# PMATH351 - Real Analysis (Class Notes)

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# Contents


**1 Lecture 1: Sep 8, 2017** **7**
- 1.1 Logistics . . . 7
- 1.2 Brief Introduction to the Course . . . 7
  - 1.2.1 Set Theory (Naive, for Real Analysis) . . . 7
- 1.3 Relations, Ordering and Zorn . . . 10

**2 Lecture 2: Sep 11, 2017** **11**
- 2.1 More on Relations . . . 11
- 2.2 Construction of the Real Numbers . . . 12
- 2.3 Dyadic representation of $\mathbb{R}$ . . . 14

**3 Lecture 3: Sep 13, 2017** **15**
- 3.1 Last Time . . . 15
- 3.2 Bounds and Completeness . . . 15
- 3.3 Chains and Zorn's Lemma . . . 17

**4 Lecture 4: Sep 15, 2017** **19**
- 4.1 Logistics . . . 19
- 4.2 Cardinal arithmetic . . . 19

**5 Lecture 5: Sep 18, 2017** **24**
- 5.1 Continuing CBS with examples . . . 24
- 5.2 Comparison Theorem . . . 25

**6 Lecture 6: Sep 20, 2017** **27**
- 6.1 Continuing ordinal arithmetic . . . 27

**7 Lecture 7: Sep 22, 2017** **31**
- 7.1 Metric Spaces . . . 31

**8 Lecture 8: Sep 25, 2017** **36**

8.1 Logistics . . . 36
8.2 Continuing Normed Vector Space . . . 36
8.3 $\ell_p$-spaces . . . 38

**9 Lecture 9: Sep 27, 2017 41**
9.1 Last Time . . . 41
9.2 Continuing with $\ell_p$ . . . 41
9.3 Topology of metric spaces . . . 44

**10 Lecture 10: Sep 27, 2017 46**
10.1 Continuing with Balls . . . 46

**11 Lecture 11: Oct 2, 2017 51**
11.1 Last time . . . 51
11.2 Continuing with Accumulation Points . . . 51
11.3 Continuous Functions . . . 53

# List of Definitions


1.2.1 Inclusion . . . 7
1.2.2 Power Set . . . 8
1.2.3 Unions and Intersections . . . 8
1.2.4 Difference Set . . . 8
1.2.5 Product Sets . . . 9
1.2.6 Function . . . 9
1.3.1 Relation . . . 10
2.1.1 More on Relations . . . 11
2.1.2 Total Order . . . 11
3.1.1 Partial Order . . . 15
3.2.1 Upper Bound, Supremum . . . 15
3.2.2 Complete . . . 16
3.2.3 Maximum . . . 16
3.2.4 Lower Bound, Infimum, Minimum . . . 16
3.3.1 Chain . . . 17
3.3.2 Maximal . . . 17
3.3.3 Linearly Independent, Spanning, Basis . . . 17
4.2.1 Injection, Surjection, Bijection . . . 19
4.2.2 Precedence . . . 21
7.1.1 Metric and Metric Space . . . 31

7.1.2 Norm, Normed Vector Space . . . . . . 33
9.2.1 The space $C[a, b]$ . . . . . . 42
9.3.1 Open and Closed Balls . . . . . . 44
9.3.2 Open and Closed Sets . . . . . . 44
10.1.1 Boundary . . . . . . 48
10.1.2 Interior . . . . . . 48
10.1.3 Convergence . . . . . . 49
10.1.4 Accumulation points/Cluster Points and Isolated Points . . . . . . 50
11.2.1 Closure . . . . . . 51
11.3.1 Continuity . . . . . . 53
11.3.2 Neighbourhood . . . . . . 53

# List of Theorems


| | | |
|---|---|---|
| Proposition 1.2.1 | De Morgan's Laws | 8 |
| Axiom 1.2.1 | Axiom of Choice | 9 |
| Proposition 1.2.2 | AC' | 9 |
| Proposition 3.2.1 | Infimum of a subset of a space | 16 |
| Axiom 3.3.1 | Zorn's Lemma | 17 |
| Theorem 3.3.1 | Vector space over $\mathbb{K}$ has a basis | 17 |
| Theorem 4.2.1 | Cantor-Bernstein-Schröder | 21 |
| Proposition 5.2.1 | Surjectivity | 25 |
| Theorem 5.2.1 | Comparison Theorem | 25 |
| Proposition 5.2.2 | Alternative Definitions of an Infinite Set | 26 |
| Corollary 6.1.1 | A set is either finite or denumerable | 28 |
| Theorem 6.1.1 | Cantor | 28 |
| Theorem 6.1.2 | Cantor's Continuum Hypothesis | 29 |
| Theorem 6.1.3 | Generalized Continuum Hypothesis | 29 |
| Theorem 6.1.4 | Cantor's Paradox | 29 |
| Axiom 6.1.1 | Well-Ordering | 29 |
| Lemma 7.1.1 | $\alpha\beta \leqslant \frac{\alpha^p}{p} + \frac{\beta^q}{q}$ | 34 |
| Theorem 7.1.1 | Holder's Inequality | 35 |
| Theorem 8.2.1 | Minkowski's Inequality | 37 |
| Corollary 8.2.1 | $\\| \cdot \\|_p$ is a norm | 38 |

| | | |
|---|---|---|
| Theorem 8.3.1 | $\ell_p$ is a $\mathbb{R}$-subspace | 38 |
| Theorem 8.3.2 | $(\ell_\infty, \|\cdot\|_\infty)$ is a normed vector space | 39 |
| Theorem 9.2.1 | Extreme Value Theorem | 42 |
| Theorem 9.2.2 | $(C[a,b], \|\cdot\|_p)$ as a normed vector space | 42 |
| Proposition 9.3.1 | Open/Closed Balls are Open/Closed Sets | 44 |
| Proposition 10.1.1 | | 47 |
| Proposition 10.1.2 | Characterizations of the Interior | 48 |
| Proposition 10.1.3 | | 50 |
| Theorem 11.2.1 | Characterization of the Closure | 51 |
| Theorem 11.3.1 | Characterization of continuity at a point | 53 |

# Chapter 1

# Lecture 1: Sep 8, 2017

## 1.1 Logistics

Course Website: http://www.math.uwaterloo.ca/~nspronk/math351/math351.html

## 1.2 Brief Introduction to the Course

### 1.2.1 Set Theory (Naive, for Real Analysis)

Sets whose existence that we shall take for granted:

$\mathbb{N} = \{1, 2, 3, ...\}$

$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$

$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}, \gcd(m, n) = 1\}$

**Definition 1.2.1 (Inclusion)**
*Given two sets $A$, $B$, write*

$$A \subseteq B, \quad A \subset B \text{ or } B \supseteq A, \quad \text{etc.} \tag{1.1}$$

*for "$B$ contains $A$", i.e. $\forall x \in A \implies x \in B$. We shall write*

$$A \subsetneq B \text{ if } A \subset B \wedge A \neq B \tag{1.2}$$

**Definition 1.2.2 (Power Set)**
*Let $X$ be a set. Let*

$$\mathcal{P}(X) := \{A : A \subseteq X\} \tag{1.3}$$

*Note that if $X = \{1, ..., n\}$, notice that $\mathcal{P}(X)$ has $2^n$ elements.*

**Definition 1.2.3 (Unions and Intersections)**
*Let $A, B \in \mathcal{P}(X)$ where $X$ is the universe, and $\{A_i\}_{i\in I} \subseteq \mathcal{P}(X)$ where $I \neq \varnothing$.*

$$A \cup B = \{x \in X : x \in A \vee x \in B\} \qquad \bigcup_{i\in I} A = \{x \in X : x \in A \textit{ for some } u \in I\}$$
$$A \cap B = \{x \in X : x \in A \wedge x \in B\} \qquad \bigcap_{i\in I} A = \{x \in X : x \in A \forall i \in I\}$$

*If we do not have A, B in a common universe, we let the "external union" be*

$$A \sqcup B = \{x : x \in A \veebar x \in B\} \tag{1.4}$$

**Example 1.2.1**
*Suppose $I \neq \varnothing$. What is the meaning of*

$$\bigcup_{i\in I} A_i, \quad \bigcap_{i\in I} A_i? \tag{1.5}$$

**Definition 1.2.4 (Difference Set)**
*If $A, b \in \mathcal{P}(X)$. Let*

$$A \backslash B = \{x \in X : x \in A \wedge x \notin B\} \tag{1.6}$$

*In particular*

$$X \backslash B = \{x \in X : x \notin B\} \textit{ (complement)} \tag{1.7}$$

**Proposition 1.2.1 (De Morgan's Laws)**
*If $X$ is a set, with $\{A_i\} \in \mathcal{P}(X)$, then*

$$X \backslash \bigcup_{i\in I} A_i = \bigcap_{i\in I} (X \backslash A_i), \quad X \backslash \bigcap_{i\in I} A_i = \bigcup_{i\in I} (X \backslash A_i) \tag{1.8}$$

The proof is straightforward and should be done in two lines.

**Definition 1.2.5 (Product Sets)**
*Let $A$, $B$ be sets.*

$$A \times B = \{(a, b) : a \in A, b \in B\} \quad \textit{(ordered pairs)} \tag{1.9}$$

**Definition 1.2.6 (Function)**
$f \subseteq A \times B$ *is called a function if*

$$\forall a \in A \quad \exists! b = f(a) \in B \tag{1.10}$$

*so that* $(a, b) \in f$.

*In practice, we write* $f : A \to B$ *and the ordered pairs are all denoted* $(a, f(a))$.

*If* $X_1, ..., X_n$ *are sets, where* $n \in \mathbb{N}$, *then*

$$X_1 \times \ldots \times X_n = \prod_{j=1}^{n} X_j = \{(x_1, ..., x_n) : x_j \in X_j \forall j \in \{1, ..., n\}\} \tag{1.11}$$

*is called the n-tuples of X.*

*IF* $\{X_i\}_{i \in I, I \neq \varnothing}$, *is a (or an infinite) family of sets*

$$\prod_{i \in I} X_i \{(x_i)_{i \in I} : x_i \in X \forall i \in I\} \tag{1.12}$$

**Axiom 1.2.1 (Axiom of Choice)**
*Given any non empty collection of nonempty sets* $\{A_i\}_{i \in I}$, *we have* $\prod_{i \in I} A_i \neq \varnothing$.

**Remark (B. Russell)**

1. $\forall n \in \mathbb{N}$, *let* $S_n = \{l_n, r_n\}$ *be a pair of shoes. Surely,* $\prod_{i \in I}^{\infty} S_n \neq \varnothing$.
2. $\forall n \in \mathbb{N}$, *let* $T_n = \{s_n, s'_n\}$ *be a pair of socks. Why do we expect* $\prod_{i \in I}^{\infty} T_n \neq \varnothing$?

**Proposition 1.2.2 (AC')**
*The AC is equivalent to* $(AC')$ *given any nonempty set A,*

$$\exists f : \mathcal{P}(A) \backslash \{\varnothing\} \to A \qquad \forall B \in P(A) \backslash \{\varnothing\} \quad f(B) \in B \tag{1.13}$$

**Proof**
$(AC) \implies (AC')$

*We assume there is*

$$(x_B)_{B\in P(A)\backslash\{\varnothing\}} \in \prod_{B\in P(A)\backslash\{\varnothing\}} B \tag{1.14}$$

*(which is nonempty by assumption).*

*Then we simply have to let* $f(B) = x_B$ *for each* $B$.

$(AC') \implies (AC)$

*Given a non-empty collection of nonempty sets* $\{A_i\}_{i\in I}$, *let*

$$A = \bigsqcup_{i\in I} A_i \quad \textit{(external product)} \tag{1.15}$$

*We have a choice function* $f : \mathcal{P}(A)\backslash\{\varnothing\} \to A, f(B) \in B$ *for each* $B$. *Then*

$$\big(f(A_i)\big)_{i\in I} \in \prod_{i\in I} A_i \tag{1.16}$$

## 1.3 Relations, Ordering and Zorn

**Definition 1.3.1 (Relation)**
*Let* $X$ *be a nonempty set. A relation on* $X$ *is any subset*

$$R \subseteq X \times X \tag{1.17}$$

*We write* $xRy$ *provided that* $(x, y) \in R$.

