STAT330S18 - Mathematical Statistics

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June 22, 2018

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Foreword

The proofs in this set of notes will be more rigourous compared to the expectations of the course. If you are not the author and is interested in reading the notes, you may skip the proofs should you have little interest in them. The rigour is required almost exclusively for the author himself, for his own practice, and because he transferred his STAT230 course from a class that is clean of proofs.

Also, many of the common mathematical notations will be heavily used both in the author's notes and proofs.

1 Lecture 1 May 1st 2018

1.1 Introduction

Definition 1 (Sample Space)

*A **sample space**, S of a random experiment is the set of all possible outcomes of the experiment.*

Example 1.1.1

The following are some random experiments and their sample space.

- *Flipping a coin*
  $S = \{H, T\}$ *where* H *denotes head and* T *tail.*
- *Rolling a 6-faced dice twice*
  $S = \{(x, y) : x, y \in \mathbb{N},\ 1 \leq x, y \leq 6\}$
- *Measuring a patient's height*
  $S = R^+ = \{x \in \mathbb{R} : x \geq 0\}$

Definition 2 (σ-field)

*Let S be a sample space. The collection of sets $\mathscr{B} \subseteq \mathbb{P}(S)$[1], is called a σ-field (or **σ-algebra**) on S if:*

1. $\emptyset \in \mathscr{B}$ *and* $S \in \mathscr{B}$;
2. $\forall A \in \mathscr{B} \quad A^C \in \mathscr{B}$; [2] *and*
3. $\forall n \in \mathbb{N} \quad \forall \{A_j\}_{j=1}^n \subseteq \mathscr{B} \quad \cup_{j=1}^n A_j \in \mathscr{B}$.

[1] The **power set** of S, $\mathbb{P}(S)$, is defined as the set that contains all subsets of S.

[2] We shall denote the compliment of a set by a superscript C in this set of notes. The supplemental notes provided in the class uses an overhead bar, e.g. $\overline{A}$, while lecture notes will use A^C and A' interchangably.

Definition 3 (Measurable Space)

*Given that S is a non-empty set, and $\mathscr{B}$ is a σ-field, $(S, \mathscr{B})$ is a **measurable space**.*[3]

[3] A measurable space is a basic object in **measure theory**.

Example 1.1.2

Consider $S = \{1,2,3,4\}$. Check if $\mathscr{B} = \{\emptyset, \{1,2,3,4\}, \{1,2\}, \{3,4\}\}$ is a σ-field on S.

1. *It is clear that $\emptyset, S \in \mathscr{B}$.*
2. *Note that $S^C = \emptyset$ and $\{1,2\}^C = \{3,4\}$.*
3. *Note that the largest possible result of any countable union of the elements of $\mathscr{B}$ is $\{1,2,3,4\}$, which is an element of $\mathscr{B}$.*

Because $(S, \mathscr{B})$ is a measurable space, we can define a measure on it.

Definition 4 (Probability Measure)

*Suppose S is a sample space of a random experiment. Let $\mathscr{B} = \{A_1, A_2, ...\} \subseteq \mathbb{P}(S)$ be the σ-field on S. The **probability set function** (or **probability measure**), $P : \mathscr{B} \to [0,1]$, is a function that satisfies the following:*[4]

[4] These conditions are also known as **Kolmogorov Axioms**, or **probability axioms**.

- $\forall A \in \mathscr{B} \;\; P(A) \geq 0$;
- $P(S) = 1$;
- $\forall \{A_j\}_{j=1}^{\infty} \subseteq \mathscr{B} \;\; \forall i \neq j \in \mathbb{N} \; A_i \cap A_j = \emptyset \implies$

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P(A_j) \tag{1.1}$$

*$(S, \mathscr{B}, P)$ is called a **probability space**.*

Example 1.1.3

Consider flipping a coin where $S = \{H, T\}$. Let P be defined as follows

$$P(\{H\}) = \frac{1}{3} \quad P(\{T\}) = \frac{2}{3} \quad P(\emptyset) = 0 \quad P(S) = 1$$

Conditions 1 and 2 of Definition 4 are met. Notice that

$$P(\{H\} \cup \{T\}) = P(S) = 1 \text{ and } P(\{H\}) + P(\{T\}) = \frac{1}{3} + \frac{2}{3} = 1.$$

Hence condition 3 is also fulfilled.

Proposition 1 (Properties of Probability Set Functions)

Let P be a probability set function and A, B be any set in $\mathscr{B}$. Prove the following:[5]

1. $P(A^C) = 1 - P(A)$
2. $P(\emptyset) = 0$
3. $P(A) \leq 1$
4. $P(A \cap B^C) = P(A) - P(A \cap B)$
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
6. $A \subseteq B \implies P(A) \leq P(B)$

[5] Many among these properties illustrate that the probability is indeed a **measure**.

Exercise 1.1.1
Prove that $A \subseteq B \iff B^C \subseteq A^C$.

Proof

Let S be the sample space for P.

1. *Note that*
   $A \in \mathscr{B} \implies A \in \mathbb{P}(S) \iff A \subseteq S$
   $A \in \mathscr{B} \iff A^C \in \mathscr{B} \implies A^C \subseteq S$. *Also, since A^C is the complement of A, it is clear that $S = A \cup A^C$.*

   $$\therefore P(S) = 1 \iff P(A \cup A^C) = 1 \overset{1}{\iff} P(A) + P(A^C) = 1$$

   where 1 is by condition 3 in Definition 4 since $A \cap A^C = \emptyset$ by definition of a complement of a set.

2. *Note that $S \cup \emptyset = S$ and $S \cap \emptyset = \emptyset$. Using a similar argument as above,*

   $$1 = P(S) = P(S \cup \emptyset) = P(S) + P(\emptyset) \implies P(\emptyset) = 0$$

3. *By 1 from above,* $P(A) = 1 - P(A^C)$. *Since* $0 \leq P(A^C) \leq 1$, *we have that* $P(A)$ *is at most* 1, *as required.*

4. *Note that* $A = (A \cap B) \cup (A \cap B^C)$. *Clearly,* $(A \cap B) \cap (A \cap B^C) = \emptyset$.[6] *Hence by condition 3 in Definition 4,*

$$P(A) = P(A \cap B) + P(A \cap B^C)$$

[6] This is an easy proof using the basic way of proving membership.

5. *Consider* $P(A \cup B) + P(A \cap B)$. *By definition,*

$$A \cup B = (A \cap B^C) \cup (A \cap B) \cup (A^C \cap B)$$

where each of the sets in brackets are disjoint from each other[7]. *By condition 3 of Definition 4, we would then have*

[7] Again, this is not hard to show

$$\begin{aligned} &P(A \cup B) + P(A \cap B) \\ &= P(A \cap B^C) + P(A \cap B) + P(A^C \cap B) + P(A \cap B) \\ &= 2P(A \cap B) + P(A) - P(A \cap B) + P(B) - P(A \cap B) \ \text{ by 4} \\ &= P(A) + P(B) \end{aligned}$$

6. *Note that* $B = B \cap S = B \cap (A^C \cup A) = (B \cap A^C) \cup A$. *Clearly,* $A \cap (B \cap A^C) \neq \emptyset$. *By condition 3 in Definition 4, we thus have that*

$$P(B) = P(B \cap A^C) + P(A). \tag{†}$$

Suppose $A \subsetneq B$. *Then* $B \cap A^C \neq \emptyset$. *I shall make the claim that* $B \cap A^C \in \mathscr{B}$. *Since* $A \subseteq B$ *we have that*

$$\begin{aligned} a \in (B \cap A^C) &\iff a \in B \wedge a \in A^C \\ &\iff a \in B \wedge a \notin A \\ &\iff a \in (B \setminus A). \end{aligned}$$

But $B \setminus A$ *is a subset of B from the above steps*[8]. *Therefore,* $(B \cap A^C) \subseteq B \in \mathscr{B}$ *as required.*

[8] This is rather obvious from the steps, since $\forall a \in (B \cap A^C)$, $a \in B$.

With that done, by condition 1 in Definition 4, $P(B \cap A^C) \geq 0$. *Hence from Equation (†), we have that*

$$\begin{aligned} P(B) &= P(B \cap A^C) + P(A) \\ &\geq P(A) \end{aligned}$$

as required. □

Definition 5 (Conditional Probability)

Suppose S is a sample space of a random experiment, and $A, B \subseteq S$. The ***conditional probability of A given B*** *is given by*

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{provided } P(B) > 0. \tag{1.2}$$

Definition 6 (Independent Events)

Suppose S is a sample space of a random experiment, and $A, B \subseteq S$. A and B are said to be ***independent of each other*** *if*

$$P(A \cap B) = P(A)P(B)$$

Proposition 2 (Boole's Inequality)

If $\{A_j\}_{j=1}^{\infty}$ is a sequence of events, then

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} P(A_j)$$

Proof

Proof shall be provided later

Proposition 3 (Bonferroni's Inequality)

If $\{A_j\}_{j=1}^{k}$ is a set of events where $k \in \mathbb{N}$, then

$$P\left(\bigcap_{j=1}^{k} A_j\right) \geq 1 - \sum_{j=1}^{k} P(A_j^C)$$

Proof

Proof shall be provided later

Proposition 4 (Continuity Property)

If $A_1 \subset A_2 \subset \ldots$ is a sequence where $A = \cup_{i=1}^{n} A_i$, then

$$\lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right) = P(A)$$

Proof

Proof shall be provided later

1.2 Random Variable

Definition 7 (Random Variable)

In a given probability space $(S, \mathscr{B}, P)$, the function $X : S \to \mathbb{R}$ is called a ***random variable***[9] *if*

$$P(X \leq x) = P\left(\{\omega \in S \,:\, X(\omega) \leq x\}\right) \tag{1.3}$$

is defined for all $x \in \mathbb{R}$[10].

[9] We shall use rv as shorthand for random variable in this set of notes.

[10] $X \leq x$ is an abbreviation for $\{\omega \in S : X(\omega) \leq x\} \in \mathscr{B}$.

Example 1.2.1

In a coin flip experiment, we have that $S = \{H, T\}$ where $\mathbb{P}(S) = \{\emptyset, S, \{H\}, \{T\}\}$. Define X : the number of heads in a flip, i.e.

$$X(\{H\}) = 1 \text{ and } X(\{T\}) = 0$$

To prove why X is a random variable given this definition, notice that

$$\begin{aligned} x < 0 &\implies P(X \leq x) = P(\{\omega \in S : X(\omega) < 0\}) = P(\emptyset) = 0 \\ x \geq 1 &\implies P(X \leq x) = P(\{\omega \in S : X(\omega) \leq x\}) = P(\{H, T\}) \\ &\quad = P(\{H\}) + P(\{T\}) = 1 \text{ by Independence} \\ 0 \leq x < 1 &\implies P(X \leq x) = P(\{\omega \in S : X(\omega) \leq x\}) = P(T) \geq 0 \end{aligned}$$

which shows that P is deiined for all $x \in \mathbb{R}$. Hence X is a random variable.

Definition 8 (Cumulative Distribution Function)

*The **cumulative distribution function (c.d.f)** of a random variable X is defined as*

$$\forall x \in \mathbb{R} \quad F(x) = P(X \leq x)$$

Note

Notice *that $F(x)$ is defined for **all** real numbers, and since it is a probability, we have $0 \leq F(x) \leq 1$.*

Proposition 5 (Properties of the cdf)

1. $\forall x_1 < x_2 \in \mathbb{R} \quad F(x_1) \leq F(x_2)$
2. $\lim_{x \to -\infty} = 0 \wedge \lim_{x \to \infty} = 1$
3. $\lim_{x \to a^+} F(x) = F(a)$ [11]
4. $\forall a < b \in \mathbb{R} \quad P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$
5. $P(X = b) = F(b) - \lim_{a \to b^-} F(a)$ [12]

[11] F is a **right-continuous** function.

[12] This is also called **the magnitude of the jump**.

Proof

Proof shall be provided later

Note

The definition and properties of the cdf hold for the rv X regardless of whether S is discrete (finite or countable) or not.

1.3 Discrete Random Variable

Definition 9 (Discrete Random Variable)

*An rv X is a **discrete random variable** when its image is finite or countably infinite, i.e. $X \in \{x_1, x_2, ...\}$. The function*

$$\forall x \in \mathbb{R} \quad f(x) := P(X = x) = F(x) - \lim_{\varepsilon \to 0^+} F(x - \varepsilon)$$

*is its probability function, commonly known as the **probability mass function** (pmf). The set $A := \{x : f(x) > 0\}$ is called the **support set** of X, and*

$$\sum_{x \in A} f(x) = \sum_{i=1}^{\infty} f(x_i) = 1. \tag{1.4}$$

Proposition 6 (Properties of pmf)

With the notation from Definition 9, prove that

1. $\forall x \in \mathbb{R} \quad f(x) \geq 0$
2. $\sum_{x \in A} f(x) = 1$

Proof

1. *This result follows from the fact that f is a pdf, a probability, i.e. $\forall x \in R$, $f(x) = 0$ is $x \notin S$ where S is the sample space, and $0 \leq f(x) \leq 1$ if $x \in S$.*
2. *Since $A = \{x : f(x) > 0\}$, we know that*

$$\sum_{x \in A} f(x) > 0.$$

If we consider all the elements of A, we have that the events $(X = x_i)$,

for $x_i \in A$, constitutes the entire sample space. Therefore,

$$\sum_{x \in A} f(x) = \sum_{x \in A} P(X = x) = P(S) = 1.$$

□

Exercise 1.3.1

Consider an urn containing r red marbles and b black marbles. Find the pmf of the rv for the following:

1. *X = number of red balls in n selections without replacement.*
2. *X = number of red balls in n selections with replacement.*
3. *X = number of black balls selected before obtaining the first red ball if sampling is done with replacement.*
4. *X = number of black balls selected before obtaining the kth red ball if sampling is done with replacement.*

Solution

1. *Let $d = \max\{n, r+b\}$. The desired pmf is therefore the pmf from the hypergeometric distribution*

$$\forall x \in \mathbb{Z}^+_{\leq r} \quad f(x) = \frac{\binom{r}{x}\binom{b}{d-x}}{\binom{r+b}{d}}.$$

2. $\forall x \in \mathbb{Z}^+ \quad f(x) = \binom{n}{x}\left(\frac{r}{r+b}\right)^x\left(\frac{b}{r+b}\right)^{n-x}$*, which is the pmf of the binomial distribution.*
3. $\forall x \in \mathbb{Z}^+ \quad f(x) = \left(\frac{b}{r+b}\right)^x\left(\frac{r}{r+b}\right)$
4. $\forall x \in \mathbb{Z}^+ \quad f(x) = \binom{x+k-1}{k-1}\left(\frac{b}{r+b}\right)^x\left(\frac{r}{r+b}\right)^k$

Example 1.3.1

Consider the function

$$f(x) = \begin{cases} \frac{C\mu^x}{x!} & x \in \mathbb{Z}^+,\ \mu > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find C such that $f(x)$ is a pmf for the rv X.

Solution

We have that

$$\begin{aligned} 1 &= \sum_{x\in\mathbb{Z}^+} \frac{C\mu^x}{x!} \\ &= C\sum_{x\in\mathbb{Z}^+} \frac{\mu^x}{x!} \\ &= Ce^{\mu} \end{aligned}$$

This gives us that $\forall x \in \mathbb{Z}^+$, $f(x) = \frac{e^{-\mu}\mu^x}{x!}$, and this is, of course, the pmf of the **Poisson distribution**.

Thus $C = e^{-\mu}$.

Exercise 1.3.2

Prove that the pdf of $X \sim \text{Poi}(\mu)$ *sums to* 1 *over all of its values.*

Solution

$$\begin{aligned} \sum_{x\in\mathbb{N}} \frac{\mu^x e^{-\mu}}{x!} &= e^{-\mu}\sum_{x\in\mathbb{N}} \frac{\mu^x}{x!} \\ &= e^{-\mu}e^{\mu} \quad \because \sum_{x\in\mathbb{N}}^{\infty} \frac{k^x}{x!} = e^k \\ &= 1 \end{aligned}$$

Exercise 1.3.3

If X *is a random variable with pmf*

$$f(x) = \frac{-(1-p)^x}{x\log p}, \;\; x = 1, 2, \ldots ; \; 0 < p < 1,$$

show that

$$\sum_{x\in\mathbb{N}} f(x) = 1$$

Solution

$$\begin{aligned} \sum_{x\in\mathbb{N}} \frac{-(1-p)^x}{x\log p} &= -\frac{1}{\log p}\sum_{x\in\mathbb{N}} \frac{(-1)^x(p-1)^x}{x} \\ &= -\frac{1}{\log p}\underbrace{\left[-(p-1) + \frac{(p-1)^2}{2} - \frac{(p-1)^3}{3} + \cdots\right]}_{\text{Taylor expansion of } -\log p} \\ &= 1 \end{aligned}$$

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2.1 Continuous Random Variable

Definition 10 (Continuous Random Variable)

Suppose X is an rv with cdf F. If F is a continuous function for all $x \in \mathbb{R}$ and F is differentiable except possibly at countably many points, then X is a ***continuous rv****. The probability function, or more commonly known as the* ***probability density function*** *(pdf), of X is $f(x) = F'(x)$ wherever F is differentiable on x and 0 otherwise.*

The set $A = \{x : f(x) > 0\}$ is called the ***support set*** *of X and*

$$\int_{x \in A} f(x)\, dx = 1$$

Proposition 7 (Properties of pdf)

Let X be a random variable and f be its pdf.

1. $\forall x \in \mathbb{R} \quad f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$
3. $f(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{P(x \leq X \leq x+h)}{h}$ *(if the limit exists)*
4. $\forall x \in \mathbb{R} \quad F(x) = \int_{-\infty}^{x} f(t)\, dt$
5. $P(a < X \leq b) = \int_a^b f(x)\, dx = F(b) - F(a)$
6. $P(X = b) = F(b) - \lim_{a \to b^-} F(a) = F(b) - F(b) = 0$

Proof

1. *The argument of this proof is similar to that provided in Proposition 6.*
2. *Same as above, except that the support set can now have complete intervals.*
3. *The first equation follows from the first principles of Calculus. The second equation follows by method of calculation using the cdf.*
4. $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt$.
5. *This follows immediately from the above property.*
6. *The first part of the equation is a way to interpret the above property. The limit equates to $F(b)$ since F is continuous.*

□

Example 2.1.1

Consider the function

$$f(x) = \begin{cases} \frac{\theta}{x^{\theta+1}} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

For what values of θ is f a pdf?

Solution

If f is a pdf, then $\theta \geq 0$. In fact, $\theta \neq 0$; otherwise f would be equivalently 0 for all $x \in \mathbb{R}$, which would imply that $\int_{\mathbb{R}} f = 0$, which is impossible. It remains to check if $\theta > 0$ is a safe choice. Now

$$\int_1^{\infty} \frac{\theta}{x^{\theta+1}}\, dx = -\frac{1}{x^{\theta}}\Big|_1^{\infty} = 1$$

Note that the above integral is valid because $\frac{1}{x^{\theta+1}} \leq \frac{1}{x}$. Therefore the choice of $\theta > 0$ is safe.

