PMATH352W18 - Complex Analysis - Topical Exercises

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Contents

Chapter 1

Complex Numbers

1.1 Basic Algebraic Properties

1. Verify that

   (i) $(\sqrt{2}-i)-i(1-\sqrt{2}i) = -2i$; (ii) $(2-3i)(-2+i) = -1+8i$; (iii) $(3+i)(3-i)(\frac{1}{5}-\frac{1}{10}i) = 2+i$.

2. Find the complex numbers which are complex conjugates of

   (i) Their own squares; (ii) Their own cubes.

3. Caclulate the following quantities:

   (i) $\frac{1+i\tan\theta}{1-i\tan\theta}$; (ii) $\frac{(1+2i)^3-(1-i)^3}{(3+2i)^3-(2+i)^2}$; (iii) $\frac{(1-i)^5-1}{(1+i)^5+1}$; (iv) $\frac{(1+i)^9}{(1-i)^7}$.

4. Find the points $z = x+iy$ such that

   (i) $|z| \leq 2$; (ii) $\operatorname{Im} z > 0$; (iii) $\operatorname{Re} z \leq \frac{1}{2}$; (iv) $\operatorname{Re}(z^2) = a$; (v) $\left|z^2-1\right| = a$; (vi) $\left|\frac{z-1}{z+1}\right| \leq 1$; (vii) $\left|\frac{z-\alpha}{z-\beta}\right| = 1$.

5. Derive the identity

$$\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) = \frac{z_1 z_2}{z_3 z_4} \quad (z_3 \neq 0, z_4 \neq 0)$$

6. Using the above identity, derive the cancellation law

$$\frac{z_1 z}{z_2 z} = \frac{z_1}{z_2} \quad (z_2 \neq 0, z \neq 0)$$

7. Using properties of moduli that has been introduced, show that when $|z_3| \neq |z_4|$,

$$\frac{\mathrm{Re}(z_1+z_2)}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}$$

8. Verify that $\sqrt{2}\,|z| \geq |\mathrm{Re}\, z| + |\mathrm{Im}\, z|$. (Hint: Reduce this inequality to $(|x|-|y|)^2 \geq 0$.)

(Jump to solutions)

1.2 Polar Form

1. Represent the following complex numbers in polar form:

   (i) $1+i$; (ii) $-1+i$; (iii) $-1-i$; (iv) $1-i$; (v) $1+\sqrt{3}i$; (vi) $-1+\sqrt{3}i$; (vii) $-1-\sqrt{3}i$; (viii) $1-\sqrt{3}i$; (ix) $2+\sqrt{3}+i$.

2. Generalize the Triangle Inequality.

3. Prove the identity

$$|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2 + |z_2|^2),$$

   for arbitrary complex numbers $z_1, z_2, ..., z_n$.

4. When do three points $z_1, z_2, z_3 \in \mathbb{C}$ lie on a straight line in the complex plane?

5. Let σ be the line segment joining two points z_1 and z_2. Find the point z dividign σ in the ratio $\lambda_1 : \lambda_2$.

6. Four points z_1, z_2, z_3, z_4 satisfy the conditions

$$z_1+z_2+z_3+z_4 = 0, \quad |z_1| = |z_2| = |z_3| = |z_4| = 1.$$

   Show that the points either lie ast the vertices of a square inscribed in the unit circle or else coincide in pairs.

7. Calculate the following quantities:

   (i) $(1+i)^{25}$; (ii) $\left(\frac{1+\sqrt{3}i}{1-i}\right)^{30}$; (iii) $\left(1-\frac{\sqrt{3}-i}{2}\right)^2 4$; (iv) $\frac{(-1+\sqrt{3}i)^{15}}{(1-i)^{30}} + \frac{(-1-\sqrt{3}i)^{15}}{(1+i)^{20}}$.

8. Use De Moivre's theorem to express $\cos nx$ and $\sim nx$ in terms of powers of $\cos x$ and $\sin x$.

9. Express $\tan 6x$ in terms of $\tan x$.

10. Write $\sqrt{1+i}$ in polar form.

1.3 Roots of a Complex Number

1. Find all the values of the following roots:

   (i) $\sqrt[3]{1}$; (ii) $\sqrt[3]{i}$; (iii) $\sqrt[4]{-1}$; (iv) $\sqrt[6]{-8}$; (v) $\sqrt[8]{1}$; (vi) $\sqrt{3+4i}$; (vii) $\sqrt[3]{-2+2i}$; (viii) $\sqrt[5]{-4+3i}$; (ix) $\sqrt[6]{\frac{1-i}{\sqrt{3}+i}}$; (x) $\sqrt[8]{\frac{1+i}{\sqrt{3}-i}}$;

2. Prove that the sum of all the distinct nth roots of unity is zero. What geometric fact does this express?

3. Let ϵ be any nth root of unity other than 1. prove that

$$1 + 2\epsilon + 3\epsilon^2 + \ldots + n\epsilon^{n-1} = \frac{n}{\epsilon - 1}$$

4. Prove that every complex number $\alpha \neq -1$ of unit modulus can be represented in the form

$$\alpha = \frac{1+it}{1-it},$$

   where $t \in \mathbb{R}$.

