PMATH433/733 - Model Theory and Set Theory

Classnotes for Fall 2018

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Foreword

This course has a ratio of about 1:3 for naive set theory to model theory.

1 Lecture 1 Sep 06th

1.1 Introduction to Set Theory

In this course, we shall focus only on **practical set theory**, which is more commonly knowned as **naive set theory**. In practical set theory, we look at set theory as a language of mathematics. Some of the examples of which we look into in this flavour of set theory are (transfinite) induction and recursion, and the measuring of the sizes of sets.

Another approach to set theory, one that is deemed required in order to learn set theory is a more formal way, is to look at set theory as the foundations of mathematics. Such an approach is more axiomatic, rigorous, and grounding as compared to practical set theory. This course will try to work around going into these topics, as they can take a life of their own, and within the context of this course, the topics that will be explored using this approach are not required.

1.2 Ordinals

1.2.1 Zermelo-Fraenkel Axioms

We use the natural numbers, i.e.

$$0, 1, 2, 3, 4, \ldots$$

to **count** finite sets. There are two related meanings attached to the word "count" here:

- enumeration; and
- measuring (of sizes)

In order to facilitate the introduction to certain axioms that we shall need, let our current goal be to develop an infinitary generalization of the natural numbers, so as to be able to enumerate and measure arbitrary sets.

We have that

$$\begin{aligned} \text{enumeration} &\to \text{ordinals} \\ \text{measuring} &\to \text{cardinals} \end{aligned}$$

where $\to$ represents "leads to" here.

To CONSTRUCT the natural numbers, we require 3 basic notions that shall remain undefined but understood:

- a set;
- membership, denoted by $\in$; and
- equality.

One such construction is

$$\begin{aligned} &0 := \emptyset, \text{ the empty set} \\ &1 := \{0\} = \{\emptyset\}, \text{ the set whose only member is } 0 \\ &2 := \{0,1\} = \{\emptyset, \{\emptyset\}\}, \text{ the set whose only members are } 0 \text{ and } 1. \end{aligned}$$

Definition 1 (Successor)

*Given a natural number n, the **successor** of n is the natural number next to n, which can be obtained by*

$$S(n) := n \cup \{n\}.$$

We can use the definition of a successor to construct the rest of the natural numbers.

Example 1.2.1

Just to verify to ourselves that the definition indeed works, observe that

$$S(1) = 2 = \{\emptyset, \{\emptyset\}\} = \{\emptyset\} \cup \{\{\emptyset\}\}.$$

So to construct the natural number 3, we see that

$$\begin{aligned} S(2) = 3 = \{0,1,2\} = 2 \cup \{2\} &= \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} \\ &= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \end{aligned}$$

Looking at these, we start wondering to ourselves: how do we know that $\emptyset$ exists in the first place? How do we know that we can use $\cup$ and what does it even mean? Now it is meaningless if we cannot take that $\emptyset$ always exists, nor is it meaningful if we cannot take the $\cup$ of sets. And so, to allow us to continue, or even start with these notions, we require axioms.

Axiom 1 (Empty Set Axiom)

There exists a set, denoted by $\emptyset$, with no members.

With this axiom, we can indeed construct 0. To get 1 from 0, we have that 1 is a set whose only member is zero, and so if we take a member from 1, that member must be 0.

Axiom 2 (Pairset Axiom)

Given set x, y, there exists a set, denoted by $\{x, y\}$, whose only members are x and y. In other words,

$$t \in \{x, y\} \leftrightarrow (t = x \vee t = y)$$

Now note that in Axiom 2, if $x = y$, then the set $\{x, y\}$ has only x as its member. For example, we realize that $1 = \{0, 0\} = \{0\}$. But why exactly does this equality make sense? What exactly does "realize" mean?

Axiom 3 (Axiom of Extension)

Given sets x, y, $x = y$ if and only if x and y have the same members.

Now, using the above 3 axioms, we are guaranteed that

$$\begin{aligned}
&0 = \emptyset \text{ exists by the Empty Set Axiom} \\
&1 = \{\emptyset\} \text{ exists by the Pairset Axiom} \\
&2 = \{\emptyset, \{\emptyset\}\} \text{ exists by the Pairset Axiom}
\end{aligned}$$

Now we've constructed 3 to be the set whose only members are $0, 1$, and 2. So far, within our axioms, there is no such thing as $\{0, 1, 2\}$, which is what our 3 is supposed to be. We now require the following axiom:

Axiom 4 (Union Set Axiom)

Given a set x, there exists a set denoted by $\cup x$, whose members are precisely the members of the members of x, i.e.

$$t \in \cup x \leftrightarrow (t \in y \text{ for some } y \in x)$$

So, by this axiom, we have that given any n, $S(n) = \cup\{n, \{n\}\}$, or in other words,

$$t \in S(n) \leftrightarrow t \in n \vee t = n.$$

With all of the above axioms, we can iteratively construct each and every natural number in a rigorous manner. However, our goal is to construct infinitely many of them.

It is tempting to simply take the infinitude of natural numbers simply as an axiom, i.e.

> *There exists a set whose members are precisely the natural numbers.*

There is a certain rule to which we set down axioms, and that is, axioms must be expressable in a "finitary" manner, i.e. they must be expressible using first-order logic.

Definition 2 (Definite Condition)

We define a ***definite condition*** *as follows:*

- *$x \in y$ and $x = y$ are definite conditions, where x and y are both indeterminants, standing for sets, or are sets themselves;*
- *if P and Q are definite conditions, then so are*
  - *not P, denoted as $\neg P$;*
  - *P and Q, denoted as $P \wedge Q$;*

- *P or Q, denoted as $P \vee Q$;*
- *for all x, P, denoted as $\forall x P$; and*
- *there exists x, P, denoted as $\exists x P$.*

Example 1.2.2

$$x \in 1, 0 \in 2, 2 \in 0$$

are all definite conditions. Note, however, that $2 \in 0$ is false.

Note

*"If P then Q", which is also written as $P \to Q$, is also a definite condition since it is "**equivalent**"[1] to the statement $\neg P \vee Q$.*

Consequently, $P if and only if Q$, which is also expressed as $P \leftrightarrow Q$, can be written as

$$(\neg P \vee Q) \wedge (\neg Q \vee P)$$

[1] We have yet to define what equivalent statements are but we shall take this for granted for now.

Now, with this definition, and first-order logic notations in mind, we can write:

- Empty Set Axiom: $\exists x \; \forall t \; \neg(t \in x)$
- Pairset Axiom: $\forall x \; \forall y \; \exists p \; \forall t \; (t \in p \leftrightarrow ((t = x) \vee (t = y)))$
- Union Set Axiom: $\forall x \; \exists z \; \forall t \; ((t \in z) \leftrightarrow (\exists y \; ((y \in x) \wedge (t \in y))))$

Exercise 1.2.1
Write Axiom 3 in first-order logic notation.

Solution

$$\forall x \; \forall y$$
$$(x = y) \leftrightarrow (\forall t \; ((t \in x) \leftrightarrow (t \in y)))$$

Note that the statement that we proposed as an axiom for the set of natural numbers in page 14 is not definite, although that itself is not obvious.

For example, we may try to write

$$\exists x \; (\forall t \; ((t \in x) \leftrightarrow ((t = 0) \vee (t = 1) \vee (t = 2) \vee \ldots)))$$

and then notice that we do not have the notion of "..." within the "tools" that we are allowed to use.

2 Lecture 2 Sep 11th

2.1 Ordinals (Continued)

2.1.1 Zermelo-Fraenkel Axioms (Continued)

We stopped at the discussion about allowing for an infinite set, so that we can construct our set of infinite natural numbers. The idea here is to take *the smallest set that contains* 0 *and is preserved by the successor function*[1]

[1] Q: Why the smallest set?

Axiom 5 (Infinity Axiom)

There exists a set I *that contains* 0 *and is preserved by the successor function. We may express this as*

$$\exists I ((0 \in I) \land \forall x (x \in I \to S(x) \in I))$$

where we have defined that $S(x) \in I$ *means*

$$\exists y (\forall t (t \in y \leftrightarrow (t \in x) \lor (t = x)) \land (y \in I))$$

We call I *the* ***successor set***.

Since we want the smallest of such successor sets, we can try taking the intersection of all successor sets. But before we can do that, we require more axiomatic statements.

Definition 3 (Subsets)

$x \subseteq y$ means that every element of x is an element of y, i.e.

$$\forall t((t \in x) \to (t \in y))$$

With a definition of a subset, we can define the Powerset Axiom.

Axiom 6 (Powerset Axiom)

Given a set x, there exists a set $\mathcal{P}(x)$ that contains all subsets of x, i.e.

$$\forall t((t \subseteq \mathcal{P}(x)) \leftrightarrow (t \subseteq x))$$

We also require the following axiom.

Axiom 7 ((Bounded) Separation Axiom)

Given a set x and a definition condition P, there exists a set whose elements are precisely the members of x that satisfies P, i.e.

$$\forall x\ \exists y\ \forall t\ ((t \in y) \leftrightarrow \forall y\ ((t \in x) \land P(t)))$$

where

$$y = \{z \in x \mid P(z)\}.$$

There are two important aspects to the Bounded Separation Axiom:

- it is bounded by the set x; and
- P is a definite condition.

Exercise 2.1.1 (Set Intersection)

Prove that given a non-empty set x, there exists a set $\cap x$ satisfying

$$\forall t\ ((t \in \cap x) \leftrightarrow \forall y\ ((y \in x) \to (t \in y)))$$

Proof

to be solved

Definition 4 (Natural Numbers)

Let I be a successor set. The set of natural numbers is[2]

$$\omega := \cap\{J \subseteq I : J \text{ is a successor set }\}$$

[2] We can also write $J \subseteq I$ as $J \in \mathcal{P}(I)$ and invoke the Bounded Separation Axiom.

Note

J being a successor set can be expressed by the definite condition

$$(0 \in J) \wedge \forall x\, (x \in J \to S(x) \in J),$$

so we can write the definite condition in the above definition by

$$\omega := \cap\{J \subseteq I : (0 \in J) \vee \forall x\, (x \in J \to S(x) \in J)\}$$

Exercise 2.1.2

Show that the definition of ω does not actually depend on I, i.e. if given I_1 and I_2 such that we have

$$\omega_1 = \cap\{J \subseteq I_1 : J \text{ is a successor set }\}$$
$$\omega_2 = \cap\{J \subseteq I_2 : J \text{ is a successor set }\}$$

we have

$$\omega_1 = \omega_2.$$

Proof

to be solved

Another useful axiom that we will use later is the following:

Axiom 8 (Replacement Axiom)

Suppose P is a binary definite condition[3] *such that for every set x, there is a unique y satisfying $P(x, y)$. Given a set A, there is a set B such that $t \in B$ if and only if there is an $a \in A$ with $P(a, t)$.*

[3] A **binary definite condition** has only two variables.

Note

The slogan for the Replacement Axiom is:

> *The image of a set under a definite operation exists.*

These eight axioms, along with another ninth axiom called the Regularity Axiom[4], constitutes the **Zermelo-Fraenkel Set Theory**.

Note that all axioms, save the Extensionality, assert the existence of sets.

[4] We shall not discuss too much about this. According to the lecture and the lecture notes, the Regularity Axiom states that every set has a minimal element. On Wikipedia, the axiom states that every set has an element that does not intersect with the set itself.

2.1.2 *Classes*

There are times where we are interested in a collection of sets that do not form a set themselves.

Example 2.1.1 (Russell's Paradox)

There is no set containing all sets.

Proof

Suppose such a set exists, and call it U. Now consider the set

$$R := \{x \in U : x \notin x\},$$

which exists by Bounded Separation. Observe that

$$\begin{gathered} R \in R \implies R \notin R \quad \lightning \\ \implies R \notin R \implies R \in R \quad \lightning \end{gathered}$$

Thus such a set U cannot exist.

To talk about such collections, that may or may not be sets, we define *classes*.

Definition 5 (Class)

A class is any collection of sets defined by definite property, i.e. given any difinite condition P,

$$\llbracket\ z \mid P(z)\ \rrbracket$$

is the class of all sets satisfying P.

Here, instead of Bounded Separation, we have what is called **unbounded separation**.

Note

We shall use $[\![\quad]\!]$ rather than $\{ \quad \}$ to emphasize that we are talking about classes, i.e. we may be talking about non-sets.

Example 2.1.2

$$\mathrm{Set} := [\![\ z \mid z = z \ ]\!]$$

is the universal class of all sets.

