PMATH450 — Lebesgue Integration and Fourier Analysis

by

Johnson Ng

BMath (Hons), Pure Mathematics major, Actuarial Science Minor

University of Waterloo

Table of Contents

List of Definitions

List of Theorems

List of Procedures

🍃 *Preface*

The pre-requisite to this course is Real Analysis. We will use a lot of the concepts introduced in Real Analysis, at times without explicitly stating it. Refer to notes on PMATH351.

This course is spiritually broken into 2 pieces:

- Lebesgue Integration; and

- Fourier Analysis,

which is as the name of the course.

In this set of notes, we use a special topic environment called **culture** to discuss interesting contents related to the course, but will not be throughly studied and not tested on exams.

1 🍃 *Lecture 1 May 07th 2019*

Since many of our results work for both $\mathbb{C}$ and $\mathbb{R}$, we shall use $\mathbb{K}$ throughout this course to represent either $\mathbb{C}$ or $\mathbb{R}$.

1.1 *Riemannian Integration*

▤ Definition 1 (Norm and Semi-Norm)

*Let V be a vector space over $\mathbb{K}$. We define a **semi-norm** on V as a function*

$$\nu : V \to \mathbb{R}$$

that satisfies

1. *(**Positive Semi-Definite**) $\nu(x) \geq 0$ for all $x \in V$;*

2. *$\nu(\kappa x) = |\kappa|\, \nu(x)$ for any $\kappa \in \mathbb{K}$ and $x \in V$; and*

3. *(**Triangle Inequality**) $\nu(x + y) \leq \nu(x) + \nu(y)$ for all $x, y \in V$.*

*If $\nu(x) = 0 \implies x = 0$, then we say that ν is a **norm**. In this case, we usually write $\|\cdot\|$ to denote the norm, instead of ν.*

❝ Note 1.1.1

- *We sometimes call a semi-norm a **pseudo-length**.*

Remark 1.1.1

Notice that we wrote $v(x) = 0 \implies x = 0$ instead of $v(x) = 0 \iff x = 0$. This is because if $z = 0 \in V$, then

$$v(z) = v(0z) = 0.$$

💬

Exercise 1.1.1

Show that if v is a semi-norm on a vector space V, then $\forall x, y \in V$,

$$|v(x) - v(y)| \leq v(x - y).$$

✏ Proof

Notice that by condition (2) and (3), we have

$$v(x - y) \leq v(x) + v(-y) = v(x) - v(y),$$

and

$$v(x - y) = -v(y - x) \geq -(v(y) - v(x)) = v(x) - v(y).$$

It follows that indeed

$$|v(x) - v(y)| \leq v(x - y).$$

□

Example 1.1.1

The absolute value $|\cdot|$ is a **norm** on $\mathbb{K}$.   🔨

Example 1.1.2 (*p*-norms)

Consider $N \geq 1$ an integer. We define a family of norms on

$$\mathbb{K}^N = \underbrace{K \times K \times \ldots \times K}_{N \text{ times}}.$$

1-norm

$$\left\| (x_n)_{n=1}^{N} \right\|_1 := \sum_{n=1}^{N} |x_n|.$$

Infinity-norm, ∞-norm

$$\left\| (x_n)_{n=1}^N \right\|_\infty := \max_{1 \leq n \leq N} |x_n|.$$

Euclidean-norm, 2-norm

$$\left\| (x_n)_{n=1}^N \right\|_2 := \left(\sum_{n=1}^N |x_n|^2 \right)^{\frac{1}{2}}$$

It is relatively easy to check that the above norms are indeed norms, except for the 2-form. In particular, the **triangle inequality** is not as easy to show [1].

[1] See Minkowski's Inequality.

Less obviously so, but true nonetheless, we can define the following p-norms on $\mathbb{K}^N$:

$$\left\| (x_n)_{n=1}^N \right\|_p := \left(\sum_{n=1}^N |x_n|^p \right)^{\frac{1}{p}},$$

for $1 \leq p < \infty$.

🏆 Culture

*Consider $V = \mathbb{M}_n(\mathbb{C})$, [2] where $n \in \mathbb{N}$ is fixed. For $T \in \mathbb{M}_n(\mathbb{C})$, we define the **singular numbers** of T to be*

[2] Note that $\mathbb{M}_n(\mathbb{C})$ is the set of $n \times n$ matrices over $\mathbb{C}$.

$$s_1(T) \geq s_2(T) \geq \ldots \geq s_n(T) \geq 0,$$

where $\sigma(T^*T) = \{s_1(T)^2, s_2(T)^2, \ldots, s_n(T)^2\}$, including multiplicity. Then we can define

$$\|T\|_p := \left(\sum_{i=1}^n s_i(T)^p \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$, which is called the p-norm of T on $\mathbb{M}_n(\mathbb{C})$.

Example 1.1.3

Let

$$V = \mathcal{C}([0,1], \mathbb{K}) = \{f : [0,1] \to \mathbb{K} \mid f \text{ is continuous }\}.$$

Then

$$\|f\|_{\text{sup}} := \sup\{|f(x)| \mid x \in [0,1]\}$$

[3] defines a norm on $\mathcal{C}([0,1], \mathbb{K})$.

A sequence $(f_n)_{n=1)^\infty}$ in V converges in this norm to some $f \in V$, i.e.

$$\lim_{n \to \infty} \|f_n - f\|_{\text{sup}} = 0,$$

which means that $(f_n)_{n=1}^\infty$ converges uniformly to f on $[0,1]$. 🔨

[3] Some authors use $\|f\|_\infty$, but we will have the notation $\|[f]\|_\infty$ later on, and so we shall use $\|f\|_{\text{sup}}$ for clarity.

📘 Definition 2 (Normed Linear Space)

A normed linear space (NLS) is a pair $(V, \|\cdot\|)$ where V is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is a norm on V.

📘 Definition 3 (Metric)

Given an NLS $(V, \|\cdot\|)$, we can define a metric d on V (called the metric induced by the norm) as follows:

$$d : V \times V \to \mathbb{R} \quad d(x,y) = \|x - y\|,$$

such that

- *$d(x,y) \geq 0$ for all $x,y \in V$ and $d(x,y) = 0 \iff x = y$;*

- *$d(x,y) = d(y,x)$; and*

- *$d(x,y) \leq d(x,z) + d(y,z)$.*

🗨 Note 1.1.2

Norms are all metrics, and so any space that has a norm will induce a metric on the space.

📘 Definition 4 (Banach Space)

We say that an NLS $(V, \|\cdot\|)$ is **complete** or is a *Banach Space* if the corresponding (V, d), where d is the metric induced by the norm, is complete [4].

Example 1.1.4

$(\mathcal{C}([0,1], \mathbb{K}), \|\cdot\|_{\sup})$ is a Banach space. 🔨

Example 1.1.5

We can define a 1-norm $\|\cdot\|_1$ on $\mathcal{C}([0,1], \mathbb{K})$ via

$$\|f\|_1 := \int_0^1 |f|.$$

Then $(\mathcal{C}([0,1], \mathbb{K}), \|\cdot\|_1)$ is an NLS. 🔨

Exercise 1.1.2

*Show that $(\mathcal{C}([0,1], \mathbb{K}), \|\cdot\|_1)$ is not complete, which will then give us an example of a **normed linear space that is not Banach**.*

Figure 1.1: Sequence of functions $(f_n)_{n=1}^{\infty}$. We show for two indices $n < m$.

