

PMATH352W18 Complex Analysis - Class Notes

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Chapter 1

Lecture 1 - Jan 3, 2018

1.1 Complex Numbers and Their Properties

Definition 1.1.1 (Complex Number, Complex Plane)

A **complex number** is a vector in \mathbb{R}^2 . The **complex plane**, denoted by \mathbb{C} , is a set of complex numbers,

$$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

In \mathbb{C} , we usually write

$$\begin{aligned} 0 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & 1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ i &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} & x &= \begin{pmatrix} x \\ 0 \end{pmatrix} \\ iy &= \begin{pmatrix} 0 \\ y \end{pmatrix} \end{aligned}$$

where $x, y \in \mathbb{R}$. Consequently, we have that

$$x + iy = x + yi = \begin{pmatrix} x \\ y \end{pmatrix}$$

If for $x, y \in \mathbb{R}$, $z = x + iy$, then x is called the real part of z and y is called the imaginary part of z , and we write

$$\operatorname{Re}(z) = x \quad \operatorname{Im}(z) = y.$$

Note

- It is easy to see how \mathbb{R} is a subset of \mathbb{C} .

- Complex Numbers of the form $\begin{pmatrix} 0 \\ y \end{pmatrix}$ where $y \in \mathbb{R}$ are called purely imaginary numbers.
- Certain authors may prefer to denote $i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Definition 1.1.2 (Sum and Product)

We define the sum of two complex numbers to be the usual vector sum, i.e.

$$\begin{aligned} (a + ib) + (c + id) &= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \begin{pmatrix} a + c \\ b + d \end{pmatrix} \\ &= (a + c) + i(b + d) \end{aligned}$$

where $a, b, c, d \in \mathbb{R}$.

We define the product of two complex numbers by setting $i^2 = -1$, and by requiring the product to be commutative, associative, and distributive over the sum. In this setup, we have that

$$\begin{aligned} (a + ib)(c + id) &= ac + iad + ibc + i^2bd \\ &= (ac - bd) + i(ad + bc) \end{aligned} \tag{1.1}$$

Note

It is interesting to note that any complex number times zero is zero, just like what we have with real numbers.

$$\begin{aligned} \forall z = x + iy \in \mathbb{C} \quad x, y \in \mathbb{R} \quad 0 \in \mathbb{C} \\ z \cdot 0 = (x + iy)(0 + i0) = 0 + i0 = 0 \end{aligned}$$

Example 1.1.1

Let $z = 2 + i, w = 1 + 3i$. Find $z + w$ and zw .

$$\begin{aligned} z + w &= (2 + i) + (1 + 3i) \\ &= 3 + 4i \end{aligned}$$

$$\begin{aligned} zw &= (2 + i)(1 + 3i) \\ &= (2 - 3) + i(6 + 1) \quad \text{By Equation (1.1)} \\ &= -1 + 7i \end{aligned}$$

Example 1.1.2

Show that every non-zero complex number has a multiplicative inverse, z^{-1} , and find a formula for this inverse.

Let $z = a + ib$ where $a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$. Then

$$\begin{aligned}
 & z(x + iy) = 1 \\
 \iff & (ax - by) + i(ay + bx) = 1 \\
 \iff & \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \iff & \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \iff & \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \iff & \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a \\ -b \end{pmatrix} \\
 \iff & x + iy = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2}
 \end{aligned}$$

Therefore, we have that the formula for the inverse is

$$(a + ib)^{-1} = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2} \quad (1.2)$$

Notation

For $z, w \in \mathbb{C}$, we write

$$\begin{aligned}
 -z &= -1z & w - z &= w + (-z) \\
 \frac{1}{z} &= z^{-1} & \frac{w}{z} &= wz^{-1}
 \end{aligned}$$

Example 1.1.3

Find $\frac{(4-i)-(1-2i)}{1+2i}$.

$$\begin{aligned}
 \frac{(4-i)-(1-2i)}{1+2i} &= \frac{3+i}{1+2i} \\
 &= (3+i)\left(\frac{1}{5} - i\frac{2}{5}\right) \\
 &= 1 - i
 \end{aligned}$$

Note

The set of complex numbers is a **field** under the operations of addition and multiplication. This means that $\forall u, v, w \in \mathbb{C}$,

$$\begin{array}{ll}
u + v = v + u & uv = vu \\
(u + v) + w = u + (v + w) & (uv)w = u(vw) \\
0 + u = u & 1u = u \\
u + (-u) = 0 & uu^{-1} = 1, \quad u \neq 0 \\
u(v + w) = uv + uw &
\end{array}$$

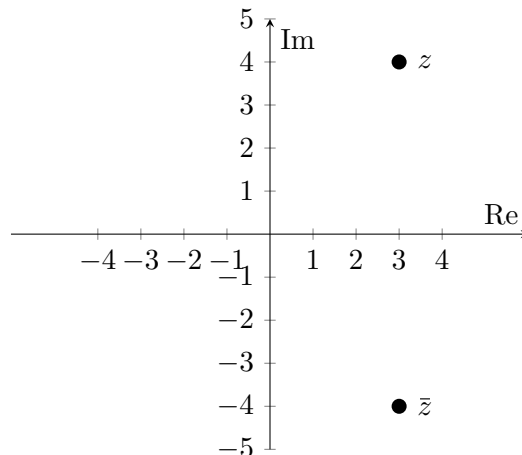
Since the distributive law holds for complex numbers, note that the binomial expansion works for $(w + z)^n$ where $w, z \in \mathbb{C}$ and $n \in \mathbb{N}$. (I did not verify if this is still true for when $n \in \mathbb{R}$.)