Note

- *Every set is a class.*

Proof

Suppose x is a set. We may write

$$x = [\![\ z \mid z \in x \ ]\!] .$$

□

- *Some classes are not sets; these are called **proper classes**. E.g. the universal class of all sets, and*

$$\mathit{Russell} := [\![\ z \mid z \notin z \ ]\!] .$$

3 Lecture 3 Sep 13th

3.1 Ordinals (Continued 2)

3.1.1 Cartesian Products and Function

Definition 6 (Ordered Pairs)

Given sets x, y, an ***ordered pair*** *of x and y is defined as*[1]

$$(x, y) = \{\{x\}, \{x, y\}\}$$

[1] This invokes the Pairset Axiom thrice. Why did we not define an ordered pair as

$$(x, y) = \{\{x\}, \{y\}\}$$

instead?

Note

Note that we must have

$$((x, y) = (x', y')) \iff (x = x' \wedge y = y').$$

Proof

The ($\impliedby$) direction is clear by Extensionality. For the other direction, we shall break it into 2 cases:

Case 1: $x = y$. Then $\{x, y\} = \{x\}$ by Extensionality, and so

$$(x, y) = \{\{x\}\}$$

Therefore, we have that

$$\{\{x\}\} = (x, y) = (x', y') = \{\{x'\}, \{x', y'\}\}$$

So we have

$$\{x\} = \{x'\} \implies x = x'$$

and

$$\{x\} = \{x', y'\} \implies y' = x = y.$$

Thus we have

$$x = x' \wedge y = y'.$$

Case 2: Suppose $x \neq y$ and $x' \neq y'$ [2] We have

[2] If any of them are equal, Case 1 would apply.

$$\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$$

Then

$$\{x\} = \{x'\} \vee \{x\} = \{x', y'\}$$

The latter leads to a contradiction, since it would imply

$$x' = x = y'.$$

Thus $x = x'$. Also, we have

$$\{x, y\} = \{x'\} \vee \{x, y\} = \{x', y'\}$$

Now the former leads to a contradiction since it would imply that

$$x = x' = y.$$

Now since $x = x'$, it must be that $y = y'$, otherwise $y = x' = x$ would contradict our assumption. Therefore, we have that

$$x = x' \wedge y = y'.$$

□

With ordered pairs, we can build Cartesian products:

Definition 7 (Cartesian Product)

Given classes X and Y, the ***Cartesian Product*** *of X and Y is defined as*

$$X \times Y := [\![\; z : z = (x, y), x \in X, y \in Y \;]\!]$$

Note

We can express this definition using definite conditions;

$$\forall x, y\Big((x \in X) \wedge (y \in Y) \wedge \Big(\exists a, b(\forall t(t \in a \leftrightarrow t = x)) \wedge \forall t(t \in b \leftrightarrow (t = x) \vee (t = y))\Big) \wedge \forall t\Big(t \in z \leftrightarrow ((t = a) \vee (t = b))\Big)\Big)$$

Note

- *A Cartesian product is a class.*
- *If A is a set and B is a class, and $B \subseteq A$, then B is also a set. This is easy to show: observe that by Extentionality,*

$$B = \{a \in A \mid a \in B\}.$$

By Bounded Separation Axiom, B is a set[3]*.*

[3] This statement can be rephrased as: subclasses of a set are subsets.

Consequently, Cartesian products of sets are sets themselves; if X and Y are sets, we want to show that $X \times Y$ is a set so it is sufficient to show that it is contained in one. Recall that

$$(x, y) = \{\{x\}, \{x, y\}\}$$

and $\{x, y\} \subset X \cup Y$ which means $\{x, y\} \in \mathcal{P}(X)$, and we observe that $\{x\} \in \mathcal{P}(X \cup Y)$. So $(x, y) \in \mathcal{P}(X \cup Y)$. Therefore, $X \times Y \subset \mathcal{P}(\mathcal{P}(X \cup Y))$, and we show to ourselves that $X \times Y$ is indeed a set.

Definition 8 (Definite Operation)

Given classes X and Y, a ***definite operation*** *$f : X \to Y$ is a subclass $\Gamma(f) \subseteq X \times Y$ such that*

$$\forall x \in X\ \exists! y \in Y\ (x, y) \in \Gamma(f).$$

Note

*We write $f(x) = y$ to mean $(x,y) \in \Gamma(f)$. We also refer to $\Gamma(f)$ as the **graph of f**.*

Example 3.1.1

The successor function $S : \mathrm{Set} \to \mathrm{Set}$ is a definite operation such that

$$S(x) = x \cup \{x\}$$

This is true since is can be expressed as

$$\forall t (t \in y \leftrightarrow (t \in x \vee t = x)).$$

To show that S is a definite operation, we need to show that S is a definite condition.

Note

If X and Y are sets and f is a definite operation, then $\Gamma(f) \subseteq X \times Y$ is a set. In such a case, we call f a function.

Definition 9 (Functions)

A function is a definite operation $f : X \to Y$ where X and Y are both sets.

We can now restate the Replacement Axiom.

Axiom 9 (Replacement Axiom (Restated))

If $f : X \to Y$ is a definite operation, and $A \subseteq X$ is a set, then $\exists B \subseteq Y$ that is a set such that $t \in B$ if and only if $t = f(a)$ for some $a \in A$.

3.1.2 The Natural Numbers

Theorem 10 (Induction Principle)

Suppose $J \subseteq \omega$, $0 \in J$ and whenever $n \in J$, $S(n) \in J$. Then $J = \omega$.

Proof

By assumption, J is a successor set, therefore $\omega \subseteq J$ by definition. Thus, sinec $J \subseteq \omega$, we have $J = \omega$. $\square$

Lemma 11 (Properties of the Natural Numbers)

Suppose $n \in \omega$. We have

1. $n \subseteq \omega$;
2. $\forall m \in n \quad m \subseteq n$;
3. $n \notin n$;
4. $n = 0 \veebar 0 \in n$; *and*
5. $y \in n \implies S(y) \in n \veebar S(y) = n$.

Proof

1. Let[4]

$$J := \{n \in \omega : n \subseteq \omega\} \subseteq \omega.$$

Note that $\emptyset \subseteq \omega$ and so $0 \subseteq \omega$. By membership, $0 \in J$.

Suppose $m \in J$. Consider $S(m) = m \cup \{m\}$. Since $J \subseteq \omega$, $m \in \omega$. Since $m \in \omega$, $\{m\} \subseteq \omega$. Therefore $S(m) = m \cup \{m\} \subseteq \omega$, and so $S(m) \in J$. So J is a successor set. And thus by Induction Principle, $J = \omega$.

2. Let

$$J := \{n \in \omega : \forall m \in n, m \subseteq n\}.$$

It is vacuously true that $0 \in J$ since $\emptyset$ is a subset of every $n \in J$. Suppose $n \in J$. Then $\forall m \in n$, we have $m \subseteq n$. Consider $S(n) = n \cup \{n\}$. Note that $n \in S(n)$ and $n \subseteq S(n)$. For $x \in S(n)$ such that $x \neq n$, we must have that $x \in n$. By assumption, $x \subseteq n \subseteq S(n)$. Therefore, $S(n) \in J$, and so J is a successor set. By the Induction Principle, $J = \omega$.

[4] We construct this J and show that it is a successor set. Note that if $J = \omega$, our proof is complete.

3. Let
$$J := \{n \in \omega : n \notin n\}.$$
We have $0 = \emptyset \notin \emptyset$. So $0 \in J$.

Let $n \in J$. Consider $S(n) = n \cup \{n\}$. In particular, note that $n \in S(n)$. Suppose, for contradiction, that $S(n) \in S(n)$. Then $S(n) = n$ or $S(n) \in n$.

$S(n) = n \implies n \in S(n) = n \lightning n \notin n$.

$S(n) \in n \implies S(n) \subseteq n$ by part 2 $\implies n \in n \lightning n \notin n$.

Thus $S(n) \notin S(n)$ and so $S(n) \in J$. So J is a successor set, and so by the Induction Principle, $J = \omega$.

4. It suffices to show that
$$\omega = \{0\} \cup \{n \in \omega : 0 \in n\}.$$
Let J = RHS. We have that $0 \in J$. Suppose $n \in J$ such that $n \neq 0$. Then $0 \in n$. Since $n \subseteq S(n) = n \cup \{n\}$, we have that $0 \in S(n)$. Therefore, $S(n) \in J$. So J is a successor set, and so by the Induction Principle, $J = \omega$ as required.

5. Let
$$J := \{n \in \omega : y \in n \implies S(y) \in n \veebar S(y) = n\}.$$
$0 \in J$ is vacuously true, since there are no $y \in 0$. Suppose $n \in J$. Let $y \in S(n) = n \cup \{n\}$. We have two choices: either $y \in n$ or $y = n$. If $y \in n$, then $S(y) \in n \veebar S(y) = n$, since $n \in J$. We have that

$S(y) \in n \subseteq S(n)$ in which case we are done; and

$(sy) \subseteq n \in S(n)$.

Otherwise, if $y \notin n$, then $y = n$. Then we simply have $S(y) = S(n)$. Thus J is a succesor set and so by the Induction Principle, $J = \omega$.

$\square$

3.1.3 Well-Orderings

Definition 10 (Strict Partially Ordered Set)

*A **strict partially ordered set** (or **strict poset**[5]) is a set E together with $R \subseteq E^2 = E \times E$ such that*

1. *(**anti-reflexive**)* $\forall a \in E \quad (a, a) \notin R$;
2. *(**anti-symmetric**)* $\forall a, b \in E \quad (a, b) \in R \wedge (b, a) \in R \implies a = b$; *and*
3. *(**transitivity**)* $\forall a, b, c \in E \quad (a, b), (b, c) \in R \implies (a, c) \in R$.

[5] This is my unofficial terminology

Definition 11 (Strict Totally Ordered Set)

*A strict poset is **total** (or **linear**) if*

$$\forall a, b \in E \quad (a, b) \in R \veebar (b, a) \in R$$

Definition 12 (Well-Order)

*A strict linear order is **well-ordered** if*

$$\forall X \subseteq E (X \neq \emptyset) \quad \exists a \in X \quad \forall b \in X (b \neq a) \quad (a, b) \in R$$

*i.e. every nonempty subset of E has a **least element**.*

We shall prove the following next lecture.[6]

[6] Anti-reflexivity and Anti-symmetry were proven in this lecture, but I am moving it to the next for ease of reading.

Proposition (ω is Strictly Well-ordered)

$(\omega, \in)$ is a strict well-ordering.

4 Lecture 4 Sep 18th

4.1 Ordinals (Continued 3)

4.1.1 Well-Orderings (Continued)

Proposition 12 (ω is Strictly Well-Ordered)

$(\omega, \in)$ is a strict well-ordering.

Lemma (Lemma 11)
Suppose $n \in \omega$. We have

1. $n \subseteq \omega$;
2. $\forall m \in n \quad m \subseteq n$;
3. $n \notin n$;
4. $n = 0 \veebar 0 \in n$; and
5. $y \in n \implies S(y) \in n \veebar S(y) = n$.

Proof
By Lemma 11, we have that $\forall n \in \omega$, $n \notin n$. (**anti-reflexivity** ✔).

$\forall n, m \in \omega$, suppose, for contradiction, that $n \in m$ and $m \in n$. Again, by Lemma 11, we have $n \subseteq m$ and $m \subseteq n$, which implies that $n = m$. Thus, we have $n \in m = n$ and $m \in n = m$, a contradiction to the fact that $n \notin n$ and $m \notin m$ (**anti-symmetry** ✔).

$\forall x, y, z \in \omega$ such that $x \in y$ and $y \in z$, by Lemma 11, $y \in z \implies y \subseteq z \implies x \in z$ (**transitivity** ✔).

To show totality of the relation, let $n \in \omega$. WTS for any $m \in \omega$, either

$$m \in n, \quad m = n, \text{ or } n \in m.$$

Let[1]

$$J = \underset{\in n}{n} \cup \underset{= n}{\{n\}} \cup \underset{> n}{\{m \in \omega : n \in m\}}.$$

<u>Case 1: $n = 0$.</u> In this case, we have[2]

$$J = \emptyset \cup \{\emptyset\} \cup \{m \in \omega : 0 \in m\}$$

As a consequence of Lemma 11 (4), we have that $J = \omega$.

[1] We construct J such that J will contain all the possible cases, and use this fact to prove that $J = \omega$ so these 3 cases are the only scenarios that can happen.
[2] Note that $0 = \emptyset$.

Case 2: $n \neq 0$. Again, by Lemma 11 (4), since $n \neq 0$, we must have $0 \in n \subseteq J$ and so $0 \in J$. Now suppose that $m \in J$.

