E_n's are pairwise disjoint, we should have

$$\infty = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) = 0,$$

which is impossible. Thus μ is **not** a measure.

Remark 3.1.2 (Finite additivity)

Given a finite set of pairwise disjoint sets $\{E_n\}_{n=1}^{N} \subseteq \mathfrak{M}$ for some σ-algebra $\mathfrak{M}$ of some set X. By the definition of a σ-algebra, we may set $E_n = \emptyset$ for $n > N$. Then

$$\mu\left(\bigcup_{n=1}^{N} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{N} \mu(E_n).$$

*We call this the **finite additivity** of a measure.*

Definition 10 (Finitivity, σ-finitivity, and Semi-finitivity of a Measure)

Let $(X, \mathfrak{M}, \mu)$ be a measure space.

1. *We say that μ is **finite** if $\mu(E) < \infty$ for every $E \in \mathfrak{M}$.*
2. *If $X = \bigcup_{n=1}^{\infty} X_n$ with $X_n \in \mathfrak{M}$, we say that μ is **σ-finite** if*

$$\mu(X_n) < \infty \text{ for every } n \in \mathbb{N}.$$

3. *We say that μ is **semi-finite** if for every $E \in \mathfrak{M}$ with $\mu(E) = \infty$,*

$\exists F \subseteq E \in \mathfrak{M}$ such that

$$0 < \mu(F) < \infty.$$

Exercise 3.1.1

1. *Show that the counting measure is finite iff the ambient space X is a finite set.*
2. *Show that δ in Example 3.1.1 is neither finite, σ-finite, nor semi-finite.*

Theorem 5 (Properties of a Measure)

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Then

1. *(Monotonicity) If $E \subseteq F$ and $E, F \in \mathfrak{M}$, then $\mu(E) \leq \mu(F)$.*
2. *(Subadditivity) If $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}$, then*
$$\mu\left(\bigcup_n E_n\right) \leq \sum_n \mu(E_n).$$
3. *(Continuity from below) If $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}$ is an increasing sequence of sets, i.e.*
$$E_1 \subseteq E_2 \subseteq \ldots \subseteq E_n \subseteq \ldots,$$
then
$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} \mu(E_n).$$
4. *(Continuity from above) If $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}$ is a decreasing sequence of sets, i.e.*
$$E_1 \supseteq E_2 \supseteq \ldots \supseteq E_n \supseteq \ldots,$$
and $\exists n_0 \in \mathbb{N}$ such that $\mu(E_{n_0}) < \infty$, then
$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} \mu(E_n).$$

Remark 3.1.3 (A comment on the condition for the 4^{th} statement)

It may seem that the extra condition of a finite measure seem extravagant. However, it is necessary, as demonstrated below.

Consider $X = \mathbb{N}$, *with* μ *as the counting measure. Then, consider the sequence of sets*

$$\begin{aligned} E_1 &= \{1, 2, 3, \ldots\}, \\ E_2 &= \{2, 3, 4, \ldots\}, \\ E_3 &= \{3, 4, 5, \ldots\}, \\ &\vdots \\ E_n &= \{n, n+1, n+2, \ldots\}, \\ &\vdots \end{aligned}$$

Then $\bigcap_{n=1}^{\infty} E_n = \emptyset$, *which then* $\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = 0$. *However,*

$$\mu(E_n) = \infty \text{ for each } n \in \mathbb{N}.$$

Homework (Homework 3)

Let $(X, \mathfrak{M}, \mu)$ *be a measure space. Let* $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}$, *and*

$$A := \{x \in X \mid x \in E_n \text{ i.o. }\}.$$

Prove that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$ *implies that* $\mu(A) = 0$.

4 Lecture 4 Sep 11th 2019

4.1 Measures (Continued 2)

We shall now prove Theorem 5.

Proof

1. Notice that
$$F = (F \cap E) \cup (F \setminus E),$$
and $F \cap E$ and $F \setminus E$ are disjoint. Thus
$$\mu(F) = \mu(F \cap E) + \mu(F \setminus E) = \mu(E) + \mu(F \setminus E).$$
Since $\mu(F \setminus E) \geq 0$, we have
$$\mu(F) \geq \mu(E).$$

2. Consider a sequence of sets defined as such: [1]
$$\begin{aligned} F_1 &= E_1 \\ F_2 &= E_2 \setminus E_1 \\ &\vdots \\ F_n &= E_n \setminus \bigcup_{j=1}^{n-1} E_j. \end{aligned}$$
First, note that $F_n \subseteq E_n$ for each $n \in \mathbb{N}$. So by the last part, we have
$$\mu(F_n) \leq \mu(E_n) \text{ for each } n \in \mathbb{N}.$$

[1] ★ This is a common technique in measure theory. We will see this repeatedly so in this course.

