

# PMATH352W18 Complex Analysis - Class Notes

Johnson Ng

January 10, 2018

# Contents

<b>1</b>	<b>Lecture 1 - Jan 3, 2018</b>	<b>4</b>
1.1	Complex Numbers and Their Properties . . . . .	4
<b>2</b>	<b>Lecture 2 - Jan 5th, 2018</b>	<b>11</b>
2.1	Complex Numbers and Their Properties (Continued) . . . . .	11
<b>3</b>	<b>Lecture 3 - Jan 8th, 2018</b>	<b>15</b>
3.1	Complex Numbers and Their Properties (Continued 2) . . . . .	15
<b>4</b>	<b>Lecture 4 - Jan 10th, 2018</b>	<b>20</b>
4.1	Examples for nth Roots of Unity . . . . .	20

# List of Definitions

1.1.1	Complex Number, Complex Plane . . . . .	4
1.1.2	Sum and Product . . . . .	5
1.1.3	Conjugate . . . . .	7
1.1.4	Modulus . . . . .	7
3.1.1	Argument of a Complex Number . . . . .	15

# List of Theorems

Proposition 1.1.1	Basic Inequalities . . . . .	8
Proposition 3.1.1	$n$ th Roots of a Complex Number . . . . .	18

# Chapter 1

## Lecture 1 - Jan 3, 2018

### 1.1 Complex Numbers and Their Properties

#### Definition 1.1.1 (Complex Number, Complex Plane)

A **complex number** is a vector in  $\mathbb{R}^2$ . The **complex plane**, denoted by  $\mathbb{C}$ , is a set of complex numbers,

$$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

In  $\mathbb{C}$ , we usually write

$$\begin{aligned} 0 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & 1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ i &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} & x &= \begin{pmatrix} x \\ 0 \end{pmatrix} \\ iy &= \begin{pmatrix} 0 \\ y \end{pmatrix} \end{aligned}$$

where  $x, y \in \mathbb{R}$ . Consequently, we have that

$$x + iy = x + yi = \begin{pmatrix} x \\ y \end{pmatrix}$$

If for  $x, y \in \mathbb{R}$ ,  $z = x + iy$ , then  $x$  is called the real part of  $z$  and  $y$  is called the imaginary part of  $z$ , and we write

$$\operatorname{Re}(z) = x \quad \operatorname{Im}(z) = y.$$

#### Note

- It is easy to see how  $\mathbb{R}$  is a subset of  $\mathbb{C}$ .

- Complex Numbers of the form  $\begin{pmatrix} 0 \\ y \end{pmatrix}$  where  $y \in \mathbb{R}$  are called purely imaginary numbers.
- Certain authors may prefer to denote  $i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ .

**Definition 1.1.2 (Sum and Product)**

We define the sum of two complex numbers to be the usual vector sum, i.e.

$$\begin{aligned} (a + ib) + (c + id) &= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \begin{pmatrix} a + c \\ b + d \end{pmatrix} \\ &= (a + c) + i(b + d) \end{aligned}$$

where  $a, b, c, d \in \mathbb{R}$ .

We define the product of two complex numbers by setting  $i^2 = -1$ , and by requiring the product to be commutative, associative, and distributive over the sum. In this setup, we have that

$$\begin{aligned} (a + ib)(c + id) &= ac + iad + ibc + i^2bd \\ &= (ac - bd) + i(ad + bc) \end{aligned} \tag{1.1}$$

**Note**

It is interesting to note that any complex number times zero is zero, just like what we have with real numbers.

$$\begin{aligned} \forall z = x + iy \in \mathbb{C} \quad x, y \in \mathbb{R} \quad 0 \in \mathbb{C} \\ z \cdot 0 = (x + iy)(0 + i0) = 0 + i0 = 0 \end{aligned}$$

**Example 1.1.1**

Let  $z = 2 + i, w = 1 + 3i$ . Find  $z + w$  and  $zw$ .

$$\begin{aligned} z + w &= (2 + i) + (1 + 3i) \\ &= 3 + 4i \end{aligned}$$

$$\begin{aligned} zw &= (2 + i)(1 + 3i) \\ &= (2 - 3) + i(6 + 1) \quad \text{By Equation (1.1)} \\ &= -1 + 7i \end{aligned}$$

**Example 1.1.2**

Show that every non-zero complex number has a multiplicative inverse,  $z^{-1}$ , and find a formula for this inverse.

Let  $z = a + ib$  where  $a, b \in \mathbb{R}$  with  $a^2 + b^2 \neq 0$ . Then

$$\begin{aligned}
 & z(x + iy) = 1 \\
 \iff & (ax - by) + i(ay + bx) = 1 \\
 \iff & \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \iff & \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \iff & \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \iff & \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a \\ -b \end{pmatrix} \\
 \iff & x + iy = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2}
 \end{aligned}$$

Therefore, we have that the formula for the inverse is

$$(a + ib)^{-1} = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2} \quad (1.2)$$

### Notation

For  $z, w \in \mathbb{C}$ , we write

$$\begin{aligned}
 -z &= -1z & w - z &= w + (-z) \\
 \frac{1}{z} &= z^{-1} & \frac{w}{z} &= wz^{-1}
 \end{aligned}$$

