PMATH433/733 - Model Theory and Set Theory

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Foreword

This course has a ratio of about 1:3 for naive set theory to model theory.

1 Lecture 1 Sep 06th

1.1 Introduction to Set Theory

In this course, we shall focus only on **practical set theory**, which is more commonly knowned as **naive set theory**. In practical set theory, we look at set theory as a language of mathematics. Some of the examples of which we look into in this flavour of set theory are (transfinite) induction and recursion, and the measuring of the sizes of sets.

Another approach to set theory, one that is deemed required in order to learn set theory is a more formal way, is to look at set theory as the foundations of mathematics. Such an approach is more axiomatic, rigorous, and grounding as compared to practical set theory. This course will try to work around going into these topics, as they can take a life of their own, and within the context of this course, the topics that will be explored using this approach are not required.

1.2 Ordinals

1.2.1 Zermelo-Fraenkel Axioms

We use the natural numbers, i.e.

$$0, 1, 2, 3, 4, \ldots$$

to **count** finite sets. There are two related meanings attached to the word "count" here:

- enumeration; and

- measuring (of sizes)

In order to facilitate the introduction to certain axioms that we
shall need, let our current goal be to develop an infinitary general-
ization of the natural numbers, so as to be able to enumerate and
measure arbitrary sets.

We have that

$$\text{enumeration} \rightarrow \text{ordinals}$$
$$\text{measuring} \rightarrow \text{cardinals}$$

where $\rightarrow$ represents "leads to" here.

To CONSTRUCT the natural numbers, we require 3 basic notions that
shall remain undefined but understood:

- a set;

- membership, denoted by $\in$; and

- equality.

  One such construction is

  $0 := \emptyset$, the empty set
  $1 := \{0\} = \{\emptyset\}$, the set whose only member is 0
  $2 := \{0, 1\} = \{\emptyset, \{\emptyset\}\}$, the set whose only members are 0 and 1.

📝 **Definition 1 (Successor)**

Given a natural number n, the **successor** *of n is the natural number next
to n, which can be obtained by*

$$S(n) := n \cup \{n\}.$$

We can use the definition of a successor to construct the rest of the
natural numbers.

Example 1.2.1

Just to verify to ourselves that the definition indeed works, observe that

$$S(1) = 2 = \{\emptyset, \{\emptyset\}\} = \{\emptyset\} \cup \{\{\emptyset\}\}.$$

So to construct the natural number 3, we see that

$$S(2) = 3 = \{0, 1, 2\} = 2 \cup \{2\} = \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\}$$
$$= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

Looking at these, we start wondering to ourselves: how do we know that $\emptyset$ exists in the first place? How do we know that we can use $\cup$ and what does it even mean? Now it is meaningless if we cannot take that $\emptyset$ always exists, nor is it meaningful if we cannot take the $\cup$ of sets. And so, to allow us to continue, or even start with these notions, we require axioms.

🛡 **Axiom 1 (Empty Set Axiom)**

There exists a set, denoted by $\emptyset$, with no members.

With this axiom, we can indeed construct 0. To get 1 from 0, we have that 1 is a set whose only member is zero, and so if we take a member from 1, that member must be 0.

🛡 **Axiom 2 (Pairset Axiom)**

Given set x, y, there exists a set, denoted by $\{x, y\}$, whose only members are x and y. In other words,

$$t \in \{x, y\} \leftrightarrow (t = x \vee t = y)$$

Now note that in 🛡 Axiom 2, if $x = y$, then the set $\{x, y\}$ has only x as its member. For example, we realize that $1 = \{0, 0\} = \{0\}$. But why exactly does this equality make sense? What exactly does "realize" mean?

🛡 **Axiom 3 (Axiom of Extension)**

Given sets x, y, $x = y$ if and only if x and y have the same members.