Secondly,

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n.$$

Also, $\{F_n\}_{n=1}^{\infty}$ is a pairwise disjoint collection of sets. It follows that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

3. Consider a sequence of sets defined as such:

$$\begin{aligned} F_1 &= E_1 \\ F_2 &= E_2 \setminus E_1 \\ F_3 &= E_3 \setminus E_2 \\ &\vdots \\ F_n &= E_n \setminus E_{n-1}. \end{aligned}$$

We see that

- $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$;
- $\bigcup_{n=1}^{N} F_n = \bigcup_{n=1}^{N} E_n = E_N$; and
- $\{F_n\}_n$ is a collection pairwise disjoint sets.

Thus we have

$$\begin{aligned} \mu\left(\bigcup_{n=1}^{\infty} E_n\right) &= \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n) \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \mu(F_n) = \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} F_n\right) \\ &= \lim_{N\to\infty} \mu(E_N). \end{aligned}$$

4. First, it is important that we notice that

$$\bigcap_{n=1}^{\infty} E_n = \bigcap_{n=m}^{\infty} E_n$$

for any $m \in \mathbb{N}$, since $\{E_n\}_n$ is a decreasing sequence of sets.

Suppose $n_0 \in \mathbb{N}$ is such that $\mu(E_{n_0}) < \infty$. Consider a sequence of sets defined as follows: for $n_0 \leq j \in \mathbb{N}$, we let $F_j = E_{n_0} \setminus E_j$. Then we have

$$\emptyset = F_{n_0} \subseteq F_{n_0+1} \subseteq \ldots \subseteq F_{n_0+k} \subseteq \ldots,$$

i.e. $\{F_n\}_{n=n_0}^{\infty}$ is an increasing sequence of sets. By the last part, we have

$$\begin{aligned}
\mu\left(\bigcup_{n=n_0}^{\infty} F_n\right) &= \lim_{n\to\infty} \mu(F_{n_0+n}) = \lim_{n\to\infty} \mu(E_{n_0} \setminus E_{n_0+n}) \\
&= \mu(E_{n_0}) - \lim_{n\to\infty} \mu(E_{n_0+n}) \\
&= \mu(E_{n_0}) - \lim_{n\to\infty} \mu(E_n).
\end{aligned}$$

Furthermore, we observe that

$$\bigcup_{n=1}^{\infty} F_n = E_{n_0} \setminus \bigcap_{n=n_0}^{\infty} E_n.$$

Thus

$$\begin{aligned}
\mu\left(\bigcup_{n=n_0}^{\infty} F_n\right) &= \mu\left(E_{n_0} \setminus \bigcap_{n=n_0}^{\infty} E_n\right) = \mu(E_{n_0}) - \mu\left(\bigcap_{n=n_0}^{\infty} E_n\right) \\
&= \mu(E_{n_0}) - \mu\left(\bigcap_{n=1}^{\infty} E_n\right).
\end{aligned}$$

It follows that indeed

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n). \qquad \square$$

Exercise 4.1.1

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Show that

1. *μ is finite iff $\mu(X) < \infty$.*
2. *μ is σ-finite implies that μ is semi-finite.*

Solution

1. This is rather simple.

   $(\implies)$ μ is finite implies that each $E \in \mathfrak{M}$ has a finite measure. In particular, $X \in \mathfrak{M}$, and so $\mu(X) < \infty$.

($\Longleftarrow$) $\forall E \in \mathfrak{M}$, $E \subseteq X$, thus by the first item in Theorem 5, we have $\mu(E) \leq \mu(X) < \infty$. Thus μ is finite.

2. μ being σ-finite means that if $X = \bigcup_{n=1}^{\infty} X_n$ where $X_n \in \mathfrak{M}$, then $\mu(X_n) < \infty$ for each n. Let $E \in \mathfrak{M}$ such that $\mu(E) = \infty$. If we take

$$E_n = X_n \cap E,$$

then $\mu(E_n) < \infty$ for each $n \in \mathbb{N}$. Then, taking a union of any finite number of these E_n's will give us a subset of E with a finite measure. Hence, μ is indeed semi-finite.