**Example 1.3.1**

1. *A function* $f \subseteq X \times X$ *is a relation.*
2. *In* $\mathbb{N} \times \mathbb{N}$, *consider*
$$mRn \iff \exists p \in \{0\} \cup \mathbb{N} \quad n = m + p \tag{1.18}$$
*We write* $m \leqslant n \iff mRn$.
3. *On* $\mathbb{Z}$, $m \leqslant n \iff n - m \in \{0\} \cup \mathbb{N}$.
4. *On* $\mathbb{Q}$, $\frac{m}{n} \leqslant \frac{\mu}{\nu} \iff m\nu \leqslant \mu n$ *in* $(\mathbb{Z}, \leqslant)$.
5. *On* $\mathcal{P}(X)$, *we have relations*
$$A \subseteq B$$
$$A \supseteq B$$

# Chapter 2

# Lecture 2: Sep 11, 2017

## 2.1 More on Relations

**Definition 2.1.1 (More on Relations)**
*A relation $R$ on $X$ is*

1. ***Symmetric*** *if* $xRy \implies yRx$.
2. ***Reflexive*** *if* $\forall x \in X \; xRx$
3. ***Transitive*** *if* $xRy \wedge yRz \implies xRz$
4. ***Anti-Symmetric*** *if* $xRy \wedge yRx \implies x = y \in X$

*(i), (ii) and (iii) makes up the* ***Equivalence Relation****. We usually use notations like* $\sim, \approx$.

*(ii), (iii) and (iv) makes up the* ***Partial Order*** *definition. We usually use notations like* $\leqslant, \geqslant$

In Example 1.3.1, (ii), (iii), (iv) and (v) are all partial orders. In (i), f is an equivalence relation only if f is an identity function.

**Definition 2.1.2 (Total Order)**
*A total order is a partial order where for x, y we have at least one of*

$$x \leqslant y \quad or \quad y \leqslant x \tag{2.1}$$

*holds.*

Notice that in Example 1.3.1, (ii), (iii) and (iv) are total orders. However, (v) is not if X has at least two elements.

If $\sim$ is an equivalence relation on X, then we denote the equivalence class by $[x] = \{y \in X : y \sim x\}$

**Example 2.1.1**
*On $\mathbb{Z} \times \mathbb{N}$, let $(m,n) \sim (\mu, v)$ if $m\nu = \mu n$ in $\mathbb{Z}$. Then equivalence classes $[(m,n)]$ are elements of $\mathbb{Q}$. Generally,*

$$\frac{m}{n} = [(m,n)] \tag{2.2}$$

## 2.2 Construction of the Real Numbers

We provide a sketch of Cantor's construction:

**Notation:** On $\mathbb{Q}$, define $\left|\frac{m}{n}\right| = \begin{cases} \frac{m}{n} & m > 0 \\ -\frac{m}{n} & m < 0 \end{cases}, n \in \mathbb{Z}$

We have the usual properties (triangle inequalities): for $p, q \in \mathbb{Q}$

$$|p+q| \leqslant |p| + |q| \tag{2.3}$$

$$\big||p| - |q|\big| \leqslant |p-q| \tag{2.4}$$

Let $\mathbb{Q}_+ = \{q \in \mathbb{Q} : q > 0\}$

$X = \{(q_n) = (q_n)_{n=1}^{\infty} \in \mathbb{Q}^{\mathbb{N}} : \forall \epsilon \in \mathbb{Q}_+ \ \exists n_\epsilon \in \mathbb{N} \ \forall n, m \geqslant n_\epsilon \ |q_n - q_m| < \epsilon\}$

(X is set of Cauchy sequences of rationals)

On X we define

$$(q_n) \sim (r_n) \text{ if } \forall \epsilon \in \mathbb{Q} \ \exists n_\epsilon \in \mathbb{N} \ |q_n - r_n| < \epsilon \text{ whenever } n \geqslant n_\epsilon \tag{2.5}$$

(tails become closer together)

Then $\sim$ is an equivalence relation (verify yourselves).

We let

$$\mathbb{R} = \{[(q_n)] : (q_n) \in X\} \tag{2.6}$$

**Note**
$\mathbb{R}$ *is a field.*

$$(q_n) \sim (s_n), (r_n) \sim (t_n) \implies (q_n + r_n) \sim (s_n + t_n), (q_n r_n) \sim (s_n t_n) \tag{2.7}$$

*(Check! To check for multiplication, observe that elements of $X$ form bounded sets in $\mathbb{Q}$).*

$(r_n) \nsim (0, 0, ...) \implies r_n = 0$ *for at most finitely many* $n$

$\implies$ *define*

$$t_n = \begin{cases} 1 & \text{if } r_n = 0 \\ \frac{1}{r_n} & \text{otherwise} \end{cases}$$

$\implies (r_n)(t_n) \sim (1, 1, 1, ...)$

We can define mutliplication, addition, etc. on $\mathbb{R}$ and it follows that $\mathbb{R}$ is a field.

**Note (Properties)**

1. *$\mathbb{Q}$ is a subfield:*
$$\mathbb{Q} \hookrightarrow \mathbb{R}, \quad q \mapsto [(q, q, ...)] \tag{2.8}$$
*(eq. class of const. seq.)*

2. *Total order: On $X$ let $(q_n) \leqslant (r_n)$ if*
$$\forall \epsilon \in \mathbb{Q}_+ \; \exists n_\epsilon \in \mathbb{N} \; \forall n \geqslant n_\epsilon \; q_n \leqslant r_n + \epsilon \tag{2.9}$$
*(Eq. $(1 - \frac{1}{n}) \leqslant (1, 1, ...)$)*

   *Then $(q_n) \leqslant (r_n), (q_n) \sim (s_n), (r_n) \sim (t_n) \implies (s_n) \leqslant (t_n)$ (check)*

   *Hence, let*

   *$[(q_n)] \leqslant [(r_n)]$ if $(q_n) \leqslant (r_n)$.*

3. *Density of $\mathbb{Q}$: (HW 1)*

   *If $[(q_n)] < [(r_n)]$ then there is $q$ in $\mathbb{Q}$ s.t.*
$$[(q_n)] < [(q, q, ...)] < [(r_n)] \tag{2.10}$$

4. *Absolute value: $|[(q_n)]| = [(|q_n|)]$*

   *This is the usual absolute value (check)*

## 2.3 Dyadic representation of $\mathbb{R}$

Like the density of $\mathbb{Q} \in \mathbb{R}$, we can show that for $[(q_n)] \in \mathbb{R}$ there is q in $\mathbb{Q}$ s.t. $[(q_n)] \leqslant [(q, q, ...)]$ (HW 1).

Let $X = [(q_n)] \in \mathbb{R}$. Suppose $x \geqslant 0$. Then there is unique $m \in \mathbb{N}$ s.t.

$$[(m, m, ...)] \leqslant x < [(m+1, m+1, ...)] \tag{2.11}$$

Call $m = \lfloor x \rfloor$.

Define

$$x_1 = \begin{cases} 0 & \text{if } x - \lfloor x \rfloor < \frac{1}{2} = [(\frac{1}{2})] \\ 1 & \text{if } x - \lfloor x \rfloor \geqslant \frac{1}{2} \end{cases} \tag{2.12}$$

$$\vdots \tag{2.13}$$

$$x_{n+1} = \begin{cases} 0 & \text{if } x - \left(\lfloor x \rfloor - \sum_{k=1}^{n} \frac{x_k}{2^k} < \frac{1}{2^{k+1}}\right) \\ 1 & \text{if } x - \left(\lfloor x \rfloor - \sum_{k=1}^{n} \frac{x_k}{2^k} \geqslant \frac{1}{2^{k+1}}\right) \end{cases} \tag{2.14}$$

Then, check that

$$x \sim \left( \lfloor x \rfloor + \sum_{k=1}^{2^n} \frac{x_k}{2^k} \right)_{n=1}^{\infty} \tag{2.15}$$

Write $x = \lfloor x \rfloor . x_1 x_2 x_3 ...$

Similarly, we have decimal (base 10) or ternary representation (base 3).

# Chapter 3

# Lecture 3: Sep 13, 2017

## 3.1 Last Time

**Definition 3.1.1 (Partial Order)**
*A partial order is a relation $\leqslant$ on $X$ which is*

- *reflexive*
- *transitive*
- *anti-symmetric*

*We write $(X, \leqslant)$ as a "partially ordered set" or a poset.*

## 3.2 Bounds and Completeness

**Definition 3.2.1 (Upper Bound, Supremum)**
*Let $X, \leqslant)$ be a partially ordered set (aka poset). Given $A \subset X$,*

- *an upper bound is any $u \in X$ s.t. $\forall x \in A$ $x \leqslant u$*
- *a supremum (aka least lower bound) is an upper bound $s$ s.t. $s \leqslant u$ for any upper bound $u$.*

**Note**

1. *A supremum need not exist.*

   *For example, in $(\mathbb{Q}, \leqslant)$,*

   - *$\mathbb{N}$ is not bounded above*

- *$A = \{q \in \mathbb{Q} : q^2 \leqslant 2\}$ is bounded above (e.g. 2 is an upper bound) but admits so supremum.*

2. *If a supremum exists, then it is unique (appeal to the anti-symmetry property of $\leqslant$), so we write $s = \sup A$.*

**Definition 3.2.2 (Complete)**
*We say that $(X, \leqslant)$ is complete if any set $A \subset X$ which admits an upper bound has a supremum, $\sup A$.*

**Example 3.2.1**

1. *$X \neq \varnothing$, consider $(\mathcal{P}(X), \subseteq)$. Given $A = \{A_i\}_{i\in I} \subseteq \mathcal{P}(X)$, we have $\sup A = \bigcup_{i\in I} A_i$, so $\mathcal{P}(X), \subseteq)$ is complete.*

2. *$(\mathbb{R}, \leqslant)$ is complete.*

   *(Sketch proof) Suppose $\varnothing \neq A \subset \mathbb{R}$ is bounded above. Based on (HW1), we can find $q_0, r_0 \in \mathbb{Q}$ $[\mathbb{Q} \hookrightarrow \mathbb{R}, q \hookrightarrow [(q, q, ...)]$ s.t.*

   - *$q_0$ is not an upper bound for $A$*
   - *$r_0$ is an upper bound for $A$*

   *Inductively, define for $n \in \{0\} \cup \mathbb{N}, (q_{n+1}, r_{n+1}) \in \mathbb{Q}^2$.*

$$(q_{n+1}, r_{n+1}) = \begin{cases} (q_n, \frac{1}{2}(q_n + r_n)) & \frac{1}{2}(q_n + r_n) \text{ is an upper bound for } A \\ (\frac{1}{2}(q_n + r_n), r_n) & \text{otherwise} \end{cases} \tag{3.1}$$

   *Fact (check): $[(q_n)_{n=1}^{\infty}] = [(r_n)_{n=1}^{\infty}]$ and is $\sup A$.*

**Definition 3.2.3 (Maximum)**
*Further, we call $m \in A(A \subset X, (X, \leqslant)$ poset a maximum of $A$ if*

- *$m = \sup A$*
- *$m \in A$*

**Definition 3.2.4 (Lower Bound, Infimum, Minimum)**
*We have symmetric defnintion for lower bounds, infimums (greatest lower bound) and minimums.*

*Note: The infimum of $A$ is unique if it exists, denoted as $\inf A$*

**Proposition 3.2.1 (Infimum of a subset of a space)**
*If $(X, \leqslant)$ is a complete partially ordered space, then any $A \subseteq X$ which is bounded below, admits an infimum.*

**Proof**

*Let $L = \{x \in X : \forall a \in A \ x \leqslant a\}$. Notice that $L \neq \varnothing$ (by assumption on $A$). Also, $L$ is bounded above, since any element of $A$ is an upper bound.*

*Then* $\sup L = \inf A$.