Now, using the above 3 axioms, we are guaranteed that

$0 = \emptyset$ exists by the Empty Set Axiom

$1 = \{\emptyset\}$ exists by the Pairset Axiom

$2 = \{\emptyset, \{\emptyset\}\}$ exists by the Pairset Axiom

Now we've constructed 3 to be the set whose only members are $0, 1$, and 2. So far, within our axioms, there is no such thing as $\{0, 1, 2\}$, which is what our 3 is supposed to be. We now require the following axiom:

🛡 **Axiom 4 (Union Set Axiom)**

Given a set x, there exists a set denoted by $\cup x$, whose members are precisely the members of the members of x, i.e.

$$t \in \cup x \leftrightarrow (t \in y \text{ for some } y \in x)$$

So, by this axiom, we have that given any n, $S(n) = \cup\{n, \{n\}\}$, or in other words,
$$t \in S(n) \leftrightarrow t \in n \vee t = n.$$

With all of the above axioms, we can iteratively construct each and every natural number in a rigorous manner. However, our goal is to construct infinitely many of them.

It is tempting to simply take the infinitude of natural numbers simply as an axiom, i.e.

> *There exists a set whose members are precisely the natural numbers.*

There is a certain rule to which we set down axioms, and that is, axioms must be expressable in a "finitary" manner, i.e. they must be expressible using first-order logic.

📓 **Definition 2 (Definite Condition)**

*We define a **definite condition** as follows:*

- *$x \in y$ and $x = y$ are definite conditions, where x and y are both indeterminants, standing for sets, or are sets themselves;*

- *if P and Q are definite conditions, then so are*

    - *not P, denoted as $\neg P$;*

    - *P and Q, denoted as $P \wedge Q$;*

- *P or Q, denoted as $P \vee Q$;*

- *for all x, P, denoted as $\forall x P$; and*

- *there exists x, P, denoted as $\exists x P$.*

Example 1.2.2

$$x \in 1, 0 \in 2, 2 \in 0$$

are all definite conditions. Note, however, that $2 \in 0$ is false.

> **❝ Note**
>
> *"If P then Q", which is also written as $P \rightarrow Q$, is also a definite condition since it is "equivalent"[1] to the statement $\neg P \vee Q$.*
>
> *Consequently, P if and only if Q, which is also expressed as $P \leftrightarrow Q$, can be written as*
> $$(\neg P \vee Q) \wedge (\neg Q \vee P)$$

[1] We have yet to define what equivalent statements are but we shall take this for granted for now.

Now, with this definition, and first-order logic notations in mind, we can write:

- Empty Set Axiom: $\exists x \, \forall t \, \neg(t \in x)$

- Pairset Axiom: $\forall x \, \forall y \, \exists p \, \forall t \, (t \in p \leftrightarrow ((t = x) \vee (t = y)))$

- Union Set Axiom: $\forall x \, \exists z \, \forall t \, ((t \in z) \leftrightarrow (\exists y \, ((y \in x) \wedge (t \in y))))$

Note that the statement that we proposed as an axiom for the set of natural numbers in page 14 is not definite, although that itself is not obvious.

For example, we may try to write

$$\exists x \, (\forall t \, ((t \in x) \leftrightarrow ((t = 0) \vee (t = 1) \vee (t = 2) \vee ...)))$$

and then notice that we do not have the notion of "..." within the "tools" that we are allowed to use.

Exercise 1.2.1
Write 🛡 Axiom 3 in first-order logic notation.

✏ **Solution**

$$\forall x \, \forall y$$
$$(x = y) \leftrightarrow (\forall t \, ((t \in x) \leftrightarrow (t \in y)))$$

2 *Lecture 2 Sep 11th*

2.1 *Ordinals (Continued)*

2.1.1 *Zermelo-Fraenkel Axioms (Continued)*

We stopped at the discussion about allowing for an infinite set, so
that we can construct our set of infinite natural numbers. The idea
here is to take *the smallest set that contains* 0 *and is preserved by the
successor function*[1]

[1] Q: Why the smallest set?

