# PMATH365 — Differential Geometry

CLASSNOTES FOR WINTER 2019

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## Preface

This course is a post-requisite of MATH 235/245 (Linear Algebra II) and AMATH 231 (Calculus IV) or MATH 247 (Advanced Calculus III). In other words, familiarity with vector spaces and calculus is expected.

The course is spiritually separated into two parts. The first part shall be called **Exterior Differential Calculus**, which allows for a natural, metric-independent generalization of **Stokes' Theorem**, **Gauss's Theorem**, and **Green's Theorem**. Our end goal of this part is to arrive at Stokes' Theorem, that renders the **Fundamental Theorem** of **Calculus** as a special case of the theorem.

The second part of the course shall be called in the name of the course: **Differential Geometry**. This part is dedicated to studying geometry using techniques from differential calculus, integral calculus, linear algebra, and multilinear algebra.

# Part I Exterior Differential Calculus

## 1 Lecture 1 Jan 07th

#### 1.1 Linear Algebra Review

#### Definition 1 (Linear Map)

Let V, W be finite dimensional real vector spaces. A map  $T: V \to W$  is called **linear** if  $\forall a, b \in \mathbb{R}$ ,  $\forall v \in V$  and  $\forall w \in W$ ,

$$T(av + bw) = aT(v) + bT(w).$$

We define L(U, W) to be the set of all linear maps from V to W.

#### 66 Note

- Note that L(U, W) is itself a finite dimensional real vector space.
- The structure of the vector space L(V,W) is such that  $\forall T,S \in L(V,W)$ , and  $\forall a,b \in \mathbb{R}$ , we have

$$aT + bS : V \rightarrow W$$

and

$$(aT + bS)(v) = aT(v) + bS(v).$$

• A special case: when W = V, we usually write

$$L(V,W) = L(V),$$

and we call this the space of linear operators on V.

Now suppose  $\dim(V) = n$  for some  $n \in \mathbb{N}$ . This means that there exists a basis  $\{e_1, \dots, e_n\}$  of V with n elements.

#### Definition 2 (Basis)

A basis  $\mathcal{B} = \{e_1, \dots, e_n\}$  of an n-dimensional vector space V is a subset of V where

1.  $\mathcal{B}$  spans V, i.e.  $\forall v \in V$ 

$$v = \sum_{i=1}^{n} v^{i} e_{i}.$$

2.  $e_1, \ldots, e_n$  are linearly independent, i.e.

$$v^i e_i = 0 \implies v^i = 0$$
 for every i.

 $^{1}$  We shall use a different convention when we write a linear combination. In particular, we use  $v^{i}$  to represent the  $i^{\text{th}}$  coefficient of the linear combination instead of  $v_{i}$ . Note that this should not be confused with taking powers, and should be clear from the context of the discussion.

#### 66 Note

We shall abusively write

$$v^i e_i = \sum_i v^i e_i$$
.

Again, this should be clear from the context of the discussion.

The two conditions that define a basis implies that any  $v \in V$  can be expressed as  $v^i e_i$ , where  $v^i \in \mathbb{R}$ .

#### Definition 3 (Coordinate Vector)

The n-tuple  $(v^1, ..., v^n) \in \mathbb{R}^n$  is called the **coordinate vector**  $[v]_{\mathcal{B}} \in \mathbb{R}^n$  of v with respect to the basis  $\mathcal{B} = \{e_1, ..., e_n\}$ .

#### 66 Note

It is clear that the coordinate vector  $[v]_{\mathcal{B}}$  is dependent on the basis  $\mathcal{B}$ . Note that we shall also assume that the basis is "ordered", which is somewhat important since the same basis (set-wise) with a different "ordering" may give us a completely different coordinate vector.

#### Example 1.1.1

Let  $V = \mathbb{R}^n$ , and  $\hat{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ , where 1 is the  $i^{th}$  compoenent of  $\hat{e}_1$ . Then

$$\mathcal{B}_{\text{std}} = \{\hat{e}_1, \dots, \hat{e}_n\}$$

is called the **standard basis** of  $\mathbb{R}^n$ .

#### 66 Note

Let  $v = (v^1, \dots, v^n) \in \mathbb{R}^n$ . Then

$$v = v^1 \hat{e}_1 + \dots v^n \hat{e}_n.$$

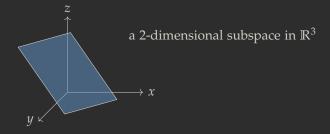
So 
$$\mathbb{R}^n \ni [v]_{\mathcal{B}_{\mathrm{std}}} = v \in V = \mathbb{R}^n$$
.

This is a privilege enjoyed by the n-dimensional vector space  $\mathbb{R}^n$ .

Now if we choose a **non-standard basis** of  $\mathbb{R}^n$ , say  $\tilde{\mathcal{B}}$ , then  $[v]_{\tilde{\mathcal{B}}} \neq$ 

#### 66 Note

It does not make sense to ask if a standard basis exists for an arbitrary space, as we have seen above. A geometrical way of wrestling with this notion is as follows:



While the subspace is embedding in a vector space of which has a standard basis, we cannot establish a "standard" basis for this 2-dimensional subspace. In laymen terms, we cannot tell which direction is up or down, positive or negative for the subspace, without making assumptions.

Figure 1.1: An arbitrary 2-dimensional subspace in a 3-dimensional space

However, since we are still in a finite-dimensional vector space, we can still make a connection to a Euclidean space of the same dimension.