Definition 1.1.3 (Conjugate)

If $z = x + iy$ where $x, y \in \mathbb{R}$, then the **conjugate of z** is given by $\bar{z} = x - iy$

Example 1.1.4

Let $z = 3 + 4i$. Then the $\bar{z} = 3 - 4i$. Represented in the complex plane, we have the following:



We observe that on the complex plane, the conjugate of a complex number is simply its reflection on the real axis.

Definition 1.1.4 (Modulus)

We define the **modulus** (length, magnitude) of $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$, to be

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}. \quad (1.3)$$

Note

Note that this definition is consistent with the notion of the absolute value in real numbers when z is a real number, since if $y = 0$, $|z| = |x + i0| = \sqrt{x^2} = \pm x$.

Note

For $z, w \in \mathbb{R}$, we have

$$\begin{aligned} \bar{\bar{z}} &= z & z + \bar{z} &= 2 \operatorname{Re}(z) & z - \bar{z} &= 2i \operatorname{Im}(z) \\ z\bar{z} &= |z|^2 & |z| &= |\bar{z}| & \overline{z \pm w} &= \bar{z} \pm \bar{w} \\ \overline{zw} &= \bar{z} - \bar{w} & |zw| &= |z| |w| \end{aligned}$$

but note that $|z + w| \neq |z| + |w|$.

Note

While inequalities such as $z_1 < z_2$, where $z_1, z_2 \in \mathbb{C}$, are meaningless unless if both of them are real, $|z_1| < |z_2|$ means that the point z_1 in the complex plane is closer to the origin than the point z_2 .

Proposition 1.1.1 (Basic Inequalities)

1. $|\operatorname{Re}(z)| \leq |z|$
2. $|\operatorname{Im}(z)| \leq |z|$
3. $|z + w| \leq |z| + |w|$ *Triangle Inequality*
4. $|z + w| \geq ||z| - |w||$ *Inverse Triangle Inequality*

Proof

Note that $|z|^2 = \operatorname{Re}(z)^2 + \operatorname{Im}(z)^2$ and that we can express $|x| = \sqrt{x^2}$ for any $x \in \mathbb{R}$. 1 and 2 immediately follows from that.

To prove 3, we have that

$$\begin{aligned} |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) \\ &= |z|^2 + |w|^2 + (w\bar{z} + \bar{w}z) \\ &= |z|^2 + |w|^2 + 2 \operatorname{Re}(w\bar{z}) \\ &\leq |z|^2 + |w|^2 + 2 |w\bar{z}| \quad \text{by 1} \\ &= |z|^2 + |w|^2 + 2 |wz| \quad \text{since } |w\bar{z}| = |w| |\bar{z}| \text{ and } |z| = |\bar{z}| \\ &= (|z| + |w|)^2 \end{aligned}$$

To prove 4, note that

$$|z| = |z + w - w| \leq |z + w| + |w| \tag{1.4}$$

$$|w| = |w + z - z| \leq |z + w| + |z| \tag{1.5}$$

Observe that

$$\text{Equation (1.4)} \implies |z| - |w| \leq |z + w|$$

$$\text{Equation (1.5)} \implies |w| - |z| \leq |z + w|$$

Thus, we have that

$$|z + w| \geq ||z| - |w||$$

as required. \square

Item 3 in Proposition 1.1.1 can be generalized by the means of mathematical induction to sums involving any finite number of terms, as:

$$|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n| \quad (1.6)$$

where $n \in \mathbb{N} \setminus \{0, 1\}$.

To note the induction proof, when $n = 2$, Equation (1.6) is just Item 3. If Equation (1.6) is true for when $n = m$ where $m \in \mathbb{N} \setminus \{0, 1\}$, $n = m + 1$ is also true since by Item 3,

$$\begin{aligned} |(z_1 + z_2 + \dots + z_m) + z_{m+1}| &\leq |z_1 + z_2 + \dots + z_m| + |z_{m+1}| \\ &\leq (|z_1| + |z_2| + \dots + |z_m|) + |z_{m+1}|. \end{aligned}$$

The distance between two points $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}, x_1, x_2, y_1, y_2 \in \mathbb{R}$ is $|z_1 - z_2|$, since $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ is our usual notion of the Euclidean distance of two points on a plane.

Also, note that

$$z_1 - z_2 = z_1 + (-z_2)$$

and thus if we apply our knowledge of vector representation, $z_1 - z_2$ is the directed line segment from the point z_2 to z_1 .

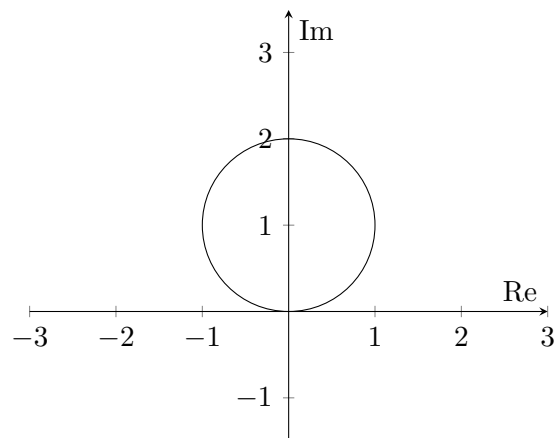
With the notion of a “distance” set on the complex plane, we can now explore upon points lying on a circle with a center z_0 and radius R , which satisfies the equation

$$|z - z_0| = R.$$

We may simply refer to this set of points as the circle $|z - z_0| = R$.