2.2 Examples of Discrete RVs

2.2.1 Binomial Distribution

Definition 11 (Binomial RV)

Consider X to be the number of successes in a sequence of n experiments where

1. *experiments are **independent**;*
2. *the outcome of each experient is a **binary** (e.g. success or failure); and*
3. *has the **probability of success,** p for each singular experiment.*

*X is called a **Binomial** rv, and we write $X \sim \text{Bin}(n, p)$ and its pmf is*

$$P(X = x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, 2, ..., n \\ 0 & \text{otherwise} \end{cases}$$

2.2.2 Geometric Distribution

Definition 12 (Geometric RV)

*Consider a sequence of independent success/failure (binary) experiments, each of which has a success probability of p. Let X be the **number of failures** before the **first success** is reached. We call X a **Geometric** rv, and we write $X \sim \text{Geo}(p)$, and its pmf is*

$$P(X = x) = \begin{cases} (1-p)^x p & x = 0, 1, 2, ..., n \\ 0 & \text{otherwise} \end{cases}$$

Note

Some authors would define the Geometric rv as:

> *Let X be the number of experiments until the first success.*
>
> *But that really is just a play of words.*

2.2.3 Poisson Distribution

Definition 13 (Poisson RV)

Suppose X is defined to be the number of occurrences of an event in a given time period. If the process on which the events occur satisfies the following:

1. *The number of occurrences in non-overlapping intervals are independent of each other;*
2. *The probability of the occurrence of an event in a short interval of length h is proportional to h;*
3. *For sufficiently short time periods of length h, the probability of 2 or more events occurring in the interval is negligible, i.e. almost zero;*

*then X is a **Poisson** rv, and we write $X \sim \text{Poi}(\lambda)$, with $\lambda > 0$, and the pmf is*

$$P(X = x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & x = 0, 1, ... \\ 0 & \text{otherwise} \end{cases}$$

2.3 Examples of Continuous RVs

2.3.1 Normal/Gaussian Distribution

Definition 14 (Normal / Gaussian RV)

*The **Normal/Gaussian** Distribution is a continuous probability distribution that is symmetric about the mean, showing that data around the mean is more frequent than data far from the mean. If X is a **Normal/Gaussian** rv, we write $X \sim \text{N}(\mu, \sigma^2)$, and its pdf is*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for } x \in \mathbb{R}.$$

Definition 15 (Standard Normal Distribution)

*The **Standard Normal Distribution** is the simplest case of a Normal Distribution. An rv Z is called the **Standard Normal** rv if $\mu = 0$ and $\sigma = 1$. We write $Z \sim \mathrm{N}(0,1)$ and its pdf is*

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{for } x \in \mathbb{R}.$$

2.3.2 Uniform Distribution

Definition 16 (Uniform RV)

*If X represents the result of drawing a real number from an interval (a,b), with $a < b$, such that all numbers in between are equally likely to be chosen, then X is called a **Uniform** rv, and we write $X \sim \mathrm{Unif}(a,b)$, and its pdf is*

$$f(x) = \begin{cases} \frac{1}{b-a} & x \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$

2.3.3 Exponential Distribution

Definition 17 (Exponential RV)

*Let X show the time between two consecutive events in a **Poisson process**, i.e. the 3 conditions in Poisson Distribution are satisfied. Then X is called an **Exponential** rv, and we write $X \sim \mathrm{Exp}(\theta)$, where $\theta > 0$, with its pdf*

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

2.3.4 Gamma Distribution

Definition 18 (Gamma RV)

Let X be the sum of n independent ***Exponential*** *rvs with some fixed θ. Then X is called a* ***Gamma*** *rv, in which we write $X \sim \Gamma(n, \theta)$, and its pdf is*

$$f(x) = \begin{cases} \frac{x^{n-1}e^{-\frac{x}{\theta}}}{\Gamma(n)\theta^n} & x > 0 \wedge \theta, n > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\Gamma(n) = \int_0^\infty e^{-y}y^{n-1}\,dy = (n-1)!$, where the last equality is true when n is an integer.

Note

The Gamma Distribution is usually used for when we are looking for the probability of the occurrence of the n-th event in the desired waiting time.

2.4 Functions of Random Variables

Consider the rv X with pdf/pmf f and cdf F. Given $Y = h(X)$ where h is some real-valued function, we are interested in finding the pdf/pmf of Y.

The following are some possible scenarios:

1. X and Y are both discrete;
2. X is continuous and Y is discrete;
3. X and Y are both continuous

We may also define $Y = h(X)$ for a continuous rv X such that Y is **neither discrete nor continuous** (e.g. discrete for some values of X while continuous for others).

2.4.1 Discrete X and Discrete Y

If X and $Y = h(X)$ are both discrete, we can derive $P(Y = y)$ by mapping values in Y onto their corresponding value through h, i.e.

$$P(Y = y) = \sum_{\{x:h(x)=y\}} P(X = x)$$

Exercise 2.4.1

Let X have the following probability function:

$$f_X(x) = \begin{cases} \frac{e^{-1}}{x!} & x = 0, 1, 2, ... \\ 0 & otherwise \end{cases}$$

Find the pmf of $Y = (X-1)^2$.

Solution

Note that since

$$\text{Dom}\, X = \{0, 1, 2, 3, 4, ...\},$$

we have that

$$\text{Dom}\, Y = \{1, 0, 1, 4, 9, ...\}.$$

With that, note that

$$\begin{aligned} P(Y=0) &= P(X=1) = \frac{e^{-1}}{1!} \\ P(Y=1) &= P(X=0 \text{ or } 2) = P(X=0) + P(X=2) \\ &= \frac{e^{-1}}{0!} + \frac{e^{-1}}{2!} = e^{-1}\left(1 + \frac{1}{2}\right) = \frac{3}{2}e^{-1} \\ P(Y=4) &= P(X=3) = \frac{e^{-1}}{3!} \\ P(Y=9) &= P(X=4) = \frac{e^{-1}}{4!}. \end{aligned}$$

Therefore, the pmf of $Y = (X-1)^2$ is

$$P(Y=y) = \begin{cases} e^{-1} & y = 0 \\ \frac{3}{2}e^{-1} & y = 1 \\ \frac{e^{-1}}{(1+\sqrt{y})!} & y = 4, 9, 16, ... \\ 0 & otherwise \end{cases}$$

2.4.2 *Continuous X and Discrete Y*

If X is continuous and Y is discrete, we can use the method that we have used in the previous subsection, and replace Σ by the integral sign $\int$, i.e. define $A := \{x : h(x) = y\}$ such that we have

$$P(Y=y) = \int_A f(x)\, dx$$

Example 2.4.1 (Example 2.9)

Suppose X is a random variable with the following probability function

$$f_X(x) = \begin{cases} 2e^{2x} & x > 0 \\ 0 & otherwise \end{cases}.$$

Suppose $Y = h(X)$ is defined as follows:

$$Y = \begin{cases} 1 & X < 1 \\ 2 & 1 \leq X \leq 2 \\ 3 & X > 2 \end{cases}$$

Find the probability function of Y.

Solution
Note that $X \sim \text{Exp}(\frac{1}{2})$. So it is clear that X is a crv and since $Y = 1, 2,$ or 3, we have that Y is discrete. Now

$$\begin{aligned} P(Y = 1) = P(X < 1) &= \int_0^1 2e^{-2x}\,dx \\ &= -e^{-2x}\Big|_0^1 = 1 - e^{-2} \\ P(Y = 2) = P(1 \leq X \leq 2) &= \int_1^2 2e^{-2x}\,dx \\ &= -e^{-2x}\Big|_1^2 = e^{-2} - e^{-4} \\ P(Y = 3) = P(X > 2) &= \int_2^\infty 2e^{-2x}\,dx \\ &= -e^{-2x}\Big|_2^\infty = e^{-4} \end{aligned}$$

Thus the pmf is

$$P(Y = y) = \begin{cases} 1 - e^{-2} & Y = 1 \\ e^{-2} - e^{-4} & Y = 2 \\ e^{-4} & Y = 3 \end{cases}$$

2.4.3 Continuous X and Continuous Y

If X and $Y = h(X)$ are both continous, start with the definition of the cdf of Y, i.e.

$$F_Y(y) = P(Y \leq y) = P(h(X) \leq y)$$

solve the inequality for X, and then obtain the cdf of Y. We will then only need to differentiate the cdf wrt y to get the pdf that we desire.

Example 2.4.2 (Example 2.10)

Let X have the following pdf:

$$f_X(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

Find the pdf of $Y = \sqrt{X}$.

Solution

We have that the range of values where $f_Y(y) \leq 0$ is $y \geq 0$. Now

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(\sqrt{X} \leq y) = P(X \leq y^2) \\ &= \int_0^{y^2} 2e^{-2x}\, dx \\ &= -e^{-2x}\Big|_0^{y^2} = 1 - e^{-2y^2} \end{aligned}$$

Therefore, the pdf of Y is

$$f_Y(y) = \begin{cases} \frac{d}{dy} 1 - e^{-2y^2} = 4ye^{-2y^2} & y \leq 0 \\ 0 & otherwise \end{cases}.$$

2.4.4 *A Formula for the Continuous Case*

Theorem 8 (One-to-One Transformation of a Random Variable)

Suppose X is a continuous random variable with pdf f_X and support set $A = \{x : f_X(x) > 0\}$ and $Y = h(X)$ where h is a real-valued function. Let f_Y be the pdf of the rv Y and let $B = \{y : f_Y(y) > 0\}$. If h is a one-to-one function from A to B and if h' is continuous, then

$$f_Y(y) = f\left(h^{-1}(y)\right) \cdot \left| \frac{d}{dy} h^{-1}(y) \right|, \quad y \in B$$

Proof

Note that since h is one-to-one, it is monotonous. Suppose h is increasing. Then h^{-1} is also an increasing function. Note that the cdf of Y is

$$F_Y(y) = P(Y \leq y) = P(X \leq h^{-1}(y)) = F_X\left(h^{-1}(y)\right).$$

Then the cdf of Y is

$$f_Y(y) = \frac{d}{dy} F_X\big(h^{-1}(y)\big) = f_X\big(h^{-1}(y)\big) \cdot \frac{d}{dy} h^{-1}(y)$$

If h is decreasing, then so is its inverse. Thus

$$F_Y(y) = P(Y \leq y) = P(X \geq h^{-1}(y)) = 1 - F_X(h^{-1}(y))$$

Thus the cdf of Y is

$$f_Y(y) = \frac{d}{dy}(1 - F_X(h^{-1}(y))) = -f_X(h^{-1}(y)) \cdot \frac{d}{dy} h^{-1}(y).$$

Note that the pdf of Y is indeed positive since h^{-1} is decreasing.

Combining the two, we have that

$$f_Y(y) = f_X(h^{-1}(y)) \cdot \left| \frac{d}{dy} h^{-1}(y) \right|,$$

as required. □

3 Lecture 3 May 08th 2018

3.1 Functions of Random Variables (Continued)

3.1.1 Special Cases

Example 3.1.1

Recall Example 2.4.1. Suppose X is a rv with the following probability function

$$f_X(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & otherwise \end{cases}.$$

Define $Y = h(X)$ as follows:

$$Y = \begin{cases} 1 & X < 1 \\ X & 1 \leq X \leq 2 \\ 3 & X > 2 \end{cases}$$

Find the cdf of Y.

Solution

Solution is given differently in the 2 sections. I am not happy with either solutions because some things don't add up. My opinion is that the definition of Y is badly given, along with a badly phrased question. As a result, there are more ways than one to interpret an already confusing information, and thus we have ourselves one hell of a mess.

3.2 Probability Integral Transformation

Theorem 9 (Probability Integral Transformation)

*If X is a contiunous rv with cdf F, then $Y = F(X) \sim \text{Unif}(0,1)$. $Y = F(X)$ is called the **probability integral transformation**.*

Note

The distribution of $Y = F(X)$ can be proven.

Proof

Let X be a continuous rv and $Y = F(X)$. Since $F(X)$ is one-to-one and increasing (i.e. monotonous), there exists $F^{-1}(Y)$ that is a real-valued and increasing function. Then

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(F_X(X) \leq y) = P(X \leq F^{-1}(y)) \\ &= F(F^{-1}(y)) = y \end{aligned}$$

*Note that $F_Y(y) = y$ is the cdf of a $\text{Unif}(0,1)$ rv, i.e. the **standard uniform random variable**. Thus $Y \sim \text{Unif}(0,1)$.*

Note

This theorem essentially states that any rv from a continuous distribution can be transformed into a standard uniform distribution.

Example 3.2.1 (Example 2.11)

Suppose $X \sim \text{Exp}(01)$. We know that $F_X(x) = 1 - e^{-10x}$ for all $x \in \mathbb{R}a$. By Probability Integral Transformation, we have that $Y = F_X(X) = 1 - e^{-10X} \sim \text{Unif}(0,1)$.

Note that the converse of Probability Integral Transformation is true:

Theorem 10 (Converse of Probability Integral Transformation)

Suppose X is a continuous rv with cdf F such that F^{-1} exists. If $U \sim \text{Unif}(0,1)$, we have that $Y = F^{-1}(U) \sim X$.

Proof

Note that

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(F^{-1}(U) \leq y) \\ &= P(U \leq F_X(y)) = F_X(y). \end{aligned}$$

□

Example 3.2.2 (Example 2.12)

Suppose $X \sim \text{Unif}(0,1)$. *Find a transformation* T *such that* $T(X) \sim \exp(\theta)$.

Solution

Let $Y = T(X) \sim \text{Exp}(\theta)$. *Note that*

$$F_Y(y) = 1 - e^{-\frac{y}{\theta}}, \quad y > 0$$

Observe that since

$$x = 1 - e^{-\frac{y}{\theta}} \implies y = -\theta \ln(1-x)$$

we have that

$$F_Y^{-1}(X) = -\theta \ln(1-X).$$

By Converse of Probability Integral Transformation 10, we have that $T = F_Y^{-1}$.

3.3 Location-Scale Families

When we look into methods for constructing confidence intervals for an unknown parameter θ. If the parameter θ is either a *scale parameter* or *location parameter*, then a confidence interval is easier to construct.

Definition 19 (Location Parameter and Family)

Suppose X *is a continuous rv with pdf* $f(x;\mu)$, *where* μ *is a parameter of the distribution of* X. *Let* $F_0(x) = F_X(x;\mu=0)$, *where* F_X *is the cdf of* X, *and* $f_0(x) = f(x;\mu=0)$. *The parameter* μ *is called a* ***location***

paramter *of the distribution if*

$$F_X(x;\mu) = F_0(x-\mu), \quad \mu \in \mathbb{R}$$

or equivalently,

$$f(x;\mu) = f_0(x-\mu), \quad \mu \in \mathbb{R}.$$

We say that F belongs to a ***location family*** *of distributions.*

Definition 20 (Scale Parameter and Family)

Suppose X is a continuous rv with pdf $f(x;\theta)$, where θ is a parameter of the distribution of X. Let $F_1(x) = F_X(x;\theta = 1)$, where F_X is the cdf of X, and $f_1(x) = (x;\theta = 1)$. The parameter θ is called a ***scale parameter*** *of the distribution if*

$$F_X(x;\theta) = F_1(\frac{x}{\theta}). \quad \theta > 0$$

or equivalently,

$$f(x;\theta) = \frac{1}{\theta} f_0(\frac{x}{\theta}), \quad \theta > 0.$$

We say that F belongs to a ***scale family*** *of distributions.*

Definition 21 (Location-Scale Family)

Suppose X is an rv with cdf $F(x;\mu,\sigma)$ where $\mu \in \mathbb{R}$ and $\sigma > 0$ are the parameters of the distribution. Let $Y = \frac{X-\mu}{\sigma}$. If the distribution of Y does not depend on μ and/or σ, then F is said to belong to a ***location-scale family*** *of distributions, with* ***location parameter*** *μ and* ***scale parameter*** *σ. In other words, F belongs to a location-scale family of distributions if*

$$F(x;\mu,\theta) = F_0\left(\frac{x-\mu}{\theta}\right),$$

where $F_0(x) = F(x;\mu = 0,\theta = 1)$, or equivalently,

$$f(x;\mu,\theta) = \frac{1}{\theta} f_0\left(\frac{x-\mu}{\theta}\right),$$

where $f_0(x) = f(x;\mu = 0,\theta = 1)$.

Example 3.3.1 (Example 2.13)

Consider $X \sim G(\mu, \sigma)$. Show that F_X belongs to a location-scale family of distributions.

We know that if $\mu = 0$ and $\sigma = 1$, then $Y = \frac{X-\mu}{\sigma} \sim G(0,1)$, and we know that $G(0,1)$ has no dependence on unknowns μ and σ. Therefore, F_X belongs to the location-scale family of distributions, with location parameter μ and scale parameter σ.

Another solution is to show that one of the equations in the definition is fulfilled. Observe that

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

So if we set $\mu = 0$ and $\sigma = 1$ to get f_0, we have that

$$f_0(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Now, note that

$$f(x) = \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{(x-\mu)}{\sigma}\right)^2}.$$

Let $y = \frac{x-\mu}{\sigma}$, and we have ourselves

$$f(x) = \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} = \frac{1}{\sigma} f_0(\frac{x-\mu}{\sigma})$$

Example 3.3.2 (Example 2.14)

Consider $X \in G(\mu, 2)$ where $\mu = E(X)$. Show that μ is a location parameter.

We can use a similar approach as before and define $Y = X - \mu$ which follows $G(0,2)$. It is clear that we then have that F_X, the cdf of X, belongs to a location family of distributions.

Example 3.3.3 (Example 2.15)

Consider $X \sim \text{Exp}(\theta)$. Show that F_X belongs to a scale family of distributions and find the scale parameter.

Note that

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = \frac{X}{\theta}$. *Then*

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(\frac{X}{\theta} \leq y) \\ &= P(X \leq \theta y) = \int_0^{\theta y} \frac{1}{\theta} e^{-\frac{x}{\theta}} \, dx \\ &= -e^{-\frac{x}{\theta}} \Big|_0^{\theta} y = 1 - e^{-y} \end{aligned}$$

and we have

$$f_Y(y) = \begin{cases} e^{-y} & y > 0 \\ 0 & otherwise \end{cases}$$

Note that if we set $\sigma = 1$ *to get* f_1, *we have*

$$f_1(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & otherwise \end{cases}.$$

Therefore, F_X *belongs to a scale family of distributions.*

3.4 Expectations

3.4.1 Expectations

Definition 22 (Expectation of A Discrete RV)

If X *is a discrete rv with pmf* f *and support set* A, *then the* ***expectation*** *of* X, *or the* ***expected value*** *of* X *is defined by*

$$E(X) = \sum_{x \in A} x f(x) \tag{3.1}$$

provided that the sum converges absolutely, i.e.

$$E(|X|) = \sum_{x \in A} |x| \, f(x) < \infty.$$

If $E(|X|)$ *does not converge, then we say that* $E(X)$ *does not exist.*

Definition 23 (Expectation of A Continuous RV)

If X *is a continuous rv with pdf* f *and support set* A, *then the* ***expecta-***

tion *of X, or the* ***expected value*** *of X is defined by*

$$E(X) = \int_{x \in A} x f(x) \tag{3.2}$$

provided that the integral converges absolutely, i.e.

$$E(|X|) = \int_{x \in A} |x| \, f(x) < \infty.$$

If $E(|X|)$ does not converge, then we say that $E(X)$ does not exist.

Example 3.4.1 (Example 2.16)

Suppose $X \sim \text{Poi}(\lambda)$. Calculate $E(X)$.

Solution
Note

$$f(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$\begin{aligned} E(X) &= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} \\ &= 0 + \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!} \\ &= e^{-\lambda}\lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \\ &= e^{-\lambda}\lambda e^{\lambda} = \lambda \end{aligned}$$

Example 3.4.2 (Example 2.18)

Suppose X is an rv with

$$f(x) = \begin{cases} \frac{1}{x^2} & 1 < x < \infty \\ 0 & \text{otherwise} \end{cases}.$$

Calculate $E(X)$.

Solution
Observe that $x \cdot \frac{1}{x^2} = \frac{1}{x}$ and the antiderivative of $\frac{1}{x}$ is $\ln x$, which would need to be evaluated at $\ln \infty$. Thus, we should instead immediately check if

the integral converges absolutely.

$$\begin{aligned} E(|X|) &= \int_1^\infty |x| \, \frac{1}{x^2} \, dx \\ &= \int_1^\infty |x| \, \frac{1}{|x| \, |x|} \, dx \\ &= \int_1^\infty \frac{1}{|x|} \, dx \\ &= \int_1^\infty \frac{1}{x} \, dx, \end{aligned}$$

and we notice that the integral would not converge. Therefore, $E(X)$ does not exist.