#### Definition 4 (Linear Isomorphism)

Let V be n-dimensional, and  $\mathcal{B} = \{e_1, \ldots, e_n\}$  be some basis of V. The map

$$v = v^i e_i \mapsto [v]_{\mathcal{B}}$$

from V to  $\mathbb{R}^n$  is a linear isomorphism of vector spaces.

#### Exercise 1.1.1

Prove that the said linear isomorphism is indeed linear and bijective<sup>2</sup>.

<sup>2</sup> i.e. we are right in calling it linear and being an isomorphism

#### 66 Note

Any n-dimensional real vecotr space is isomorphic to  $\mathbb{R}^n$ , but not canonically so, as it requires the knowledge of the basis that is arbitrarily chosen. In other words, a different set of basis would give us a different isomorphism.

#### 1.2 Orientation

Consider an n-dimensional vector space V. Recall that for any linear operator  $T \in L(V)$ , we may associate a real number  $\det(T)$ , called the **determinant** of T, such that T is said to be **invertible** iff  $\det(T) \neq 0$ .

#### Definition 5 (Same and Opposite Orientations)

Let

$$\mathcal{B} = \{e_1, \dots, e_n\}$$
 and  $\tilde{\mathcal{B}} = \{\tilde{e}_1, \dots, \tilde{e}_n\}$ 

be two ordered bases of V. Let  $T \in L(V)$  be the linear operator defined by

$$T(e_i) = \tilde{e}_i$$

for each i = 1, 2, ..., n. This mapping is clearly invertible, and so

 $\det(T) \neq 0$ , and  $T^{-1}$  is also linear, such that  $T^{-1}(\tilde{e}_i) = e_i$ , for each

We say that  $\mathcal{B}$  and  $\tilde{\mathcal{B}}$  determine the same orientation if det(T) > 0, and we say that they determine the **opposite orientations** if det(T) <

#### 66 Note

- This notion of orientation only works in real vector spaces, as, for instance, in a complex vector space, there is no sense of "positivity" or "negativity".
- Whenever we talk about same and opposite orientation(s), we are usually talking about 2 sets of bases. It makes sense to make a comparison to the standard basis in a Euclidean space, and determine that the compared (non-)standard basis is "positive" (same direction) or "negative" (opposite), but, again, in an arbitrary space, we do not have this convenience.

#### Exercise 1.2.1 (A1Q1)

Show that any n-dimensional real vector space V admits exactly 2 orientations.

#### Example 1.2.1

On  $\mathbb{R}^n$ , consider the standard basis

$$\mathcal{B}_{\text{std}} = \{\hat{e}_1, \dots, \hat{e}_n\}.$$

The orientation determined by  $\mathcal{B}_{std}$  is called the standard orientation of  $\mathbb{R}^n$ .

#### Dual Space

#### Definition 6 (Dual Space)

Let V be an n-dimensional vector space. Then  $\mathbb{R}$  is a 1-dimensional real vector space. Thus we have that  $L(V,\mathbb{R})$  is also a real vector space <sup>3</sup>. The

<sup>&</sup>lt;sup>3</sup> Note that  $L(V, \mathbb{R})$  is also finite dimensional since both the domain and codomain are finite dimensional.

**dual space**  $V^*$  of V is defined to be

$$V^* := L(V, \mathbb{R}).$$

Let  $\mathcal{B}$  be a basis of V. For all i = 1, 2, ..., n, let  $e^i \in V^*$  such that

$$e^i(e_j) = \delta^i_j = egin{cases} 1 & i = j \ 0 & i 
eq j \end{cases}.$$

This  $\delta^i_j$  is known as the **Kronecker Delta**.

In general, we have that for every  $v=v^je_j\in V$ , where  $v^i\in\mathbb{R}$ , by the linearity of  $e^i$ , we have

$$e^{i}(v) = e^{i}(v^{j}e_{j}) = v^{j}e^{i}(e_{j}) = v_{j}\delta_{j}^{i} = v^{i}.$$

So each of the  $e^i$ , when applied on v, gives us the  $i^{th}$  component of  $[v]_{\mathcal{B}}$ , where  $\mathcal{B}$  is a basis of V.

## 2 Lecture 2 Jan 09th

#### 2.1 Dual Space (Continued)

#### • Proposition 1 (Dual Basis)

The set

$$\mathcal{B}^* := \left\{ e^1, \dots, e^n \right\}$$

<sup>1</sup> is a basis of  $V^*$ , and is called the **dual basis** of  $\mathcal{B}$ , where  $\mathcal{B}$  is a basis of V. In particular, dim  $V^* = n = \dim V$ .

 $^{\scriptscriptstyle 1}$  Note that the  $e^{i}$ 's are defined as in the last part of the last lecture.

#### Proof

 $\mathcal{B}^*$  spans  $V^*$  Let  $\alpha \in V^*$ . Let  $v = v^j e_j \in V$ , where we note that

$$\mathcal{B} = \{e_i\}_{i=1}^n$$

We have that

$$\alpha(v) = \alpha(v^j e_j) = v^j \alpha(e_j).$$

Now for all j = 1, 2, ..., n, define  $\alpha_j = \alpha(e_j)$ . Then

$$\alpha(v) = \alpha_j v^j = \alpha_j e^j(v),$$

which holds for all  $v \in V$ . This implies that  $\alpha = \alpha_j e^j$ , and so  $\mathcal{B}^*$  spans  $V^*$ .