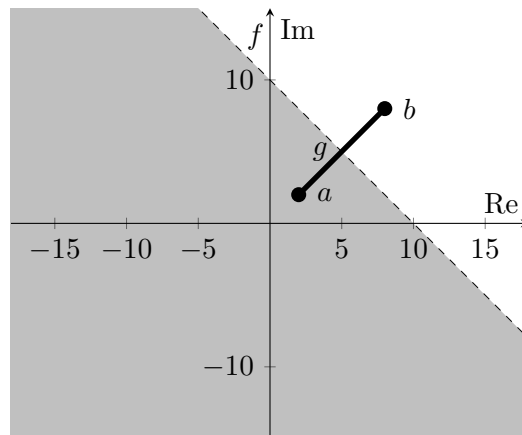
Example 1.1.5

We may describe a set $\{z \in \mathbb{C} : |z - i| = 1\}$ as follows:



Let $a, b \in \mathbb{C}$ describe the set $\{z \in \mathbb{C} : |z - a| < |z - b|\}$.

Suppose the following coordinates for a and b are arbitrary,



In the above, g is the line segment that connects the points a and b on the complex plane, while f is the perpendicular bisector of the line segment g . The area described by the set $\{z \in \mathbb{C} : |z - a| < |z - b|\}$ is the shaded area which is below f .

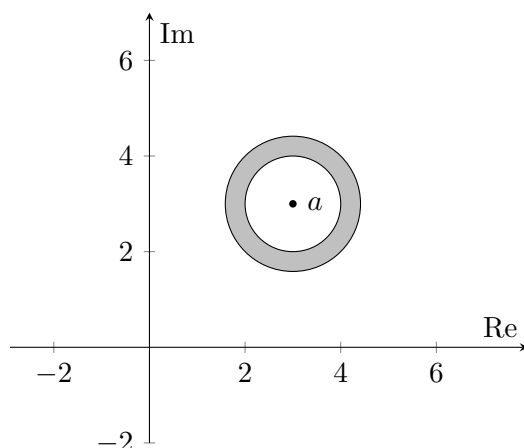
Chapter 2

Lecture 2 - Jan 5th, 2018

2.1 Complex Numbers and Their Properties (Continued)

Example 2.1.1

Let $a \in \mathbb{C}$. Describe the set $\{z \in \mathbb{C} : 1 < |z - a| < 2\}$.



Example 2.1.2

Show that every non-zero complex number has exactly two complex square roots, and find a formula for the square roots.

Let $z = x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$, and let $w = u + iv$, $u, v \in \mathbb{R}$. Then

$$\begin{aligned}
w^2 = z &\iff (u + iv)^2 = x + iy \\
&\iff (u^2 - v^2) + i(2uv) = x + iy \\
&\iff x = u^2 + v^2 \quad \text{and}
\end{aligned} \tag{2.1}$$

$$y = 2uv \tag{2.2}$$

Square both sides of Equation (2.2), and thus we have $y^2 = 4u^2v^2$.

Multiply Equation (2.1) by $4u^2$, and we get

$$\begin{aligned}
4u^2x &= 4u^4 - 4u^2v^2 = 4u^4 - y^2 \\
\iff 0 &= 4u^4 - 4u^2x - y^2 \\
\iff u^2 &= \frac{4x \pm \sqrt{16x^2 + 16y^2}}{8} \\
&= \frac{x \pm \sqrt{x^2 + y^2}}{2}
\end{aligned}$$

Suppose $y \neq 0$. Note that $x < \sqrt{x^2 + y^2}$. Thus $u^2 = \frac{x + \sqrt{x^2 + y^2}}{2} \implies u = \left(\frac{x + \sqrt{x^2 + y^2}}{2} \right)^{\frac{1}{2}}$.

Similarly, we can get

$$v = \pm \left(\frac{-x + \sqrt{x^2 + y^2}}{2} \right)^{\frac{1}{2}}$$

Note that all four choices of signs satisfy Equation (2.1). If $y > 0$, then u and v are either both positive or both negative by Equation (2.2).

Suppose $y = 0$. Then we have

$$w^2 = z = x$$

Therefore, we get

$$w = \begin{cases} \pm \left[\left(\frac{x + \sqrt{x^2 + y^2}}{2} \right)^{\frac{1}{2}} + i \left(\frac{-x + \sqrt{x^2 + y^2}}{2} \right)^{\frac{1}{2}} \right] & y > 0 \\ \pm \left[\left(\frac{x + \sqrt{x^2 + y^2}}{2} \right)^{\frac{1}{2}} - i \left(\frac{-x + \sqrt{x^2 + y^2}}{2} \right)^{\frac{1}{2}} \right] & y < 0 \\ \pm \sqrt{x} & y = 0, x > 0 \\ \pm i\sqrt{x} & y = 0, x < 0 \end{cases}$$

Remark

Let $z \in \mathbb{C}$. The notation \sqrt{z} may represent either one of the square roots of z or both of the square roots, i.e. it is possible that \sqrt{z} represents a set.

Exercise 2.1.1

Is it always okay for complex numbers such that $\sqrt{zw} = \sqrt{z}\sqrt{w}$, for $z, w \in \mathbb{C}$?

No. For example, consider $z = w = -1$. Then we have

$$\sqrt{zw} = \sqrt{1} = \pm 1$$

while

$$\sqrt{z}\sqrt{w} = i \cdot i = -1$$

and thus

$$\sqrt{zw} \neq \sqrt{z}\sqrt{w}.$$

Example 2.1.3

Find the values of $\sqrt{3 - 4i}$.

