PMATH352W18 Complex Analysis - Class Notes

Johnson Ng

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Chapter 1

Lecture 1 Jan 3rd 2018

1.1 Complex Numbers and Their Properties

Definition 1.1.1 (Complex Number, Complex Plane)
A ***complex number*** *is a vector in* $\mathbb{R}^2$*. The* ***complex plane****, denoted by* $\mathbb{C}$*, is a set of complex numbers,*

$$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

In $\mathbb{C}$*, we usually write*

$$\begin{aligned} 0 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & 1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ i &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} & x &= \begin{pmatrix} x \\ 0 \end{pmatrix} \\ iy &= \begin{pmatrix} 0 \\ y \end{pmatrix} \end{aligned}$$

where $x, y \in \mathbb{R}$*. Consequently, we have that*

$$x + iy = x + yi = \begin{pmatrix} x \\ y \end{pmatrix}$$

If for $x, y \in \mathbb{R}$*,* $z = x + iy$*, then* x *is called the* ***real part*** *of* z *and* y *is called the* ***imaginary part*** *of* z*, and we write*

$$\mathrm{Re}(z) = x \quad \mathrm{Im}(z) = y.$$

Note

- *It is easy to see how* $\mathbb{R}$ *is a subset of* $\mathbb{C}$*.*

- *Complex Numbers of the form* $\binom{0}{y}$ *where* $y \in \mathbb{R}$ *are called* ***purely imaginary numbers***.
- *Certain authors may prefer to denote* $i = \binom{0}{1}$.

Definition 1.1.2 (Sum and Product)
We define the sum of two complex numbers to be the usual vector sum, i.e.

$$\begin{aligned}(a+ib)+(c+id) &= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \begin{pmatrix} a+c \\ b+d \end{pmatrix} \\ &= (a+c)+i(b+d)\end{aligned}$$

where $a, b, c, d \in \mathbb{R}$.

We define the product of two complex numbers by setting $i^2 = -1$, *and by requiring the product to be* ***commutative, associative, and distributive*** *over the sum. In this setup, we have that*

$$\begin{aligned}(a+ib)(c+id) &= ac + iad + ibc + i^2bd \\ &= (ac-bd)+i(ad+bc) \qquad (1.1)\end{aligned}$$

Note
It is interesting to note that ***any complex number times zero is zero***, *just like what we have with real numbers.*

$$\forall z = x+iy \in \mathbb{C}\ x, y \in \mathbb{R}\ \ 0 \in \mathbb{C}$$
$$z \cdot 0 = (x+iy)(0+i0) = 0+i0 = 0$$

Example 1.1.1
Let $z = 2+i, w = 1+3i$. *Find* $z+w$ *and* zw.

$$\begin{aligned}z+w &= (2+i)+(1+3i) \\ &= 3+4i\end{aligned}$$

$$\begin{aligned}zw &= (2+i)(1+3i) \\ &= (2-3)+i(6+1) \quad \textit{By Equation (1.1)} \\ &= -1+7i\end{aligned}$$

Example 1.1.2
Show that every non-zero complex number has a ***multiplicative inverse****, z^{-1}, and find a formula for this inverse.*

Let $z = a + ib$ where $a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$. Then

$$\begin{aligned} & z(x+iy) = 1 \\ \iff & (ax - by) + i(ay + bx) = 1 \\ \iff & \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \iff & \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \iff & \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \iff & \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2+b^2} \begin{pmatrix} a \\ -b \end{pmatrix} \\ \iff & x + iy = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2} \end{aligned}$$

Therefore, we have that the formula for the inverse is

$$(a+ib)^{-1} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2} \tag{1.2}$$

Notation
For $z, w \in \mathbb{C}$, we write

$$\begin{array}{cc} -z = -1z & w - z = w + (-z) \\ \frac{1}{z} = z^{-1} & \frac{w}{z} = wz^{-1} \end{array}$$

Example 1.1.3
Find $\frac{(4-i)-(1-2i)}{1+2i}$.

$$\begin{aligned} \frac{(4-i)-(1-2i)}{1+2i} &= \frac{3+i}{1+2i} \\ &= (3+i)(\frac{1}{5} - i\frac{2}{5}) \\ &= 1 - i \end{aligned}$$

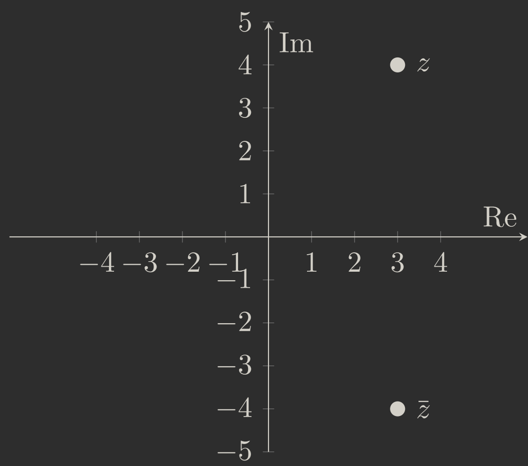
Note
The set of complex numbers is a ***field*** *under the operations of addition and multiplication. This means that $\forall u, v, w \in \mathbb{C}$,*

$$
\begin{aligned}
u + v &= v + u & uv &= vu \\
(u + v) + w &= u + (v + w) & (uv)w &= u(vw) \\
0 + u &= u & 1u &= u \\
u + (-u) &= 0 & uu^{-1} &= 1,\ u \neq 0 \\
u(v + w) &= uv + uw & &
\end{aligned}
$$

Since the distributive law holds for complex numbers, note that the ***binomial expansion works*** *for $(w+z)^n$ where $w, z \in \mathbb{C}$ and $n \in \mathbb{N}$. (I did not verify if this is still true for when $n \in \mathbb{R}$.)*

Definition 1.1.3 (Conjugate)
If $z = x + iy$ where $x, y \in \mathbb{R}$, then the ***conjugate of z*** *is given by $\bar{z} = x - iy$*

Example 1.1.4
Let $z = 3 + 4i$. Then the $\bar{z} = 3 - 4i$. Represented in the complex plane, we have the following:



We observe that on the complex plane, the conjugate of a complex number is simply its reflection on the real axis.