 $\mathcal{B}^*$  is linearly independent Suppose  $\alpha_j e^j = 0 \in V^*$ . Applying  $\alpha_j e^j$  to each of the vectors  $e_k$  in  $\mathcal{B}$ , we have

$$\alpha_j e^j(e_k) = 0(e_k) = 0 \in \mathbb{R}$$

and

$$\alpha_j e^j(e_k) = \alpha_j \delta_k^j = \alpha_k.$$

By A1Q2, we have that  $a_k = 0$  for all k = 1, 2, ..., n, and so  $\mathcal{B}^*$  is linearly independent.

#### Remark

Let  $\mathcal{B} = \{e_1, \dots, e_n\}$  be a basis of V, with dual space  $\mathcal{B}^* = \{e^1, \dots, e^n\}$ . Then the map  $T: V \to V^*$  such that

$$T(e_i) = e^i$$

is a vector space isomorphism. And so we have that  $V \simeq V^*$ , but not cannonically so since we needed to know what the basis is in the first place.

We will see later that if we impose an **inner product** on V, then it will induce a canonical isomorphism from V to  $V^*$ .

#### Definition 7 (Natural Pairing)

The function

$$\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$$

given by

$$\langle \alpha, v \rangle \mapsto \alpha(v)$$

is called a **natural pairing** of  $V^*$  and V.

#### 66 Note

A natural pairing is bilinear, i.e. it is linear in  $\alpha$  and linear in v, which means that

$$\langle \alpha, t_1 v_1 + t_2 v_2 \rangle = t_1 \langle \alpha, v_1 \rangle + t_2 \langle \alpha, v_2 \rangle$$

and

$$\langle t_1 \alpha_1 + t_2 \alpha_2, v \rangle = t_1 \langle \alpha_1, v \rangle + t_2 \langle \alpha_2, v \rangle,$$

respectively.

#### • Proposition 2 (Natural Pairings are Nondegenerate)

For a finite dimensional real vector space V, a natural pairing is said to be nondegenerate if

This is A1Q2.

$$\forall v \in V \ \langle \alpha, v \rangle = 0 \iff \alpha = 0$$

and

$$\forall \alpha \in V^* \ \langle \alpha, v \rangle = 0 \iff v = 0.$$

#### Example 2.1.1

Fix a basis  $\mathcal{B} = \{e_1, \dots, e_n\}$  of V. Given  $T \in L(V)$ , there is an associated  $n \times n$  matrix  $A = [T]_{\mathcal{B}}$  defined by

The defined by 
$$T(e_i) = A_i^j e_j$$
.

Tow index  $T(e_i) = A_i^j e_j$ .

In particular,

$$A = \overbrace{\left[ [T(e_1)]_{\mathcal{B}} \quad \dots \quad [T(e_n)]_{\mathcal{B}} \right]}^{\text{block matrix}}$$

and

$$A_i^k = e^k(T(e_i)) = \langle e^k, T(e_i) \rangle.$$

#### Definition 8 (Double Dual Space)

The set

$$V^{**} = L(V^*, \mathbb{R})$$

is called the double dual space.

#### • Proposition 3 (The Space and Its Double Dual Space)

Let V be a finite dimensional real vector space and  $V^{**}$  be its double dual space. There exists a linear map  $\xi$  such that

$$\xi: V \to V^{**}$$

#### Proof

Let  $v \in V$ . Then  $\xi(v) \in V^{**} = L(V^*, \mathbb{R})$ , i.e.  $\xi(v) : V^* \to \mathbb{R}$ . Then

As messy as this may seem, this is really a follow your nose kind of proof. Since we are proving that a map exists, we need to construct it. Since  $\xi: V \to V^{**} = L(V^*, \mathbb{R})$ , for any  $v \in V$ , we must have  $\xi(v)$  as some linear map from  $V^*$  to  $\mathbb{R}$ .

for any  $\alpha \in V^*$ ,

$$(\xi(v))(\alpha) \in \mathbb{R}.$$

Since  $\alpha \in V^*$ , i.e.  $\alpha : V \to \mathbb{R}$ , and  $\alpha$  is linear, let us define

$$\xi(v)(\alpha) = \alpha(v).$$

To verify that  $\xi(v)$  is indeed linear, notice that for any  $t,s \in \mathbb{R}$ , and for any  $\alpha,\beta \in V^*$ , we have

$$\xi(v)(t\alpha + s\beta) = (t\alpha + s\beta)(v)$$
$$= t\alpha(v) + s\beta(v)$$
$$= t\xi(v)(\alpha) + s\xi(v)(\beta).$$

It remains to show that  $\xi$  itself is linear: for any  $t,s \in \mathbb{R}$ , any  $v,w \in V$ , and any  $\alpha \in V^*$ , we have

$$\xi(tv + sw)(\alpha) = \alpha(tv + sw) = t\alpha(v) + s\alpha(w)$$
$$= t\xi(v)(\alpha) + s\xi(v)(\alpha)$$
$$= [t\xi(v) + s\xi(w)](\alpha)$$

by addition of functions.