## 3.3 Chains and Zorn's Lemma

**Definition 3.3.1 (Chain)**

*Let $(X, \leqslant)$ be a poset. A chain is any subset $C \subseteq X$ s.t. $(C, \leqslant)$ is totally ordered.*

*(Note: Strictly, we should have $(C, \leqslant \restriction_{C \times C})$.)*

**Definition 3.3.2 (Maximal)**

*We say an element $m \in X$ is maximal if we have that $\forall x \in X \ m \leqslant x \implies x = m$.*

**Axiom 3.3.1 (Zorn's Lemma)**

*Suppose in a poset $(X, \leqslant)$ every chain $C \subseteq X$ admits an upper bound, i.e.*

$$\exists u \in X \ \forall x \in C \ x \leqslant u \tag{3.2}$$

*Then $(X, \leqslant)$ admits a maximal element.*

**Definition 3.3.3 (Linearly Independent, Spanning, Basis)**

*Let $V$ be a vector space over a field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}$). A subset $L \subseteq V$ is **linearly independent (aka lin. ind.)** if for each finite $\{v_1, ..., v_n\} \subseteq L$,*

$$\forall \alpha_n \in \mathbb{K} \ \ 0 = \sum_{i=1}^{n} \alpha_i v_i \implies \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$$

*A subset $S \subset V$ is **spanning** if for each $v \in V$ there are finite $\{v_1, ..., v_n\} \subseteq S, \{\alpha_1, ..., \alpha_n\} \subseteq \mathbb{K}$ s.t.*

$$v = \sum_{i=1}^{n} \alpha_i v_i$$

*A **basis** is a set $B \subset V$ which is both linearly independent and spanning.*

**Theorem 3.3.1 (Vector space over $\mathbb{K}$ has a basis)**

*A vector space $V$ over $\mathbb{K}$ always admits a basis.*

**Proof**

*Let $\mathcal{L} = \{L \subset V : L \text{ is linearly independent}\}$. We note that $(\mathcal{L}, \subseteq)$ is a poset.*

*Furthermore,* $\big\{\{v\} : v \in V\backslash\{0\}\big\} \subseteq \mathcal{L}$. *So* $\mathcal{L} \neq \varnothing$.

*Let* $\mathcal{C} = \{L_i\}_{i\in I}$ *be a chain in* $\mathcal{L}$, *and consider* $L = \bigcup_{i\in I} L_i$. *If* $\{v_1, ...v_n\} \subseteq L$, *we have* $v_k \in L_{i_k}$ *for some* $k \in [0, n]$, *and since* $\mathcal{C}$ *is a chain, we may relate so* $L_{i_1} \subseteq L_{i_2} \subseteq \ldots \subseteq L_{i_k}$. *Thus* $\{v_1, ..., v_n\} \subseteq L_{i_n}$ *and is lin. ind. It follows* $L$ *is lin. ind. Hence, Axiom 3.3.1 tells us that* $\mathcal{L}$ *admits a maximal element* $B$.

*WTP* $B$ *is spanning. Suppose* $B$ *is not spanning. Then there is* $v_o \in V$ *which cannot be written as a linear combination of finitely many vectors from* $B$. *Consider*

$$0 = \alpha_0 v_0 + \sum_{i=1}^{n} \alpha_i v_i \tag{3.3}$$

*for* $\{v_1, ..., v_n\} \subseteq B$, *and* $\alpha_1, ..., \alpha_n \in \mathbb{K}$. *If we can have* $\alpha_n \neq 0$, *then*

$$v_0 = \sum_{i=1}^{n} \left(-\frac{\alpha_i}{\alpha_n} v_i\right) \tag{3.4}$$

*which contradicts our assumption on* $v_o$. *Hence* $\alpha_n = 0$, *and thus* $0 = \sum_{i=1}^{n} \alpha_i v_i$, *so* $\alpha_1 = \ldots = \alpha_n = 0$, *as well. Hence* $B \cup \{v_o\} \in \mathcal{L}$. *But* $B \subseteq B \cup \{v_0\}$, *contradicting maximality.*

**Remark**

*An easy modification of the proof shows that any* $L = \mathcal{L}$ *is a subset of a basis.*

# Chapter 4

# Lecture 4: Sep 15, 2017

## 4.1 Logistics

**Office Hours**

- today: 1430 - 1520
- Wed, next week: 1430 - 1630

## 4.2 Cardinal arithmetic

**Definition 4.2.1 (Injection, Surjection, Bijection)**
*Given nonempty sets $X, Y$, a function $f : X \to Y$ is called a(n)*

- ***injection*** $x_1 \neq x_2 \in X \implies f(x_1) \neq f(x_2)$
- ***surjection*** $\forall y \in Y\ \exists x \in X\ f(x) = y$
- ***bijection*** *if it is both an injection and a surjection (aka invertible)*

*Of course, if $f : X \to Y$ is a bijection then we can define $f^{-1} : Y \to X$ by $f^{-1}(f(x)) = x$.*

*We write $X \sim Y$ if there exists a bijection $f : X \to Y$.*

*Sometimes, we write*

$$X \underset{f}{\sim} Y$$

**Note ($\sim$ as an equivalence relation)**

- *(reflexitivity)* $X \underset{id}{\sim} X$ *($id : X \to X$ is the identity function)*

- *(symmetry)* $X \underset{f}{\sim} Y \implies Y \underset{f^{-1}}{\sim} X$
- *(transitivity)* $X \underset{f}{\sim} Y \wedge Y \underset{g}{\sim} Z \implies X \underset{gf}{\sim} Z$

*Hence $\sim$ is an equivalence relation on any given family of sets. We let $|X|$ denote the equivalence class. We call this cardinality of X.*

*Note:* $|\varnothing| = 0, \ \ |\{1, ..., n\}| = n \in \mathbb{N}$

**Example 4.2.1**

*1.*

$$\mathbb{N} \sim \mathbb{Z} \quad \because f(n) = \begin{cases} \frac{n}{2} & n \textit{ is even} \\ \frac{1}{n}(1-n) & n \textit{ is odd} \end{cases}$$

*2.*

$$\mathbb{R} \underset{f}{\sim} (-1,1) \quad \because f(x) = \frac{x}{|x|+1}$$

*Execise: exhibit* $f^{-1}$

*Answer:* $f^{-1}(x) = \frac{x}{1-|x|}$

*3.* $a < b \in \mathbb{R}$ $(0,1) \underset{g}{\sim} (a,b), \ g(x) = a + x(b-a)$

**Note (Notation)**

$$\aleph_0 = |\mathbb{N}| \ \textit{(“aleph-naught”)} \quad c = |\mathbb{R}| \ \textit{(“continuum”)}$$

**Note (Arithmetic)**
*Let $A, B$ be sets.*

$$|A| + |B| = |A \sqcup B|$$
$$|A||B| = |A \times B|$$
$$|A|^{|B|} = |A^B| \quad (B \neq \varnothing, \ A^B = \{f : B \to A \mid f \textit{ is a fucntion}\})$$

**Note (Properties)**

- *(commutativity)*

$$|A| + |B| = |B| + |A|$$
$$|A||B| = |B||A|$$

- *(distributivity)*

$$|A|(|B| + |C|) = |A||B| + |A||C|$$
$$(A \times (B \sqcup C) \sim (A \times B) \sqcup (A \times C))$$

- *(exponential laws)*

$$(B \neq \varnothing \neq C)$$
$$(1) \quad |A|^{|B|+|C|} = |A|^{|B|}|A|^{|C|} \qquad (2) \quad |A|^{|B||C|} = \left(|A|^{|B|}\right)^{|C|}$$

$$(1) \quad \left(A^{B \sqcup C} \sim A^B \times A^C \text{ via } \phi \mapsto (\phi|_B,\ \phi|_C)\right)$$
$$(2) \quad A^{B \times C} \sim (A^B)^C \text{ via } \phi \mapsto (\phi(b, \cdot) : C \to A)_{b \in B}$$

**Definition 4.2.2 (Precedence)**
*For sets $A, B$, define*

$$A \preceq B \text{ if there is an injection } f : A \to B$$

*We sometimes write the above as $A \underset{f}{\preceq} B$.*

- *(reflexivity) $A \preceq A$*
- *(transitivity) $A \preceq B,\ B \preceq C \implies A \preceq C$*

We are one property short of making $\preceq$ as an order relation.

**Note**
*It seems reasonable to write $|A| \leqslant |B|$, in this case, our question is: Is $\leqslant$ in cardinal numbers anti-symmetric?*

**Theorem 4.2.1 (Cantor-Bernstein-Schröder)**
*If, for non-empty sets $A$, $B$, we have*

$$A \preceq B \ \wedge \ B \preceq A \implies A \sim B \tag{4.1}$$

*i.e.*

$$|A| \leqslant |B| \wedge |B| \leqslant |A| \implies |A| = |B| \tag{4.2}$$

**Proof**
*Our assumption is that we have injections*

$$A \underset{\phi}{\preceq} B, \quad B \underset{\psi}{\preceq} A \tag{4.3}$$

*To avoid triviality, let us suppose that neither $\phi$ or $\psi$ is surjective. Thus*

$$\phi(A) \subsetneq B \quad \psi \circ \phi(A) \subsetneq \psi(B) \subsetneq A \tag{4.4}$$

*Let $A_0 = A,\ A_1 = \psi(B),\ A_2 = \psi \circ \phi(A)$ and we inductively define*

$$A_{n+1} = g(A_n) \text{ where } g = \psi \circ \phi \tag{4.5}$$

*Then $A_2 \subsetneq A_1 \subsetneq A_0$, so by applying injection $g$,*

$$\begin{gathered} A_4 \subsetneq A_3 \subsetneq A_2 \\ \vdots \\ A_{n+1} \subsetneq A_n \subsetneq A_{n-1} \end{gathered}$$

*Hence, we may decompose*

$$\begin{aligned} A = A_0 &= (A_0 \backslash A_1) \cup A_1 \\ &= (A_0 \backslash A_1) \cup (A_1 \backslash A_2) \cup A_2 \\ &\vdots \\ &= \bigcup_{n=1}^{\infty} (A_{n-1} \backslash A_n) \cup A_\infty \end{aligned}$$

*where $A_\infty = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=2}^{\infty} A_n$, we likewise observe*

$$A_i = \bigcup_{n=2}^{\infty} (A_{n-1} \backslash A_n) \cup A_\infty$$

*Using definitions of the sets $A_n$ $(n \geqslant 2)$ we have*

$$g(A_{n=1} \backslash A_n) = A_{n+1} \backslash A_{n+2}$$

*Define*

$$h : A_0 \to A_1 \quad h(x) = \begin{cases} g(x) & x \in A_{n-1} \backslash A_n \text{ is odd} \\ x & \text{otherwise} \end{cases} \tag{4.6}$$

*Then $h$ is a bijection.*

*Thus $A = A_0 \underset{h}{\sim} A_1 - \phi(B),\ B \underset{\phi}{\sim} \psi(B)$ so we conclude that $A \sim B$.* □

**Example 4.2.2**

1. *Let* $a < b \in \mathbb{R}$. *Then*

$$\begin{aligned} [a,b) &\preceq \mathbb{R} \qquad \textit{obvious} \\ \mathbb{R} \sim (-1,1) \sim (0,1) \sim (a,b) &\preceq [a,b) \end{aligned}$$

*i.e.* $[a,b) \preceq \mathbb{R}$ *and* $\mathbb{R} \preceq [a,b)$ *so* $\mathbb{R} \sim [a,b)$.

