PMATH347S18 - Groups & Rings

Johnson Ng

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1 Lecture 1 May 02nd 2018

1.1 Introduction

1.1.1 Numbers

The following are some of the number sets that we are already familiar with:

$$\mathbb{N} = \{1, 2, 3, ...\} \qquad \mathbb{Z} = \{.., -2, -1, 0, 1, 2, ...\}$$
$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N} \right\} \qquad \mathbb{R} = \text{ set of real numbers}$$
$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i = \sqrt{-1}\} = \text{ set of complex numbers}$$

For $n \in \mathbb{Z}$, let $\mathbb{Z}_n$ denote the set of integers modulo n, i.e.

$$\mathbb{Z}_n = \{[0], [1], ..., [n-1]\}$$

where the $[r]$, $0 \leq r \leq n-1$, are the congruence classes, i.e.

$$[r] = \{z \in \mathbb{Z} : z \equiv r \mod n\}$$

These sets share some common properties, e.g. $+$ and $\times$. Let's try to break that down to make further observation.

Note that for $R = \mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Z}_n$, R has 2 operations, i.e. addition and multiplication.

Addition If $r_1, r_2, r_3 \in R$, then

- (**closure**) $r_1 + r_2 \in R$
- (**associativity**) $r_1 + (r_2 + r_3) = (r_1 + r_2) + r_3$

Also, if $R \neq \mathbb{N}$, then $\exists 0 \in R$ (the **additive identity**) such that

$$\forall r \in R \quad r + 0 = r = 0 + r.$$

Also, $\forall r \in R$, $\exists (-r) \in R$ such that

$$r + (-r) = 0 = (-r) + r.$$

Multiplication For $r_1, r_2, r_3 \in R$, we have

- (**closure**) $r_1 r_2 \in R$
- (**associativity**) $r_1(r_2 r_3) = (r_1 r_2) r_3$

Also, $\exists 1 \in R$ (a.k.a the **mutiplicative identity**), such that

$$\forall r \in R \quad r \cdot 1 = r = 1 \cdot r.$$

Finally, for $R = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, $\forall r \in R$, $\exists r^{-1} \in R$ such that

$$r \cdot r^{-1} = 1 = r^{-1} \cdot r.$$

Note that for $R = \mathbb{Z}_n$, where $n \in \mathbb{Z}$, not all $[r] \in \mathbb{Z}_n$ have a multiplicative inverse. For example, for $[2] \in \mathbb{Z}_4$, there is no $[x] \in \mathbb{Z}_4$ such that $[2][x] = [1]$.[1]

[1] This is best proven using techniques introduced in MATH135/145.

1.1.2 *Matrices*

For $n \in \mathbb{N} \setminus \{1\}$, an $n \times n$ matrix over $\mathbb{R}$ [2] is an $n \times n$ array that can be expressed as follows:

[2] $\mathbb{R}$ can be replaced by $\mathbb{Q}$ or $\mathbb{C}$.

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

where for $1 \leq i, j \leq n$, $a_{ij} \in \mathbb{R}$. We denote $M_n(\mathbb{R})$ as the set of all $n \times n$ matrices over $\mathbb{R}$.

As in Section 1.1.1, we can perform **addition and multiplication** on $M_n(\mathbb{R})$.

Matrix Addition Given $A = [a_{ij}], B = [b_{ij}], C = [c_{ij}] \in M_n(\mathbb{R})$, we define matrix addition as

$$A + B = [a_{ij} + b_{ij}],$$

which immediately gives the **closure property**, since $a_{ij} + b_{ij} \in \mathbb{R}$ and hence $A + B \in M_n(\mathbb{R})$. Also, by this definition, we also immediately obtain the **associativity property**, i.e.

$$A + (B + C) = (A + B) + C.$$

We define the zero matrix as

$$0 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then we have that 0 is the **additive identity**, i.e.

$$A + 0 = A = 0 + A.$$

Finally, $\forall A \in M_n(\mathbb{R}), \exists(-A) \in M_n(\mathbb{R})$ (the **additive inverse**) such that

$$A + (-A) = 0 - (-A) + A.$$

Note that in this case, we also have that that the operation is **commutative**, i.e.