# **6** Proposition 4 (Isomorphism Between The Space and Its Dual Space)

*The linear map in* **♦** *Proposition 3 is an isomorphism.* 

#### Proof

From  $\bullet$  Proposition 3,  $\xi$  is linear. Let  $v \in V$  such that  $\xi(v) = 0$ , i.e.  $v \in \ker(\xi)$ . Then by the same definition of  $\xi$  as above, we have

$$0 = (\xi(v))(\alpha) = \alpha(v)$$

for any  $\alpha \in V^*$ . By  $\bullet$  Proposition 2, we must have that v=0, i.e.  $\ker(\xi)=\{0\}$ . Thus by  $\bullet$  Proposition A.2,  $\xi$  is injective.

Now, since

$$V^{**} = L(V^*, \mathbb{R}) = L(L(V, \mathbb{R}), \mathbb{R}),$$

we have that

$$\dim(V^{**}) = \dim(V^*) = \dim(V).$$

Thus, by the Rank-Nullity Theorem  $^2$ , we have that  $\xi$  is surjective.

<sup>2</sup> See Appendix A.1, and especially • Proposition A.3.

The above two proposition shows to use that we may identify Vwith  $V^{**}$  using  $\xi$ , and we can gleefully assume that  $V = V^{**}$ .

Consequently, if  $v \in V = V^{**}$  and  $\alpha \in V^*$ , we have

$$\alpha(v) = v(\alpha) = \langle \alpha, v \rangle.$$
 (2.1)

#### 2.2 Dual Map

#### Definition 9 (Dual Map)

Let  $T \in L(V, W)$ , where V, W are finite dimensional real vector spaces. Let

$$T^*:W^*\to V^*$$

be defined as follows: for  $\beta \in W^*$ , we have  $T^*(\beta) \in V^*$ . Let  $v \in V$ , and so  $(T^*(\beta))(v) \in \mathbb{R}$  3. From here, we may define

$$(T^*(\beta))(v) = \beta(T(v)).$$

The map  $T^*$  is called **the dual map**.

<sup>3</sup> It shall be verified here that  $T^*(\beta)$ is indeed linear: let  $v_1, v_2 \in V$  and  $c_1, c_2 \in \mathbb{R}$ . Indeed

$$T^*(\beta)(c_1v_1 + c_2v_2)$$
  
=  $c_1T^*(\beta)(v_1) + c_2T^*(\beta)(v_2)$ 

#### Exercise 2.2.1

Prove that  $T^* \in L(W^*, V^*)$ , i.e. that  $T^*$  is linear.

#### Proof

Let  $\beta_1, \beta_2 \in W^*$ ,  $t_1, t_2 \in \mathbb{R}$ , and  $v \in V$ . Then

$$T^*(t_1\beta_1 + t_2\beta_2)(v) = (t_1\beta_1 + t_2\beta_2)(Tv)$$

$$= t_1\beta_1(Tv) + t_2\beta_2(Tv)$$

$$= t_1T^*(\beta_1)(v) + t_2T^*(\beta_2)(v).$$

#### 66 Note

Note that in  $\blacksquare$  Definition 9, our construction of  $T^*$  is canonical, i.e. its construction is independent of the choice of a basis.

Also, notice that in the language of pairings, we have

$$\langle T^*\beta, v \rangle = (T^*(\beta))(v) = \beta(T(v)) = \langle \beta, T(v) \rangle,$$

where we note that

$$T^*(\beta) \in V^* \quad v \in V$$
  
 $\beta \in W^* \quad T(v) \in W.$ 

# 3 Lecture 3 Jan 11th

#### 3.1 Dual Map (Continued)

#### 66 Note

Elements in  $V^*$  are also called **co-vectors**.

Recall from last lecture that if  $T \in L(V, W)$ , then it induces a dual map  $T^* \in L(W^*, V^*)$  such that

$$(T^*\beta)(v) = \beta(T(v)).$$

#### • Proposition 5 (Identity and Composition of the Dual Map)

Let V and W be finite dimensional real vector spaces.

1. Suppose V = W and  $T = I_V \in L(V)$ , then

$$(I_V)^* = I_{V^*} \in L(V^*).$$

2. Let  $T \in L(V, W)$ ,  $S \in L(W, U)$ . Then  $S \circ T \in L(V, U)$ . Moreover,

$$L(U^*,V^*)\ni (S\circ T)^*=T^*\circ S^*.$$

#### Proof

1. Observe that for any  $\beta \in V^*$ , and any  $v \in V$ , we have

$$((I_V)^*(\beta))(v) = \beta((I_V)(v)) = \beta(v).$$

Therefore  $(I_V)^* = I_{V^*}$ .

2. Observe that for  $\gamma \in U^*$  and  $v \in V$ , we have

$$\begin{split} ((S \circ T)^*(\gamma))(v) &= \gamma((S \circ T)(v)) \\ &= \gamma(S(T(v))) \\ &= S^*(\gamma T(v)) \\ &= (T^* \circ S^*)(\gamma)(v), \end{split}$$

and so  $(S \circ T)^* = T^* \circ S^*$  as required.

Let  $T \in L(V)$ , and the dual map  $T^* \in L(V^*)$ . Let  $\mathcal{B}$  be a basis of V, with the dual basis  $\mathcal{B}^*$ . We may write

$$A = [T]_{\mathcal{B}} \text{ and } A^* = [T^*]_{\mathcal{B}^*}.$$

Note that

$$T(e_i) = A_i^j e_j$$
 and  $T^*(e^i) = (A^*)_i^i e^j$ .