Case 2(a): $m \in n$. Then by Lemma 11 (5), $S(m) \in n$ or $S(m) = n$.
$S(m) \in n \implies S(m) \in J$
$S(m) \in n \implies S(m) \in J$

Case 2(b): $m = n$. Then $S(m) = S(n) = n \cup \{n\}$. And so $n \in S(m)$, which implies $S(m) \in J$.

Case 2(c): $n \in m$ Then since $S(m) = m \cup \{m\}$, we have that $m \in m \subseteq S(m)$. Therefore $S(m) \in J$.

Therefore, J is a sucessor subset of ω. Thus by the Induction Principle, $J = \omega$. (**totality ✔**)

To prove that $\in$ is a well-ordering, suppose $X \subseteq \omega$ is non-empty. Suppose, for contradiction, that X has no $\in$-least element. Now consider

$$J = \{n \in \omega : S(n) \cap X = \emptyset\}$$

Claim: J is a successor set.[3]

[3] Since we want to prove that $\in$ is a well-ordering, we can suppose that there is a non-empty subset of ω that is not empty, and has no $\in$-least element. The core idea here is that, by the construction of J, if $J = \omega$, then all elements of ω would be disjoint from X, forcing X to be the empty set.

By Lemma 11 (4), 0 is the $\in$-least element of ω. If $0 \in X$, then 0 would be $\in$-least in X, contradicting our supposition. Thus $0 \notin X$, And so

$$S(0) \cap X = (0 \cup \{0\}) \cap X = \{0\} \cap X = \emptyset$$

since $0 \notin X$. Thus $0 \in J$.

Suppose $n \in J$. By construction of J, we have $S(n) \cap X = \emptyset$. Observe that

$$S(S(n)) \cap X = (S(n) \cup \{S(n)\}) \cap X.$$

Now if RHS of the above is non-empty (aiming for contradiction), then we may have $S(n) \in X$. Then $S(n)$ would be the $\in$-least element in X, a contradiction. If $m \in S(n)$, we have that $m \notin X$ since $S(n) \cap X = \emptyset$. Thus $SS(n) \cap X = \emptyset$ and so $S(n) \in J$. Therefore, by the Induction Principle, $J = \omega$.

We observe that $\forall n \in \omega$,

$$\emptyset = S(n) \cap X) = (n \cup \{n\}) \cap X$$

$\implies n \notin X$, and so we must have $X = \emptyset$ (**well-ordered ✔**). $\square$

Note

Given $n, m \in \omega$, we often write $n < m$ to mean $n \in m$.

Definition 13 (Ordinals)

*An **ordinal** is a set α satisfying:*

1. $x \in \alpha \implies x \subseteq \alpha$;
2. $(\alpha, \in)$ *is a strict well-ordering.*

Example 4.1.1

ω is an ordinal: $\forall n \in \omega$, by Lemma 11, $n \subseteq \omega$, and ω is proven to have a strict well-ordering under $\in$.

Example 4.1.2

Every natural number is an ordinal (*finite ordinals*): by Lemma 11 (2), the first property is satisfied; well-ordering follows from the property of ω.

Let Ord denote the class of all ordinals. We shall show later that Ord is a proper class.

Exercise 4.1.1

Verify that for a set to be an ordinal is a definite condition.

Observe that

$$\forall t (t \in \mathrm{Ord} \leftrightarrow ((x \in t \implies x \subseteq t) \wedge ((t, \in) \text{ is a strict well-ordering })))$$

where $(x \in t \implies x \subset t)$ is the definite condition

$$\forall x (x \in t \to \forall a (a \in x \to a \in t))$$

and $(t, \in)$ is a strict well-ordering is the definite condition

$$\forall s (s \subseteq t \wedge s \neq \emptyset \to \exists a (a \in s \to \forall b (b \in s \wedge b \neq a \to (a, b) \in (\in))))$$

Lemma 13 (Proper Subsets of an Ordinal Are Its Elements)

If $\alpha, \beta \in \mathrm{Ord}$ and $\alpha \subsetneq \beta$, then $\alpha \in \beta$.

Proof

We shall prove that α is the least element in β that is not in α itself.[4]

Let $D := \beta \setminus \alpha = \{x \in \beta : x \notin \alpha\} \subset \beta$[5]. Since $\alpha \subsetneq \beta$, $D \neq \emptyset$. Since $\beta \in \mathrm{Ord}$, $(\beta, \in)$ has a strict well-ordering, and so D has a least element, d. Note that $d \in \beta$, and since $\beta \in \mathrm{Ord}$, $d \subseteq \beta$.

Claim: $\alpha = d$.[6] WTS $\alpha \subseteq d$. $\forall x \in \alpha$, we have $x, d \in \beta$. Then since $(\beta, \in)$ is a strict well-ordering, we have either

$$x < d, \quad x = d, \quad \text{or } d < x$$

Note that $x \neq d$, otherwise $x = d \in D = \beta \setminus \alpha$.

[7] If $d < x$, then $d \in x$ (by our notation). Now since $\alpha \in \mathrm{Ord}$, $x < \alpha \implies x \in \subseteq$, and so $d \in \alpha$, which is yet another contradiction ($d \in D = \beta \setminus \alpha$).

Thus we must have $x < d$, i.e. $x \in d$. So $\alpha \subseteq d$.

WTS $d \subseteq \alpha$. Suppose not. Then let $x \in d \setminus \alpha$. Then since $d \in D = \beta \setminus \alpha$, we have $x \in \beta \setminus \alpha$, which then contradicts the minimality of d. Therefore, $d = \alpha$ as required. $\square$

[4] We shall construct a subset of $\beta \setminus \alpha$ and show that α is its element.

[5] Exists by Bounded Separation Axiom.

[6] If $\alpha = d$, then α is the said least element.

[7] This is an errorneous proof.

Warning

$$d < x \wedge x < \alpha \underset{\text{transitivity}}{\implies} d < \alpha \implies d \in \alpha$$

This argument is errorneous because we do not yet know if $\alpha \in \beta$.

Proposition 14 (Properties of Ordinals)

1. *Every member of an ordinal is an ordinal.*
2. $\alpha \in \mathrm{Ord} \implies \alpha \notin \alpha$.
3. $\alpha \in \mathrm{Ord} \implies S(\alpha) \in \mathrm{Ord}$.
4. $\alpha, \beta \in \mathrm{Ord} \implies \alpha \cap \beta \in \mathrm{Ord}$.
5. $\alpha, \beta \in \mathrm{Ord} \implies \alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha$.
6. $E \subseteq \mathrm{Ord}$ *a subset* $\implies (E, \in)$ *is a strict well-ordering.*[8]

Some of proofs of these properties are available in the course notes.

Exercise 4.1.2
Prove Item 3, Item 4, and Item 5 of Proposition 14.

[8] **I think** that such an E need not be an ordinal itself. For example, $E = \{1, 5, 10\} \subset \mathrm{Ord}$, but $4 \in 5$ and $4 \notin E$, and so $5 \in E$ but $5 \not\subseteq E$.

7. Ord *is a proper class.*

Proof

1. Suppose $x \in \beta \in \mathrm{Ord}$. WTS $x \in \mathrm{Ord}$, and we shall show that x satisfies Definition 13.

   Since $\beta \in \mathrm{Ord}$, $x \in \beta \implies x \subseteq \beta$. Thus $(x, \in)$ is a strict well-ordering (through inheriting the property). So it suffices to show that $y \in x \implies y \subseteq x$. So let $y \in x$, and let $t \in x$ [9]. Observe that

   $$\begin{aligned} t \in y &\implies t < y \\ y \in x &\implies y < x \end{aligned}$$

   and $t, y, x \in \beta \in \mathrm{Ord}$. Therefore, by transitivity, we have $t < y < x \implies t \in x$.

2. Suppose not, i.e. $\alpha \in \alpha$. Then $\alpha \subseteq \alpha \in \mathrm{Ord}$, and so $(\alpha, \in)$ is a strict well-ordering, i.e. $\alpha \notin \alpha$, a contradiction.

6. Suppose $A \subseteq E$ and $A \neq \emptyset$. Let $\alpha \in A$.

   Case 1: $\alpha \cap A = \emptyset$. Then $\forall \beta \in \alpha \implies \beta \notin A$. Therefore α is $\in$-least in A.

   Case 2: $a \cap A \neq \emptyset$. Let $A' = \alpha \cap A \subseteq \alpha$. Since $\alpha \in A \subseteq E \subseteq \mathrm{Ord}$, we have $(\alpha, \in)$ is a strict well-ordering, and so A' has a strict well-ordering as well, and thus it must have a $\in$-least element, x. Then x is the $\in$-least element in A.

7. If Ord is a set, then by Item 6, $(\mathrm{Ord}, \in)$ is a strict well-ordering. Also, by Item 1, every element of Ord is a subset of Ord. Therefore, Ord satisfies Definition 13, and so $\mathrm{Ord} \in \mathrm{Ord}$, which contradicts Item 2. Therefore $\mathrm{Ord} \notin \mathrm{Set}$.

$\square$

[9] To show that $y \subseteq x$, we need to show that $\forall t \in y$, $t \in x$.

5 Lecture 5 Sep 20th

5.1 Ordinals (Continued 4)

Note

If $A, B \in \mathrm{Ord}$, we will write $A < B$ to mean $A \in B$.

Proposition 15 (Properties of Ordinals 2)

1. *If $\alpha \in \mathrm{Ord}$, then $\alpha < S(\alpha)$, and there is nothing in between.*
2. *Let $E \subseteq \mathrm{Ord}$, where $E \neq \emptyset$ is a set, and $\sup E := \cup E$. Then $\sup E \in \mathrm{Ord}$, and it is a least upper bound for E.*[1]
3. *If $E \subseteq \mathrm{Ord}$ is a subset, then there is a least ordinal that is not in E.*

[1] I noted down from the lectures that this is "not necessarily strict", but I do not remember what it means now. **(Clarification required.)**
Perhaps this related to my question; can $E = \sup E$?

This is not necessarily true. If $E \notin \mathrm{Ord}$, then $E \neq \sup E$.

Proof

1. Since $S(\alpha) = \alpha \cup \{\alpha\}$, $\alpha \in S(\alpha)$ and so $\alpha < S(\alpha)$.

   It suffices to show that $\forall x < S(\alpha)$, we have $x \leq \alpha$. Let $x < S(\alpha)$, i.e. $x \in S(\alpha)$. So $x \in \alpha$ or $x = \alpha$, i.e. $x < \alpha$ or $x = \alpha$.

2. By definition, $\forall x \in E \subseteq \mathrm{Ord}$, we have that $x \subseteq \mathrm{Ord}$. Since $\cup E \subseteq E$, we have that $\cup E \subseteq \mathrm{Ord}$ is a subset. Thus by Proposition 14 Item 6, $(\cup E, \in)$ is a strict well-ordering.

   [2] Suppose $\alpha \in \cup E$, then $\exists e \in E$ such that $\alpha \in e \subseteq E \subseteq \mathrm{Ord}$. So e is an ordinal and so $\alpha \subseteq e$. [3] $\forall x \in \alpha$, we have $x \in e \in E$, and so $x \in \cup E$ by definition. Therefore $\alpha \subseteq \cup E$.

   And so, we have shown that $\cup E = \sup E \in \mathrm{Ord}$.

Exercise 5.1.1

Prove Proposition 15 Item 3.

Recommended strategy: $\alpha \in \mathrm{Ord}$ such that $E \subsetneq \alpha$ and take the least element of $\alpha \setminus E$ (which is non-empty). Prove that this least element is the least ordinal that is not in E.
You can take $\alpha = SS(\sup E)$. Verify that this works.

[2] This part shows that $\cup E$ is also an ordinal.
[3] Now we show that $\alpha \subseteq \cup E$.

Claim 1: $\sup E$ is an upper bound for E.

Suppose, for contradiction, that $\exists e \in E$ such that $\sup E < e$. Then since $\sup E$ and e are both ordinals, we have $\sup E \in e \in E$. Then by definition of $\cup$, we have that $\sup E \in \cup E = \sup E$, but by Proposition 14 Item 2, $\sup E \notin \sup E$, a contradiction.

Thus $\sup E$ is an upper bound as claimed.

Claim 2: $\sup E$ is the supremum (least upper bound).

$\forall \alpha < \sup E$, we have that $\alpha \in \sup E = \cup E$, and so $\exists e \in E$ such that $\alpha \in e$. Then $\alpha < e \in E$, i.e. α is not an upper bound of E.

Definition 14 (Successor Ordinal)

*The **successor ordinal** is an ordinal of the form $S(\alpha)$ for some $\alpha \in \mathrm{Ord}$.*

Definition 15 (Limit Ordinal)

*A **limit ordinal** is an ordinal that is not a successor.*

Example 5.1.1

0 and ω are both limit ordinals; 0 is vacuosly a limit ordinal, and ω is not a successor of any $\alpha \in Ord$ [4].