$$A + B = B + A.$$

Matrix Multiplication Given $A = [a_{ij}], B = [b_{ij}], C = [c_{ij}] \in M_n(\mathbb{R})$, we define the matrix multiplication as

$$AB = [d_{ij}] \text{ where } c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \in \mathbb{R}.$$

Clearly, $AB \in M_n(\mathbb{R})$, i.e. it is **closed under matrix multiplication**. Also, we have that, under such a defintion, matrix multiplication is **associative**, i.e.

$$A(BC) = (AB)C.$$

Define the identity matrix, $I \in M_n(\mathbb{R})$, as follows:

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Then we have that I is the **multiplicative identity**, since

$$AI = A = IA.$$

However, contrary to matrix addition, $\forall A \in M_n(\mathbb{R})$, it is not always true that $\exists A^{-1} \in M_n(\mathbb{R})$ such that

This is especially true if the **determinant** of A is 0.

$$AA^{-1} = I = A^{-1}A.$$

Also, we can always find some $A, B \in M_n(\mathbb{R})$ such that

$$AB \neq BA,$$

i.e. matrix multiplication is not always commutative.

The common properties of the operations from above: **closure, associativity, and existence of an inverse**, are not unique to just addition and multiplication. We shall see in the next lecture that there are other operations where these properties will continue to hold, e.g. **permutations**.

2 Lecture 2 May 04th 2018

2.1 Introduction (Continued)

2.1.1 Permutations

Definition 2.1.1 (Injectivity)
*Let $f : X \to Y$ be a function. We say that f is **injective** (or **one-to-one**) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.*

Definition 2.1.2 (Surjectivity)
*Let $f : X \to Y$ be a function. We say that f is **surjective** (or **onto**) if $\forall y \in Y \;\; \exists x \in X \;\; f(x) = y$.*

Definition 2.1.3 (Bijectivity)
*Let $f : X \to Y$ be a function. We say that f is **bijective** if it is both **injective** and **surjective**.*

Definition 2.1.4 (Permutations)
Given a non-empty set L, a permutation of L is a bijection from L to L. The set of all permutations of L is denoted by S_L.

Example 2.1.1
Consider the set $L = \{1, 2, 3\}$, which has the following 6 different permuta-

tions:

$$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} \quad \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} \quad \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} \quad \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} \quad \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$$

Note

$$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$$

indicates the bijection $\sigma : \{1,2,3\} \to \{1,2,3\}$ *with* $\sigma(1) = 1$, $\sigma(2) = 3$ *and* $\sigma(3) = 2$.

FOR $n \in \mathbb{N}$, we denote $S_n := S_{\{1,2,...,n\}}$, the set of all permutations of $\{1,2,...,n\}$. Example 2.1.1 shows the elements of the set S_3.

Definition 2.1.5 (Order)
The ***order*** *of a set* A*, denoted by* $|A|$*, is the cardinality of the set.*

Example 2.1.2
We have seen that the order of S_3, $|S_3|$ *is* $6 = 3!$.

Proposition 2.1.1
$|S_n| = n!$

Proof
$\forall \sigma \in S_n$, *there are* n *choices for* $\sigma(1)$, $n-1$ *choices for* $\sigma(2)$, ..., 2 *choices for* $\sigma(n-1)$, *and finally* 1 *choice for* $\sigma(n)$. $\square$

Do elements of S_n *share the same properties as what we've seen in the numbers?* Given $\sigma, \tau \in S_n$, we can **compose** the 2 together to get a third element in S_n, namely $\sigma\tau$ (wlog), where $\sigma\tau : \{1,...,n\} \to \{1,...,n\}$ is given by $\forall x \in \{1,...,n\}$, $x \mapsto \sigma(\tau(x))$.

It is important to note that $\because$ σ, τ are **both bijective**, $\sigma\tau$ is also bijective. Thus, together with the fact that $\sigma\tau : \{1,...,n\} \to \{1,...,n\}$, we have that $\sigma\tau \in S_n$ by definition of S_n.

$\therefore \forall \sigma, \tau \in S_n$, $\sigma\tau, \tau\sigma \in S_n$, but $\sigma\tau \neq \tau\sigma$ in general. The following is an example of the stated case:

Example 2.1.3
Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \text{ and } \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}.$$

Compute $\sigma\tau$ and $\tau\sigma$ to show that they are not equal.

Solution

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \text{ but } \tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

Perhaps what is interesting is the question of: **when does commutativity occur?** One such case is when σ and τ have support sets that are disjoint[1].