Definition 11 (Null Set of a Measure)

Let $(X, \mathfrak{M}, \mu)$ be a measure space. The set

$$\mathcal{N} := \{N \in \mathfrak{M} : \mu(N) = 0\}$$

*is called the **μ-null set**, or the **null set** of the measure μ.*

Remark 4.1.1

1. *If $N_j \in \mathcal{N}$, then $\bigcup_{n=1}^{\infty} N_j \in \mathcal{N}$.* [2]
2. *If $N \in \mathcal{N}$, and $E \in \mathfrak{M}$ and $E \subseteq N$, then $E \in \mathcal{N}$.*

   It is important to note there that the highlighted condition is required, since not all subsets of N are measurable.
3. *$\mathcal{N}$ is **not** a σ-algebra. If we picked an X such that $\mu(X) \neq 0$, then $\emptyset \in \mathcal{N}$ but $X \notin \mathcal{N}$.*

[2] Requires elab

Definition 12 (Complete Measure Space)

*Let $(X, \mathfrak{M}, \mu)$ be a measure space. We say that the space is **complete** if $N \in \mathcal{N}$ and $E \subseteq N$, then $E \in \mathfrak{M}$. In this case, we also say that μ is a **complete measure** on $\mathfrak{M}$.*

Remark 4.1.2

By the first item in Theorem 5, we have that if $\mu(E) = 0$, and so $E \in \mathcal{N}$ as well.

Theorem 6 (Extending the Measurable Sets)

Let $(X, \mathfrak{M}, \mu)$ be a measure space and

$$\mathcal{N} := \{N \in \mathfrak{M} \mid \mu(N) = 0\}.$$

Consider

$$\overline{\mathfrak{M}} := \{E \cup F \mid E \in \mathfrak{M},\, F \subseteq N \in \mathcal{N}\}.$$

Then $\overline{\mathfrak{M}}$ is a σ-algebra which contains $\mathfrak{M}$. Furthermore, if we define $\overline{\mu} : \overline{\mathfrak{M}} \to [0, \infty]$ as

$$\overline{\mu}(E \cup F) = \mu(E),$$

then $\overline{\mu}$ is a well-defined measure on $\overline{\mathfrak{M}}$.

Moreover, if $\nu : \overline{\mathfrak{M}} \to [0, \infty]$ is any measure such that $\nu(E) = \mu(E)$ for all $E \in \mathfrak{M}$, then $\nu = \overline{\mu}$.

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5 Lecture 5 Sep 13th 2019

5.1 Measures (Continued 3)

Proof (Extending the Measurable Sets)

$\overline{\mathfrak{M}}$ is a σ-algebra Since $\emptyset \in \mathfrak{M}$ and $\emptyset \subseteq N$ for any $N \in \mathcal{N}$, it is clear that $\emptyset \in \overline{\mathfrak{M}}$.

Now, for $E \cup F \in \overline{\mathfrak{M}}$, if we suppose $F \subseteq N \in \mathcal{N}$, then

$$(E \cup F)^C = (E \cup N)^C \cup (N \setminus E \cup F) \in \overline{\mathfrak{M}}$$

since $E \cup N \in \mathfrak{M}$ and $N \setminus (E \cup F) \in \mathcal{N}$.

Let $\{E_n \cup F_n\}_{n=1}^{\infty} \subseteq \overline{\mathfrak{M}}$. Then we observe that

$$\bigcup_{n=1}^{\infty}(E_n \cup F_n) = \underbrace{\bigcup_{n=1}^{\infty} E_n}_{\in \mathfrak{M}} \cup \underbrace{\bigcup_{n=1}^{\infty} F_n}_{\in \mathcal{N}} \in \overline{\mathfrak{M}}.$$

Well-definedness of $\overline{\mu}$ Let $E_1 \cup F_1 = E_2 \cup F_2 \in \overline{\mathfrak{M}}$. Suppose $F_1 \subseteq N_1, F_2 \subseteq N_2 \in \mathcal{N}$. WTS

$$\mu(E_1) = \overline{\mu}(E_1 \cup F_1) = \overline{\mu}(E_2 \cup F_2) = \mu(E_2)$$

Notice that

$$E_1 \subseteq E_1 \cup F_1 = E_2 \cup F_2 \subseteq E_2 \cup N_2,$$

and

$$E_2 \subseteq E_2 \cup F_2 = E_1 \cup F_1 \subseteq E_1 \cup N_1.$$

By Theorem 5, in particular, by subadditivity, we have that

$$\mu(E_1) \leq \mu(E_2 \cup N_2) \leq \mu(E_2) + 0 = \mu(E_2)$$

and

$$\mu(E_2) \leq \mu(E_1 \cup N_1) \leq \mu(E_1) + 0 = \mu(E_1).$$

It follows that $\mu(E_1) = \mu(E_2)$, as required.