On the other hand, for $n \in \omega$ such that $n \neq 0$, $\exists \cup n \in \omega$ such that $S(\cup n) = n$ [5].

Exercise 5.1.2

Prove that $S(\omega)$ is a successor ordinal.

Solution

We have that $\omega \in \mathrm{Ord}$, and so $S(\omega)$ is a successor ordinal.

5.1.1 *Transfinite Induction & Recursion*

Theorem 16 (Transfinite Induction Theorem v1)

[4] Need a more careful proof, which I cannot do. The idea is to show that any such ordinal α will be an element of ω, and so will its successor $S(\alpha)$, and $\omega \notin \omega$.

[5] See A1.

Suppose P is a definite condition, with the property

$$\forall \alpha \in \mathrm{Ord} \wedge (\forall \beta < \alpha \; P(\beta)) \implies P(\alpha). \tag{5.1}$$

Then P is true of all ordinals.

Proof

$P(0)$ is vacuously true, since there are no elements that are less than 0. Suppose $P(\alpha)$ is false for some $\alpha \in \mathrm{Ord}$ such that $\alpha > 0$. By the Bounded Separation Axiom,

$$D := \{\beta \leq \alpha : \neg P(\beta)\}$$

is a set [6]. Note that $D \neq \emptyset$, since $\alpha \in D$. Since $\alpha \in \mathrm{Ord}$, we have $D \subseteq \alpha \subseteq \mathrm{Ord}$, and so $(D, \in)$ has a strict well-ordering. Let $\alpha_0 \in D$ be $\in$-least. Then $\forall \beta < \alpha_0$, we have that $\neg P(\beta)$, which contradicts the assumption Equation (5.1). Thus $P(\alpha)$ is true for all ordinals. $\square$

[6] Note that $\beta \leq \alpha \iff \beta < \alpha \vee \beta = \alpha \iff \beta \in S(\alpha)$

Theorem 17 (Transfinite Induction Theorem v2)

Suppose P is a definite condition satisfying

1. $P(0)$;
2. $\forall \beta \in \mathrm{Ord} \; P(\beta) \implies P(S(\beta))$; *and*
3. *If $\alpha \in \mathrm{Ord}$ is a limit ordinal and $\forall \beta < \alpha$, $P(\beta)$, then $P(\alpha)$.*

Then P is true of all ordinals.

This statement strongly resembles the Induction Princple that we have learnt in the earlier years of university. In contrast, v1 resembles Strong Induction Principle. It can be shown that v1 $\iff$ v2. v1 $\implies$ v2 is proven in this lecture.

Exercise 5.1.3
Prove that Theorem 17 $\implies$ Theorem 16.

Proof

It suffices to show that P satisfies Equation (5.1), i.e. $\forall \alpha \in \mathrm{Ord}$, we want to prove that $\forall \beta < \alpha$, if $P(\beta)$, then $P(\alpha)$.

When $\alpha = 0$, we have $P(0)$ and so Equation (5.1) is satisfies. When $\alpha > 0$ is a limit ordinal, our assumption immediately satisfies Equation (5.1). Now suppose $\alpha > 0$ is a successor ordinal, and suppose that $\alpha = S(\gamma)$ for some $\gamma \in \mathrm{Ord}$. By the assumption in

Equation (5.1), we have that

$$\forall \beta < \gamma \; P(\beta) \implies P(\gamma)$$

and so by condition (2), we have $P(S(\gamma))$ since $\gamma \in \mathrm{Ord}$. Thus we have $P(\alpha) = P(S(\gamma))$. $\square$

We shall prove the following in the next lecture:

Theorem (Transfinite Recursion)

Let X be a class of all definite operations whose domain is an ordinal. Given a definite operation

$$G : X \to \mathrm{Set}$$

$\exists! F : \mathrm{Ord} \to \mathrm{Set}$, *a definite operation, such that* $F(\alpha = F(F \restriction_\alpha))$, *for all* $\alpha \in \mathrm{Ord}$.

We want to use Transfinite Recursion to construct definite operations on ordinals such that they have properties that we are familiar with (and hence desire).

Note (Notation - Restriction)

Let $H : U \to Y$ be a definite operation on classes U, Y, and $Z \subseteq U$ a subclass. $H \restriction_Z$ is the definite operation

$$H \restriction_Z : Z \to Y$$

obtained by restricting H onto Z.

Note

In the theorem, we stated that F has its domain on Ord. *We know that for* $\alpha \in \mathrm{Ord}$, $\alpha \subseteq \mathrm{Ord}$, *and so $F \restriction_\alpha$ makes sense; in particular,*

$$F \restriction_\alpha : \alpha \to \mathrm{Set}.$$

Note that $F \restriction_\alpha \in X$, and so $G(F \restriction_\alpha)$ is valid and makes sense.

6 Lecture 6 Sep 25th

6.1 Ordinals (Continued 5)

6.1.1 Transfinite Induction & Recursion (Continued)

Theorem 18 (Transfinite Recursion v1)

Let X be a class of all definite operations whose domain is an ordinal. Given a definite operation

$$G : X \to \mathrm{Set}$$

$\exists! F : \mathrm{Ord} \to \mathrm{Set}$, *a definite operation, such that* $F(\alpha) = G(F \restriction_\alpha))$, *for all* $\alpha \in \mathrm{Ord}$.

Before proving the theorem, we shall note the following definition.

Definition 16 (α-function)

Using definitions in Theorem 18, a function t with domain in the ordinals is called an ***α-function*** *defined by G if*

$$\forall \beta < \alpha \quad t(\beta) = G(t \restriction_\beta).$$

Proof

We shall first prove for **uniqueness**. Suppose F and F' are two definition operations such that

$$F : \mathrm{Ord} \to \mathrm{Set} \quad F' : \mathrm{Ord} \to \mathrm{Set}$$
$$F(\alpha) = G(F \restriction_\alpha) \quad F'(\alpha) = G(F' \restriction_\alpha)$$

[1]Suppose $\forall \beta < \alpha \in \mathrm{Ord}$, we have $F(\beta) = F'(\beta)$. Note that

$$F(\beta) = F'(\beta) \iff F \restriction_\alpha = F' \restriction_\alpha$$
$$\implies F(\alpha) = G(F \restriction_\alpha) = G(F' \restriction_\alpha) = F'(\alpha)$$

[1] Here, we want to use Transfinite Induction v1 to show that they are unique.

Thus, uniqueness of F is guaranteed.

To prove existence, firstly, we note that the α-functions defined in Definition 16 are approximations to the F that we want. However, before going further, we need to show that they are also unique and that we can form a chain of extensions on these functions over Ord.

Uniqueness of t_α Let t, t' be a α-functions defined by G. WTS $\forall \beta < \alpha$, $t(\beta) = t'(\beta)$. If an α-function defined by G exists, we shall denote it as t_α.

Consider the definite condition

$$P(x) := (x \geq \alpha) \vee (t(x) = t'(x)).$$

Suppose that $\forall \gamma < \beta$, $P(\gamma)$ holds, i.e. $\gamma \geq \alpha$ or $t(\gamma) = t'(\gamma)$, which implies that $t \restriction_\beta = t' \restriction_\beta$. Therefore $t(\beta) = t'(\beta)$, i.e. $P(\beta)$ holds. Thus t_α is unique if it exists by Transfinite Induction.

t_α as a chain of extensions Now $\forall \beta < \alpha \in \mathrm{Ord}$, we have that $\beta \subseteq \alpha$. If t_α and t_β exist, then

$$\Gamma(t_\beta) \subseteq \Gamma(t_\alpha),$$

or in other words

$$t_\alpha \restriction_\beta = t_\beta.$$

We shall denote this relation as $t_\beta \subseteq t_\alpha$.

Existence of t_α The existence of t_α is a definite condition: by the Replacement Axiom, a function that maps $\alpha \mapsto t_\alpha$ is definite, and by Bounded Separation Axiom, the set

$$\Gamma(t_\alpha) = \{(\beta, G(t_\alpha \restriction_\beta) \mid \beta < \alpha, t_\alpha(\beta) = G(t_\alpha \restriction_\beta)\} \subseteq \mathrm{Ord} \times \mathrm{Set}$$

exists. [2]Now $t_0 = t_\emptyset$ is vacuously true. Suppose for any successor ordinal α, t_α exists. Since $\alpha \in \mathrm{Ord}$, we have that there is nothing between α and its successor $S(\alpha)$, and $\alpha < S(\alpha)$. Thus

$$t_{S(\alpha)} = t_\alpha \cup \{(\alpha, G(t_\alpha))\},$$

which exists by Union Set Axiom. We see that $t_{S(\alpha)}$ extends t_α onto α itself.

[2] We use Transfinite Induction v2 to CTP.

Suppose $\alpha > 0$ is a limit ordinal. Since t_α is a definite condition by Replacement, by Bounded Separation, we have that[3]

[3] **Verify the motivation in using or the reason behind getting this.**

$$t_\alpha = \bigcup_{\beta<\alpha} t_\beta = \cup\{t_\beta : \beta < \alpha\}.$$

Note that t_α is, indeed, an α-function defined by G:

- t_α is a function on α: we have that $\forall \beta < \alpha$,the t_β's form a chain of extensions;
- t_α is an α-function defined by G: $\forall \beta < \alpha$, since α is a limit ordinal, $S(\beta) < \alpha$, and so

$$t_\alpha(\beta) = t_{S(\beta)}(\beta) = G(t_{S(\beta)} \restriction_\alpha) = G(t_\beta \restriction_\alpha).$$

And so by Transfinite Induction, $\forall \alpha \in Ord$, t_α exists as required.

Construction of F Now for any $\beta < \alpha$, we have a chain of extensions

$$t_0 \subseteq t_1 \subseteq t_2 \subseteq \ldots \subseteq t_\beta \subseteq t_\alpha \subseteq \ldots$$

Let[4]

$$F := \bigcup_{\alpha\in\mathrm{Ord}} t_\alpha = \cup [\![\ t_\alpha \mid \alpha \in \mathrm{Ord}\ ]\!]$$

[4] I will leave the proof unfinished here. **Need to verify my understanding**

Corollary 19 (Transfinite Recursion v2)

Given $G_1 \in \mathrm{Set}$, $G_2 : \mathrm{Set} \to \mathrm{Set}$ a definite operation, $G_3 : X \to \mathrm{Set}$ a definite operation, where X is the class of all definite opeartion whose domain is an ordinal. Then

$$\exists! F : \mathrm{Ord} \to \mathrm{Set}$$

such that

1. $F(0) = G_1$;
2. $\forall \alpha \in \mathrm{Ord} \quad F(S(\alpha)) = G_2(F(\alpha))$; and
3. $\forall \beta > 0$ a limit ordinal, $F(\beta) = G_3(F \restriction_\beta)$.

Proof

The result is clear by Theorem 16 with $G : X \to Set$ defined by

$$G(f) = \begin{cases} G_1 & f = \emptyset \\ G_2(f(\alpha)) & \mathrm{Dom}(f) = S(\alpha) \\ G_3(f) & \mathrm{Dom}(f) > 0 \text{ a limit ordinal} \end{cases}$$

□

6.1.2 Ordinal Arithmetric

6.1.2.1 Ordinal Addition

Definition 17 (Ordinal Addition)

Let $\beta \in \mathrm{Ord}$. For any $\alpha \in \mathrm{Ord}$, we define

$$\beta + \alpha$$

using Transfinite Recursion[5] on α as follows:

- $\beta + 0 := \beta$;
- *if α is a successor ordinal, then $\beta + S(\alpha) := S(\beta + \alpha)$; and*
- *if $\alpha > 0$ is a limit ordinal, then $\beta + \alpha := \sup\{\beta + \gamma : \gamma < \alpha\}$.*

[5] Note that we are using Corollary 19 with

$$G_1 = \beta$$
$$G_2 = S : \mathrm{Set} \to \mathrm{Set}$$
$$G_3 : X \to \mathrm{Set} \text{ by } G_3(f) = \sup \mathrm{Im}(f)$$

Exercise 6.1.1

Using both the Induction Principle and Transfinite Induction, prove that $\beta + \alpha \in \mathrm{Ord}$.