[1] This is proven in A1

On the other hand, the associative property holds[2], i.e.

$$\forall \sigma, \tau, \mu \in S_n \;\; \sigma(\tau\mu) = (\sigma\tau)\mu$$

[2]

Exercise 2.1.1
Prove this as an exercise.

The set S_n also has an identity element[3], namely

$$\varepsilon = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}$$

[3]

Exercise 2.1.2
Verify that the given identity element is indeed the identity, i.e.

$$\forall \sigma \in S_n \;\; \sigma\varepsilon = \sigma = \varepsilon\sigma.$$

Finally, $\forall \sigma \in S_n$, since σ is a bijection, we have that its inverse function, σ^-1 is also a bijection, and thus satisfies the requirements to be in S_n. We call $\sigma^{-1} \in S_n$ to be the **inverse permutation** of σ, such that

$$\forall x, y \in \{1, ..., n\} \quad \sigma^{-1}(x) = y \iff \sigma(y) = x.$$

It follows, immediately, that

$$\sigma\big(\sigma^{-1}(x)\big) = x \wedge \sigma^{-1}\big(\sigma(y)\big) = y.$$

$\therefore$ We have that

$$\sigma\sigma^{-1} = \varepsilon = \sigma^{-1}\sigma.$$

Example 2.1.4
Find the inverse of

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 273 & \end{pmatrix}$$

Solution
By rearranging the image in ascending order, using them now as the object

and their respective objects as their image, construct

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}.$$

It can easily (although perhaps not so prettily) be shown that

$$\sigma\tau = \varepsilon = \tau\sigma.$$

With all the above, we have for ourselves the following proposition:

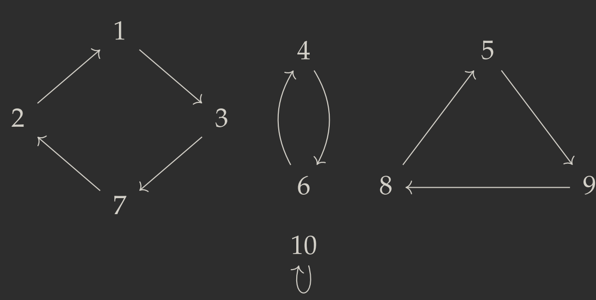
Proposition 2.1.2 (Properties of S_n)
We have

1. $\forall \sigma, \tau \in S_n \;\; \sigma\tau, \tau\sigma \in S_n.$
2. $\forall \sigma, \tau, \mu \in S_n \;\; \sigma(\tau\mu) = (\sigma\tau)\mu.$
3. $\exists \varepsilon \in S_n \;\; \forall \sigma \in S_n \;\; \sigma\varepsilon = \sigma = \varepsilon\sigma.$
4. $\forall \sigma \in S_n \;\; \exists! \sigma^{-1} \in S_n \;\; \sigma\sigma^{-1} = \varepsilon = \sigma^{-1}\sigma.$

CONSIDER

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 7 & 6 & 9 & 4 & 2 & 5 & 8 & 10 \end{pmatrix} \in S_{10}$$

If we represent the action of σ geometrically, we get



We observe that σ can be **decomposed** into one 4-cycle, $\begin{pmatrix} 1 & 3 & 7 & 2 \end{pmatrix}$, one 2-cycle, $\begin{pmatrix} 4 & 6 \end{pmatrix}$, one 3-cycle, $\begin{pmatrix} 5 & 9 & 8 \end{pmatrix}$, and one 1-cycle, $\begin{pmatrix} 10 \end{pmatrix}$.

Note that these cycles are (pairwise) **disjoint**, and we can write[4]

[4] We generally do not include the 1-cycle and assume that by excluding them, it is known that any number that is supposed to appear loops back to themselves.

$$\sigma = \begin{pmatrix} 1 & 3 & 7 & 2 \end{pmatrix} \begin{pmatrix} 4 & 6 \end{pmatrix} \begin{pmatrix} 5 & 9 & 8 \end{pmatrix}$$

Note that we may also write

$$\begin{aligned} \sigma &= \begin{pmatrix} 4 & 6 \end{pmatrix} \begin{pmatrix} 5 & 9 & 8 \end{pmatrix} \begin{pmatrix} 1 & 3 & 7 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 4 \end{pmatrix} \begin{pmatrix} 9 & 8 & 5 \end{pmatrix} \begin{pmatrix} 7 & 2 & 1 & 3 \end{pmatrix} \end{aligned}$$

It is interesting to note that the cycles can rotate their "elements" in a **cyclic** manner, i.e.

$$\begin{pmatrix} 1 & 3 & 7 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 2 & 1 & 3 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 7 & 3 \end{pmatrix}.$$

Although the decomposition of the cycle notation is not unique (i.e. you may rearrange them), each individual cycle is unique, and is proven below[5].