### Example 1.1.3

Find  $\frac{(4-i)-(1-2i)}{1+2i}$ .

$$\begin{aligned}
 \frac{(4-i)-(1-2i)}{1+2i} &= \frac{3+i}{1+2i} \\
 &= (3+i)\left(\frac{1}{5} - i\frac{2}{5}\right) \\
 &= 1 - i
 \end{aligned}$$

### Note

The set of complex numbers is a **field** under the operations of addition and multiplication. This means that  $\forall u, v, w \in \mathbb{C}$ ,

$$\begin{array}{ll}
u + v = v + u & uv = vu \\
(u + v) + w = u + (v + w) & (uv)w = u(vw) \\
0 + u = u & 1u = u \\
u + (-u) = 0 & uu^{-1} = 1, \quad u \neq 0 \\
u(v + w) = uv + uw &
\end{array}$$

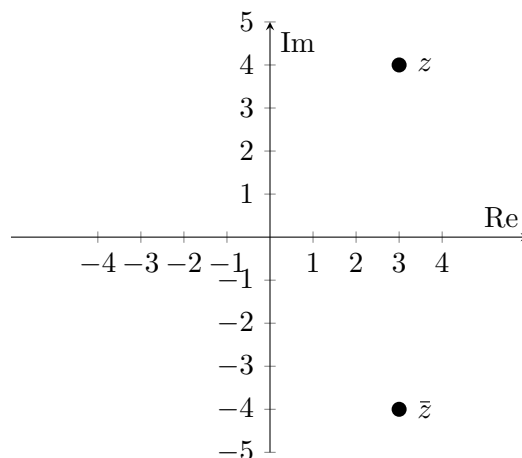
Since the distributive law holds for complex numbers, note that the binomial expansion works for  $(w + z)^n$  where  $w, z \in \mathbb{C}$  and  $n \in \mathbb{N}$ . (I did not verify if this is still true for when  $n \in \mathbb{R}$ .)

### Definition 1.1.3 (Conjugate)

If  $z = x + iy$  where  $x, y \in \mathbb{R}$ , then the **conjugate of  $z$**  is given by  $\bar{z} = x - iy$

### Example 1.1.4

Let  $z = 3 + 4i$ . Then the  $\bar{z} = 3 - 4i$ . Represented in the complex plane, we have the following:



We observe that on the complex plane, the conjugate of a complex number is simply its reflection on the real axis.

### Definition 1.1.4 (Modulus)

We define the **modulus** (length, magnitude) of  $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$ , to be

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}. \quad (1.3)$$

### Note

Note that this definition is consistent with the notion of the absolute value in real numbers when  $z$  is a real number, since if  $y = 0$ ,  $|z| = |x + i0| = \sqrt{x^2} = \pm x$ .



**Note**

For  $z, w \in \mathbb{R}$ , we have

$$\begin{aligned} \bar{\bar{z}} &= z & z + \bar{z} &= 2 \operatorname{Re}(z) & z - \bar{z} &= 2i \operatorname{Im}(z) \\ z\bar{z} &= |z|^2 & |z| &= |\bar{z}| & \overline{z \pm w} &= \bar{z} \pm \bar{w} \\ \overline{zw} &= \bar{z} - \bar{w} & |zw| &= |z| |w| \end{aligned}$$

but note that  $|z + w| \neq |z| + |w|$ .

**Note**

While inequalities such as  $z_1 < z_2$ , where  $z_1, z_2 \in \mathbb{C}$ , are meaningless unless if both of them are real,  $|z_1| < |z_2|$  means that the point  $z_1$  in the complex plane is closer to the origin than the point  $z_2$ .

**Proposition 1.1.1 (Basic Inequalities)**

1.  $|\operatorname{Re}(z)| \leq |z|$
2.  $|\operatorname{Im}(z)| \leq |z|$
3.  $|z + w| \leq |z| + |w|$      *Triangle Inequality*
4.  $|z + w| \geq ||z| - |w||$      *Inverse Triangle Inequality*

**Proof**

Note that  $|z|^2 = \operatorname{Re}(z)^2 + \operatorname{Im}(z)^2$  and that we can express  $|x| = \sqrt{x^2}$  for any  $x \in \mathbb{R}$ . 1 and 2 immediately follows from that.

To prove 3, we have that

$$\begin{aligned} |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) \\ &= |z|^2 + |w|^2 + (w\bar{z} + \bar{w}z) \\ &= |z|^2 + |w|^2 + 2 \operatorname{Re}(w\bar{z}) \\ &\leq |z|^2 + |w|^2 + 2 |w\bar{z}| \quad \text{by 1} \\ &= |z|^2 + |w|^2 + 2 |wz| \quad \text{since } |w\bar{z}| = |w| |\bar{z}| \text{ and } |z| = |\bar{z}| \\ &= (|z| + |w|)^2 \end{aligned}$$

To prove 4, note that

$$|z| = |z + w - w| \leq |z + w| + |w| \tag{1.4}$$

$$|w| = |w + z - z| \leq |z + w| + |z| \tag{1.5}$$

Observe that

$$\text{Equation (1.4)} \implies |z| - |w| \leq |z + w|$$

$$\text{Equation (1.5)} \implies |w| - |z| \leq |z + w|$$

Thus, we have that

$$|z + w| \geq ||z| - |w||$$

as required.  $\square$

Item 3 in Proposition 1.1.1 can be generalized by the means of mathematical induction to sums involving any finite number of terms, as:

$$|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n| \quad (1.6)$$

where  $n \in \mathbb{N} \setminus \{0, 1\}$ .

