PMATH467 — Algebraic Geometry

Classnotes for Winter 2019

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Preface

The basic goal of the course is to be able to find **algebraic invariants**, which we shall use to classify topological spaces up to homeomorphism.

Other questions that we shall also look into include a uniqueness problem about manifolds; in particular, how many manifolds exist for a given invariant up to homeomorphism? We shall see that for a **2-manifold**, the only such manifold is the 2-dimensional sphere S^2. For a 4-manifold, it is the 4-dimensional sphere S^4. In fact, for any other n-manifold for $n > 4$, the unique manifold is the respective n-sphere. The problem is trickier with the 3-manifold, and it is known as the Poincaré Conjecture, solved in 2003 by Russian Mathematician Grigori Perelman. Indeed, the said manifold is homeomorphic to the 3-sphere.

For this course, you are expected to be familiar with notions from **real analysis**, such as topology, and concepts from **group theory**.

The following topics shall be covered:

1. Point-Set Topology
2. Introduction to Topological Manifolds
3. Simplicial complexes & Introduction to Homology
4. Fundamental Groups & Covering Spaces
5. Classification of Surfaces

Basic Logistics for the Course

I shall leave this here for my own notes, in case something happens to my hard copy.

- OH: (Tue) 1630 - 1800, (Fri) 1245 - 1320
- OR: MC 6457
- EM: aaleyasin

Part I

Point-Set Topology

1 Lecture 1 Jan 07th

We will not be too rigorous in this part.

1.1 Euclidean Space

For any $(x_1, \ldots, x_m) \in \mathbb{R}^m$, we can measure its distance from the origin 0 using either

- $\|x\|_\infty = \max\{|x_i|\}$ (the supremum-norm);
- $\|x\|_2 = \sqrt{\sum (x_j)^2}$ (the 2-norm); or
- $\|x\|_p = \left(\sum |x_j|^p \right)^{\frac{1}{p}}$ (the p-norm),

where we may define a “distance” by

$$d_p(x, y) = \|x - y\|_p .$$

Definition 1 (Metric)

Let X be an arbitrary space. A function $d : X \times X \to \mathbb{R}$ is called a ***metric*** *if it satisfies*

1. *(symmetry) $d(x, y) = d(y, x)$ for any $x, y \in X$;*
2. *(positive definiteness) $d(x, y) \geq 0$ for any $x, y \in X$, and $d(x, y) = 0 \iff x = y$; and*
3. *(triangle inequality) $\forall x, y, z \in X$*

$$d(x, y) \leq d(x, z) + d(y, z).$$

Definition 2 (Open and Closed Sets)

Given a space X with a metric d, and $r > 0$, the set

$$B(x, r) := \{w \in X \mid d(x, w) < r\}$$

is called the ***open ball*** *of radius r centered at x. An* ***open set*** *A is such that $\forall a \in A, \exists r > 0$ such that*

$$B(a, r) \subseteq A.$$

We say that a set is ***closed*** *if its complement is open.*

Definition 3 (Continuous Map)

A function

$$f : (X, d_1) \to (Y, d_2)$$

is said to be ***continuous*** *if the preimage of an open set in Y is open in X.*

See notes on Real Analysis for why we defined a continuous map in such a way.

Warning

This definition does not imply that a continuous map f maps open sets to open sets.

Exercise 1.1.1

Contruct a function on $[0, 1]$ which assumes all values between its maximum and minimum, but is not continuous.

Solution

Consider the piecewise function

$$f(x) = \begin{cases} x & 0 \leq x < \frac{1}{2} \\ x - \frac{1}{2} & x \geq \frac{1}{2}. \end{cases}$$

It is clear that the maximum and minimum are $\frac{1}{2}$ and 0 respectively, and f assumes all values between 0 and $\frac{1}{2}$. However, a piecewise function is not continuous.

Definition 4 (Homeomorphism)

*A function f is a **homeomorphism** if it is a bijection and both f and f^{-1} are continuous.*

Example 1.1.1

The function

$$g : [0, 2\pi) \to \mathbb{R}^2 \text{ given by } \theta \mapsto (\cos\theta, \sin\theta)$$

is not homeomorphic, since if we consider an alternating series that converges to 0 on the unit circle on $\mathbb{R}^2$, we have that the preimage of the series does not converge and f^{-1} is in fact discontinuous.

Now, we want to talk about topologies without referring to a metric.

Definition 5 (Topology)

*Let X be a space. We say that the set $\mathcal{T} \subseteq \mathcal{P}(X)$ is a **topology** if*

1. $X, \emptyset \in \mathcal{T}$;
2. *if* $\{x_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ *for an arbitrary index set* A, *then*
$$\bigcup_{\alpha \in A} x_\alpha \in \mathcal{T}; \text{ and}$$
3. *If* $\{x_\beta\}_{\beta \in B} \subset \mathcal{T}$ *for some finite index set* B, *then*
$$\bigcap_{\beta \in B} x_\beta \in \mathcal{T}.$$

2 Lecture 2 Jan 09th

2.1 Euclidean Space (Continued)

In the last lecture, from metric topology, we generalized the notion to a more abstract one that is based solely on open sets.