4 Lecture 4 May 10th 2018

4.1 Expectations (Continued)

4.1.1 Expectations (Continued)

Theorem 11 (Expectation from the cdf)

Suppose X is a non-negative continuous rv with cdf F, and $E(X) < \infty$. Then

$$E(X) = \int_0^\infty [1 - F(x)]\, dx = \int_0^\infty P(X \geq x)\, dx \tag{4.1}$$

If X is a discrete rv with cdf F, and $E(X) < \infty$, then

$$E(X) = \sum_{x=0}^{\infty} [1 - F(x)] = \sum_{x=0}^{\infty} P(X \geq x) \tag{4.2}$$

Proof

Note that for a continuous rv X, we have

$$1 - F(x) = P(X \geq x) = \int_x^\infty f(t)\, dt$$

Therefore,

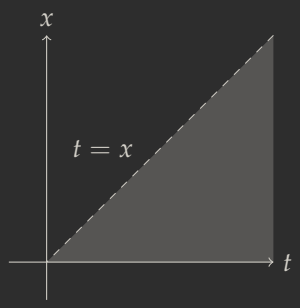
$$\int_0^\infty [1 - F(x)]\, dx = \int_0^\infty \int_x^\infty f(t)\, dt\, dx.$$

Since $1 - F(x)$ is a finite value, so is $\int_0^\infty f(t)\, dt$, and thus we can apply ***Fubini's Theorem***[1]*:*

$$\int_0^\infty [1 - F(x)]\, dx = \int_0^\infty \int_x^\infty f(t)\, dt\, dx = \int_0^\infty \int_0^t f(t)\, dx\, dt$$

Note that the limits of the integral utilizes the following figure:

[1] Condition for Fubini's Theorem to hold is that the integrand of the double integral must be absolutely convergent. See Wikipedia.



With that, note that

$$\int_0^t f(t)\,dx = xf(t)\Big|_0^t = tf(t)$$

Since t is just a dummy variable, we can indeed let $t = x$, and thus we have

$$\int_0^\infty [1 - F(x)]\,dx = \int_0^\infty xf(x)\,dx = E(X)$$

as required.

□

Work on the discrete case as an exercise.

Exercise 4.1.1
For a non-negative discrete rv X with cdf F and $E(X) < \infty$, prove that

$$E(X) = \sum_{x=0}^{\infty} [1 - F(x)]$$

Example 4.1.1 (Example 2.20)

Suppose $X \sim \text{Exp}(\theta)$. Use Theorem 11 to calculate $E(X)$.

Solution
Note that X is a non-negative rv. The cdf of $X \operatorname{Im} \text{Exp}(\theta)$ is

$$F_X(x) = 1 - e^{-\frac{x}{\theta}}.$$

Then

$$\begin{aligned} E(X) &= \int_0^\infty 1 - F_X(x)\,dx = \int_0^\infty e^{-\frac{x}{\theta}}\,dx \\ &= -\theta e^{-\frac{x}{\theta}}\Big|_0^\infty = \theta \end{aligned}$$

□

Theorem 12 (Expected Value of a Function of X)

Suppose $h(x)$ is a real-valued function.

If X is a discrete rv with pmf f and support set A, then

$$E[h(x)] = \sum_{x \in A} h(x) f(x) \tag{4.3}$$

provided that the sum converges absolutely.

If X is a continuous rv with pdf f, then

$$E[h(x)] = \int_{-\infty}^{\infty} h(x) f(x)\, dx \tag{4.4}$$

provided that the integral converges absolutely.

The proof is, unfortunately, not trivial. One would have to look into Lesbesgue integrals (or at the very least, Riemann-Stieltjes integrals) in order to prove this statement. This "theorem" is also called **The Law of the Unconscious Statistician** [Reference - Wikipedia]. An idea of the proof is given on Math SE.

Example 4.1.2

Suppose $X \sim \text{Unif}(0, \theta)$. Calculate $E(X^2)$.

Solution

$$E(X^2) = \int_0^\theta \frac{x^2}{\theta}\, dx = \frac{1}{\theta} \frac{x^3}{3}\Big|_{x=0}^{\theta} = \frac{\theta^2}{3}$$

Exercise 4.1.2

Find the pdf of $Y = X^2$ and find $E(Y)$ by evaluating $\int_{-\infty}^{\infty} y f_Y(y)\, dy$

Theorem 13 (Linearity of Expectation)

Suppose X is an rv with pf f. Let $a_i, b_i \in \mathbb{R}$, for $i = 1, ..., n$, be constants, and $g_i(x)$, for $i = 1, ..., n$, are real-valued functions. Then

$$E\left[\sum_{i=1}^{n} (a_i g_i(X) + b_i)\right] = \sum_{i=1}^{n} (a_i E[g_i(X)] + b_i) \tag{4.5}$$

provided that $E[g_i(X)] < \infty$ for $i = 1, ..., n$.

This theorem essentially states that the expectation is a linear operator.

Proof

Suppose X is a discrete rv with support set A. Then

$$\begin{aligned}
E\left[\sum_{i=1}^{n}(a_i g_i(X)+b_i)\right] &= \sum_{x\in A}\left[\sum_{i=1}^{n}(a_i g_i(x)+b_i)\right]f(x) \quad \because \textit{Theorem } 12\\
&= \sum_{x\in A}\sum_{i=1}^{n}[a_i g_i(x)f(x)+b_i f(x)]\\
&= \sum_{i=1}^{n}\sum_{x\in A}[a_i g_i(x)f(x)+b_i f(x)] \qquad (*)\\
&= \sum_{i=1}^{n}\left[a_i\sum_{x\in A}g_i(x)f(x)+b_i\sum_{x\in A}f(x)\right]\\
&= \sum_{i=1}^{n}(a_i E[g_i(X)]+b_i)
\end{aligned}$$

where note that $(*)$ is valid because a_i, b_i are constants, and $g_i(x), f(x)$ are finite real-valued functions.

Note

In general, $E(g(X)) \neq g(E(X))$ unless if g is a linear function. For example, for $a, b \in \mathbb{R}$, we have

$$E(aX+b) = aE(X)+b$$

4.1.2 *Moments and Variance*

Since these concepts were introduced in STAT230 and were given little treatment in the lecture, we shall only cover over them briefly.

Definition 24 (Variance)

*The expectation tof the squared deviation of an rv from its mean is called the **variance**, i.e. for an rv X with mean $\mu = E(X)$,*

$$\sigma^2 = \mathrm{Var}(X) = E[(X-\mu)^2] = E(X^2) - E(X)^2$$

Definition 25 (Moments)

Let X be an rv with mean μ.

*The k^{th} **moment about the origin** is defined as:*

$$E(X^k)$$

*The k^{th} **moment about the mean** is defined as:*

$$E[(X-\mu)^k]$$

*The k^{th} **factorial moment** is defined as:*

$$E[X^{(k)}] = E[X(X-1)\dots(X-k+1)] = E\left[\frac{X!}{(X-k)!}\right]$$

Theorem 14 (Variance of a Linear Function)

Suppose X is an rv with pf f and $a, b \in \mathbb{R}$. Then

$$\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$$

Proof

Observe that

$$\begin{aligned}\mathrm{Var}(aX+b) &= E[(aX+b)^2] - E(aX+b)^2 \\ &= E[a^2X^2 + 2abX + b^2] - (aE(X)+b)^2 \\ &= a^2E(X^2) + 2abE(X) + b^2 - (a^2E(X)^2 + 2abE(X) + b^2) \\ &= a^2E(X^2) - a^2E(X)^2 = a^2\,\mathrm{Var}(X)\end{aligned}$$

□

Example 4.1.3 (Example 2.22 (course notes - 2.6.10 (1)))

If $X \sim \mathrm{Poi}(\theta)$, then $E[X^{(k)}] = \theta^k$ for $k = 1, 2, \dots$.

Solution

Note

$$f_X(x) = \begin{cases} \frac{e^{-\theta}\theta^x}{x!} & x = 0, 1, 2, \dots \\ 0 & otherwise \end{cases}$$

So

$$\begin{aligned} E[X^{(k)}] &= E(X(X-1)(X-2)\dots(X-k+1)) \\ &= \sum_{x=0}^{\infty} x(x-1)(x-2)\dots(x-k+1)\frac{e^{-\theta}\theta^x}{x!} \\ &= 0 + \sum_{x=k}^{\infty} x(x-1)(x-2)\dots(x-k+1)\frac{e^{-\theta}\theta^x}{x!} \qquad (*) \\ &= \sum_{x=k}^{\infty} \frac{x!}{(x-k)!}\frac{e^{-\theta}\theta^x}{x!} \qquad \because x(x-1)\dots(x-k+1) = \frac{x!}{(x-k)!} \\ &= e^{-\theta}\theta^k \sum_{x=k}^{\infty} \frac{\theta^{x-k}}{(x-k)!} \\ &= e^{-\theta}\theta^k \sum_{y=0}^{\infty} \frac{\theta^y}{y!} \qquad let\ y = x-k \\ &= e^{-\theta}\theta^k e^{\theta} = \theta^k \end{aligned}$$

where for $(*)$ we have that $\sum_{x=0}^{k-1} x(x-1)\dots(x-k+1)A = 0$ for any $A \in \mathbb{R}$.

Note that it is not necessarily true that

$$x(x-1)\dots(x-k+1) = \frac{x!}{(x-k)!}$$

for $0 \leq x \leq k-1$. And so we can only say that the equality is true for $x \geq k$, and hence we have the approach that we use in $(*)$.

4.2 Inequalities

4.2.1 Markov/Chebyshev Style Inequalities

Theorem 15 (Markov's Inequality)

If X is a non-negative rv and $a > 0$, then the probability that X is no less than a is no greater than the expectation of X divided by a, i.e.

$$P(X \geq a) \leq \frac{E(X)}{a} \tag{4.6}$$

Proof

We shall prove for the discrete case. Suppose X is a non-negative discrete rv with pf f. Let $A \subset S$, where S is the sample space, such that

$A = \{w \in S : X(w) \geq a\}$.

$$\begin{aligned} E(X) &= \sum_{x \in S} xf(x) \\ &= \sum_{x \in A} xf(x) + \sum_{x \notin A} xf(x) \\ &\geq \sum_{x \in A} xf(x) \quad \because \sum_{x \notin A} xf(x) \geq 0 \\ &\geq \sum_{x \in A} af(x) \\ &= a \sum_{x \in A} f(x) = a \cdot P(A) \\ &= a \cdot P(\{w \in S : X(w) \geq a\}) = aP(X \geq a). \end{aligned}$$

□

Exercise 4.2.1
Prove Markov's Inequality for a continuous rv.

Theorem 16 (Markov's Inequality 2)

If X is a non-negative rv and $a, k > 0$, then the probability that X is no less than a is no greater than the expectation of X divided by a, i.e.

$$P(|X| \geq a) \leq \frac{E(|X|^k)}{a^k} \tag{4.7}$$

Proof

We shall, again, prove for the discrete case. Suppose X is a non-negative discrete rv with pf f. $A := \{w \in S : |X(w)| \geq a\} \subseteq S$. Then

$$\begin{aligned} E(|X|^k) &= \sum_{x \in S} |x|^k f(x) \\ &= \sum_{x \in A} |x|^k f(x) + \sum_{x \notin A} |x|^k f(x) \\ &\geq \sum_{x \in A} |x|^k f(x) \geq \sum_{x \in A} af(x) \\ &= a^k P(A) = a^k P(|X| \geq a). \end{aligned}$$

□

Question: Can we write

$P(\{w \in S : |X(w)| \geq a\}) = P(|X| \geq a)$?

Exercise 4.2.2
Prove for the continuous case.

Theorem 17 (Chebyshev's Inequality)

Suppose X is an rv with finite mean μ and finite variance σ^2. Then for any $k > 0$,

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2} \tag{4.8}$$

Proof

By Theorem 16,

$$P(|X - \mu| \geq k\sigma) \leq \frac{E(|X - \mu|^2)}{(k\sigma)^2} = \frac{1}{k^2}$$

since $E(|X - \mu|^2) = \mathrm{Var}(X) = \sigma^2$. □

Example 4.2.1 (Example 2.23)

A post office handles, on average, 10000 letters a day. What can be said about the probability that it will handle at least 15000 letters tomorrow?

Solution

$X :=$ number of letters handled in a day. Note that by its definition, X is a non-negative discrete rv. Then, using Theorem 15, since $E(X) = 10000$

$$P(X \geq 15000) \leq \frac{10000}{15000} = \frac{2}{3}.$$

Thus, we know that there is less than two-third of chance that the post office will handle more than 15000 tomorrow.

5 Lecture 5 May 15th 2018

5.1 Inequalities (Continued)

5.1.1 Markov/Chebyshev Style Inequalities (Continued)

Example 5.1.1 (Example 2.24)

A post office handles 10000 *letters per day with a variance of* 2000 *letters. What can be said about the probability that this post office handles between* 8000 *and* 12000 *letters tomorrow? What about the probability that more than* 15000 *letters come in (use Theorem 17)?*

1. *Probability that this post office handles between* 8000 *and* 12000 *letters tomorrow:*

$$\begin{aligned}
&P(8000 < X < 12000) \\
&= P(-2000 < X - 10000 < 2000) \\
&= P(|X - 10000| < 2000) = 1 - P(|X - 10000| \geq 2000) \\
&\geq 1 - \frac{1}{(\sqrt{2000})^2} \quad \because \textit{Theorem 17} \wedge k = \frac{2000}{\sigma} = \sqrt{2000} \\
&= \frac{1999}{2000}
\end{aligned}$$

2. *Probability that more than 15000 letters come in:*

$$\begin{aligned}
P(X > 15000) &= P(X - 10000 > 15000 - 10000) \\
&= P(X - 10000 > 5000) \\
&\leq P(X - 10000 > 5000) + P(X - 10000 < 5000) \\
&\leq P(|X - 10000| > 5000) \\
&\leq \frac{1}{\left(\frac{5000}{\sqrt{2000}}\right)^2} = \frac{2000}{5000^2}
\end{aligned}$$

5.2 Moment Generating Function

Moment generating functions are important because they uniquely define the distribution of an rv.

Definition 26 (Moment Generating Function)

*If X is an rv, then $M_X(t) = E(e^{tx})$ is called the **moment generating function** (mgf) of X provided this expectation exists for all $t \in (-h, h)$ for some $h > 0$.*

Note

*When determining the mgf of an rv, the values of t for which the expectation exists must always be stated. The range of t where the expectation is defined is "essentially" the **radius of convergence**.*

Exercise 5.2.1 (Example 2.25 (2.9.2 (1) of the course notes))

Find the mgf of $X \sim \Gamma(\alpha, \beta)$. Make sure you specify the domain on which the mgf is defined.

Solution

Note that the pdf of the Gamma distribution is:

$$f(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$\begin{aligned} M_X(t) = E(e^{tx}) &= \int_0^\infty e^{tx} \frac{1}{\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} \, dx \\ &= \frac{1}{\beta^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha)} x^\alpha e^{-x\left(\frac{1}{\beta}-t\right)} \, dx \\ &= \frac{\left(\frac{\beta}{1-t\beta}\right)^\alpha}{\beta^\alpha} \underbrace{\int_0^\infty \frac{1}{\left(\frac{\beta}{1-t\beta}\right)^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\frac{\beta}{1-t\beta}}} \, dx}_{\text{sum over all values for pdf of } \Gamma(\alpha, \frac{\beta}{1-t\beta}) = 1} \quad \text{for } \frac{1}{\beta} - t > 0 \\ &= (1-t\beta)^{-\alpha} \qquad \text{for } t < \frac{1}{\beta} \end{aligned}$$

Definition 27 (Indicator Function)

*The function $\mathbb{1}_A$ is called the **indicator function** of the set A, i.e.*

$$\mathbb{1}_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A^C \text{ occurs} \end{cases} \tag{5.1}$$

Example 5.2.1

The pdf

$$f(x) = \begin{cases} \frac{1}{\theta} & 0 \leq x \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

can be represented as

$$f(x) = \frac{1}{\theta}\mathbb{1}_{\{0 \leq x \leq \theta\}}$$

Example 5.2.2 (Example 2.26)

Find the mgf of $X \sim \text{Poi}(\lambda)$. Make sure you specify the domain on which the mgf is defined.

Solution

Note that the pmf of X is

$$f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}\mathbb{1}_{\{0,1,2,\ldots\}}$$

The mgf is thus

$$\begin{aligned} M_X(t) = E(e^{tX}) &= \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!} \\ &= e^{-\lambda}\sum_{x=0}^{\infty}\frac{(e^t\lambda)^x}{x!} = e^{-\lambda}e^{e^t\lambda} \\ &= e^{\lambda(e^t-1)} \qquad \forall t \in \mathbb{R} \end{aligned}$$

Proposition 18 (Properties of the MGF)

Suppose X is an rv. Then

1. $M_X(0) = 1$
2. *Suppose the derivatives $M_X^{(k)}(t)$, for $k = 1, 2, \ldots$, exists for $t \in (-h, h)$*

for some $h > 0$, then the ***Maclaurin Series***[1] *of $M_X(t)$ is*

$$M_X(t) = \sum_{k=0}^{\infty} \frac{M_X^{(k)}(t)\Big|_{t=0}}{k!} t^k$$

[1] The Maclaurin series is the Taylor expansion around 0.

3. *If the mgf exists, then the k^{th} moment of X is:*

$$E(X^k) = \frac{d^k M_X(t)}{dt^k}\Big|_{t=0}$$

4. *Putting 2 and 3 together, we have*

$$M_X(t) = \sum_{k=0}^{\infty} \frac{E(X^k)}{k!} t^k$$

The final item shows why $M_X(t)$ is called the ***moment generating function****.*

Proof

1. $M_X(t)\Big|_{t=0} = E(e^{tX})\Big|_{t=0} = E(e^0) = 1$
2. *This is simply a result of using the Maclaurin series.*
3. *Note that*

$$\begin{aligned} E(e^{tX}) &= E\left[1 + tX + \frac{1}{2}(tX)^2 + \frac{1}{3!}(tX)^3 + \dots\right] \\ &= 1 + tE(X) + \frac{t^2}{2}E(X^2) + \frac{t^3}{3!}E(X^3) + \dots. \end{aligned}$$

So

$$\frac{d^k}{dt^k}E(e^{tX})\Big|_{t=0} = \frac{k!}{k!}E(X^k) + \underbrace{\frac{k! \cdot t}{(k+1)!}E(X^{k+1}) + \dots}_{=0 \text{ when } t=0}\Big|_{t=0} = E(X^k)$$

□

Example 5.2.3 (Example 2.27)

A discrete random variable X has the pmf

$$f(x) = \left(\frac{1}{2}\right)^{x+1} \mathbb{1}_{\{0,1,2,\ldots\}}$$

Derive the mgf of X and use it calculate its mean and variance.

$$\begin{aligned} M_X(t) &= \sum_{x=0}^{\infty} e^{tx} \left(\frac{1}{2}\right)^{x+1} \\ &= \frac{1}{2} \cdot \sum_{x=0}^{\infty} \left(\frac{e^t}{2}\right)^x \\ &= \frac{1}{2} \cdot \frac{1}{1-\frac{e^t}{2}} \quad \text{for } \left|\frac{e^t}{2}\right| < 1 \text{ or } t < \ln 2 \\ &= \frac{1}{2-e^t} \end{aligned}$$

To get the first two moments,

$$\begin{aligned} E(X) &= \frac{d}{dt} M_X(t)\Big|_{t=0} \\ &= \frac{e^t}{(2-e^t)^2}\Big|_{t=0}^{=} 1 \\ E(X^2) &= \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} \\ &= \frac{e^t}{(2-e^t)^2} + \frac{2e^t}{(2-e^t)^3}\Big|_{t=0} \\ &= 1+2=3 \end{aligned}$$

Thus we have that the expected value and variance are

$$\begin{aligned} E(X) &= 1 \\ \text{Var}(X) &= E(X^2) - E(X)^2 = 3-1=2 \end{aligned}$$

respectively.

