UW W17 PMATH333: Definitions and Theorems

Johnson Ng

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Appendix A

Zermelo-Fraenkel Set Theory and the Axiom of Choice

A.1 Introduction

Example A.1.1 (Russel's Paradox)
Let X be the set of all sets, and let $S = \{A \in X | A \notin A\}$.
Note for example that $Z \notin Z \implies Z \in S$, and $X \in X \implies X \notin S$.
Thus we have $S \in S \iff S \notin S$.

To ensure that mathematical paradoxes (like the above) can no longer arise, mathematicians considered the following questions, and with these questions, rough answers are provided:

1. What exactly is an allowable mathematical object?
   A: Every mathematical object is a mathematical set, and a mathematical set can be constructed using certain rules, for e.g. the now widely accepted Zermelo-Fraenkel Set Theory and the Axiom of Choice. While the Axiom of Choice is still highly criticized even today (e.g. the highly controversial Banach-Tarski Paradox), the Zermelo-Fraenkel Set Theory is widely welcomed, but not without critics. We shall call the Zermelo-Fraenkel Set Theory and the Axiom of Choice as the ZFC Axioms of Set Theory.

2. What exactly is an allowable mathematical statement?
   A: Every mathematical statement can be expressed in a formal symbolic language, which uses symbols rather than words from any spoken language.

3. What exactly is allowable in a mathematical proof?
   A: Every mathematical proof is a finite list of ordered pairs $(\mathscr{S}_n, \mathscr{F}_n)$ (which we can think of as proven theorems), where each $\mathscr{S}_n$ is a finite set of formulas (called the *premises*) and each $\mathscr{F}_n$ is a single formula (called the *conclusion*), which that each pair $(\mathscr{S}_n, \mathscr{F}_n)$ can be obtained from previous pairs $(\mathscr{S}_i, \mathscr{F}_i)$ with $i < n$, using certain proof rules.

In the remainder of this appendix, we shall look more into the first 2 questions.

Definition A.1.1 (Mathematical Symbols)
We allow ourselves to use only the following symbols from the following symbol set:

$\neg$	*not*
$\wedge$	*and*
$\vee$	*or*
$\implies$	*implies*
$\iff$	*if and only if*
$=$	*equals*
$\in$	*is an element of*
$\forall$	*for all*
$\exists$	*there exists*
$()$ $\{\}$ $[]$	*parenthesis*

along with some variable symbols such as $x, y, z, u, v, w, ...$ *or* $x_1, x_2, x_3, ...$

Definition A.1.2 (Formula)
A formula (in the formal symbolic language of first order set theory) is a non-empty finite string of symbols, from the above list, which can be obtained using finitely many applications following the three rules below:

1. *If x and y are variable symbols, then each of the following strings are formulas.*

$$x = y, \quad x \in y$$

2. *If F and G are formulas then each of the following strings are formulas.*

$$\neg F, \quad (F \wedge G), \quad (F \vee G), \quad (F \implies G), \quad (F \iff G)$$

3. *If x is a variable symbol and F is a formula then each of the following is a formula.*

$$\forall x \in F, \quad \exists x \in F$$

Definition A.1.3 (Free or Bounded Variable)
Let x be a variable symbol and let F be a formula. For each occurrence of the symbol x, which does not immediately follow a quantifier, in the formula F, we define whether the occurrence of x is free or bound inductively as follows:

1. *If F is a formula of one of the forms $y = z$ or $y \in z$, where y and z are variable symbols (possibly equal to x), then every occurence of x in F is free, and no occurrence is bound.*

2. *If F is a formula of one of the forms $\neg H, (H \wedge G), (H \vee G), (H \implies G), (H \iff G)$, where G and H are formulas, then each occurence of the symbol x is either an occurrence in the formula G or an occurrence in the formula H, and each free (respectively, bound) occurence of x in G remains free (respectively, bound) in F, and similarly for each free (or bound) occurrence of x in G. In other words, wlog, if x is bounded in G, then it is bounded in F, and vice versa.*

3. *If F is a formula of one of the forms $\forall y \in G$ or $\exists y \in G$, where G is a formula and y is a variable symbol. If y is different from x, then each free (or bound) occurrence of x in G remains free (or bound) in the formula G, and if $y = x$ then every free occurrence of x in G becomes bound in F, and every bound occurrence of x in G remains bound in F.*

Definition A.1.4 (Is Bound By and Binds)
When a quantifier symbol occurs in a given formula F, and is followed by the variable symbol x and then by the formula G, any free occurence of x in G will become bound in the given formula F (by the 3rd definition above). We shall say that the occurrence of x is bound by (that occurrence of) the quantifier symbol, or that (the occurrence of) the quantifier symbol binds the occurence of x.