$\overline{\mu}$ is a measure

1. Since $\emptyset \in \mathfrak{M}$ and $\emptyset \in \mathcal{N}$, $\overline{\mu}$ is defined for $\emptyset$, and

$$\overline{\mu}(\emptyset) = \mu(\emptyset) = 0.$$

2. Let $\{E_n \cup F_n\}_{n=1}^{\infty} \subseteq \overline{\mathfrak{M}}$ be a pairwise disjoint collection. We observe that

$$\begin{aligned} \overline{\mu}\left(\bigcup_{n=1}^{\infty}(E_n \cup F_n)\right) &= \overline{\mu}\left(\bigcup_{n=1}^{\infty} E_n \cup \bigcup_{n=1}^{\infty} F_n\right) \\ &= \mu\left(\bigcup_{n=1}^{\infty} E_n\right) \\ &= \sum_{n=1}^{\infty} \mu(E_n), \end{aligned}$$

and

$$\sum_{n=1}^{\infty} \overline{\mu}(E_n \cup F_n) = \sum_{n=1}^{\infty} \mu(E_n).$$

Hence

$$\overline{\mu}\left(\bigcup_{n=1}^{\infty}(E_n \cup F_n)\right) = \sum_{n=1}^{\infty} \overline{\mu}(E_n \cup F_n).$$

$\nu = \overline{\mu}$ Let $E \cup F \in \overline{\mathfrak{M}}$. Suppose $F \subseteq N \in \mathfrak{M}$ By monotonicity,

$$\overline{\mu}(E \cup F) = \mu(E) = \nu(E) \leq \nu(E \cup F).$$

By subadditivity,

$$\nu(E \cup F) \leq \nu(E) + \nu(F) \leq \mu(E) + \nu(N) \leq \overline{\mu}(E \cup F) + \mu(N) = \overline{\mu}(E \cup F) + 0.$$

Thus, indeed,

$$\nu(E \cup F) = \overline{\mu}(E \cup F).$$

5.2 The Outer Measure

In this section, we will show that one way we can construct a measure is by using an outer measure.

Definition 13 (Outer Measure)

Given a set X, a function

$$\mu^* : \mathcal{P}(X) \implies [0, \infty]$$

is called an ***outer measure*** *if*

1. $\mu^*(\emptyset) = 0$;
2. (*monotonicity*) *if $E \subseteq F$, then $\mu^*(E) \leq \mu^*(F)$; and*
3. (*countable subadditivity*) *if $\{A_n\}_n \subseteq \mathcal{P}(X)$, then*

$$\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

Coming from PMATH450, we have seen an example of an outer measure.

Proposition 7 (Lebesgue's Outer Measure)

Given $E \subseteq \mathbb{R}$, consider

$$\mu^*(E) := \inf \left\{ \sum_{n=1}^{\infty} (b_n - a_n) : E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n) \right\}.$$

μ^* is Lebesgue's outer measure.

Proof

1. It is clear that $\mu^*(\emptyset) = \emptyset$, since we can pick all $(a_n, b_n) = \emptyset$.

2. Suppose $A \subseteq B \subseteq \mathbb{R}$. It is clear that any collection of intervals whose union contain B will contain A, but there are such collections for A that do not contain B. This means that

$$\mu^*(A) \leq \mu^*(B)$$

by the property of the infimum.

3. Let $E = \bigcup_{i=1}^{\infty} E_i$. WTS $\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$.

Now if $\mu^*(E_i) = \infty$ for any i, then the inequality is trivially true. Thus, wma $\mu^*(E_i) < \infty$ for all i.

[1] Let $\varepsilon > 0$. By the definition of the infimum, for each i, we can pick a countable sequence $\{(a_n^i, b_n^i)\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$ such that $E_1 \subseteq \bigcup_{n=1}^{\infty} (a_n^i, b_n^i)$ and

[1] This is also a common trick in measure theory.

$$\sum_{n=1}^{\infty} (b_n^i - a_n^i) \leq \mu^*(E_i) + \frac{\varepsilon}{2^i}.$$

Then

$$E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} (a_n^i, b_n^i).$$

And so it follows that

$$\begin{aligned}
\mu^*(E) &\leq \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} (b_n^i - a_n^i) \\
&\leq \sum_{i=1}^{\infty} \mu^*(E_i) + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} \\
&= \sum_{i=1}^{\infty} \mu^*(E_i) + \varepsilon.
\end{aligned}$$

Since ε was arbitrary, it follows that

$$\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i). \qquad \square$$

Exercise 5.2.1

Show that had we defined Lebesgue's outer measure with closed intervals, i.e.

$$\tilde{\mu}^*(E) := \inf \left\{ \sum_{n=1}^{\infty} (b_n - a_n) : E \subseteq \bigcup_{n=1}^{\infty} [a_n, b_n] \right\},$$

$\tilde{\mu}^*$ is still an outer measure.