5.2.1 MGF of a Linear Transformation

Theorem 19 (MGF of a Linear Transformation)

Suppose the rv X has an mgf $M_X(t)$ defined for $t \in (-h,h)$ for some

$h > 0$. Let $Y = aX + b$, where $a, b \in \mathbb{R}$ and $a \neq 0$. Then the mgf of Y is

$$M_Y(t) = e^{bt} M_X(at), \quad |t| \leq \frac{h}{|a|}. \tag{5.2}$$

Proof

Observe that

$$M_Y(t) = E(e^{tY}) = E(e^{t(aX+b)}) = E(e^{atX} e^{tb}) = e^{bt} M_X(at).$$

The range of t is

$$|at| < h \overset{a \neq 0}{\iff} |t| < \frac{h}{|a|}$$

Example 5.2.4 (Example 2.28)

Consider $X \sim \text{Unif}(\theta_1, \theta_2)$. Find the mgf of $Y = 5X + 3$.

Solution

Note that

$$\begin{aligned} M_X(t) &= \int_{\theta_1}^{\theta_2} \frac{e^{tx}}{\theta_2 - \theta_1} \, dx \\ &= \begin{cases} \frac{e^{tx}}{t(\theta_2 - \theta_1)} \Big|_{\theta_1}^{\theta_2} & t \neq 0 \\ 1 & t = 0 \end{cases} \\ &= \begin{cases} \frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)} & t \neq 0 \\ 1 & t = 0 \end{cases} \end{aligned}$$

Thus by Theorem 19,

$$M_Y(t) = e^{3t} M_X(5t) = \begin{cases} e^{3t} \frac{e^{5t\theta_2} - e^{5t\theta_1}}{5t(\theta_2 - \theta_1)} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

5.2.2 *Uniqueness of the MGF*

Theorem 20 (Uniqueness of the MGF)

Suppose the rv X has mgf $M_X(t)$ and the rv Y has mgf $M_Y(t)$. Suppose also that $M_X(y) = M_Y(t)$ for all $t \in (-h, h)$ for some $h > 0$. Then X

and Y have the same distribution, that is, $\forall s \in \mathbb{R}$,

$$P(X \leq s) = F_X(s) = F_Y(s) = P(Y \leq s)$$

Proof

The proof of this theorem is not trivial. See this comment on Math SE for information. It appears that the 2nd bullet point points to a material that I might be able to understand. If I can find that material, and understand it, I may change this proof section to become my own notes.

Example 5.2.5 (Example 2.29)

Suppose $X \sim \text{Unif}(0,1)$. Define $Y = -2\log X$, and use the mgf method to show that $Y \sim \chi^2_2$.
(Hint: Find mgf of χ_2 and show that Y has the same mgf)

Solution

Let $Z = \chi^2_2$. The pdf of Z is therefore

$$f_Z(z) = \frac{1}{2} e^{-\frac{z}{2}} \mathbb{1}_{\{z>0\}}.$$

Then

$$\begin{aligned} M_Z(t) = E(e^{tZ}) &= \int_0^\infty e^{tz} \frac{1}{2} e^{-\frac{z}{2}} \, dz \\ &= \frac{1}{2} \int_0^\infty e^{(t-\frac{1}{2})z} \, dz \\ &= \begin{cases} \frac{1}{2} \frac{1}{t-\frac{1}{2}} e^{(t-\frac{1}{2})z} \Big|_{z=0}^{\infty} & t \neq \frac{1}{2} \\ \infty & t = \frac{1}{2} \end{cases} \\ &= \frac{1}{2t-1} \qquad t \neq \frac{1}{2} \end{aligned}$$

6 Lecture 6 May 17th 2018

6.1 Joint Distributions

6.1.1 Introduction to Joint Distributions

Note (Motivation)

Most studies collect information for multiple variables per subject rather than just one variable. Because these variables may interfere/interact with each other and hence give us results that may not be fully reliant on a single variable, it is in our interest to study the interaction of these variables.

To start off with the basics, we will first look at the bivariate case of a joint distribution.

6.1.2 Joint and Marginal CDFs

Definition 28 (Joint CDF)

*Suppose X and Y are rvs defined on a sample space S. The **joint cdf** of X and Y is given by*

$$\forall (x,y) \in \mathbb{R}^2 \qquad F(x,y) = P(X \leq x, Y \leq y).$$

Note

- *Depending on whether X and Y are both discrete or both continuous, we can derive the joint pmf or joint pdf of (X, Y), respectively.*
- *Definition 28 only concerns two variables (a bivariate case), but we can certainly extend the idea to a k-dimensional joint cdf for the rvs $X_1, X_2, ..., X_k$ as $\forall (x_1, x_2, ..., x_k) \in \mathbb{R}^k$,*

$$F(x_1, x_2, ..., x_k) = P(X_1 \leq x_1, X_2 \leq x_2, ..., X_k \leq x_k).$$

Proposition 21 (Properties of Joint CDF)

Suppose X, Y are rvs, either both continuous or discrete, and has a joint cdf F. Then

1. *F is non-decreasing in x for fixed y.*
2. *F is non-decreasing in y for fixed x.*
3. *$\lim_{x \to -\infty} F(x, y) = 0$ and $\lim_{y \to -infty} F(x, y) = 0$.*
4. *$\lim_{(x,y) \to (-\infty, -\infty)} F(x, y) = 0$ and $\lim_{(x,y) \to (\infty, \infty)} F(x, y) = 1$*

Proof

1. *Suppose not, i.e. that we have instead that F is decreasing for x. Then for $x_1 < x_2 \in \mathbb{R}$, we would have*

$$\begin{aligned} & F(x_1, y) > F(x_2, y) \\ \implies & P(X \leq x_1, Y \leq y) > P(X \leq x_2, Y \leq y) \end{aligned}$$

In other words,

$$\begin{aligned} & P(\{(w, v) : (w, v) \in S,\ X(w) \leq x_1,\ Y(v) \leq y\}) \\ & > P(\{(w, v) : (w, v) \in S,\ X(w) \leq x_2,\ Y(v) \leq y\}) \end{aligned}$$

However, note that for fixed y, since $x_1 < x_2$, we must have that

$$\begin{aligned} & \{(w, v) \in S : X(w) \leq x_1, Y(v) \leq y\} \\ & \subseteq \{(w, v) \in S : X(w) \leq x_2, Y(v) \leq y\}. \end{aligned}$$

By Proposition 1, we have that

$$\begin{aligned} &P(\{(w,v) : (w,v) \in S,\ X(w) \leq x_1,\ Y(v) \leq y\}) \\ &\leq P(\{(w,v) : (w,v) \in S,\ X(w) \leq x_2,\ Y(v) \leq y\}). \end{aligned}$$

This is clearly a contradiction.

2. *The proof for this statement is similar to the above.*

3. *Note that*

$$\begin{aligned} \lim_{x \to -\infty} F(x,y) &= \lim_{x \to -\infty} P(X \leq x, Y \leq y) \\ &= P(X \leq -\infty, Y \leq y) \\ &= P([X \leq -\infty] \cap [Y \leq y]) \\ &= P(\emptyset \cup [Y \leq y]) = P(\emptyset) = 0 \end{aligned}$$

The proof for the case where $y \to -\infty$ is similar.

4. *This is simply a consequence of 3.*

Note

We say that F is a joint cdf if it satisfies all the conditions in Proposition 21.[1]

[1] Many literature actually claims this, and it does look like it will be assumed so for this class.

Example 6.1.1 (Example 3.1)

Consider the following joint cdf of two rvs (X_1, X_2):

$$F(x_1,x_2) = \begin{cases} 0 & x_1 < 0 \vee x_2 < 0 \\ 0.49 & 0 \leq x_1 < 1 \wedge 0 \leq x_2 < 1 \\ 0.7 & 0 \leq x_1 < 1 \wedge x_2 > 1 \\ 0.7 & x_1 \geq 1 \wedge 0 \leq x_2 < 1 \\ 1 & x_1 \geq 1 \wedge x_2 \geq 1 \end{cases}$$

Flipping an unfair coin with $P(\{H\}) = 0.3$ twice independently, we define for $i = 1, 2$

$$X_i = \begin{cases} 1 & \text{if the } i^{th} \text{ flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

The joint cdf of (X_1, X_2) is the given F above. Verify that under this experiment, F is indeed a cdf.

Solution

Note that conditions 3 and 4 of Proposition 21 are automatically satisfied by the definition of F.

incomplete example

Definition 29 (Marginal CDF)

*For the rvs X, Y with joint cdf F, the **marginal cdf** of X is*

$$F_X(x) = P(X \leq x) = \lim_{y \to \infty} F(x, y) = F(x, \infty) \quad \forall x \in \mathbb{R}$$

*and the **marginal cdf** of Y is*

$$F_Y(y) = P(Y \leq y) = \lim_{x \to \infty} F(x, y) = F(\infty, y) \quad \forall y \in \mathbb{R}$$

Note that the marginal cdf is defined for both discrete and continuous cases.

Example 6.1.2

Based on Example 6.1.1, derive $F_{X_i}(x_i)$ for $i = 1, 2$.

Solution

$$\begin{aligned} F_{X_1}(x_1) &= \lim_{x_2 \to \infty} F(x_1, x_2) \\ &= \begin{cases} 0 & x_1 < 0 \\ 0.7 & 0 \leq x_1 < 1 \\ 1 & x_1 \geq 1 \end{cases} \end{aligned}$$

The solution for $F_{X_2}(x_2)$ is similar.

6.1.3 Joint Discrete RVs

Definition 30 (Joint Discrete RV)

Suppose X amd Y are rvs defined on a sample space S. If S is discrete

then X and Y are discrete rvs. The ***joint pmf*** *of X and Y is given by*

$$\forall (x,y) \in \mathbb{R}^2 \quad f(x,y) = P(X = x, Y = y).$$

The set $A = \{(x,y) : f(x,y) > 0\}$ is called the ***support set*** *of (X,Y).*

Proposition 22 (Properties of Joint PMF)

Suppose X, Y are discrete rvs with joint pmf f and support set A. Then

1. $\forall (x,y) \in \mathbb{R}^2 \qquad f(x,y) \geq 0$
2. $\sum\limits_{(x,y)\in A} \sum f(x,y) = 1$
3. $\forall R \subset \mathbb{R}^2$,
$$P[(X,Y) \in R] = \sum_{(x,y)\in R} \sum f(x,y)$$

The proof is analogous to the univariate case as seen in Proposition 6

Example 6.1.3 (Example 3.2)

Consider the following joint pmf where the numbers inside the table show $P(X = x, Y = y)$. Find c. Then, calculate $P(X + Y \leq 2)$.

	x = -2	x = 0	x = 2
y = 0	0.05	0.1	0.15
y = 1	0.07	0.11	c
y = 2	0.02	0.25	0.05

Solution

Since the sum of all the probabilities must be 1*, thus*

$$c = 1 - 0.05 - 0.07 - 0.02 - \ldots - 0.15 - 0.05 = 0.2.$$

Notice that the only cases where $X + Y > 2$ is when

- $X = 2, Y = 1$*; and*
- $X = 2, Y = 2$.

Thus

$$\begin{aligned} P(X+Y \leq 2) &= 1 - P(X=2, Y=1) - P(X=2, Y=2) \\ &= 1 - 0.2 - 0.05 = 0.75 \end{aligned}$$

Example 6.1.4 (Example 3.3)

A small college has 90 male and 30 female professors. An ad hoc committee of 5 is selected at random to write the vision and mission of the college. Let X and Y be the number of men and women in this committee, respectively. Derive the joint distribution of (X, Y).

Solution

Observe that the support set of this distribution is

$$A = \{(x, y) : x + y = 5, x, y = 0, 1, 2, 3, 4, 5\}.$$

We have that the distribution is

$$P(X = x, Y = y) = \begin{cases} \frac{\binom{90}{x}\binom{30}{y}}{\binom{120}{5}} & \substack{x,y=0,\,1,\,2,\,3,\,4,\,5 \\ x+y=5} \\ 0 & \text{otherwise} \end{cases}$$

Definition 31 (Marginal Distribution - Discrete Case)

*Suppose X and Y are discrete rvs with joint pf f. Then the **marginal pf** of X is*

$$\forall x \in \mathbb{R}^2 \quad f_X(x) = P(X = x) = \sum_{y \in \mathbb{R}} f(x, y),$$

*and the **marginal pf** of Y is*

$$\forall y \in \mathbb{R}^2 \quad f_Y(y) = P(Y = Y) = \sum_{x \in \mathbb{R}} f(x, y).$$

Example 6.1.5 (Example 3.4)

Consider the joint pmf from Example 6.1.3. Find the marginal distributions, i.e. marginal pmfs of X and Y.

	$x = -2$	$x = 0$	$x = 2$
$y = 0$	0.05	0.1	0.15
$y = 1$	0.07	0.11	0.2
$y = 2$	0.02	0.25	0.05

Solution

Using the definition, we have that

$$f_X(x) = \sum_{y\in\mathbb{R}} f(x,y) = \begin{cases} 0.14 & x = -2 \\ 0.46 & x = 0 \\ 0.40 & x = 2 \end{cases}$$

and

$$f_Y(y) = \sum_{x\in\mathbb{R}} f(x,y) = \begin{cases} 0.3 & y = 0 \\ 0.38 & y = 1 \\ 0.32 & y = 2 \end{cases}$$

Example 6.1.6 (Example 3.5)

Suppose that a penny and a nickel are each tossed 10 times so that every pair of sequences of tosses (n tosses in each sequence) is equally likely to occur. Let X be the number of heads obtained with the penny, and Y be the number of heads obtained with the nickel. It can be shown that (show it!) the joint pmf of X and Y is as follows.

$$P(X = x, Y = y) = \begin{cases} \binom{10}{x}\binom{10}{y}\left(\frac{1}{2}\right)^{20} & x, y = 0, ..., 10 \\ 0 & \text{otherwise} \end{cases}$$

Solution

Note that the support set of X and Y are the same, i.e.

$$A_X = A_Y = \{0, 1, ..., 10\}.$$

We may assume that the penny and the nickel are fair coins, i.e. if we let p_x and p_y be the probability of getting a head for a penny and nickel, respectively, then $p_x = p_y = \frac{1}{2}$. Since there are 10 ways to get x heads with the penny, and similarly so for the nickel, we have that

$$\begin{aligned} P(X = x, Y = y) &= \begin{cases} \binom{10}{x}\binom{10}{y}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{10} & x, y = 0, 1, ..., 10 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \binom{10}{x}\binom{10}{y}\left(\frac{1}{2}\right)^{20} & x, y = 0, 1, ..., 10 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

as required.

Note

It is interesting to observe that the two rvs in the last example have seemingly no relationship with one another in terms of the experiment conducted, since they do not affect each other. This leads us to introducing the next concept.

6.1.4 Independence of Discrete RVs

Definition 32 (Independence of Discrete RVs)

*Two rvs X and Y with joint cdf F are said to be **independent** if and only if*

$$\forall x, y \in \mathbb{R} \quad F(x,y) = F_X(x)F_Y(y)$$

Theorem 23 (Independence by PF)

Suppose X and Y are rvs with joint cdf F, joint pf f, marginal cdf F_X and F_Y respectively, and marginal pf f_X and f_Y respectively. Also, suppose that $A_X = \{x : f_X(x) > 0\}$ is the support set of X and $A_Y = \{y : f_Y(y) > 0\}$ is the support set of Y. Then X and Y are independent rvs if and only if either

$$\forall (x,y) \in A_X \times A_Y \quad f(x,y) = f_X(x)f_Y(y)$$

holds, or

$$\forall x, y \in \mathbb{R} \quad F(x,y) = F_X(x)F_Y(y)$$

I am not certain as to why this is presented as a theorem that repeats the definition. As so, the prove for the 2nd equation will not be shown.

Proof

*The ($\implies$) direction is simply a result of **Clairaut's Theorem**[2]. While the ($\impliedby$) direction is a direct result of applying double integrals.* □

[2] Work needs to be done to show that our statement actually satisfies the condition for Clairaut's Theorem to apply. Clairaut's Theorem states that:

Theorem 24 (Clairaut's Theorem)
If (x_0, y_0) is a point in the domain of a function f with

- *f is defined on all points in an open disk centered at (x_0, y_0);*
- *the first partial derivatives, f_{xy} and f_{yx} are all continuous for all points in the open disk.*

Then $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.

Example 6.1.7 (Example 3.6)

Suppose X and Y are discrete rvs with joint pf

$$f(x,y) = \frac{\theta^{x+y}e^{-2\theta}}{x!y!}\mathbb{1}_{\{x,y=0,1,\ldots\}}.$$

Are X and Y independent of each other?

Solution

Note that we may write f as

$$f(x,y) = \left(\frac{\theta^x e^{-\theta}}{x!}\cdot\frac{\theta^y e^{-\theta}}{y!}\right)\mathbb{1}_{\{x,y=0,1,\ldots\}}$$

and so this suggests that we can indeed break down f into two parts, each only affected by x and y respectively, "indenpdent" of each other. Indeed, since

$$\begin{aligned}
f_X(x) &= \sum_{y=0}^{\infty}\frac{\theta^{x+y}e^{-\theta}}{x!y!}\mathbb{1}_{\{x,y=0,1,\ldots\}} \\
&= \sum_{y=0}^{\infty}\left(\frac{\theta^x e^{-\theta}}{x!}\cdot\frac{\theta^y e^{-\theta}}{y!}\right)\mathbb{1}_{\{x=0,1,\ldots\}} \\
&= \frac{\theta^x e^{-\theta}}{x!}\underbrace{\sum_{y=0}^{\infty}\frac{\theta^y e^{-\theta}}{y!}}_{\text{sum of pmf of } \mathrm{Poi}(\theta)=1} \\
&= \frac{\theta^x e^{-\theta}}{x!}
\end{aligned}$$

Similarly, we can obtain

$$f_Y(y) = \frac{\theta^y e^{-\theta}}{y!}$$

Multiplying $f_X(x)$ and $f_Y(y)$ together, we indeed get back to the original joint pmf.

7 Lecture 7 May 24th 2018

7.1 Joint Distributions (Continued)

7.1.1 Independence of Discrete RVs (Continued)

Example 7.1.1 (Example 3.7)

Consider the joint pmf below from Example 6.1.3. Are X and Y independent? Prove or disprove.

	$x = -2$	$x = 0$	$x = 2$	$P(Y = y)$
$y = 0$	0.05	0.1	0.15	0.3
$y = 1$	0.07	0.11	0.2	0.38
$y = 2$	0.02	0.25	0.05	0.32
$P(X = x)$	0.14	0.46	0.4	

Solution

Note that

$$P(X = -2, Y = 0) = 0.5 \text{ but}$$
$$P(X = -2)P(Y = 0) = 0.14 \cdot 0.3 = 0.042 \neq 0.5.$$

Thus X and Y are not independent.

7.1.2 Joint Continuous RVs

Definition 33 (Joint Continuous RVs)

*Two random variables X and Y are said to be **jointly continuous** if there exists a function $f(x, y)$ such that the joint cdf of X and Y can be written*

as

$$\forall (x,y) \in \mathbb{R}^2 \quad F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(t_1,t_2) d_{t_2} d_{t_1}.$$

The function f is called the ***joint density function*** *of X and Y. It follows from the above defintiion that when the second partial derivative exists, we have*

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$$

The set $\{(x,y) : f(x,y) > 0\}$ is called the ***support set*** *of (X,Y).*

Note (Convention)

Define $f(x,y) = 0$ when $\frac{\partial^2}{\partial x \partial y} F(x,y)$ does not exist.

Example 7.1.2 (Example 3.8)

Suppose X and Y have joint pdf $f(x,y) = \mathbb{1}_{\{0<x,y<1\}} = \mathbb{1}_{\{0<x<1, 0<y<1\}}$. Calculate the joint cdf of X and Y.