✏️ Proof

Consider the sequence $(f_n)_{n=1}^{\infty}$ of continuous functions given by

$$f_n(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2} \\ n\left(x + \frac{1}{2}\right) & \frac{1}{2} \le x \le \frac{1}{2} + \frac{1}{n} \\ 1 & \text{otherwise} \end{cases}$$

Note that the sequence $(f_n)_{n=1}^{\infty}$ is indeed **Cauchy**: let $\varepsilon > 0$ and $|n - m| < \frac{\varepsilon}{|x - \frac{1}{2}|}$, and then we have

$$|f_n(x) - f_m(x)| = \left| n\left(x - \frac{1}{2}\right) - m\left(x - \frac{1}{2}\right) \right|$$
$$= \left| (n - m)\left(x - \frac{1}{2}\right) \right| = |n - m|\left|x - \frac{1}{2}\right| < \varepsilon.$$

However, it is clear that the sequence $(f_n)_{n=1}^{\infty}$ converges to the
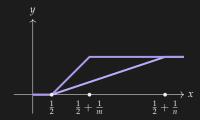
piecewise function (in particular, a non-continuous function)

$$f(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}.$$

□

Example 1.1.6

If $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ are NLS's, and if $T : \mathfrak{X} \to \mathfrak{Y}$ is a linear map, we define the **operator norm** of T to be

$$\|T\| := \sup\{\|T(x)\|_{\mathfrak{Y}} \mid \|x\|_{\mathfrak{X}} \leq 1\}.$$

We set

$$B(\mathfrak{X}, \mathfrak{Y}) := \{T : \mathfrak{X} \to \mathfrak{Y} \mid T \text{ is linear }, \|T\| < \infty\}.$$

Note that for any such linear map T, $\|T\| < \infty \iff T$ is continuous. Thus $B(\mathfrak{X}, \mathfrak{Y})$ is the set of all continuous functions from $\mathfrak{X}$ into $\mathfrak{Y}$.

Then $(B(\mathfrak{X}, \mathfrak{Y}), \|\cdot\|)$ is an NLS.   🔨

It is likely that we have seen this in Real Analysis.

Exercise 1.1.3

Show that $(B(\mathfrak{X}, \mathfrak{Y}), \|\cdot\|)$ is complete iff $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ is complete.

❝ Note 1.1.3

*One example of the last example is when $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}}) = (\mathbb{K}, |\cdot|)$. In this case, $B(\mathfrak{X}, \mathbb{K})$ is known as the **dual space** of $\mathfrak{X}$, or simple the **dual** of $\mathfrak{X}$.*

We are interested in integrating over Banach spaces.

📄 Definition 5 (Partition of a Set)

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a,b] \to \mathfrak{X}$ a function, where $a < b \in \mathbb{R}$. A partition P of $[a,b]$ is a finite set

$$P = \{a = p_0 < p_1 < \ldots < p_N = b\}$$

for some $N \geq 1$. The set of all partitions of $[a,b]$ is denoted by $\mathcal{P}[a,b]$.

📄 Definition 6 (Test Values)

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a,b] \to \mathfrak{X}$ a function, where $a < b \in \mathbb{R}$. Let $P \in \mathcal{P}[a,b]$. A set

$$P^* := \{p_k^*\}_{k=1}^N$$

satisfying

$$p_{k-1} \leq p_k^* \leq p_k, \text{ for } 1 \leq k \leq n$$

is called a set of test values for P.

📄 Definition 7 (Riemann Sum)

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a,b] \to \mathfrak{X}$ a function, where $a < b \in \mathbb{R}$. Let $P \in \mathcal{P}[a,b]$ and P^* its corresponding set of test values. We define the Riemann sum as

$$S(f, P, P^*) = \sum_{k=1}^N f(p_k^*)(p_k - p_{k-1}).$$

Remark 1.1.2

1. *Note that because* 📄 *Definition 5, $p_k - p_{k-1} > 0$.*

2. *When $(\mathfrak{X}, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, then this is the usual Riemann sum from first-year calculus.*

3. *In general, note that*

$$\frac{1}{b-a} S(f, P, P^*) = \sum_{k=1}^N \lambda_k f(p_k^*),$$

where $0 < \lambda_k = \frac{p_k - p_{k-1}}{b-a} < 1$ and [5]

$$\sum_{k=1}^N \lambda_k = 1.$$

[5] via the fact that the λ_k's form a telescoping sum

So $\frac{1}{b-a}S(f, P, P^*)$ is an *averaging* of f over $[a, b]$. We call $\frac{1}{b-a}S(f, P, P^*)$ the *convex combination* of the $f(p_k^*)$'s.

Example 1.1.7 (Silly example)

Let $(\mathfrak{X} = \mathcal{C}([-\pi, \pi], \mathbb{K}), \|\cdot\|_{\text{sup}})$. Let

$$f : [0, 1] \to \mathfrak{X} \text{ such that } x \mapsto e^{2\pi x} \sin 7\theta + \cos x \cos(12\theta),$$

where $\theta \in [-\pi, \pi]$. Now if we consider the partition

$$P = \left\{ -\pi, \frac{1}{10}, \frac{1}{2}, \pi \right\}$$

and its corresponding test value

$$P^* = \left\{ 0, \frac{1}{3}, 2 \right\},$$

then

$$S(f, P, P^*) = f(0)\left(\frac{1}{10} + \pi\right) + f\left(\frac{1}{3}\right)\left(\frac{1}{2} - \frac{1}{10}\right) + f(2)\left(\pi - \frac{1}{2}\right)$$

$$= (\sin 7\theta + \cos 12\theta)\left(\pi + \frac{1}{10}\right)$$

$$+ \left(e^{\frac{2\pi}{3}} \sin 7\theta + \cos \frac{1}{3} \cos 12\theta\right)\left(\frac{2}{5}\right)$$

$$+ (e^{4\pi} \sin 7\theta + \cos 2 \cos 12\theta)\left(\pi - \frac{1}{2}\right)$$

📑 **Definition 8 (Refinement of a Partition)**

Let $a < b \in \mathbb{R}$, and $P \in \mathcal{P}[a, b]$. We say Q is a *refinement* of P is $Q \in \mathcal{P}[a, b]$ and $P \subseteq Q$.

❝ **Note 1.1.4**

In simpler words, Q is a "finer" partition that is based on P.

📑 **Definition 9 (Riemann Integrable)**

Let $a < b \in \mathbb{R}$, $(\mathfrak{X}, \|\cdot\|_\mathfrak{X})$ be a Banach space and $f : [a,b] \to \mathfrak{X}$ be a function. We say that f is Riemann integrable *over $[a,b]$ if $\exists x_0 \in \mathfrak{X}$ such that*

$$\forall \varepsilon > 0 \quad \exists P \in \mathcal{P}[a,b],$$

such that if Q is any refinement of P, and Q^* is any set of test values of Q, then

$$\|x_0 - S(f, Q, Q^*)\|_\mathfrak{X} < \varepsilon.$$

In this case, we write

$$\int_a^b f = x_0.$$

💧 Proposition 1 (Uniqueness of the Riemann Integral)

If f is Riemann integrable over $[a,b]$, then the value of $\int_a^b f$ is unique.

✏️ Proof

Suppose not, i.e.

$$\int_a^b f = x_0 \text{ and } \int_a^b f = y_0$$

for some $x_0 \neq y_0$. Then, let

$$\varepsilon = \frac{\|x_0 - y_0\|}{2},$$

which is > 0 since $\|x_0 - y_0\| > 0$. Let $P_{x_0}, P_{y_0} \in \mathcal{P}[a,b]$ be partitions corresponding to x_0 and y_0 as in the definition of Riemann integrability.

Then, let $R = P_{x_0} \cup P_{y_0}$, so that R is a **common refinement** of P_{x_0} and P_{y_0}. If Q is any refinement of R, then Q is also a common refinement of P_{x_0} and P_{y_0}. Then for any test values Q^* of Q, we have

$$2\varepsilon = \|x_0 - y_0\|$$
$$\leq \|x_0 - S(f, Q, Q^*)\| + \|S(f, Q, Q^*) - y_0\| < \varepsilon + \varepsilon = 2\varepsilon,$$

which is a contradiction.