In fact, we can do so for half-open intervals.

Example 5.2.1 (Lebesgue-Stieltjes Outer Measure)

Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function that is continuous from the right. Let

$$\mu^*(E) := \inf \left\{ \sum_{n=1}^{\infty} (F(b_n) - F(a_n)) : E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n] \right\}.$$

Then μ^* is an outer measure.

Remark 5.2.1

Again, we could have defined the above outer measure using open or closed intervals.

Example 5.2.2 (Lebesgue's Outer Measure on $\mathbb{R}^2$)

Let $E \subseteq \mathbb{R}^2$, and

$$\mu^*(E) := \inf \left\{ \sum_{n=1}^{\infty} A(R_n) : E \subseteq \bigcup_{n=1}^{\infty} R_n \right\},$$

where A is the 'area' function, and $R_n = (a_n, b_n) \times (c_n, d_n)$ are open rectangles. Then μ^* is an outer measure.

Remark 5.2.2

1. *Again, we can define the above outer measure using closed rectangles, or half-open rectangles.*
2. *We can continue defining an outer measure for $\mathbb{R}^3$ using cubes, for $\mathbb{R}^4$ using hypercubes, and so on.*

We want to now show that given an outer measure, we can always construct a measure. This is known as Carathéodory's Theorem.

This requires the following definition:

Definition 14 (μ^*-measurability)

A set $A \subseteq X$ is said to be ***μ^*-measurable*** *if $\forall E \subseteq X$,*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C).$$

Remark 5.2.3

1. *By subadditivity, we always have*

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^C),$$

since $E = (E \cap A) \cup (E \cap A^C)$.

2. *Note that $E \cap A^C = E \setminus A$. In a sense, A is said to be μ^*-measurable if it can slice any subset of X such that we have additivity of the sliced parts. We may also say that A is a 'universal slicer'.*

Theorem 8 (Carathéodory's Theorem)

If μ^* is an outer measure on a set X, let

$$\mathfrak{M} := \{A \subseteq X : A \text{ is } \mu^*\text{-measureable}\}.$$

Then $\mathfrak{M}$ is a σ-algebra, and we set

$$\mu : \mathfrak{M} \to [0, \infty]$$

such that

$$\mu(A) = \mu^*(A).$$

Then μ is a complete measure on $\mathfrak{M}$.

6 Lecture 6 Sep 16th 2019

6.1 The Outer Measure (Continued)

Homework (Homework 4)

Let $\mathfrak{M}$ be an algebra of sets on X, and whenever $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{M}$ is a disjoint collection of sets, then $\bigcup_n A_n \in \mathfrak{M}$. Then $\mathfrak{M}$ is a σ-algebra.

Homework (Homework 5)

Recall that Lebesgue's Outer Measure on $\mathbb{R}$ is defined as

$$\mu^*(E) \coloneqq \inf\left\{\sum_{n=1}^{\infty}(b_n - a_n) : E \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n)\right\}.$$

Prove that we can equivalently define

$$\mu^*(E) \coloneqq \inf\left\{\sum_{n=1}^{\infty}(b_n - a_n) : E \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n]\right\}.$$

Similarly, Lebesgue's Outer Measure on $\mathbb{R}^2$ is defined as

$$\mu_2^*(E) = \inf\left\{\sum_{n=1}^{\infty}(b_n - a_n)(d_n - c_n) : E \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n) \times (c_n, d_n)\right\}.$$

Prove that we can equivalently define

$$\mu_2^*(E) = \inf\left\{\sum_{n=1}^{\infty}(b_n - a_n)(d_n - c_n) : E \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n] \times (c_n, d_n]\right\}.$$

Definition 15 (Metric Outer Measure)

Let (X, d) be a metric space, and $A, B \subseteq X$, and

$$d(A, B) = \inf \left\{ d(x, y) : x \in A,\ y \in B \right\}.$$

An outer measure, μ^*, on X is called a ***metric outer measure*** *if whenever $d(A, B) > 0$, then*

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

Homework (Homework 6)

Prove that Lebesgue's Outer Measure on $\mathbb{R}$ is a metric outer measure.