Definition 1.1.4 (Modulus)
We define the ***modulus*** *(length, magnitude) of $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$, to be*

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}. \tag{1.3}$$

Note
Note that this definition is consistent with the notion of the absolute value in real numbers when z is a real number, since if $y = 0$, $|z| = |x + i0| = \sqrt{x^2} = \pm x$.

Note
For $z, w \in \mathbb{C}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} \bar{\bar{z}} &= z & z + \bar{z} &= 2\operatorname{Re}(z) & z - \bar{z} &= 2i\operatorname{Im}(z) \\ z\bar{z} &= |z|^2 & |z| &= |\bar{z}| & \overline{z \pm w} &= \bar{z} \pm \overline{w} \\ \overline{zw} &= \bar{z}\overline{w} & |zw| &= |z|\,|w| & \bar{z}^n &= \overline{z^n} \end{aligned}$$

but note that $|z + w| \neq |z| + |w|$.

Also, note that the last equation is a generalization of the ***highlighted equation****.*

Note
While inequalities such as $z_1 < z_2$, where $z_1, z_2 \in \mathbb{C}$, are meaningless unless if both of them are real, $|z_1| < |z_2|$ means that the point z_1 in the complex plane is closer to the origin than the point z_2.

Proposition 1.1.1 (Basic Inequalities)

1. $|\operatorname{Re}(z)| \leq |z|$
2. $|\operatorname{Im}(z)| \leq |z|$
3. $|z + w| \leq |z| + |w|$ *Triangle Inequality*
4. $|z + w| \geq \big|\, |z| - |w| \,\big|$ *Inverse Triangle Inequality*

Proof
Note that $|z|^2 = \operatorname{Re}(z)^2 + \operatorname{Im}(z)^2$ and that we can express $|x| = \sqrt{x^2}$ for any $x \in \mathbb{R}$. 1 and 2 immediately follows from that.

To prove 3, we have that

$$\begin{aligned} |z + w|^2 &= (z + w)(\bar{z} + \overline{w}) \\ &= |z|^2 + |w|^2 + (w\bar{z} + \overline{w}z) \\ &= |z|^2 + |w|^2 + 2\operatorname{Re}(w\bar{z}) \\ &\leq |z|^2 + |w|^2 + 2\,|w\bar{z}| \quad \textit{by 1} \\ &= |z|^2 + |w|^2 + 2\,|wz| \quad \textit{since } |w\bar{z}| = |w|\,|\bar{z}| \textit{ and } |z| = |\bar{z}| \\ &= (|z| + |w|)^2 \end{aligned}$$

To prove 4, note that

$$|z| = |z + w - w| \leq |z + w| + |w| \tag{1.4}$$
$$|w| = |w + z - z| \leq |z + w| + |z| \tag{1.5}$$

Observe that

$$\begin{aligned} \text{Equation (1.4)} &\implies |z| - |w| \leq |z + w| \\ \text{Equation (1.5)} &\implies |w| - |z| \leq |z + w| \end{aligned}$$

Thus, we have that

$$|z + w| \geq |\, |z| - |w| \,|$$

as required. □

Item 3 in Proposition 1.1.1 can be generalized by the means of mathematical induction to sums involving any finite number of terms, as:

$$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n| \tag{1.6}$$

where $n \in \mathbb{N} \setminus \{0, 1\}$.

To note the induction proof, when $n = 2$, Equation (1.6) is just Item 3. If Equation (1.6) is true for when $n = m$ where $m \in \mathbb{N} \setminus \{0, 1\}$, $n = m + 1$ is also true since by Item 3,

$$\begin{aligned} |(z_1 + z_2 + \ldots + z_m) + z_{m+1}| &\leq |z_1 + z_2 + \ldots + z_m| + |z_{m+1}| \\ &\leq (|z_1| + |z_2| + \ldots + |z_m|) + |z_{m+1}| \,. \end{aligned}$$

The distance between two points $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}, x_1, x_2, y_1, y_2 \in \mathbb{R}$ is $|z_1 - z_2|$, since $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 (y_1 - y_2)^2}$ is our usual notion of the Euclidean distance of two points on a plane.

Also, note that

$$z_1 - z_2 = z_1 + (-z_2)$$

and thus if we apply our knowledge of vector representation, $z_1 - z_2$ is the directed line segment from the point z_2 to z_1.

