PMATH467 — Algebraic Geometry

Classnotes for Winter 2019

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List of Theorems

Preface

The basic goal of the course is to be able to find **algebraic invariants**, which we shall use to classify topological spaces up to homeomorphism.

Other questions that we shall also look into include a uniqueness problem about manifolds; in particular, how many manifolds exist for a given invariant up to homeomorphism? We shall see that for a **2-manifold**, the only such manifold is the 2-dimensional sphere S^2. For a 4-manifold, it is the 4-dimensional sphere S^4. In fact, for any other n-manifold for $n > 4$, the unique manifold is the respective n-sphere. The problem is trickier with the 3-manifold, and it is known as the Poincaré Conjecture, solved in 2003 by Russian Mathematician Grigori Perelman. Indeed, the said manifold is homeomorphic to the 3-sphere.

For this course, you are expected to be familiar with notions from **real analysis**, such as topology, and concepts from **group theory**.

Basic Logistics for the Course

I shall leave this here for my own notes, in case something happens to my hard copy.

- OH: (Tue) 1630 - 1800, (Fri) 1245 - 1320
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Part I

Point-Set Topology

1 Lecture 1 Jan 07th

1.1 Euclidean Space

For any $(x_1, \dots, x_m) \in \mathbb{R}^m$, we can measure its distance from the origin 0 using either

- $\|x\|_\infty = \max\{|x_i|\}$ (the supremum-norm);
- $\|x\|_2 = \sqrt{\sum (x_j)^2}$ (the 2-norm); or
- $\|x\|_p = \left(\sum |x_j|^p\right)^{\frac{1}{p}}$ (the p-norm),

where we may define a "distance" by

$$d_p(x, y) = \|x - y\|_p .$$

Definition 1 (Metric)

Let X be an arbitrary space. A function $d : X \times X \to \mathbb{R}$ is called a ***metric*** *if it satisfies*

1. *(symmetry)* $d(x, y) = d(y, x)$ *for any* $x, y \in X$;
2. *(positive definiteness)* $d(x, y) \geq 0$ *for any* $x, y \in X$, *and* $d(x, y) = 0 \iff x = y$; *and*
3. *(triangle inequality)* $\forall x, y, z \in X$

$$d(x, y) \leq d(x, z) + d(y, z).$$

Definition 2 (Open and Closed Sets)

Given a space X with a metric d, and $r > 0$, the set

$$B(x, r) := \{w \in X \mid d(x, w) < r\}$$

is called the ***open ball*** *of radius r centered at x. An* ***open set*** *A is such that $\forall a \in A$, $\exists r > 0$ such that*

$$B(a, r) \subseteq A.$$

We say that a set is ***closed*** *if its complement is open.*

Definition 3 (Continuous Map)

A function

$$f : (X, d_1) \to (Y, d_2)$$

is said to be ***continuous*** *if the preimage of an open set in Y is open in X.*

See notes on Real Analysis for why we defined a continuous map in such a way.

This definition does not imply that a continuous map f maps open sets to open sets.

Exercise 1.1.1

Contruct a function on $[0, 1]$ which assumes all values between its maximum and minimum, but is not continuous.

Solution

Consider the piecewise function

$$f(x) = \begin{cases} x & 0 \leq x < \frac{1}{2} \\ x - \frac{1}{2} & x \geq \frac{1}{2}. \end{cases}$$

It is clear that the maximum and minimum are $\frac{1}{2}$ and 0 respectively, and f assumes all values between 0 and $\frac{1}{2}$. However, a piecewise function is not continuous.

Definition 4 (Homeomorphism)

A function f is a ***homeomorphism*** *if it is a bijection and both f and f^{-1} are continuous.*

Example 1.1.1

The function

$$g : [0, 2\pi) \to \mathbb{R}^2 \text{ given by } \theta \mapsto (\cos\theta, \sin\theta)$$

is not homeomorphic, since if we consider an alternating series that converges to 0 on the unit circle on $\mathbb{R}^2$, we have that the preimage of the series does not converge and f^{-1} is in fact discontinuous.

Now, we want to talk about topologies without referring to a metric.

Definition 5 (Topology)

*Let X be a space. We say that the set $T \subseteq \mathcal{P}(X)$ is a **topology** if*

1. *$X, \emptyset \in T$;*
2. *if $\{x_\alpha\}_{\alpha \in A} \subseteq T$ for an arbitrary index set A, then*
$$\bigcup_{\alpha \in A} x_\alpha \in T; \text{ and}$$
3. *If $\{x_\beta\}_{\beta \in B} \subset T$ for some finite index set B, then*
$$\bigcap_{\beta \in B} x_\beta \in T.$$

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