Proof (Carathéodory's Theorem)

$\mathfrak{M}$ is a σ-algebra

$\emptyset \in \mathfrak{M}$ Given any $E \subseteq X$, we observe that

$$\begin{aligned} \mu^*(E \cap \emptyset) + \mu^*(E \cap \emptyset^C) &= \mu^*(\emptyset) + \mu^*(E \cap X) \\ &= 0 + \mu^*(E) = \mu^*(E). \end{aligned}$$

$A \in \mathfrak{M} \implies A^C \in \mathfrak{M}$ Observe that given any $E \subseteq X$,

$$\mu^*(E \cap A^C) + \mu^*(E \cap (A^C)^C) = \mu^*(E \cap A^C) + \mu^*(E \cap A) = \mu^*(E).$$

Thus $A^C \in \mathfrak{M}$.

[1] To show that $\mathfrak{M}$ is closed under countable unions, we break the work into several steps.

[1] For a deep dive, see Appendix A.1.

$A, B \in \mathfrak{M} \implies A \cup B \in \mathfrak{M}$ Since $A \in \mathfrak{M}$, we have

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C).$$

Since $B \in \mathfrak{M}$,

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^C) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^C) \\ &\quad + \mu^*(E \cap A^C \cap B) + \mu^*(E \cap A^C \cap B^C) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^C) \\ &\quad + \mu^*(E \cap A^C \cap B) + \mu^*(E \cap (A \cup B)^C)\end{aligned}$$

Notice that

$$E \cap (A \cup B) = [E \cap A \cap B] \mathbin{\dot\cup} [E \cap A^C \cap B] \mathbin{\dot\cup} [E \cap A \cap B^C].$$

Thus

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^C) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^C) \\ &\quad + \mu^*(E \cap A^C \cap B) + \mu^*(E \cap (A \cup B)^C) \\ &\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^C).\end{aligned}$$

Thus $A \cup B \in \mathfrak{M}$.

Consequently, by induction, we have that $\forall \{A_n\}_n \subseteq \mathfrak{M}$,

$$\bigcup_{n=1}^{N} A_n \in \mathfrak{M}$$

for all $N \in \mathbb{N}$.

Now $\mathfrak{M}$ is an algebra of sets. By Homework 4, it suffices for us to prove the following to show that $\mathfrak{M}$ is a σ-algebra of sets.

$\forall \{A_n\}_n \subseteq \mathfrak{M}$ disjoint, $\implies \dot\bigcup_n A_n \in \mathfrak{M}$ Let $B_N = \dot\bigcup_{n=1}^N A_n$. We first require the following lemma:

$\forall E \subseteq X,\ \mu^*(E \cap B_N) = \sum_{n=1}^N \mu^*(E \cap A_n)$ Notice that for any $n \in \mathbb{N}$, $A_n \in \mathfrak{M}$, and so

$$\begin{aligned}\mu^*(E \cap B_N) &= \mu^*(E \cap B_N \cap A_n) + \mu^*(E \cap B_N \cap A_n^C) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{N-1}).\end{aligned}$$

The desired result follows by induction. ⊣

Let $B = \dot\bigcup_{n=1}^{\infty} A_n$. Then

$$\mu^*(E \cap B) \leq \sum_{n=1}^{\infty} \mu^*(E \cap A_n)$$

by subadditivity.

Now $B_N \subseteq B$ for each $N \in \mathbb{N}$. This implies that $B_N^C \supseteq B^C$, and so by monotonicity,

$$\mu^*(E \cap B_N^C) \geq \mu^*(E \cap B^C).$$

Thus, for every $N \in \mathbb{N}$,

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap B_N) + \mu^*(E \cap B_N^C) \\ &\geq \sum_{n=1}^{N} \mu^*(E \cap A_n) + \mu^*(E \cap B^C). \end{aligned}$$

It follows that

$$\begin{aligned} \mu^*(E) &\geq \sum_{n=1}^{\infty} \mu^*(E \cap A_n) + \mu^*(E \cap B^C) \\ &\geq \mu^*(E \cap B) + \mu^*(E \cap B^C). \end{aligned}$$

With Homework 4, $\mathfrak{M}$ is a σ-algebra.