To note the induction proof, when  $n = 2$ , Equation (1.6) is just Item 3. If Equation (1.6) is true for when  $n = m$  where  $m \in \mathbb{N} \setminus \{0, 1\}$ ,  $n = m + 1$  is also true since by Item 3,

$$\begin{aligned} |(z_1 + z_2 + \dots + z_m) + z_{m+1}| &\leq |z_1 + z_2 + \dots + z_m| + |z_{m+1}| \\ &\leq (|z_1| + |z_2| + \dots + |z_m|) + |z_{m+1}|. \end{aligned}$$

The distance between two points  $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}, x_1, x_2, y_1, y_2 \in \mathbb{R}$  is  $|z_1 - z_2|$ , since  $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$  is our usual notion of the Euclidean distance of two points on a plane.

Also, note that

$$z_1 - z_2 = z_1 + (-z_2)$$

and thus if we apply our knowledge of vector representation,  $z_1 - z_2$  is the directed line segment from the point  $z_2$  to  $z_1$ .

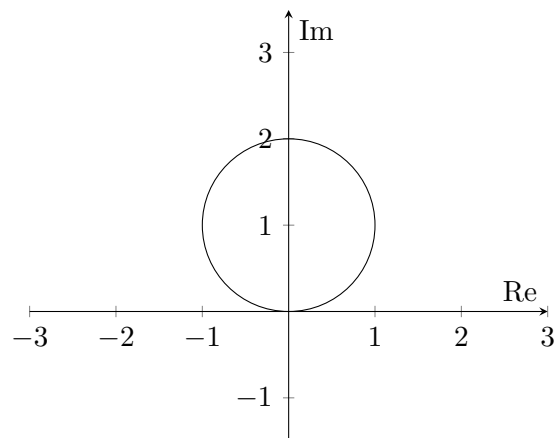
With the notion of a “distance” set on the complex plane, we can now explore upon points lying on a circle with a center  $z_0$  and radius  $R$ , which satisfies the equation

$$|z - z_0| = R.$$

We may simply refer to this set of points as the circle  $|z - z_0| = R$ .

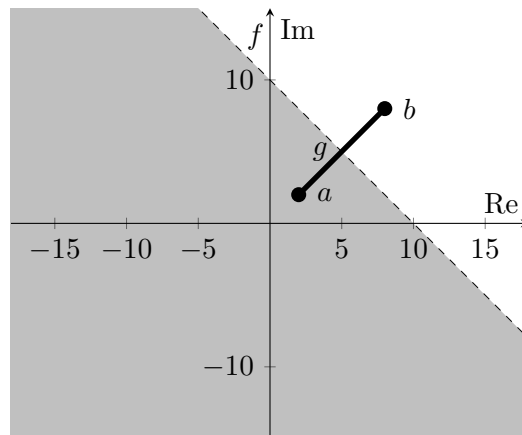
### Example 1.1.5

We may describe a set  $\{z \in \mathbb{C} : |z - i| = 1\}$  as follows:



Let  $a, b \in \mathbb{C}$  describe the set  $\{z \in \mathbb{C} : |z - a| < |z - b|\}$ .

Suppose the following coordinates for  $a$  and  $b$  are arbitrary,



In the above,  $g$  is the line segment that connects the points  $a$  and  $b$  on the complex plane, while  $f$  is the perpendicular bisector of the line segment  $g$ . The area described by the set  $\{z \in \mathbb{C} : |z - a| < |z - b|\}$  is the shaded area which is below  $f$ .

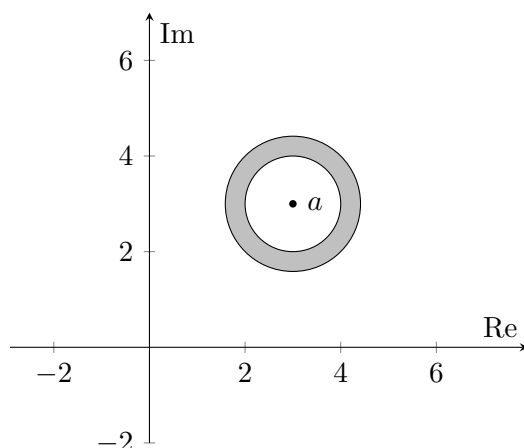
## Chapter 2

### Lecture 2 - Jan 5th, 2018

#### 2.1 Complex Numbers and Their Properties (Continued)

**Example 2.1.1**

Let  $a \in \mathbb{C}$ . Describe the set  $\{z \in \mathbb{C} : 1 < |z - a| < 2\}$ .



**Example 2.1.2**

Show that every non-zero complex number has exactly two complex square roots, and find a formula for the square roots.