Example 6.1.1

We have that

$$0, 1, 2, ..., \omega, \omega+1, \omega+2, ..., \omega+\omega$$

Observe that

$$\omega + 1 = \omega + S(0) = S(\omega + 0) = S(\omega)$$

and so $\omega + 1$ is a successor to ω. In general, we have that $\forall \alpha \in \text{Ord}$, $\alpha + 1 = S(\alpha)$. For example,

$$\omega + 2 = \omega + S(1) = S(\omega + 1).$$

On the other hand, note that

$$\omega + \omega = \sup\{\omega + n : n \in \omega\}.$$

Unlike regular addition, ordinal addition is not commutative. For instance, while $\omega + 1 = S(\omega)$,

$$1 + \omega = \sup\{1 + n : n \in \omega\} = \omega.$$

Exercise 6.1.2

Prove that ordinal addition is only commutative for "finite" ordinals.

6.1.2.2 *Ordinal Multiplication*

Definition 18 (Ordinal Multiplication)

Let $\beta \in \text{Ord}$. For any $\alpha \in$ Ord, we define

$$\beta \cdot \alpha$$

using Transfinite Recusion[6] *as follows:*

- $\beta \cdot 0 := 0$;
- *if α is a successor ordinal, $\beta \cdot S(\alpha) := \beta\alpha + \beta$*
- *if $\alpha > 0$ is a limit ordinal, $\beta \cdot \alpha := \sup\{\beta \cdot \gamma : \gamma < \alpha\}$.*

[6] Here, we use

$$G_1 = 0$$
$$G_2 : \text{Set} \to \text{Set by } G_2(x) = x + \beta$$
$$G_3(f) = \sup \text{Im}(f)$$

Example 6.1.2

We have

$$\omega \cdot 1 = \omega \cdot S(0) = \omega \cdot 0 + \omega = 0 + \omega = \omega.$$

Exercise 6.1.3

Prove that in general, $\forall \beta \in \mathrm{Ord}$, *we have* $\beta \cdot 1 = \beta$.

Exercise 6.1.4

Prove that $\forall \alpha, \beta \in \mathrm{Ord}$, $\alpha \cdot \beta \in \mathrm{Ord}$.

Example 6.1.3

We have

$$\omega \cdot 2 = \omega \cdot S(1) = \omega \cdot 1 + \omega = \omega + \omega.$$

Exercise 6.1.5

Prove that in general, $\forall \beta \in \mathrm{Ord}$, *we have* $\beta \cdot 2 = \beta + \beta$.

Note that ordinal multiplication, like its addition counterpart, is not necessarily commutative.

Example 6.1.4

While we have

$$1 \cdot \omega = \sup\{1 \cdot n \mid n \in \omega\} = \omega = \omega \cdot 1,$$

observe that

$$2 \cdot \omega = \sup\{2 \cdot n \mid n \in \omega\} = \omega \neq \omega + \omega = \omega \cdot 2.$$

Proposition 20 (Properties of Ordinal Addition and Ordinal Multiplication)

Let $\alpha, \beta, \delta \in \mathrm{Ord}$.

- $\alpha < \beta \iff \delta + \alpha < \delta + \beta$;
- $\alpha = \beta \iff \delta + \alpha = \delta + \beta$;
- (***(associativity)***) $(\alpha + \beta) + \delta = \alpha + (\beta + \delta)$;
- *if* $\delta \neq 0$, *then* $\alpha < \beta \iff \delta\alpha < \delta\beta$;
- *if* $\delta \neq 0$, *then* $\alpha = \beta \iff \delta\alpha = \delta\beta$;
- $(\alpha\beta)\delta = \alpha(\beta\delta)$.

Exercise 6.1.6
Prove Proposition 20.

6.1.2.3 Ordinal Exponentiation

Definition 19 (Ordinal Exponentiation)

Let $\beta \in \mathrm{Ord}$. For any $\alpha \in \mathrm{Ord}$, define

$$\beta^\alpha$$

using Transfinite Recursion by

- $\beta^0 := 1$;
- *if α is a successor ordinal, then $\beta^{S(\alpha)} := \beta^\alpha \cdot \beta$;*
- *if $\alpha > 0$ is a limit ordinal, then $\beta^\alpha := \sup\{\beta^\gamma \mid \gamma < \alpha\}$.*

In the next lecture, we shall study the following theorem:

Theorem (Strict Well-Ordered Sets are Isomorphic to a Unique Ordinal)
Every strict well-ordering is isomorphic to an ordinal. Both the ordinal and the isomorphism are unique.

The definition of isomorphism is:

Definition 20 (Isomorphism)

Let E, F be sets and R, S be relations defined on each set respectively so. We say that (E, R) and (F, S) are ***isomorphic****, which we denote by*

$$(E, R) \simeq (F, S),$$

if $\exists f : E \to F$, a bijection, such that

$$e_1 R e_2 \iff f_1 S f_2.$$

Such an f is called an ***isomorphism****.*

7 Lecture 7 Sep 27th

7.1 Ordinals (Continued 6)

7.1.1 Well-Orderings and Ordinals

Before proving the theorem stated at the end of last lecture, we require the following 2 lemmas:

Lemma 21 (Rigidity of Well-Orderings)

Well-orderings are rigid, i.e. the only automorphism[1] is the identity.

[1] An **automorphism** is an isomorphism from a set to itself.

Proof

Suppose $(E, <)$ is a well-ordering, and $f : E \to E$ an automorphism. Let[2]

$$D = \{x \in E \mid f(x) \neq x\}.$$

[2] We look at the fellas that were 'moved'.

Suppose for contradiction that f is not the identity map, i.e. $D \neq \emptyset$. Then, since $D \subseteq E$, D has a well-ordering, and so we can pick $\alpha \in D$ to be the least element.

Case 1: $f(a) < a$. Then $f(a) \notin D$ since a is least. But then that would mean

$$f(f(a)) = f(a) \implies f(a) = a$$

since f is a bijection. This contradicts the choice that $a \in D$.

Case 2: $a < f(a)$. Since f is a bijection, its inverse exists, and so

$$\begin{aligned} f^{-1}(a) < a &\implies f^{-1}(a) \notin D \\ &\iff ff^{-1}(a) = f^{-1}(a) \iff a = f^{-1}(a) \end{aligned}$$

which contradicts the choice that $a \in D$, yet again. Thus there is no such element $a \in D$, forcing $D = \emptyset$, and so f must be the identity map. $\square$

Lemma 22 (Strict Well-Ordering $\not\simeq$ Any of Its Proper Initial Segment)

A strict well-ordering is not isomorphic to any proper initial segment[3] of itself.

[3] **Definition 21 (Initial Segment)** *For a strict well-order $(E, <)$, an **initial segment** is a subset of the form*

$$\{x \in E : x < b\}$$

for some $b \in E$.

Proof
Let $(E, <)$ be a strict well-ordering. $\forall b \in E$,

$$I_b := \{x \in E : x < b\}$$

has an induced well-order, in particular $(I_b, <)$.

Suppose for contradiction that there exists an isomorphism $f : E \to I_b$. Let

$$D = \{x \in E \mid f(x) \neq x\}$$

$b \notin I_b$ and so $b \notin D$. **Proof is left incomplete until I verify the proof with the prof.**

Theorem 23 (Strict Well-Ordered Sets are Isomorphic to a Unique Ordinal)

Every strict well-ordering is isomorphic to an ordinal. Both the ordinal and the isomorphism are unique.

Proof
Uniqueness

Let (E, R) be a strict well-ordering. Suppose that

$$(E, R) \simeq (\alpha, \in) \text{ and } (E, R) \simeq (\beta, \in) \tag{7.1}$$

where $\alpha, \beta \in \mathrm{Ord}$. Then, either

$$\alpha < \beta, \quad \alpha = \beta \quad \text{or} \quad \beta < \alpha.$$

Suppose $\alpha < \beta$ (this argument also works for $\beta < \alpha$, by simply swapping the inequality on α and β). Then α is a proper initial segment of β. From Equation (7.1), we have that $(\alpha, \in) \simeq (\beta, \in)$ via (E, R), but this contradicts Lemma 22. Thus we must have $\alpha = \beta$.

Existence

If $E = \emptyset$, then we can choose $0 \in \mathrm{Ord}$ to be the ordinal of which E is isomorphic to. Suppose $E \neq \emptyset$. Denote an initial segment of E by

$$I_x := \{y \in E \mid y < x\}.$$

Let

$$A = \{x \in E \mid \exists \beta \in \mathrm{Ord} \;\; (I_x, R) \simeq (\beta, \in)\}.$$

Notice that $(I_x, R) \simeq (\beta, \in)$ is a definite condition: the both I_x and β are sets, and so by Replacement, there is a graph, $\Gamma(f)$, from I_x to β; injectivity of an element in $\Gamma(f)$ is expressible as

$$\forall y_1 \forall y_2 \forall \beta_1 \forall \beta_2 (y_1, y_2 \in I_x \land \beta_1, \beta_2 \in \mathrm{Ord}((y_1, \beta_1) = (y_2, \beta_2) \leftrightarrow y_1 = y_2));$$

surjectivity is expressible as

$$\forall \beta (\beta \in \alpha \to \exists y (y \in I_x \to (y, \beta) \in \Gamma(f))).$$

Thus by Bounded Separation, A is a set. Also, A is nonempty, since the least element of E will be isomorphic to 0.

By our uniqueness proof above, let f be a function on A, where $f(x)$ is the unique ordinal that is isomorphic to (I_x, R). By Replacement,

$$\mathrm{Im}(f) = \{f(x) \in \mathrm{Ord} \mid x \in A\} \subset \mathrm{Ord}$$

is a set. By Proposition 15 Item 3, $\exists \alpha \in \mathrm{Ord} \setminus \mathrm{Im}(f)$ that is $\in$-least. We want to show that $f : A \to \mathrm{Im}(f)$ is an isomorphism between (E, R) and $(\alpha, \in)$.[4]

[4] This is why we need to show that

1. f is order-preserving, which is one of the requirements of an isomorphism (by our definition in Definition 20);
2. f is injective;
3. $\alpha = \mathrm{Im}(f)$;
4. $A = E$,

where the last 2 items will force f to be surjective.

f is order-preserving: We shall also show here that A is **downward closed**. $\forall x, y \in E$, we want to show that $xRy \land y \in A \implies x \in A$. By the assumption, we have that

$$f(x) \overset{h}{\simeq} I_X \subsetneq I_y \overset{h}{\simeq} f(y)$$

Since $f(x), f(y) \in \mathrm{Ord}$, we have either

$$f(x) < f(y), \quad f(x) = f(y), \quad \text{or} \quad f(y) < f(x).$$

If $f(x) = f(y)$, then $I_x \simeq I_y$ which contradicts Lemma 22. If $f(y) < f(x)$, then

$$h(I_x) \subseteq h(I_y) = f(y) < f(x) = h(I_x),$$

which is a contradiction. Thus we must have $f(x) < f(y)$, i.e. f preserves order as claimed, and $f(x)$ is an initial segment of $f(y)$, i.e. $f(x) \in \mathrm{Ord}$, which implies that $x \in A$.

$\underline{\alpha = \mathrm{Im}(f)}$: Suppose $\beta \in \alpha \implies \beta < \alpha \implies \beta \in \mathrm{Im}(f)$ by choice of α being the least. Therefore, $\alpha \in \mathrm{Im}(f)$.

Now suppose that $\beta \in \mathrm{Im}(f)$. Then $\exists x \in A$ such that $(I_x, R) \simeq (\beta, \in)$. Again, we have 3 possibilities; either

$$\alpha < \beta, \quad \alpha = \beta \quad \text{or} \quad \beta < \alpha.$$

Now $\alpha < \beta \implies \alpha \in \beta \implies \alpha = h(I_y)$ where yRx and $I_y \subset I_X$. Since A is downward closed, $\alpha \in \mathrm{Im}(f)$, a contradiction. We also have that $\alpha \neq \beta$ since $\beta \in \mathrm{Im}(f)$ and $\alpha \in \mathrm{Ord} \setminus \mathrm{Im}(f)$, i.e. $\alpha \notin \mathrm{Im}(f)$. Thus $\beta < \alpha$, and so $\beta \in \alpha$. Therefore $\alpha = \mathrm{Im}(f)$.

$\underline{f \text{ is injective}}$: Suppose that $f(x) = f(y)$. $xRy \implies I_x$ is an initial segment of I_y, which contradicts Lemma 22. The argument is similar for if yRx. Thus we must have $x = y$.

$\underline{A = E}$: It suffices to show that $E \setminus A = \emptyset$. Suppose for contradiction that $E \setminus A \neq \emptyset$, i.e. $E = A$. Then $\exists x \in E \setminus A$, since $E \setminus A \subset E$ which has a strict well-ordering. Since f preserves order, for any $y \in E$, $xRy \implies y \notin A$. On the other hand, $yRx \implies y \in A$. Thus $I_x = A$. However, since f is an isomorphism between (A, R) and $(\alpha, \in)$ (since we assume that $A = E$), we have that $I_x = A \simeq \alpha$, i.e. $x \in A$, which is a contradiction to the choice that $x \in E \setminus A$.