By [Example 2.1.2](#),

$$\begin{aligned} \sqrt{3 - 4i} &= \pm \left(\sqrt{\frac{3 + \sqrt{9 + 16}}{2}} - i \sqrt{\frac{-3 + \sqrt{9 + 16}}{2}} \right) \\ &= \pm(2 - i) \end{aligned}$$

Remark

The quadratic formula holds for complex polynomials, i.e.

$$\forall a, b, c \in \mathbb{C} \quad a \neq 0 \quad \forall z \in \mathbb{C} \quad az^2 + bz + c = 0,$$

the solution for z is given by

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (2.3)$$

The following is a short proof.

Proof

$$\begin{aligned}
az^2 + bz + c = 0 &\iff z^2 + \frac{b}{a}z + \frac{c}{a} = 0 \\
&\iff z^2 + \frac{b}{a}z + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \\
&\iff \left(z + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2} \\
&\iff z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

(Personal Note: where did the $-$ for the supposed \pm go? Or should it really be \pm ?)

Example 2.1.4

Solve $iz^2 - (2 + 3i)z + 5(1 + i) = 0$.

$$\begin{aligned}
z &= \frac{2 + 3i + \sqrt{(2 + 3i)^2 - 4i[5(1 + i)]}}{2i} \\
&= \frac{2 + 3i + \sqrt{-5 + 12i - 20i + 20}}{2i} \\
&= \frac{2 + 3i + \sqrt{15 + 8i}}{2i}
\end{aligned}$$

Note that by [Example 2.1.2](#),

$$\begin{aligned}
\sqrt{15 + 8i} &= \pm \left[\sqrt{\frac{15 + \sqrt{225 + 64}}{2}} - i\sqrt{\frac{-15 + \sqrt{225 + 64}}{2}} \right] \\
&= \pm \left[\sqrt{\frac{15 + 17}{2}} - i\sqrt{\frac{-15 + 17}{2}} \right] \\
&= \pm(4 - i)
\end{aligned}$$

Thus we have

$$\begin{aligned}
z &= \frac{2 + 3i + \sqrt{15 + 8i}}{2i} \\
&= \frac{2 + 3i \pm (4 - i)}{2i} \\
&= (6 + 2i) \left(-\frac{1}{2}i\right) \text{ or } (-2 + 4i) \left(-\frac{1}{2}i\right) \quad \text{by [Example 1.1.2](#)} \\
&= (1 - 3i) \text{ or } (2 + i)
\end{aligned}$$

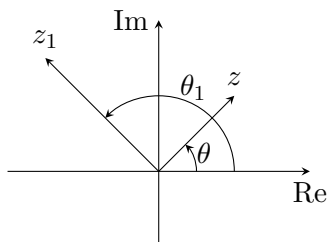
Chapter 3

Lecture 3 - Jan 8th, 2018

3.1 Complex Numbers and Their Properties (Continued 2)

Definition 3.1.1 (Argument of a Complex Number)

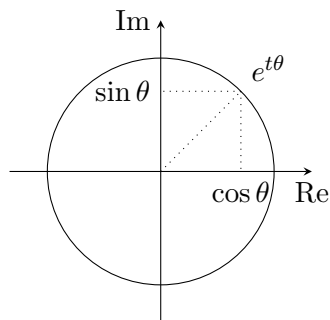
Let $z \in \mathbb{C} \setminus \{0\}$. The **argument** (or the angle) of z , denoted by $\arg z$, $\text{Arg } z$, or simply $\theta = \theta(z)$, is the angle modulo 2π (i.e. $0 \leq \theta < 2\pi$) between the vector defining z and the positive real axis (in the counterclockwise direction).



Notation

Let $e^{i\theta} := \cos \theta + i \sin \theta$. Note that this definition can be derived by extending the Taylor expansion of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for when $x \in \mathbb{C}$ (the sum of the real parts of the expansion is the Taylor expansion of cosine while the imaginary part for sine).

Now $e^{i\theta}$ is on the unit circle.


Example 3.1.1

Some examples of $\theta \in [0, 2\pi)$:

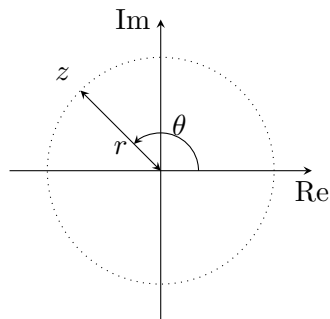
$$\begin{aligned} e^{i\frac{\pi}{4}} &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\frac{\pi}{2}} &= i \\ e^{i\frac{3\pi}{4}} &= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\pi} + 1 &= 0 \end{aligned}$$

Remark

$$\forall k \in \mathbb{Z} \quad \forall \theta \in \mathbb{R} \quad e^{i\theta} = e^{i(\theta+2\pi k)}$$

Remark

The complex number $re^{i\theta}$, where $r > 0, \theta \in [0, 2\pi)$, represents the complex number with modulus r and argument θ .