Example 2.1.1

Let X be a set. The following two are uninteresting examples of topologies:

1. The **trivial topology** $\mathcal{T} = \{\emptyset, X\}$.
2. The **discrete topology** $\mathcal{T} = \mathcal{P}(X)$.

We shall now continue with looking at more concepts that we shall need down the road.

Definition 6 (Closure of a Set)

*Let A be a set. Its **closure**, denoted as $\overline{A}$, is defined as*

$$\overline{A} = \bigcap_{C \supset A}^{C\text{: closed}} C.$$

It is the smallest closed set that contains A.

Note

In metric topology, one typically defines the closure of a set by taking the union of A and its limit points.

Definition 7 (Interior of a Set)

*Let A be a set. Its **interior**, denoted either as $\mathrm{Int}\,(A)$, A° or $\mathring{A}$, is defined as*

$$\mathring{A} = \bigcup_{G \subseteq A}^{G:\text{ open}} G.$$

Definition 8 (Boundary of a Set)

*Let A be a set. Its **boundary**, denoted as ∂A, is defined as*

$$\partial A = \overline{A} \setminus \mathring{A}.$$

Exercise 2.1.1

Let A be a set. Prove that ∂A is closed.

Proof

Notice that

$$(\partial A)^C = (\overline{A} \setminus \mathring{A})^C = X \setminus \overline{A} \cup \mathring{A} = X \cap \overline{A}^C \cup \mathring{A}$$

which is open. □

Exercise 2.1.2

Let A be a set. Show that

$$\partial(\partial A) = \partial A.$$

Proof

First, notice that $\mathring{\partial A} = \emptyset$. Since ∂A is closed, $\overline{\partial A} = \partial A$. Then

$$\partial(\partial A) = \overline{\partial A} \setminus \mathring{\partial A} = \partial A \setminus \emptyset = \partial A$$

Example 2.1.2

We know that $\mathbb{Q} \subseteq \mathbb{R}$, and $\overline{\mathbb{Q}} = \mathbb{R}$. We say that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 9 (Dense)

We say that a subset A of a set X is dense if

$$\overline{A} = X.$$

Example 2.1.3

From the last example, we have that $\mathring{\mathbb{Q}} = \emptyset$.

Definition 10 (Limit Point)

We say that $p \in X \supseteq A$ is a limit point of A if any neighbourhood of p has a nontrivial intersection with A.

Example 2.1.4 (A Topologist's Circle)

Consider the function

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

on the interval $\left[-\frac{1}{2\pi}, \frac{1}{2\pi}\right]$. Extend the function on both ends such that we obtain Figure 2.1 (See also: Desmos).

The limit points of the graph includes all the points on the straight line from $(0, -1)$ to $(0, 1)$, including the endpoints. This is the case because for any of the points on this line, for any neighbourhood around the point, the neighbourhood intersects the graph f infinitely many times.

Going back to continuity, given a function f, how do we know if f^{-1} maps an open set to an open set?

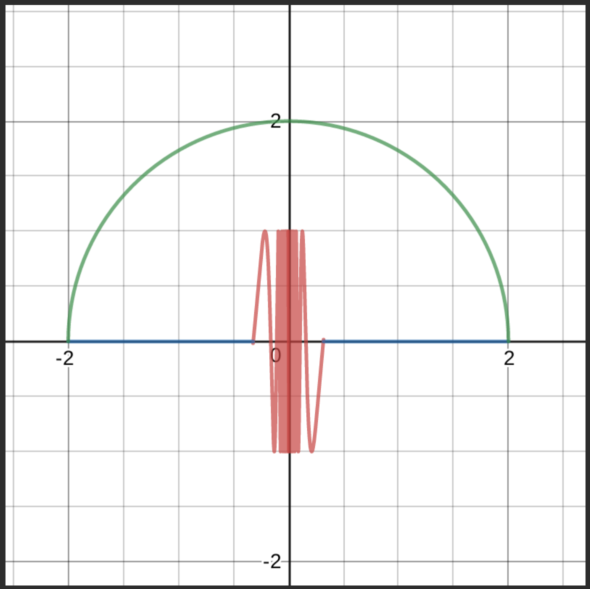


Figure 2.1: A Topologist's Circle

We can actually reduce the problem to only looking at open balls. But why are we allowed to do that?

Definition 11 (Basis of a Topology)

*Given a topology $\mathcal{T}$, we say that $\mathcal{B} = \{B_\alpha\}_{\alpha \in I}$ is a **basis** if $\forall T \in \mathcal{T}$, there exists $J \subset I$ such that*

$$T = \bigcup_{\alpha \in J} B_\alpha.$$

Note that while the definition is similar to that of a cover, we are now "covering" over sets and not points.