Solution

$$F(x,y) = \begin{cases} 0 & x \leq 0, \vee y \leq 0 \\ \int_0^x \int_0^y 1 \, ds \, dt = xy & 0 < x < 1 \wedge 0 < y < 1 \\ \int_0^1 \int_0^y 1 \, ds \, dt = y & x \geq 1 \wedge 0 < y < 1 \\ \int_0^x \int_0^1 1 \, ds \, dt = x & 0 < x < 1 \wedge y \geq 1 \\ \int_0^1 \int_0^1 1 \, ds \, dt = 1 & x \geq 1 \wedge y \geq 1 \end{cases}$$

Proposition 25 (Properties of Joint PDF)

1. $\forall (x,y) \in \mathbb{R}^2 \quad f(x,y) \geq 0$
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \, dx \, dy = 1$
3. $\forall B \subset \mathbb{R}^2$,

$$P[(X,Y) \in B] = \int\int_{(x,y) \in B} f(x,y) \, dx \, dy$$

Proof

Example 7.1.3 (Example 3.9)

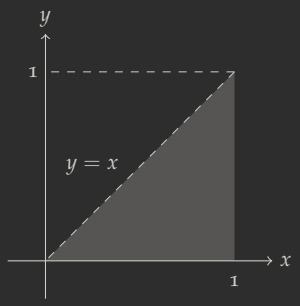
Suppose that $f(x,y) = Kxy \cdot \mathbb{1}_{\{0<x\,y<1\}}$ for some constant $K > 0$. Find K so that f is a valid joint pdf. If X and Y have the joint density f, calculate $P(X > Y)$.

Solution

Note that

$$1 = \int_0^1 \int_0^1 Kxy\,dx\,dy = \frac{K}{4}.$$

Thus $K = 4$. To solve the next part, observe that for $X > Y$, we have the diagram to the right to show the support set of the joint distribution. The shaded region is the support set. We then have



$$\begin{aligned} P(X > Y) &= \int_0^1 \int_0^x 4xy\,dy\,dx = \int_0^1 2xy^2\Big|_0^x dx \\ &= \int_0^1 2x^3\,dx = \frac{1}{2}x^3\Big|_0^1 = \frac{1}{2} \end{aligned}$$

Example 7.1.4 (Example 3.10)

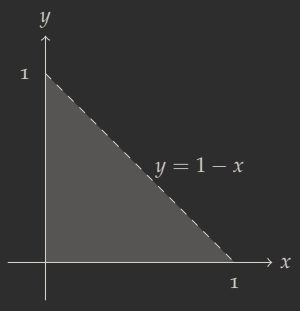
Suppose that

$$f(x,y) = \begin{cases} Cxy & 0 < x, y < 1,\ x + y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find C so that $f(x,y)$ is a valid joint probability density function, and calculate $P(Y^2 < X)$.

Solution

Note that the diagram on the right shows the support set of (X,Y). To find C,

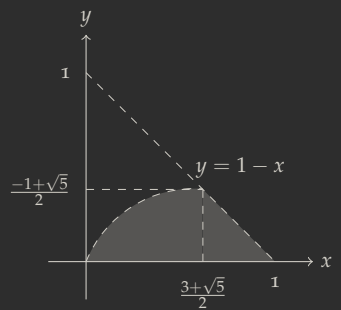


$$\begin{aligned} 1 &= \int_0^1 \int_0^{1-x} Cxy\,dy\,dx = \int_0^1 \frac{C}{2}xy^2\Big|_0^{1-x} dx \\ &= C\int_0^1 \frac{1}{2}x(x^2 - 2x + 1)\,dx = C\int_0^1 \frac{1}{2}(x^3 - 2x^2 + x)\,dx \\ &= C\left(\frac{1}{8}x^4 - \frac{1}{3}x^3 + \frac{1}{4}x^2\right)\Big|_0^1 = C\left(\frac{3}{24} - \frac{8}{24} + \frac{6}{24}\right) = \frac{C}{24}. \end{aligned}$$

And so $C = 24$.

To calculate $P(Y^2 < X)$, note the diagram to the right. Then

$$\begin{aligned} P(Y^2 < X) &= \int_0^{\frac{3+\sqrt{5}}{2}} \int_0^{\sqrt{x}} 24xy\,dy\,dx + \int_{\frac{3+\sqrt{5}}{2}}^1 \int_0^{1-x} 24xy\,dy\,dx \\ &= \int_0^{\frac{3+\sqrt{5}}{2}} 12xy^2\Big|_0^{\sqrt{x}}\,dx + \int_{\frac{3+\sqrt{5}}{2}}^1 12xy^2\Big|_0^{1-x}\,dx \\ &= 4x^3\Big|_0^{\frac{3+\sqrt{5}}{2}} + \int_{\frac{3+\sqrt{5}}{2}}^1 24(x^3 - 2x^2 + x)\,dx \\ &= 4\left(\frac{3+\sqrt{5}}{2}\right)^3 + 24\left[\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2\right]\Big|_{\frac{3+\sqrt{5}}{2}}^1 = \dots \end{aligned}$$



Solve for $y = 1 - x$ and $y^2 = x$ to get the intersection.

We shall not proceed to get the final solution since it is a messy process and the result is not important.

7.1.3 Marginal Distribution (Continuous)

Definition 34 (Marginal PDF)

Suppose X and Y are continuous rvs with joint pdf f. Then the marginal pdf of X is given by

$$\forall x \in \mathbb{R} \quad f_X(x) = \int_{-\infty}^{\infty} f\,dy,$$

and the marginal pdf of Y is

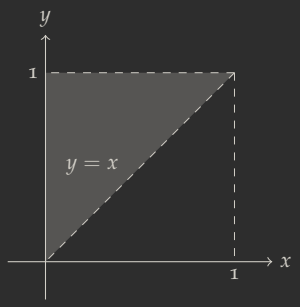
$$\forall y \in \mathbb{R} \quad f_Y(y) = \int_{-\infty}^{\infty} f\,dx.$$

Example 7.1.5 (Example 3.11)

Suppose X and Y have joint pdf $f(x,y) = K(x+y)\mathbb{1}_{0\le x<y\le 1}$ for some constant K. Find K. Then, calculate the marginal density of X.

Solution

A diagram showing the region of the support set is on the right.



To get K,

$$\begin{aligned} 1 &= \int_0^1 \int_x^1 K(x+y)\,dy\,dx = \int_0^1 \left(Kxy + \frac{1}{2}Ky^2\right)\Big|_x^1 dx \\ &= \int_0^1 Kx + \frac{K}{2} - Kx^2 - \frac{1}{2}Kx^2\,dx \\ &= \frac{K}{2}\left(x^2 + x - x^3\right)\Big|_0^1 = \frac{K}{2} \end{aligned}$$

Thus $K = 2$.

To get the marginal density of X, note that our joint pdf is now the following:

$$f(x,y) = 2(x+y)\mathbb{1}_{\{0\le x<y\le 1\}}$$

Thus

$$\int_x^1 2(x+y)\,dy = 2xy + y^2\Big|_x^1 = 2x + 1 - 3x^2$$

And hence

$$f_X(x) = \begin{cases} -3x^2 + 2x + 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

7.1.4 Independence of Continuous RVs

Definition 35 (Independence of Continuous RVs)

Two random variables X and Y with joint cdf F and joint pdf f are independent iff

$$\forall x, y \in \mathbb{R} \quad F(x,y) = F_X(x)F_Y(y)$$

or[1]

$$\forall x, y \in \mathbb{R} \quad f(x,y) = f_X(x)f_Y(y).$$

[1] It's really an "AND"

Note

A necessary, but insufficient, condition for X and Y to be independent is that

$$\text{supp}(X,Y) = \text{supp}(X) \times \text{supp}(Y)$$

Example 7.1.6 (Example 3.12)

Are random variables X and Y introduced in Example 7.1.5 independent? Explain.

Solution

Recall that the pdf was given as

$$f(x,y) = 2(x+y)\mathbb{1}_{\{1\leq x<y\leq 1\}}.$$

We derived the marginal pdf of X in the earlier example:

$$f_X(x) = (-3x^2+2x+1)\mathbb{1}_{\{0\leq x\leq 1\}}.$$

To get the marginal pdf of Y, note

$$\int_0^y 2(x+y)\,dx = x^2+2xy\Big|_0^y = 3y^2.$$

Thus

$$f_Y(y) = \begin{cases} 3y^2 & 0\leq y\leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$f_X(x)f_Y(y) = -9x^2y^2+6xy^2+3y^2 \quad 0\leq x<y\leq 1$$

which is not equal to f. Thus, X and Y are not independent.

8 Lecture 8 May 29th 2018

8.1 Joint Distributions (Continued 2)

8.1.1 Independence of Continuous RVs (Continued)

Example 8.1.1 (Example 3.12 (3.4.8 course note))

Suppose X and Y are continuous with joint pdf

$$f(x,y) = \frac{3}{2}y(1-x^2)\mathbb{1}_{\{-1\le x\le 1\}}\mathbb{1}_{\{0\le y\le 1\}}$$

Are X and Y independent?

Solution

The marginal pdf of X is

$$\begin{aligned} f_X(x) &= \int_0^1 \frac{3}{2}y(1-x^2)\,dy = \frac{3}{4}y^2(1-x^2)\Big|_0^1 \\ &= \begin{cases} \frac{3}{4}(1-x^2) & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The marginal pdf of Y is

$$\begin{aligned} f_Y(x) &= \int_{-1}^1 \frac{3}{2}y(1-x^2)\,dx = \frac{3}{2}y\left(x-\frac{1}{3}x^3\right)\Big|_{-1}^1 \\ &= \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Clearly, we have

$$f_X(x)f_Y(y) = \frac{3}{2}y(1-x^2) = f(x,y) \quad -1 \le x \le 1,\ 0 \le y \le 1.$$

Thus X and Y are independent.

Theorem 26 (Factorization Theorem for Independence)

Suppose X and Y are rvs with joint pf f, and marginal pf f_X and f_Y, respectively. Suppose also that

$$\begin{aligned} A &= \{(x,y) : f(x,y) > 0\} \text{ is the support set of } (X,Y) \\ A_X &= \{x : f_X(x) > 0\} \text{ is the support set of } X, \text{ and} \\ A_Y &= \{y : f_Y(y) > 0\} \text{ is the support set of } Y \end{aligned}$$

Then X and Y are independent rvs iff $A = A_X \times A_Y$ and there exist non-negative functions g and h such that

$$f(x,y) = g(x)h(y)$$

for all $(x,y) \in A_X \times A_Y$.

Proof

The $\implies$ direction is straightforward: Since X and Y are independent, we have that $f = f_X f_Y$, and so clearly, $A = A_X \times A_Y$ and so $\forall (x,y) \in A = A_x \times A_Y$, we have that f_X and f_Y are non-negative.

For the $\impliedby$ direction, note that

$$\begin{aligned} f_Y(y) &= \int_{x \in A_X} g(x)h(y)\,dx = h(y) \int_{x \in A_X} g(x)\,dx \\ f_X(x) &= \int_{y \in A_Y} g(x)h(y)\,dy = g(x) \int_{y \in A_Y} h(x)\,dy. \end{aligned}$$

Thus,

$$\begin{aligned} f_X(x)f_Y(y) &= g(x)h(y) \int_{x \in A_X} g(x)\,dx \int_{y \in A_Y} h(y)\,dy \\ &= g(x)h(x) \int_{x \in A_X} \int_{y \in A_Y} g(x)h(y)\,dy\,dx = g(x)h(y) \end{aligned}$$

*where line 2 is by **linearity of integration**. Thus $f(x,y) = f_X(x)f_Y(y)$. Thus X and Y are independent.* □

Note

1. *If Theorem 26 holds, then f_X will be proportional to g and f_Y will be*

proportional to h. Clearly so, since

$$g(x) \cdot h(y) = f_X(x) f_Y(y)$$
$$g(x) \propto f_X(x) \wedge h(y) \propto f_Y(y)$$

2. *The definitions and theorems can be easily extended to the random vector $(X_1, X_2, ..., X_n)$. Indeed, if we apply mathematical induction on the proof above, we will be able to get our desired result.*[1]

[1] I wonder if this statement is equivalent to the Fisher-Neyman Factorization Theorem.

8.1.2 Conditional Distributions

Definition 36 (Conditional Distributions)

Suppose X and Y are rvs with joint pf f, and marginal pfs f_X and f_Y, respectively. Suppose also that $A = \{(x,y) : f(x,y) > 0\}$. The ***conditional pf*** *of X given $Y = y$ is given by*

$$f_X(x|y) = \frac{f(x,y)}{f_Y(y)}$$

for $(x,y) \in A$ provided that $f_Y(y) \neq 0$. The ***conditional pf*** *of Y given $X = x$ is given by*

$$f_Y(y|x) = \frac{f(x,y)}{f_X(x)}$$

for $(x,y) \in A$ provided that $f_X(x) \neq 0$.

Remark

If X and Y are discrete rvs then

$$f_X(x|y) = P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{f(x,y)}{f_Y(y)}$$

and

$$\sum_x f_X(x|y) = \sum_x \frac{f(x,y)}{f_Y(y)} = \frac{1}{f_Y(y)} \sum_x f(x,y) = \frac{f_Y(y)}{f_Y(y)} = 1,$$

and similarly so for $f_Y(y|x)$. Similarly, if X and Y are both continuous rvs,

then

$$\int_{-\infty}^{\infty} f_X(x|y)\,dx = \int_{-\infty}^{\infty} \frac{f(x,y)}{f_Y(y)}\,dx = \frac{1}{f_Y(y)}\int_{-\infty}^{\infty} f(x,y)\,dx = \frac{f_Y(y)}{f_Y(y)} = 1$$

Now consider if X is a continuous rv such that $f_X(x) \neq P(X = x)$ and $P(X = x) = 0$ for all x. Then to justify the definition of the conditional pdf of Y given $X = x$, when X and Y are both continuous rvs, we consider $P(Y \leq y|X = x)$ using the limit approach:

$$\begin{aligned}
P(Y \leq y|X = x) &= \lim_{h\to 0} P(Y \leq y|x \leq X \leq x+h) \\
&= \lim_{h\to 0} \frac{\int_x^{x+h}\int_{-\infty}^{y} f(u,v)\,dv\,du}{\int_x^{x+h} f_X(u)\,du} \\
&= \lim_{h\to 0} \frac{\frac{d}{dh}\int_x^{x+h}\int_{-\infty}^{y} f(u,v)\,dv\,du}{\frac{d}{dh}\int_x^{x+h} f_X(u)\,du} \quad \textit{by } \textbf{L'Hôpital's Rule} \\
&= \lim_{h\to 0} \frac{\int_{-\infty}^{y}\frac{d}{dh}\int_x^{x+h} f(u,v)\,du\,dv}{\frac{d}{dh}\int_x^{x+h} f_X(u)\,du} \quad (1) \\
&= \lim_{h\to 0} \frac{\int_{-\infty}^{y} f(x+h,v)\,dv}{f_X(x+h)} \quad (2) \\
&= \frac{\int_{-\infty}^{y} f(x,v)\,dv}{f_X(x)}
\end{aligned}$$

where (1) is by assuming that the integrands are all convergent so that we may interchange the integral signs and the differential operator, and (2) by the ***Fundamental Theorem of Calculus****. If we differentiate the last line with respect to y, by the Fundamental Theorem of Calculus, we have*

$$\frac{d}{dy}P(Y \leq y|X = x) = \frac{f(x,y)}{f_X(x)}$$

which justifies the using of our definition

$$f_Y(y|x) = \frac{f(x,y)}{f_X(x)}.$$

Example 8.1.2

A fair coin is flipped 10 times independently.

1. *What is the distribution of Y, the number of heads in 10 flips?*

2. *Suppose the first 4 flips have all landed on tails. What is the distribution of Y given this information?*

Solution

1. *Clearly, we know that* $Y \sim \text{Bin}\left(10, \frac{1}{2}\right)$.

2. *Since each flip is independent of each other and the first four flips have already been determined, the range of values for* Y *changes from* $\{0, ..., 10\}$ *to* $\{0, ..., 6\}$. *Since the experiment is still essentially the same, we have that*

$$Y \mid \textit{first 4 flips are tails} \sim \text{Bin}\left(6, \frac{1}{2}\right).$$

Example 8.1.3 (Example 3.13)

Consider the experiment carried out in Example 8.1.2. Let

$$\begin{aligned} X &:= \textit{number of heads in the first 4 flips} \\ Y &:= \textit{number of heads in 10 flips} \end{aligned}$$

Derive the conditional distribution of Y *given that the first 4 flips landed on heads, i.e. derive the distribution for* $Y|X = 4$.

Solution

Let W *be the number of heads in the last 6 flips. Then* W *has the same distribution as in part 2 of our earlier example. Also,* $X \sim \text{Bin}\left(4, \frac{1}{2}\right)$ *Clearly,* $Y = X + W$. *We proceed to derive the joint pf of* X *and* Y:

$$\begin{aligned} P(X = x, Y = y) &= P(X = x, X + W = y) = P(X = x, W = y - x) \\ &= P(X = x)P(W = y - x) \quad \textit{by } \textbf{\textit{Independence}} \\ &= \binom{4}{x}\left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x} \cdot \binom{6}{y-x}\left(\frac{1}{2}\right)^{y-x}\left(\frac{1}{2}\right)^{6-y+x} \\ &= \binom{4}{x}\binom{6}{y-x}\left(\frac{1}{2}\right)^{10} \end{aligned}$$

Then

$$\begin{aligned} P(Y|X = 4) &= \frac{P(X = 4, Y = y)}{P(X = 4)} \\ &= \frac{\binom{4}{4}\binom{6}{y-4}\left(\frac{1}{2}\right)^{10}}{\binom{4}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^0} \\ &= \binom{6}{y-4}\left(\frac{1}{2}\right)^6 \quad y \in \{4, 5, ..., 10\}. \end{aligned}$$

We may also re-label the conditional distribution to have

$$\binom{6}{y^*}\left(\frac{1}{2}\right)^6 \quad y^* \in \{0, ..., 6\}$$

Example 8.1.4 (Example 3.14)

From Example 7.1.5, we had that

$$f(x,y) = 2(x+y)\mathbb{1}_{¶0\leq x<y\leq 1\}}$$

and the marginal density of X is

$$f_X(x) = (2x - 3x^2 + 1)\mathbb{1}_{\{0\leq x<1\}}.$$

Derive the conditional distribution of $Y|X = \frac{1}{2}$.

Solution
Observe that

$$f(y|X = \frac{1}{2}) = \frac{f\left(\frac{1}{2}, y\right)}{f_X\left(\frac{1}{2}\right)} = \frac{2\left(\frac{1}{2} + y\right)}{2\left(\frac{1}{2}\right) - 3\left(\frac{1}{2}\right)^2 + 1} = \frac{8}{13}(1+2y)$$

for $\frac{1}{2} < y \leq 1$.

Proposition 27 (Properties of Conditional Distributions)

Let X and Y be rvs. If both X and Y are discrete, then

- $\sum_x f(x|y) = 1$;
- $F(x|y) = \sum_{\{w:w\leq x\}} f(w|y)$; *and*
- $f(x|y) = F(x|y) - F(x^-|y)$.

If X and Y are both continuous, then

- $\int_x f(x|y)\,dx = 1$;
- $F(x|y) = \int_{-\infty}^{x} f(t|y)\,dt$; *and*
- $f(x|y) = \frac{\partial}{\partial x}F(x|y)$

Exercise 8.1.1
Prove Proposition 27.

Theorem 28 (Product Rule)

Suppose X and Y are rvs with joint pf f, marginal pfs $f_X(x)$ *and* $f_Y(y)$ *respectively, and conditional pfs* $f_X(x|y)$ *and* $f_Y(y|x)$ *respectively. Then*

$$f(x,y) = f_X(x|y)f_Y(y) = f_Y(y|x)f_X(x).$$

Proof

Notice once and for all that by rearranging the definition of conditional distribution

$$f_X(x|y)f_Y(y) = f(x,y) = f_Y(y|x)f_X(x)$$

□

Proposition 29 (Independence from Conditionality)

Suppose X and Y are rvs with marginal pfs $f_X(x)$ and $f_Y(y)$ respectively, and conditional pfs $f_X(x|y)$ and $f_Y(y|x)$ respectively. Let $A_X = \{x : f_X(x) > 0\}$ and $A_Y = \{y : f_Y(y) > 0\}$. X and Y are independent rvs iff either of the following holds:

$$\forall x \in A_X \quad f_X(x|y) = f_X(x)$$

or

$$\forall y \in A_Y \quad f_Y(y|x) = f_Y(y).$$

Proof

Suppose that X and Y are independent rvs. Then

$$f(x,y) = f_X(x)f_Y(y).$$

Then

$$f_X(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_X(x)$$
$$f_Y(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_Y(y).$$

for $x \in A_X$ and $y \in A_Y$.