This completes the proof. $\square$

7.2 Cardinals

While ordinals allow us to enumerate, we cannot use it to "measure". For example, $\omega \neq \omega + 1$, but they have the same size.

Definition 22 (Equinumerous)

Two sets A and B have the same size, or ***equinumerous****, if there is a bijection from A to B. We denote this relation by $|A| = |B|$.*[5]

[5] Note that we have yet to define $|\cdot|$.

The following is a well-known theorem that makes proving equinumerosity a lot easier.

Lemma 24 (Schröder-Bernstein Theorem)

Given two sets A and B, $|A| = |B|$ if and only if there exists injections in both directions, i.e. an injection from A to B, and an injection from B to A.

Proof

The ($\implies$) direction is easy, since a bijection exists. So it suffices to show the ($\impliedby$) direction. We shall use $A \hookrightarrow B$ to say that there is an injection from A to B.

Suppose

$$A \hookrightarrow B \overset{g}{\hookrightarrow} A$$

Then $\exists f : A \to A$ an injective map, and we would have

$$f(A) \subseteq g(B) \subseteq A. \tag{7.2}$$

From here, it suffices to show that for an injective ap $f : X \to X$, if we have

$$f(X) \subseteq Y \subseteq X,$$

then $|Y| = |X|$. From our observation in Equation (7.2), we have

$$X \supseteq Y \supseteq f(X) \supseteq f(Y) \supseteq f^2(X) \supseteq f^2(Y) \supseteq f^3(X) \supseteq \dots$$

Let

$$Z = X \setminus Y \ \dot{\cup}\ f(X) \setminus f(Y) \ \dot{\cup}\ f^2(X) \setminus f^2(Y) \ \dot{\cup}\ \dots$$

and

$$W = X \setminus Z$$

Then

$$X = Z \ \dot{\cup}\ W$$

<u>Claim:</u> For sets A and B such that $B \subseteq A$, we have that[6]

$$f(A \setminus B) = f(A) \setminus f(B).$$

Note that since f is injective, $f : B \to f(B)$ is a bijective map. Suppose $\exists x \in A \setminus B$ such that $f(x) \in f(B)$. Since $f : B \to f(B)$ is bijective, $\exists b \in B$ such that $f(b) = f(x)$, but f is injective. Thus the claim is true.

[6] Forgive me if the proof of this claim is a little sloppy.

Using a similar argument, it can be shown that $f(A \ \dot{\cup}\ B) = f(A) \ \dot{\cup}\ f(B)$.

Observe that

$$\begin{aligned} f(Z) &= f(X \setminus Y) \ \dot{\cup}\ f(f(X) \setminus f(Y)) \ \dot{\cup}\ f(f^2(X) \setminus f^2(Y)) \ \dot{\cup}\ \dots \\ &= f(X) \setminus f(Y) \ \dot{\cup}\ f^2(X) \setminus f^2(Y) \ \dot{\cup}\ f^3(X) \setminus f^3(Y) \ \dot{\cup}\ \dots \end{aligned}$$

and note that

$$f(Z) = Z \setminus (X \setminus Y).$$

Since $W = X \setminus Z$, we have $W \subseteq Y$. Since $Z \cap W = \emptyset$, we still have $f(Z) \cap W = \emptyset$. Also, note that $(X \setminus Y) \cap W = \emptyset$. Thus, we have

$$Y = X \setminus (X \setminus Y) = (Z \ \dot{\cup}\ W) \setminus (X \setminus Y) = f(Z) \ \dot{\cup}\ W$$

Let $g : X \to Y$ such that

$$g(A) = \begin{cases} A & A \subseteq W \\ f(A) & A \subseteq Z \end{cases}$$

Clearly so, g is bijective. $\square$

Example 7.2.1

We claimed that $|\omega| = |\omega + 1|$.

We can simply use the identity map from $\omega \to \omega + 1$. For $\omega + 1 \to \omega$, consider the mapping

$$f(\alpha) = \begin{cases} S(\alpha) & \alpha \in \omega \\ 0 & \alpha = \omega \end{cases}$$

This map is clearly injective by properties of elements of ω.

Definition 23 (Cardinal)

*A **cardinal** is an ordinal α with the property that $\forall \beta < \alpha$, $|\alpha| \neq |\beta|$*

In the next lecture, we shall see that the collection of cardinals is a proper class, and is a subclass of the ordinals.

Definition 24 (Finite & Countable)

*A set A is **finite** if $|A| = |n|$ for some $n \in \omega$. A is **countable** if A is finite or $|A| = |\omega|$.*

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8 Lecture 8 Oct 02nd

8.1 Cardinals (Continued)

Note

If $\kappa \in \mathrm{Card}$ *and* κ *is infinite, then* κ *is a limit ordinal. In other words, successor ordinals are either finite or are not Cardinals. This is true since* $\forall \alpha \in \mathrm{Ord}$ *such that* $\alpha \geq \omega$*, clearly we have* $\alpha \hookrightarrow S(\alpha)$*, and we can define a function* $f : S(\alpha) \to \alpha$ *such that*

$$f(\beta) = \begin{cases} S(\beta) & \beta < \omega \\ \beta & \omega \leq \beta < \alpha \\ 0 & \beta = \alpha \end{cases},$$

which is injective, and so by Schröder-Bernstein, $|S(\alpha)| = |\alpha|$.

Proposition 25 (The Least Cardinality Not Equinumerous to Subsets of a Set)

$\forall E \in \mathrm{Set}\ \exists \alpha \in \mathrm{Ord}\ \forall e \subseteq E$

$$|e| \neq |\alpha|$$

and there exists a least such $h(E) \in \mathrm{Ord}$.

Note

Note that $h(E) \in \mathrm{Card}$.

Proof

Suppose not, i.e. $\exists \alpha \in Ord$ such that $|h(E)| = |\alpha|$ and $\alpha < h(E)$. Since $h(E)$ is the least, we must have $|\alpha| = |A|$ or some $A \subseteq E$. Then $|h(E)| = |A|$, which is a contradiction to the definition of $h(E)$. $\square$

In particular, we have that $h(\omega)$ is an uncountable cardinal.

Now to prove Proposition 25.

Proof

It suffices to prove the existence of $h(E)$. Consider the class

$$H = [\![\ \alpha \in \mathrm{Ord} \mid \exists e \subseteq E \ \ |e| = |\alpha| \ ]\!] .$$

If we can show that H is a set, then the least element not in H shall be our $h(E)$. Consider

$$W := \{ (A, R) \mid A \subseteq E, \ (A, R) \text{ is a strict well-ordering } \}$$

which is a set by Replacement, i.e.

$$W \subseteq \mathcal{P}(E) \times \mathcal{P}(E \times E).$$

Note that $W \neq \emptyset$, since $\emptyset \subseteq E$ and the empty relation would be a well-ordering of $\emptyset$. By Theorem 23, $\exists f : W \to \mathrm{Ord}$ such that $f(A, R)$ is a unique ordinal. By Replacement, $\mathrm{Im}(f) \subseteq \mathrm{Ord}$ is a set.

<u>Claim: $\mathrm{Im}(f) = H$</u>: It is clear that $\mathrm{Im}(f) \subseteq H$, since all elements of $\mathrm{Im}(f)$ are isomorphic to some subset of E by definition. Now let $\alpha \in H$. Then $\exists g : \alpha \to A$ a bijection, for some $A \subseteq E$. Then define $\prec$ on A by

$$a \prec b \iff g^{-1}(a) < g^{-1}(b).$$

Then $(\alpha, <) \underset{\simeq}{\xrightarrow{g}} (A, \prec)$. Therefore, $(A, \prec) \in W$ and $f(A, \prec) = (\alpha, <)$, i.e. $H \subseteq \mathrm{Im}(f)$. Thus $H = \mathrm{Im}(f)$ and so H is a set as required. $\square$

Remark

This revelation tells us that Card *is a proper class.*

To use Card to measure all sets, we need every set to be equinumerous with a cardinal. In particular, every set would then have to be equinumerous to an ordinal. This would then require evrey set to have a strict well-ordering, which is something that we cannot prove with our axioms thus far.

8.1.1 Axiom of Choice

Definition 25 (Choice Function)

Suppose $\mathcal{F}$ is a set. A ***choice function*** *on $\mathcal{F}$ is a function*

$$c : \mathcal{F} \to \cup\mathcal{F} \text{ such that } \forall F \in \mathcal{F} \;\; c(F) \in F.$$

Note

If $\emptyset \in \mathcal{F}$, then $\mathcal{F}$ has no choice function, since nothing belongs in $\emptyset$.

Axiom 26 (Axiom of Choice)

Every $\mathcal{F} \in \mathrm{Set}$ such that $\emptyset \notin \mathcal{F}$ admits a choice function.[1]

[1] This is, again, an existential axiom.

Note

Unlike the other axioms, while the Axiom of Choice asserts the existence of choice functions on sets, the choice function need not be unique. Recall that in other axioms, the sets of which we assert their existence are unique.

Theorem 27 (Axiom of Choice and Its Equivalents)

TFAE

1. *Axiom of Choice*
2. *Well-ordering Principle: Every set admits a well-ordering.*

3. *Zorn's Lemma: If (E, R) is a strict poset with the property that* ***every totally ordered subset of*** E ***has an upper bound****, i.e.*

$$\forall A \subset E(\forall a, b \in A\ aRb \vee bRa \vee a = b)\ \forall a \in A \exists e \in E(aRe \vee a = e).$$

Then (E, R) has a maximal elements, i.e. $\exists z \in E$ such that $\forall x \in E$, $\neg zRx$.

Proof

(1) $\implies$ (2): Let $A \in$ Set. If $A = \emptyset$, then there is nothing to do and the statement is vacuously true. So suppose $A \neq \emptyset$. By the asusmption, fix a choice function c on $\mathcal{F} := \mathcal{P}(A) \setminus \{\emptyset\}$. Let $\theta \in \mathrm{Ord} \setminus A$. Define a definite operation $F : \mathrm{Ord} \to \mathrm{Set}$ such that

$$F(\alpha) = \begin{cases} c(A \setminus \mathrm{Im}(F \restriction_\alpha)) & A \setminus \mathrm{Im}(F \restriction_\alpha) \neq \emptyset \\ \theta & \text{otherwise} \end{cases}$$

Note that F exists by Transfinite Recursion[2].

[2] I am not sure how.

Claim 1: F halts, i.e. $\exists \alpha \in \mathrm{Ord}$ such that $F(\alpha) = \theta$ and $\forall \beta \in \mathrm{Ord}$ such that $\alpha < \beta$, $F(\beta) = \theta$.

Suppose not. Then F must be injective and has codomain $\cup\mathcal{F} = \cup(\mathcal{P}(A) \setminus \{\emptyset\}) = A$, i.e. we have that $F : \mathrm{Ord} \to A$ injective. Then by Proposition 25, there exists $h(A)$ that is the least ordinal that is not equinumerous with any subset of A. We may consider $F \restriction_{h(A)}: h(A) \to A$ since $h(A) \subset \mathrm{Ord}$. Now $\forall \alpha < \beta \in h(A)$, by our supposition, $F(\beta) \neq \theta$, and so $F(\beta) = c(A \setminus \mathrm{Im}(F \restriction_\beta)) \in (A \setminus \mathrm{Im}(F \restriction_\beta))$. Thus $F(\beta) \notin \mathrm{Im}(F \restriction_\beta)$. But since $\alpha < \beta$, it must be that $F(\alpha) \in \mathrm{Im}(F \restriction_\beta)$. Then $F(\alpha) \neq F(\beta)$. Consequently, since $\forall \beta \in h(A)$, we have that $F(\beta) \neq \theta$, and so we created an injection from $h(A)$ to A. This is impossible by the definition of $h(A)$. Thus it must be the case that $\exists \beta \in h(A)$ such that $F(\beta) = \theta$.

Let α be the least such β. The previous paragraph showed that $F \restriction_\alpha$ is an injection from α to A. It remains to show that the map is surjective. Suppose $\mathrm{Im}(F \restriction_\alpha) \neq A$. Then $A \setminus \mathrm{Im}(F \restriction_\alpha) \neq \emptyset$. Then

$$F(\alpha) = c(A \setminus \mathrm{Im}(F \restriction_\alpha)) \in A$$

which is a contradiction since $F(\alpha) = \theta$. So $A = \mathrm{Im}(F \restriction_\alpha)$ and so

$F \restriction \alpha$ is surjective.

Therefore $F \restriction_\alpha$ is a bijection from α to A, and so A has an induced strict well-ordering from α.