μ is a measure

- $\mu(\emptyset) = \mu^*(\emptyset) = 0$.
- Let $\{A_n\}_n \subseteq \mathfrak{M}$ be a disjoint collection of sets, and $B = \dot\bigcup_{n=1}^{\infty} A_n$. Then by a similar argument as the end of the last 'part',

$$\begin{aligned} \mu(B) &= \mu^*(B) \\ &\geq \sum_{n=1}^{\infty} \mu^*(B \cap A_n) + \mu^*(B \cap B^C) \\ &= \sum_{n=1}^{\infty} \mu^*(B \cap A_n) + 0 \\ &= \sum_{n=1}^{\infty} \mu^*(A_n) = \sum_{n=1}^{\infty} \mu(A_n). \end{aligned}$$

Thus

$$\mu(B) = \mu\left(\dot{\bigcup_{n=1}^{\infty}} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

μ is complete Let $A \in \mathcal{N}$ and $B \subseteq A$. By monotonicity, $\mu(B) = \mu^*(B) \leq \mu^*(A) = 0$. Then

$$\mu^*(E \cap B) + \mu^*(E \cap B^C) = 0 + \mu^*(E \cap B^C) \leq \mu^*(E)$$

by monotonicity. Thus $B \in \mathfrak{M}$. Thus μ is complete. □

We would like to make sure that

1. there are many sets that are measurable; and
2. the notion of a measure covers our notion of length.

We shall see this with the Metric Outer Measure, and that the measurable sets is at least the **Borel set**.

6.2 *The Lebesgue-Stieltjes Outer Measure*

This outer measure is motivated by probability theory. The idea is that we consider the measure space $(\Omega, \mathfrak{M}, P)$, where Ω is the sample space set, $\mathfrak{M}$ is a σ-algebra on Ω, and P is the probability measure, i.e. $P(\Omega) = 1$.

We then define a **random variable**, which is a function $X : \Omega \to \mathbb{R}$. The **cumulative distribution function** (cdf) is defined as

$$F_X(t) := P(\{\omega : X(\omega) \leq t\}),$$

and it has these properties:

1. F_X is increasing; and
2. F_X is right-continuous.

Example 6.2.1

Let $\Omega = \{H, T\}$, and define the probability measure as

$$P(\{H\}) = \frac{1}{2} = P(\{T\}).$$

We can define

$$X(T) = 0 \text{ and } X(H) = 1.$$

Then

$$P(\{\omega : X(\omega) = 1\}) = P(\{H\}) = \frac{1}{2}$$

and

$$P(\{\omega : X(\omega) = 0\}) = P(\{T\}) = \frac{1}{2}.$$

In the context of probability, we often see the shorthand

$$P(X = t) = P(\{\omega : X(\omega) = t\}).$$

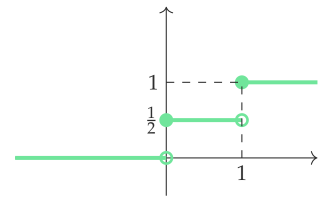

Figure 6.1: Simple example of a cdf

Definition 16 (Lebesgue-Stieltjes Outer Measure)

Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function that is continuous from the right. Let

$$\mu^*(E) := \inf \left\{ \sum_{n=1}^{\infty} (F(b_n) - F(a_n)) : E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n] \right\}.$$

Then μ^* is an outer measure.

Remark 6.2.1

We showed that the above is indeed an outer measure in Example 5.2.1.

A Deep Dives into Proofs

A.1 Proving that $\mathfrak{M}$ is closed under countable unions in Carathéodory's Theorem

This section is created in reference to the proof for Carathéodory's Theorem.

We have

$$\mathfrak{M} = \{A \subseteq X : A \text{ is } \mu^*\text{-measurable }\}$$

where μ^* is an outer measure. We wanted to show that $\mathfrak{M}$ is a σ-algebra. In particular, the hard problem was to show that $\mathfrak{M}$ is closed under countable unions.

Consider $\{A_n\}_n \subseteq \mathfrak{M}$. Thinking from behind, WTS $\forall E \subseteq X$,

$$\begin{aligned}\mu^*(E) &\geq \mu^*\left(E \cap \bigcup_n A_n\right) + \mu^*\left(E \cap \left(\bigcup_n A_n\right)^C\right) \\ &= \mu^*\left(E \cap \bigcup_n A_n\right) + \mu^*\left(E \cap \left(\bigcap_n A_n^C\right)\right).\end{aligned}$$

For simplicity, write $B = \bigcup_n A_n$. WTS

$$\mu^*(E) \geq \mu^*(E \cap B) + \mu^*(E \cap B^C). \tag{$*$}$$

Also thinking from behind, if $\mathfrak{M}$ is a σ-algebra, [1] then it must be an algebra (of sets). We showed that $\mathfrak{M}$ is closed under complementation.