Suppose WLOG that $f_X(x|y) = f_X(x)$. Thus by Theorem 28

$$f(x,y) = f_X(x|y)f_Y(y) = f_X(x)f_Y(y) \text{ for } x \in A_X,\ y \in A_Y.$$

□

Example 8.1.5 (Example 3.15)

In a game of chance, a random number is generated from $P \sim \text{Beta}(\alpha, \beta)$. Given $P = p$, a coin with $P(\{H\}) = p$ is flipped independently n times, where the player is rewarded the same amount of dollars as the number of heads in n. Calculate the probability that a random player earns at least \$1 in this game.

Solution

Let X be the number of heads that appear in n flips, which equates to the total amount of \$1 earned. Then

$$X \,|P = p \sim \text{Bin}(n, p)$$

However, note that

$$P(X \geq 1) = P(\text{earn at least \$1}) = 1 - P(\text{earn nothing}) = 1 - P(X = 0)$$

To get $P(X = x)$, we need to do the following: note that the support set of P is from 0 to 1, then

$$\begin{aligned}
Pr(X = x) &= \int_0^1 Pr(X = x, P = p)\, dp \\
&= \int_0^1 Pr(X = x|P = p) Pr(P = p)\, dp \qquad \text{by Theorem 28} \\
&= \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} \frac{p^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}\, dp \\
&= \binom{n}{x} \frac{1}{B(\alpha,\beta)} \int_0^1 p^{\alpha+x-1}(1-p)^{\beta+n-x-1}\, dp \\
&= \binom{n}{x} \frac{B(\alpha+x, \beta+n-x)}{B(\alpha,\beta)} \int_0^1 \underbrace{\frac{p^{\alpha+x-1}(1-p)^{\beta+n-x-1}}{B(\alpha+x,\beta+n-x)}}_{pdf\ of\ \text{Beta}(\alpha+x,\beta+n-x)}\, dp \\
&= \binom{n}{x} \frac{B(\alpha+x, \beta+n-x)}{B(\alpha,\beta)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
P(X \geq 1) = 1 - P(X = 0) &= 1 - \binom{n}{0} \frac{\Gamma(\alpha)\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \\
&= 1 - \frac{\Gamma(\alpha+\beta)\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)\Gamma(\beta)}
\end{aligned}$$

Definition 37 (Beta Distribution)
If $X \sim \text{Beta}(\alpha, \beta)$, then

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$$

where

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

8.1.3 Joint Expectations

Definition 38 (Joint Expectation)

Suppose $h(x,y)$ is a real-valued function. If X and Y are discrete rvs with joint pf f and support A, then

$$E[h(x,y)] = \sum_{(x,y)\in A}\sum h(x,y)f(x,y).$$

provided that the joint sum converges absolutely.

If X and Y are continuous rvs with joint pf f, then

$$E[h(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)f(x,y)\,dx\,dy$$

provided that the joint integral converges absolutely.

This is also known as the ***Law of the Unconscious Statistician*** *for two rvs.*

Example 8.1.6 (Example 3.16)

Consider X and Y with the following joint probability distribution. Calculate $E(XY)$.

	$x = -2$	$x = 0$	$x = 2$
$y = 0$	0.05	0.1	0.15
$y = 1$	0.07	0.1	0.2
$y = 2$	0.02	0.25	0.05

Solution

$$\begin{aligned} E(XY) &= \sum_x\sum_y xyf(x,y) \\ &= -2(1)(0.07) - 2(2)(0.02) + 2(1)(0.2) + 2(2)(0.05) \\ &= 0.38 \end{aligned}$$

9 Lecture 9 May 31st 2018

9.1 Joint Distributions (Continued 3)

9.1.1 Joint Expectations (Continued)

Theorem 30 (Linearity of Expectation in Bivariate Case)

Suppose X and Y are two rvs with joint pf f, a_i, b_i, for $i = 1, ..., n$, are constants, and $g_i(x, y)$, for $i = 1, ..., n$, are real-valued functions. Then

$$E\left[\sum_{i=1}^{n}(a_i g_i(X,Y) + b_i)\right] = \sum_{i=1}^{n}(a_i E[g_i(X,Y)]) + \sum_{i=1}^{n} b_i$$

provided that $E[g_i(X,Y)]$ is finite for $i = 1, ..., n$.

Proof

This is simply an extension of Theorem 13.

Theorem 31 (Implication of Independence on Joint Expectation)

If X and Y are independent rvs with joint pf f, and $g(x)$ and $h(y)$ are real valued funcctions, then

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

Proof

We shall prove for the discrete case and leave the continuous case for future exercises.

Observe that

$$\begin{aligned} E[g(X)h(Y)] &= \sum_x \sum_y g(x)h(y)f(x,y) \quad \because \textit{Definition 38} \\ &= \sum_x \sum_y g(x)h(y)f_X(x)f_Y(y) \\ &= \sum_x g(x)f_X(x) \sum_y h(y)f_Y(y) \\ &= E[g(X)]E[h(Y)] \end{aligned}$$

where $f_X(x)$ and $f_Y(y)$ are the marginal pfs of X and Y respectively. □

Exercise 9.1.1
Prove Theorem 31 for the continuous case.

We may repeatedly apply the above proof for n rvs through induction and get the following result.

Theorem 32 (Generalized Implication of Independence on Joint Expectation)

If $X_1, X_2, ..., X_n$, for some $n \in \mathbb{N}$, are independent rvs and $h_1, h_2, ..., h_n$ are real valued functions, then

$$E\left[\prod_{i=1}^{n} h_i(X_i)\right] = \prod_{i=1}^{n} E[h_i(X_i)].$$

9.1.2 Covariance

Independence of two rvs X and Y implies that knowledge of the value of X does not provide any information whatsoever about the distribution of Y. Essentially, we can say that there is no "relationship" between X and Y. In statistics, **linear relationships** are often the subject of interest. The strength of a linear relationship is related to **covariance** and measured by the **correlation coefficient**, usually denoted by ρ.

It can be shown that when X and Y have no linear relationship iff their covariance is 0.

On a related thought, does covariance relate to independence? If so, how?

Definition 39 (Covariance)

*The **covariance** of rvs X and Y is given by*

$$\text{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E(XY) - \mu_X\mu_Y. \tag{9.1}$$

where $\mu_X = E[X]$ and $\mu_Y = E[Y]$.

*If $\text{Cov}(X,Y) = 0$, then X and Y are called **uncorrelated** rvs.*

Note

Note that the 2nd and 3rd term are equivalent in Equation (9.1) since

$$\begin{aligned} E[(X-\mu_X)(Y-\mu_Y)] &= E[XY - \mu_X Y - \mu_Y X + \mu_X\mu_Y] \\ &= E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X\mu_Y \because \textit{Theorem 30} \\ &= E[XY] - \mu_X\mu_Y - \mu_X\mu_Y + \mu_X\mu_Y = E[XY] - \mu_X\mu_Y \end{aligned}$$

From here, it is easy to see from Theorem 31, that since $E[XY] = E[X]E[Y] = \mu_X\mu_Y$, we have that the independence of X from Y will imply that $\text{Cov}(X,Y) = 0$.

*However, the converse of the above is **not true**.*

Example 9.1.1

Source: Stats SE

Let X be an rv that it is -1 or 1 with probability 0.5. Then let Y be an rv such that $Y = 0$ if $X = -1$, and Y is randomly -1 or 1 with probability 0.5 if $X = 1$.

Clearly X and Y are highly dependent (since knowing Y allows me to

perfectly know X). They both have zero mean:

$$E[X] = -1\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) = 0$$
$$E[Y] = -1\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) = 0$$

and

$$\begin{aligned} E[XY] &= (-1)\cdot 0P(X=-1,Y=0) + 1(-1)\cdot P(X=1,Y=-1) \\ &\quad + 1(1)P(X=1,Y=1) \\ &= -\frac{1}{4} + \frac{1}{4} = 0 \end{aligned}$$

Thus $\mathrm{Cov}(X,Y) = 0$

Or more generally, take any distribution $P(X)$ and any $P(Y|X)$ such that $P(Y=a|X) = P(Y=-a|X)$ for all X (i.e., a joint distribution that is symmetric around the x axis), and you will always have zero covariance. But you will have non-independence whenever $P(Y|X) \neq P(Y)$, i.e., the conditionals are not all equal to the marginal, and vice versa for symmetry around the y axis.

Note

- *If* $\mathrm{Cov}(X,Y) = 0$*, then X and Y are called* ***uncorrelated*** *rvs.*
- *By definition,* $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$*, since*

$$\mathrm{Cov}(X,X) = E[(X-\mu_X)^2] = \mathrm{Var}(X)$$

Example 9.1.2 (Example 3.17)

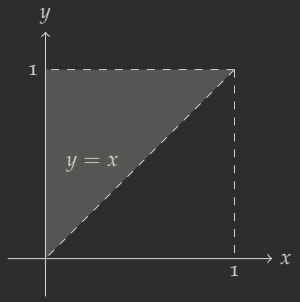
Consider the rvs X and Y with the joint pdf

$$f(x,y) = \begin{cases} 2 & 0 \leq x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Calculate $\mathrm{Cov}(X,Y)$.

Solution

Observe the diagram of the support set of X and Y to our right.



Then we can calculate

$$\begin{aligned} E[XY] &= \int_0^1 \int_x^1 2xy\,dy\,dx = \int_0^1 xy^2\Big|_x^1 dx \\ &= \int_0^1 x - x^3\,dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \\ f_X(x) &= \int_x^1 2\,dy = \begin{cases} 2-2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \\ E[X] &= \int_0^1 x(2-2x)\,dx = 1 - \frac{2}{3} = \frac{1}{3} \\ f_Y(y) &= \int_0^y 2\,dx = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \\ E[Y] &= \int_0^1 2y^2\,dy = \frac{2}{3} \end{aligned}$$

Thus

$$\text{Cov}(X,Y) = \frac{1}{4} - \frac{1}{3}\left(\frac{2}{3}\right) = \frac{1}{36}$$

We observe that the covariance is positive. This implies a ***positive linear relationship****. However, we cannot tell from this value the strength of the relationship between X and Y.*

Example 9.1.3 (Example 3.18)

Consider rvs X and Y with the joint pf

$$f(x,y) = \frac{xy}{7}\mathbb{1}_{\{y=1,2\}}\mathbb{1}_{\{x=0,\ldots,y\}}.$$

Calculate $\text{Cov}(X,Y)$.

Solution

The following table captures all the probabilities that can be found from the given pf

	$x=0$	$x=1$	$x=2$
$y=1$	0	$\frac{1}{7}$	0
$y=2$	0	$\frac{2}{7}$	$\frac{4}{7}$

Observe that we thus have

$$f_X(x) = \begin{cases} \frac{3}{7} & x = 1 \\ \frac{4}{7} & x = 2 \end{cases}$$
$$E[X] = \frac{3}{7} + 2\left(\frac{4}{7}\right) = \frac{11}{7}$$
$$f_Y(y) = \begin{cases} \frac{1}{7} & y = 1 \\ \frac{6}{7} & y = 2 \end{cases}$$
$$E[Y] = \frac{1}{7} + 2\left(\frac{6}{7}\right) = \frac{13}{7}$$

Also,

$$E[XY] = \frac{1}{7} + 2\left(\frac{2}{7}\right) + 4\left(\frac{4}{7}\right) = 3$$

Therefore,

$$\operatorname{Cov}(X, Y) = E[XY] - E[X]E[Y] = 3 - \frac{11}{7}\frac{13}{7} = \frac{4}{49}$$

Theorem 33 (Variance of Linear Combinations)

Suppose X and Y are rvs and a, b, c are real constants. Then

$$\operatorname{Var}(aX + bY + c) = a^2 \operatorname{Var}(X) + b^2 \operatorname{Var}(Y) + 2ab \operatorname{Cov}(X, Y)$$

Proof

Let $E[aX + bY + c] = \mu$. Observe that

$$\begin{aligned}
&\operatorname{Var}[aX + bY + c] \\
&= E\left[[(aX + bY + c) - \mu]^2\right] \\
&= E\left[(aX + bY + c)^2 - 2\mu(aX + bY + c) + \mu^2\right] \\
&= E\left[a^2X^2 + abXY + acX + abXY + b^2Y^2 + bcY + c^2 \right. \\
&\qquad \left. -2\mu(aX + bY + c) + \mu^2\right] \\
&= a^2E[X^2] + 2abE[XY] + acE[X] + b^2E[Y^2] + bcE[Y] + c^2 - \mu^2
\end{aligned}$$

Note that

$$\begin{aligned}\mu^2 &= E\left[aX + bY + c\right]^2 \\ &= \left(aE[X] + bE[Y] + c\right)^2 \\ &= a^2E[X]^2 + b^2E[Y]^2 + 2abE[X]E[Y] + acE[X] + bcE[Y] + c^2\end{aligned}$$

Therefore, we have that

$$\begin{aligned}&\text{Var}[aX + bY + c] \\ &= a^2E[X^2] + a^2E[X]^2 + b^2E[Y^2] - b^2E[Y]^2 + 2abE[XY] - 2abE[X]E[Y] \\ &= a^2\left(E[X^2] - E[X]^2\right) + b^2\left(E[Y^2] - E[Y]^2\right) + 2ab\left(E[XY] - E[X]E[Y]\right) \\ &= a^2\,\text{Var}(X) + b^2\,\text{Var}\,Y + 2ab\,\text{Cov}(X, Y)\end{aligned}$$

as required. □

By applying Theorem 33 repeatedly, we have the following generalized theorem.

Theorem 34 (Generalized Variance of Linear Combinations)

Suppose $X_1, X_2, ..., X_n$ are rvs with $\text{Var}(X_i) = \sigma_i^2$, and $a_1, a_2, ..., a_n$ are real constants. Then

$$\text{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2\sigma_i^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i a_j\,\text{Cov}(X_i, X_j)$$

Note that the prove the above, we have to also use the fact that

$$\text{Cov}(X_i, X_j) = \text{Cov}(X_j, X_i)$$

Note

Note that in Theorem 34, if the rvs are independent rvs, then $\text{Cov}(X_i, X_j) = 0$ for $i \neq j$, thus wiping off the 2nd term in the equation, leaving us with

$$\text{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2\sigma_i^2$$

Example 9.1.4 (Example 3.19)

To build a ship engine piece, suppose two pole-shaped components A and B are attached at one end to each other to make one long pole-shaped component C. Suppose the length of part A is an rv with a mean of 3 inches and a variance of 0.25 $inch^2$. Similarly, the length of component B is an rv with a mean of 25 inches and a variance of 0.5 $inch^2$.

Find the mean and the variance of the length of part C if

1. *the lengths of A and B are independent;*
2. *the covariance between lengths of A and B is -0.3 $inch^2$.*

Solution

Note that we are given that $C = A + B$

1. *We have that*

$$E(C) = E(A + B) = E(A) + E(B) = 3 + 25 = 28.$$

For variance, since A and B are independent, $\mathrm{Cov}(A, B) = 0$, thus

$$\begin{aligned}\mathrm{Var}(C) = \mathrm{Var}(A + B) &= \mathrm{Var}(A) + \mathrm{Var}(B) + 2\,\mathrm{Cov}(A, B)\\ &= 0.25 + 0.5 + 0 = 0.75\end{aligned}$$

2. *Since C is a linear equation, the covariance does not affect the expectation and thus we still have*

$$E(C) = 28.$$

Now, given that $\mathrm{Cov}(A, B) = -0.3$, we have

$$\mathrm{Var}(C) = \mathrm{Var}(A) + \mathrm{Var}(B) + 2\,\mathrm{Cov}(A, B) = 0.75 - 0.6 = 0.15.$$

9.1.3 Correlation

The **covariance** is a real number which depends on the units of measurement of X and Y. The information part of a covariance is its **sign**, unless if it is used as the context.

To use the covariance as the context, and to quantitatively measure the strength of a linear relationship, which we have discussed and desired before, we use the **correlation coefficient**.

Definition 40 (Correlation Coefficient)

*The **correlation coefficient** of rvs X and Y is given by*

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

where $\sigma_X = \sqrt{\text{Var}(X)}$ and $\sigma_Y = \sqrt{\text{Var}(Y)}$.

Note that this is the definition of the **Pearson Correlation Coefficient**. There are other correlation coefficients but we will be using only Pearson, at it seems.

Proposition 35 (Properties of the Correlation Coefficient)

Let X and Y be rvs, and $\rho(X,Y)$ the correlation coefficient of X and Y. Then

1. $|\rho(X,Y)| \leq 1$;
2. *(**perfect positive linear relationship**)*
   $\rho(X,Y) = 1 \iff Y = aX + b$ *for some* $a > 0$;
3. *(**perfect inverse linear relationship**)*
   $\rho(X,Y) = -1 \iff Y = aX + b$ *for some* $a < 0$.

Proof

1. *This is somewhat beyond the scope of what we can cover now but we shall use this result presented on Wikipedia:*

$$|\text{Cov}(X,Y)| \leq \sqrt{\text{Var}(X)\,\text{Var}(Y)}.$$

Then given the formula of $\rho(X,Y)$, the proof is complete:

$$|\rho(X,Y)| = \left| \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}} \right| \leq \frac{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}} = 1$$

Proving 2 and 3 is currently outside of my abilities. Refer to this Math SE Q&A for a hint on how to prove this statement.

Example 9.1.5 (Example 3.20)

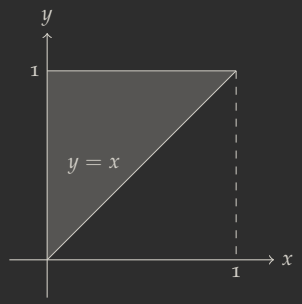
Consider rvs X and Y with the joint pdf

$$f(x,y) = \begin{cases} 2 & 0 \leq x \leq y \leq 1 \\ 0 & otherwise \end{cases}$$

Calculate $\rho(X,Y)$.

Solution

The diagram to the right is an illustration of the region of support for X and Y. We now calculate the following values:



$$E(XY) = \int_0^1 \int_x^1 2xy\,dy\,dx = \int_0^1 x - x^3\,dx = \frac{1}{4}$$
$$f_X(x) = \int_x^1 2\,dy = 2 - 2x \quad 0 \leq x \leq 1$$
$$f_Y(y) = \int_0^y 2\,dx = 2y \quad 0 \leq y \leq 1$$
$$E(X) = \int_0^1 2x - 2x^2\,dx = \left(x^2 - \frac{2}{3}x^3\right)\Big|_0^1 = \frac{1}{3}$$
$$E(X^2) = \int_0^1 2x^2 - 2x^3\,dx = \left(\frac{2}{3}x^3 - \frac{1}{2}x^4\right)\Big|_0^1 = \frac{1}{6}$$
$$E(Y) = \int_0^1 2y^2\,dy = \frac{2}{3}$$
$$E(Y^2) = \int_0^1 2y^3\,dy = \frac{1}{2}$$
$$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$$
$$\text{Var}(Y) = E(Y^2) - E(Y)^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$$

Therefore we have that

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}$$
$$\sqrt{\text{Var}(X)\,\text{Var}(Y)} = \frac{1}{18}$$

and so

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}} = \frac{\frac{1}{36}}{\frac{1}{18}} = \frac{1}{2}$$

9.1.4 Conditional Expectation

Definition 41 (Conditional Expectation)

Let G be a real-valued function. The ***conditional expectation*** *of $g(Y)$ given $X = x$, denoted as $g(Y) \mid (X = x)$ is given by*

$$E\left[g(Y) \mid x\right] = \sum_y g(y) f_Y(y \mid x)$$

if $Y|(X = x)$ is a discrete rv and

$$E\left[g(Y) \mid x\right] = \int_y g(y) f_Y(y \mid x)$$

if $Y|(X = x)$ is a continuous rv. This definition only holds provided that the sum and the integral converges absolutely. The conditional expectation of $h(X)$ given $Y = y$, where h is a real-valued function, is defined in a similar manner.

