PMATH352W18 Complex Analysis - Class Notes

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Chapter 1

Lecture 1 - Jan 3, 2018

1.1 Complex Numbers and Their Properties

Definition 1.1.1 (Complex Number, Complex Plane)
*A **complex number** is a vector in $\mathbb{R}^2$. The **complex plane**, denoted by $\mathbb{C}$, is a set of complex numbers,*

$$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

In $\mathbb{C}$, we usually write

$$\begin{aligned} 0 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & 1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ i &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} & x &= \begin{pmatrix} x \\ 0 \end{pmatrix} \\ iy &= \begin{pmatrix} 0 \\ y \end{pmatrix} \end{aligned}$$

where $x, y \in \mathbb{R}$. Consequently, we have that

$$x + iy = x + yi = \begin{pmatrix} x \\ y \end{pmatrix}$$

If for $x, y \in \mathbb{R}$, $z = x + iy$, then x is aclled the real part of z and y is called the imaginary part of z, and we write

$$\mathrm{Re}(z) = x \quad \mathrm{Im}(z) = y.$$

Note

- *It is easy to see how $\mathbb{R}$ is a subset of $\mathbb{C}$.*

- *Complex Numbers of the form* $\binom{0}{y}$ *where* $y \in \mathbb{R}$ *are called purely imaginary numbers.*
- *Certain authors may prefer to denote* $i = \binom{0}{1}$.

Definition 1.1.2 (Sum and Product)
We define the sum of two complex numbers to be the usual vector sum, i.e.

$$\begin{aligned}(a+ib)+(c+id) &= \begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}\\ &= \begin{pmatrix}a+c\\b+d\end{pmatrix}\\ &= (a+c)+i(b+d)\end{aligned}$$

where $a, b, c, d \in \mathbb{R}$.

We define the product of two complex numbers by setting $i^2 = -1$, *and by requiring the product to be commutative, associative, and distributive over the sum. In this setup, we have that*

$$\begin{aligned}(a+ib)(c+id) &= ac+iad+ibc+i^2bd\\ &= (ac-bd)+i(ad+bc) \qquad (1.1)\end{aligned}$$

Note
It is interesting to note that any complex number times zero is zero, just like what we have with real numbers.

$$\forall z = x+iy \in \mathbb{C}\ x, y \in \mathbb{R}\ \ 0 \in \mathbb{C}$$
$$z \cdot 0 = (x+iy)(0+i0) = 0+i0 = 0$$

Example 1.1.1
Let $z = 2+i, w = 1+3i$. *Find* $z+w$ *and* zw.

$$\begin{aligned}z+w &= (2+i)+(1+3i)\\ &= 3+4i\end{aligned}$$

$$\begin{aligned}zw &= (2+i)(1+3i)\\ &= (2-3)+i(6+1) \quad \text{By Equation (1.1)}\\ &= -1+7i\end{aligned}$$

Example 1.1.2
Show that every non-zero complex number has a multiplicative inverse, z^{-1}, *and find a formula for this inverse.*

Let $z = a + ib$ where $a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$. Then

$$\begin{aligned}
& z(x+iy) = 1 \\
\iff & (ax - by) + i(ay + bx) = 1 \\
\iff & \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\iff & \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\iff & \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\iff & \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a \\ -b \end{pmatrix} \\
\iff & x + iy = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}
\end{aligned}$$

Therefore, we have that the formula for the inverse is

$$(a + ib)^{-1} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2} \tag{1.2}$$

Notation
For $z, w \in \mathbb{C}$, we write

$$\begin{array}{ll} -z = -1z & w - z = w + (-z) \\ \frac{1}{z} = z^{-1} & \frac{w}{z} = wz^{-1} \end{array}$$

Example 1.1.3
Find $\frac{(4-i)-(1-2i)}{1+2i}$.

$$\begin{aligned}
\frac{(4-i)-(1-2i)}{1+2i} &= \frac{3+i}{1+2i} \\
&= (3+i)(\frac{1}{5} - i\frac{2}{5}) \\
&= 1 - i
\end{aligned}$$