(3) $\implies$ (1): Let $\mathcal{F}$ be a set such that $\emptyset \notin \mathcal{F}$. Let Λ be the set of all partial choice functions on $\mathcal{F}$, identified with their graphs, i.e. $\forall f \in \Lambda$, $f : \mathcal{G} \to \cup\mathcal{G}$ such that $f(G) \in G$ for all $G \in \mathcal{G}$, and $\mathcal{G} \subseteq \mathcal{F}$. Note that Λ is indeed a set since the graphs exist by Replacement, and Λ is therefore a set from Bounded Separation. $\Lambda \neq \emptyset$, since the function $f(F) = x$ exists for $F \in \mathcal{F} \neq \emptyset$, and $F \neq \emptyset$.

Now $(\Lambda, \subseteq)$ is a poset, where we order the functions by extensions. For every Θ that is a totally ordered subset of Λ, we have that the union, $\cup\Theta$ is the upper bound of Θ. Thus the assumptions for Zorn's Lemma are satisfied, and so there exists a maximal function

$$f : \mathcal{G} \to \cup\mathcal{G} \text{ in } \Lambda.$$

To prove that this f is a choice function on $\mathcal{F}$, we want to prove that $f : \mathcal{F} \to \cup\mathcal{F}$, i.e. we need to show that $\mathrm{Dom}(f) = \mathcal{F}$. Suppose not, i.e. $\exists F \in \mathcal{F}$ such that $F \notin \mathrm{Dom}(f)$. Then since $F \neq \emptyset$, $\exists x \in F$, and so $f \cup \{(F, x)\}$ is a larger partial choice function on $\mathcal{F}$, contradicting the maximality of f in Λ. Thus $\mathrm{Dom}(f) = \mathcal{F}$ as claimed, and so f is a choice function on $\mathcal{F}$, proving the Axiom of Choice.

(2) $\implies$ (3): Suppose (E, R) is a strict poset. By (2), let $<$ be a strict well-ordering on E. Now by Theorem 23, $\exists! \alpha \in \mathrm{Ord}$ such that $(E, R) \simeq (\alpha, \in)$ through a unique isomorphism. Therefore, we may assume that $E \in \mathrm{Ord}$. Suppose that (E, R) satisfies the assumptions of Zorn's Lemma.

Assume, to the contrary, that (E, R) does not have an R-maximal element. From Proposition 25, $\exists h(E) \in Ord$ such that $\forall \beta < h(E)$, $\forall A \subseteq E$, $|\beta| \neq |A|$[3]. Recursively so, define $F : h(E) \to E$ by

$$F(0) = e \text{ for some } e \in E$$
$$F(S(\beta)) = < \text{-least element } \gamma \text{ of } E \text{ such that } F(\beta) R \gamma$$

[3] The strategy here is to use, once again, Proposition 25 to arrive at a contradiction that is similar to when we were proving (1) $\implies$ (2).

and for $\beta > 0$ a limit ordinal,

$$F(\beta) = \begin{cases} < \text{-least } \gamma \text{ such that } F(\zeta) R \gamma \text{ for all } \zeta < \beta \text{ if such } \gamma \text{ exists} \\ e \qquad \text{otherwise} \end{cases}$$

Note that this function is well-defined, since $F(\beta)R\gamma$ properly distinguishes the ordinals, for there are no R-maximal element in E, and $<$ is a strict well-ordering on E.

Now since $h(E) \in \text{Card}$, it is a limit ordinal, to show that F is injective, it suffices to show that $\forall \beta < h(E)$, $F \restriction_\beta$ is strictly order-preserving, i.e. $\forall x < y < \beta$, $F(x)RF(y)$. We shall prove this by Transfinite Induction.

$\beta = 0$ is vacuously true. If $\beta > 0$ is a limit ordinal, then $\beta = \bigcup_{\gamma<\beta} \gamma$, and so the strict ordering is preserved as given by the induction hypothesis. For β a successor ordinal, consider $F \restriction_{S(\beta)}$. $\forall x < y < S(\beta)$, if $y \neq \beta$, then we are done by the induction hypothesis. Suppose $y = \beta$. Since β is a successor ordinal, $\exists \gamma \in \text{Ord}$ such that $S(\gamma) = \beta$. Since $\gamma < S(\gamma)$ (by Proposition 15 Item 1), either $x = \gamma$ or $x < \gamma$. If $x < \gamma$, then our proof is complete by the induction hypothesis. If $x = \gamma$, then since $x = \gamma < S(\gamma) = \beta$, regardless if γ is a limit ordinal or successor ordinal, we have that $F(\gamma)RF(\beta)$.

Thus, by Transfinite Induction, we have that $F \restriction_\beta$ is strictly order-preserving for any $\beta < h(E)$, implying that $F : h(E) \hookrightarrow E$, hence contradicting the definition of $h(E)$. Thus, an R-maximal must exist. $\square$

9 Lecture 9 Oct 04th

9.1 Cardinals (Continued 2)

9.1.1 Axiom of Choice (Continued)

From hereon, unless stated otherwise, we shall assume AC.

Proposition 28 (Using Cardinals to Measure Sets)

Assume AC. Every set is equinumerous with a cardinal.

Proof

Let $A \in \text{Set}$. By the Well-Ordering Principle, A is well-orderable, i.e. there is a strict well-ordering on A. By Theorem 23, $\exists! \alpha \in \text{Ord}$ such that $(A, <) \simeq (\alpha, \in)$. Let

$$S = \{\beta \leq \alpha \mid |\beta| = |\alpha|\},$$

which is a set by Bounded Separation. Note that $S \neq \emptyset$ since $\alpha \in S$. Let β be the least such ordinal in S. By this minimal choice, $\beta \in \text{Card}$, $|\beta| = |\alpha| = |A|$. $\square$

And now our notation of $|A| = |B|$ makes sense provided the following definition.

Definition 26 (Cardinality)

Let $A \in \text{Set}$. $|A|$, in which we shall call the ***cardinality*** *of A, is the (unique) cardinal which is equinumerous with A.*

Proposition 29 (Lesser Cardinality)

$\forall A, B \in \mathrm{Set} \quad |A| \leq |B| \iff \exists f : A \hookrightarrow B.$

Proof

Let $\kappa = |A|$ and $\lambda = |B|$. If $\kappa \leq \lambda$, then

$$A \overset{\text{bijection}}{\to} \kappa \overset{id}{\hookrightarrow} \lambda \overset{\text{bijection}}{\to} B$$

By composition of the 3 functions, $A \hookrightarrow B$.

Conversely, suppose $\exists h : A \hookrightarrow B$. Suppose to the contrary that $\lambda < \kappa$. Then $\lambda \subsetneq \kappa$. Then

$$\kappa \overset{\text{bijection}}{\to} A \overset{h}{\hookrightarrow} B \overset{\text{bijection}}{\to} \lambda$$

and so there exists an injection from $\kappa \to \lambda$. By Schröder-Bernstein, we have $|\kappa| = |\lambda|$, which contradicts the fact that $\kappa \in \mathrm{Card}$. Thus $\kappa \leq \lambda$. $\square$

Corollary 30 (Cardinalities are Always Comparable)

$\forall A, B \in \mathrm{Set} \quad A \hookrightarrow B \vee B \hookrightarrow A.$

Proof

WLOG, suppose $\neg(B \hookrightarrow A)$. Then $\neg(|B| \leq |A|)$, i.e. $|A| < |B|$, i.e. (not by Proposition 29), $\exists f : A \hookrightarrow B$. $\square$

Proposition 31 (Functions are "Lossy Compressions")

Suppose $f : A \to B$. *Then* $|\mathrm{Im}(f)| \leq |A|$.

Proof

Let

$$\mathcal{F} = \{f^{-1}(y) \mid y \in \operatorname{Im}(f)\}$$

be the set of **fibres**[1] of f. Note that $\emptyset \notin \mathcal{F}$, as otherwise we would be saying that $\exists y \in \operatorname{Im}(f)$ that has no pre-image. Let $h : \operatorname{Im}(f) \to A$ by

$$h(y) := c(f^{-1}(y)) \in A$$

where $c : \mathcal{F} \to \cup\mathcal{F}$ is a choice function that exists by AC. Clearly so, h is injective, and so by Proposition 29, $|\operatorname{Im}(f)| \leq |A|$. $\square$

[1]

Definition 27 (Fibres)
*Let $f : A \to B$. For $y \in \operatorname{Im}(f)$, the **fibre** of y is defined as*

$$f^{-1}(y) = \{x \in A \mid f(x) = y\}.$$

*The fibres are also commonly called the **pullback**.*

Proposition 32 (Countable Union of Countable Sets is Countable)

Let $A \in \mathrm{Set}$ be countable, and every $a \in A$ is also countable. Then $\cup A$ is countable.

Proof

to be added

9.1.2 Hierarchy of Infinite Cardinals

Note (Notation)

Let $\kappa \in \mathrm{Card}$. Let $\kappa^+ := h(\kappa)$, which is the least ordinal not equinumerous with any subset of κ. We proved that $\forall E \in \mathrm{Set}$, $h(E) \in \mathrm{Card}$. Therefore, $\kappa^+ \in \mathrm{Card}$.

Remark

κ^+ is the least cardinal that contains κ. This follows immediately from the definition of $h(\kappa)$.

And so we observe that $\kappa \mapsto \kappa^+$ is a "successor" operation on cardinals.[2]

[2] Note that $\kappa + 1$ is not necessarily κ^+.

Definition 28 (Cardinal Numbers)

Using Transfinite Recursion, define the following ordinal-enumerated collection of cardinals

- $\aleph_0 = \omega$;
- $\forall \alpha \in \mathrm{Ord}$ *that is a successor ordinal,* $\aleph_{S(\alpha)} = \aleph_{\alpha+1} = \aleph_\alpha^+$; *and*
- $\forall \alpha > 0$ *that is a limit ordinal,* $\aleph_\alpha := \sup\{\aleph_\beta \mid \beta < \alpha\}$.

The $\aleph_\alpha$*'s are called* ***cardinal numbers****.*

Lemma 33 (Cardinal Numbers are Cardinals)

If $\alpha \in \mathrm{Ord}$, *then* $\aleph_\alpha \in \mathrm{Card}$.

Proof

We shall use Transfinite Induction. The result is clear for $\alpha = 0$, since $\aleph_0 = \omega$ and $\forall n \in \omega, |n| \neq |\omega|$. For successor ordinals α, since $h(\alpha)$ is an ordinal, we have that $\aleph_\alpha^+ = h(\aleph_\alpha)$ is also a cardinal. Now for $\alpha > 0$ a limit ordinal, suppose that $\beta < \aleph_\alpha$. Since $\aleph_\alpha = \sup\{\aleph_\gamma \mid \gamma < \alpha\}$, $\exists \gamma < \alpha$ such that $\beta < \aleph_\gamma$. Since $\aleph_\gamma$ is a cardinal by the Inductive Hypothesis, and $\beta < \aleph_\gamma < \aleph_\alpha$, we have that

$$|\beta| < |\aleph_\gamma| \leq |\aleph_\alpha|.$$

Thus $\aleph_\alpha$ is not equinumerous with any lesser ordinal, i.e. it is a cardinal. $\square$

Lemma 34 (Ordinals Index the Cardinal Numbers)

$\forall \alpha < \beta \in \mathrm{Ord}$, *we have* $\aleph_\alpha < \aleph_\beta$.

Proof

We shall use Transfinite Induction on β. $\beta = 0$ is true vacuously so. For β a successor ordinal, suppose $\forall \alpha < \beta, \aleph_\alpha < \aleph_\beta$. Now $\aleph_{\beta+1} = \aleph_\beta^+ = h(\aleph_\beta)$, which we have that $\aleph_\beta < h(\aleph_\beta)$ as proven before.