Thus $x_0 = y_0$ as required.

□

📖 Theorem 2 (Cauchy Criterion of Riemann Integrability)

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space, $a < b \in \mathbb{R}$ and $f : [a,b] \to \mathfrak{X}$ be a function. TFAE:

1. *f is Riemann integrable over $[a,b]$;*

2. *$\forall \varepsilon > 0$, $R \in \mathcal{P}[a,b]$, if P,Q is any refinement of R, and P^* (respectively Q^*) is any test values of P (respectively Q), then*

$$\|S(f,P,P^*) - S(f,Q,Q^*)\|_{\mathfrak{X}} < \varepsilon.$$

✏️ Proof

⟹ This is a rather straightforward proof. Suppose $P, Q \in \mathcal{P}[a,b]$ is some refinement of the given partition $R \in \mathcal{P}[a,b]$, and P^*, Q^* any test values for P, Q, respectively. Then by assumption and 💧 Proposition 1, $\exists x_0 \in \mathfrak{X}$ such that

$$\|x_0 - S(f,P,P^*)\|_{\mathfrak{X}} < \frac{\varepsilon}{2} \text{ and } \|x_0 - S(f,Q,Q^*)\|_{\mathfrak{X}} < \frac{\varepsilon}{2}.$$

It follows that

$$
\begin{aligned}
&\|S(f,P,P^*) - S(f,Q,Q^*)\|_{\mathfrak{X}} \\
&\leq \|x_0 - S(f,P,P^*)\|_{\mathfrak{X}} + \|x_0 - S(f,Q,Q^*)\|_{\mathfrak{X}} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

⟸ By hypothesis, wma $\varepsilon = \frac{1}{n}$ for some $n \geq 1$, such that if P,Q are any refinements of the partition $R_n \in \mathcal{P}[a,b]$, and P^*, Q^* are the respective arbitrary test values, then

$$\|S(f,P,P^*) - S(f,Q,Q^*)\|_{\mathfrak{X}} < \frac{1}{n}$$

Now for each $n \geq 1$, define

$$W_n := \bigcup_{k=1}^{n} R_k \in \mathcal{P}[a,b],$$

so that W_n is a common refinement for $R_1, R_2, \ldots, R_n$. For each $n \geq 1$, let W_n^* be an arbitrary set of test values for W_n. For simplicity, let us write

$$x_n = S(f, W_n, W_n^*), \text{ for each } n \geq 1.$$

6

Claim: $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence If $n_1 \geq n_2 > N \in \mathbb{N}$, then

$$\|x_{n_1} - x_{n_2}\|_{\mathfrak{X}} = \left\|S(f, W_{n_1}, W_{n_1}^*) - S(f, W_{n_2}, W_{n_2}^*)\right\| < \frac{1}{N}$$

by our assumption, since W_{n_1}, W_{n_2} are refinements of R_N. Then by picking $N = \frac{1}{\varepsilon}$ for any $\varepsilon > 0$, we have that $(x_n)_{n=1}^{\infty}$ is indeed a Cauchy sequence in $\mathfrak{X}$.

Since $\mathfrak{X}$ is a Banach space, it is complete, and so $\exists x_0 := \lim_{n \to \infty} x_n \in \mathfrak{X}$. It remains to show that, indeed,

$$x_0 = \int_a^b f.$$

Let $\varepsilon > 0$, and choose $N \geq 1$ such that

- $\frac{1}{N} < \frac{\varepsilon}{2}$; and

- $k \geq N$ implies that $\|x_k - x_0\| < \frac{\varepsilon}{2}$.

Then suppose that V is any refinement of W_N, and V^* is an arbitrary set of test values of V. Then we have

$$
\begin{aligned}
\|x_0 - S(f, V, V^*)\|_{\mathfrak{X}} &\leq \|x_0 - x_N\|_{\mathfrak{X}} + \|x_N - S(f, V, V^*)\|_{\mathfrak{X}} \\
&< \frac{\varepsilon}{2} + \|S(f, W_N, W_N^*) - S(f, V, V^*)\|_{\mathfrak{X}} \\
&< \frac{\varepsilon}{2} + \frac{1}{N} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

It follows that

$$\int_a^b f = x_0,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

[6] Note that it would be nice if for the finer and finer partitions that we have constructed, i.e. the W_n's, give us a convergent sequence of Riemann sums, since it makes sense that this convergence will give us the final value that we want.

In first-year calculus, all continuous functions over $\mathbb{R}$ are integrable. A similar result holds in Banach spaces as well. In the next lecture, we shall prove the following theorem.

☕ **Theorem (Continuous Functions are Riemann Integrable)**

Let $(\mathfrak{X}, \|\cdot\|)$ be a Banach space and $a < b \in \mathbb{R}$. If $f : [a,b] \to \mathfrak{X}$ is continuous, then f is Riemann integrable over $[a,b]$.

2.1   *Riemannian Integration (Continued)*

We shall now prove the last theorem stated in class.

🖥 Theorem 3 (Continuous Functions are Riemann Integrable)

Let $(\mathfrak{X}, \|\cdot\|)$ be a Banach space and $a < b \in \mathbb{R}$. If $f : [a,b] \to \mathfrak{X}$ is continuous, then f is Riemann integrable over $[a,b]$.

✈ Strategy

This is rather routine should one have gone through a few courses on analysis, and especially on introductory courses that involves Riemannian integration.

We shall show that if $P_N \in \mathcal{P}[a,b]$ is a partition of $[a,b]$ into 2^N subintervals of equal length $\frac{b-a}{2^N}$, and if we use $P_N^* = P_n \setminus \{a\}$ as the set of test values for P_N, which consists of the right-endpoints of each the subintervals in P_N, then the sequence $(S(f, P_N, P_N^*))_{N=1}^{\infty}$ converges in $\mathfrak{X}$ to $\int_a^b f$.

Note that this choice of partition is a valid move, since any of these P_N's, for different N's, is a refinement of some other partition of $[a,b]$, and if we choose a different set of test values, then we may as well consider an even finer partition.

✏ Proof

First, note that since $[a,b]$ is closed and bounded in $\mathbb{R}$, it is compact. Also, we have that X is a metric space (via the metric induced by the norm). This means that **any continuous function f on $[a,b]$ is uniformly continuous on $[a,b]$**. In other words,

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x,y \in [a,b]$$

$$|x - y| < \delta \implies \|f(x) - f(y)\| < \frac{\varepsilon}{2(b-a)}.$$

$\boxed{\text{Claim: } (S(f, P_N, P_N^*))_{N=1}^{\infty} \text{ is Cauchy}}$ Now by picking $P_N \in \mathcal{P}[a,b]$ and set of test values P_N^* as described in the strategy above, we proceed by picking $M > 0$ such that $\frac{b-a}{2^M} < \delta$. Then for any $K \geq L \geq M$, since each of the subintervals have length $\frac{b-a}{2^L}$ and $\frac{b-a}{2^K}$ for P_L and P_K respectively, if we write

$$P_L = \{a = p_0 < p_1 < \ldots < p_{2^L} = b\}$$

and

$$P_K = \{a = q_0 \leq q_1 < \ldots < q_{2^K} = b\},$$

then $p_j = q_j 2^{K-L}$ [1] for all $0 \leq j \leq 2^L$. By uniform continuity, for $1 \leq j \leq 2^L$, wma

$$\left\| f(p_j^*) - f(q_s^*) \right\| < \frac{\varepsilon}{2(b-a)}, \text{ where } (j-1)2^{K-L} < s \leq j2^{K-L}.$$

[1] This is not immediately clear on first read. Think of a as 0.