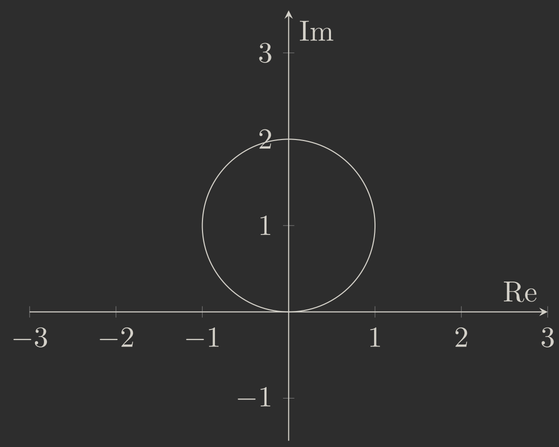
With the notion of a "distance" set on the complex plane, we can now explore upon points lying on a circle with a center z_0 and radius R, which satisfies the equation

$$|z - z_0| = R.$$

We may simply refer to this set of points as the circle $|z - z_0| = R$.

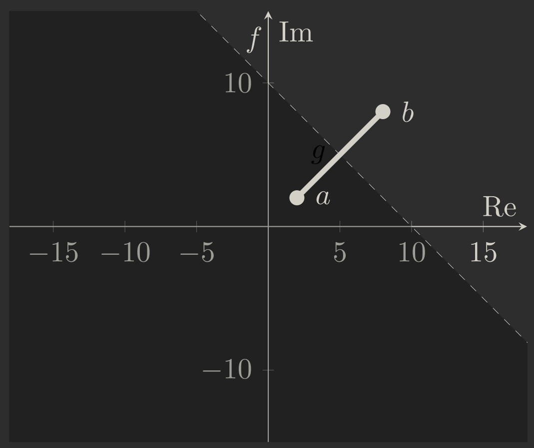
Example 1.1.5

We may describe a set $\{z \in \mathbb{C} : |z - i| = 1\}$ *as follows:*



Let $a, b \in \mathbb{C}$ *describe the set* $\{z \in \mathbb{C} : |z - a| < |z - b|\}$.

Suppose the following coordinates for a *and* b *are arbitrary,*



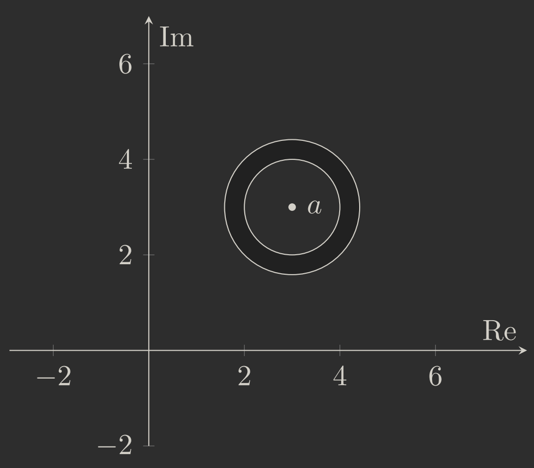
In the above, g *is the line segment that connects the points* a *and* b *on the complex plane, while* f *is the perpendicular bisector of the line segment* g*. The area described by the set* $\{z \in \mathbb{C} : |z - a| < |z - b|\}$ *is the shaded area which is below* f*.*

Chapter 2

Lecture 2 Jan 5th 2018

2.1 Complex Numbers and Their Properties (Continued)

Example 2.1.1
Let $a \in \mathbb{C}$. Describe the set $\{z \in \mathbb{C} : 1 < |z - a| < 2\}$.



Example 2.1.2
Show that every non-zero complex number has exactly two complex square roots, and find a formula for the square roots.

Let $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$, and let $w = u + iv, u, v \in \mathbb{R}$. Then

$$\begin{aligned} w^2 = z &\iff (u+iv)^2 = x+iy \\ &\iff (u^2-v^2)+i(2uv) = x+iy \\ &\iff x = u^2+v^2 \quad and \qquad (2.1) \\ & y = 2uv \qquad (2.2) \end{aligned}$$

Square both sides of Equation (2.2), and thus we have $y^2 = 4u^2v^2$.

Multiply Equation (2.1) by $4u^2$*, and we get*

$$\begin{aligned} 4u^2x &= 4u^4 - 4u^2v^2 = 4u^4 - y^2 \\ \iff 0 &= 4u^4 - 4u^2x - y^2 \\ \iff u^2 &= \frac{4x \pm \sqrt{16x^2+16y^2}}{8} \\ &= \frac{x \pm \sqrt{x^2+y^2}}{2} \end{aligned}$$

Suppose $y \neq 0$. *Note that* $x < \sqrt{x^2+y^2}$. *Thus* $u^2 = \frac{x+\sqrt{x^2+y^2}}{2} \implies u = \left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}$.

Similarly, we can get

$$v = \pm \left(\frac{-x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}$$

Note that all four choices of signs satisfy Equation (2.1). If $y > 0$*, then* u *and* v *are either both positive or both negative by Equation (2.2).*

Suppose $y = 0$. *Then we have*

$$w^2 = z = x$$

Therefore, we get

$$w = \begin{cases} \pm\left[\left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}} + i\left(\frac{-x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}\right] & y > 0 \\ \pm\left[\left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}} - i\left(\frac{-x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}\right] & y < 0 \\ \pm\sqrt{x} & y = 0, x > 0 \\ \pm i\sqrt{x} & y = 0, x < 0 \end{cases}$$

Remark
Let $z \in \mathbb{C}$. The notation $\sqrt{z}$ may represent either one of the square roots of z or both of the square roots, i.e. ***it is possible that $\sqrt{z}$ represents a set****.*

Exercise 2.1.1
Is it always okay for complex numbers such that $\sqrt{zw} = \sqrt{z}\sqrt{w}$, for $z, w \in \mathbb{C}$?