# Chapter 5

# Lecture 5: Sep 18, 2017

## 5.1 Continuing CBS with examples

**Example 5.1.1**

*2.* $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$, *i.e.* $|\mathcal{P}(\mathbb{N})| = c$

$$\mathcal{P}(\mathbb{N}) \sim \{0,1\}^{\mathbb{N}} \text{ via } A \mapsto \chi(A) \tag{5.1}$$

*where*

$$\chi_A(n) = \begin{cases} 1 & n \in A \\ 0 & n \notin A \end{cases} \tag{5.2}$$

*is the "characteristic indicator".*

$$\{0,1\}^{\mathbb{N}} \preceq [0,1) \text{ via } (x_k)_{k=1}^{\infty} \mapsto \sum_{k=1}^{\infty} \frac{x_k}{3^k} = 0.x_1x_2x_3... \quad \text{is the ternary rep'n} \tag{5.3}$$

*which is injective.*

*Claim* $[0,1) \preceq \{0,1\}^{\mathbb{N}}$, $\quad 0.x_1x_2x_3... = \sum_{k=1}^{\infty} \frac{x_k}{2^k} \mapsto (x_k)_{k=1}^{\infty}$ *which is the binary rep'n. Note that this representation doesn't allow* $0.1111... = 1$ *(see Lecture 2).*

$\mathcal{P}(\mathbb{N}) \sim \{0,1\}^{\mathbb{N}} \preceq [0,1) \preceq \{0,1\}^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N})$

*Thus by Theorem 4.2.1,*

$$|\mathcal{P}(\mathbb{N})| = |[0,1)| = c = |\mathbb{R}| \tag{5.4}$$

*3.* $\mathbb{N} \sim \mathbb{Q} \sim \mathbb{N}^2$

- $\mathbb{N} \preceq \mathbb{Q}$ *(obvious)*

- $\mathbb{Q} \preceq \mathbb{Z} \times \mathbb{N}$, *which we pick* $\frac{m}{n} \mapsto (m, n)$ *with* $\gcd(m, n) = 1$ *where* $m \in \mathbb{Z}, n \in \mathbb{N}$.
- $\mathbb{Z} \times \mathbb{N} \sim \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ *as* $\mathbb{Z} \sim \mathbb{N}$
- $\mathbb{N}^2 \sim \mathbb{N}$ *via* $(m, n) \mapsto 2^m 3^n$

*Therefore*

$$\mathbb{N} \preceq \mathbb{Q} \preceq \mathbb{Z} \times \mathbb{N} \sim \mathbb{N}^2 \preceq \mathbb{N} \tag{5.5}$$

*Thus by Theorem 4.2.1*, $\mathbb{N} \sim \mathbb{Q} \sim \mathbb{N}^2$.

**Note (Notation)**
*We say that a set $A$ is*

- ***countable*** *if* $A \preceq \mathbb{N}$, *i.e.* $|A| \leqslant \aleph_0$
- ***denumerable*** *if* $A \sim \mathbb{N}$, *i.e.* $|A| = \aleph_0$

## 5.2 Comparison Theorem

**Proposition 5.2.1 (Surjectivity)**
*Suppose $X$ and $Y$ are non-empty sets and there is a surjection $g : X \to Y$. Then $Y \preceq X$.*

**Proof**
*Let* $f : \mathcal{P}(X)\backslash\{\varnothing\} \to X$ *be a choice function (by Axiom 1.2.1 AC). For each* $y \in Y$, *we have* $g^{-1}(\{y\}) = \{x \in X : g(x) = y\} \neq \varnothing$, *as $g$ is surjective. Define* $h : Y \to X$ *be given by* $h(y) = f(g^{-1}(\{y\})$ *and $h$ is injective, as if* $y_1 \neq y_2, \{y_1\} \cap \{y_2\} = \varnothing$, *so we see that*

$$g^{-1}(\{y_1\}) \cap g^{-1}(\{y_2\}) = \varnothing \tag{5.6}$$

*too.* □

**Theorem 5.2.1 (Comparison Theorem)**
*Let $X$ and $Y$ be sets. Then either $X \preceq Y$ or $Y \preceq X$.*

**Proof**
*If* $X = \varnothing$ *then* $X \preceq Y$; *likewise if* $Y = \varnothing$. *Hence, assume* $X \neq \varnothing \neq Y$. *Let*

$$\Delta = \{(A, f) : A \in \mathcal{P}(X)\backslash\{\varnothing\},\ f \in Y^A \text{ is an injection}\} \tag{5.7}$$

*We observe that* $\Delta \neq \varnothing$. *If* $x \in X, y \in Y$, *then* $(\{x\},\ x \mapsto y) \in \Delta$.

*On $\Delta$ let*

$$(A, f) \preceq (B, g) \iff {}^{A \subseteq B \subseteq X}_{g|_A = f} \tag{5.8}$$

*Notice that $\preceq$ is reflexive, anti-symmetric, and transitive. Thus $\preceq$ is a partial order on $\Delta$.*

*Let $\Gamma = \{(A_i, f_i)\}_{i\in I}$ be a chain in $(\Delta, \preceq)$. We let*

$$A = \bigcup_{i\in I} A_i \tag{5.9}$$

*and $f \in Y^A$ be given by $f(x) = f_i(x)$ provided $x \in A_i$.*

*Notice that $f$ is well-defined. Say $x \in A_i$ and $x \in A_j$, then since $\Gamma$ is a chain, without loss of generality, $A_i \subseteq A_j$, and $f_j|_{A_i} = f_i$.*

*Furthermore, if $x_1 \neq x_2 \in A$, then $x_1 \in A_{i_1}, x_2 \in A_{i_2}$, and we may suppose $A_{i_1} \subseteq A_{i_2}$. Then $f(x_1) = f_{i_1}(x_1) = f_{i_2}(x_1) \neq f_{i_2}(x_2) = f(x_2)$.*

*So $f$ is an injection. Thus $(A, f) \in \Delta$ and is an upper bound for $\Gamma$.*

*Thus there is a maximal element $(M, g) \in \Delta$, by Axiom 3.3.1 Zorn's Lemma.*

1. *Case 1: $M = X$. Then $X = M \underset{g}{\preceq} Y$.*
2. *Case 2: $M \subsetneq X$. We wish to see that $g$ is surjective.*

   *Suppose not, i.e. $\exists y_0 \in Y \backslash g(M)$. Since $M \subsetneq X$, $\exists x_0 \in X \backslash M$. Define $h : M \cup \{x_0\} \to Y$ by*

   $$h(x) = \begin{cases} g(x) & x \in M \\ y_0 & x = x_0 \end{cases} \tag{5.10}$$

   *which is injective.*

   *Then $(M \cup \{x_0\}, h) \in \Delta$, and $(M, g) \precneqq (M \cup \{x_0\}, h)$, contradicting the maximality of $(M, g) \in \Delta$. Thus $g$ is surjective as desired.*

*Therefore, $Y \underset{g^{-1}}{\preceq} X$.* □

**Proposition 5.2.2 (Alternative Definitions of an Infinite Set)**
*Let $A$ be a set. Then TFAE:*

1. *$n \leqslant |A|$ for all $n \in \mathbb{N}$.*
2. *$\aleph_0 \leqslant |A|$, i.e. $A$ is infinite*
3. *$\exists B \subsetneq A$ s.t. $|B| = |A|$.*
4. *$1 + |A| = |A|$ (Hilbert hotel)*
5. *$\aleph_0 + |A| = |A|$*

# Chapter 6

# Lecture 6: Sep 20, 2017

## 6.1 Continuing ordinal arithmetic

**Proof**

*1.* $1 \implies 2$

*We have that for each $n \in \mathbb{N}$ there is an injection $\phi_n : \{1, ..., n\} \to A$. Inductively, define $f : \mathbb{N} \to A$ by*

$$\begin{aligned} f(1) &= \phi_1(1) \\ &\vdots \\ f(n+1) &= \phi_{n+1}(k) \quad \textit{where } k = \min\{j \in \{1, ..., n+1\} : \phi_{n+1}(j) \notin \{f(1), ..., f(n)\}\} \end{aligned}$$

*The $f$ is injective by construction, i.e. $\mathbb{N} \underset{f}{\lesssim} A$ or $\aleph_0 \leqslant |A|$*

*2.* $2 \implies 3$

*We have $\mathbb{N} \underset{f}{\lesssim} A$. Let $B = A \backslash \{f(1)\}$.*

*Define $g : A \to B$ by*

$$g(x) = \begin{cases} f(n+1) & x = f(n),\ n \in \mathbb{N} \\ x & \textit{otherwise} \end{cases} \tag{6.1}$$

*Then $A \underset{g}{\sim} B$, i.e. $|A| = |B|$.*

*3.* $3 \implies 4$

*We suppose that there is* $x_0 \in A \backslash B$ *and* $B \sim A$. *Thus,*

$$A \sim B \preceq B \cup \{x_0\} \preceq A \tag{6.2}$$

*Then by Theorem 4.2.1,* $A \sim B$ *and furthermore* $A \sim B \cup \{x_0\} \sim A \sqcup \{1\}$, *i.e.* $|A| = |A| + 1$.

4. $4 \implies 5$

   *We have* $\{1\} \sqcup A \underset{\phi}{\sim} A$. *Then* $\phi(A) \subsetneq A$. *Thus* $\phi \circ \phi(A) \subsetneq \phi(A) \subsetneq A$, *and by induction*

$$\underbrace{\phi^{\circ n}}_{\phi \text{ composed with itself } n \text{ times}}(A) \subsetneq \phi^{\circ(n-1)}(A) \subsetneq \ldots \subsetneq A \tag{6.3}$$

   *Hence* $|A| \geqslant |A \backslash \phi^{\circ n}(A)| \geqslant n$ *(at each stage above, we gain at least one point).*

5. $2 \implies 5$

   *We have* $\mathbb{N} \underset{f}{\preceq} A$. *Let*

$$g : \mathbb{N} \sqcup A \to A,\ g(x) = \begin{cases} f(2n) & x = n,\ n \in \mathbb{N} \\ f(2n+1) & x = f(n) \in A,\ n \in \mathbb{N} \\ x & \text{otherwise} \end{cases} \tag{6.4}$$

6. $5 \implies 2$

   $\aleph_0 \leqslant \aleph_0 + |A| \underset{\text{by assumption}}{=} |A|$.

**Note**
*Any set satisfying 1 to 5 of the above is called infinite.*

**Corollary 6.1.1 (A set is either finite or denumerable)**
*If* $A \in \mathcal{P}(\mathbb{N})$, *then either* $A$ *is finite or denumerable.*

**Proof**
*Either* $n \leqslant |A|$ *for all* $n \in \mathbb{N}$, *or* $|A| < n$ *for some* $n \in \mathbb{N}$.

**Theorem 6.1.1 (Cantor)**
*For any set* $X$

$$|X| \lneqq |\mathcal{P}(X)|,\ \text{i.e. } X \preceq \mathcal{P}(X) \wedge X \nsim \mathcal{P}(X) \tag{6.5}$$

**Proof**
*If* $X = \varnothing, 0 = |\varnothing| \lneqq 1 = |\{\varnothing\}|$.

*If $X \neq \emptyset$, then $x \mapsto \{x\} : X \to \mathcal{P}(X)$ shows $X \preceq \mathcal{P}(X)$.*

*Now suppose $X \neq \emptyset$, $f : X \to \mathcal{P}(X)$. We will show that f cannot be surjective. Let*

$$E = \{x \in X : x \notin f(x)\} \tag{6.6}$$

*i.e. E is a set that is not in the range of f.*

*If we had $E \subseteq f(X)$, i.e. $E = f(x)$ for some $x \in X$, then either*

- *$x \in E$, i.e. $x \notin f(x)$, which means that $E \neq f(x)$, or*
- *$x \notin E = f(x)$, so $x \in E$.*

*These contradictions show that $E \nsubseteq f(X)$.*

*Hence there is no surjection $f : X \to \mathcal{P}(X)$.*

**Example 6.1.1**
$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(bbN)| = |\mathbb{R}| = c$

**Theorem 6.1.2 (Cantor's Continuum Hypothesis)**
*This is no set A such that*

$$\aleph_0 < |A| < c \tag{6.7}$$

**Remark**
*This theorem has recently been proven (about a month ago from Sep 20, 2017). This theorem is independent of ordinary set theory.*

**Theorem 6.1.3 (Generalized Continuum Hypothesis)**
*Given an infinite set C, there is no set A such that*

$$|C| < |A| < |\mathcal{P}(C)| \tag{6.8}$$

**Theorem 6.1.4 (Cantor's Paradox)**
*There is no "set" of all sets.*

*Suppose there was a universal set $\mathcal{U}$, i.e. any set $A \subseteq \mathcal{U}$. But then,*

$$|\mathcal{U}| < |\mathcal{P}(\mathcal{U})|, \text{ so } \mathcal{P}(\mathcal{U}) \npreceq \mathcal{U} \tag{6.9}$$