We can see that

$$\|S(f, P_L, P_L^*) - S(f, P_K, P_K^*)\|$$

$$= \left\| \sum_{j=1}^{2^L} \sum_{s=(j-1)2^{K-L}+1}^{j2^{K-L}} (f(p_j) - f(q_s))(q_s - q_{s-1}) \right\|$$

$$\leq \sum_{j=1}^{2^L} \sum_{s=(j-1)2^{K-L}+1}^{j2^{K-L}} \|f(p_j) - f(q_s)\| (q_s - q_{s-1})$$

$$\leq \sum_{j=1}^{2^L} \sum_{s=(j-1)2^{K-L}+1}^{j2^{K-L}} \frac{\varepsilon}{b-a}(q_s - q_{s-1})$$

$$= \frac{\varepsilon}{b-a} \sum_{s=1}^{2^K} (q_s - q_{s-1})$$

$$= \frac{\varepsilon}{2(b-a)}(b-a) = \frac{\varepsilon}{2}.$$

This proves our claim.

Since $\mathfrak{X}$ is a Banach space, and hence complete, we have that the sequence $(S(f, P_N, P_N^*))_{N=1}^{\infty}$ has a limit $x_0 \in \mathfrak{X}$.

It remains to show that $\int_a^b f = x_0$. [2]

Let $\varepsilon > 0$, and choose $T \geq 1$ such that $\frac{b-a}{2^T} < \delta$ [3], so that we have

$$\|x_0 - S(f, P_T, P_T^*)\| < \frac{\varepsilon}{2}.$$

Now let $R = \{a = r_0 < r_1 < \ldots < r_J = b\} \in \mathcal{P}[a,b]$ such that $P_T \subseteq R$. Then there exists a sequence

$$0 = j_0 < j_1 < \ldots < j_{2^T} = J$$

such that

$$r_{j_k} = p_k, \text{ where } 0 \leq k \leq 2^T.$$

Let R^* be any set of test values of R. Note that for $j_{k-1} \leq s \leq j_k$, it is clear that

$$|p_k^* - r_s^*| \leq |p_k - p_{k-1}| = \frac{b-a}{2^T} < \delta.$$

Thus

$$\|S(f, P_T, P_T^*) - S(f, R, R^*)\|$$
$$\leq \sum_{k=1}^{2^T} \sum_{s_{j_{k-1}+1}}^{j_k} \|f(p_k^*) - f(r_s^*)\| \, (r_s - r_{s-1})$$
$$< \frac{\varepsilon}{2(b-a)} \sum_{k=1}^{2^T} \sum_{s_{j_{k-1}+1}}^{j_k} (r_s - r_{s-1})$$
$$= \frac{\varepsilon}{2(b-a)} (b-a) = \frac{\varepsilon}{2}.$$

Putting everything together, we have

$$\|x_0 - S(f, R, R^*)\|$$
$$\leq \|x_0 - S(f, P_T, P_T^*)\| + \|S(f, P_T, P_T^*) - S(f, R, R^*)\|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We can also find another refinement of P_T, say Q, that works similarly as in the case of R. It follows from ☕Theorem 2 that

$$x_0 = \int_a^b f,$$

i.e. that f is indeed Riemann integrable over $[a, b]$. □

The following is a corollary whose proof shall be left as an exercise.

🏹 Corollary 4 (Piecewise Functions are Riemann Integrable)

*A **piecewise continuous** function is also Riemann integrable: if f : $[a, b] \to \mathcal{X}$ is piecewise continuous, then f is Riemann integrable.*

Exercise 2.1.1

Prove 🏹*Corollary 4.*

LET US exhibit a function that is not Riemann integrable.

📃 Definition 10 (Characteristic Function)

*Given a subset E of a set $\mathbb{R}$, we define the **characteristic function** of E as a function $\chi_E : \mathbb{R} \to \mathbb{R}$ given by*

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

Example 2.1.1

Consider the set $E = \mathbb{Q} \cap [0, 1] \subseteq \mathbb{R}$. Let $P \in \mathcal{P}[0, 1]$ such that

$$P = \{0 = p_0 < p_1 < \ldots < p_N = 1\},$$

and let
$$P^* = \{p_k^*\}_{k=1}^N \text{ and } P^{**} = \{p_k^{**}\}_{k=1}^N$$

be 2 sets of test values for P, such that we have

$$p_k^* \in \mathbb{Q} \text{ and } p_k^{**} \in \mathbb{R} \setminus \mathbb{Q}.$$

Then we have

$$\begin{aligned}
S(\chi_E, P, P^*) &= \sum_{k=1}^N \chi_E(p_k^*)(p_k - p_{k-1}) \\
&= \sum_{k=1}^N 1 \cdot (p_k - p_{k-1}) \\
&= p_N - p_0 = 1 - 0 = 1,
\end{aligned}$$

and

$$\begin{aligned}
S(\chi_E, P, P^{**}) &= \sum_{k=1}^N \chi_E(p_k^{**})(p_k - p_{k-1}) \\
&= \sum_{k=1}^N 0 \cdot (p_k - p_{k-1}) \\
&= 0.
\end{aligned}$$

It is clear that the Cauchy criterion fails for χ_E. This shows that χ_E is not Riemann integrable.

Remark 2.1.1

*Let us once again consider $E = \mathbb{Q} \cap [0,1]$. Note that E is **denumerable** [4]. We may thus write*

$$E = \{q_n\}_{n=1}^\infty.$$

Now, for $k \geq 1$, define

$$f_k(x) = \sum_{n=1}^k \chi_{\{q_n\}}(x).$$

In other words, $f_k = \chi_{\{q_1,\dots,q_k\}}$. Furthermore, we have that

$$f_1 \leq f_2 \leq f_3 \dots \leq \chi_E.$$

[4] This means that E is countably infinite.

Moreover, we have that $\forall x \in [0, 1]$,

$$\chi_E(x) = \lim_{k \to \infty} f_k(x),$$

and

$$\int_0^1 f_k = 0 \text{ for all } k \geq 1.$$

And yet, we have that $\int_0^1 \chi_E$ *does not exist!*   💬

WE WANT TO develop a different integral that will 'cover' for this 'pathological' behavior of where the Riemann integral fails.

The   rough idea   is as follows.

In Riemann integration, when integrating over an interval $[a, b]$, we partitioned $[a, b]$ into subintervals. This happens on the x-axis.
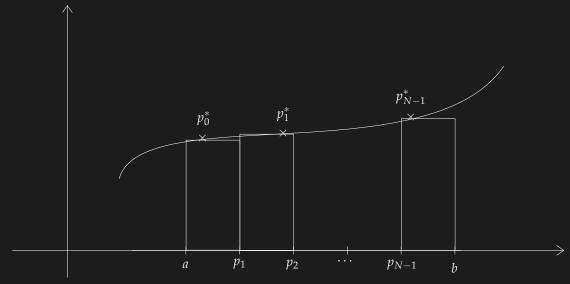
Figure 2.1: Rough illustration of how Riemann's integration works
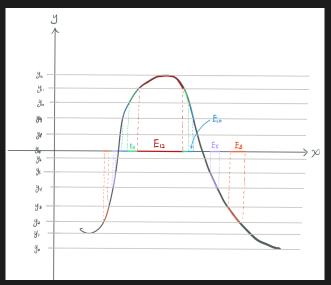
In each of the subintervals of the partition, we pick out a **test value** p_i^*, and basically draw a rectangle with base at $[p_i, p_{i+1}]$ and height from 0 to p_i^*.

What we shall do now is that we **partition the range of f on the y-axis**, instead of the x-axis as we do in Riemannian integration.