No. For example, consider $z = w = -1$. Then we have

$$\sqrt{zw} = \sqrt{1} = \pm 1$$

while

$$\sqrt{z}\sqrt{w} = i \cdot i = -1$$

and thus

$$\sqrt{zw} \neq \sqrt{z}\sqrt{w}.$$

Example 2.1.3
Find the values of $\sqrt{3 - 4i}$.

By Example 2.1.2,

$$\begin{aligned} \sqrt{3 - 4i} &= \pm \left(\sqrt{\frac{3 + \sqrt{9 + 16}}{2}} - i \sqrt{\frac{-3 + \sqrt{9 + 16}}{2}} \right) \\ &= \pm (2 - i) \end{aligned}$$

Remark
The quadratic formula holds for complex polynomials, i.e.

$$\forall a, b, c \in \mathbb{C} \quad a \neq 0 \quad \forall z \in \mathbb{C} \; az^2 + bz + c = 0,$$

the solution for z is given by

$$z_{1,2} = \frac{-b + \sqrt{b^2 - 4ac}}{b} \tag{2.3}$$

The following is a short proof.

Proof

$$\begin{aligned}
az^2 + bz + c = 0 &\iff z^2 + \frac{b}{a}z + \frac{c}{a} = 0 \\
&\iff z^2 + \frac{b}{a}z + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \\
&\iff \left(z + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2} \\
&\iff z = \frac{-b + \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

(Personal Note: where did the $-$ for the supposed $\pm$ go? Or should it really be $\pm$?)

Example 2.1.4
Solve $iz^2 - (2+3i)z + 5(1+i) = 0$.

$$\begin{aligned}
z &= \frac{2 + 3i + \sqrt{(2+3i)^2 - 4i[5(1+i)]}}{2i} \\
&= \frac{2 + 3i + \sqrt{-5 + 12i - 20i + 20}}{2i} \\
&= \frac{2 + 3i + \sqrt{15 + 8i}}{2i}
\end{aligned}$$

Note that by Example 2.1.2,

$$\begin{aligned}
\sqrt{15 - 8i} &= \pm\left[\sqrt{\frac{15 + \sqrt{225 + 64}}{2}} - i\sqrt{\frac{-15 + \sqrt{225 + 64}}{2}}\right] \\
&= \pm\left[\sqrt{\frac{15 + 17}{2}} - i\sqrt{\frac{-15 + 17}{2}}\right] \\
&= \pm(4 - i)
\end{aligned}$$

Thus we have

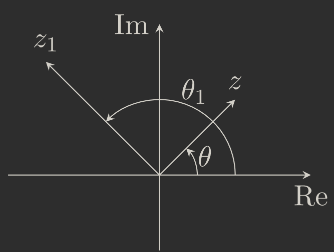
$$\begin{aligned}
z &= \frac{2 + 3i + \sqrt{15 + 8i}}{2i} \\
&= \frac{2 + 3i \pm (4 - i)}{2i} \\
&= (6 + 2i)\left(-\frac{1}{2}i\right) \ \textit{or}\ (-2 + 4i)\left(-\frac{1}{2}i\right) \quad \textit{by Example } 1.1.2 \\
&= (1 - 3i) \ \textit{or}\ (2 + i)
\end{aligned}$$

Chapter 3

Lecture 3 Jan 8th 2018

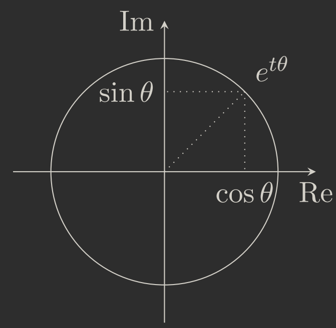
3.1 Complex Numbers and Their Properties (Continued 2)

Definition 3.1.1 (Argument of a Complex Number)
Let $z \in \mathbb{C} \setminus \{0\}$. The ***argument*** *(or the angle) of z, denoted by $\arg z$, $\operatorname{Arg} z$, or simply $\theta = \theta(z)$, is the angle modulo 2π (i.e. $0 \leq \theta < 2\pi$) between the vector defining z and the positive real axis (in the counterclockwise direction).*



Notation
Let $e^{i\theta} := \cos\theta + i\sin\theta$. Note that this definition, called ***Euler's formula****, can be derived by the extending the Taylor expansion of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for when $x \in \mathbb{C}$ (the sum of the real parts of the expansion is the Taylor expansion of cosine while the imaginary part for sine).*

Now $e^{i\theta}$ is on the unit circle.



Remark
If $z = 0$, the coordinate θ is undefined, and so it is implied that $z \neq 0$ whenever we use the polar form.

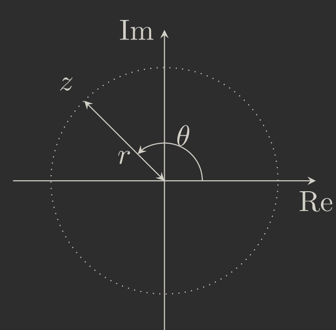
Example 3.1.1
Some examples of $\theta \in [0, 2\pi)$:

$$\begin{aligned} e^{i\frac{\pi}{4}} &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\frac{\pi}{2}} &= i \\ e^{i\frac{3\pi}{4}} &= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\pi} + 1 &= 0 \end{aligned}$$

Remark

$$\forall k \in \mathbb{Z} \ \forall \theta \in \mathbb{R} \ \ e^{i\theta} = e^{i(\theta + 2\pi k)}$$

Remark
The complex number $re^{i\theta}$, where $r > 0, \theta \in [0, 2\pi)$, represents the complex number with modulus r and argument θ.