Consequently, we have

$$\langle e^k, T(e_i) \rangle = A_i^k \text{ and } \langle T^*(e^i), e_k \rangle = (A^*)_k^i.$$

From here, notice that by applying  $e_k \in V = V^{**}$  to both sides, we have

$$(A^*)_k^i = e_k(T^*(e^i)) = \langle T^*(e^i), e_k \rangle \stackrel{(*)}{=} \langle e^i, T(e_k) \rangle = A_k^i.$$

Thus  $A^*$  is the transpose of A, and

$$[T^*]_{\mathcal{B}^*} = [T]_{\mathcal{B}}^t \tag{3.1}$$

where  $M^t$  is the transpose of the matrix M.

#### 3.1.1 *Application to Orientations*

Let  $\mathcal{B}$  be a basis of V. Then  $\mathcal{B}$  determines an orientation of V. Let  $\mathcal{B}^*$  be the dual basis of  $V^*$ . So  $\mathcal{B}^*$  determines an orientation for  $V^*$ .

#### Example 3.1.1

Suppose  $\mathcal{B}$  and  $\tilde{\mathcal{B}}$  determines the same orientation of V. Does it

follow that the dual bases  $\mathcal{B}^*$  and  $\tilde{\mathcal{B}}^*$  determine the same orientation of  $V^*$ ?

#### Proof

Let

$$\mathcal{B} = \{e_1, \dots, e_n\}$$
  $\qquad \qquad \tilde{\mathcal{B}} = \{\tilde{e}_1, \dots, \tilde{e}_n\}$   $\qquad \qquad \tilde{\mathcal{B}}^* = \{\tilde{e}^1, \dots, \tilde{e}^n\}$ 

Let  $T \in L(V)$  such that  $T(e_i) = \tilde{e}_i$ . By assumption, det T > 0. Notice that

$$\delta_j^i = \tilde{e}^i(\tilde{e}_j) = \tilde{e}^i(Te_j) = (T^*(\tilde{e}^i))(e_j),$$

and so we must have  $T^*(\tilde{e}^i) = e^i$ . By Equation (3.1), we have that

$$\det T^* = \det T > 0$$

as well. This shows that  $\mathcal{B}^*$  and  $\tilde{\mathcal{B}}^*$  determines the same orientation.

#### 3.2 The Space of k-forms on V

#### Definition 10 (k-Form)

Let V be an ndimensional vector space. Let  $k \geq 1$ . A k-form on V is a тар

$$\alpha: \underbrace{V \times V \times \ldots \times V}_{k \text{ times}} \to \mathbb{R}$$

such that

1. (k-linearity / multi-linearity) if we fix all but one of the arguments of  $\alpha$ , then it is a linear map from V to  $\mathbb{R}$ ; i.e. if we fix

$$v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_k\in V$$
,

then the map

$$u \mapsto \alpha(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_k)$$

is linear in u.

2. (alternating property)  $\alpha$  is alternating (aka totally skewed-symmetric) in its k arguments; i.e.

$$\alpha(v_1,\ldots,v_i,\ldots,v_j,\ldots,v_k)=\alpha(v_1,\ldots,v_j,\ldots,v_i,\ldots,v_k).$$

#### Example 3.2.1

The following is an example of the second condition: if k = 2, then  $\alpha : V \times V \to \mathbb{R}$ . Then  $\alpha(v, w) = -\alpha(w, v)$ .

If k = 3, then  $\alpha : V \times V \times V \to \mathbb{R}$ . Then we have

$$\alpha(u,v,w) = -\alpha(v,u,w) = -\alpha(w,v,u) = -\alpha(u,w,v)$$
$$= \alpha(v,w,u) = \alpha(w,u,v).$$

#### 66 Note

Note that if k = 1, then condition 2 is vacuous. Therefore, a 1-form of V is just an element of  $V^* = L(W, \mathbb{R})$ .

#### Remark (Permutations)

From the last example, we notice that the 'sign' of the value changes as we permute more times. To be precise, we are performing transpositions on the arguments <sup>1</sup>, i.e. we only swap two of the arguments in a single move. Here are several remarks about permutations from group theory:

<sup>1</sup> See PMATH 347.

- A permutation  $\sigma$  of  $\{1, 2, ..., k\}$  is a bijective map.
- Compositions of permutations results in a permutation.
- The set  $S_k$  of permutations on the set  $\{1, 2, ..., k\}$  is called a group.
- *There are k! such permutations.*
- For each transposition, we may assign a parity of either -1 or 1, and the parity is determined by the number of times we need to perform a transposition to get from (1, 2, ..., k) to  $(\sigma(1), \sigma(2), ..., \sigma(k))$ . We usually denote a parity by  $sgn(\sigma)$ .

The following is a fact proven in group theory: let  $\sigma, \tau \in S_k$ . Then

$$\begin{split} \operatorname{sgn}(\sigma \circ \tau) &= \operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\tau) \\ \operatorname{sgn}(\operatorname{id}) &= 1 \\ \operatorname{sgn}(\tau) &= \operatorname{sgn}(\tau^{-1}). \end{split}$$

Using the above remark, we can rewrite condition 2 as follows:

### 66 Note (Rewrite of condition 2 for Definition 10)

 $\alpha$  is alternating, i.e.

$$\alpha(v_{\sigma(1)},\ldots,v_{\sigma(k)}) = \operatorname{sgn}(\sigma) \cdot \alpha(v_1,\ldots,v_k),$$

where  $\sigma \in S_k$ .