Let  $z = x + iy \in \mathbb{C}$ ,  $x, y \in \mathbb{R}$ , and let  $w = u + iv$ ,  $u, v \in \mathbb{R}$ . Then

$$\begin{aligned}
w^2 = z &\iff (u + iv)^2 = x + iy \\
&\iff (u^2 - v^2) + i(2uv) = x + iy \\
&\iff x = u^2 + v^2 \quad \text{and}
\end{aligned} \tag{2.1}$$

$$y = 2uv \tag{2.2}$$

Square both sides of Equation (2.2), and thus we have  $y^2 = 4u^2v^2$ .

Multiply Equation (2.1) by  $4u^2$ , and we get

$$\begin{aligned}
4u^2x &= 4u^4 - 4u^2v^2 = 4u^4 - y^2 \\
\iff 0 &= 4u^4 - 4u^2x - y^2 \\
\iff u^2 &= \frac{4x \pm \sqrt{16x^2 + 16y^2}}{8} \\
&= \frac{x \pm \sqrt{x^2 + y^2}}{2}
\end{aligned}$$

Suppose  $y \neq 0$ . Note that  $x < \sqrt{x^2 + y^2}$ . Thus  $u^2 = \frac{x + \sqrt{x^2 + y^2}}{2} \implies u = \left( \frac{x + \sqrt{x^2 + y^2}}{2} \right)^{\frac{1}{2}}$ .

Similarly, we can get

$$v = \pm \left( \frac{-x + \sqrt{x^2 + y^2}}{2} \right)^{\frac{1}{2}}$$

Note that all four choices of signs satisfy Equation (2.1). If  $y > 0$ , then  $u$  and  $v$  are either both positive or both negative by Equation (2.2).

Suppose  $y = 0$ . Then we have

$$w^2 = z = x$$

Therefore, we get

$$w = \begin{cases} \pm \left[ \left( \frac{x + \sqrt{x^2 + y^2}}{2} \right)^{\frac{1}{2}} + i \left( \frac{-x + \sqrt{x^2 + y^2}}{2} \right)^{\frac{1}{2}} \right] & y > 0 \\ \pm \left[ \left( \frac{x + \sqrt{x^2 + y^2}}{2} \right)^{\frac{1}{2}} - i \left( \frac{-x + \sqrt{x^2 + y^2}}{2} \right)^{\frac{1}{2}} \right] & y < 0 \\ \pm \sqrt{x} & y = 0, x > 0 \\ \pm i\sqrt{x} & y = 0, x < 0 \end{cases}$$

**Remark**

Let  $z \in \mathbb{C}$ . The notation  $\sqrt{z}$  may represent either one of the square roots of  $z$  or both of the square roots, i.e. it is possible that  $\sqrt{z}$  represents a set.

**Exercise 2.1.1**

Is it always okay for complex numbers such that  $\sqrt{zw} = \sqrt{z}\sqrt{w}$ , for  $z, w \in \mathbb{C}$ ?

No. For example, consider  $z = w = -1$ . Then we have

$$\sqrt{zw} = \sqrt{1} = \pm 1$$

while

$$\sqrt{z}\sqrt{w} = i \cdot i = -1$$

and thus

$$\sqrt{zw} \neq \sqrt{z}\sqrt{w}.$$

**Example 2.1.3**

Find the values of  $\sqrt{3 - 4i}$ .

By [Example 2.1.2](#),

$$\begin{aligned} \sqrt{3 - 4i} &= \pm \left( \sqrt{\frac{3 + \sqrt{9 + 16}}{2}} - i \sqrt{\frac{-3 + \sqrt{9 + 16}}{2}} \right) \\ &= \pm(2 - i) \end{aligned}$$

**Remark**

The quadratic formula holds for complex polynomials, i.e.

$$\forall a, b, c \in \mathbb{C} \quad a \neq 0 \quad \forall z \in \mathbb{C} \quad az^2 + bz + c = 0,$$

the solution for  $z$  is given by

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2.3}$$

The following is a short proof.

**Proof**

$$\begin{aligned}
az^2 + bz + c = 0 &\iff z^2 + \frac{b}{a}z + \frac{c}{a} = 0 \\
&\iff z^2 + \frac{b}{a}z + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \\
&\iff \left(z + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2} \\
&\iff z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

(Personal Note: where did the  $-$  for the supposed  $\pm$  go? Or should it really be  $\pm$ ?)

**Example 2.1.4**

Solve  $iz^2 - (2 + 3i)z + 5(1 + i) = 0$ .

$$\begin{aligned}
z &= \frac{2 + 3i + \sqrt{(2 + 3i)^2 - 4i[5(1 + i)]}}{2i} \\
&= \frac{2 + 3i + \sqrt{-5 + 12i - 20i + 20}}{2i} \\
&= \frac{2 + 3i + \sqrt{15 + 8i}}{2i}
\end{aligned}$$

Note that by [Example 2.1.2](#),

$$\begin{aligned}
\sqrt{15 + 8i} &= \pm \left[ \sqrt{\frac{15 + \sqrt{225 + 64}}{2}} - i\sqrt{\frac{-15 + \sqrt{225 + 64}}{2}} \right] \\
&= \pm \left[ \sqrt{\frac{15 + 17}{2}} - i\sqrt{\frac{-15 + 17}{2}} \right] \\
&= \pm(4 - i)
\end{aligned}$$