In particular, given a function $f : [a, b] \to \mathbb{R}$, we first partition the

range of f into subintervals $[y_{k-1}, y_k]$, where $1 \leq k \leq N$. Then, we set

$$E_k = \{x \in [a,b] : f(x) \in [y_{k-1}, y_k]\} \text{ for } 1 \leq k \leq N.$$

Figure 2.2: A sketch of what's happening with the construction of the E_k's
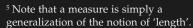
This will then allow us to estimate the integral of f over $[a,b]$ by the expression

$$\sum_{k=1}^{N} y_k m E_k,$$

where each of the $y_k m E_k$ are called **simple functions**. In the expression, mE_k denotes a "measure" [5] of E_k.

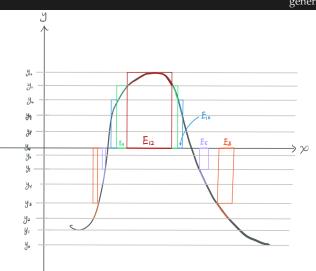[5] Note that a measure is simply a generalization of the notion of 'length'.

Figure 2.3: Drawing out the rectangles of $y_k m E_k$ from Figure 2.2.

We observe that E_k need not be a particularly well-behaved set.

However, note that we may rearrange the possibly scattered pieces of each E_k together, so as to form a 'continuous' base for the rectangle. We need our definition of a measure to be able to capture this.

The following is an analogy from Lebesgue himself on comparing Lebesgue integration and Riemann integration [6]:

> *I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral.*
>
> *But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral.*

The insight here is that one can freely arrange the values of te functions, all the while preserving the value of the integral.

- This requires us to have a better understanding of what a measure is.

- This process of rearrangement converts certain functions which are extremely difficult to deal with, or outright impossible, with the Riemann integral, into easily digestible pieces using Lebesgue integral.

2.2 *Lebesgue Outer Measure*

Goals of the section

1. Define a "measure of length" on as many subsets of $\mathbb{R}$ as possible.

2. The definition should agree with our intuition of what a 'length' is.

📄 Definition 11 (Length)

*For $a \leq b \in \mathbb{R}$, we define the *length* of the interval (a, b) to be $b - a$, and*

we write

$$\ell((a,b)) := b - a.$$

We also define

- $\ell(\varnothing) = 0$; *and*

- $\ell((a,\infty)) = \ell((-\infty, b)) = \ell((-\infty, \infty)) = \infty.$

📘 Definition 12 (Cover by Open Intervals)

Let $E \subseteq \mathbb{R}$. A countable collection $\{I_n\}_{n=1}^{\infty}$ of open intervals is said to be a cover of E by open intervals *if $E \subseteq \bigcup_{n=1}^{\infty} I_n$.*

🔖 Note 2.2.1

In this course, the only covers that we shall use are open intervals *, and so we shall henceforth refer to the above simply as covers of E.*

Before giving what immediately follows from the above, I shall present the following notion of an outer measure.

📘 Definition 13 (Outer Measure)

Let $\varnothing \neq X$ be a set. An outer measure μ on X *is a function*

$$\mu : \mathcal{P}(X) \to [0,\infty] := [0,\infty) \cup \{\infty\}$$

which satisfies

1. $\mu\varnothing = 0$;

2. (*monotone increment* or *monotonicity*) $E \subseteq F \subseteq X \implies \mu E \leq \mu F$; *and*

3. (*countable subadditivity* or *σ-subadditivity*) $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu E_n.$$

❝ Note 2.2.2

Note that by the monotonicity, the σ-subadditivity condition is equivalent to: given $\{E_n\}_{n=1}^\infty \subseteq \mathcal{P}(X)$ and $F \subseteq \bigcup_{n=1}^\infty E_n$, we have that

$$\mu(F) \leq \sum_{n=1}^\infty \mu(E_n).$$

▤ Definition 14 (Lebesgue Outer Measure)

We define the Lebesgue outer measure as a function $m^* : \mathcal{P}(X) \to \mathbb{R}$ such that

$$m^*E := \inf\left\{ \sum_{n=1}^\infty \ell(I_n) : E \subseteq \bigcup_{n=1}^\infty I_n \right\}.$$

We cheated a little bit by calling the above an outer measure, so let us now justify our cheating.

💧 Proposition 5 (Validity of the Lebesgue Outer Measure)

m^* is indeed an outer measure.

✏ Proof

$\boxed{\mu\varnothing = 0}$ We consider a sequence of sets $\{I_n\}_{n=1}^\infty$ such that $I_n = \varnothing$ for each $n = 1, \ldots, \infty$. It is clear that $\varnothing \subseteq \bigcup_{n=1}^\infty I_n$. Also, we have that $\ell(I_n) = 0$ for all $n = 1, \ldots, \infty$. It follows that

$$0 \leq m^*(\varnothing) \leq \sum_{n=1}^\infty m^*(I_n) = \sum_{n=1}^\infty 0 = 0,$$

where the inequality is simply by the definition of m^* being an infimum, **not to be confused with σ-subadditivity**. We thus have that

$$m^*(\varnothing) = 0.$$

Monotonicity Suppose $E \subseteq F \subseteq \mathbb{R}$, and $\{I_n\}_{n=1}^{\infty}$ a cover of F. Then

$$E \subseteq F \subseteq \bigcup_{n=1}^{\infty} I_n.$$

In particular, all covers of F are also covers of E, i.e.

$$\left\{ \{J_m\}_{m=1}^{\infty} : E \subseteq \bigcup_{m=1}^{\infty} J_m \right\} \subseteq \left\{ \{I_n\}_{n=1}^{\infty} : F \subseteq \bigcup_{n=1}^{\infty} I_n \right\}.$$

It follows that

$$m^* E \le m^* F.$$

σ-subaddivitity Consider $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$ such that $E \subseteq \bigcup_{n=1}^{\infty} E_n$. WTS

$$m^* E \le \sum_{n=1}^{\infty} m^* E_n.$$

Now if the sum of the RHS is infinite, i.e. if any of the $m^* E_n$ is infinite, then the inequality comes for free. Thus WMA $\sum_{n=1}^{\infty} E_n < \infty$, and in particular that $m^* E_n < \infty$ for all $n = 1, \dots, \infty$.

To do this, let $\varepsilon > 0$. Since $m^* E_n < \infty$ for all n, we can find covers $\left\{ I_k^{(n)} \right\}_{k=1}^{\infty}$ for each of the E_n's such that

$$\sum_{k=1}^{\infty} \ell \left(I_k^{(n)} \right) < m^* E_n + \frac{\varepsilon}{2^n}.$$

Then, we have that

$$E \subseteq \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} I_k^{(n)}.$$

Then by $m^* E$ being the infimum of the sum of lengths of the covering intervals, we have that

$$\begin{aligned}
m^* E &\le \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \ell \left(I_k^{(n)} \right) \\
&\le \sum_{n=1}^{\infty} \left(m^* E_n + \frac{\varepsilon}{2^n} \right) \\
&= \sum_{n=1}^{\infty} m^* E_n + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} \\
&= \sum_{n=1}^{\infty} m^* E_n + \varepsilon.
\end{aligned}$$

Since ε was arbitrary, we have that

$$m^* E_n \leq \sum_{n=1}^{\infty} m^* E_n,$$

as desired. □

▶ Corollary 6 (Lebesgue Outer Measure of Countable Sets is Zero)

If $E \subseteq \mathbb{R}$ is countable, then $m^* E = 0$.

✏ Proof

We shall prove for when E is denumerable, for the finite case follows a similar proof. Let us write $E = \{x_n\}_{n=1}^{\infty}$. Let $\varepsilon > 0$ and

$$I_n = \left(x_n - \frac{\varepsilon}{2^{n+1}}, \; x_n + \frac{\varepsilon}{2^{n+1}} \right).$$

Then it is clear that $\{I_n\}_{n=1}^{\infty}$ is a cover of E.