🛡 **Axiom 5 (Infinity Axiom)**

There exists a set I that contains 0 *and is preserved by the successor
function. We may express this as*

$$\exists I((0 \in I) \land \forall x(x \in I \to S(x) \in I))$$

where we have defined that $S(x) \in I$ *means*

$$\exists y(\forall t(t \in y \leftrightarrow (t \in x) \lor (t = x)) \land (y \in I))$$

We call I the **successor set**.

Since we want the smallest of such successor sets, we can try tak-
ing the intersection of all successor sets. But before we can do that,
we require more axiomatic statements.

📖 **Definition 3 (Subsets)**

$x \subseteq y$ *means that every element of x is an element of y, i.e.*

$$\forall t((t \in x) \to (t \in y))$$

With a definition of a subset, we can define the Powerset Axiom.

🛡 **Axiom 6 (Powerset Axiom)**

Given a set x, there exists a set $\mathcal{P}(x)$ that contains all subsets of x, i.e.

$$\forall t((t \subseteq \mathcal{P}(x)) \leftrightarrow (t \subseteq x))$$

We also require the following axiom.

🛡 **Axiom 7 ((Bounded) Separation Axiom)**

Given a set x and a definition condition P, there exists a set whose elements are precisely the members of x that satisfies P, i.e.

$$\forall x \, \exists y \, \forall t \, ((t \in y) \leftrightarrow \forall y \, ((t \in x) \wedge P(t)))$$

where

$$y = \{z \in x \mid P(z)\}.$$

There are two important aspects to the Bounded Separation Axiom:

- it is bounded by the set x; and
- P is a definite condition.

Exercise 2.1.1 (Set Intersection)

Prove that given a non-empty set x, there exists a set $\cap x$ satisfying

$$\forall t \, ((t \in \cap x) \leftrightarrow \forall y \, ((y \in x) \to (t \in y)))$$

✏ **Proof**
to be solved

📘 **Definition 4 (Natural Numbers)**

Let I be a successor set. The set of natural numbers is[2]

$$\omega := \cap\{J \subseteq I : J \text{ is a successor set }\}$$

📖 Note

J being a successor set can be expressed by the definite condition

$$(0 \in J) \wedge \forall x \ (x \in J \to S(x) \in J),$$

so we can write the definite condition in the above definition by

$$\omega := \cap\{J \subseteq I : (0 \in J) \vee \forall x \ (x \in J \to S(x) \in J)\}$$

Exercise 2.1.2

Show that the definition of ω does not actually depend on I, i.e. if given I_1 and I_2 such that we have

$$\omega_1 = \cap\{J \subseteq I_1 : J \text{ is a successor set }\}$$
$$\omega_2 = \cap\{J \subseteq I_2 : J \text{ is a successor set }\}$$

we have

$$\omega_1 = \omega_2.$$

✏️ Proof
to be solved

Another useful axiom that we will use later is the following:

🛡 Axiom 8 (Replacement Axiom)

Suppose P is a binary definite condition[3] such that for every set x, there is a unique y satisfying $P(x,y)$. Given a set A, there is a set B such that $t \in B$ if and only if there is an $a \in A$ with $P(a,t)$.

> ❝ **Note**
>
> *The slogan for the Replacement Axiom is:*
>
> > | *The image of a set under a definite operation exists.*

These eight axioms, along with another ninth axiom called the Regularity Axiom[4], constitutes the **Zermelo-Fraenkel Set Theory**.

Note that all axioms, save the Extensionality, assert the existence of sets.

[4] We shall not discuss too much about this. According to the lecture and the lecture notes, the Regularity Axiom states that every set has a minimal element. On Wikipedia, the axiom states that every set has an element that does not intersect with the set itself.

2.1.2   *Classes*

There are times where we are interested in a collection of sets that do not form a set themselves.

Example 2.1.1 (Russell's Paradox)

There is no set containing all sets.