We also call $E[Y \mid X = x]$ the ***conditional mean****, which may be denoted as $E(Y \mid x)$, and $\text{Var}(Y \mid X = x)$ the* ***conditional variance****, which may be denoted as $\text{Var}(Y \mid x)$.*

Note

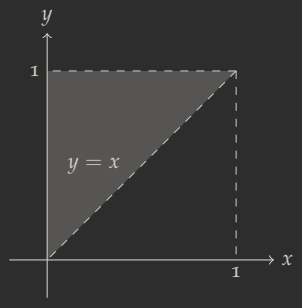
Note that there is also the notation $E(Y \mid X)$, which is an rv, and hence different from $E(Y \mid x)$.

Example 9.1.6 (Example 3.21)

Consider $f(x, y) = 8xy\mathbb{1}_{\{0<x<y<1\}}$. Calculate the conditional mean and the conditional variance of $X \mid \left(Y = \frac{1}{2}\right)$.

Solution

The diagram to the right illustrates the region of support for X and Y. To derive the conditional distribution, we first need $f_Y(y)$.



$$f_Y(y) = \int_0^y 8xy\,dx = 4x^2y\Big|_0^y = 4y^3$$

Thus

$$f_X\left(X \mid Y = \frac{1}{2}\right) = \frac{f\left(x, \frac{1}{2}\right)}{f_Y\left(\frac{1}{2}\right)} = \frac{4x}{\frac{1}{2}} = 8x \quad 0 < x < \frac{1}{2}$$

Therefore the conditional mean is

$$E\left[X \mid \frac{1}{2}\right] = \int_0^{\frac{1}{2}} 8x^2\,dx = \frac{8}{3} \cdot \frac{1}{2^3} = \frac{1}{3},$$

the second conditional moment is

$$E\left[X^2 \mid \frac{1}{2}\right] = \int_0^{\frac{1}{2}} 8x^3\,dx = 2 \cdot \frac{1}{2^4} = \frac{1}{8},$$

and so the conditional variance is

$$\text{Var}\left(X^2 \mid \frac{1}{2}\right) = E\left[X^2 \mid \frac{1}{2}\right] + E\left[X \mid \frac{1}{2}\right]^2 = \frac{1}{8} - \frac{1}{9} = \frac{1}{72}$$

10 Lecture 10 Jun 05th 2018

10.1 Joint Distribution (Continued 4)

10.1.1 Conditional Expectation (Continued)

Example 10.1.1 (Example 3.22)

Given the joint distribution below, calculate $\mathrm{Var}(X \mid Y = 1)$ *and compare it to* $\mathrm{Var}(X)$.

	$x = -2$	$x = 0$	$x = 2$	$P(Y = y)$
$y = 0$	0.05	0.1	0.15	0.3
$y = 1$	0.07	0.11	0.2	0.38
$y = 2$	0.02	0.25	0.05	0.32
$P(X = x)$	0.14	0.46	0.4	

Solution

Note that

$$\begin{aligned}\mathrm{Var}(X) &= E(X^2) - E(X)^2 \\ &= 4 \cdot 0.14 + 4 \cdot 0.4 - (-2 \cdot 0.14 + 2 \cdot 0.4)^2 = 1.8896\end{aligned}$$

To get $\mathrm{Var}(X \mid Y = 1)$, *we first need*

$$f(X \mid Y = 1) = \frac{P(X = x, Y = 1)}{P(Y = 1)} = \begin{cases} \frac{0.07}{0.38} = 0.1842 & x = -2 \\ \frac{0.11}{0.38} = 0.2895 & x = 0 \\ \frac{0.2}{0.38} = 0.5263 & x = 2 \end{cases}$$

Thus

$$\begin{aligned}\text{Var}(X \mid Y=1) &= E[X^2 \mid Y=1] - E[X \mid Y=1]^2 \\ &= \frac{14}{19} + \frac{40}{19} - (\frac{13}{19})^2 = \frac{857}{361} = 2.3740\end{aligned}$$

Proposition 36 (Independence on Conditional Expectation)

If X and Y are independent rvs then $E[g(Y) \mid x] = E[g(Y)]$ and $E[h(X) \mid y] = E[h(X)]$.

Proof

We shall prove one of the above for the other will follow a similar argument. Also, we shall prove the continuous case and leave the discrete case as an exercise.

Observe that

$$\begin{aligned}E[g(Y) \mid X=x] &= \int_y g(y)\frac{f(x,y)}{f_X(x)}\,dy \\ &= \int_y g(y)\frac{f_X(x)f_Y(y)}{f_X(x)}\,dy \quad \because \textit{ independence} \\ &= \int_y g(y)f_Y(y)\,dy = E[g(Y)]\end{aligned}$$

□

Exercise 10.1.1
Prove the discrete case for Proposition 36.

Theorem 37 (Law of Total Expectation)

Suppose X and Y are rvs, then

$$E\left(E[g(Y) \mid X]\right) = E[g(Y)]$$

If g is the identity function, we have $E(E[Y \mid X]) = E(Y)$.

Proof

We shall prove for the discrete case and leave the continuous case as an exercise. Observe that

Exercise 10.1.2
Prove the continuous case for Theorem 37.

$$
\begin{aligned}
E[g(Y) \mid X] &= \sum_y [g(y) \cdot P(Y = y \mid X)] \\
E[E[g(Y) \mid X]] &= \sum_x \left[\sum_y [g(y) \cdot P(Y = y \mid X)] \right] P(X = x) \\
&= \sum_x \sum_y g(y) \cdot P(X = x, Y = y) \\
&= \sum_y g(y) \sum_x P(X = x, Y = y) \\
&= \sum_y [g(y) \cdot P(Y = y)] = E[g(Y)]
\end{aligned}
$$

□

Example 10.1.2 (Example 3.23 - A Classical Example)

This is a very classical example to illustrate the power of the **Law of Total Expectation**.

A man is lost in a mine, and 3 paths are in front of him. If he takes path 1, after 3 hours, he will be back at his current place. If he takes path 2, the time to get out of the mine (in hours) follows an $\text{Exp}(1)$ *distriburion. If he takes the 3rd path, he will be back to his current place after 2 hours. Suppose that the man cannot recognize which path he took every time he comes back to the original spot (after going through either path 1 or 3), and so he randomly chooses a path every time he comes back to this original spot. What is the expected time that he will take to get out of the mine?*

Solution

Let X *an rv that represents the path number, i.e.* $X = 1, 2$ *or* 3*, and let* Y *represent the total time that the man takes to exit the mine. We are given that*

$$P(X = 1) = P(X = 2) = P(X = 3) = \frac{1}{3}.$$

We are also given that

$$
\begin{aligned}
E[Y \mid X = 1] &= 3 + E[Y] \\
E[Y \mid X = 2] &= 1 \quad \because Y \mid (X = 2) \sim \text{Exp}(1) \\
E[Y \mid X = 3] &= 2 + E[Y].
\end{aligned}
$$

Therefore, to get the expected time, by Theorem 37,

$$\begin{aligned} E[Y] &= E[E[Y \mid X]] \\ &= \frac{1}{3} \cdot E[Y \mid X = 1] + \frac{1}{3} \cdot E[Y \mid X = 2] + \frac{1}{3} \cdot E[Y \mid X = 3] \\ &= \frac{1}{3} \cdot (3 + E[Y]) + \frac{1}{3} \cdot 1 + \frac{1}{3}(2 + E[Y]) \\ &= 2 + \frac{2}{3}E[Y] \end{aligned}$$

and hence

$$E[Y] = 6$$

Theorem 38 (Law of Total Variance)

Suppose X and Y are rvs. Then

$$\mathrm{Var}(Y) = E[\mathrm{Var}(Y \mid X)] + \mathrm{Var}[E(Y \mid X)]$$

Proof

Note that

$$\begin{aligned} \mathrm{Var}(Y|X) &= E(Y^2|X) - E(Y|X)^2 \\ E[\mathrm{Var}(Y|X)] &= E[E(Y^2|X) - E(Y|X)^2] \\ &= E[Y^2] - E[E(Y|X)^2] \\ &= E[Y^2] - \left[\mathrm{Var}(E(Y|X)) + E\left(E(Y|X)\right)^2\right] \\ &= \mathrm{Var}(Y) - \mathrm{Var}(E(Y|X)) \end{aligned}$$

By rearranging the above, we get

$$\mathrm{Var}(Y) = E[\mathrm{Var}(Y|X)] + \mathrm{Var}[E(Y|X)]$$

□

Example 10.1.3 (Example 3.24 (Course Note 3.7.11))

Suppose $P \sim \mathrm{Unif}(0, 0.1)$ and $Y \mid P = p \sim \mathrm{Bin}(10, p)$. Find $E(Y)$ and $\mathrm{Var}(Y)$.

Solution

$$
\begin{aligned}
E[Y] &= E[E[Y|P]] = E[10P] = 10E[P] \\
&= 10 \cdot \int_0^{0.1} \frac{p}{0.1}\, dp = \frac{10}{0.2} p^2 \Big|_0^{0.1} = 0.05 \\
\text{Note: } E[P^2] &= \int_0^{0.1} \frac{p^2}{0.1}\, dp = \frac{p^3}{0.3}\Big|_0^{0.1} = \frac{0.001}{0.3} = \frac{1}{300} \\
\text{Var}(Y) &= E[\text{Var}(V|P)] + \text{Var}[E[Y|P]] \\
&= E[10P(1-P)] + \text{Var}[10P] \\
&= 10E[P] - 10E[P^2] + 100\,\text{Var}(P) \\
&= 0.05 - \frac{1}{30} + 100\left[E[P^2] - E[P]^2\right] \\
&= \frac{1}{60} + 100\left[\frac{1}{300} - 0.05^2\right] = \frac{1}{60} + 100\left[\frac{1}{300} - \frac{1}{400}\right] \\
&= \frac{1}{60} + \frac{1}{12} = \frac{1}{10}
\end{aligned}
$$

10.1.2 *Joint Moment Generating Functions*

Definition 42 (Joint Moment Generating Functions)

*The **joint moment generating function** of two rvs X and Y is defined as*

$$M(t_1, t_2) = E\left(e^{t_1 X + t_2 Y}\right)$$

if the expectation exists for all $t_1 \in (-h_1, h_1)$ and $t_2 \in (-h_2, h_2)$ for some $h_1, h_2 > 0$.

More generally, if $X_1, X_2, ..., X_n$ are rvs, then

$$M(t_1, t_2, ..., t_n) = E\left[\exp\left(\sum_{i=1}^{n} t_i X_i\right)\right]$$

*is called the **joint mgf** of $X_1, X_2, ..., X_n$ is the expectation exists for all $t_i \in (-h_i, h_i)$ for some $h_i > 0$, where $i = 1, ..., n$.*

Definition 43 (Joint Moments and Marginal MGF)

Given the joint mgf $M(t_1, t_2)$, we can calculate the joint moments. In

particular,

$$E\left(X^jY^k\right) = \frac{\partial^{j+k}}{\partial t_i^j \partial t_2^k} M(t_1,t_2)\Big|_{(t_1,t_2)=(0,0)}$$

If $M(t_1,t_2)$ exists for all $t_1 \in (-h_1,h_1)$ and $t_2 \in (-h_2,h_2)$ for some $h_1,h_2 > 0$, then the mdf of X is given by

$$M_X(t) = E\left(e^{tX}\right) = M(t,0) \quad t \in (-h_1,h_1)$$

and the mgf of Y is given by

$$M_Y(t) = E\left(e^{tY}\right) = M(0,t) \quad t \in (-h_2,h_2).$$

Example 10.1.4 (Example 3.25)

Given the joint distribution below, calculate the joint mgf $M(t_1,t_2)$, the first joint moment, $E[XY]$, from the joint mgf, and the marginal mgf of X and that of Y.

	x = -1	x = 1
y = 1	0.5	0.3
y = 2	0.1	0.1

Solution

Since all probabilities are provided,

$$\begin{aligned}
M(t_1,t_2) &= E\left(e^{t_1X+t_2Y}\right) = \sum_x \sum_y e^{t_1x+t_2y} P(X=x,Y=y) \\
&= 0.5e^{-t_1+t_2} + 0.3e^{t_1+t_2} + 0.1e^{-t_1+2t_2} + 0.1e^{t_1+2t_2} \\
E(XY) &= \frac{\partial^2}{\partial t_1 \partial t_2} M(t_1,t_2)\Big|_{t_1=0,\,t_2=0} \\
&= \frac{\partial}{\partial t_1}\left[0.5e^{-t_1+t_2} + 0.3e^{t_1+t_2} + 0.2e^{-t_1+2t_2} + 0.2e^{t_1+2t_2}\right]\Big|_{t_1=0,t_2=0} \\
&= -0.5e^{-t_1+t_2} + 0.3e^{t_1+t_2} - 0.2e^{-t_1+2t_2} + 0.2e^{t_1+2t_2}\Big|_{t_1=0,t_2=0} \\
&= -0.2 \\
M_X(t_1) &= M(t_1,0) = 0.5e^{-t_1} + 0.3e^{t_1} + 0.1e^{-t_1} + 0.1^{t_1} \\
&= 0.6e^{-t_1} + 0.4e^{t_1} \\
M_Y(t_2) &= M(0,t_2) = 0.5e^{t_2} + 0.3e^{t_2} + 0.1e^{2t_2} + 0.1e^{2t_2} \\
&= 0.8e^{t_2} + 0.2e^{2t_2}
\end{aligned}$$

Proposition 39 (Independence on Joint MGF)

Suppose X and Y are rvs with joint mgf $M(t_1, t_2)$ which exists $\forall t_1 \in (-h_1, h_1)$, $t_2 \in (-h_2, h_2)$, for some $h_1, h_2 > 0$. Then X and Y are independent rvs iff

$$\forall t_1 \in (-h_1, h_1),\ t_2 \in (-h_2, h_2) \quad M(t_1, t_2) = M_X(t_1) M_Y(t_2)$$

where $M_X(t_1) = M(t_1, 0)$ and $M_Y(t_2) = M(0, t_2)$.

Proof

to be proven later

Example 10.1.5 (Example 3.26 (Course Note 3.8.5))

Suppose X and Y are continous rvs with joint pdf

$$f(x, y) = e^{-y} \quad 0 < x < y < \infty$$

Find the joint mdf of X and Y. Are X and Y independent rvs? What is the marginal mgf of X and Y?

Solution

$$\begin{aligned}
M(t_1, t_2) &= E[e^{t_1 X + t_2 Y}] = \int_0^\infty \int_0^y e^{t_1 x + t_2 y} e^{-y}\, dx\, dy \\
&= \int_0^\infty \frac{1}{t_1} e^{t_1 x + t_2 y - y} \Big|_0^y \, dy \\
&= \int_0^\infty \frac{1}{t_1} \left[e^{y(t_2 - 1)} - e^{y(t_1 + t_2 - 1)} \right] dy \\
&= \frac{1}{t_1} \left[\frac{1}{t_2 - 1} e^{y(t_2 - 1)} - \frac{1}{t_1 + t_2 - 1)} e^{y(t_1 + t_2 - 1)} \right] \Big|_0^\infty \\
&= \frac{1}{t_1} \left[\frac{t_1}{(t_2 - 1)(t_1 + t_2 - 1)} \right] \\
&= \frac{1}{(t_2 - 1)(t_1 + t_2 - 1)} \quad t_2 < 1 \wedge t_1 + t_2 < 1 \\
M_X(t_1) &= M(t_1, 0) = \frac{1}{t_1 - 1} \quad t_1 < 1 \\
M_Y(t_2) &= M(0, t_2) = \frac{1}{(t_2 - 1)^2} \quad t_2 < 1
\end{aligned}$$

Observe that

$$M_X(t_1)M_Y(t_2) = \frac{1}{(t_1-1)(t_2-1)^2} \neq M(t_1,t_2)$$

and so by Proposition 39, X and Y are not independent.

Example 10.1.6 (Example 3.27)

Investigate the independence of X and Y in Example 10.1.4 using the mgf method.

Solution

We had that

$$M_X(t_1) = 0.6^{-t_1} + 0.4e^{t_1} \quad t_1 \in \mathbb{R}$$
$$M_Y(t_2) = 0.8e^{t_2} + 0.2e^{2t_2} \quad t_2 \in \mathbb{R}.$$

Since

$$M_X\left(\frac{1}{2}\right) M_Y\left(\frac{1}{2}\right) \neq M\left(\frac{1}{2},\frac{1}{2}\right)$$

we have that X and Y are not independent.

11 Lecture 11 Jun 07th 2018

11.0.1 Working with Multivariate Cases

Almost everything that has been introduced above can be extended to cases where we have more than just 2 rvs. For example:

Definition 44 (k-variate CDF)

*The **k-variate CDF**, $k > 2$, rvs $X_1, ..., X_k$ is defined as*

$$F(x_1, ..., x_k) = P(X_1 \leq x_1, X_2 \leq x_2, ..., X_k \leq x_k).$$

In the continuous case, we may write

$$f(x_1, ..., x_k) = \frac{\partial^k}{\partial x_1 \dots \partial x_k} F(x_1, ..., x_k).$$

Definition 45 (k-variate Support Set)

The support set of the distribution for $X_1, X_2, ..., X_k$ is

$$\{(x_1, ..., x_k) : f(x_1, ..., x_k) > 0\}$$

We also have the following:

Proposition 40 (Law of Total Probability - Multivariate)

If $X_1, ..., X_k$ are continuous rvs, then

$$\int_{x_1} \dots \int_{x_k} f(x_1, ..., x_k)\, dx_1 \dots dx_k = 1.$$

Should they by discrete, then

$$\sum_{x_1} \dots \sum_{x_k} f(x_1, ..., x_k) = 1$$

Definition 46 (k-Variate Marginal Distribution)

To get the marginal distribution of a subset of m variables from $X_1, ..., X_k$ ($1 \leq m \leq k$), we will sum or integrate over the other ones if they are discrete or continuous, respectively. For example,

$$f(x_1, x_2, x_3) = \int_{x_4} \dots \int_{x_k} f(x_1, ..., x_k)\, dx_4 \dots dx_k$$

Definition 47 (k-Variate Joint MGF)

The joint mgf of $X_1, ..., X_k$ is defined as

$$M(t_1, t_2, ..., t_k) = E\left(e^{t_1 X_1 + ... + t_k X_k}\right)$$

Proposition 41 (Independence for Multivariate Cases)

If $X_1, ..., X_k$ are independent, then

$$f(x_1, ..., x_k) = \prod_{i=1}^{k} f_{X_i}(x_i) \qquad F(x_1, ..., x_k) = \prod_{i=1}^{k} F_{X_i}(x_i)$$

$$M(t_1, ..., t_k) = \prod_{i=1}^{k} M_{X_i}(t_i)$$

There are many different examples of multivariate distributions. We shall discuss two:

- Multinomial Distribution
- Multivariate Normal Distribution

The **multinomial distribution** is an extension of the binomial distribution to cases where there are more categories than two results. For a multinomial distribution, we have that

- the experiment involves n trials, each with k categories
- the outcome of trials are independent of each other
- the probability of each category, p_i, remains the same across n trials
- $X = (X_1, ..., X_k) \sim \text{Mult}(n, p_1, ..., p_k)$ counts the number of elements in each category among the n trials.

Definition 48 (Mutlinomial Distribution)

Suppose $X_1, ..., X_k$ are discrete rvs with joint pf

$$f(x_1, ..., x_k) = \frac{n!}{x_1! x_2! \dots x_k! x_{k+1}!} p_1^{x_1} p_2^{x_2} \dots P_k^{x_k} p_{k+1}^{x_{k+1}}$$

where $x_i = 0, ..., n,$, $x_{k+1} = n - \sum_{i=1}^{k} x_i$, $0 < p_i < 1$, $p_{k+1} = 1 - \sum_{i=1}^{k} p_i$, for $i = 1, ..., k+1$.

Under these conditions, $(X_1, ..., X_k)$ is said to have a multinomial distribution, and we write $(X_1, ..., X_k) \sim \text{Mult}(n, p_1, ..., p_k)$.

Note

Observe that $\text{Bin}(n, p) = \text{Mult}(n, p, p)$.