Example 2.1.5

Let $\mathcal{T}$ be the Euclidean topology on $\mathbb{R}$. Then we can take

$$\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}, a \leq b\}.$$

Note that $\mathcal{B}$ is **uncountable**. We can, in fact, have [1]

$$\mathcal{B}_1 = \{(a, b) \mid a, b \in \mathbb{Q}, a \leq b\},$$

which is countable, as a basis for $\mathbb{R}$. Furthermore, we can consider the set

$$\mathcal{B}_2 = \left\{ (a, b) \;\middle|\; a \leq b, \, a = \frac{m}{2^p}, \, b = \frac{n}{2^q}, \, m, n, p, q \in \mathbb{Z} \right\},$$

[1] Recall from PMATH 351 that we can write $\mathbb{R}$ as a disjoint union of open intervals with rational endpoints.

which is also a countable basis for $\mathbb{R}$. Notice that

$$\mathcal{B}_2 \subseteq \mathcal{B}_1 \subseteq \mathcal{B}.$$

Example 2.1.6

In $\mathbb{R}^2$, we can do a similar construction of $\mathcal{B}$, $\mathcal{B}_1$, and $\mathcal{B}_2$ as in the last example and use them as a basis for $\mathbb{R}^2$. In particular, we would have

$$\mathcal{B} = \{(a_1, b_1) \times (a_2, b_2) \mid a_1, a_2, b_1, b_2 \in \mathbb{R}\}.$$

This is called a **dyadic partitioning** of $\mathbb{R}^2$.

Example 2.1.7

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be two topological spaces. Then the Cartesian product $X_1 \times X_2$ has topology induced from $\mathcal{T}_1$ and $\mathcal{T}_2$ by taking the set

$$\mathcal{B} = \{\beta_1 \times \beta_2 \mid \beta_1 \in \mathcal{T}_1,\ \beta_2 \in \mathcal{T}_2\}$$

as the basis.

Exercise 2.1.3

Prove that

1. *β_1 and β_2 can be taken to be elements of bases $\mathcal{B}_1 \subset \mathcal{T}_1$ and $\mathcal{B}_2 \subset \mathcal{T}_2$, respectively.*
2. *the product topology on $\mathbb{R}^2$ is the same as the Euclidean topology.*

3 Lecture 3 Jan 11th

3.1 Euclidean Space (Continued 2)

Let $\tilde{X}$ be a metric space, and $p, q \in \tilde{X}$ with $p \neq q$. Then we have that $d(p,q) = r > 0$.

The student is recommended to do a quick review for the first 3 chapters of the recommended text.

Then we must have that

$$B\left(p, \frac{r}{3}\right) \cap B\left(q, \frac{r}{3}\right) = \emptyset.$$

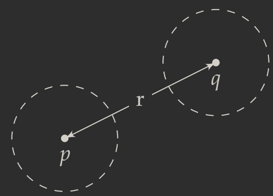


Figure 3.1: Idea of separation

Exercise 3.1.1

Prove that the above claim is true. (Use the triangle inequality)

Proof

Suppose $\exists x \in B\left(p, \frac{r}{3}\right) \cap B\left(q, \frac{r}{3}\right)$. Then

$$d(p,x) + d(q,x) < \frac{2r}{3} < r = d(p,q),$$

which violates the triangle inequality. □

We observe here that the two open sets (or balls) "separate" p and q.

Definition 12 (Hausdorff / T_2)

Let X be a topological space. X is said to be ***Hausdorff*** *or T_2 iff any 2 distinct points can be separated by disjoint open sets.*

Note

1. *The Hausdorff space (or T_2 space) is an important space; we can only define a metric on spaces that are T_2.*
2. *A space is called T_1 is for any $p, q \in X$ with $p \neq q$, $\exists U \ni p$ open such that $q \notin U$ and $\exists V \ni q$ open such that $p \notin v$. It is worth noting that a T_2 space is also T_1.*

Example 3.1.1 (The Discrete Topology)

Suppose X is a metric space. For any $x \in X$, we have that $\{x\}$ is open. Thus for any $x_1, x_2 \in X$, if $x_1 \neq x_2$, then the open sets $\{x_1\}$ and $\{x_2\}$ separates x_1 and x_2.

This is true as we can define the following metric on the space: let $d : X \times X \to \mathbb{R}$ such that

$$d(x_1, x_2) = \begin{cases} 0 & x_1 = x_2 \\ 1 & x_1 \neq x_2 \end{cases}$$

This topology is called a **discrete topology**, and it is a metric space.

Let X be a metric space and $A \subseteq X$. Then there is a metric induced by X on A, and this in turn induces a topology on A.

More generally, if $A \subset X$ where X is some arbitrary topological space, then a set $U \subseteq A$ is open iff $U = A \cap V$ for some $V \subseteq X$ that is open. In other words, a subset U of A is said to be open iff we can find an open set V in X such that the intersection of A and V gives us U.

Exercise 3.1.2

Prove that the construction above gives us a topology.

Proof

Let $A \subseteq X$. We shall show that τ_A is a topological space induced by the topological space τ of X. It is clear that $\emptyset \in \tau_A$, since it is open in X, and so $A \cap \emptyset = \emptyset$. Since X is open, we have $A \cap X = A$, and so $A \in \tau_A$.