*so $\mathcal{U}$ cannot exist.*

**Axiom 6.1.1 (Well-Ordering)**
*Given a non-empty set X, a **well-order** is a partial order on X such that any $\emptyset \neq A \subseteq X$ admits a minimal element, i.e.*

$$\exists m_A \in A \; \forall a \in A \; m_A \leqslant a \tag{6.10}$$

**Remark**
*Well-order VS total order:* $x, y \in X$ *consider* $A = \{x, y\}$.

**Example 6.1.2**

1. $(\mathbb{N}, \leqslant)$ *is well-ordered (principle of mathematical induction).*
2. $\mathbb{N}^2$. *Let us consider two well-orders.*

   *(pyramid)* $(m, n) \leqslant (\mu, \nu) \iff$

$$\begin{cases} \text{either } m + n < \mu + \nu \\ m + n = \mu + \nu \text{ and } m \leqslant \mu \end{cases} \tag{6.11}$$

   *(lexicographic)* $(m, n) \underset{l}{\leqslant} (\mu, \nu) \iff$

$$\begin{cases} \text{either } m < \mu \text{ or} \\ m = \mu \text{ and } n \leqslant \nu \end{cases} \tag{6.12}$$

   *Notice that* $(m, n) \underset{l}{\leqslant} (\mu, \nu) \iff 2m - \frac{1}{n} \leqslant 2\mu - \frac{2}{\nu} \in (\mathbb{Q}, \leqslant)$

# Chapter 7

# Lecture 7: Sep 22, 2017

## 7.1 Metric Spaces

**Note**
*We can use $\mathbb{R}$ in any reasonable manner.*

**Definition 7.1.1 (Metric and Metric Space)**
*Let $X$ be a nonempty set. A metric $d : X \times X \to \mathbb{R}$ is a function which satisfies, for $x, y, z \in X$*

- ***(non-negativity)*** $d(x, y) \geqslant 0$
- ***(non-degeneracy)*** $d(x, y) = 0 \iff x = y$
- ***(symmetry)*** $d(x, y) = d(y, x)$
- ***(triangle inequality)*** $d(x, z) \leqslant d(x, y) + d(y, z)$

*We often call the pair $(X, d)$ a metric space.*

**Example 7.1.1**

1. *On $\mathbb{R}, d(x, y) = |x - y|$*
2. *Let $X \neq \varnothing$ any set. Define the "discrete" metric*

$$d : X \times X \to \{0, 1\} \subseteq \mathbb{R}, \;\; d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases} \tag{7.1}$$

*Note that non-degeneracy and symmetry are obvious. The triangle inequality is sat-*

*isfied since*

$$\textit{Case: } x \neq y \neq z \neq x$$
$$1 = d(x,z) \leqslant 2 = d(x,y) + d(y,z)$$

3. *Let $f : \mathbb{R} \to \mathbb{R}$ be strictly increasing. Let*

$$d_f : \mathbb{R}^2 \to [0,\infty) \;\; d_f(x,y) = |f(x) - f(y)| \tag{7.2}$$

*E.g. $f(x) = \frac{x}{|x|+1}$.*

*Exercise: check for its properties.*

**Proof**

*By definition of $d_f$, it is non-negative and symmetric.*
*If $x = y$, then $d_f(x,y) = |f(x) - f(y)| = |f(x) - f(x)| = 0$. Suppose $x \neq y$. Since $f$ is strictly increasing, without loss of generality, suppose $f(x) < f(y)$. Then $d_f(x,y) > 0$ since $f(y) - f(x) > 0$. Thus $d_f$ is non-degenerate.*
*Let $x, y, z \in \mathbb{R}^2$.*

$$\begin{aligned} d_f(x,z) &= |f(x) - f(z)| \\ &= |f(x) - f(y) + f(y) - f(z)| \\ &\leqslant |f(x) - f(y)| + |f(y) - f(z)| \\ &= d_f(x,y) + d_f(y,z) \end{aligned}$$

4. *(French railroad metric) Suppose we have a set $X \neq \varnothing$, and a function $f : X \to [0,\infty)$ which satisfies $f^{-1}(\{0\}) = \{p_0\}$. Notice that $f(x) > 0$ if $x \in X \backslash \{p_0\}$.*

$$d_f : X \times X \to [0,\infty) \;\; d_f(x,y) = \begin{cases} 0 & x = y \\ f(x) + f(y) & x \neq y \end{cases} \tag{7.3}$$

*Easy exercise: This is a metric.*

**Proof**
*Non-negativity and non-degeneracy are embeded in the function, since* $\forall x, y \in X$*, since* $f(x), f(y) \in [0, \infty)$*, we have that* $d_f(x, y) = f(x) + f(y) \geqslant 0$*, and if* $x = y$*,* $d_f(x, y) = 0$*.*
*The function is also symmetric, since*

$$\begin{gathered}\forall x, y \in X \\ x \neq y \implies d_f(x,y) = f(x) + f(y) = f(y) + f(x) = d_f(y,x) \\ x = y \implies d_f(x,y) = 0 = d_f(y,x)\end{gathered}$$

*To prove the triangle inequality, let* $x, y, z \in X$*. If* $x = y = z$*,* $d_f$ *is trivially a metric. Without loss of generality, suppose* $x = y \neq z$*, then* $d(x, z) = f(x) + f(z) \stackrel{(1)}{=} f(y) + f(z) = d(x, y) + d(y, z)$*, where (1) is since* $f(x) = f(y)$*, and* $d(x, y) = 0$*. Suppose* $x \neq y \neq z$*, then*

$$\begin{aligned} d_f(x,z) &= f(x) + f(z) \\ &\leqslant f(x) + f(y) + f(y) + f(z) \quad \text{since } f(y) \geqslant 0 \\ &= d_f(x,y) + d_f(y,z) \end{aligned}$$

**Definition 7.1.2 (Norm, Normed Vector Space)**
*Let* $V$ *be a vector space over* $\mathbb{R}$*. A* ***norm*** *is a function* $\|\cdot\| : V \to \mathbb{R}$ *which satisfies, for* $x, y \in V$*,* $\alpha \in \mathbb{R}$

1. ***(non-negativity)*** $\|x\| \geqslant 0$
2. ***(non-degeneracy)*** $\|x\| = 0 \iff x = 0$
3. ***($\|\cdot\|$-homogeneity)*** $\|\alpha x\| = |\alpha| \, \|x\|$
4. ***(subadditivity)*** $\|x + y\| \leqslant \|x\| + \|y\|$

*We call the pair* $(V, ||\cdot||)$ *a* ***normed vector space****.*

**Note**
*If* $(V, \|\cdot\|)$*is a normed vector space, then*

$$d : V \times V \to [0, \infty) \;\; d(x, y) = \|x - y\| \tag{7.4}$$

*is always a metric on* $V$*. Everything is easy to check; subadditivity of* $\|\cdot\| \implies$ *triangle inequality of* $d$*.*

**Example 7.1.2**
1. $(\mathbb{R}, |\cdot|)$ *is a normed vector space.*

2. *On $\mathbb{R}^n$, for $x = (x_1, ..., x_n)$*

$$\|x\|_2 = \sqrt{x_1^2 + \ldots + x_n^2} \tag{7.5}$$

*This is the Euclidean norm.*

*Consider, also*

$$\|x\|_1 = |x_1| + \ldots + |x_n|$$
$$\|x\|_\infty = \max\{|x_1|, ..., |x_n|\}$$

**Note**
*non-degeneracy and $|\cdot|$-homogeneity are obvious for $\|\cdot\|_1,\ \|\cdot\|_\infty$*

*Let us consider subadditivity*

$$\begin{aligned}\|x+y\|_1 &= |x_1 + y_1| + \ldots + |x_n + y_n| \\ &\leqslant |x_1| + |y_1| + \ldots + |x_n| + |y_n| \\ &= |x_1| + \ldots + |x_n| + |y_1| + \ldots + |y_n| \\ &= \|x\|_1 + \|y\|_1\end{aligned}$$

$$\begin{aligned}\|x+y\|_\infty &= \max\{|x_i + y_i| : i = 1, ..., n\} \\ &= \max\{|x_i| + |y_i| : i = 1, ..., n\} \\ &= \max\{|x_i| + |y_j| : i, j = 1, ..., n\} \\ &= \max\{|x_i| : i = 1, ..., n\} + \max\{|y_j| : j = 1, ..., n\} \\ &= \|x\|_\infty + \|y\|_\infty\end{aligned}$$

*Now for $1 < p < \infty$ consider*

$$x^p = \begin{cases} e^{p \log x} & x > 0 \\ 0 & x = 0 \end{cases} \tag{7.6}$$

$$\|x\|_p = (|x_1|^p + \ldots + |x_n|^p)^{\frac{1}{p}}$$

**Remark (Cauchy-Bunyakovsky-Schwartz)**
$|x \cdot y| \leqslant \|x\|_2 \|y\|_2$

**Lemma 7.1.1 ($\boldsymbol{\alpha\beta \leqslant \frac{\alpha^p}{p} + \frac{\beta^q}{q}}$)**
*Let $\alpha, \beta \leqslant 0 \in \mathbb{R},\ 1 < p < \infty$ and $q$ is chosen such that $\frac{1}{p} + \frac{1}{q} = 1$ (i.e. $q = \frac{p}{p-1}$) then*

$$\alpha\beta \leqslant \frac{\alpha^p}{p} + \frac{\beta^q}{q} \tag{7.7}$$

*with the equality when* $\alpha^p = \beta^q$.

**Proof**

*Consider the graph of* $y = x^{p-1}$ *(assume* $p \geqslant 2$*). Then*

$$\begin{aligned}\alpha\beta &\leqslant \int_0^\alpha x^{p-1}dx + \int_0^b y^{q-1}dy \\ &= \frac{\alpha^p}{p}\frac{\beta^q}{q}\end{aligned}$$

*Equality holds only if* $\beta = \alpha^{p-1} \implies \beta^{\frac{1}{p-1}} = \alpha \implies \beta^q = \alpha^p$

**Theorem 7.1.1 (Holder's Inequality)**

*Let* $x, y \in \mathbb{R}^n$, $1 < p < \infty$ *and* $q$ *be so* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\left|\sum_{j=1}^n x_j y_j\right| \leqslant \sum_{j=1}^n |x_j||y_j| \leqslant \left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^n |y_j|^q\right)^{\frac{1}{q}} = \|x\|_p \|y\|_q \tag{7.8}$$

# Chapter 8

# Lecture 8: Sep 25, 2017

## 8.1 Logistics

Expect assignment 2 to be up tonight!