It follows that

$$0 \leq m^* E \leq \sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Thus as $\varepsilon \to 0$, we have that

$$m^* E = 0,$$

as expected. □

▶ Corollary 7 (Lebesgue Outer Measure of $\mathbb{Q}$ is Zero)

We have that $m^* \mathbb{Q} = 0$.

IN THE PROOFS above that we have looked into, and based on the

intuitive notion of the length of an open interval, it is compelling to simply conclude that

$$m^*(a,b) = \ell(a,b) = b - a.$$

However, looking back at ▤ Definition 14, we know that that is not how $m^*(a,b)$ is defined.

This leaves us with an interesting question:

> *how does our notion of measure $m^*(a,b)$ of an interval compare with the notion of the length of an interval?*

By taking $I_1 = (a,b)$ and $I_n = \varnothing$ for $n \geq 2$, it is rather clear that $\{I_n\}_{n=1}^{\infty}$ is a cover of (a,b), and so we have

$$m^*(a,b) \leq \ell(a,b) = b - a. \tag{2.1}$$

However, the other side of the game is not as easy to confirm: we would have to consider **all** possible covers of (a,b), which is a lot.

Another question that we can ask ourselves seeing Equation (2.1) is why can't $m^*(a,b)$ be something that is strictly less than the length to give us an even more 'precise' measurement?

To answer these questions, it is useful to first consider the outer measure of a closed and bounded interval, e.g. $[a,b]$, since these intervals are **compact** under the **Heine-Borel Theorem**. This will give us a finite subcover for every infinite cover of the compact interval, which is easy to deal with.

We shall see that with the realization of the outer measure of a compact interval, we will also be able to find the outer measure of intervals that are neither open nor closed.

We shall prove the following proposition in the next lecture. Note that for the sake of presentation, I shall abbreviate the Lebesgue Outer Measure as LOM.

💧 **Proposition (LOM of Arbitrary Intervals)**

Suppose $a < b \in \mathbb{R}$. Then

1. $m^*([a,b]) = b - a$; *and therefore*

2. $m^*((a,b]) = m^*([a,b)) = m^*((a,b)) = b - a$.

Suppose $a < b \in \mathbb{R}$. Then

3 🍃 *Lecture 3 May 14th 2019*

3.1 *Lebesgue Outer Measure Continued*

💧 **Proposition 8 (LOM of Arbitrary Intervals)**

Suppose $a < b \in \mathbb{R}$. Then

1. $m^*([a,b]) = b - a$; *and therefore*

2. $m^*((a,b]) = m^*([a,b)) = m^*((a,b)) = b - a$.

✏️ **Proof**

1. Consider $a < b \in \mathbb{R}$. Let $\varepsilon > 0$, and let

$$I_1 = \left(a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2} \right)$$

and $I_n = \varnothing$ for $n \geq 2$. Then $\{I_n\}_{n=1}^{\infty}$ is a cover of $[a,b]$. This means that

$$m^*([a,b]) \leq \sum_{n=1}^{\infty} \ell(I_n) = b - a + \varepsilon.$$

So for all $\varepsilon \to 0$, we have that

$$m^*([a,b]) \leq b - a.$$

[1] Conversely, if $[a,b]$ is covered by open intervals $\{I_n\}_{n=1}^{\infty}$, then by compactness of $[a,b]$ (via the **Heine-Borel Theorem**), we know that we can cover $[a,b]$ by finitely many of these intervals, and let us denote these as $\{I_n\}_{n=1}^{N}$, for some $1 \leq N < \infty$.

[1] For the converse, we know that $m^*([a,b]) = \inf \bigstar$, where $\bigstar$ is just a placeholder for you-know-what. So $m^*([a,b])$ is one of the sums. So if we can show that for an arbitrary sum, $\geq$ holds, our work is done.

WTS

$$\sum_{n=1}^{N} \ell(I_n) \geq b - a.$$

If LHS $= \infty$, then our work is done. Thus wlog, WMA each $I_n = (a_n, b_n)$ is a finite interval. Note that we have

$$[a, b] \subseteq \bigcup_{n=1}^{N} (a_n, b_n).$$

In particular, $a \in \bigcup_{n=1}^{N} I_n$. Thus, $\exists 1 \leq n_2 \leq N$ such that $a \in I_{n_1}$. Now if $b_{n_1} > b$, we shall stop this process for our work is done, since then $[a, b] \subseteq I_{n_1}$. Otherwise, if $b_{n_1} \leq b$, then $b_{n_1} \in [a, b] \subseteq \bigcup_{n=1}^{N} I_n$, which means that $\exists 1 \leq n_2 \leq N$ such that $b_{n_1} \in I_{n_2}$. Notice that $n_1 \neq n_2$, since $b_{n_1} \notin I_{n_1}$ but $b_{n_1} \in I_{n_2}$.

Now once again, if $b_{n_2} > b$, then we shall stop this process since our work is done. Otherwise, we have $a < b_{n_2} \leq b$, and so $\exists 1 \leq n_3 \leq N$, $n_3 \neq n_1, n_2$, such that $b_{n_2} \in I_3$...

We continue with the above process for as long as $b_{n_k} \leq b$. We can thus find, for each k, $I_{n_{k+1}}$, where $n_{k+1} \in \{1, \ldots, N\} \setminus \{n_1, n_2, \ldots, n_k\}$, such that $b_{n_k} \in I_{n_{k+1}}$.

However, since each of the I_{n_k}'s are different, and since we only have N such intervals, there must exists a $K \leq N$ such that

$$b_{n_{K-1}} \leq b \text{ and } b_{n_K} > b.$$

It now suffices for us to show that

$$\sum_{j=1}^{K} \ell(I_{n_j}) \geq b - a.$$

Observe that

$$\sum_{j=1}^{K} \ell(I_{n_j}) = (b_{n_K} - a_{n_K}) + (b_{n_{K-1}} - a_{n_{K-1}}) + \ldots$$

$$+ (b_{n_2} - a_{n_2}) + (b_{n_1} - a_{n_1})$$

$$= b_{n_K} + \underset{\geq 0}{(b_{n_{K-1}} - a_{n_K})} + \underset{\geq 0}{(b_{n_{K-2}} - a_{n_{K-1}})} + \dots$$

$$+ \underset{\geq 0}{(b_{n_1} - a_{n_2})} - a_{n_1}$$

$$\geq b_{n_K} - a_{n_1} \geq b - a.$$

Thus

$$\sum_{n=1}^{\infty} \ell(I_n) \geq \sum_{n=1}^{N} \ell(I_n) \geq \sum_{j=1}^{K} \ell(I_{n_j}) \geq b - a,$$

whence

$$m^*([a, b]) \geq b - a.$$

It follows that, indeed,

$$m^*([a, b]) = b - a.$$

2. First, note that

$$m^*((a, b)) \leq m^*([a, b]) \leq b - a.$$

On the other hand, notice that $\forall 0 < \varepsilon < \frac{b-a}{2}$, we have that

$$[a + \varepsilon, b - \varepsilon] \subseteq (a, b),$$

and so by monotonicity,

$$(b - a) - 2\varepsilon = m^*([a + \varepsilon, b - \varepsilon]) \leq m^*((a, b)).$$

As $\varepsilon \to 0$, we have that

$$b - a \leq m^*((a, b)) \leq b - a.$$

So

$$m^*((a, b)) = b - a$$

as desired.