> ✏️ **Proof**
>
> *Suppose such a set exists, and call it U. Now consider the set*
>
> $$R := \{x \in U : x \notin x\},$$
>
> *which exists by Bounded Separation. Observe that*
>
> $$R \in R \implies R \notin R \quad \text{⚡}$$
> $$\implies R \notin R \implies R \in R \quad \text{⚡}$$
>
> *Thus such a set U cannot exist.*

To talk about such collections, that may or may not be sets, we define *classes*.

> 📕 **Definition 5 (Class)**
>
> *A class is any collection of sets defined by definite property, i.e. given any difinite condition P,*
>
> $$⟦ \; z \mid P(z) \; ⟧$$

is the class of all sets satisfying P.

Here, instead of Bounded Separation, we have what is called **unbounded separation**.

❝ Note

We shall use $[\![\quad]\!]$ *rather than* $\{ \quad \}$ *to emphasize that we are talking about classes, i.e. we may be talking about non-sets.*

Example 2.1.2

$$Set := [\![\; z \mid z = z \;]\!]$$

is the universal class of all sets.

❝ Note

- *Every set is a class.*

    ✏ Proof

    Suppose x is a set. We may write

    $$x = [\![\; z \mid z \in x \;]\!].$$

    $\square$

- *Some classes are not sets; these are called **proper classes**. E.g. the universal class of all sets, and*

    $$Russell := [\![\; z \mid z \notin z \;]\!].$$

3 Lecture 3 Sep 13th

3.1 Ordinals (Continued 2)

3.1.1 Cartesian Products and Function

📘 **Definition 6 (Ordered Pairs)**

Given sets x, y, an **ordered pair** of x and y is defined as[1]

$$(x, y) = \{\{x\}, \{x, y\}\}$$

[1] This invokes the Pairset Axiom thrice. Why did we not define an ordered pair as
$$(x, y) = \{\{x\}, \{y\}\}$$
instead?

❝ **Note**

Note that we must have

$$((x, y) = (x', y')) \iff (x = x' \land y = y').$$

✏️ **Proof**

The ($\Longleftarrow$) direction is clear by Extensionality. For the other direction, we shall break it into 2 cases:

Case 1: $x = y$. Then $\{x, y\} = \{x\}$ by Extensionality, and so

$$(x, y) = \{\{x\}\}$$

Therefore, we have that

$$\{\{x\}\} = (x, y) = (x', y') = \{\{x'\}, \{x', y'\}\}$$

So we have

$$\{x\} = \{x'\} \implies x = x'$$

and

$$\{x\} = \{x', y'\} \implies y' = x = y.$$

Thus we have

$$x = x' \wedge y = y'.$$

Case 2: *Suppose* $x \neq y$ *and* $x' \neq y'$ [2] *We have*

$$\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$$

Then

$$\{x\} = \{x'\} \vee \{x\} = \{x', y'\}$$

The latter leads to a contradiction, since it would imply

$$x' = x = y'.$$

Thus $x = x'$. *Also, we have*

$$\{x, y\} = \{x'\} \vee \{x, y\} = \{x', y'\}$$

Now the former leads to a contradiction since it would imply that

$$x = x' = y.$$

Now since $x = x'$, *it must be that* $y = y'$, *otherwise* $y = x' = x$ *would contradict our assumption. Therefore, we have that*

$$x = x' \wedge y = y'.$$

$\square$

[2] If any of them are equal, Case 1 would apply.

With ordered pairs, we can build Cartesian products:

📄 **Definition 7 (Cartesian Product)**

Given classes X *and* Y, *the* **Cartesian Product** *of* X *and* Y *is defined as*

$$X \times Y := [\![\ z : z = (x, y), x \in X, y \in Y \ ]\!]$$

> **❝ Note**
>
> *We can express this definition using definite conditions;*

$$\forall x, y \Bigg((x \in X) \wedge (y \in Y) \wedge \Big(\exists a, b(\forall t(t \in a \leftrightarrow t = x)) \wedge \forall t(t \in b \leftrightarrow (t = x) \vee (t = y))\Big) \wedge$$

$$\forall t \Big(t \in z \leftrightarrow \big((t = a) \vee (t = b)\big)\Big)\Bigg)$$

> **❝ Note**
>
> - *A Cartesian product is a class.*
>
> - *If A is a set and B is a class, and $B \subseteq A$, then B is also a set. This is easy to show: observe that by Extentionality,*
>
> $$B = \{a \in A \mid a \in B\}.$$
>
> *By Bounded Separation Axiom, B is a set[3].*

[3] This statement can be rephrased as: subclasses of a set are subsets.