Therefore, $\forall z \in \mathbb{C}$, we can express

$$z := |z| e^{i \operatorname{Arg} z}. \quad (3.1)$$

With that, we now have two representations of a complex number:

- Cartesian representation: $z = x + iy$ where $x = \operatorname{Re}(z)$ and $y = \operatorname{Im}(z)$
- Polar representation: $z = re^{i\theta}$ where $r = |z|$ and $\theta = \operatorname{Arg} z \in [0, 2\pi)$

To convert between the two representations, we have the following equations:

Polar \rightarrow Cartesian:

$$x = r \cos \theta \quad y = r \sin \theta \quad (3.2)$$

Cartesian \rightarrow Polar:

$$\begin{aligned} r &= |z| \\ x \neq 0 &\implies \tan \theta = \frac{y}{x} \\ x = 0 &\implies \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \end{aligned} \quad (3.3)$$

On another note,

$$z = re^{i\theta} \implies \bar{z} = re^{-i\theta}$$

and

$$z \neq 0 \implies \frac{1}{z} = \frac{1}{r}e^{-i\theta}$$

Remark

$$\begin{aligned} \forall r_1, r_2 \in \mathbb{R} \quad \forall \theta_1, \theta_2 \in [0, 2\pi) \\ z_1 := r_1 e^{i\theta_1} \quad z_2 := r_2 e^{i\theta_2} \end{aligned}$$

Then

$$z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

Note that $e^{ix} e^{iy} = e^{i(x+y)}$ is true for all $x, y \in \mathbb{R}$ since

$$\begin{aligned} e^{ix} e^{iy} &= (\cos x + i \sin x)(\cos y + i \sin y) \\ &= (\cos x \cos y - \sin x \sin y) + i(\cos x \sin y + \cos y \sin x) \\ &= \cos(x + y) + i \sin(x + y) \\ &= e^{i(x+y)}. \end{aligned}$$

Generalizing the above, we get that

$$\forall n \in \mathbb{Z} \quad (re^{in}) = r^n e^{in\theta}$$

which is commonly known as **deMoivre's Law**.

Proposition 3.1.1 (nth Roots of a Complex Number)

$$\begin{aligned}
\forall z = re^{i\theta} \in \mathbb{C} \quad r = |z| \in \mathbb{R} \quad \theta \in [0, 2\pi) \\
\exists w = se^{i\tau} \in \mathbb{C} \quad s \in \mathbb{R} \quad \tau \in [0, 2\pi) \\
\forall n \in \mathbb{Z} \\
w^n = (se^{i\tau})^n = z = re^{i\theta}
\end{aligned}$$

The n th roots of z is described by the set

$$\left\{ r^{\frac{1}{n}} e^{i\left(\frac{\theta+2\pi k}{n}\right)} : k = 0, 1, \dots, n-1 \right\} \quad (3.4)$$

Proof

$$\begin{aligned}
s^n = r &\iff s = r^{\frac{1}{n}} \\
e^{in\theta} = e^{i\tau} &\iff \theta = \frac{\tau + 2\pi k}{n}
\end{aligned}$$

Therefore, the set that describes the n th roots of z is

$$\left\{ w = r^{\frac{1}{n}} e^{i\left(\frac{\theta+2\pi k}{n}\right)} : k = 0, 1, \dots, n-1 \right\}$$

Remark (nth Roots of Unity)

The n th roots of unity is a direct consequence of [Proposition 3.1.1](#) where we solve for the equation $z^n = 1$ for any $z \in \mathbb{C}, n \in \mathbb{Z}$.

The set that describes the n th roots of unity is

$$\left\{ e^{i\theta} : \theta = \frac{2\pi k}{n}, k = 0, 1, \dots, n-1 \right\} \quad (3.5)$$

It is easy to see how the n th roots of unity partitions the unit circle into n parts.

Example 3.1.2

Find the cubic roots of $-2 + 2i$.

Let $z = -2 + 2i$. Note that $|z| = 2\sqrt{2}$ and $\text{Arg } z = \frac{3\pi}{4}$.

Therefore, in polar form, $z = 2\sqrt{2}e^{i\frac{3\pi}{4}}$.

Let $w = re^{i\theta}$, where $\theta \in [0, 2\pi)$, and $w^3 = z$. Then

$$r = (2\sqrt{2})^{\frac{1}{3}}$$

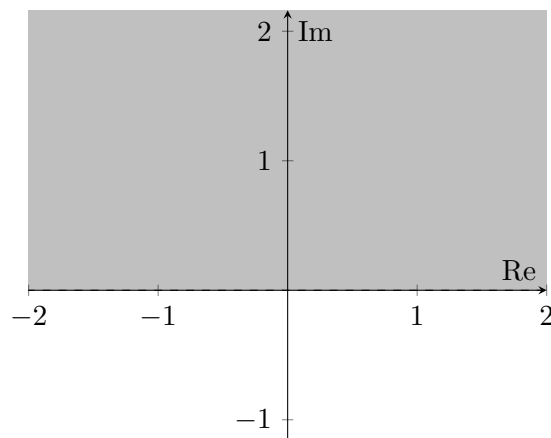
$$\theta = \frac{\frac{3\pi}{4} + 2\pi k}{3}, \quad k = 0, 1, 2$$

The set that describes the cubic root of $-2 + 2i$ is thus

$$\left\{ (2\sqrt{2})^{\frac{1}{3}} e^{i\theta} : \theta = \frac{\frac{3\pi}{4} + 2\pi k}{3}, k = 0, 1, 2 \right\}$$

Example 3.1.3

Describe the set $\{z \in \mathbb{C} : |\operatorname{Arg} z - \frac{\pi}{2}| < \frac{\pi}{2}\}$. (Note: $\operatorname{Arg} z \in [0, 2\pi)$)



Exercise 3.1.1

Solve

1. $z^4 = -1$

Let $z = re^{i\theta}$

$$r = |-1| = 1 \quad \theta = \frac{\pi + 2\pi k}{4} = \frac{(2k+1)\pi}{4}, \quad k = 0, 1, 2, 3$$

2. $z^4 = -1 + \sqrt{3}i$

Let $z = re^{i\theta}$

$$r = |-1 + \sqrt{3}i| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\theta = \frac{\frac{2\pi}{3} + 2\pi k}{4} = \frac{(2k + \frac{2}{3})\pi}{4}, \quad k = 0, 1, 2, 3$$

Chapter 4

Lecture 4 - Jan 10th, 2018

4.1 Examples for n th Roots of Unity

Recall that the n th roots of unity are given by $e^{i\frac{2\pi k}{n}}, k = 0, 1, \dots, n-1$.