Thus we have

$$\begin{aligned}
z &= \frac{2 + 3i + \sqrt{15 + 8i}}{2i} \\
&= \frac{2 + 3i \pm (4 - i)}{2i} \\
&= (6 + 2i) \left(-\frac{1}{2}i\right) \text{ or } (-2 + 4i) \left(-\frac{1}{2}i\right) \quad \text{by [Example 1.1.2](#)} \\
&= (1 - 3i) \text{ or } (2 + i)
\end{aligned}$$

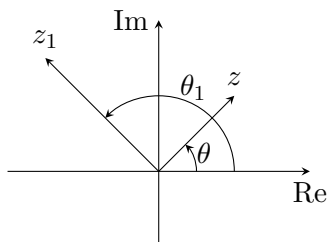
# Chapter 3

## Lecture 3 - Jan 8th, 2018

### 3.1 Complex Numbers and Their Properties (Continued 2)

#### Definition 3.1.1 (Argument of a Complex Number)

Let  $z \in \mathbb{C} \setminus \{0\}$ . The **argument** (or the angle) of  $z$ , denoted by  $\arg z$ ,  $\text{Arg } z$ , or simply  $\theta = \theta(z)$ , is the angle modulo  $2\pi$  (i.e.  $0 \leq \theta < 2\pi$ ) between the vector defining  $z$  and the positive real axis (in the counterclockwise direction).

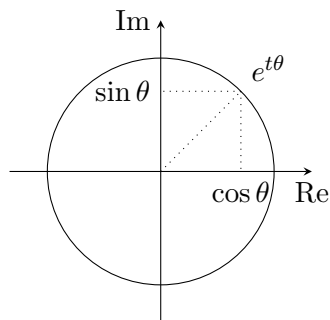


#### Notation

Let  $e^{i\theta} := \cos \theta + i \sin \theta$ . Note that this definition can be derived by extending the Taylor expansion of  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  for when  $x \in \mathbb{C}$  (the sum of the real parts of the expansion is the Taylor expansion of cosine while the imaginary part for sine).

Now  $e^{i\theta}$  is on the unit circle.



**Example 3.1.1**

Some examples of  $\theta \in [0, 2\pi)$ :

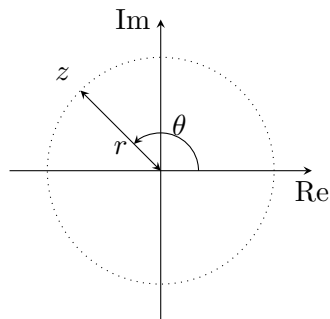
$$\begin{aligned} e^{i\frac{\pi}{4}} &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\frac{\pi}{2}} &= i \\ e^{i\frac{3\pi}{4}} &= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\pi} + 1 &= 0 \end{aligned}$$

**Remark**

$$\forall k \in \mathbb{Z} \quad \forall \theta \in \mathbb{R} \quad e^{i\theta} = e^{i(\theta + 2\pi k)}$$

**Remark**

The complex number  $re^{i\theta}$ , where  $r > 0, \theta \in [0, 2\pi)$ , represents the complex number with modulus  $r$  and argument  $\theta$ .



Therefore,  $\forall z \in \mathbb{C}$ , we can express

$$z := |z| e^{i \operatorname{Arg} z}. \quad (3.1)$$

With that, we now have two representations of a complex number:

- Cartesian representation:  $z = x + iy$  where  $x = \operatorname{Re}(z)$  and  $y = \operatorname{Im}(z)$
- Polar representation:  $z = re^{i\theta}$  where  $r = |z|$  and  $\theta = \operatorname{Arg} z \in [0, 2\pi)$

To convert between the two representations, we have the following equations:

Polar  $\rightarrow$  Cartesian:

$$x = r \cos \theta \quad y = r \sin \theta \quad (3.2)$$

Cartesian  $\rightarrow$  Polar:

$$\begin{aligned} r &= |z| \\ x \neq 0 &\implies \tan \theta = \frac{y}{x} \\ x = 0 &\implies \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \end{aligned} \quad (3.3)$$

On another note,

$$z = re^{i\theta} \implies \bar{z} = re^{-i\theta}$$

and

$$z \neq 0 \implies \frac{1}{z} = \frac{1}{r}e^{-i\theta}$$

**Remark**

$$\begin{aligned} \forall r_1, r_2 \in \mathbb{R} \quad \forall \theta_1, \theta_2 \in [0, 2\pi) \\ z_1 := r_1 e^{i\theta_1} \quad z_2 := r_2 e^{i\theta_2} \end{aligned}$$

Then

$$z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

Note that  $e^{ix} e^{iy} = e^{i(x+y)}$  is true for all  $x, y \in \mathbb{R}$  since

$$\begin{aligned} e^{ix} e^{iy} &= (\cos x + i \sin x)(\cos y + i \sin y) \\ &= (\cos x \cos y - \sin x \sin y) + i(\cos x \sin y + \cos y \sin x) \\ &= \cos(x + y) + i \sin(x + y) \\ &= e^{i(x+y)}. \end{aligned}$$

Generalizing the above, we get that

$$\forall n \in \mathbb{Z} \quad (re^{in}) = r^n e^{in\theta}$$

which is commonly known as **deMoivre's Law**.