Consequently, Cartesian products of sets are sets themselves; if X and Y are sets, we want to show that $X \times Y$ is a set so it is sufficient to show that it is contained in one. Recall that

$$(x, y) = \{\{x\}, \{x, y\}\}$$

and $\{x, y\} \subset X \cup Y$ which means $\{x, y\} \in \mathcal{P}(X)$, and we observe that $\{x\} \in \mathcal{P}(X \cup Y)$. So $(x, y) \in \mathcal{P}(X \cup Y)$. Therefore, $X \times Y \subset \mathcal{P}(\mathcal{P}(X \cup Y))$, and we show to ourselves that $X \times Y$ is indeed a set.

▨ Definition 8 (Definite Operation)

Given classes X and Y, a **definite operation** $f : X \to Y$ is a subclass $\Gamma(f) \subseteq X \times Y$ such that

$$\forall x \in X \; \exists! y \in Y \; (x, y) \in \Gamma(f).$$

> **❝ Note**
>
> *We write $f(x) = y$ to mean $(x, y) \in \Gamma(f)$. We also refer to $\Gamma(f)$ as the graph of f.*

Example 3.1.1

The successor function $S : Set \rightarrow Set$ is a definite operation such that

$$S(x) = x \cup \{x\}$$

This is true since is can be expressed as

$$\forall t(t \in y \leftrightarrow (t \in x \lor t = x)).$$

To show that S is a definite operation, we need to show that S is a definite condition.

> **❝ Note**
>
> *If X and Y are sets and f is a definite operation, then $\Gamma(f) \subseteq X \times Y$ is a set. In such a case, we call f a function.*

> **▤ Definition 9 (Functions)**
>
> *A function is a definite operation $f : X \rightarrow Y$ where X and Y are both sets.*

We can now restate the Replacement Axiom.

> **⛉ Axiom 9 (Replacement Axiom (Restated))**
>
> *If $f : X \rightarrow Y$ is a definite operation, and $A \subseteq X$ is a set, then $\exists B \subseteq Y$ that is a set such that $t \in B$ if and only if $t = f(a)$ for some $a \in A$.*

3.1.2 *The Natural Numbers*

> **🖳 Theorem 10 (Induction Principle)**

Suppose $J \subseteq \omega$, $0 \in J$ and whenever $n \in J$, $S(n) \in J$. Then $J = \omega$.

✏️ Proof

By assumption, J is a successor set, therefore $\omega \subseteq J$ by definition. Thus, sinec $J \subseteq \omega$, we have $J = \omega$. ☐

🌲 Lemma 11 (Properties of the Natural Numbers)

Suppose $n \in \omega$. We have

1. $n \subseteq \omega$;

2. $\forall m \in n \quad m \subseteq n$;

3. $n \notin n$;

4. $n = 0 \veebar 0 \in n$; and

5. $y \in n \implies S(y) \in n \veebar S(y) = n$.

✏️ Proof

1. Let[4]
$$J := \{n \in \omega : n \subseteq \omega\} \subseteq \omega.$$

   Note that $\varnothing \subseteq \omega$ and so $0 \subseteq \omega$. By membership, $0 \in J$.

   Suppose $m \in J$. Consider $S(m) = m \cup \{m\}$. Since $J \subseteq \omega$, $m \in \omega$. Since $m \in \omega$, $\{m\} \subseteq \omega$. Therefore $S(m) = m \cup \{m\} \subseteq \omega$, and so $S(m) \in J$. So J is a successor set. And thus by *Induction Principle*, $J = \omega$.