Proposition 42 (Properties of Multinomial Distribution)

Suppose $(X_1, ..., X_k) \sim \text{Mult}(n, p_1, ..., p_k)$, then

1. *$\forall (t_1, ..., t_k) \in \mathbb{R}^k$, the random vector $(X_1, ..., X_k)$ has joint mgf*

$$M(t_1, ..., t_k) = E\left(e^{t_1 X_1 + ... t_k X_k}\right) = (p_1 e^{t_1} + \dots + p_k e^{t_k} + p_{k+1})^n$$

2. *Any subset of $X_1, ..., X_{k+1}$ also has a multinomial distribution. In particular, $X_i \sim \text{Bin}(n, p_i)$, $i = 1, ..., k+1$.*
3. *If $T = X_i + X_j$, for $i \neq j$, then $T \sim \text{Bin}(n, p_i + p_j)$*

4. $\text{Cov}(X_i, X_j) = -np_ip_j$, *for* $i \neq j$
5. *The conditional distribution of any subset of* $(X_1, ..., X_{k+1})$ *given the rest of the coordinates is a multinomial distribution. In particular, the conditional pf of* X_i *given* $X_j = x_j$, $i \neq j$, *is*

$$X_i \mid X_j = x_j \sim \text{Bin}\left(n - x_j, \frac{p_i}{1 - p_j}\right)$$

6. *The conditional distribution of* X_i *given* $T = X_i + X_j = t$, *for* $i \neq j$, *is*

$$X_i \mid X_i + X_j = t \sim \text{Bin}\left(t, \frac{p_i}{p_i + p_j}\right)$$

We shall look at the bivariate normal distribution so that it is easier to be explained. The same idea can be extended to a multivariate Normal distribution.

Definition 49 (Bivariate Normal Distribution)

Let X_1 *and* X_2 *be rvs with joint pdf*

$$\begin{aligned} f(x_1, x_2) &= \frac{1}{2\pi \left|\Sigma\right|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right\} \\ &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2}\right.\right. \\ &\quad \left.\left. + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right]\right\} \end{aligned}$$

where $(x_1, x_2) \in \mathbb{R}^2$ *and*

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

where Σ *is a nonsingular matrix. Then* $X = (X_1, X_2)^T$ *is said to have a bivariate normal distribution, and we write* $X \sim \text{BVN}(\mu, \Sigma)$.

Proposition 43 (Properties of Bivariate Normal Distribution)

Suppose $X \sim \text{BVN}(\mu, \Sigma)$, *where*

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

1. *X has a joint mgf of*

   $$M(t_1, t_2) = E[\exp\left(t^T X\right)] = E\left(e^{t_1 X_1 + t_2 X_2}\right) = \exp\left(\mu^T t + \frac{1}{2} t^T \Sigma t\right)$$

   $\forall (t_1, t_2) \in \mathbb{R}^2$.

2. $X_1 \sim N(\mu_1, \sigma_1^2)$ *and* $X_2 \sim N(\mu_2, \sigma_2^2)$

3. $\text{Cov}(X_1, X_2) = \rho\sigma_1\sigma_2$ *and* $\text{Corr}(X_1, X_2) = \rho$ *where* $|rho| \leq 1$

4. X_1 *and* X_2 *are independent rvs iff* $\rho = 0$

5. $c = (c_1, c_2)^T$ *is a nonzero vector of constants* $\implies$

   $$c^T X = \sum_{i=1}^{2} c_i X_i \sim N\left(c^T \mu, c^T \Sigma c\right).$$

6. *If A is a* 2×2 *nonsingular matrix and b is a* 2×1 *vector, then* $Y = AX + b \sim \text{BVN}\left(A\mu + b, A\Sigma A^T\right)$.

7. $$X_2 \mid X_1 = x_1 \sim N\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1 - \mu_1), \sigma_2^2(1 - \rho^2)\right)$$
   $$X_1 \mid X_2 = x_2 \sim N\left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2 - \mu_2), \sigma_1^2(1 - \rho^2)\right)$$

8. $(X - \mu)^T \Sigma^{-1} (X - \mu \sim \chi^2(2))$

Definition 50 (Random Sample (IID))

Rvs $X_1, ..., X_2$ *are said to form a* ***simple random sample*** *or are said to be* ***independent and identically distributed*** *(IID) if* $X_1, ..., X_n$ *are independent, and* $f_{X_i} = f_{X_j}, \forall i \neq j$.

Example 11.0.1 (Example 3.28)

Let $X_1, ..., X_{10}$ *be a random sample of standard normal distribution. Let*

Y_1 denote the number of these variables that are between -1 and 1, let Y_2 denote the number that have absolute value between 1 and 2, and let Y_3 denote the number that have absolute value larger than 2. Calculate:

1. $P(Y_1 \leq 2)$
2. $E[Y_2 \mid Y_1 = 5]$

12 Lecture 12 Jun 12th 2018

12.1 Functions of Random Variables

12.1.1 Transformation of Two or More Random Variables

In earlier lectures we discussed about basic transformations from one random variable to another, for example, from a continuous rv X to $Y = g(X)$. In particular, we methods were presented:

- The direct method, i.e. $P(Y \leq y) = P(g(X) \leq y)$, and taking the derivative of $P(Y \leq y)$ with respect to y.
- Using the mgf of Y, and then translate it as the mgf of X.

Note (Recall)

In Section 2.4.4, we used the following idea to obtain the result that we desire: for rvs X and $Y = g(X)$ where g is some continuous and injective function

$$P(Y \leq y) = P(g(X) \leq y) = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$$
$$f_Y(y) = \frac{d}{dy} F_X(g^{-1}(y)) = (g^{-1}(y))' f_X(g^{-1}(y))$$

In this chapter, we will now study the case where we have more than one rv involved. In particular, let X and Y be two continuous rvs with joint pdf $f(x, y)$. Our questions are:

1. What is the distribution of $U = h_1(X, Y)$?
2. What is the joint distribution of $U = h_1(X, Y)$ and $V = h_2(X, Y)$?

To answer the first question, we can actually still employ the direct method:

Example 12.1.1 (Example 4.1 (Course Notes 4.1.1))

Suppose X and Y are continuous rvs with joint pdf

$$f(x,y) = 3y\mathbb{1}_{0\le x\le y\le 1}$$

Find the pdf of $T = XY$.

Solution

First, note that[1]

$$P(T \le t) = P(XY \le t) = P\left(Y \le \frac{t}{X}\right)$$

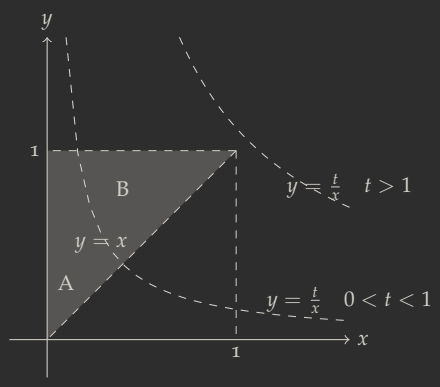
The diagram to the right shows us the support of the joint probability. We observe that if $t \le 0$, then $P(T \le t) = 0$, and if $t \ge 1$, then $P(T \le t) = 1$. Now if $0 < t < 1$, the region that we are looking for is the shaded region with the label A, and so we consider

$$\begin{aligned}
P(T \le t) &= 1 - P(B) = 1 - \int\int_B f(x,y)\,dx\,dy \\
&\overset{(1)}{=} 1 - \int_{\sqrt{t}}^1 \int_{\frac{t}{y}}^{\sqrt{t}} f(x,y)\,dx\,dy - \int_{\sqrt{t}}^1 \int_{\sqrt{t}}^{y} f(x,y)\,dx\,dy \\
&\overset{(2)}{=} 1 - \int_{\sqrt{t}}^1 \int_{\frac{t}{y}}^{y} f(x,y)\,dx\,dy \\
&= 1 - \int_{\sqrt{t}}^1 \int_{\frac{t}{y}}^{y} 3y\,dx\,dy \\
&= 1 - \int_{\sqrt{t}}^1 3y\left(y - \frac{t}{y}\right)dy \quad \because \textit{FTC} \\
&= 1 - \frac{\sqrt{t}}{1}\left(3y^2 - 3t\right)dy \\
&= 1 - \left[y^3 - 3ty\right]\Big|_{\sqrt{t}}^1 = 1 - (1 - 3t - \sqrt{t}^3 + 3t\sqrt{t}) \\
&= 3t - 2t\sqrt{t} \text{ for } 0 < t < 1
\end{aligned}$$

where for step (1), we broke B into two parts, in particular at $x = \sqrt{t}$ where $y = x$ and $y = \frac{t}{x}$ coincide, and step (2) is true by linearity of integration. With that, i.e. with the CDF of T, we can then obtain

$$f_T(t) = \begin{cases} 3 - 3\sqrt{t} & 0 < t < 1 \\ 0 & \textit{otherwise} \end{cases}$$

[1] I wonder if the last step is actually valid. The support of X definitely includes 0, so the division would not make sense with $\frac{t}{X}$. We can, however, still make sense of the event in the 2nd term, which we would have $P(0 \le t)$. Should we be concerned about $X = 0$, or can we neglect that single point given that X is a continuous rv?

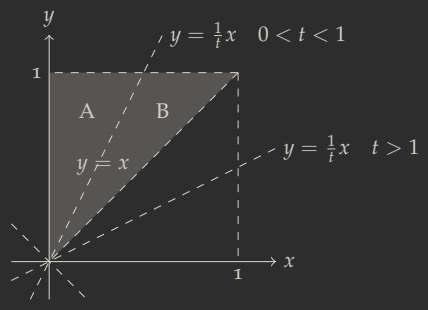


Example 12.1.2 (Example 4.2 (Course Note 4.1.2))

Using the info in Example 12.1.1, find the pdf of $T = \frac{X}{Y}$.

Solution

The diagram to the right shows the support of X, Y *and the function* $t = \frac{x}{y}$. *We observe that if* $t = 0$, *then* $x = 0$, *and we would have the line on the axis, and so* $P(T \leq t) = 0$. *If* $t < 0$, *we would have* $y = mx$ *where* $m = \frac{1}{t} < 0$, *which, regardless of what* $t < 0$ *is, will not interact with the support of* X *and* Y. *So for* $t < 0$, $P(T \leq t) = 0$. *Now if* $t > 0$, *we have*



$$P(T \leq t) = P\left(\frac{X}{Y} \leq t\right) = P\left(Y \geq \frac{1}{t}X\right).$$

Consider the case where $t \geq 1$, *we have that the event would still cover the entire support set of* X *and* Y, *and so* $P(T \leq t) = 1$ *for* $t \geq 1$. *With that, the only remaining case is when* $0 < t < 1$. *In this case,*

$$P(T \leq t) = \int_0^1 \int_0^{ty} 3y \, dx \, dy = \int_0^1 3ty^2 \, dy = t$$

and so the pdf of T *is*

$$f_T(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & otherwise \end{cases}$$

Example 12.1.3 (Example 4.3 (Course Note 4.1.3) - Order Statistics)

Suppose $X_1, ..., X_n$ *are IID samples, each from a continuous distribution, and with pdf* f *and cdf* F. *Find the pdf of*

1. $T = \min(X_1, ..., X_n) = X_{(1)}$
2. $Y = \max(X_1, ..., X_n) = X_{(n)}$

Solution

1. *For* T, *we have that its cdf is*[2]

$$\begin{aligned} P(T \leq t) &= 1 - P(T > t) = 1 - P(\min(X_1, ..., X_n) > t) \\ &= 1 - P(X_1 > t, X_2 > t, ..., X_n > t) \\ &= 1 - \prod_{i=1}^{n} P(X_i > t) \qquad \because \textit{independence} \\ &= 1 - \prod_{i=1}^{n} P(X_1 > t) \qquad \because \textit{identical distribution} \\ &= 1 - P(X_1 > t)^n = 1 - [1 - F_{X_1}(t)]^n \end{aligned}$$

[2] The use the **Law of Total Probability** here so that we can use the following argument: "the *smallest* rv is larger than t, and so rest of the rvs must be the same."

and so its pdf is

$$f_T(t) = -\frac{d}{dt}[1 - F_{X_1}(t)]^n = -n(-F_{X_1}(t))'[1 - F_{X_1}(t)]^{n-1}$$
$$= nf_{X_1}(t)[1 - F_{X_1}(t)]^{n-1}.$$

Since T relies entirely on X_1 (due to IID), and since we did not have to condition on the values of t, we have that

$$\text{supp}(T) = \text{supp}(X_1) = \text{supp}(X_i) \quad \text{for } i = 1, ..., n.$$

2. *For Y, we have that its cdf is*[3]

$$P(Y \leq t) = P(\max(X_1, ..., X_n) \leq y) = P(X_1 \leq y, ..., X_n \leq y)$$
$$= \prod_{i=1}^{n} P(X_i \leq y) \quad \because \textit{independence}$$
$$= \prod_{i=1}^{n} P(X_1 \leq y) \quad \because \textit{identical distribution}$$
$$= P(X_1 \leq t)^n = F_{X_1}(t)^n$$

and therefore its pdf is

$$f_Y(y) = \frac{d}{dy} F_{X_1}(y)^n = nf_{X_1}(y)F_{X_1}(y)^{n-1}.$$

[3] This time, we do not have to employ the **Law of Total Probability**, because we simply have that "the *largest* rv is smaller than t, and so must the rest of the rvs."

Exercise 12.1.1

From Example 12.1.3, find the joint distribution of $X_{(1)}$ and $X_{(n)}$.

12.1.2 One-to-One Bivariate Transformations

Definition 51 (One-to-One Bivariate Transformation)

Let X and Y be rvs, and $R_{XY} = \text{supp}[(X, Y)] \in \mathbb{R}^2$. We define

$$U = h_1(X, Y) \quad V = h_2(X, Y)$$
$$S : R_{XY} \to \mathbb{R}^2 \text{ by } (x, y) \mapsto (h_1(x, y), h_2(x, y))$$

*The mapping S is called a **one-to-one mapping** if and only if*[4] *$\forall (u, v) \in R_{UV}$, $\exists! (x, y) \in R_{XY}$, $\exists w_1, w_2$ that are functions such that*

$$x = w_1(u, v) \quad y = w_2(u, v)$$

[4] There is nothing magnificent about this definition, since this is simply the definition of a one-to-one function.

i.e. $\exists S^{-1} : R_{UV} \to R_{XY}$ *such that* $(u,v) \mapsto (x,y)$. *The* ***Jacobian*** *of the transformation* S^{-1} *is*

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \left| \frac{\partial(u,v)}{\partial(x,y)} \right|^{-1}$$

where $\frac{\partial(u,v)}{\partial(x,y)}$ *is the Jacobian of the transformation* S.

Theorem 44 (One-to-One Bivariate Transformations)

This is a generalization of Theorem 8. Let X *and* Y *be continuous rvs with joint pdf* f_{XY} *and let* $R_{XY} = \{(x,y) : f(x,y) > 0\}$ *be the support set of* (X,Y), *and* R_{UV} *be the support set of* (U,V). *Suppose the transformation* $S : R_{XY} \to R_{UV}$ *defined by*

$$U = h_1(X,Y) \quad V = h_2(X,Y)$$

is a one-to-one transformation, with inverse transformation

$$X = w_1(U,V) \quad Y = w_2(U,V).$$

Then $g(u,v)$, *the joint pdf of* U *and* V, *is given by*

$$\forall (u,v) \in R_{UV} \quad g(u,v) = f(w_1(u,v), w_2(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|.$$

Proof

Let the inverse transformation be labelled as $S^{-1} : R_{UV} \supset B \to A \subset R_{XY}$. *Then*

$$\begin{aligned} \int_B \int g(u,v)\, dv\, du &= P[(U,V) \in B] = P[(X,Y) \in A] \\ &= \int_A \int f(x,y)\, dx\, dy \\ &= \int_B \int f(w_1(u,v), w_2(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv \end{aligned}$$

where the last step is by the Change of Variables Theorem. And so by

comparing integrands, we have

$$\forall (u,v) \in R_{UV} \quad g(u,v) = f(w_1(u,v), w_2(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

as required. □

Proposition 45 (Properties of the Jacobian)

Given the setup in Definition 51, we have that

1. *if S is a linear transformation, i.e. $\exists a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}$ such that $u(x,y) = a_1x + b_1y + c_1$ and $v(x,y) = a_2x + b_2y + c_2$, then the Jacobian is a constant;*
2. *if S is a one-to-one transformation, then $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| \neq 0$*

Proof

1. *We have*

$$\frac{\partial u}{\partial x} = a_1 \qquad \frac{\partial u}{\partial y} = b_1$$
$$\frac{\partial v}{\partial x} = a_2 \qquad \frac{\partial v}{\partial y} = b_2$$

   and so

$$|J| = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

   which is a constant, as required.

2. ***I have no idea how to prove this.***

Example 12.1.4 (Example 4.4 (Course Notes 4.2.4))

Suppose $X \sim \text{Gam}(a,1)$ and $Y \sim \text{Gam}(b,1)$ independently. Find the joint pdf of $U = X + Y$ and $V = \frac{X}{X+Y}$. Show that $U \sim \text{Gam}(a+b,1)$ and $V \sim \text{Beta}(a,b)$, independently. Find $E(V)$.

Solution

Given $U = X + Y$ and $V = \frac{X}{X+Y}$, rearranging variables, we have

$$X = UV \text{ and } Y = U(1-V)$$

In order for U to have a Gamma distribution, we need U to be non-negative, which we do since both X and Y have Gamma distributions. Note that the transformation is indeed one to one, since $\forall (u_1, v_1), (u_2, v_2) \in R_{UV}$ with

$$\begin{aligned} u_1 &= x_1 + y_1 \quad v_1 = \frac{x_1}{x_1 + y_1} \\ u_2 &= x_2 + y_2 \quad v_2 = \frac{x_2}{x_2 + y_2}, \end{aligned}$$

we have that, if we let ϕ denote the transformation,

$$\begin{aligned} &\phi(u_1, v_1) = \phi(u_2, v_2) \\ \implies &\left(x_1 + y_1, \frac{x_1}{x_1 + y_1}\right) = \left(x_2 + y_2, \frac{x_2}{x_2 + y_2}\right) \end{aligned}$$

which then

$$\begin{aligned} x_1 + y_1 &= x_2 + y_2 \qquad (12.1) \\ \frac{x_1}{x_2 + y_2} &= \frac{x_2}{x_2 + y_2} \\ \overset{\text{Equation (12.1)}}{\implies} x_1 &= x_2 \\ \overset{\text{Equation (12.1)}}{\implies} y_1 &= y_2. \end{aligned}$$

We shall now get the Jacobian so that we may use Theorem 44, so that we may consequently get the distributions for U and V.

$$\begin{aligned} J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1 - v & -u \end{vmatrix} &= -vu - u(1 - v) = -u \\ |J| &= u \end{aligned}$$

By Theorem 44, and since X and Y are independent, we have

$$\begin{aligned} f_{U,V}(u, v) &= f_{X,Y}(x, y) \cdot |J| = f_X(x) f_Y(y) \cdot |J| \\ &= \frac{x^{a-1} e^{-x}}{\Gamma(a)} \frac{y^{b-1} e^{-y}}{\Gamma(b)} \cdot |J| \\ &= \frac{e^{-a} e^{-b}}{\Gamma(a)\Gamma(b)} (uv)^{a-1} u^{b-1} (1 - v)^{b-1} u \\ &= \underbrace{\frac{u^{a+b-1} e^{-(a+b)}}{\Gamma(a + b)}}_{\text{pdf of } \mathrm{Gam}(a+b,1)} \cdot \underbrace{\frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} v^{a-1} (1 - v)^{b-1}}_{\text{pdf of } (a,b)} \end{aligned}$$

We have already shown that U is a non-negative rv, and so $U \sim \mathrm{Gam}(a + b, 1)$ as required. Note that for V, we have $X + Y > X > 0 \implies 1 > \frac{X}{X+Y} > 0 \quad \because X + Y \neq 0$ and so $0 < V < 1$. Therefore $V \sim \mathrm{Beta}(a, b)$.

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