Exercise 4.1.1

Let z be any n th root of unity other than 1. Show that

$$z^{n-1} + z^{n-2} + \dots + z + 1 = 0 \quad (4.1)$$

Proof

By the Sum of Finite Geometric Terms,

$$z^{n-1} + z^{n-2} + \dots + z + 1 = \frac{1 - z^n}{1 - z}.$$

Since $z^n = 1$, RHS is thus zero, which in turn completes the proof.

As an aside, if we wish to remove the restriction that z can also be 1, we may consider that

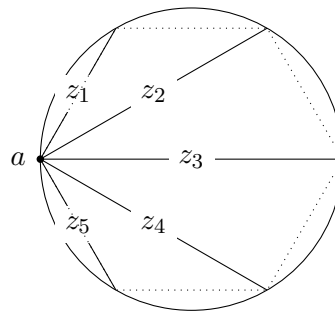
$$z^n - 1 = (z - 1)(1 + z + \dots + z^{n-1})$$

Since $z^n = 1$, LHS is zero. Then either $z = 1$ or $(1 + z + \dots + z^{n-1}) = 0$.

Exercise 4.1.2

Consider the $n-1$ diagonals of a regular n -gon, inscribed in a circle of radius 1, obtained by connecting one vertex on the n -gon to all its other vertices.

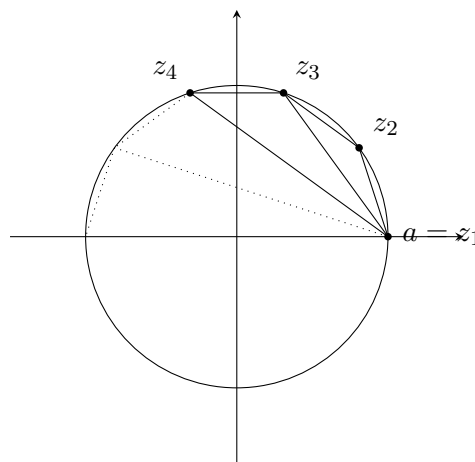
For example, if we are given $n = 6$, we obtain the following diagram.

Figure 4.1: $n = 6$, where a is an arbitrary vertex on the hexagon

Show that the product of the lengths of these diagonals is equal to n .

Proof

Note that Figure 4.1 can be translated into Figure 4.2.

Figure 4.2: A regular n -gon with the roots of unity on its vertices

Thus the equation that we wish to prove becomes

$$|1 - z_2| |1 - z_3| \dots |1 - z_n| = n \quad (4.2)$$

Note that z_2, \dots, z_n are the n th roots of unity other than 1.

Let z be a variable and consider the polynomial

$$P(z) := 1 + z + z^2 + \dots + z^{n-1} \quad (4.3)$$

Since the roots of $P(z)$ are the n th roots of unity other than 1, we can factorize [Equation \(4.3\)](#) into

$$P(z) = (z - z_2)(z - z_3) \dots (z - z_n)$$

Now let $z = 1$ and take the modulus of $P(z)$, and we get [Equation \(4.2\)](#).

Exercise 4.1.3

Let $n \in \mathbb{N}$. Show that $\sum_{j=0}^n \binom{3n}{3j} = \frac{2^{3n} + 2(-1)^n}{3}$.

Proof

Let $\alpha = e^{i\frac{2\pi}{3}}$. Then α is a cubic root of unity, i.e. $\alpha^3 = 1$, and from [Exercise 4.1.1](#), $1 + \alpha + \alpha^2 = 0$.

Consider

$$\begin{aligned} (1 + 1)^{3n} &= \binom{3n}{0} + \binom{3n}{1} + \binom{3n}{2} + \binom{3n}{3} + \binom{3n}{4} \\ &\quad + \binom{3n}{5} + \binom{3n}{6} + \dots + \binom{3n}{3n} \end{aligned} \tag{4.4}$$

$$\begin{aligned} (1 + \alpha)^{3n} &= \binom{3n}{0} + \binom{3n}{1}\alpha + \binom{3n}{2}\alpha^2 + \binom{3n}{3} + \binom{3n}{4}\alpha \\ &\quad + \binom{3n}{5}\alpha^2 + \binom{3n}{6} + \dots + \binom{3n}{3n} \end{aligned} \tag{4.5}$$

$$\begin{aligned} (1 + \alpha^2)^{3n} &= \binom{3n}{0} + \binom{3n}{1}\alpha^2 + \binom{3n}{2}\alpha + \binom{3n}{3} + \binom{3n}{4}\alpha^2 \\ &\quad + \binom{3n}{5}\alpha + \binom{3n}{6} + \dots + \binom{3n}{3n} \end{aligned} \tag{4.6}$$