## 8.2 Continuing Normed Vector Space

**Proof (Holder's Inequality)**
$\|x\|_p\|y\|_q = 0 \implies (x = 0 \vee y = 0) \wedge$ *the inequality is trivial. Let us assume* $\|x\|_p\|y\|_q \neq 0$.
*For* $j = 1, ..., n$

$$\alpha_j = \frac{|x_j|}{\|x\|_p}, \quad \beta_j = \frac{|y_j|}{\|y\|_q}$$

*Then*

$$\begin{aligned}
\frac{1}{\|x\|_p\|y\|_q}\sum_{j=1}^{n}|x_j||y_j| &= \sum_{j=1}^{n}\alpha_j\beta_j \overset{(1)}{\leqslant} \sum_{j=1}^{n}\left(\frac{\alpha_j^p}{p}+\frac{\beta_j^q}{q}\right) \\
&= \frac{1}{p}\sum_{j=1}^{n}\alpha_j^p + \frac{1}{q}\sum_{j=1}^{n}\beta_j^q \\
&= \frac{1}{p\|x\|_p^p}\sum_{j=1}^{n}|x_j|^p + \frac{1}{q\|y\|_q^q}\sum_{j=1}^{n}|y_j|^q \\
&= \frac{1}{p\|x\|_p^p}\|x\|_p^p + \frac{1}{q\|y\|_q^q}\|y\|_q^q = \frac{1}{p}+\frac{1}{q} \overset{(2)}{=} 1
\end{aligned}$$

*where (1) is by Lemma 7.1.1 and (2) is by choice of $q$.*

*Hence, we multiply by $\|x\|_p\|y\|_q$ and see that*

$$\sum_{j=1}^{n} |x_j||y_j| \leqslant \|x\|_p\|y\|_q \tag{8.1}$$

□

**Theorem 8.2.1 (Minkowski's Inequality)**
*Let $x, y \in \mathbb{R}^n$ and $1 < p < \infty$. Then*

$$\|x+y\|_p \leqslant \|x\|_p + \|y\|_p \tag{8.2}$$

**Proof**
*If $x+y=0$, this is trivial, hence suppose $x+y \neq 0$. Compute*

$$\begin{aligned}
\|x+y\|_p^p = \sum_{j=1}^{n} |x_j+y_j|^p &= \sum_{j=1}^{n} |x_j+y_j||x_j+y_j|^{p-1} \\
&= \sum_{j=1}^{n} \left(|x_j|+|y_j|\right)|x_j+y_j|^{p-1} \\
&= \sum_{j=1}^{n} |x_j||x_j+y_j|^{p-1} + \sum_{j=1}^{n} |y_j||x_j+y_J|^{p-1} \\
&\leqslant \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |x_j+y_j|^{(p-1)q}\right)^{\frac{1}{q}} \\
&\quad + \left(\sum_{j=1}^{n} |y_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |x_j+y_j|^{(p-1)q}\right)^{\frac{1}{q}} \\
&= \left(\|x\|_p + \|y\|_p\right) \left(\sum_{j=1}^{n} |x_j+y_j|^{(p-1)q}\right)^{\frac{1}{q}}
\end{aligned}$$

*We have $\frac{1}{p}+\frac{1}{q}=1 \implies \frac{1}{q}=1-\frac{1}{p}=\frac{p-1}{p} \implies p=q(p-1)$, and thus*

$$\begin{aligned}
\|x+y\|_p^p &\leqslant \left(\|x\|_p + \|y\|_p\right) \left(\sum_{j=1}^{n} |x_j+y_j|^p\right)^{\frac{1}{q}} \\
&= \left(\|x\|_p + \|y\|_p\right)\|x+y\|_p^{\frac{p}{q}}
\end{aligned}$$

*Now divide* $\|x+y\|_p^{\frac{p}{q}} \neq 0$*, we get*

$$\|x+y\|_p = \|x+y\|_p^{p-\frac{p}{q}} \leqslant \|x\|_p + \|y\|_p \quad (\textit{since } p - \frac{p}{q} = p(1-\frac{1}{q}) = \frac{p}{p} = 1) \tag{8.3}$$

**Corollary 8.2.1 ($\|\cdot\|_p$ is a norm)**
*Given* $1 < p < \infty, \|\cdot\|_p$ *is a norm on* $\mathbb{R}^n$*.*

**Proof**
*Clearly,* $\|\cdot\|_p$ *is non-negative and non-degenerate. If* $\alpha \in \mathbb{R}, x \in \mathbb{R}^n$ *then*

$$\begin{aligned}\|\alpha x\|_p = \left(\sum_{j=1}^{n} |\alpha x_j|_p^p\right)^{\frac{1}{p}} &= \left(\sum_{j=1}^{n} |\alpha|^p |x_j|^p\right)^{\frac{1}{p}} \\ &= |\alpha| \left(\sum_{j=1}^{n} |x_) j|^p\right)^{\frac{1}{p}} = |\alpha| \|x\|_p\end{aligned}$$

*Finally, subadditivity is provided by Theorem 8.2.1.*

## 8.3 $\ell_p$-spaces

Consider $\mathbb{R}^n = \{x = (x_k)_{k=1}^{\infty} \ : \ x_k \in \mathbb{R}\}$ which is a $\mathbb{R}$-vector space:

$$(x_k)_{k=1}^{\infty} + (y_k)_{k=1}^{\infty} = (x_k + y_k)_{k=1}^{\infty}, \quad \alpha(x_k)k = 1^{\infty} = (\alpha x_k)k = 1^{\infty} \tag{8.4}$$

We let, for $1 \leqslant p < \infty$,

- $\ell_p = \{x = (x_k)_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \sum_{k=1}^{\infty} |x_k|^p = \lim_{n\to\infty} \sum_{k=1}^{n} |x_k|^p < \infty\}$

  and

  $\ell_{\infty} = \{x = (x_k)_{k=1}^{\infty} : \sup_{k\in\mathbb{N}} |x_k| < \infty\}$

  On $\ell_p$ we define

$$\|x\|_p = \begin{cases} \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} & 1 \leqslant p < \infty \\ \sup_{k\in\mathbb{N}} |x_k| & p = \infty \end{cases} \tag{8.5}$$

**Theorem 8.3.1 ($\ell_p$ is a $\mathbb{R}$-subspace)**
*Let* $1 \leqslant p < \infty$*. Then* $\ell_p$ *is a* $\mathbb{R}$*-subspace of* $\mathbb{R}^{\mathbb{N}}$ *and* $\|\cdot\|_p$ *is a norm.*

**Proof**
*We shall prove these statements together. Suppose that $x, y \in \ell_p$. Then*

$$
\begin{aligned}
\|x+y\|_p &= \left(\sum_{k=1}^{\infty} |x_k + y_k|^p\right)^{\frac{1}{p}} \qquad \textit{(may be } \infty, \infty^{\frac{1}{p}} = \infty\textit{)} \\
&= \left(\lim_{n\to\infty} \sum_{k=1}^{n} |x_k + y_k|^p\right)^{\frac{1}{p}} \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{\frac{1}{p}} \qquad \left(\substack{x\mapsto x^{\frac{1}{p}} \text{ is cts on } [0,\infty) \\ x\to\infty \implies x^{\frac{1}{p}}\to\infty}\right) \\
&\leqslant \lim_{n\to\infty} \left[\left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{\frac{1}{p}}\right] \qquad \textit{by Theorem 8.2.1 on each } n \\
&= \left(\lim_{n\to\infty} \sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} + \left(\lim_{n\to\infty} \sum_{k=1}^{n} |y_k|^p\right)^{\frac{1}{p}} \qquad \textit{cty again} \\
&= \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{\frac{1}{p}} = \|x\|_p + \|y\|_p < \infty
\end{aligned}
$$

*Thus $x + y \in \ell_p$, and we get subadditivity of $\|\cdot\|_p$.*

*We note that non-negativity and non-degeneracy of $\|\cdot\|_p$ are obvious properties. Liekwise, the $|\cdot|$-homogeneity is straightforward.* □

**Theorem 8.3.2 ($(\ell_\infty, \|\cdot\|_\infty)$ is a normed vector space)**
*$(\ell_\infty, \|\cdot\|_\infty)$ is a normed vector space.*

**Proof**
$x, y \in \ell_\infty \implies$

$$
\begin{aligned}
\|x+y\|_\infty = \sup_{k\in\mathbb{N}} |x_k + y_k| &\leqslant \sup_{k\in\mathbb{N}} (|x_k| + y_k|) \\
&\leqslant \sup_{j,k\in\mathbb{N}} (|x_j| + |y_k|) \\
&= \sup_{j\in\mathbb{N}} |x_j| + \sup_{k\in\mathbb{N}} |y_k| = \|x\|_\infty + \|y\|_\infty
\end{aligned}
$$

*Other properties are easy (exercise).*

*Note that the norm must be non-negative since* $\forall x \in \ell_\infty, \|x\|_\infty = \max\{|x_1|, |x_2|, ..., |x_n|\} > 0$.
*The norm is also non-degenerate, since if* $x = 0$, *then* $\|x\|_\infty$ *is trivially zero, and if* $\|x\|_\infty = 0$, *then each* $|x_k| = 0$ *for all* $k$, *thus* $x = 0$.
*The norm is clearly* $\|\cdot\|$*-homogenous, since given* $\alpha x \in \ell_\infty$,

$$\begin{aligned} \|\alpha x\|_\infty &= \max\{|\alpha x_1|, |\alpha x_2|, ..., |\alpha x_n|\} \\ &= \alpha \max\{|x_1|, |x_2|, ..., |x_n|\} \\ &= \alpha\|x\|_\infty \end{aligned}$$

$\square$

# Chapter 9

# Lecture 9: Sep 27, 2017

## 9.1 Last Time

**Note**

$$1 \leqslant p < \infty$$

$$\ell_p = \left\{ x = (x_k)_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \|x\|_p = \left( \sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} \right\}$$

$$\ell_\infty = \left\{ x = (x_k)_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \|x\|_\infty = \sup_{k \in \mathbb{N}} |x_k| \right\}$$

## 9.2 Continuing with $\ell_p$

Define

$$c_0 = \{x = (x_k)_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \lim_{k \to \infty} x_k = 0\}$$

Note that $c_0$ is a $\mathbb{R}$-subspace of $\mathbb{R}^{\mathbb{N}}$ : $x, y \in c_0$ and $\alpha \in \mathbb{R}$, then

$$x + y = (x_k + y_k)_{k=1}^{\infty} \in c_0 \left[ x_k + y_k \overset{k \to \infty}{\to} 0 \right], \ \alpha x \in c_0$$

Also $(0) = (0, 0, ...) \in c_0$. Also, $c_l \subset \ell_\infty$. Indeed, let $n_1 \in \mathbb{N}$ such that

$$n \geqslant n_1 \implies |x_n - 0| = |x_k| < 1 \quad \text{(here, } \epsilon = 1)$$

Then for $h \in \mathbb{N}$,

$$|x_k| \leqslant \max\{x_1|, ..., |x_{n_1-1}|, 1\} = M$$

i.e. $\|x\|_\infty = \sup_{h\in\mathbb{N}} |x_k| \leqslant M$.

**Definition 9.2.1 (The space $C[a, b]$)**
*Let $a < b \in \mathbb{R}$, and*

$$C[a, b] = \{f \in \mathbb{R}^{[a,b]} : f \text{ is continuous } \} \tag{9.1}$$

*Note that $C[a, b]$ is a $\mathbb{R}$-vector space $f, g \in C[a, b]$, $\alpha \in \mathbb{R}$, define $f + g, \alpha f \in \mathbb{R}^{[a,b]}$ by*

$$(f + g)(t) = f(t) + g(t),\ (\alpha f)(t) = \alpha f(t) \tag{9.2}$$

*for all $t \in [a, b]$*

**Theorem 9.2.1 (Extreme Value Theorem)**
*if $f \in C[a, b]$ then there exists $t_{\min}, t_{\max} \in [a, b]$ for which*

$$f(t_{\min}) \leqslant f(t) \leqslant f(t_{\max}) \quad \text{for all} t \in [a, b] \tag{9.3}$$

Consequently from the Extreme Value Theorem (Theorem 9.2.1), if $f \in C[a, b]$, $|f(\cdot)| \in C[a, b]$ and there is $t_{\max} \in [a, b]$ for which $|f(t)| \leqslant |f(t_{\max})|$ for $r \in [a, b]$. Define, for $f \in C[a, b]$, $\|f\|_\infty = \max_{t\in[a,b]} |f(t)|$.

Just like for $(\ell_\infty, \|\cdot\|_\infty)$, we have that $(C[a, b], \|\cdot\|_\infty)$ is a normed vector space.