#### Remark

If  $\alpha$  is a k-form on V, notice that

$$\alpha(v_1,\ldots,v_k)=0$$

if any 2 of the arguments are equal.

## 4 Lecture 4 Jan 14th

#### 4.1 The Space of k-forms on V (Continued)

#### **■** Definition 11 (Space of *k*-forms on *V*)

The space of k-forms on V, denoted as  $\wedge^k(V^*)$ , is the set of all k-forms on V, made into a vector space by setting

$$(t\alpha + s\beta)(v_1, \ldots, v_k) := t\alpha(v_1, \ldots, v_k) + s\beta(v_1, \ldots, v_k),$$

for  $\alpha\beta \in \wedge^k(V^*)$ ,  $t,s \in \mathbb{R}$ .

#### 66 Note

By convention, we define  $\wedge^0(V^*)=\mathbb{R}$ . The reasoning shall we shown later.

#### 66 Note

By the note on page 26, observe that  $\wedge^{1}(V^{*}) = V^{*}$ .

# **♦** Proposition 6 (A *k*-form is equivalently 0 if its arguments are linearly dependent)

Let  $\alpha$  be a k-form. Then if  $v_1, \ldots, v_k$  are linearly dependent, then

$$\alpha(v_1,\ldots,v_k)=0.$$

#### Proof

Suppose one of the  $v_1, \ldots, v_k$  is a linear combination of the rest of the other vectors; i.e.

$$v_j = c_1 v_1 + \ldots + c_{j-1} v_{j-1} + c_{j+1} v_{j+1} + \ldots + c_k v_k.$$

Then since  $\alpha$  is multilinear, and by the last remark in Chapter 3, we have

$$\alpha(v_1,\ldots,v_{j-1},v_j,v_{j+1},\ldots,v_k)=0.$$

► Corollary 7 (k-forms of even higher dimensions)

$$\wedge^k (V^*) = \{0\} \text{ if } k > n = \dim V.$$

#### Proof

Any set of k > n vectors is necessarily linearly dependent.

#### 66 Note

ightharpoonup Corollary 7 implies that  $\wedge^k(V^*)$  can only be non-trivial for  $0 \le k \le n = \dim V$ .

#### 4.2 Decomposable k-forms

There is a simple way to construct a k-form on V using k-many 1-forms from V, i.e. k-many elements from  $V^*$ . Let  $\alpha^1, \ldots, \alpha^k \in V^*$ . Define a map

$$\alpha^1 \wedge \ldots \wedge \alpha^k : \underbrace{V \times V \times \ldots \times V}_{k \text{ copies}} \to \mathbb{R}$$

by

$$\left(\alpha^{1} \wedge \ldots \wedge \alpha^{k}\right)(v_{1}, \ldots, v_{k}) := \sum_{\sigma \in S_{k}} (\operatorname{sgn} \sigma) \alpha^{\sigma(1)}(v_{1}) \alpha^{\sigma(2)}(v_{2}) \ldots \alpha^{\sigma(k)}(v_{k}).$$

$$(4.1)$$

We need, of course, to verify that the above formula is, indeed, a *k*-form. Before that, consider the following example:

#### Example 4.2.1

If k = 2, we have

$$(\alpha^1 \wedge \alpha^2) (v_1, v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^2(v_1)\alpha^1(v_2).$$

and if k = 3, we have

$$\begin{split} \left(\alpha^1 \wedge \alpha^2 \wedge \alpha^3\right)(v_1, v_2, v_3) &= \alpha^1(v_1)\alpha^2(v_2)\alpha^3(v_3) + \alpha^2(v_1)\alpha^3(v_2)\alpha^1(v_1) \\ &+ \alpha^3(v_1)\alpha^1(v_2)\alpha^2(v_3) - \alpha^1(v_1)\alpha^3(v_2)\alpha^2(v_3) \\ &- \alpha^2(v_1)\alpha^1(v_1)\alpha^3(v_3) - \alpha^3(v_1)\alpha^2(v_2)\alpha^1(v_3). \end{split}$$

Now consider a general case of k. It is clear that Equation (4.1) is k-linear: if we fix any one of the arguments, then Equation (4.1) is reduced to a linear equation.

For the alternating property, let  $\tau \in S_k$ . WTS

$$\left(\alpha^1\wedge\ldots\wedge\alpha^k\right)\left(v_{\tau(1)},\ldots,v_{\tau(k)}\right)=\left(\operatorname{sgn}\tau\right)\left(\alpha^1\wedge\ldots\wedge\alpha^k\right)\left(v_1,\ldots,v_k\right).$$