Now if $\{U_\alpha\}_{\alpha \in I} \subseteq \tau_A$, then $\exists V_\alpha \subseteq X$ such that $U_\alpha = A \cap V_\alpha$.

Then

$$\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} A \cap V_\alpha = A \cap \bigcup_{\alpha \in I} V_\alpha,$$

and $\bigcup_{\alpha \in I} V_\alpha$ is open in X by the properties of open sets. Thus $\bigcup_{\alpha \in I} U_\alpha \in \tau_A$.

If $\{U_i\}_{i=1}^n \subset \tau_A$, then again, by the properties of open sets, finite intersection of open sets is open, and so $\bigcap_{i=1}^n U_i \in \tau_A$.

□

Note

We can say the same can be said about closed sets of A.

Example 3.1.2

Let $A \subseteq X$ and consider the function

$$\iota_A : A \to X \text{ given by } x \mapsto x,$$

which is the inclusion map.

Then ι_A is continuous when the topology on A is chosen to be the induced subspace topology. This is rather clear; notice that the inverse of the inclusion map brings open sets to open sets.

Let Y be an arbitrary topological space. Then let

$$\begin{tikzcd} & X \\ Y \arrow[ur, "\iota_A \circ f"] \arrow[r, "f"'] & A \arrow[u, "\iota_A"'] \end{tikzcd}$$

Figure 3.2: Composition of a function and the inclusion map

where f is continuous. Then $\iota_A \circ f$ is continuous.

The converse is also true: if $\iota_A \circ f$ is continuous, then f is continuous. However, we will not prove this. This property is known as the **characteristic property of the subspace topology**.

Lemma 1 (Restriction of a Continuous Map is Continuous)

Let $X \xrightarrow{f} Y$ be continuous, and $A \subseteq X$ [1]. Then

[1] Here, A is equipped with the subspace topology

$$f\restriction_A : A \to Y$$

is also continuous.

3.2 Connected Spaces

Consider the real line $\mathbb{R}$, and consider two disjoint intervals on $\mathbb{R}$.

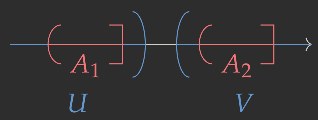


Figure 3.3: Motivation for Connectedness

Observe that we may find two open subsets U and V of $\mathbb{R}$ such that $A_1 \subseteq U$ and $A_2 \subseteq V$, which effectively separates the two intervals on the space $\mathbb{R}$.

Definition 13 (Disconnectedness)

A space X is said to be ***disconnected*** *iff X can be written as a disjoint union*

$$X = A_1 \coprod A_2$$

where $A_1, A_2 \subseteq X$, $A_1 = A_2^C$, that they are both non-empty and open [2]*.*

[2] It goes without saying that the two sets are also simultaneously closed.

Definition 14 (Connctedness)

A space X is said to be ***connected*** *if it is not disconnected.*

Note

By the above definitions, we have that X is connected iff for any partition $X = A \coprod A^C$ with A being open, either A is $\emptyset$ or A is X.

Example 3.2.1

The space $\mathbb{R} \setminus \{0\}$ is disconnected; our disjoint sets are $(-\infty, 0)$ and $(0, \infty)$.

However, $\mathbb{R}^2 \setminus \{0\}$ is connected, but it is not easy to describe why.

Definition 15 (Path)

Require clarification

Lemma 2 (Path Connectedness implies Connectedness)

If a space X is path connected, then it is connected.

Theorem 3 (From Connected Space to Connected Space)

If $X \xrightarrow{f} Y$ is continuous and X is connected, then $\mathrm{Img}(f)$ is connected.

4 *Lecture 4 Jan 14th*

4.1 *Connected Spaces (Continued)*

Definition 16 (Locally Connected)

We say that X is ***locally connected*** *at x if for every open set V containing x there exists a connected, open set U with $x \in U \subseteq V$. We say that the space X is* ***locally connected*** *if it is locally connected $\forall x \in X$.*

Example 4.1.1

The space S generated by the function $\sin \frac{1}{x}$ with 0 at $x = 0$, on the $\mathbb{R}^2$ is not locally connected: consider $(0, y) \in S, y \neq 0$. Then any small open ball at this point will contain infinitely many line segments from S. This cannot be connected, as each one of these constitutes a component, within the neighborhood.

Definition 17 (Connected Component)

The maximal connected subsets of any topological space X are called ***connected components*** *of the space. The components form a partition of the space.*

4.2 *Compactness*

Definition 18 (Sequential Compactness)

*For $A \subseteq X$, where X is a topological space, if $\{x_i\}_{i \in I} \subseteq A$, arbitrary sequence in A, has a convergent subsequence, we say that A is **sequentially compact**.*

Definition 19 (Compactness)

*We say that a topological space X is **compact** if every **open cover** of X has a finite **subcover**.*

Lemma 4 (Compactness implies Sequential Compactness)

Compactness implies sequential compactness.