We note that $\|\cdot\|_\infty$ is not the only norm on $C[a, b]$. Let $1 \leqslant p < \infty$ and let, for $f \in C[a, b]$

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} \quad \text{(good ol' Riemann integral)} \tag{9.4}$$

**Theorem 9.2.2 ($(C[a, b], \|\cdot\|_p)$ as a normed vector space)**
*$(C[a, b], \|\cdot\|_p)$, $(1 \leqslant p < \infty)$ is a normed vector space.*

**Proof**
*First, let us recall right endpoint Riemann sums: $f, g \in C[a, b]$, then*

$$\int_a^b g(t)dt = \lim_{n\to\infty} \sum_{k=1}^{n} g\left(a + \frac{k}{n}(b - a)\right) \frac{b - a}{n} \tag{9.5}$$

*Hence if* $f \in C[a,b]$*, then*

$$\begin{aligned}\|f\|_p &= \left(\lim_{n\to\infty}\sum_{k=1}^{n}|f(b_k)|^p\frac{b-a}{n}\right) \quad \textit{where } b_k = a+\frac{k}{n}(b-a)\\ &= \lim_{n\to\infty}\left(\sum_{k=1}^{n}|f(b_k)|^p\right)^{\frac{1}{p}}\left(\frac{b-a}{n}\right)^{\frac{1}{p}}\end{aligned}$$

*Now, suppose,* $f,g \in C[a,b]$

$$\begin{aligned}\|f+g\|_p &= \lim_{n\to\infty}\left(\sum_{k=1}^{n}|f(b_k)+g(b_k)|^p\right)\left(\frac{b-a}{n}\right)^{\frac{1}{p}}\\ &\leqslant \lim_{n\to\infty}\left[\left(\sum_{k=1}^{n}|f(b_k)|^p\right)^{\frac{1}{p}}\right.\\ &\qquad\left.+\left(\sum_{k=1}^{n}|g(b_k)|^p\right)^{\frac{1}{p}}\right]\left(\frac{b-s}{n}\right)^{\frac{1}{p}} \quad \textit{Minkowski's Theorem 8.2.1}\\ &= \lim_{n\to\infty}\left(\sum_{k=1}^{n}|f(b_k)|^p\right)^{\frac{1}{p}}\left(\frac{b-a}{n}\right)^{\frac{1}{p}}+\lim_{n\to\infty}\left(\sum_{k=1}^{n}|g(b_k)|^p\right)^{\frac{1}{p}}\left(\frac{b-a}{n}\right)^{\frac{1}{p}}\\ &= \|f\|_p+\|g\|_p\end{aligned}$$

*hence we have subadditivity of* $\|\cdot\|_p$*. It is routine to verify that for* $\alpha \in \mathbb{R}$*,* $f \in C[a,b]$ *we have*

$$\|\alpha f\|_p = |\alpha|\|f\|_p \tag{9.6}$$

*and* $\|f\|_p \geqslant 0$ *as* $|f(\cdot)|^p \geqslant 0$ *and finally*

$$\|f\|_p = 0 \iff \int_a^b |f(t)|^p dx = 0 \overset{(1)}{\iff} |f(t)|^p = 0 \textit{ for all } t \in [a,b] \iff f = 0 \tag{9.7}$$

*((1) as* $|f(t)|^p \geqslant 0$ *for all* $t$*).*

**Note (Summary thus far about Normed Vector Spaces)**

$$\begin{gathered}(\mathbb{R}, |\cdot|)\\ (\mathbb{R}^{\mathbb{N}}, \|\cdot\|_p),\ 1 \leqslant p < \infty\\ (\ell_p, \|\cdot\|_p),\ 1 \leqslant p < \infty\\ (c_0, \|\cdot\|_\infty)\\ (C[a,b], \|\cdot\|_p),\ 1 \leqslant p < \infty\end{gathered}$$

## 9.3 Topology of metric spaces

**Definition 9.3.1 (Open and Closed Balls)**
*Let $(X, d)$ be a metric space, $x_0 \in X$, and $\epsilon > 0$. We define*

- *(open ball)* $B(x_0, \epsilon) = \{x \in X : d(x_0, x) < \epsilon\}$

- *(closed ball)* $B[x, \epsilon] = \{x \in X : d(x_0, x) \leqslant \epsilon\}$

**Example 9.3.1**
*In $\mathbb{R}$ we have for $a < b$*

$$(a, b) = B\left(\frac{1}{2}(a+b), \frac{1}{2}(b-a)\right)$$
$$[a, b] = B\left[\frac{1}{2}(a+b), \frac{1}{2}(b-a)\right]$$

**Definition 9.3.2 (Open and Closed Sets)**
*Let $X, d$ be a metric space.*

- *A set $U \subseteq X$ is open if*

$$\forall x \in U \; \exists \epsilon_x > 0 \; B(x, \epsilon_x) \subseteq U \tag{9.8}$$

- *A set $F \subseteq X$ is closed if $X \backslash F$ is open.*

**Proposition 9.3.1 (Open/Closed Balls are Open/Closed Sets)**
*Let $(X, d), x_0, \epsilon$ as above.*

1. $B(x_0, \epsilon)$ *is open.*
2. $B[x_0, \epsilon]$ *is closed.*

**Proof**

1. *Let $x \in B(x_0, \epsilon)$. Let $\epsilon_x = \epsilon - d(x_0, x) > 0$. Then for $y \in B(x, \epsilon_x)$ and we have*

$$\begin{aligned} d(x_0, y) \leqslant d(x_0, x) + d(y, x) &< d(x_0, x) + \epsilon_x \\ &= d(x_0, x) + \epsilon - d(x_0, x) = \epsilon \end{aligned}$$

   *So $y \in B(x_0, \epsilon)$, i.e. $B(x, \epsilon_x) \subseteq B(x_0, \epsilon)$.*

2. *Let* $x \in X \backslash B[x_0, \epsilon]$, *and let* $\epsilon_x = d(x, x_0) - \epsilon > 0$. *Now if* $y \in B(x, \epsilon_x)$ *then*

$$\begin{aligned} d(x, x_0) &\leqslant d(x, y) + d(y, x_0) \\ &< \epsilon_x + d(y, x_0) \\ &= d(x, x_0) - \epsilon + d(y, x_0) \end{aligned}$$

$\implies \epsilon < d(y, x_0)$, *i.e.* $y \notin B[x_0, \epsilon]$, *i.e.* $y \in X \backslash b[x_0, \epsilon]$, *so* $B(x, \epsilon_x) \subseteq X \backslash B[x_0, \epsilon]$.

**Remark**
*We may let*

$$B[x_0, 0] = \{x \in X : d(x_0, x) \leqslant 0\} = \{x_0\} \tag{9.9}$$

*As above, singleton sets* $\{x_0\}$ *are closed.*

# Chapter 10

# Lecture 10: Sep 27, 2017

## 10.1 Continuing with Balls

**Note (Recall)**
*$(X, d)$ be a metric space, $x_0 \in X$, $\epsilon > 0$*

$$B(x_0, \epsilon) = \{x \in X : d(x_0, x) < \epsilon\}$$
$$B[x_0, \epsilon] = \{x \in X : d(x_0, x) \leqslant \epsilon\}$$

**Example 10.1.1**

1. *$X \neq \varnothing$, $|X| \geqslant 2$, the discrete metric*

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

*We have for $x_0 \in X$,*

$$B(x_0 \epsilon) = \begin{cases} \{x_0\} & 0 < \epsilon \leqslant 1 \\ X & \epsilon > 1 \end{cases}$$
$$B[x_0, \epsilon] = \begin{cases} \{x_0\} & 0 < \epsilon < 1 \\ X & \epsilon \geqslant 1 \end{cases}$$

2. *(Geometry of balls in $\mathbb{R}^2$)*

$$1 \leqslant p < \infty,\ B_p(0, 1) = \{x \in \mathbb{R}^2, d_p(0, x) = ||x||_p < 1\}$$

*Pictures*

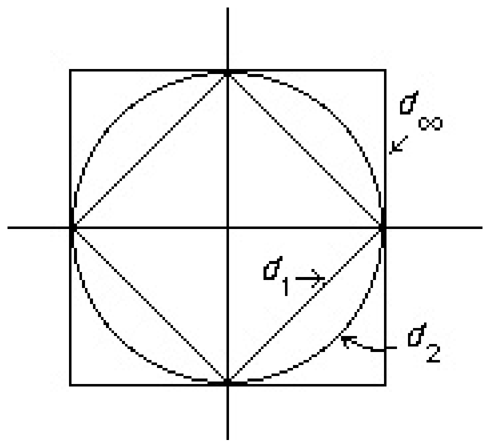
$B_1(0,1) : x_1 + x_2 = 1$ *is a diamond-shaped "ball"*

$B_2(0,1)$ *is a round "ball"*

$B_\infty(0,1)$ *is a squared "ball"*

$B_p(0,1)$ $1 < p < 2$ *the "ball" is inscribed inside the circle*

$B_p(0,1)$ $2 < p < \infty$ *: circle is inscribed within (a square with rounded corners)*




**Proposition 10.1.1**
*Let $(X, d)$ be a metric space.*

1. $X, \varnothing$ *are both open and closed.*
2. *If $\{U\}_{i\in I}$ is a family of open sets, then*

$$\bigcup_{i\in I} U_i \quad \textit{is open} \tag{10.1}$$

3. *If $\{U_1, ..., U_n\}$ is a finite family of open sets, then*

$$\bigcap_{i=1}^{n} U_i \quad \textit{is open} \tag{10.2}$$

4. *If $\{F_i\}_{i\in I}$ is a family of closed sets, then*

$$\bigcap_{i\in I} F_i \quad \textit{is closed} \tag{10.3}$$

5. *Of* $\{F_1, ..., F_n\}$ *is a finite family of closed sets, then*

$$\bigcup_{i=1}^{n} F_i \quad \textit{is closed} \tag{10.4}$$

*[Recall that singleton sets are closed, hence (5) implies that finite sets are closed]*

**Proof**

1. *Let* $x \in X$*. Then* $x \in B(x, 1) \subseteq X$*, so* $X$ *is open. The test for openness of* $\varnothing$ *is vacuously true (i.e. there are no points to speak of: there are no* $x \in \varnothing$ *at all, hence for any such* $x$*, we have* $x$ *is "contained" in a ball in* $\varnothing$*).*

   *We have* $\varnothing = X \backslash X$, $X = X \backslash \varnothing$ *are closed.*

2. *Let* $x \in U = \bigcup_{i \in I} U_i$*. Then there is some* $i_0 \in I$ *so* $x \in U_{i_0}$*, which is open, so there is an* $\epsilon_x > 0$ *such that*

$$x \in B(x, \epsilon_x) \subseteq U_{i_0} \subseteq U \tag{10.5}$$

3. *Let* $x \in V = \bigcap_{i=1}^{n} U_i$*. Then for each* $i = 1, ..., n$*, there is* $\epsilon_i > 0$ *so* $B(x, \epsilon_i) \subseteq U_i$*. Let* $\epsilon = \min\{\epsilon_1, ..., \epsilon_n\} > 0$ *and* $B(x, \epsilon) \subseteq \bigcap_{i=1}^{n} B(x, \epsilon_i) \subseteq V$

*For (4) and (5), use De Morgan's Laws and (2) & (3) from above.*

**Definition 10.1.1 (Boundary)**
*Given a metric space* $(X, d)$*,* $A \subseteq X$*, we define the boundary of* $A$ *as*

$$\partial A = \{x \in X : \forall \epsilon > 0 \; B(x, \epsilon) \cap A \neq \varnothing, \; \underbrace{B(x, \epsilon) \backslash A}_{B(x,\epsilon) \cap (X \backslash A)} \neq \varnothing\} \tag{10.6}$$

**Remark**
$\partial A = \partial(X \backslash A)$

**Definition 10.1.2 (Interior)**
*We let the interior of* $A$

$$A^\circ = \bigcup \{U \subseteq X : U \subseteq A \wedge U \textit{ is open}\} \tag{10.7}$$

**Proposition 10.1.2 (Characterizations of the Interior)**
*If* $(X, d)$*,* $A$ *are as above, then*

$$A^\circ = \{x \in X : \exists \epsilon_x > 0 \; B(x, \epsilon_x) \subseteq A\} \tag{10.8}$$
$$= A \backslash \partial A \tag{10.9}$$

**Proof**

*Let $x \in A$. Then we have either*

- *for some* $\epsilon_x > 0,\ x \in \underbrace{B(x, \epsilon_x)}_{open} \subseteq A \implies x \in A^\circ$*; or*
- $\forall \epsilon > 0,\ B(x.\epsilon)\backslash A \neq \varnothing \implies$ *since* $x \in A \cap B(x, \epsilon)$*, we have* $x \in \partial A$*. Since* $A^\circ \subseteq A$*, we see that the the two equalities in Equation 10.9 coincide.*