Therefore, $\forall z \in \mathbb{C}$, we can express

$$z := |z|\, e^{i \operatorname{Arg} z}. \tag{3.1}$$

With that, we now have two representations of a complex number:

- **Cartesian representation**: $z = x + iy$ where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$
- **Polar representation**: $z = re^{i\theta}$ where $r = |z|$ and $\theta = \mathrm{Arg}\, z \in [0, 2\pi)$

To convert between the two representations, we have the following equations:

Polar $\to$ Cartesian:

$$x = r\cos\theta \quad y = r\sin\theta \tag{3.2}$$

Cartesian $\to$ Polar:

$$\begin{gathered} r = |z| \\ x \neq 0 \implies \tan\theta = \frac{y}{x} \\ x = 0 \implies \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \end{gathered} \tag{3.3}$$

On another note,

$$z = re^{i\theta} \implies \bar{z} = re^{-i\theta}$$

and

$$z \neq 0 \implies \frac{1}{z} = \frac{1}{r}e^{-i\theta} \tag{3.4}$$

Remark

$$\begin{gathered} \forall r_1, r_2 \in \mathbb{R}\ \forall \theta_1, \theta_2 \in [0, 2\pi) \\ z_1 := r_1 e^{i\theta_1} \quad z_2 := r_2 e^{i\theta_2} \end{gathered}$$

Then

$$z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

Note that $e^{ix}e^{iy} = e^{i(x+y)}$ *is true for all* $x, y \in \mathbb{R}$ *since*

$$\begin{aligned} e^{ix}e^{iy} &= (\cos x + i\sin x)(\cos y + i\sin y) \\ &= (\cos x\cos y - \sin x\sin y) + i(\cos x\sin y + \cos y\sin x) \\ &= \cos(x+y) + i\sin(x+y) \\ &= e^{i(x+y)}. \end{aligned}$$

Generalizing the above, we get that

$$\forall n \in \mathbb{Z} \;\; z = (re^{in}) = r^n e^{in\theta} \tag{3.5}$$

which is commonly known as ***deMoivre's Law****. Note that by simply generalizing the above, all we have is that $n \in \mathbb{Z}^+$. But by Equation (3.4), we can have that for $n \in \mathbb{Z}^-$, let $m = -n$, and thus*

$$z^n = \left[\frac{1}{r}e^{i(-\theta)}\right]^m = \left(\frac{1}{r}\right)^m e^{im(-\theta)} = \left(\frac{1}{r}\right)^{-n} e^{i(-n)(-\theta)} = r^n e^{i\theta}$$

This proves that deMoivre's Law also holds for when $n \in \mathbb{Z}^-$.

Observe that if $r = 1$, Equation (3.5) becomes

$$(e^{i\theta})^n = e^{in\theta} \quad \textit{for all } n \in \mathbb{Z} \setminus \{0\} \tag{3.6}$$

When written in the form

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta \quad (n \in \mathbb{Z} \setminus \{0\}) \tag{3.7}$$

this is known as ***deMoivre's formula****.*

Example 3.1.2

Equation (3.7) with $n = 2$ tells us that

$$(\cos\theta + i\sin\theta)^n = \cos 2\theta + i\sin 2\theta$$

or we can express the equation as

$$\cos^2\theta - \sin^2\theta + i2\sin\theta\cos\theta = \cos 2\theta + i \sin 2\theta$$

Equating real and imaginary parts, we have the familiar double angle trigonometric identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

3.1.1 Roots of Complex Numbers

Proposition 3.1.1 (nth Roots of a Complex Number)

$$\forall z = re^{i\theta} \in \mathbb{C} \;\; r = |z| \in \mathbb{R} \;\; \theta \in [0, 2\pi)$$
$$\exists w = se^{i\tau} \in \mathbb{C} \;\; s \in \mathbb{R} \;\; \tau \in [0, 2\pi)$$
$$\forall n \in \mathbb{Z}$$
$$w^n = \left(se^{i\tau}\right)^n = z = re^{i\theta}$$

The nth roots of z is described by the set

$$\left\{ r^{\frac{1}{n}} e^{i\left(\frac{\theta+2\pi k}{n}\right)} : k = 0, 1, ..., n-1 \right\} \tag{3.8}$$

Proof

$$s^n = r \iff s = r^{\frac{1}{n}}$$
$$e^{in\theta} = e^{i\tau} \iff \theta = \frac{\tau + 2\pi k}{n}$$

Therefore, the set that describes the nth roots of z is

$$\left\{ w = r^{\frac{1}{n}} e^{i\left(\frac{\theta+2\pi k}{n}\right)} : k = 0, 1, ..., n-1 \right\}$$

Remark (nth Roots of Unity)
The ***nth roots of unity*** *is a direct consequence of Proposition 3.1.1 where we solve for the equation $z^n = 1$ for any $z \in \mathbb{C}, n \in \mathbb{Z}$.*

The set that describes the nth roots of unity is

$$\left\{ e^{i\theta} : \theta = \frac{2\pi k}{n}, k = 0, 1, ..., n-1 \right\} \tag{3.9}$$

It is easy to see how the nth roots of unity **partitions the unit circle into n parts**.