Chapter 2

Complex Functions

2.1 Limits and Continuity

1. Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$, $z_0 \in \Omega$. $\forall z \in \Omega$, prove that

$$(z \to z_0 \implies f(z) \to \infty) \iff (z \to z_0 \implies \psi(z) = \frac{1}{f(z)} \to 0)$$

2. The **Cauchy Convergence Criterion for sequences** states that a complex sequence z_n is convergent iff

$$\forall \epsilon > 0 \, \exists N = N(\epsilon) > 0 \, \forall m, n > N$$
$$|z_m - z_n| < \epsilon.$$

Prove the generalization of the criterion: The function $f(z)$ approaches a limit as $z \to z_0$ iff

$$\forall \epsilon > 0 \, \exists \delta = \delta(\epsilon) > 0$$
$$(0 < \left|z' - z_0\right| < \delta \, \wedge \, 0 < \left|z'' - z\right| < \delta) \implies \left|f(z') - f(z'')\right| < \epsilon$$

3. Let $f(z)$ be a rational function, i.e., a ratio

$$f(z) = \frac{a_0 + a_1 z + \ldots a_m z^m}{b_0 + b_1 z + \ldots + b_n z^n} \;\; (a_m \neq 0, b_n \neq 0) \tag{2.1}$$

of two polynomials. Discuss the possible values of $\lim_{z\to\infty} f(z)$.

4. Where is the function Equation (2.1) continuous?

5. Prove that if $f(z)$ is continuous in a region Ω, then so is $|f(z)|$.
6. Is the function
$$f(zz) = \frac{1}{1-z}$$
continuous in the open disk $|z| < 1$?

Answers

Chapter 1

Basic Algebraic Properties

(Jump to: Section 1.1)

1.
2. (a) $(0,0),(1,0),(-\frac{1}{2},\pm\sqrt{\frac{3}{4}})$

   (b) $(0,0),(1,0),(-1,0),(0,i),(0,-i)$
3. (a) $(\cos\theta - i\sin\theta)^2$

   (b)

   (c) $\frac{9-40i}{41}$

   (d) 2
4. (a) $\{(x,y) : x^2+y^2 \leq 4\}$

   (b) $\{(x,y) : y > 0\}$

   (c) $\{(x,y) : x \leq \frac{1}{2}\}$

   (d) $x^2 - y^2 = a$. Also, refer to this graph on Desmos: https://www.desmos.com/calculator/buwtyobjrn

   (e) Refer to graph on Desmos: https://www.desmos.com/calculator/a3pnbwueja

5.

$$\begin{aligned}
\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) &= z_1\left(\frac{\overline{z_3}}{|z_3|^2}\right)z_2\left(\frac{\overline{z_4}}{|z_4|^2}\right) \quad \text{since } z\bar{z} = |z|^2 \\
&= z_1 z_2\left(\frac{\overline{z_3}\,\overline{z_4}}{|z_3|^2\,|z_4|^2}\right) \\
&= z_1 z_2\left(\frac{\overline{z_3 z_4}}{|z_3 z_4|^2}\right) \quad \text{since } \bar{z}\overline{w} = \overline{zw} \text{ and } |z|\,|w| = |zw| \\
&= \frac{z_1 z_2}{z_3 z_4} \quad \text{since } z\bar{z} = |z|^2
\end{aligned}$$

6. $\frac{z_1 z}{z_2 z} = \left(\frac{z_1}{z_2}\right)\left(\frac{z}{z}\right) = \frac{z_1}{z_2}$

7. By the Triangle Inequality,

$$|z_1| + |z_2| \geq |z_1 + z_2| \geq \sqrt{\mathrm{Re}(z_1+z_2)^2 + \mathrm{Im}(z_1+z_2)^2} \geq \sqrt{\mathrm{Re}(z_1+z_2)} = \mathrm{Re}(z_1+z_2)$$

and by the Reversed Triangle Inequality,

$$\begin{gathered}
|z_3 + z_4| \geq ||z_3| - |z_4|| \\
\implies \frac{1}{|z_3+z_4|} \leq \frac{1}{||z_3| - |z_4||}
\end{gathered}$$

Thus

$$\frac{\mathrm{Re}(z_1+z_2)}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}$$