Finally, we have that

$$b - a = m^*((a, b)) \leq m^*((a, b]) \leq m^*([a, b]) = b - a,$$

and similarly

$$b - a = m^*((a,b)) \leq m^*([a,b)) \leq m^*([a,b]) = b - a.$$

Thus

$$m^*((a,b)) = m^*((a,b]) = m^*([a,b)) = b - a$$

as required. □

💧 **Proposition 9 (LOM of Infinite Intervals)**

We have that $\forall a, b \in \mathbb{R}$,

$$m^*((a,\infty)) = m^*([a,\infty))$$
$$= m^*((-\infty,b)) = m^*((-\infty,b])$$
$$= m^*\mathbb{R} = \infty.$$

✏️ **Proof**

Observe that

$$(a, a+n) \subseteq (a,\infty)$$

for all $n \geq 1$. Thus

$$n = m^*((a,a+n)) \leq m^*((a,\infty))$$

for all $n \geq 1$. Hence

$$m^*((a,\infty)) = \infty$$

by definition.

All other cases follow similarly. □

📢 **Corollary 10 (Uncountability of $\mathbb{R}$)**

$\mathbb{R}$ *is uncountable.*

✏️ Proof

We have that

$$m^*\mathbb{R} = \infty \neq 0,$$

and so it follows from 🖝Corollary 6, we must have that $\mathbb{R}$ is uncountable. □

📙 Definition 15 (Translation Invariant)

Let μ be an outer measure on $\mathbb{R}$. We say that μ is translation invariant if $\forall E \subseteq \mathbb{R}$,

$$\mu(E) = \mu(E + \kappa)$$

for all $\kappa \in \mathbb{R}$, where

$$E + \kappa := \{x + \kappa : x \in E\}.$$

💧 Proposition 11 (Translation Invariance of the LOM)

The Lebesgue outer measure is translation invariant.

✏️ Proof

Let $E \subseteq \mathbb{R}$ and $\kappa \in \mathbb{R}$. Note that E is covered by open intervals $\{I_n\}_{n=1}^\infty$ iff $E + \kappa$ is covered by $\{I_n + \kappa\}_{n=1}^\infty$.

Claim: $\forall n \geq 1, \ell(I_n + \kappa) = \ell(I_n)$ Write

$$I_n = (a_n, b_n).$$

Then

$$I_n + \kappa = (a_n + \kappa, b_n + \kappa).$$

Observe that

$$\ell(I_n + \kappa) = b_n + \kappa - (a_n - \kappa) = b_n - a_n = \ell(I_n),$$

as claimed. ⊣

By the claim, it follows that

$$
\begin{aligned}
m^*(E) &= \inf\left\{ \sum_{n=1}^{\infty} \ell(I_n) : E \subseteq \bigcup_{n=1}^{\infty} \right\} \\
&= \inf\left\{ \sum_{n=1}^{\infty} \ell(I_n + \kappa) : E + \kappa \subseteq \bigcup_{n=1}^{\infty} (I_n + \kappa) \right\} \\
&= m^*(E + \kappa).
\end{aligned}
$$

□

Remark 3.1.1

Suppose $E \subseteq \mathbb{R}$ and $E = \bigcup_{n=1}^{\infty} E_n$, where

$$E_i \cap E_j = \emptyset \text{ if } i \neq j.$$

Now by σ-subadditivity of m^*, we have that

$$m^* E \leq \sum_{n=1}^{\infty} m^* E_n.$$

However, equality is not guaranteed. Consider the following case: if $E = [0, 1]$, we may have $E_n = [0, 1]$ for all $n \geq 1$, in which case $E = \bigcup_{n=1}^{\infty} E_n = [0, 1]$, but

$$m^* E = m^*[0, 1] = 1 < \infty = \sum_{n=1}^{\infty} m^* E_n.$$

It would be desirable to have

$$m^* E = \sum_{n=1}^{\infty} m^* E_n,$$

when the E_i's are pairwise disjoint, i.e. $E = \bigcup_{n=1}^{\infty} E_n$. In fact, this would agree with our intuition, that if the outer measure is going to be our 'length'. Consider the example $A = [0, 2] \cup [5, 7]$. Then we would expect $m^* A = 2 + 2 = 4$.

However, this is actually impossible for an arbitrary number of collections. 💬

📖 Theorem 12 (Non-existence of a sensible Translation Invariant Outer Measure that is also σ-additive)

There does not exist a translation-invariant outer measure μ on $\mathbb{R}$ that satisfies

1. $\mu(\mathbb{R}) > 0$;

2. $\mu[0,1] < \infty$; and

3. μ is σ-**additive**; i.e. if $\{E_n\}_{n=1}^\infty$ is a countable collection of disjoint subsets of $\mathbb{R}$ that covers $E \subseteq \mathbb{R}$, then

$$\mu E = \sum_{n=1}^\infty \mu E_n.$$

*Consequently, the Lebesgue outer measure m^* is **not** σ-additive.*

✏️ Proof

Suppose to the contrary that such a μ exists.

Step 1 Consider the relation $\sim$ on $\mathbb{R}$ such that $x \sim y$ if $x - y \in \mathbb{Q}$.

Claim: $\sim$ is an equivalence relation

- **(reflexivity)** We know that $0 \in \mathbb{Q}$ and $x - x = 0$. Thus $x \sim x$.

- **(symmetry)** Since $\mathbb{Q}$ is a **field**, it is closed under multiplication, and $-1 \in \mathbb{Q}$. Thus if $x \sim y$, then $x - y \in \mathbb{Q}$, and so $(-1)(x - y) = y - x \in \mathbb{Q}$, which means $y \sim x$.

- **(transitivity)** Again, since $\mathbb{Q}$ is a field, it is closed under (this time) addition. Thus

$$\begin{aligned} x \sim y \wedge y \sim z &\implies (x - y), (y - z) \in \mathbb{Q} \\ &\implies (x - y) + (y - z) = x - z \in \mathbb{Q}. \end{aligned}$$

Thus $x \sim z$.

This proves the claim. $\dashv$

Let

$$[x] := x + \mathbb{Q} := \{x + q : q \in \mathbb{Q}\}$$

denote the equivalence class of x wrt $\sim$. Note that the set of equivalence classes, which we shall represent as

$$\mathcal{F} := \{[x] : x \in \mathbb{R}\},$$

partitions $\mathbb{R}$, i.e.

- $[x] = [y] \iff x - y \in \mathbb{Q}$; and

- $[x] \cap [y] = \emptyset$ otherwise.

Note that since $\mathbb{Q}$ is **dense** in $\mathbb{R}$, we have that $[x] = x + \mathbb{Q}$ is also dense in $\mathbb{R}$, for all $x \in \mathbb{R}$. Then for each [2] $F \in \mathcal{F}$, $\exists x_F \in F$ such that

[2] Notice that here, we have invoked the Axiom of Choice .

$$0 \le x_F \le 1.$$

Now consider the set

$$\mathbb{V} := \{x_F : F \in \mathcal{F}\} \subseteq [0, 1],$$

which is called **Vitali's Set**.