**Proposition 3.1.1 (nth Roots of a Complex Number)**

$$\begin{aligned}
\forall z = re^{i\theta} \in \mathbb{C} \quad r = |z| \in \mathbb{R} \quad \theta \in [0, 2\pi) \\
\exists w = se^{i\tau} \in \mathbb{C} \quad s \in \mathbb{R} \quad \tau \in [0, 2\pi) \\
\forall n \in \mathbb{Z} \\
w^n = (se^{i\tau})^n = z = re^{i\theta}
\end{aligned}$$

The  $n$ th roots of  $z$  is described by the set

$$\left\{ r^{\frac{1}{n}} e^{i\left(\frac{\theta+2\pi k}{n}\right)} : k = 0, 1, \dots, n-1 \right\} \quad (3.4)$$

**Proof**

$$\begin{aligned}
s^n = r &\iff s = r^{\frac{1}{n}} \\
e^{in\theta} = e^{i\tau} &\iff \theta = \frac{\tau + 2\pi k}{n}
\end{aligned}$$

Therefore, the set that describes the  $n$ th roots of  $z$  is

$$\left\{ w = r^{\frac{1}{n}} e^{i\left(\frac{\theta+2\pi k}{n}\right)} : k = 0, 1, \dots, n-1 \right\}$$

**Remark (nth Roots of Unity)**

The  $n$ th roots of unity is a direct consequence of [Proposition 3.1.1](#) where we solve for the equation  $z^n = 1$  for any  $z \in \mathbb{C}, n \in \mathbb{Z}$ .

The set that describes the  $n$ th roots of unity is

$$\left\{ e^{i\theta} : \theta = \frac{2\pi k}{n}, k = 0, 1, \dots, n-1 \right\} \quad (3.5)$$

It is easy to see how the  $n$ th roots of unity partitions the unit circle into  $n$  parts.

**Example 3.1.2**

Find the cubic roots of  $-2 + 2i$ .

Let  $z = -2 + 2i$ . Note that  $|z| = 2\sqrt{2}$  and  $\text{Arg } z = \frac{3\pi}{4}$ .

Therefore, in polar form,  $z = 2\sqrt{2}e^{i\frac{3\pi}{4}}$ .

Let  $w = re^{i\theta}$ , where  $\theta \in [0, 2\pi)$ , and  $w^3 = z$ . Then

$$r = (2\sqrt{2})^{\frac{1}{3}}$$

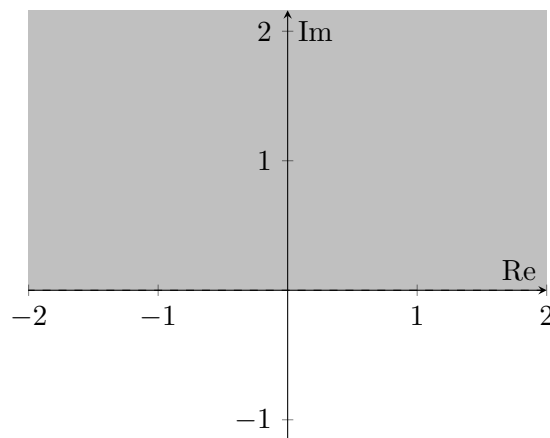
$$\theta = \frac{\frac{3\pi}{4} + 2\pi k}{3}, \quad k = 0, 1, 2$$

The set that describes the cubic root of  $-2 + 2i$  is thus

$$\left\{ (2\sqrt{2})^{\frac{1}{3}} e^{i\theta} : \theta = \frac{\frac{3\pi}{4} + 2\pi k}{3}, k = 0, 1, 2 \right\}$$

### Example 3.1.3

Describe the set  $\{z \in \mathbb{C} : |\operatorname{Arg} z - \frac{\pi}{2}| < \frac{\pi}{2}\}$ . (Note:  $\operatorname{Arg} z \in [0, 2\pi)$ )



### Exercise 3.1.1

Solve

1.  $z^4 = -1$

Let  $z = re^{i\theta}$

$$r = |-1| = 1 \quad \theta = \frac{\pi + 2\pi k}{4} = \frac{(2k + 1)\pi}{4}, \quad k = 0, 1, 2, 3$$

2.  $z^4 = -1 + \sqrt{3}i$

Let  $z = re^{i\theta}$

$$r = |-1 + \sqrt{3}i| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\theta = \frac{\frac{2\pi}{3} + 2\pi k}{4} = \frac{(2k + \frac{2}{3})\pi}{4}, \quad k = 0, 1, 2, 3$$

## Chapter 4

# Lecture 4 - Jan 10th, 2018

### 4.1 Examples for $n$ th Roots of Unity

Recall that the  $n$ th roots of unity are given by  $e^{i\frac{2\pi k}{n}}, k = 0, 1, \dots, n-1$ .

#### Exercise 4.1.1

Let  $z$  be any  $n$ th root of unity other than 1. Show that

$$z^{n-1} + z^{n-2} + \dots + z + 1 = 0 \quad (4.1)$$

#### Proof

By the Sum of Finite Geometric Terms,

$$z^{n-1} + z^{n-2} + \dots + z + 1 = \frac{1 - z^n}{1 - z}.$$

Since  $z^n = 1$ , RHS is thus zero, which in turn completes the proof.