Example 3.1.3
Find the cubic roots of $-2 + 2i$.

Let $z = -2 + 2i$. Note that $|z| = 2\sqrt{2}$ and $\operatorname{Arg} z = \frac{3\pi}{4}$.

Therefore, in polar form, $z = 2\sqrt{2}e^{i\frac{3\pi}{4}}$.

Let $w = re^{i\theta}$, where $\theta \in [0, 2\pi)$, and $w^3 = z$. Then

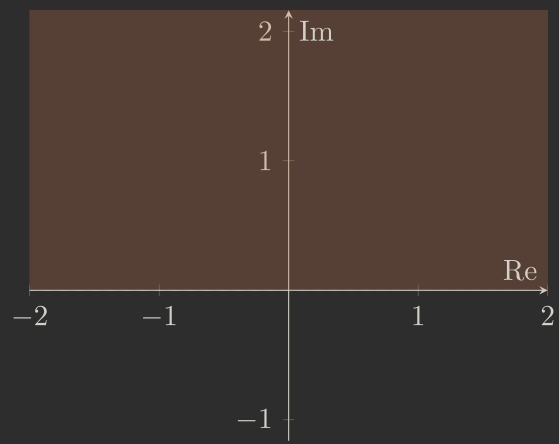
$$r = (2\sqrt{2})^{\frac{1}{3}}$$
$$\theta = \frac{\frac{3\pi}{4} + 2\pi k}{3}, \quad k = 0, 1, 2$$

The set that describes the cubic root of $-2 + 2i$ is thus

$$\left\{ (2\sqrt{2})^{\frac{1}{3}} e^{i\theta} : \theta = \frac{\frac{3\pi}{4} + 2\pi k}{3}, k = 0, 1, 2 \right\}$$

Example 3.1.4

Describe the set $\{z \in \mathbb{C} : \left|\operatorname{Arg} z - \frac{\pi}{2}\right| < \frac{\pi}{2}\}$. *(Note:* $\operatorname{Arg} z \in [0, 2\pi)$*)*



Exercise 3.1.1

Solve

1. $z^4 = -1$

$$\textit{Let } z = re^{i\theta}$$

$$r = |-1| = 1 \quad \theta = \frac{\pi + 2\pi k}{4} = \frac{(2k+1)\pi}{4}, \quad k = 0, 1, 2, 3$$

2. $z^4 = -1 + \sqrt{3}i$

$$\textit{Let } z = re^{i\theta}$$

$$r = \left|-1 + \sqrt{3}i\right| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\theta = \frac{\frac{2\pi}{3} + 2\pi k}{4} = \frac{(2k + \frac{2}{3})\pi}{4}, \quad k = 0, 1, 2, 3$$

Chapter 4

Lecture 4 Jan 10th 2018

4.1 Examples for nth Roots of Unity

Recall that the nth roots of unity are given by $e^{i\frac{2\pi k}{n}}, k = 0, 1, ..., n-1$.

Exercise 4.1.1
Let z be any nth root of unity other than 1. Show that

$$z^{n-1} + z^{n-2} + \ldots + z + 1 = 0 \tag{4.1}$$

Proof
By the Sum of Finite Geometric Terms,

$$z^{n-1} + z^{n-2} + \ldots + z + 1 = \frac{1 - z^n}{1 - z}.$$

Since $z^n = 1$, RHS is thus zero, which in turn completes the proof.

As an aside, if we wish to remove the restriction that z can also be 1, we may consider that

$$z^n - 1 = (z-1)(1 + z + \ldots + z^{n-1})$$

Since $z^n = 1$, LHS is zero. Then either $z = 1$ or $(1 + z + \ldots + z^{n-1}) = 0$.

Exercise 4.1.2
Consider the $n-1$ diagonals of a regular n-gon, inscribed in a circle of radius 1, obtained by connecting one vertex on the n-gon to all its other vertices.

For example, if we are given $n = 6$, we obtain the following diagram.

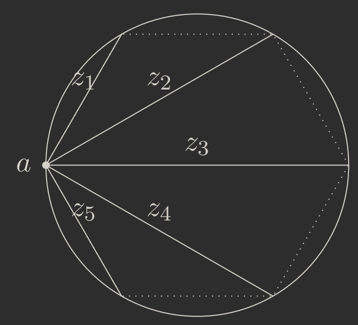


Figure 4.1: $n = 6$, where a is an arbitrary vertex on the hexagon

Show that the product of the lengths of these diagonals is equal to n.

Proof

Note that Figure 4.1 can be translated into Figure 4.2.

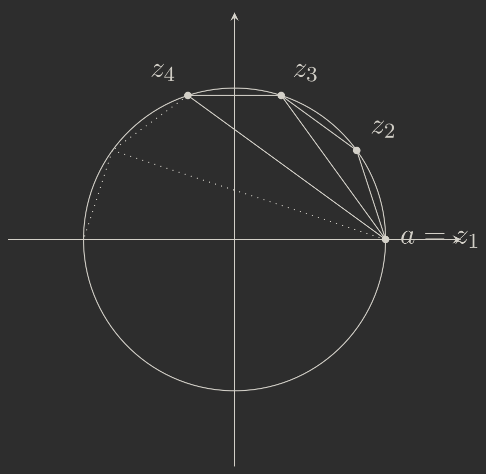


Figure 4.2: A regular n-gon with the roots of unity on its vertices

Thus the equation that we wish to prove becomes

$$|1 - z_2|\,|1 - z_3| \ldots |1 - z_n| = n \tag{4.2}$$

Note that $z_2, ..., z_n$ are the nth roots of unity other than 1.