Observe that

$$\begin{split} &\left(\alpha^{1}\wedge\ldots\wedge\alpha^{k}\right)\left(v_{\tau(1)},\ldots,v_{\tau(k)}\right) \\ &= \sum_{\sigma\in S_{k}}\left(\operatorname{sgn}\sigma\right)\alpha^{\sigma(1)}\left(v_{\tau(1)}\right)\ldots\alpha^{\sigma(k)}\left(v_{\tau(k)}\right) \\ &= \sum_{\sigma\in S_{k}}\left(\operatorname{sgn}\sigma\circ\tau^{-1}\right)\left(\operatorname{sgn}\tau\right)\alpha^{\left(\sigma\circ\tau^{-1}\right)\left(\tau(1)\right)}\left(v_{\tau(1)}\right)\ldots\alpha^{\left(\sigma\circ\tau^{-1}\right)\left(\tau(k)\right)}\left(v_{\tau(k)}\right) \\ &= \left(\operatorname{sgn}\tau\right)\sum_{\sigma\circ\tau^{-1}\in S_{k}}\left(\operatorname{sgn}\sigma\circ\tau^{-1}\right)\alpha^{\left(\sigma\circ\tau^{-1}\right)\left(1\right)}\left(v_{1}\right)\ldots\alpha^{\left(\sigma\circ\tau^{-1}\right)\left(k\right)}\left(v_{k}\right) \\ &= \left(\operatorname{sgn}\tau\right)\sum_{\sigma\in S_{k}}\alpha^{\sigma(1)}(v_{1})\ldots\alpha^{\sigma(k)}(v_{k}) \quad \because \text{ relabelling} \\ &= \left(\operatorname{sgn}\tau\right)\left(\alpha^{1}\wedge\ldots\alpha^{k}\right)\left(v_{1},\ldots,v_{k}\right), \end{split}$$

as claimed.

#### Definition 12 (Decomposable *k*-form)

The k-form as discussed above is called a **decomposable** k-form, which for ease of reference shall be re-expressed here:

$$\left(\alpha^1 \wedge \ldots \wedge \alpha^k\right)(v_1, \ldots, v_k) := \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \, \alpha^{\sigma(1)}(v_1) \alpha^{\sigma(2)}(v_2) \ldots \alpha^{\sigma(k)}(v_k).$$

#### 66 Note

Not all k-forms are decomposable. If k = 1, n - 1 and n, but not for 1 < k < n - 1.

In A1Q5(c), we will show that there exists a 2-form in n=4 that is not decomposable.

#### • Proposition 8 (Permutation on *k*-forms)

Let  $\tau \in S_k$ . Then

$$\alpha^{\tau(1)} \wedge \ldots \wedge \alpha^{\tau(k)} = (\operatorname{sgn} \tau) \alpha^1 \wedge \ldots \wedge \alpha^k$$

#### Proof

Firstly, note that  $\operatorname{sgn} \tau = \operatorname{sgn} \tau^{-1}$ . Then for any  $(v_1, \ldots, v_k) \in V^k$ , we have

$$\begin{split} & \alpha^{\tau(1)} \wedge \ldots \wedge \alpha^{\tau(k)}(v_1, \ldots, v_k) \\ & = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \alpha^{\sigma \circ \tau(1)}(v_1) \ldots \alpha^{\sigma \circ \tau(k)}(v_k) \\ & = \sum_{\sigma \circ \tau S_k} (\operatorname{sgn} \sigma \circ \tau) \left( \operatorname{sgn} \tau^{-1} \right) \alpha^{\sigma \circ \tau(1)}(v_1) \ldots \alpha^{\sigma \circ \tau(k)}(v_k) \\ & = (\operatorname{sgn} \tau) \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \alpha^{\sigma(1)}(v_1) \ldots \alpha^{\sigma(k)}(v_k) \\ & = (\operatorname{sgn} \tau) (\alpha^1 \wedge \ldots \wedge \alpha^k). \end{split}$$

This completes our proof.

#### • Proposition 9 (Alternate Definition of a Decomposable k-form)

Another way we can define a decomposable k-form is

$$(\alpha^1 \wedge \ldots \wedge \alpha^k)(v_1, \ldots, v_k) = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \alpha^1(v_{\sigma(1)}) \ldots \alpha^k(v_{\sigma(k)}).$$

#### $\blacksquare$ Theorem 10 (Basis of $\Lambda^k(V^*)$ )

Let  $\mathcal{B} = \{e_1, \dots, e_n\}$  be a basis of V, a n-dimensional real vector space, and the dual basis  $\mathcal{B}^* = \{e^1, \dots, e^n\}$  of  $V^*$ . Then the set

$$\left\{ e^{j_1} \wedge \ldots \wedge e^{j_k} \mid 1 \leq j_1 < j_2 < \ldots < j_k \leq n \right\}$$

is a basis of  $\Lambda^k(V^*)$ .

#### ightharpoonup Corollary 11 (Dimension of $\Lambda^k(V^*)$ )

The dimension of  $\Lambda^k(V^*)$  is  $\binom{n}{k}=\binom{n}{n-k}$ , which is also the dimension of  $\Lambda^{n-k}(V^*)$ . This also works for k=0 1.

<sup>1</sup> This is why we wanted  $\Lambda^0(V^*) = \mathbb{R}$ .

#### Proof ( Theorem 10)

Firstly, let  $\alpha$  be an arbitrary k-form, and let  $v_1, \ldots, v_k \in V$ . We may write

$$v_i = v_i^j e_j,$$

where  $v_i^j \in \mathbb{R}$ . Then

$$\alpha(v_1,\ldots,v_k) = \alpha\left(v_1^{j_1}e_{j_1},\ldots,v_k^{j_k}e_{j_k}\right)$$
$$= v_1^{j_1}\ldots v_k^{j_k}\alpha(e_{j_1},\ldots,e_{j_k})$$

by multilinearity and totally skew-symmetry of  $\alpha$ , where  $j_i \in$  $\{1, \ldots, n\}$ . Let

$$\alpha(e_{j_1},\ldots,e_{j_k})=\alpha_{j_1,\ldots,j_k},\tag{4.2}$$

represent the scalar. Then

$$\alpha(v_1,\ldots,v_k) = \alpha_{j_1,\ldots,j_k} v_1^{j_1} \ldots v_k^{j_k}$$
  
=  $\alpha_{j_1,\ldots,j_k} e^{j_1}(v_1) \ldots e^{j_k}(v_k)$ .