**Definition 10.1.3 (Convergence)**

*Let $(X, d)$ be a metric space, $(x_n)_{n=1}^\infty \subseteq X$ and $x_0 \in X$. Then we say that $(x_n)_{n=1}^\infty$ converges to the limit $x_0$, written*

$$x_0 = \lim_{n\to\infty} x_n \tag{10.10}$$

*or*

$$x_n \underset{n\to\infty}{\to} x_0 \tag{10.11}$$

*if*

$$\begin{gathered}\forall \epsilon > 0\ \exists n_\epsilon \in \mathbb{N}\\ n \geqslant n_\epsilon \implies d(x_0, x_n) < \epsilon\end{gathered}$$

**Remark**

*The limit, if it exists, is unique. Indeed, since*

$$x_0 = \lim_{n\to\infty} x_n \wedge y_0 = \lim_{n\to\infty} x_n$$

*then*

$$\begin{gathered}\forall \epsilon > 0\ \exists n_\epsilon, n'_\epsilon \in \mathbb{N}\\ n \geqslant n_\epsilon \implies d(x_0, x_n) < \frac{\epsilon}{2}\\ n \geqslant n'_\epsilon \implies d(y_0, x_n) < \frac{\epsilon}{2}\end{gathered}$$

*But then if $n \geqslant \max\{n_\epsilon, n'_\epsilon\}$ we have*

$$d(x_0, y_0) \leqslant d(x_0, x_n) + d(x_n, y_0) < \epsilon$$

*If this holds for all $\epsilon > 0,\ d(x_0, y_0) = 0$ so $x_0 = y_0$.*

**Example 10.1.2**

*Let $(V, \|\cdot\|)$ be a normed vector space. A subset $\{e_n\}_{n=1}^\infty \subseteq V$ is a **Schauder basis** provided that*

$$\begin{gathered}\forall x \in V\ \exists! \{x_n\}_{n=1}^\infty\\ x = \lim_{n\to\infty} \sum_{k=1}^n x_k e_k \quad \in V\end{gathered}$$

*Example: In* $\ell_p$ $(1 \leqslant p < \infty)$, *let* $e_n = (0, ..., 0, \underset{n\text{-th place}}{1}, 0, ..)$

**Definition 10.1.4 (Accumulation points/Cluster Points and Isolated Points)**
*We let* $(X, d)$ *is a metric space,* $A \subseteq X$ *as above, the set of accumulation points (or cluster points) be given*

$$A' = \{x \in X : \forall \epsilon > 0 \ (B(x, \epsilon) \backslash \{x\}) \cap A \neq \varnothing\} \tag{10.12}$$

*(aka a punctured ball).*

*Furthermore, we call elements of* $A \backslash A'$ *as isolated points.*

**Proposition 10.1.3**
*Given* $(X, d)$ *as a metric space,* $A \subseteq X$ *as above, the set of all accumulation points*

$$A' = \{x \in X : x = \lim_{n \to \infty} x_n, \ \text{where } (x_n)_{n=1}^{\infty} \subseteq A \backslash \{x\}\}$$

**Proof**
*If* $x \in A'$, *let* $x_1 \in (B(x, 1) \backslash \{x\}) \cap A$, *and inductively let*

$$x_{n+1} \in (B(x, \epsilon_n) \backslash \{x\}) \cap A$$

*where* $\epsilon + m = \min\{\frac{1}{n}, d(x, x_n)$.

*Then we have (exercise) that* $x = \lim_{n \to \infty} x_n$, *while* $(x_n)_{n=1}^{\infty} \subseteq A \backslash \{x\}$. *[Notice the points* $x_1, x_2, ..., x_n$ *are distinct]*

*The converse inclusion just uses the definition of limits.* □

# Chapter 11

# Lecture 11: Oct 2, 2017

## 11.1 Last time

**Note (3 Descriptions of interior $A^\circ$)**

$$\bigcup\{U \subseteq A : U \text{ open in } X\}, \{x \in X : \exists \epsilon_x > 0,\ B(x, \epsilon_x) \subseteq A\}$$

## 11.2 Continuing with Accumulation Points

$$A' = \{x \in X : \forall \epsilon > 0, (B(x\epsilon)\backslash\{x\}) \cap A \neq \varnothing\}$$
$$\text{Also } A' = \{x \in X : x = \lim_{n\to\infty} x_n,\ (x_n)_{n=1}^{\infty} \subset A\backslash\{x\}\}$$

**Definition 11.2.1 (Closure)**
*Given a metric space $(X, d)$ and $A \subseteq X$, define the closure of $A$ by*

$$\bar{A} = \cap\{F \subseteq X : A \subseteq F,\ F \text{ is closed in } X\} \tag{11.1}$$

*Of course, $A^\circ \subseteq A \subseteq \bar{A}$.*

**Theorem 11.2.1 (Characterization of the Closure)**
*Given a metric space $(X, d)$, $A \subseteq X$, the following sets are the same*

$$\bar{A} \quad A \cup \partial A \quad A \cup A' \tag{11.2}$$

*("meet" set) $A_m = \{x \in X : \forall \epsilon > 0,\ B(x, \epsilon) \cap A \neq \varnothing\}$.*

*("limit" set)* $A_L = \{x \in X : x = \lim_{n\to\infty} x_n, \quad \textit{where } (x_n)_{n=1}^{\infty} \subseteq A\}$

*[The notations $A_L, A_m$ will not be used afterwards, we shall use $\bar{A}$.]*

**Proof**
*We have*

$$\begin{aligned}
\bar{A} &= \bigcap\{F \subseteq X : A \subseteq F,\ F \textit{ closed}\} \\
&= \bigcap\{X\backslash U : U \subseteq X\backslash A,\ U \textit{ open in } X\} \\
&= X\backslash \bigcup\{U : U \subseteq X\backslash A\ U \textit{ open in } X\} \quad \textit{De Morgon's Law} \\
&= X\backslash[(X\backslash A)^\circ] \quad \textit{(complement of interior)} \\
&= X\backslash[(X\backslash A)\backslash\partial(X\backslash A)] \quad \textit{(definition of } (X\backslash A)^\circ\textit{)} \\
&= X\backslash[(X\backslash A)\backslash\partial A] \\
&= A \cup \partial A
\end{aligned}$$

*We thus have that* $\bar{A} = A \cup \partial A$.

*Now if* $x \in A \cup \partial A$, *then* $\forall \epsilon > 0, B(x,\epsilon) \cap A \neq \varnothing$ *[i.e. either* $x \in A \cap B(x,\epsilon)$, *or* $x \in \partial A$, *so that* $B(x,\epsilon) \cap A \neq \varnothing$*]. Thus* $A \cup \partial A \subseteq A_m$. *Conversely, if* $x \in A_m$, *then, either*

- $\exists \epsilon > 0,\ B(x,\epsilon) \subset A \implies x \in A^\circ \subseteq A$, *or*
- $\forall \epsilon > 0,\ B(x,\epsilon) \cap A \neq \varnothing$, *in which case* $x \in \partial A$.

*Hence,* $x \in A_m \implies x \in A \cup \partial A$ *so* $A_m \subseteq A \cup \partial A$.

*If* $x \in A \cup A'$, *then* $\forall \epsilon > 0,\ B(x,\epsilon) \cap A \neq \varnothing$. *Indeed, as above, either* $x \in A$, *so* $\forall \epsilon > 0, x \in B(x,\epsilon) \cap A$, *or* $x \in A'$ *so that* $B(x,\epsilon) \cap A \supseteq (B(x,\epsilon)\backslash\{x\}) \cap A \neq \varnothing$. *Hence* $A \cup A' \subseteq A_m$.

*The definition of a limit of a sequence shows that* $A_m = A_L$.

*Finally, consider*

$$\begin{aligned}
X\backslash(A \cup A') &\subseteq \{x \in X : \exists \epsilon_x > 0,\ B(x,\epsilon_x) \cap A = \varnothing( \implies B(x,\epsilon_x) \subseteq X\backslash A)\} \\
&= (X\backslash A)^\circ \implies X\backslash[(X\backslash A)^\circ] \subseteq X\backslash[X\backslash(A \cup A')]
\end{aligned}$$

*Hence*

$$\begin{aligned}
\bar{A} = X\backslash[(X\backslash A)^\circ] &\subseteq X\backslash[X\backslash(A \cup A')] \\
&= A \cup A'
\end{aligned}$$

*Hence* $\bar{A} \subseteq A \cup A' \subseteq A_m = \bar{A}$, *so* $\bar{A} = A \cup A'$. □

**Note**
*The "limit" set is going to help in A2*

## 11.3 Continuous Functions

**Definition 11.3.1 (Continuity)**
*Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Let $f : X \to Y$ and $x_0 \in X$. We say that $f$ is continuous at $x_0$ if*

$$\begin{aligned} &\forall \epsilon > 0 \quad \delta > 0 \\ d_X(x, x_0) < \delta &\implies d_Y(f(x), f(x_0)) < \epsilon \end{aligned} \tag{11.3}$$

*We say that $f$ is continuous at the domain $X$ if it is continuous at each point in $X$.*

**Note**

$$\begin{aligned} \text{Equation 11.3} &\iff f(B(x,\delta) \subseteq B(f(x),\epsilon)) \\ &\iff B(x,\delta) \subseteq f^{-1}B((f(x),\epsilon)) \end{aligned}$$

**Definition 11.3.2 (Neighbourhood)**
*In a metric space, a set $N$ is a **neighbourhood** of a point $x_0$, if $x_0 \in N^\circ$ (interior).*

**Theorem 11.3.1 (Characterization of continuity at a point)**
*If $(X, d_X), (Y, d_Y), f : X \to Y, x \in X$ are as above, then TFAE:*

1. *$f$ is continuous at $x_0$*
2. *$\forall N$ of $f(x_0) \in (Y, d_Y)$, we have $f^{-1}(N)$ is a neighbourhood of $x_0$ in $(X, d_X)$.*
3. *$x_0 = \lim_{n\to\infty} x_n \in (X, d_X) \implies f(x_0) = \lim_{n\to\infty} f(x_n) \in (Y, d_Y)$*

**Proof**
(1) $\implies$ (2)

*Given a neighbourhood $N$ of $f(x_0)$, $\exists \epsilon > 0,\ B(f(x_0), \epsilon) \subseteq N$. By assumption of (1), $\exists \delta > 0,\ B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \epsilon) \subseteq f^{-1}(N)$. Thus $f^{-1}(N)$ is a neighbourhood of $x_0$.*

(2) $\implies$ (1) $\implies$ (3)

*Given $\epsilon > 0$, $B(f(x_0), \epsilon)$ is a neighbourhood of $f(x_0)$, so $f^{-1}(B(f(x_0), \epsilon))$ is a neighbourhood of $x_0$, and hence $\exists \delta > 0,\ B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \epsilon))$, which proves (1).*

*Now if $x_0 = \lim_{n\to\infty} x_n \in (X, d_X)$, then by definition, $\exists n_\delta \in \mathbb{N}$ such that if $n \geqslant n_\delta$, $x_n \in B(x_0, \delta)$. But then for $n \geqslant n_\delta$, we have*

$$f(x_n) \in f(B(x, \delta)) \subseteq B(f(x_0), \epsilon)$$

*and hence $f(x_0) = \lim_{n\to\infty} f(x_n)$.*

(3) $\implies$ (1) *which we shall prove by contrapositive, i.e.* $\neg(1) \implies \neg(3)$

*If* $\neg(1)$, *then* $\exists\epsilon > 0$, $\forall\delta > 0$

$$B(x_0, \delta) \not\subset f^{-1}(B(f(x_0), \epsilon)).$$

*Hence for each* $n \in \mathbb{N}$, *we may find*

$$x_n \in B(x_0, \frac{1}{n}) \backslash f^{-1}(B(f(x_0), \epsilon)$$

*Given* $\epsilon' > 0$, $\exists n_{\epsilon'} \in \mathbb{N}$, $\forall n_{\epsilon'} \geqslant \frac{1}{\epsilon'}$. *Then for* $n \geqslant n_{\epsilon'}$, $x_n \in B(x_0, \epsilon')$ *thus* $\lim_{n\to\infty} x_n = x_0$. *However, each* $f(x_n) \notin B(f(x_0), \epsilon)$, *so* $f(x_n) \underset{n\to\infty}{\not\to} f(x_0)$.