Adding [Equation \(4.4\)](#), [Equation \(4.5\)](#) and [Equation \(4.6\)](#), we observe that the terms with coefficients $\binom{3n}{k}$ where k is not a multiple of 3 sums to 0 as given by $1 + \alpha + \alpha^2 = 0$, and

therefore we obtain

$$\begin{aligned}
 2^{3n} + (1 + \alpha)^{3n} + (1 + \alpha^2)^{3n} &= 3 \sum_{j=0}^n \binom{3n}{3j} \\
 \frac{1}{3} [2^{3n} + (1 + \alpha)^{3n} + (1 + \alpha^2)^{3n}] &= \sum_{j=0}^n \binom{3n}{3j} \\
 \frac{1}{3} [2^{3n} + (-\alpha^2)^{3n} + (-\alpha)^{3n}] &= \sum_{j=0}^n \binom{3n}{3j} \quad \text{since } 1 + \alpha + \alpha^2 = 0 \\
 \frac{1}{3} [2^{3n} + (-1)^n + (-1)^n] &= \sum_{j=0}^n \binom{3n}{3j} \quad \text{since } \alpha^3 = 1 \\
 \frac{2^{3n} + 2(-1)^n}{3} &= \sum_{j=0}^n \binom{3n}{3j}
 \end{aligned}$$

as required.

Exercise 4.1.4

Note that we can define $\text{Arg } z$ in any interval of length 2π , i.e. it is not necessary that $\text{Arg } z \in [0, 2\pi)$.

For example, if we restrict $\text{Arg } z \in [-\pi, \pi]$, then we can write

$$\text{Arg} \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) = -\frac{3\pi}{4}$$

Let z be on the unit circle and $\text{Arg } z \in [-\pi, \pi]$. Suppose that $z \notin \mathbb{R}$, i.e. $z \neq 1, z \neq -1$. Show that

$$\text{Arg} \left(\frac{z-1}{z+1} \right) = \begin{cases} \frac{\pi}{2} & \text{Im } z > 0 \\ -\frac{\pi}{2} & \text{Im } z < 0 \end{cases}$$

Proof

Note that $\forall w_1, w_2 \in \mathbb{C}$, where $\text{Arg } w_1 = \tau_1, \text{Arg } w_2 = \tau_2$ for τ_1, τ_2 in the same 2π -interval,

$$\text{Arg} \frac{w_1}{w_2} = \frac{e^{i\tau_1}}{e^{i\tau_2}} \equiv e^{i(\tau_1 - \tau_2)} = \text{Arg } w_1 - \text{Arg } w_2 \quad (4.7)$$

in modulo 2π .

Suppose $\text{Im } z > 0$. Let $\theta_1 = \text{Arg}(z-1)$ and $\theta_2 = \text{Arg}(z+1)$. Consider [Figure 4.3](#). Note that since both $\theta_1, \theta_2 \in [0, \pi]$, we have that $\theta_1 - \theta_2 \in [-\pi, \pi]$, and thus [Equation \(4.7\)](#) holds

true without the need of the condition of being in modulo 2π . We observe that

$$\begin{aligned}\frac{\pi}{2} &= \theta_2 + \pi - \theta_1 \\ \theta_1 - \theta_2 &= \frac{\pi}{2}\end{aligned}$$

as desired.

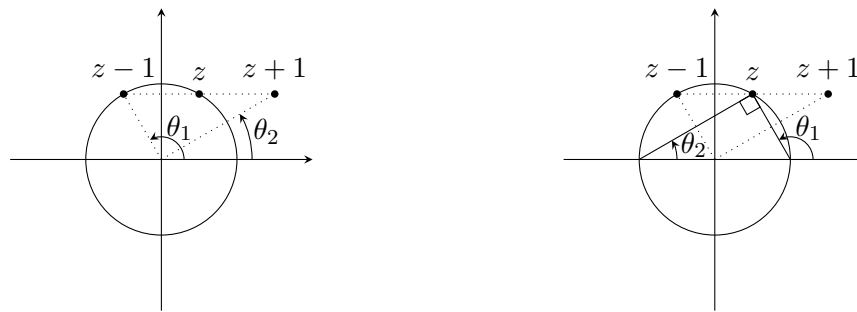


Figure 4.3: (Right) Depicted question, (Left) Translated Angles

Similarly, we can obtain $\theta_1 - \theta_2 = -\frac{\pi}{2}$ for when $\text{Im } z < 0$. This completes the proof.

Exercise 4.1.5

Let $f(z) = e^z$ for $z \in \mathbb{C}$. Let $A = \{z = x + iy \in \mathbb{C} : x \leq 1, y \in [0, \pi]\}$. Describe the image of $f(A)$.

Solution

Firstly, note that

$$\begin{aligned}e^z &= e^{x+iy} \\ e^x &\in (0, e] \\ y &\in [0, \pi]\end{aligned}$$

Figure 4.4: (Right) Domain of $f(A)$, (Left) Image of $f(A)$

It is clear that the image will be in on the positive side of the imaginary-axis. Also, since $e^x \in (0, e]$, we get the right graph represented in [Figure 4.4](#). The image of $f(A)$ is described in the left image of [Figure 4.4](#).

Chapter 5

Lecture 5 - Jan 12, 2018

5.1 Complex Functions

5.1.1 Limits

Definition 5.1.1 (Convergence)

A sequence of complex numbers z_1, z_2, z_3, \dots converges to $z \in \mathbb{C}$ if

$$\lim_{n \rightarrow \infty} |z_n - z| = 0 \quad (5.1)$$

or we may say

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \quad |z_n - z| < \epsilon \quad (5.2)$$

Note

If $\{z_n\}_{n \in \mathbb{N}}$ converges to z , we may write $\lim_{n \rightarrow \infty} z_n = z$ or $z_n \rightarrow z$ (as $n \rightarrow \infty$).

