List of Theorems

14 Lecture 14 Jun 01 2018

14.1 Isomorphism Theorems (Continued 2)

14.1.1 Isomorphism Theorems (Continued)

Note (Recall)

In First Isomorphism Theorem 38, we had that for a group homomorphism $\alpha : G \to H$ where G and H are groups,

$$G/\ker\alpha \cong \operatorname{im}\alpha$$

Now let $\alpha : G \to H$ be a group homomorphism, $K = \ker\alpha$, $\phi : G \to G/K$ be the coset map, and $\bar{\alpha}$ be as defined in the proof of First Isomorphism Theorem 38. We then have the following commutative diagram to illustrate the relationship between the three groups.

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\phi"'] & H \\ G/K \arrow[ur, "\bar{\alpha}"'] & \end{tikzcd}$$

A natural question to ask after seeing the relationship is: Is $\bar{\alpha}\phi = \alpha$? If it is, is the definition of $\bar{\alpha}$ unique? The answer is: **YES!** on both accounts.

Proof

Let $g \in G$. Then

$$\bar{\alpha}\phi(g) = \bar{\alpha}\big(\phi(g)\big) = \bar{\alpha}(Kg) = \alpha(g)$$

Suppose $\alpha = \beta\phi$ where $\beta : {}^{G}\!/_{K} \to H$. Then

$$\beta(Kg) \overset{(1)}{=} \beta(\phi(g)) = \beta\phi(g) = \alpha(g) = \bar{\alpha}(Kg)$$

where (1) is because ϕ is surjective by Proposition 35. Therefore, we observe that $\beta = \bar{\alpha}$ for any $Kg \in {}^{G}\!/_{K}$. This proves that $\bar{\alpha}$ is the unique homomorphism such that ${}^{G}\!/_{K} \to H$ satisfying $\alpha = \bar{\alpha}\phi$. □

With that, we have the following proposition.

Proposition 39

Let $\alpha : G \to H$ be a group homomorphism, where G and H are groups. Let $K = \ker\alpha$. Then α factors uniquely as $\alpha = \bar{\alpha}\phi<$ where $\phi : G \to {}^{G}\!/_{K}$ is the coset map and $\bar{\alpha} : GK \to H$ is defined by

$$\bar{\alpha}(Kg) = \alpha(g).$$

Note that ϕ is surjective and $\bar{\alpha}$ is injective.

*In such a scenario, we also say that α **factors through** ϕ.*[1]

[1] Reference for the terminology: https://math.stackexchange.com/questions/68941/terminology-a-homomorphism-factors.

Example 14.1.1

Let $G = \langle\, g \,\rangle$ be a cyclic group. Consider $\alpha : \mathbb{Z} \to G$, defined as

$$\forall k \in \mathbb{Z} \quad \alpha(k) = g^k,$$

which is a group homomorphism. By definition, α is surjective. Note that

$$\ker\alpha = \{k \in \mathbb{Z} : g^k = 1\}.$$

We have, therefore, two cases to consider.

- *G is an infinite group*
  This would imply that $\ker\alpha = \{0\}$ since only $g^0 = 1$. Then by First Isomorphism Theorem 38, we have that

$$ {}^{\mathbb{Z}}\!/_{\ker\alpha} \cong G$$

  Note that[2]

$$ {}^{\mathbb{Z}}\!/_{\ker\alpha} = \{(\ker\alpha)k : k \in \mathbb{Z}\} = \{0 + k : k \in \mathbb{Z}\} = \mathbb{Z}.$$

[2] We are assuming that the group $\mathbb{Z}$ here works under the operation of addition, otherwise, if we employ multiplication, then $\mathbb{Z}$ would not be a group and α would not be a group homomorphism.

Therefore

$$\mathbb{Z} \cong G$$

- *G is a finite group*

  Suppose that $|G| = o(g) = n \in \mathbb{N}$*, which is valid by Corollary 24. Then*

  $$\ker \alpha = n\mathbb{Z}$$

  Then by the First Isomorphism Theorem 38, we have

  $$\mathbb{Z}/n\mathbb{Z} \cong G.$$

  Observe that

  $$\mathbb{Z}/n\mathbb{Z} = \{n\mathbb{Z} + k : k \in \mathbb{Z}\} = \mathbb{Z}_n$$

  since the set in the middle is the definition of the set of integers modulo n.[3] *Therefore,*

  $$\mathbb{Z}_n \cong G$$

  Therefore, we have that

  $$\mathbb{Z} \cong G \text{ or } \mathbb{Z}_{o(g)} \cong G$$

[3] This is why we often see texts from various authors using $\mathbb{Z}/n\mathbb{Z}$ to represent the set of integers modulo n.

Theorem 40 (Second Isomorphism Theorem)

Let H and K be the subgroups of a group G with $K \triangleleft G$*. Then*

- HK *is a subgroup of* G*;*
- $K \triangleleft HK$*;*
- $H \cap K \triangleleft H$*; and*
- $HK/K \cong H/H \cap K$.

Proof

Since $K \triangleleft G$*, by Lemma 29 and Proposition 30, we have that* $HK = KH$ *is a subgroup of* G*. Consequently, we have* $K \triangleleft HK$*, since* K *is clearly a subgroup of* HK *and* $K \triangleleft G$*, and so* $\forall x \in HK \subseteq G$ *we have that* $gK = Kg$.

Consider $\alpha : H \to HK/K$*, defined by*[4]

[4] Note that $Kh \in HK/K$ since $h \in H \subseteq HK$.