[5] See bonus question of A1. Proof will be included in the notes once the assignment is over.

Theorem 2.1.1 (Cycle Decomposition Theorem)
If $\sigma \in S_n$, $\sigma \neq \varepsilon$, then σ is a product of (one or more) disjoint cycles of length at least 2. This factorization is unique up to the order of the factors.

Note (Convention)
Every permutation in S_n can be regarded as a permutation of S_{n+1} by fixing the permutation of $n+1$. Therefore, we have that

$$S_1 \subseteq S_2 \subseteq \ldots \subseteq S_n \subseteq S_{n+1} \subseteq \ldots$$

3 Lecture 3 May 07th 2018

3.1 Groups

3.1.1 Groups

Definition 3.1.1 (Groups)
*Let G be a set and $*$ an operation on $G \times G$. We say that $G = (G, *)$ is a **group** if it satisfies*[1]

1. ***Closure***: $\forall a, b \in G \quad a * b \in G$
2. ***Associativity***: $\forall a, b, c \in G \quad a * (b * c) = (a * b) * c$
3. ***Identity***: $\exists e \in G \;\; \forall a \in G \quad a * e = a = e * a$
4. ***Inverse***: $\forall a \in G \;\; \exists b \in G \quad a * b = e = b * a$

[1] If you wonder why the uniqueness is not specified for **Identity** and **Inverse**, see Proposition 3.1.1.

Definition 3.1.2 (Abelian Group)
A group G is said to be abelian if $\forall a, b \in G$, we have $a * b = b * a$.

Proposition 3.1.1 (Group Identity and Group Element Inverse)
Let G be a group and $a \in G$.

1. *The identity of G is unique.*
2. *The inverse of a is unique.*

Proof

1. *If $e_1, e_2 \in G$ are both identities of G, then we have*

$$e_1 \overset{(1)}{=} e_1 * e_2 \overset{(2)}{=} e_2$$

where (1) is because e_2 is an identity and (2) is because e_1 is an identity.

2. *Let $a \in G$. If $b_1, b_2 \in G$ are both the inverses of a, then we have*

$$b_1 = b_1 * e = b_1 * (a * b_2) \overset{(1)}{=} e * b_2 = b_2$$

where (1) is by associativity.

Example 3.1.1

The sets $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, and $(\mathbb{C}, +)$ are all abelian, wehre the additive identity is 0, and the additive inverse of an element r is $(-r)$.

Note

$(\mathbb{N}, +)$ is not a group for neither does it have an identity nor an inverse for any of its elements.

Example 3.1.2

*The sets $(\mathbb{Q}, \cdot)$, $(\mathbb{R}, \cdot)$ and $(\mathbb{C}, \cdot)$ are **not** groups, since 0 has no multiplicative inverse in $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.*

We may define that for a set S, let $S^* \subseteq S$ contain all the elements of S that has a multiplicative inverse. For example, $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$. Then, $(\mathbb{Q}, \cdot), (\mathbb{R}, \cdot)$ and $(\mathbb{C}, \cdot)$ are groups and are in fact abelian, where the multiplicative identity is 1 and the multiplicative of an element r is $\frac{1}{r}$.

Example 3.1.3

The set $(M_n(\mathbb{R}), +)$ is an abelian group, where the additive identity is the zero matrix, $0 \in M_n(\mathbb{R})$, and the additive inverse of an element $M = [a_{ij}] \in M_n(\mathbb{R})$ is $-M = [-a_{ij}] \in M_n(\mathbb{R})$.

CONSIDER the set $M_n(\mathbb{R})$ under the matrix mutiplication operation that we have introduced in Lecture 1 May 02nd 2018. We found that the identity matrix is

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \in M_n(\mathbb{R}).$$

But since not all elements of $M_n(\mathbb{R})$ have a multiplicative inverse[2], $(M_n(\mathbb{R}), \cdot)$ is not a group.