Now since  $\alpha_{j_1,...,j_k}$  is totally skew-symmetric,  $\alpha=0$  if any of the  $j_k$ 's are equal to one another. Thus we only need to consider the terms where the  $j_k$ 's are distinct. Now for any set of  $\{j_1,\ldots,j_k\}$ , there exists a unique  $\sigma \in S_k$  such that  $\sigma$  rearranges the  $j_i$ 's so that  $j_1,\ldots,j_k$  is strictly increasing. Thus

$$\begin{split} \alpha(v_1,\ldots,v_k) &= \sum_{j_1 < \ldots < j_k} \sum_{\sigma \in S_k} \alpha_{j_{\sigma 1(),\ldots,\sigma(k)}} e^{j_{\sigma(1)}}(v_1) \ldots e^{j_{\sigma(k)}}(v_k) \\ &= \sum_{j_1 < \ldots < j_k} \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma) \alpha_{j_1,\ldots,j_k} e^{j_{\sigma(1)}}(v_1) \ldots e^{j_{\sigma(k)}}(v_k) \\ &= \sum_{j_1 < \ldots < j_k} \alpha_{j_1,\ldots,j_k} \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma) e^{j_{\sigma(1)}}(v_1) \ldots e^{j_{\sigma(k)}}(v_k) \\ &= \underbrace{\sum_{j_1 < \ldots < j_k} \alpha_{j_1,\ldots,j_k} \left( e^{j_1} \wedge \ldots \wedge e^{j_k} \right)}_{\alpha} (v_1,\ldots,v_k). \end{split}$$

Thus we have that

$$\alpha = \sum_{j_1 < \dots < j_k} \alpha_{j_1, \dots, j_k} e^{j_1} \wedge \dots \wedge e^{j_k}.$$

Hence  $e^{j_1} \wedge \ldots \wedge e^{j_k}$  spans  $\Lambda^k(V^*)$ .

Now suppose that

$$\sum_{j_1 < \dots < j_k} \alpha_{j_1, \dots, j_k} e^{j_1} \wedge \dots \wedge e^{j_k}$$

is the zero element in  $\Lambda^k(V^*)$ . Then the scalar in Equation (4.2) must be 0 for any  $j_1, \ldots, j_k$ . Thus

$$\left\{ e^{j_1} \wedge \ldots \wedge e^{j_k} \mid 1 \leq j_1 < j_2 < \ldots < j_k \leq n \right\}$$

is linearly independent.

# A Review of Earlier Contents

#### A.1 Rank-Nullity Theorem

#### Definition A.1 (Kernel and Image)

Let V and W be vector spaces, and let  $T \in L(V, W)$ . The **kernel** (or **null** space) of T is defined as

$$\ker(T) := \{ v \in V \mid Tv = 0 \},$$

i.e. the set of vectors in V such that they are mapped to 0 under T.

The *image* (or range) of T is defined as

$$\operatorname{Img}(T) = \{ Tv \mid v \in V \},\,$$

that is the set of all images of vectors of V under T.

It can be shown that for a linear map  $T \in L(V, W)$ , ker(T) and Img(T) are subspaces of V and W, respectively. As such, we can define the following:

#### Definition A.2 (Rank and Nullity)

Let V, W be vector spaces, and let  $T \in L(V, W)$ . If ker(T) and Img(T) are finite-dimensional  $^1$ , then we define the **nullity** of T as

$$nullity(T) := \dim \ker(T)$$
,

<sup>&</sup>lt;sup>1</sup> In this course, this is always the case, since we are only dealing with finite dimensional real vector spaces.

and the rank of T as

$$rank(T) := dim Img(T)$$
.

#### 66 Note

From the action of a linear transformation, we observe that the *larger the nullity, the smaller the rank*. Put in another way, the more vectors are sent to 0 by the linear transformation, the smaller the range.

Similarly, the larger the rank, the smaller the nullity.

This observation gives us the Rank-Nullity Theorem.

#### **■** Theorem A.1 (Rank-Nullity Theorem)

Let V and W be vector spaces, and  $T \in L(V, W)$ . If V is finite-dimensional, then

$$\operatorname{nullity}(T) + \operatorname{rank}(T) = \dim(V).$$

From the Rank-Nullity Theorem, we can make the following observations about the relationships between injection and surjection, and the nullity and rank.

#### • Proposition A.2 (Nullity of Only 0 and Injectivity)

Let V and W be vector spaces, and  $T \in L(V, W)$ . Then T is injective iff  $\operatorname{nullity}(T) = \{0\}.$ 

Surjection and injectivity come hand-in-hand when we have the following special case.

# • Proposition A.3 (When Rank Equals The Dimension of the Space)

Let V and W be vector spaces of equal (finite) dimension, and let  $T \in$ L(V,W). TFAE

- 1. T is injective;
- 2. T is surjective;
- 3.  $\operatorname{rank}(T) = \dim(V)$ .

Note that the proof for **6** Proposition A.3 requires the understanding that  $ker(T) = \{0\}$  implies that nullity(T) = 0. See this explanation on Math SE.

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