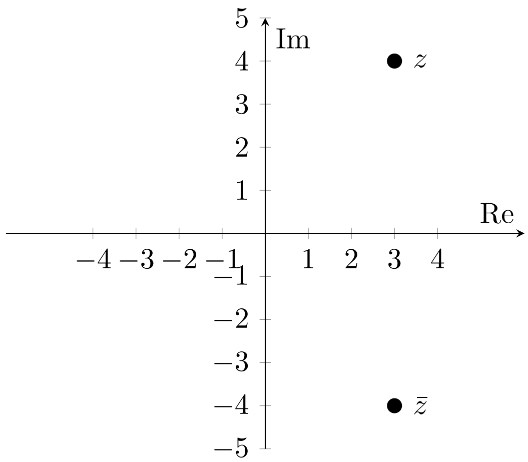
Note
The set of complex numbers is a ***field*** *under the operations of additiona and multiplication. This means that $\forall u, v, w \in \mathbb{C}$,*

$$\begin{array}{cc} u + v = v + u & uv = vu \\ (u + v) + w = u + (v + w) & (uv)w = u(vw) \\ 0 + u = u & 1u = u \\ u + (-u) = 0 & uu^{-1} = 1,\ u \neq 0 \\ u(v + w) = uv + uw & \end{array}$$

Since the distributive law holds for complex numbers, note that the binomial expansion works for $(w+z)^n$ where $w, z \in \mathbb{C}$ and $n \in \mathbb{N}$. (I did not verify if this is still true for when $n \in \mathbb{R}$.)

Definition 1.1.3 (Conjugate)
If $z = x + iy$ where $x, y \in \mathbb{R}$, then the ***conjugate of z*** *is given by $\bar{z} = x - iy$*

Example 1.1.4
Let $z = 3 + 4i$. Then the $\bar{z} = 3 - 4i$. Represented in the complex plane, we have the following:



We observe that on the complex plane, the conjugate of a complex number is simply its reflection on the real axis.

Definition 1.1.4 (Modulus)
We define the ***modulus*** *(length, magnitude) of $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$, to be*

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}. \tag{1.3}$$

Note
Note that this definition is consistent with the notion of the absolute value in real numbers when z is a real number, since if $y = 0$, $|z| = |x + i0| = \sqrt{x^2} = \pm x$.

Note
For $z, w \in \mathbb{R}$, we have

$$\begin{aligned} \bar{\bar{z}} &= z & z + \bar{z} &= 2\,\mathrm{Re}(z) & z - \bar{z} &= 2i\,\mathrm{Im}(z) \\ z\bar{z} &= |z|^2 & |z| &= |\bar{z}| & \overline{z \pm w} &= \bar{z} \pm \bar{w} \\ \overline{zw} &= \bar{z} - \bar{w} & |zw| &= |z|\,|w| \end{aligned}$$

but note that $|z + w| \neq |z| + |w|$.

Note
While inequalities such as $z_1 < z_2$, where $z_1, z_2 \in \mathbb{C}$, are meaningless unless if both of them are real, $|z_1| < |z_2|$ means that the point z_1 in the complex plane is closer to the origin than the point z_2.

Proposition 1.1.1 (Basic Inequalities)

1. $|\mathrm{Re}(z)| \leq |z|$
2. $|\mathrm{Im}(z)| \leq |z|$
3. $|z + w| \leq |z| + |w|$ *Triangle Inequality*
4. $|z + w| \geq |\,|z| - |w|\,|$ *Inverse Triangle Inequality*

Proof
Note that $|z|^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$ and that we can express $|x| = \sqrt{x^2}$ for any $x \in \mathbb{R}$. 1 and 2 immediately follows from that.

To prove 3, we have that

$$\begin{aligned} |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) \\ &= |z|^2 + |w|^2 + (w\bar{z} + \bar{w}z) \\ &= |z|^2 + |w|^2 + 2\,\mathrm{Re}(w\bar{z}) \\ &\leq |z|^2 + |w|^2 + 2\,|w\bar{z}| \quad \textit{by 1} \\ &= |z|^2 + |w|^2 + 2\,|wz| \quad \textit{since } |w\bar{z}| = |w|\,|\bar{z}| \textit{ and } |z| = |\bar{z}| \\ &= (|z| + |w|)^2 \end{aligned}$$

To prove 4, note that

$$|z| = |z + w - w| \leq |z + w| + |w| \tag{1.4}$$
$$|w| = |w + z - z| \leq |z + w| + |z| \tag{1.5}$$

Observe that

$$\begin{aligned} \textit{Equation (1.4)} &\implies |z| - |w| \leq |z + w| \\ \textit{Equation (1.5)} &\implies |w| - |z| \leq |z + w| \end{aligned}$$

Thus, we have that

$$|z + w| \geq |\,|z| - |w|\,|$$

as required. □

Item 3 in Proposition 1.1.1 can be generalized by the means of mathematical induction to sums involving any finite number of terms, as:

$$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n| \tag{1.6}$$

where $n \in \mathbb{N} \setminus \{0, 1\}$.