[2] The multiplicative inverse of a matrix does not exist if its determinant is 0.

But we can try to do something similar as to what we did before: by excluding the elements that do not have an inverse. In this case, we exclude elements whose determinant is 0. Define the set

$$GL_n(\mathbb{R}) := \{M \in M_n(\mathbb{R}) \,:\, \det M \neq 0\}$$

Note that $\because \det I = 1 \neq 0$, we have that $I \in GL_n(\mathbb{R})$.

Also, $\forall A, B \in GL_n(\mathbb{R})$, we have that $\because \det A \neq 0 \wedge \det B \neq 0$,

$$\det AB = \det A \det B \neq 0,$$

and therefore $AB \in GL_n(\mathbb{R})$. Finally, $\forall M \in GL_n(\mathbb{R})$, $\exists M^{-1} \in GL_n(\mathbb{R})$ such that

$$MM^{-1} = I = M^{-1}M$$

since $\det M \neq 0$. $\therefore$ $(GL_n(\mathbb{R}), \cdot)$ is a group, and is in fact called the **general linear group of degree n over $\mathbb{R}$**.

Since we have introduced permutations in Lecture 2 May 04th 2018, we shall formalize the purpose of its introduction below.

Example 3.1.4

*Consider S_n, the set of all permutations on $\{1, 2, ..., n\}$. By Proposition 2.1.2, we know that S_n is a group. We call S_n the **symmetry group of degree n**. For $n \geq 3$, the group S_n is not abelian[3].*

[3] Let us make this an exercise.

Exercise 3.1.1
For $n \geq 3$, prove that the group S_n is not abelian.

Now that we have a fairly good idea of the basic concept of a group, we will now proceed to look into handling multiple groups. One such operation is known as the **direct product**.

Example 3.1.5

Let G and H be groups. Their direct product is the set $G \times H$ with the *component-wise operation defined by*

$$(g_1, h_1) * (g_2, h_2) = (g_1 *_G g_2, h_1 *_H h_2)$$

where $g_1, g_2 \in G$, $h_1, h_2 \in H$, $*_G$ is the operation on G, and $*_H$ is the operation on H.

*The **closure** and **associativity** property follow immediately from the definition of the operation. The identity is $(1_G, 1_H)$ where 1_G is the identity of G and 1_H is the identity of H. The inverse of an element $(g_1, h_1) \in G \times H$ is (g_1^{-1}, h_1^{-1}).*

By induction, we can show that if $G_1, G_2, ..., G_n$ are groups, then so is $G_1 \times G_2 \times \ldots \times G_n$.

To facilitate our writing, use shall use the following notations:

Notation

Given a group G and $g_1, g_2 \in G$, we often denote its identity by 1, and write $g_1 * g_2 = g_1 g_2$. Also, we denote the unique inverse of an element $g \in G$ as g^{-1}.

We will write $g^0 = 1$. Also, for $n \in \mathbb{N}$, we define

$$g^n = \underbrace{g * g * \ldots * g}_{n \text{ times}}$$

and

$$g^{-n} = (g^{-1})^n$$

With the above notations,

Proposition 3.1.2

Let G be a group and $g, h \in G$. We have

1. $(g^{-1})^{-1} = g$
2. $(gh)^{-1} = h^{-1} g^{-1}$
3. $g^n g^m = g^{n+m}$ *for all* $n, m \in \mathbb{Z}$
4. $(g^n)^m = g^{nm}$ *for all* $n, m \in \mathbb{Z}$

Exercise 3.1.2
Prove Proposition 3.1.2 as an exercise.

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Warning

In general, it is not true that if $g, h \in G$, then $(gh)^n = g^n h^n$. For example,

$$(gh)^2 = ghgh \quad \text{but} \quad g^2h^2 = gghh.$$

The two are only equal if and only if G is abelian.

Proposition 3.1.3 (Cancellation Laws)

Let G be a group and $g, h, f \in G$. Then

1. *(**Left and Right Cancellation**)*
   (a) $gh = gf \implies h = f$
   (b) $hg = fg \implies h = f$
2. *The equation $ax = b$ and $ya = b$ have unique solution for $x, y \in G$.*

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