1 Complex Numbers and Their Properties

Complex Plane as a Set
$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$

Real and Imaginary Part
$\forall z = x + iy \in \mathbb{C}\ x, y \in \mathbb{R}$
$\mathrm{Re}(z) = x\ \ \mathrm{Im}(z) = y$

Product
$\forall z = a + ib, w = c + id \in \mathbb{C}\ a, b, c, d \in \mathbb{R}$
$zw = (ac - bd) + i(ad + bc)$

Inverse of a Complex Number
$\forall z = a + ib \in \mathbb{C}\ a, b \in \mathbb{R}$
$\exists z^{-1} = \frac{a}{aa^2+b^2} - i\frac{b}{a^2+b^2} \in \mathbb{C}$

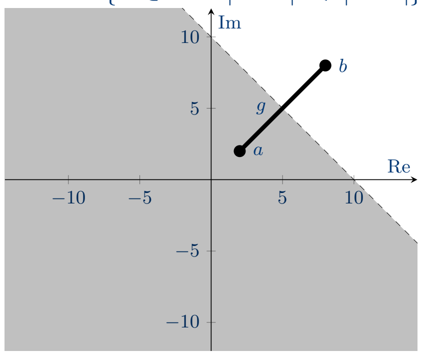
Conjugate
$\forall z = a + ib \in \mathbb{C}\ a, b \in \mathbb{R}$
$\exists \bar{z} = a - ib \in \mathbb{C}$

Modulus
$\forall z = x = iy \in \mathbb{C}\ x, y \in \mathbb{R}$
$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}$

Basic Inequalities
$\forall z, w \in \mathbb{C}$,

1. $|\mathrm{Re}(z)| \leq |z|$
2. $|\mathrm{Im}(z)| \leq |z|$
3. $|z + w| \leq |z| + |w|$
4. $|z + w| \geq |\ |z| + |w|\ |$

Region of a set of Complex Numbers
Describe $\{z \in \mathbb{C} : |z - a| < |z - b|\}$.



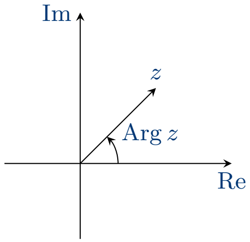
Every complex number has exactly 2 roots
$\forall z = x = iy \in \mathbb{C}\ x, y \in \mathbb{R}$
$\exists w_{1,2} = u + iv \in \mathbb{C}\ u, v \in \mathbb{R}$

$$w = \begin{cases} \pm\left[\left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}} + i\left(\frac{-x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}\right] & y > 0 \\ \pm\left[\left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}} - i\left(\frac{-x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}\right] & y < 0 \\ \pm\sqrt{x} & y = 0, x > 0 \\ \pm i\sqrt{x} & y = 0, x < 0 \end{cases}$$

Quadratic Formula
$\forall a, b, c \in \mathbb{C}\ a \neq 0\ az^2 + bz + c = 0$

$$z = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Argument



Polar Form

$\forall z \in \mathbb{C}\ \exists r, \theta \in \mathbb{R}\ \theta \in [0, 2\pi)$
$z = re^{i\theta}$

Polar to Cartesian

$x = r\cos\theta \quad y = r\sin\theta$

Cartesian to Polar

$r = |z| \quad \tan\theta = \frac{x}{y}$

Conjugate in Polar Form

$z = re^{i\theta} \iff \bar{z} = re^{-i\theta}$

Inverse in Polar Form

$z = re^{i\theta} \land z \neq 0$
$\implies z^{-1} = \frac{1}{r}e^{-i\theta}$

Product in Polar Form

- $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

- $\forall n \in \mathbb{Z}\ (re^{in}) = r^n e^{in\theta}$

nth Roots of a Complex Number

$\left\{ r^{\frac{1}{n}} e^{i\left(\frac{\theta + 2\pi k}{n}\right)} : k = 0, 1, ..., n - 1 \right\}$

nth Roots of Unity

$\left\{ e^{i\left(\frac{2\pi k}{n}\right)} : k = 0, 1, ..., n - 1 \right\}$