Step 2 Since $\mathcal{F}$ partitions $\mathbb{R}$, we have that

$$\mathbb{R} = \bigcup_{F \in \mathcal{F}} F = \bigcup_{F \in \mathcal{F}} [x_F]$$

$$= \bigcup_{F \in \mathcal{F}} x_F + \mathbb{Q}$$

$$= \mathbb{V} + \mathbb{Q} := \{x + q : q \in \mathbb{Q}, x \in \mathbb{V}\}.$$

Step 3 Claim: $p \ne q \in \mathbb{Q} \implies (\mathbb{V} + p) \cap (\mathbb{V} + q) = \emptyset$ Suppose not, and suppose $\exists y \in (\mathbb{V} + p) \cap (\mathbb{V} + q)$. Then $\exists F_1, F_2 \in \mathcal{F}$ such that

$$y = x_{F_1} + p = x_{F_2} + q. \tag{3.1}$$

Then we may rearrange the above equation to get

$$x_{F_1} - x_{F_2} = q - p \in \mathbb{Q}.$$

This implies that

$$[x_{F_1}] = [x_{F_2}] \implies F_1 = F_2$$

since $\mathbb{V}$ consists of one unique representative from each of the equivalence classes. However, this would mean that

$$x_{F_1} = x_{F_2}.$$

Since $p \neq q$, we have that

$$x_{F_1} + p \neq x_{F_2} + q,$$

which contradicts Equation (3.1). Thus

$$(\mathbb{V} + p) \cap (\mathbb{V} + q) = \emptyset,$$

as claimed. ⊣

This in turn means that the $\mathbb{V} + q$, for each $q \in \mathbb{Q}$, also partitions $\mathbb{R}$. In other words, if we write $\mathbb{Q} = \{p_n\}_{n=1}^{\infty}$, then

$$\mathbb{R} = \mathbb{V} + \mathbb{Q} = \biguplus_{n=1}^{\infty} \mathbb{V} + p_n.$$

Now, note that

$$0 \neq \mu\mathbb{R} \overset{(1)}{=} \sum_{n=1}^{\infty} \mu(\mathbb{V} + p_n) \overset{(2)}{=} \sum_{n=1}^{\infty} \mu(\mathbb{V}),$$

where (1) is by μ being σ-additive and (2) is by μ being translation invariant, both directly from our assumptions. This means that

$$\mu\mathbb{V} > 0.$$

Step 4 Now consider $S = \mathbb{Q} \cap [0,1]$ such that S is denumerable. Write

$$S = \{s_n\}_{n=1}^{\infty}.$$

Note that for all $n \geq 1$,

$$\mathbb{V} \subseteq [0,1] \implies \mathbb{V} + s_n \subseteq [0,2],$$

and as proven above

$$i \neq j \implies (\mathbb{V} + s_i) \cap (\mathbb{V} + s_j) = \emptyset.$$

Thus it follows that

$$\mu \left(\bigcup_{n=1}^{\infty} \mathbb{V} + s_n \right) = \sum_{n=1}^{\infty} \mu(\mathbb{V} + s_n) = \sum_{n=1}^{\infty} \mu(\mathbb{V}) = \infty.$$

Also,

$$\begin{aligned}
\mu \left(\bigcup_{n=1}^{\infty} \mathbb{V} + s_n \right) &= \sum_{n=1}^{\infty} \mu(\mathbb{V} + s_n) \\
&\leq \mu([0,2]) = \mu([0,1] \cup ([0,1]+1)) \\
&\leq \mu[0,1] + \mu([0,1]+1) \\
&= 2\mu([0,1]) = 2 < \infty,
\end{aligned}$$

contradicting what we have right above.

Therefore, no such μ exists. $\square$

With the realization of ☕Theorem 12, we find ourselves facing a losing dilemma: we may either

1. be happy with the Lebesgue outer measure m^* for all subsets $E \subseteq \mathbb{R}$, which would agree with our intuitive notion of length, at the price of σ-additivity; or

2. restrict the domain of our function m^* to some family of subsets of $\mathbb{R}$, where m^* would have σ-additivity.

We shall adopt the second approach. We shall call the collection of sets where m^* has σ-additivity as the collection of **Lebesgue measurable sets**.

3.2 *Lebesgue Measure*

We shall first introduce **Carathéodory's** definition of a Lebesgue measurable set.

📘 Definition 16 (Lebesgue Measureable Set)

A set $E \subseteq \mathbb{R}$ is said to be Lebesgue measurable if, $\forall X \subseteq \mathbb{R}$,

$$m^* X = m^*(X \cap E) + m^*(X \setminus E).$$

We denote the collection of all Lebesgue measurable sets as $\mathfrak{M}(\mathbb{R})$.

Remark 3.2.1

Since we shall almost exclusively focus on the Lebesgue measure, we shall hereafter refer to "Lebesgue measurable sets" as simply "measurable sets". 💬

❝ Note 3.2.1

I shall quote and paraphrase this remark from our course notes [3]:

> *Informally, we see that a set $E \subseteq \mathbb{R}$ is measurable provided that it is a "universal slicer", that it "slices" every other set X into two **disjoint** sets, into where the Lebesgue outer measure is σ-**additive**.*

[3] Marcoux, L. W. (2018). *PMath 450 Introduction to Lebesgue Measure and Fourier Analysis.* (n.p.)

Also, note that we get the following inequality for free, simply from σ-subadditivity of m^*:

$$m^* X \leq m^*(X \cap E) + m^*(X \setminus E).$$

Thus, it suffices for us to check if the reverse inequality holds for all sets $X \subseteq \mathbb{R}$.

Before ploughing forward to getting out hands dirty with examples, let us first study a result on a structure of $\mathfrak{M}(\mathbb{R})$ that is rather

interesting. [4]

📘 **Definition 17 (Algebra of Sets)**

A collection $\Omega \subseteq \mathcal{P}(\mathbb{R})$ is said to be an *algebra of sets* *if*

1. $\mathbb{R} \in \Omega$;

2. (*closed under complementation*) $E \in \Omega \implies E^C \in \Omega$; *and*

3. (*closed under finite union*) *given $N \geq 1$ and $\{E_n\}_{n=1}^N \subseteq \Omega$, then*

$$\bigcup_{n=1}^{N} E_n \in \Omega.$$

We say that Ω is a *σ-algebra of sets* *if*

1. *Ω is an algebra of sets; and*

2. (*closed under countable union*) *if $\{E_n\}_{n=1}^\infty \subseteq \Omega$, then*

$$\bigcup_{n=1}^{\infty} E_n \in \Omega.$$

❝ **Note 3.2.2**

We often call a σ-algebra of sets as simply a σ-algebra.

🖥 **Theorem 13 ($\mathfrak{M}(\mathbb{R})$ is a σ-algebra)**

The collection $\mathfrak{M}(\mathbb{R})$ of Lebesgue measurable sets in $\mathbb{R}$ is a σ-algebra.

Due to time constraints, we shall prove the first 2 requirements in this lecture and prove the last requirement next time (which is also really long).

✏ **Proof**

$\boxed{\mathbb{R} \in \mathfrak{M}(\mathbb{R})}$ Observe that $\forall X \subseteq \mathbb{R}$,

$$m^*X = m^*X + 0 = m^*X + m^*\emptyset = m^*(X \cap \mathbb{R}) + m^*(X \setminus \mathbb{R}) \qquad \square$$

$\boxed{E \in \mathfrak{M}(\mathbb{R}) \implies E^C \in \mathfrak{M}(\mathbb{R})}$ Observe that $\forall X \subseteq \mathbb{R}$, since $E \in \mathfrak{M}(\mathbb{R})$, we have

$$
\begin{aligned}
m^*X &= m^*(X \cap E) + m^*(X \setminus E) \\
&= m^*(X \cap (E^C)^C) + m^*(X \cap E^C) \\
&= m^*(X \setminus E^C) + m^*(X \cap E^C) \\
&= m^*(X \cap E^C) + m^*(X \setminus E^C)
\end{aligned}
$$

$\left.\right\} A \setminus B = A \cap B^C$

$\left.\right\}$ *rearrangement*

Thus $E^C \in \mathfrak{M}(\mathbb{R})$.

🍃 *Bibliography*

Marcoux, L. W. (2018). *PMath 450 Introduction to Lebesgue Measure and Fourier Analysis.* (n.p.).

Siegmund-Schultze, R. (2008). *Henri Lesbesgue, in Timothy Gowers, June Barrow-Green, Imre Leader (eds.), Princeton Companion to Mathematics.* Princeton University Press.

Index