Example 4.2.1

$[0,1)$ is not compact: consider the open cover $\left\{ \left[0, 1 - \frac{1}{n}\right) \right\}_{n \in \mathbb{N}}$, which contains $[0,1)$ as $n \to \infty$. But whenever n is finite, we have $1 - \frac{1}{n} < 1$, and so any finite collection of the $\left[0, \frac{1}{n}\right)$ is not a cover of $[0,1)$.

Theorem 5 (Continuous Maps map Compact Sets to Compact Images)

Let $f : X \to Y$ be continuous, where X is compact. Then $f(X)$ is compact.

Proof

Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $f(X)$. Since f is continuous, we have that $f^{-1}(U_\alpha)$ is open for each $\alpha \in I$. Since f is bijective between the image set and its domain, we have that $\{f^{-1}(U_\alpha)\}_{\alpha \in I}$ is an open cover of X. Since X is compact, this cover has a finite subcover, say $\{f^{-1}(U_i)\}_{i=1}^n$. Thus

$$X = \bigcup_{i=1}^n f^{-1}(U_i).$$

Thus

$$f(X) = f\left(\bigcup_{i=1}^{n} f^{-1}(U_i)\right) = \bigcup_{i=1}^{n} U_i.$$

Hence $\{U_\alpha\}_{\alpha \in I}$ has a finite subcover and so $f(X)$ is compact. □

Corollary 6 (Homeomorphic Maps map Compact Sets to Compact Sets)

Let $X \xrightarrow{f} Y$ be a homeomorphism. Then X is compact iff Y is compact.

Note

Compactness is a topological property.

Lemma 7 (Properties of Compact Sets)

1. *A closed subset of a compact space is compact.*
2. *A compact subset of a topological space is closed provided that the space is Hausdorff.*
3. *In a metric space, a compact set is bounded.*
4. *Finite (Cartesian) product of compact sets is compact.*

The proof for the first item is simple: consider an open cover of the closed subset, and union them with the complement of the closed subset. This covers the entire space, and so it has a finite subcover. We just need to then remove that complement set, and that would be a finite subcover for the closed subset.

Example 4.2.2

The subset $[-a, b]$, $a, b \in \mathbb{R}$, is compact.

Example 4.2.3

$[0,1]^{\mathbb{N}}$ is not compact: the space is equivalent to ℓ_∞.

Theorem 8 (Heine-Borel)

Let $X \subseteq \mathbb{R}^n$. Then X is compact iff X is closed and bounded.

Proof

$(\implies)$ We say that compactness impliess boundedness. Also, since $\mathbb{R}^n$ is Hausdorff, X is closed.[1].

$(\impliedby)$ X is bounded implies that $X \subseteq [-R, R]^n$ with R sufficiently large. Since X is closed, and $[-R, R]^n$ is compact, X is necessarily compact by Lemma 7. $\square$

[1] Both from Lemma 7

Theorem 9 (Bolzano-Weierstrass)

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Exercise 4.2.1

Prove Theorem 9 as an exercise.

We shall start the next part this lecture.

4.3 Manifolds

Definition 20 (Locally Homeomorphic)

*A space is said to be **locally homeomorphic** to $\mathbb{R}^n$ provided that $\forall x \in X$, $\exists U \ni x$ open such that U is homeomorphic to $\mathbb{R}^n$.*

Definition 21 (Manifold)

*An n-dimensional **manifold** is a **second countable**[2], **Hausdorff** topological space that is **locally homeomorphic** to $\mathbb{R}^n$.*

[2] a topological space is said to be **second countable** if its basis is countable.

Note

One can give an equivalent definition of locally homeomorphic by requiring that U be homeomorphic to an open ball $B^n \subseteq \mathbb{R}^n$. Notice that B^n is homeomorphic to $\mathbb{R}^n$ [3]

[3] By scaling, really.

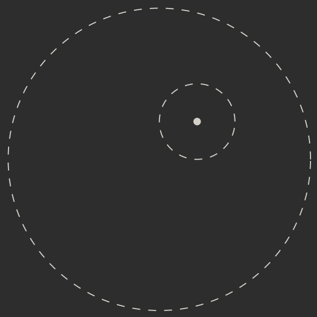


Figure 4.1: Open ball in an open set in $\mathbb{R}^2$

Example 4.3.1

Let $B^n = B^n(0, 1) \subseteq \mathbb{R}^n$. Then B^n is homeomorphic to $\mathbb{R}^n$.

Figure 4.2: Open ball on a point on the boundary of a closed set

Example 4.3.2

Now consider the closed ball $\overline{B}^n = \overline{B}^n(0, 1) \subset \mathbb{R}^n$. This is actually not a manifold, but we are not yet there to prove this. This sort of a structure motivates us to the next definition.

Definition 22 (Manifold on a Boundary)

An n-dimensional space that is second countable and Hausdorff, such that $\forall x \in X$, there exists a neighbourhood either homeomorphic to $B^n \subseteq \mathbb{R}^n$ or $B^n \cap \mathbb{H}^n$.