2. Let
$$J := \{n \in \omega : \forall m \in n, m \subseteq n\}.$$

   It is vacuously true that $0 \in J$ since $\varnothing$ is a subset of every $n \in J$. Suppose $n \in J$. Then $\forall m \in n$, we have $m \subseteq n$. Consider $S(n) = n \cup \{n\}$. Note that $n \in S(n)$ and $n \subseteq S(n)$. For $x \in S(n)$ such that $x \neq n$, we must have that $x \in n$. By assumption, $x \subseteq n \subseteq S(n)$. Therefore, $S(n) \in J$, and so J is a successor set. By the Induction Principle, $J = \omega$.

[4] We construct this J and show that it is a successor set. Note that if $J = \omega$, our proof is complete.

3. *Let*
$$J := \{n \in \omega : n \notin n\}.$$

We have $0 = \emptyset \notin \emptyset$. So $0 \in J$.

Let $n \in J$. Consider $S(n) = n \cup \{n\}$. In particular, note that $n \in S(n)$. Suppose, for contradiction, that $S(n) \in S(n)$. Then $S(n) = n$ or $S(n) \in n$.

$S(n) = n \implies n \in S(n) = n \notin n \notin n.$

$S(n) \in n \implies S(n) \subseteq n$ by part 2 $\implies n \in n \notin n \notin n.$

Thus $S(n) \notin S(n)$ and so $S(n) \in J$. So J is a successor set, and so by the Induction Principle*, $J = \omega$.*

4. *It suffices to show that*
$$\omega = \{0\} \cup \{n \in \omega : 0 \in n\}.$$

Let $J = $ RHS. We have that $0 \in J$. Suppose $n \in J$ such that $n \neq 0$. Then $0 \in n$. Since $n \subseteq S(n) = n \cup \{n\}$, we have that $0 \in S(n)$. Therefore, $S(n) \in J$. So J is a successor set, and so by the Induction Principle, $J = \omega$ as required.

5. *Let*
$$J := \{n \in \omega : y \in n \implies S(y) \in n \veebar S(y) = n\}.$$

$0 \in J$ is vacuously true, since there are no $y \in 0$. Suppose $n \in J$. Let $y \in S(n) = n \cup \{n\}$. We have two choices: either $y \in n$ or $y = n$. If $y \in n$, then $S(y) \in n \veebar S(y) = n$, since $n \in J$. We have that

$S(y) \in n \subseteq S(n)$ in which case we are done; and

$(sy) \subseteq n \in S(n).$

Otherwise, if $y \notin n$, then $y = n$. Then we simply have $S(y) = S(n)$. Thus J is a succesor set and so by the Induction Principle, $J = \omega$.

$\square$

3.1.3 *Well-Orderings*

📘 **Definition 10 (Strict Partially Ordered Set)**

A **strict partially ordered set** (or **strict poset**[5]) is a set E together with $R \subseteq E^2 = E \times E$ such that

1. (**anti-reflexive**) $\forall a \in E \quad (a, a) \notin R$;

2. (**anti-symmetric**) $\forall a, b \in E \quad (a, b) \in R \wedge (b, a) \in R \implies a = b$; and

3. (**transitivity**) $\forall a, b, c \in E \quad (a, b), (b, c) \in R \implies (a, c) \in R$.

🗂 Definition 11 (Strict Totally Ordered Set)

A strict poset is **total** (or **linear**) if

$$\forall a, b \in E \quad (a, b) \in R \veebar (b, a) \in R$$

🗂 Definition 12 (Well-Order)

A strict linear order is **well-ordered** if

$$\forall X \subseteq E(X \neq \varnothing) \quad \exists a \in X \quad \forall b \in X(b \neq a) \quad (a, b) \in R$$

i.e. every nonempty subset of E has a **least element**.

🔻 Proposition (ω is Strictly Well-ordered)

$(\omega, \in)$ is a strict well-ordering.

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