To note the induction proof, when $n = 2$, Equation (1.6) is just Item 3. If Equation (1.6) is true for when $n = m$ where $m \in \mathbb{N} \setminus \{0, 1\}$, $n = m + 1$ is also true since by Item 3,

$$\begin{aligned} |(z_1 + z_2 + \ldots + z_m) + z_{m+1}| &\leq |z_1 + z_2 + \ldots + z_m| + |z_{m+1}| \\ &\leq (|z_1| + |z_2| + \ldots + |z_m|) + |z_{m+1}|\,. \end{aligned}$$

The distance between two points $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}, x_1, x_2, y_1, y_2 \in \mathbb{R}$ is $|z_1 - z_2|$, since $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 (y_1 - y_2)^2}$ is our usual notion of the Euclidean distance of two points on a plane.

Also, note that

$$z_1 - z_2 = z_1 + (-z_2)$$

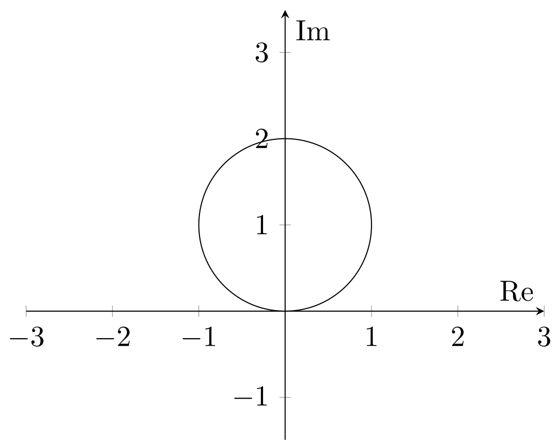
and thus if we apply our knowledge of vector representation, $z_1 - z_2$ is the directed line segment from the point z_2 to z_1.

With the notion of a “distance” set on the complex plane, we can now explore upon points lying on a circle with a center z_0 and radius R, which satisfies the equation

$$|z - z_0| = R.$$

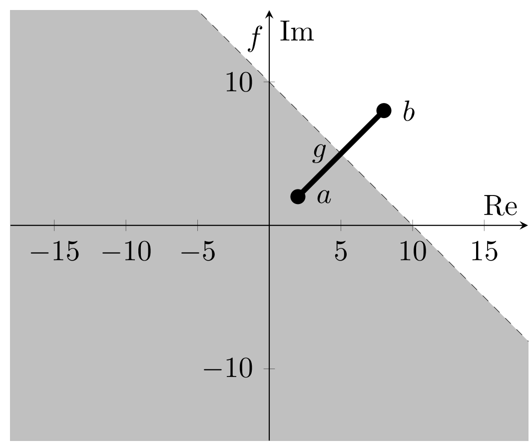
We may simply refer to this set of points as the circle $|z - z_0| = R$.

Example 1.1.5
We may describe a set $\{z \in \mathbb{C} : |z - i| = 1\}$ *as follows:*



Let $a, b \in \mathbb{C}$ *describe the set* $\{z \in \mathbb{C} : |z - a| < |z - b|\}$.

Suppose the following coordinates for a *and* b *are arbitrary,*



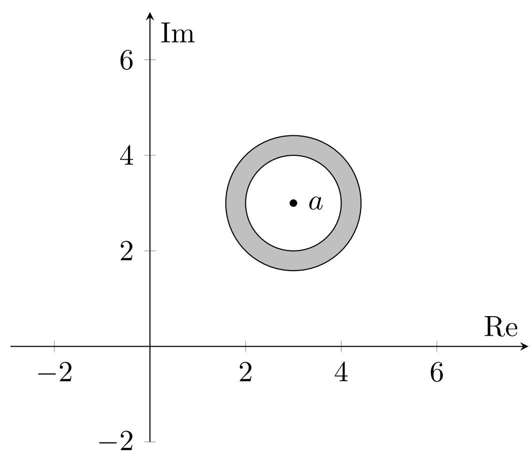
In the above, g *is the line segment that connects the points* a *and* b *on the complex plane, while* f *is the perpendicular bisector of the line segment* g. *The area described by the set* $\{z \in \mathbb{C} : |z - a| < |z - b|\}$ *is the shaded area which is below* f.

Chapter 2

Lecture 2 - Jan 5th, 2018

2.1 Complex Numbers and Their Properties (Continued)

Example 2.1.1
Let $a \in \mathbb{C}$. Describe the set $\{z \in \mathbb{C} : 1 < |z - a| < 2\}$.



Example 2.1.2
Show that every non-zero complex number has exactly two complex square roots, and find a formula for the square roots.