As an aside, if we wish to remove the restriction that  $z$  can also be 1, we may consider that

$$z^n - 1 = (z - 1)(1 + z + \dots + z^{n-1})$$

Since  $z^n = 1$ , LHS is zero. Then either  $z = 1$  or  $(1 + z + \dots + z^{n-1}) = 0$ .

#### Exercise 4.1.2

Consider the  $n-1$  diagonals of a regular  $n$ -gon, inscribed in a circle of radius 1, obtained by connecting one vertex on the  $n$ -gon to all its other vertices.

For example, if we are given  $n = 6$ , we obtain the following diagram.

Show that the product of the lengths of these diagonals is equal to  $n$ .

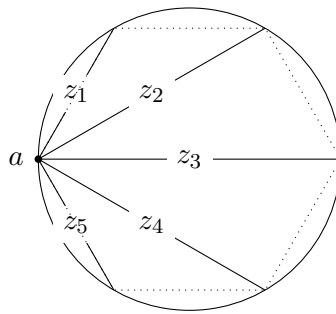


Figure 4.1:  $n = 6$ , where  $a$  is an arbitrary vertex on the hexagon

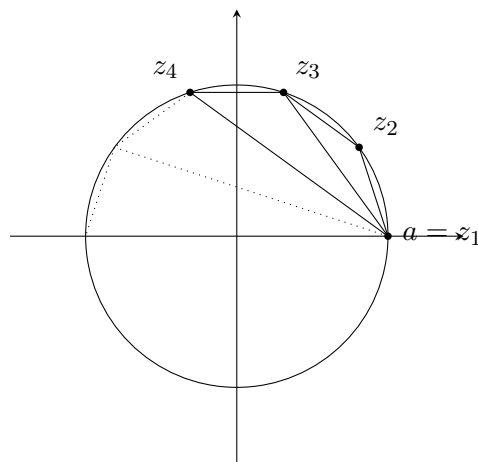


Figure 4.2: A regular  $n$ -gon with the roots of unity on its vertices

### Proof

Note that [Figure 4.1](#) can be translated into [Figure 4.2](#).

Thus the equation that we wish to prove becomes

$$|1 - z_2| |1 - z_3| \dots |1 - z_n| = n \quad (4.2)$$

Note that  $z_2, \dots, z_n$  are the  $n$ th roots of unity other than 1.

Let  $z$  be a variable and consider the polynomial

$$P(z) := 1 + z + z^2 + \dots + z^{n-1} \quad (4.3)$$

Since the roots of  $P(z)$  are the  $n$ th roots of unity other than 1, we can factorize [Equation \(4.3\)](#) into

$$P(z) = (z - z_2)(z - z_3) \dots (z - z_n)$$

Now let  $z = 1$  and take the modulus of  $P(z)$ , and we get [Equation \(4.2\)](#).

**Exercise 4.1.3**

Let  $n \in \mathbb{N}$ . Show that  $\sum_{j=0}^n \binom{3n}{3j} = \frac{2^{3n} + 2(-1)^n}{3}$ .

**Proof**

Let  $\alpha = e^{i\frac{2\pi}{3}}$ . Then  $\alpha$  is a cubic root of unity, i.e.  $\alpha^3 = 1$ , and from [Exercise 4.1.1](#),  $1 + \alpha + \alpha^2 = 0$ .

Consider

$$(1+1)^{3n} = \binom{3n}{0} + \binom{3n}{1} + \binom{3n}{2} + \binom{3n}{3} + \binom{3n}{4} + \binom{3n}{5} + \binom{3n}{6} + \dots + \binom{3n}{3n} \quad (4.4)$$

$$(1+\alpha)^{3n} = \binom{3n}{0} + \binom{3n}{1}\alpha + \binom{3n}{2}\alpha^2 + \binom{3n}{3}\alpha + \binom{3n}{4}\alpha^2 + \binom{3n}{5}\alpha^2 + \binom{3n}{6} + \dots + \binom{3n}{3n} \quad (4.5)$$

$$(1+\alpha^2)^{3n} = \binom{3n}{0} + \binom{3n}{1}\alpha^2 + \binom{3n}{2}\alpha + \binom{3n}{3}\alpha^2 + \binom{3n}{4}\alpha^2 + \binom{3n}{5}\alpha + \binom{3n}{6} + \dots + \binom{3n}{3n} \quad (4.6)$$

Adding [Equation \(4.4\)](#), [Equation \(4.5\)](#) and [Equation \(4.6\)](#), we observe that the terms with coefficients  $\binom{3n}{k}$  where  $k$  is not a multiple of 3 sums to 0 as given by  $1 + \alpha + \alpha^2 = 0$ , and therefore we obtain

$$\begin{aligned} 2^{3n} + (1+\alpha)^{3n} + (1+\alpha^2)^{3n} &= 3 \sum_{j=0}^n \binom{3n}{3j} \\ \frac{1}{3} [2^{3n} + (1+\alpha)^{3n} + (1+\alpha^2)^{3n}] &= \sum_{j=0}^n \binom{3n}{3j} \\ \frac{1}{3} [2^{3n} + (-\alpha^2)^{3n} + (-\alpha)^{3n}] &= \sum_{j=0}^n \binom{3n}{3j} \quad \text{since } 1 + \alpha + \alpha^2 = 0 \\ \frac{1}{3} [2^{3n} + (-1)^n + (-1)^n] &= \sum_{j=0}^n \binom{3n}{3j} \quad \text{since } \alpha^3 = 1 \\ \frac{2^{3n} + 2(-1)^n}{3} &= \sum_{j=0}^n \binom{3n}{3j} \end{aligned}$$

as required.