Now suppose that $\beta > 0$ is a limit ordinal. Then $\aleph_\beta = \sup\{\aleph_\alpha \mid \alpha < \beta\}$ and $\forall \gamma < \alpha < \beta, \aleph_\gamma < \aleph_\alpha$. Since β is a limit ordinal, $\exists \zeta < \beta$ such that $\alpha < \zeta$. By the Induction Hypothesis, we have that $\aleph_\alpha < \aleph_\zeta$, and $\aleph_\zeta \leq \aleph_\beta$. Thus $\aleph_\zeta \subseteq \aleph_\beta$, and so $\aleph_\alpha < \aleph_\beta$. $\square$

Lemma 35 (Infinite Cardinals are Distant)

$\forall \alpha \in \mathrm{Ord}$, $\alpha \leq \aleph_\alpha$. *The inequality is strict if α is a successor ordinal.*

Proof

Again, we shall use Transfinite Induction on α. For $\alpha = 0$, we have $0 < \aleph_0 = \omega$, and so $\alpha = 0$ holds. For α a successor ordinal, suppose $\alpha \leq \aleph_\alpha$. Thus $\alpha + 1 \leq \aleph_\alpha + 1$. By definition of $h(\aleph_\alpha)$ and definition of cardinal numbers

$$\alpha + 1 \leq \aleph_\alpha + 1 < \aleph_\alpha^+ = \aleph_{\alpha+1}.$$

Thus the statement holds for $\alpha + 1$, and indeed, the inequality is strict for successor ordinals. For $\alpha > 0$ a limit ordinal, suppose that $\forall \beta < \alpha$, we have $\beta \leq \aleph_\beta$. By Lemma 34, we have

$$\beta \leq \aleph_\beta < \aleph_\alpha.$$

Thus

$$\alpha = \sup\{\beta \mid \beta < \alpha\} < \aleph_\alpha$$

as required. $\square$

Proposition 36 (All Infinite Cardinals are Indexed by the Ordinals)

Every infinite cardinal is of the form $\aleph_\alpha$ for some $\alpha \in$ Ord.

Proof

Let $\kappa \in \text{Card}$. By Lemma 35, we have $\kappa \leq \aleph_\kappa < \aleph_{\kappa+1}$. And so we can show that $\forall \beta \in \text{Ord}$, $\forall \kappa < \aleph_\beta$, $\exists \alpha < \beta$ such that $\kappa = \aleph_\alpha$. We shall use Transfinite Induction on β.

Since there are no infinite cardinals strictly below ω, $\beta = 0$ is trivially true. Suppose $\beta = \gamma + 1$ is a successor ordinal, where $\gamma \in \text{Ord}$, and suppose $\kappa < \aleph_\beta$. Since $\gamma < \beta$, Lemma 34 implies that $\aleph_\gamma < \aleph_\beta = \aleph_{\gamma+1} = \aleph_\gamma^+$, and by definition, there are no cardinals between $\aleph_\gamma$ and $\aleph_\beta$. Thus $\kappa \leq \aleph_\gamma$. We thus have that either $\kappa = \aleph_\gamma$ or, by the Induction Hypothesis, $\exists \alpha < \gamma$ such that $\kappa = \aleph_\alpha$. Thus the statement holds for successor ordinals.

Let $\beta > 0$ be a limit ordinal and $\kappa < \aleph_\beta = \sup\{\aleph_\gamma \mid \gamma < \beta\}$. Then $\exists \gamma < \beta$ such that $\kappa < \aleph_\gamma$, and so by the Induction Hypothesis, $\exists \alpha < \gamma$ such that $\kappa = \aleph_\alpha$. $\square$

Consequently, we have ourselves an *ordinal-valued, order-preserving complete indexing* of the infinite cardinals.

Exercise 9.1.1

For the inequality in Lemma 35, show that the equality can occur. In particular, consider the sequence of ordinals defined recursively by $\alpha_0 = 0$, *and* $\alpha_{n+1} = \aleph_{\alpha_n}$, *and verify that* $\alpha = \aleph_\alpha$. *In fact, this works if we start with any ordinal* α_0, *not just* 0.

10 Lecture 10 Oct 11th

10.1 Cardinals (Continued 3)

10.1.1 Cardinal Arithmetic

Contents in this lecture should be extended upon. A lot of the contents need to be explored for it was presented tersely so without all the details of which we need.

10.1.1.1 Cardinal Summation

Definition 29 (Cardinal Sum)

$\forall \kappa_1, \kappa_2 \in \mathrm{Card}$. *Let the* ***cardinal sum***

$$\kappa_1 + \kappa_2 := |X_1 \cup X_2|$$

where $X_1, X_2 \in \mathsf{Set}$ *such that*

$$|X_1| = \kappa_1 \text{ and } |X_2| = \kappa_2$$

and $X_1 \cap X_2 = \emptyset$.

Remark

This definition does not depend on the choice of X_1 *and* X_2*, i.e. if we have another* X_1' *and* X_2' *such that*

$$|X_i'| = |X_i| = \kappa_i \quad i = 1, 2$$

and $X_1' \cap X_2' = \emptyset$*, then*

$$|X_1' \cup X_2'| = |X_1 \cup X_2|.$$

This shows that the cardinal summation is well-defined.

Exercise 10.1.1
Prove this remark.

Note

If $X, Y \in \text{Set}$ are arbitarily chosen, then

$$|X \cup Y| \leq |X| + |Y|$$

Warning

The cardinal summation is different from ordinal summation. For example, $\aleph_0 + \aleph_0$, where the $+$ represents the ordinal summation, is not a cardinal, as we have shown that $|\omega + \omega| = |\omega|$.

Therefore, there is a need to explicitly mention the context of which $+$ is used.

10.1.1.2 *Cardinal Product*

Definition 30 (Cardinal Product)

$\forall \kappa_1, \kappa_2 \in \text{Card}$, *the* ***cardinal product***

$$\kappa_1 \kappa_2 := |X_1 \times X_2|$$

where $X_1, X_2 \in \text{Set}$ with $|X_i| = \kappa_i$, for $i = 1, 2$.

Exercise 10.1.2

Prove that Definition 30 is well-defined.

Remark

We may as well choose

$$X_1 = \kappa_1 \text{ and } X_2 = \kappa_2,$$

and so $\kappa_1 \kappa_2 = |\kappa_1 \times \kappa_2|$.[1]

[1] We cannot do so for cardinal sums, for $\kappa_1 \cap \kappa_2 \neq \emptyset$.

Exercise 10.1.3

Prove that the cardinal sum and product agrees with ordinal sum and products on the finite ordinals. This is the usual arithmetic on natural numbers.

Theorem 37 (Dominance of the Larger Cardinal)

Let $\kappa_1, \kappa_2 \in \mathrm{Card}$ not both finite. Then

1. $\kappa_1 + \kappa_2 = \max\{\kappa_1, \kappa_2\}$; *and*
2. *if neither κ_1 or κ_2 is 0, then $\kappa_1 \kappa_2 = \max\{\kappa_1, \kappa_2\}$.*

Exercise 10.1.4
Prove/read Theorem 37.

We can generalize the notions of cardinal sum and cardinal product. But first, a definition.

Definition 31 (I-sequence)

Let $I \in \mathrm{Set}$. By an ***I-sequence*** *of sets, we mean a definite operation*[2]

$$f : I \to \mathrm{Set}\,.$$

We write such sequences as

$$(x_i : i \in I)$$

where $x_i := f(i)$.

[2] Note that $f : I \to \mathrm{Im}(f)$ is a function by the Replacement Axiom.

Definition 32 (Generalized Cardinal Sum)

Suppose $(\kappa_i : i \in I)$ is a sequence of cardinals. We define the ***(generalized) cardinal sum*** *to be*

$$\Sigma_{i \in I} \kappa_i := \left| \bigcup_{i \in I} X_i \right|$$

where $(X_i : i \in I)$ is a sequence of ***pairwise disjoint sets*** *with $|X_i| = \kappa_i$.*

Exercise 10.1.5

Check that Definition 32 is well-defined.

Note

Observe that

$$\bigcup_{i \in I} X_i = \cup\{X_i \mid i \in I\} = \cup \operatorname{Im}(f)$$

where $(X_i : i \in I)$ is a sequence of functions and $f : I \to \mathrm{Set}$ is given by $f(i) = X_i$.

Theorem 38 (Properties of Cardinal Sum)

Let $I \in \mathrm{Set}$ be infinite, and $(\kappa_i : i \in I)$ a sequence of cardinals not all zero. Then

1. $\sup_{i \in I} \kappa_i \in \mathrm{Card}$; *and*
2. $\sum_{i \in I} \kappa_i = \max\{|I|, \sup_{i \in I} \kappa_i\}$.

Exercise 10.1.6
Prove/read Theorem 38

Example 10.1.1

We have that

$$\sum_{0 < n \in \omega} n = \max\{|\omega \setminus \{0\}|, \sup_{0 < n \in \omega} n\} = \max\{\aleph_0, \omega\} = \aleph_0.$$

Definition 33 (Generalized Cardinal Product)

Suppose $(\kappa_i : i \in I)$ is a sequence of cardinals. Then the ***cardinal product*** *is defined as*

$$\prod_{i \in I} \kappa_i := |X_1 \times X_2 \times \ldots| = \left| \bigtimes_{i \in I} X_i \right|$$

where $(X_i : i \in I)$ is a sequence of sets with $|X_i| = \kappa_i$.

Exercise 10.1.7

Check that Definition 33 is well-defined.

Note

Note that $\bigtimes_{i \in I} X_i$ *is properly defined, as*

$$i \in I \times X_i := \{(a_i : i \in I) \mid \forall i \in I, a_i \in X_i\}$$
$$= \left\{ f : I \to \bigcup_{i \in I} X_i, f(i) = X_i \right\}.$$

Remark

Once again, we might as well take $X_i = \kappa_i$*, just as we did when we defined pair products.*

Example 10.1.2

Suppose $\kappa_i = 2$ for all $i \in I$. Define a function such that

$$\bigtimes_{i \in I} 2 \to \mathcal{P}(I)$$

given by

$$(a_i : i \in I) \mapsto \{i \in I : a_i = 1\}.$$

Note that $2 = \{0, 1\}$, and so each $a_i = 0$ or 1. Clearly so, this is a bijection[3]. Consequently, we have

[3] This is the usual correspondence between subsets and characteristic functions.

$$\prod_{i \in I} 2 = \left| \bigtimes_{i \in I} 2 \right| = |\mathcal{P}(I)|$$

We claim that

$$|I| < |\mathcal{P}(I)|.$$

10.1.2 An Interlude on the Continuum Hypothesis

Theorem 39 (Cantor's Diagonalization)

$\forall I \in \mathrm{Set}$, *we have* $|I| < |\mathcal{P}(I)|$.

Proof

Clearly we have that $I \hookrightarrow \mathcal{P}(I)$ through the map $i \mapsto \{i\}$. Thus $|I| \leq |\mathcal{P}(I)|$.

Suppose to the contrary that there exists a bijection $f : I \to \mathcal{P}(I)$.

Let[4]

$$\Delta = \{i \in I : i \notin f(i)\} \subseteq I.$$

which is a set by Bounded Separation. Thus $\Delta \in \mathcal{P}(I)$. Then $\exists i_0 \in I$ such that $f(i_0) = \Delta$. Now if $i_0 \in \Delta$, then $i_0 \in f(i_0) = \Delta$, but this contradicts the membership condition which states that $i_0 \notin f(i_0)$. If $i_0 \notin \Delta$, then $i_0 \notin f(i_0) = \Delta$, but by the membership condition, it must be that $i_0 \in \Delta$, yet another contradiction. Thus such a bijection does not exist. $\square$

[4] This definition looks awfully familiar to Russell's Paradox.

This proves our earlier claim. In fact, so long as $|I| > 2$,

$$\prod_{i \in I} 2 \neq \max\{|I|, \sup_{i \in I} 2\} = \max\{|I|, 2\},$$

since $RHS = |I|$ while $LHS = |\mathcal{P}(I)|$.

Definition 34 (Cardinal Exponentiation)

$\forall \kappa, \lambda \in \mathrm{Card}$, *the* ***cardinal exponentiation*** *is defined as*

$$\kappa^\lambda := |\mathrm{Fun}(\lambda, \kappa)|,$$

where $\mathrm{Fun}(\lambda, \kappa)$ *is the set of all functions from* λ *to* κ.

Exercise 10.1.8

Prove that

$$\prod_{i < \lambda} \kappa = \kappa^\lambda.$$

As a consequence, we have that $2^\lambda = |\mathcal{P}(\lambda)|$. Then what is $|\mathcal{P}(\aleph_0)|$?

Axiom 40 (Continuum Hypothesis)

We have that

$$\aleph_0 < 2^{\aleph_0} = |\mathcal{P}(\aleph_0)|,$$

i.e. $2^{\aleph_0} = \aleph_1$.

More generally,

Axiom 41 (Generalized Continuum Hypothesis)

$\forall \kappa \in \text{Card}$, *we have*

$$2^\kappa = \kappa^+.$$

In this course, we will not assume the Continuum Hypothesis (nor for the general case).

It has been proven by Paul Cohen (1963)[5] that the Continuum Hypothesis is independent from ZFC.

[5] Cohen, P. J. (1963). The independence of the continuum hypothesis. https://www.ncbi.nlm.nih.gov/pmc/articles/PMC221287/

Bibliography

Cohen, P. J. (1963). The independence of the continuum hypothesis. https://www.ncbi.nlm.nih.gov/pmc/articles/PMC221287/.

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