Let $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$, and let $w = u + iv, u, v \in \mathbb{R}$. Then

$$\begin{aligned}
w^2 = z \iff & (u+iv)^2 = x+iy \\
\iff & (u^2 - v^2) + i(2uv) = x + iy \\
\iff & x = u^2 + v^2 \quad \textit{and} \qquad (2.1) \\
& y = 2uv \qquad (2.2)
\end{aligned}$$

Square both sides of Equation (2.2), and thus we have $y^2 = 4u^2v^2$.

Multiply Equation (2.1) by $4u^2$*, and we get*

$$\begin{aligned}
& 4u^2x = 4u^4 - 4u^2v^2 = 4u^4 - y^2 \\
\iff & 0 \quad\;\; = 4u^4 - 4u^2x - y^2 \\
\iff & u^2 \quad = \frac{4x \pm \sqrt{16x^2 + 16y^2}}{8} \\
& \qquad\;\; = \frac{x \pm \sqrt{x^2+y^2}}{2}
\end{aligned}$$

Suppose $y \neq 0$*. Note that* $x < \sqrt{x^2+y^2}$*. Thus* $u^2 = \frac{x+\sqrt{x^2+y^2}}{2} \implies u = \left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}$.

Similarly, we can get

$$v = \pm\left(\frac{-x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}$$

Note that all four choices of signs satisfy Equation (2.1). If $y > 0$*, then* u *and* v *are either both positive or both negative by Equation (2.2).*

Suppose $y = 0$*. Then we have*

$$w^2 = z = x$$

Therefore, we get

$$w = \begin{cases}
\pm\left[\left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}} + i\left(\frac{-x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}\right] & y > 0 \\
\pm\left[\left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}} - i\left(\frac{-x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}\right] & y < 0 \\
\pm\sqrt{x} & y = 0, x > 0 \\
\pm i\sqrt{x} & y = 0, x < 0
\end{cases}$$

Remark
Let $z \in \mathbb{C}$. The notation $\sqrt{z}$ may represent either one of the square roots of z or both of the square roots, i.e. it is possible that $\sqrt{z}$ represents a set.

Exercise 2.1.1
Is it always okay for complex numbers such that $\sqrt{zw} = \sqrt{z}\sqrt{w}$, for $z, w \in \mathbb{C}$?

No. For example, consider $z = w = -1$. Then we have

$$\sqrt{zw} = \sqrt{1} = \pm 1$$

while

$$\sqrt{z}\sqrt{w} = i \cdot i = -1$$

and thus

$$\sqrt{zw} \neq \sqrt{z}\sqrt{w}.$$

Example 2.1.3
Find the values of $\sqrt{3-4i}$.

By Example 2.1.2,

$$\begin{aligned}\sqrt{3-4i} &= \pm\left(\sqrt{\frac{3+\sqrt{9+16}}{2}} - i\sqrt{\frac{-3+\sqrt{9+16}}{2}}\right) \\ &= \pm(2-i)\end{aligned}$$

Remark
The quadratic formula holds for complex polynomials, i.e.

$$\forall a, b, c \in \mathbb{C} \quad a \neq 0 \quad \forall z \in \mathbb{C}\; az^2 + bz + c = 0,$$

the solution for z is given by

$$z_{1,2} = \frac{-b + \sqrt{b^2 - 4ac}}{b} \tag{2.3}$$

The following is a short proof.

Proof

$$\begin{aligned}
az^2 + bz + c = 0 &\iff z^2 + \frac{b}{a}z + \frac{c}{a} = 0 \\
&\iff z^2 + \frac{b}{a}z + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \\
&\iff \left(z + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2} \\
&\iff z = \frac{-b + \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

(Personal Note: where did the $-$ for the supposed $\pm$ go? Or should it really be $\pm$?)

Example 2.1.4
Solve $iz^2 - (2+3i)z + 5(1-i) = 0$.

$$\begin{aligned}
z &= \frac{2 + 3i + \sqrt{(2+3i)^2 - 4i[5(1-i)]}}{2i} \\
&= \frac{2 + 3i + \sqrt{-5 + 12i - 20i - 20}}{2i} \\
&= \frac{2 + 3i + \sqrt{-25 - 8i}}{2i}
\end{aligned}$$

$$\sqrt{-25 - 8i} = \pm\left[\left(\frac{-25 + \sqrt{625 + 64}}{2}\right)^{\frac{1}{2}} - i\left(\frac{25 + \sqrt{625 + 64}}{2}\right)^{\frac{1}{2}}\right]$$

(Personal note: temporarily stuck, seeing that there's no "nice" solution)

Chapter 3

Lecture 3 - Jan 8th, 2018