[1] Useful links: Algebra of Sets, σ-Algebra of Sets.

If $\mathfrak{M}$ is closed under finite unions, [2] then for each $N \in \mathbb{N}$,

[2] Unproved point 1

$$\mu^*(E) = \mu^*\left(E \cap \bigcup_{n=1}^{N} A_n\right) + \mu^*\left(E \cap \left(\bigcup_{n=1}^{N} A_n\right)^C\right).$$

Let $B_N := \bigcup_{n=1}^{N} A_n \in \mathfrak{M}$. Then

$$\mu^*(E) = \mu^*(E \cap B_N) + \mu^*(E \cap B_N^C) \tag{†}$$

for each $N \in \mathbb{N}$.

Notice that

$$B_N = \bigcup_{n=1}^{N} A_n \subseteq \bigcup_{n=1}^{\infty} A_n = B.$$

Consequently,

$$\implies B^C \supseteq B_N^C \implies \mu^*(B^C) \leq \mu^*(B_N^C)$$

by the monotonicity of the outer measure.

As a result, looking at Equation (∗) and Equation (†), we see that

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap B_N) + \mu^*(E \cap B_N^C) \\ &\geq \mu^*(E \cap B_N) + \mu^*(E \cap B^C) \end{aligned}$$

for each $N \in \mathbb{N}$.

We are in quite the predicament at this point. We need to do something about $\mu^*(E \cap B_N)$ and somehow relate it to $\mu^*(E \cap B)$. We can try and see that

$$\mu^*(E \cap B_N) \leq \sum_{n=1}^{N} \mu^*(E \cap A_n).$$

Notice that in the case of **equality**, we would have

$$\mu^*(E) \geq \sum_{n=1}^{N} \mu^*(E \cap A_n) + \mu^*(E \cap B^C)$$

for all $N \in \mathbb{N}$. Since $\{\sum_{n=1}^{N} \mu^*(E \cap A_n)\}_N$ is an increasing sequence in $\mathbb{R}$, we have

$$\mu^*(E) \geq \sum_{n=1}^{\infty} \mu^*(E \cap A_n) + \mu^*(E \cap B^C),$$

and

$$\sum_{n=1}^{\infty} \mu^*(E \cap A_n) \geq \mu^*(E \cap B)$$

by subadditivity since $B = \bigcup_{n=1}^{\infty} A_n$.

Unfortunately, the equality does not always hold. But, since μ^* is an outer measure, we can make an educated guess [3] that given $\{A_n\}_n$ a disjoint collection of sets,

$$\mu^* \left(\bigcup_{n=1}^{N} A_n \right) = \sum_{n=1}^{N} \mu^*(A_n).$$

[3] Unproved point 2

Our work becomes even easier with the realization of Homework 4. Proving that all of our above argument works for the case of $\{A_n\}_n \subseteq \mathfrak{M}$ being disjoint, is sufficient to prove that $\mathfrak{M}$ is indeed a σ-algebra.

B Common Themes and Tricks

B.1 Re-represent an arbitrary union using disjoint sets

A common trick in measure theory, especially when it comes to a collection of sets, is to represent its union as a disjoint union of sets. This is a useful trick because measures simply add over disjoint sets, instead of just having **subadditivity**.

Example B.1.1

Given a collection $\{A_n\}_n$ of sets, we may define a collection of disjoint sets whose union is $\bigcup_n A_n$ as such:

$$\begin{aligned} F_1 &= A_1 \\ F_2 &= A_2 \setminus A_1 \\ F_3 &= A_3 \setminus (A_1 \cup A_2) \\ &\vdots \\ F_n &= A_n \setminus \bigcup_{i=1}^{n-1} A_i \\ &\vdots \end{aligned}$$

Example B.1.2

Given an increasing collection $\{A_n\}_n$ of sets, i.e.

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots,$$

we may represent the countable union of the A_n's as such: let

$$\begin{aligned} F_1 &= A_1 \\ F_2 &= A_2 \setminus A_1 \\ F_3 &= A_3 \setminus A_2 \\ &\vdots \\ Fn &= A_n \setminus A_{n-1} \\ &\vdots \end{aligned}$$

Remark B.1.1

The reason why we simply consider $F_n = A_n \setminus A_{n-1}$ instead of having to take a union up to the $(n-1)^{th}$ set in Example B.1.2 is because

$$\bigcup_{i=1}^{n-1} A_i = A_{n-1}. \tag{B.1}$$

The reader may also notice that Example B.1.2 is an application of Example B.1.1 just because of Equation (B.1).

Bibliography

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