Let z be a variable and consider the polynomial

$$P(z) := 1 + z + z^2 + \ldots + z^{n-1} \tag{4.3}$$

Since the roots of $P(z)$ are the nth roots of unity other than 1, we can factorize Equation (4.3) into

$$P(z) = (z - z_2)(z - z_3)\dots(z - z_n)$$

Now let $z = 1$ and take the modulus of $P(z)$, and we get Equation (4.2).

Exercise 4.1.3
Let $n \in \mathbb{N}$. Show that $\sum_{j=0}^{n} \binom{3n}{3j} = \frac{2^{3n} + 2(-1)^n}{3}$.

Proof
Let $\alpha = e^{i\frac{2\pi}{3}}$. Then α is a cubic root of unity, i.e. $\alpha^3 = 1$, and from Exercise 4.1.1, $1 + \alpha + \alpha^2 = 0$.

Consider

$$\begin{aligned}(1+1)^{3n} = &\binom{3n}{0} + \binom{3n}{1} + \binom{3n}{2} + \binom{3n}{3} + \binom{3n}{4} \\ &+ \binom{3n}{5} + \binom{3n}{6} + \dots + \binom{3n}{3n}\end{aligned} \tag{4.4}$$

$$\begin{aligned}(1+\alpha)^{3n} = &\binom{3n}{0} + \binom{3n}{1}\alpha + \binom{3n}{2}\alpha^2 + \binom{3n}{3} + \binom{3n}{4}\alpha \\ &+ \binom{3n}{5}\alpha^2 + \binom{3n}{6} + \dots + \binom{3n}{3n}\end{aligned} \tag{4.5}$$

$$\begin{aligned}(1+\alpha^2)^{3n} = &\binom{3n}{0} + \binom{3n}{1}\alpha^2 + \binom{3n}{2}\alpha + \binom{3n}{3} + \binom{3n}{4}\alpha^2 \\ &+ \binom{3n}{5}\alpha + \binom{3n}{6} + \dots + \binom{3n}{3n}\end{aligned} \tag{4.6}$$

Adding Equation (4.4), Equation (4.5) and Equation (4.6), we observe that the terms with coefficients $\binom{3n}{k}$ where k is not a multiple of 3 sums to 0 as given by $1 + \alpha + \alpha^2 = 0$, and *therefore we obtain*

$$\begin{aligned}
2^{3n} + (1+\alpha)^{3n} + (1+\alpha^2)^{3n} &= 3\sum_{j=0}^{n}\binom{3n}{3j} \\
\frac{1}{3}\left[2^{3n} + (1+\alpha)^{3n} + (1+\alpha^2)^{3n}\right] &= \sum_{j=0}^{n}\binom{3n}{3j} \\
\frac{1}{3}\left[2^{3n} + (-\alpha^2)^{3n} + (-\alpha)^{3n}\right] &= \sum_{j=0}^{n}\binom{3n}{3j} \quad \textit{since } 1+\alpha+\alpha^2 = 0 \\
\frac{1}{3}\left[2^{3n} + (-1)^{n} + (-1)^{n}\right] &= \sum_{j=0}^{n}\binom{3n}{3j} \quad \textit{since } \alpha^3 = 1 \\
\frac{2^{3n} + 2(-1)^n}{3} &= \sum_{j=0}^{n}\binom{3n}{3j}
\end{aligned}$$

as required.

Exercise 4.1.4

Note that we can define $\operatorname{Arg} z$ *in any interval of length* 2π*, i.e. it is not necessary that* $\operatorname{Arg} z \in [0, 2\pi)$*.*

For example, if we restrict $\operatorname{Arg} z \in [-\pi, \pi]$*, then we can write*

$$\operatorname{Arg}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -\frac{3\pi}{4}$$

Let z *be on the unit circle and* $\operatorname{Arg} z \in [-\pi, \pi]$*. Suppose that* $z \notin \mathbb{R}$*, i.e.* $z \neq 1, z \neq -1$*. Show that*

$$\operatorname{Arg}\left(\frac{z-1}{z+1}\right) = \begin{cases} \frac{\pi}{2} & \operatorname{Im} z > 0 \\ -\frac{\pi}{2} & \operatorname{Im} z < 0 \end{cases}$$

Proof

Note that $\forall w_1, w_2 \in \mathbb{C}$*, where* $\operatorname{Arg} w_1 = \tau_1, \operatorname{Arg} w_2 = \tau_2$ *for* τ_1, τ_2 *in the same* 2π*-interval,*

$$\operatorname{Arg}\frac{w_1}{w_2} = \frac{e^{i\tau_1}}{e^{i\tau_2}} \equiv e^{i(\tau_1 - \tau_2)} = \operatorname{Arg} w_1 - \operatorname{Arg} w_2 \tag{4.7}$$

in modulo 2π*.*

Suppose $\operatorname{Im} z > 0$*. Let* $\theta_1 = \operatorname{Arg}(z-1)$ *and* $\theta_2 = \operatorname{Arg}(z+1)$*. Consider Figure 4.3. Note that since both* $\theta_1, \theta_2 \in [0, \pi]$*, we have that* $\theta_1 - \theta_2 \in [-\pi, \pi]$*, and thus Equation (4.7) holds* *true without the need of the condition of being in modulo 2π. We observe that*

$$\frac{\pi}{2} = \theta_2 + \pi - \theta_1$$
$$\theta_1 - \theta_2 = \frac{\pi}{2}$$

as desired.