Note

Note that $\mathbb{H}^n$ is defined as

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) : x_n \geq 0\}.$$

For instance, $\mathbb{H}^2$ has the following graph:

Figure 4.3: Graph of $\mathbb{H}^2$

Part II

Introduction to Topological Manifolds

5 Lecture 5 Jan 16th

5.1 Manifolds (Continued)

Definition 23 (Interior Point)

*A point $x \in M$ is called an **interior point** if there is a local homeomorphism*

$$\phi : \mathcal{U} \to \mathbb{B} \subseteq \mathbb{R}^n,$$

where $\mathcal{U}$ is open.

In the last lecture we asked ourselves the following: how do we know if the idea of 'being on a boundary' is a well-defined notion? In particular, how do well tell the difference between the following two graphs, mathematically?

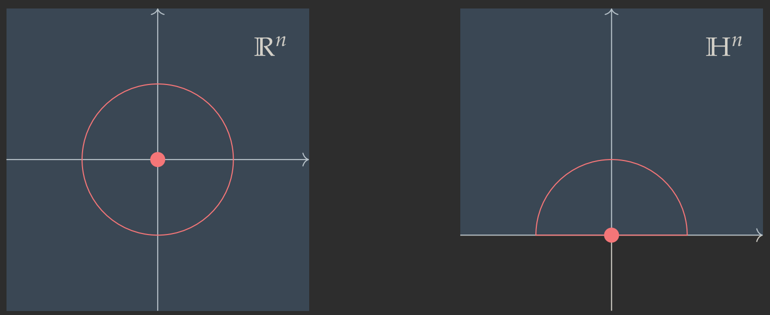


Figure 5.1: How do we tell the difference between the two graphs?

Definition 24 (Boundary Point)

A point x is on the boundary of M, denoted as $x \in \partial M$, if there exists

$\mathcal{U} \ni x$ that is open, and a homeomorphism

$$\psi : \mathcal{U} \to \mathbb{B}_{0,1} \cap \mathbb{H}^n.$$

Note

The ϕ in Definition 23 and ψ in Definition 24 are called ***local charts****.*

Also, our definitions do not rule out, e.g.

$$\phi_2 : \mathcal{U} \to \mathbb{B}^2 \subseteq \mathbb{R}^2$$
$$\phi_5 : \mathcal{U} \to \mathbb{B}^5 \subseteq \mathbb{R}^5.$$

We shall later on prove that a point cannot simultaneously be a boundary point and an interior point. This property is called **the invariance of the boundary**. [1]

[1] This also means that $\mathring{M} \cap \partial M = \emptyset$. Also, by Definition 21, $\mathring{M} \cup \partial M = M$.

Note

$\mathring{M}$ is open. Thus, we can use the same definition about open sets as before using ϕ.

Note

In contrast, ∂M is closed; thanks to the invariance of the boundary we have that $\partial M = M \setminus \mathring{M}$ and $\mathring{M}$ is open, and so $(\partial M)^C = \mathring{M}$.

We shall also prove the following theorem later on:

Theorem (Invariance of the Dimension)

The n in $\mathbb{R}^n$ is well-defined.

Example 5.1.1

Consider the equation

$$x^2 - y^2 - z^2 = 0. \tag{5.1}$$

Note that we may write

$$x = \pm\sqrt{y^2 + z^2},$$

and so the graph generated by Equation (5.1) is as shown in Figure 5.2.

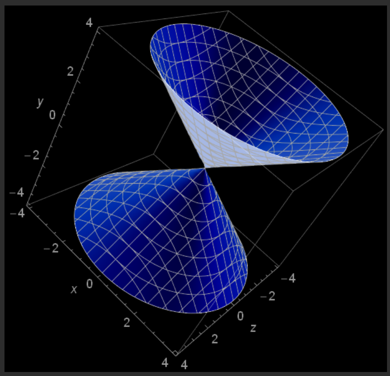


Figure 5.2: A 3D cone in $\mathbb{R}^3$, from WolframAlpha

However, this is not a manifold: if we assume that a ball arounnd the origin is homeomorphic to $\mathbb{R}^2$, then by removing the point at the origin in the cone, the cone becomes two disconnected components, but the ball in $\mathbb{R}^2$ homeomorphic to the aforementioned ball is still connected.

> **Note**
> *An open subset of a manifold is a manifold, by restriction.*

5.1.1 *The 1-Sphere S^1*

From Example 2.1.5, we have

- $[0, 1) \simeq [0, \infty)$ is a manifold with boundary;
- $(0, 1) \simeq \mathbb{R}$ is a manifold; and
- $[0, 1]$ is a manifold with boundary.