Example 5.1.1

For $|z| > 1$, does $\{\frac{1}{z^n}\}_{n=1}^{\infty}$ converge? Explain.

Solution

We claim that the limit is 0. Since $|z| > 1$, we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{1}{z^n} - 0 \right| &= \lim_{n \rightarrow \infty} \left| \frac{1}{z} \right|^n \\ &= 0 \end{aligned}$$

Another way to prove this, since $|z| > 1 \implies 0 < \left|\frac{1}{z}\right| < 1$,

$$\forall \epsilon = \left|\frac{1}{z}\right| > 0$$

$$\left|\frac{1}{z^n} - 0\right| = \left|\frac{1}{z}\right|^n < \left|\frac{1}{z}\right| = \epsilon$$

Definition 5.1.2 (Convergence for Complex Functions)

$\forall \Omega \subseteq \mathbb{C}$, let $f : \Omega \rightarrow \mathbb{C}$. We say that

$$\lim_{z \rightarrow z_0} f(z) = L \quad (5.3)$$

for some $L \in \mathbb{C}$ if for every sequence $\{z_n\}_n \subseteq \Omega$, we have that

$$z_n \rightarrow z_0 \implies f(z_n) \rightarrow L \quad (5.4)$$

Note that L need not be in Ω . (I copied z instead of L in class. Needs further confirmation.)

Example 5.1.2

Let $f(z) = \frac{\bar{z}}{z}$, $z \in \mathbb{C} \setminus \{0\}$. Find $\lim_{z \rightarrow 0} f(z)$.

Solution

Suppose $z = x \in \mathbb{R} \setminus \{0\}$. Then $f(z) = f(x) = \frac{x}{x} = 1$.

Suppose $z = iy$, $y \in \mathbb{R} \setminus \{0\}$. Then $f(z) = f(iy) = \frac{-iy}{iy} = -1$.

Therefore, the limit $\lim_{z \rightarrow 0} f(z)$ does not exist.

Exercise 5.1.1

Show that $z_n \rightarrow z \iff \text{Re}(z_n) \rightarrow \text{Re}(z) \wedge \text{Im}(z_n) \rightarrow \text{Im}(z)$.

(Hint: $|\text{Re}(z)|, |\text{Im}(z)| \leq |z| \leq |\text{Re}(z)| + |\text{Im}(z)|$)

Solution

Suppose $z_n \rightarrow z$. Then $\forall \epsilon_0 > 0 \exists N \in \mathbb{N} \forall n > N |z_n - z| < \epsilon$. Note once and for all that

$$\begin{aligned} \text{Re}(z_n - z) &= \text{Re}(z_n) - \text{Re}(z) \\ \text{Im}(z_n - z) &= \text{Im}(z_n) - \text{Im}(z). \end{aligned}$$

Thus

$$\begin{aligned} |\text{Re}(z_n) - \text{Re}(z)| &= |\text{Re}(z_n - z)| \\ &\leq |z_n - z| < \epsilon \\ |\text{Im}(z_n) - \text{Im}(z)| &= |\text{Im}(z_n - z)| \\ &\leq |z_n - z| < \epsilon \end{aligned}$$

For the other direction,

$$\begin{aligned} \forall \frac{\epsilon}{2} > 0 \quad \exists N_0 \in \mathbb{N} \quad \forall n > N_0 \quad |\operatorname{Re}(z_n) - \operatorname{Re}(z)| < \frac{\epsilon}{2} \\ \forall \frac{\epsilon}{2} > 0 \quad \exists N_1 \in \mathbb{N} \quad \forall n > N_1 \quad |\operatorname{Im}(z_n) - \operatorname{Im}(z)| < \frac{\epsilon}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} |z_n - z| &= |\operatorname{Re}(z_n) + i\operatorname{Im}(z_n) - \operatorname{Re}(z) - i\operatorname{Im}(z)| \\ &\leq |\operatorname{Re}(z_n) - \operatorname{Re}(z)| + |\operatorname{Im}(z_n) - \operatorname{Im}(z)| \\ &\leq \epsilon \end{aligned}$$

□

5.1.2 Continuity

Definition 5.1.3 (Continuity)

$\forall \Omega \subseteq \mathbb{C}$, let $f : \Omega \rightarrow \mathbb{C}$. We say that f is continuous at $z_0 \in \Omega$ if

1. $\forall \{z_n\}_{n \in \mathbb{N}}$
 $z_n \rightarrow z_0 \implies f(z_n) \rightarrow f(z_0)$
2. $\forall \epsilon > 0 \quad \exists \delta > 0$
 $|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$

Remark

1. f is continuous on Ω if it is continuous on every point in Ω .
2. We may split f into its real and imaginary parts, i.e.

$$f(z) = f(x, y) = u(x, y) + iv(x, y)$$

where $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Example 5.1.3

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ and for $z \in \mathbb{C}$, $f(z) = \frac{\bar{z}}{z}$. To split f into real and imaginary parts:

$$\begin{aligned} f(z) &= \frac{\bar{z}}{z} \\ &= (x + iy) \left(\frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2} \right) \\ &= \frac{x^2 - y^2}{x^2 + y^2} + i \frac{(-2xy)}{x^2 + y^2} \end{aligned}$$

and we get

$$u(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$
$$v(x, y) = -\frac{2xy}{x^2 + y^2}$$