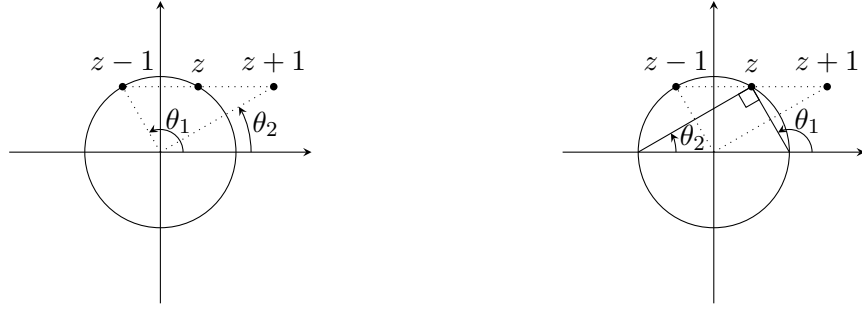


Figure 4.3: (Right) Depicted question, (Left) Translated Angles

**Exercise 4.1.4**

Note that we can define  $\text{Arg } z$  in any interval of length  $2\pi$ , i.e. it is not necessary that  $\text{Arg } z \in [0, 2\pi)$ .

For example, if we restrict  $\text{Arg } z \in [-\pi, \pi]$ , then we can write

$$\text{Arg} \left( -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) = -\frac{3\pi}{4}$$

Let  $z$  be on the unit circle and  $\text{Arg } z \in [-\pi, \pi]$ . Suppose that  $z \notin \mathbb{R}$ , i.e.  $z \neq 1, z \neq -1$ . Show that

$$\text{Arg} \left( \frac{z-1}{z+1} \right) = \begin{cases} \frac{\pi}{2} & \text{Im } z > 0 \\ -\frac{\pi}{2} & \text{Im } z < 0 \end{cases}$$

**Proof**

Note that  $\forall w_1, w_2 \in \mathbb{C}$ , where  $\text{Arg } w_1 = \tau_1, \text{Arg } w_2 = \tau_2$  for  $\tau_1, \tau_2$  in the same  $2\pi$ -interval,

$$\text{Arg} \frac{w_1}{w_2} = \frac{e^{i\tau_1}}{e^{i\tau_2}} \equiv e^{i(\tau_1 - \tau_2)} = \text{Arg } w_1 - \text{Arg } w_2$$

in modulo  $2\pi$ .

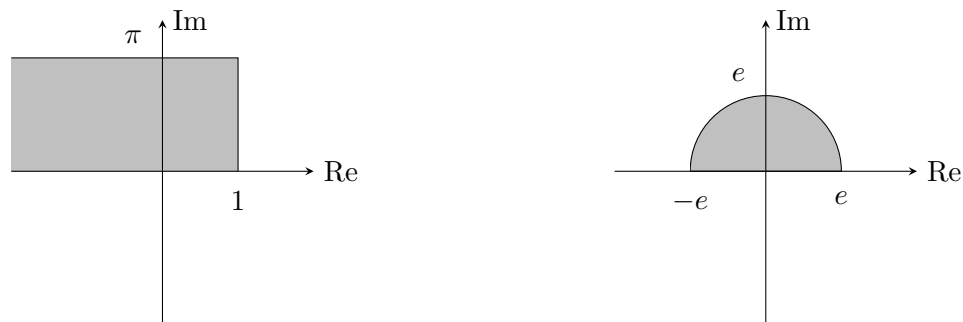
Suppose  $\text{Im } z > 0$ . Let  $\theta_1 = \text{Arg}(z-1)$  and  $\theta_2 = \text{Arg}(z+1)$ . Consider [Figure 4.3](#). We observe that

$$\begin{aligned} \frac{\pi}{2} &= \theta_2 + \pi - \theta_1 \\ \theta_1 - \theta_2 &= \frac{\pi}{2} \end{aligned}$$

as desired.

Similarly, we can obtain  $\theta_1 - \theta_2 = -\frac{\pi}{2}$  for when  $\text{Im } z < 0$ . This completes the proof.



Figure 4.4: (Right) Domain of  $f(A)$ , (Left) Image of  $f(A)$ **Exercise 4.1.5**

Let  $f(z) = e^z$  for  $z \in \mathbb{C}$ . Let  $A = \{z = x + iy \in \mathbb{C} : x \leq 1, y \in [0, \pi]\}$ . Describe the image of  $f(A)$ .

**Solution**

Firstly, note that

$$e^z = e^{x+iy}$$

$$e^x \in (0, e]$$

$$y \in [0, \pi]$$

It is clear that the image will be in on the positive side of the imaginary-axis. Also, since  $e^x \in (0, e]$ , we get the right graph represented in [Figure 4.4](#). The image of  $f(A)$  is described in the left image of [Figure 4.4](#).