Example 5.1.2 (S^1 is a manifold)

Consider the function $f : [0, 2\pi) \to e^{i\theta}$. The image of f is Consider

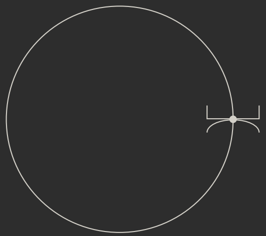


Figure 5.3: S^1 as a manifold

the following two functions $(0, 2\pi) \to \mathbb{C}^2$ by

$$\theta_1 \to e^{i\theta_1} \text{ and } \theta_2 \to e^{i\theta_2 + \pi},$$

which gives us the graphs:

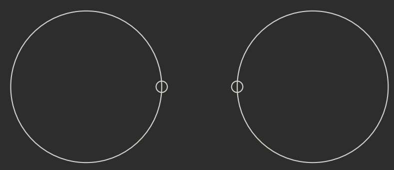


Figure 5.4: Basis for S^1

respectively. Note that the image of both these functions are not compact. Regardless, this gives us a basis for S^1, which we notice is countable, Hausdorff, and locally homeomorphic to $\mathbb{R}^2$.

Theorem 10 (1-Dimensional Manifolds Determined by Its Compactness)

Let M be a connected component of a 1-dimensional manifold. Then either

1. *M is compact, in which case if it is*
   - *without a boundary, then M is homeomorphic to S^1.*
   - *with a boundary, then M is homeomorphic to $[0, 1]$.*
2. *M is not compact, in which case if it is*
   - *without a boundary, then M is homeomorphic to $[0, 1)$.*
   - *with a boundary, then M is homeomorphic to $(0, 1)$.*

6 Lecture 6 Jan 18th

6.1 Manifolds (Continued 2)

6.1.1 The 1-Sphere S^1 (Continued)

Set theoretic view of S^1 We showed that S^1 is a manifold. We can, in fact, set theoretically, look at S^1 as $A = [0, 1]$ glued at the endpoints, i.e. we **identify** the points 0 and 1 as 'the same', and label this notion as $0 \sim 1$.

Topological view of S^1 Topologically, for $0 \sim 1$ in A, we can construct an open set around the point such that the open set is properly contained in A.

Figure 6.1: Topological representation of A

But how can we describe this notion mathematically so?

Consider the real line as follows:

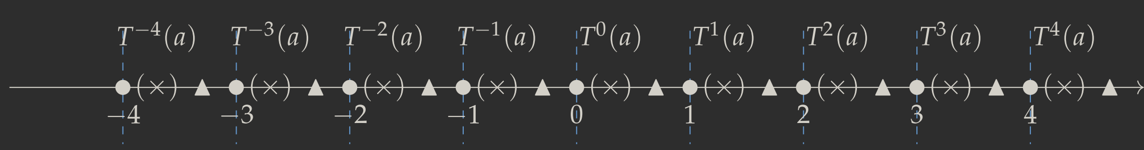


Figure 6.2: Breaking down the real line into parts

Let's define $T : \mathbb{R} \to \mathbb{R}$ such that $x \mapsto x + 1$. Clearly so, T is bijective and, in particular, has an inverse $x \mapsto x - 1$.

Notice that within each interval $[x, x + 1]$, we can find a $\times$ and $\blacktriangle$ at the same distance from x. Also, notice that we can use the same radius for $\times$ such that the open ball around $\times$ sits in $[x, x + 1]$ for each $x \in \mathbb{Z}$.

Thus, instead of studying the entire real line at once, we can reduce our attention only to $[0,1]$, and simply scale the interval with a 'scalar multiplication' to get to wherever we want on the real line.

Now let

$$G := \left\{ T^k \;\middle|\; k \in \mathbb{Z} \right\},$$

which is evidently a **group**. Furthermore, every element in G is a homeomorphism to $\mathbb{R}$. Let G **act on** $\mathbb{R}$, and for $a \in \mathbb{R}$, consider the **orbit** of a, which is denoted as

$$G \cdot a := \left\{ T^k(a) \;\middle|\; k \in \mathbb{Z} \right\}.$$

Then

$$S^1 \simeq \text{ the space of all orbits of } G \text{ acting on } \mathbb{R} =: \mathbb{R}/G,$$

where $\simeq$ represents homeomorphism. [1] Also, notice that here, G is effectively $\mathbb{Z}$.

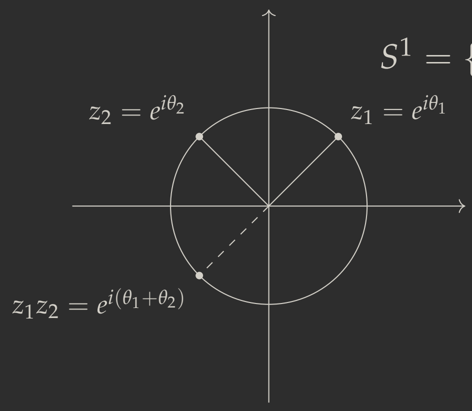
[1] $\mathbb{R}/G$ is the **quotient space** of $\mathbb{R}$ over G.

This realization implies the existence of some topology on S^1.

Thus far, we have seen that we may look at S^1

- set theoretically: as $A = [0,1]$ with glued endpoints; and
- topologically: as $\mathbb{R} \setminus \mathbb{Z}$.

S^1 as a topological group Since $\mathbb{C} \simeq \mathbb{R}^2$, we may think of S^1 as a sphere on the complex plane. We see that this 'group' takes on the



$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\} = \left\{ e^{i\theta} \mid \theta \in \mathbb{R} \right\}$$

Figure 6.3: S^1 on the complex plane

operation of adding the indices of the exponents. Thus $G = (S^1, \cdot)$. Notice that G is indeed a group equipped with said operation, and for each $z_1 \in G$, there exists $z_1^{-1} = \frac{1}{z_1} \in G$ such that $z_1 \cdot \frac{1}{z_1} = 1$.

Furthermore, the function

$$\iota : S^1 \to S^1 \text{ given by } z \mapsto \frac{1}{z}$$
$$\text{which is } e^{i\theta} \mapsto e^{-i\theta}$$

is continuous.

Also, the function

$$P : S^1 \times S^1 \to S^1 \text{ given by } (z_1, z_2) \mapsto z_1 z_2$$

is continuous.

Definition 25 (Topological Group)

If G is a group, and functions ι and P as defined above, if both ι and P are continuous, then we say that G is a ***topological group****.*

S^1 as a moduli space

Definition 26 (Moduli Space)

A ***moduli space*** *is the space of all lines passing through the origin.*

On $\mathbb{R}^2$, we have

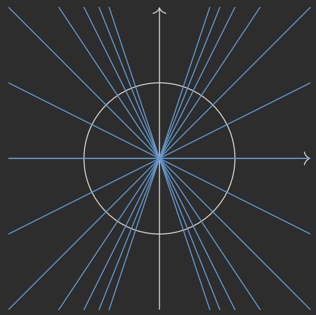


Figure 6.4: The moduli space on $\mathbb{R}^2$

First, how can we understand 'closeness' in a moduli space? We can actually look at the difference in the radians of each line, or really just $x/360$ and compare the x's.

Also, notice that each line passes through S^1 twice. We can indeed avoid this problem by shifting S^1 to one side, as shown in Figure 6.5.

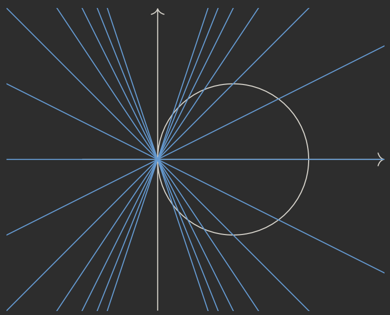


Figure 6.5: Shifted S^1 for the moduli space

Now each line intersects S^1 at the origin and another point on S^1, and this intersection is in fact unique.

6.1.1.1 *The space of $S^1 \times S^1$*

Observe that the product of two manifolds is a manifold [2].

[2] This is called a **product space**.

Consider using the set theoretical viewpoint, with credits to **Felix Klein**, the following figure By joining the sides with $>$, where we

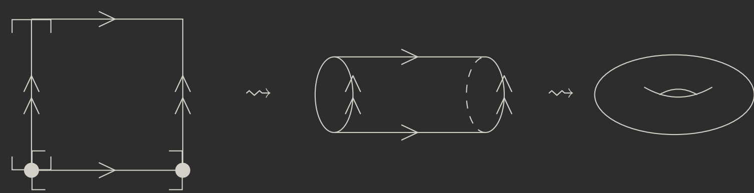


Figure 6.6: $S^1 \times S^1$ becomes a cylinder by identifying the edges

identify the endpoints, we can go from the figure introduced by Klein to a cylinder. Then by identifying the sides with $>>$, we get a **torus**.

Now on a Cartesian plane, observe that

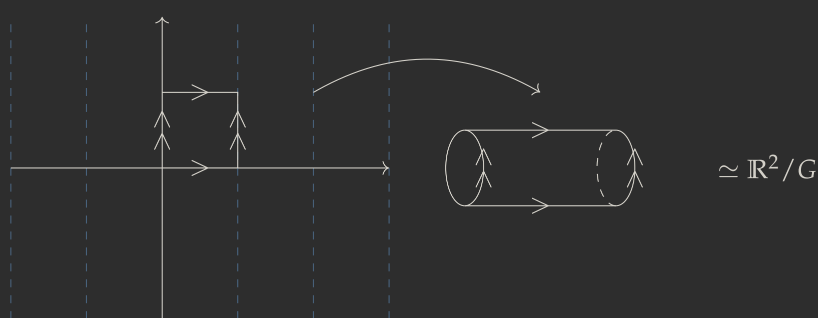


Figure 6.7: Klein's figure on a Cartesian plane to a cylinder

where we define $G = \left\{ T^k \;\middle|\; k \in \mathbb{Z} \right\}$ as before, for $T((x, y)) = (x + 1, y)$. Now on a similar note, define $R : \mathbb{R} \to \mathbb{R}$ by $R((x, y)) = (x, y + 1)$. Then let $G_2 = \left\{ R^k \;\middle|\; k \in \mathbb{Z} \right\}$. Thus

$$\mathbb{R}^2 / G \oplus G_2 \simeq \mathbb{R}^2 / \mathbb{Z} \oplus \mathbb{Z}.$$

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