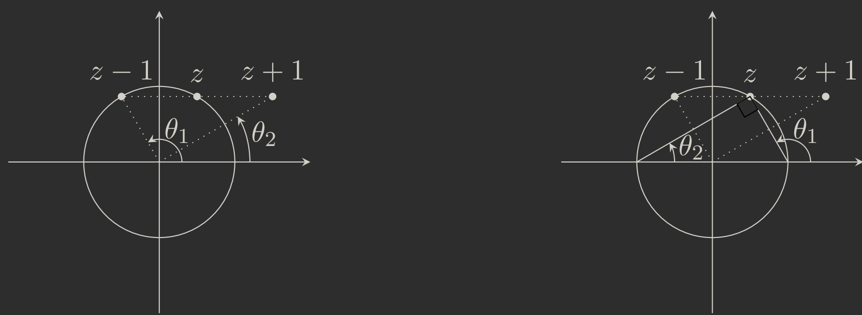


Figure 4.3: (Right) Depicted question, (Left) Translated Angles

Similarly, we can obtain $\theta_1 - \theta_2 = -\frac{\pi}{2}$ for when $\operatorname{Im} z < 0$. This completes the proof.

Exercise 4.1.5
Let $f(z) = e^z$ for $z \in \mathbb{C}$. Let $A = \{z = x + iy \in \mathbb{C} : x \leq 1, y \in [0, \pi]\}$. Describe the image of $f(A)$.

Solution
Firstly, note that

$$e^z = e^{x+iy}$$
$$e^x \in (0, e]$$
$$y \in [0, \pi]$$

Figure 4.4: (Right) Domain of $f(A)$, (Left) Image of $f(A)$

It is clear that the image will be in on the positive side of the imaginary-axis. Also, since $e^x \in (0, e]$, *we get the right graph represented in Figure 4.4. The image of* $f(A)$ *is described in the left image of Figure 4.4.*

Chapter 5

Lecture 5 Jan 12 2018

5.1 Complex Functions

5.1.1 Limits

Definition 5.1.1 (Convergence)
A sequence of complex numbers $z_1, z_2, z_3, \ldots$ ***converges*** *to $z \in \mathbb{C}$ if*

$$\lim_{n\to\infty} |z_n - z| = 0 \tag{5.1}$$

or we may say

$$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n > N \quad |z_n - z| < \varepsilon \tag{5.2}$$

Note
If $\{z_n\}_{n\in\mathbb{N}}$ converges to z, we may write $\lim_{n\to\infty} z_n = z$ or $z_n \to z$ (as $n \to \infty$).

Example 5.1.1
For $|z| > 1$, does $\{\frac{1}{z^n}\}_{n=1}^{\infty}$ converge? Explain.

Solution
We claim that the limit is 0. Since $|z| > 1$, we have that

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{1}{z^n} - 0\right| &= \lim_{n\to\infty}\left|\frac{1}{z}\right|^n \\ &= 0\end{aligned}$$

Another way to prove this, since $|z| > 1 \implies 0 < \left|\frac{1}{z}\right| < 1$,

$$\forall \varepsilon = \left|\frac{1}{z}\right| > 0$$
$$\left|\frac{1}{z^n} - 0\right| = \left|\frac{1}{z}\right|^n < \left|\frac{1}{z}\right| = \varepsilon$$

Definition 5.1.2 (Convergence for Complex Functions)
$\forall \Omega \subseteq \mathbb{C}$, *let* $f : \Omega \to \mathbb{C}$. *We say that*

$$\lim_{z \to z_0} f(z) = L \tag{5.3}$$

for some $L \in \mathbb{C}$ *if for every sequence* $\{z_n\}_n \subseteq \Omega$ *(not including* z_0 *if it is in* Ω*), we have that*

$$z_n \to z_0 \implies f(z_n) \to L \tag{5.4}$$

Note that L *need not be in* Ω.

Example 5.1.2
Let $f(z) = \frac{\bar{z}}{z}, z \in \mathbb{C} \setminus \{0\}$. *Find* $\lim_{z\to 0} f(z)$.

Solution
Suppose $z = x \in \mathbb{R} \setminus \{0\}$. *Then* $f(z) = f(x) = \frac{x}{x} = 1$.

Suppose $z = iy, y \in \mathbb{R} \setminus \{0\}$. *Then* $f(z) = f(iy) = \frac{-iy}{iy} = -1$.

Therefore, the limit $\lim_{z\to 0} f(z)$ *does not exist.*

Exercise 5.1.1
Show that $z_n \to z \iff \mathrm{Re}(z_n) \to \mathrm{Re}(z) \wedge \mathrm{Im}(z_n) \to \mathrm{Im}(z)$.
(Hint: $|\mathrm{Re}(z)|, |\mathrm{Im}(z)| \leq |z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$*)*

Solution
Suppose $z_n \to z$. *Then* $\forall \varepsilon_0 > 0\ \exists N \in \mathbb{N}\ \forall n > N\ \ |z_n - z| < \varepsilon$. *Note once and for all that*

$$\mathrm{Re}(z_n - z) = \mathrm{Re}(z_n) - \mathrm{Re}(z)$$
$$\mathrm{Im}(z_n - z) = \mathrm{Im}(z_n) - \mathrm{Im}(z).$$

Thus

$$\begin{aligned}
|\mathrm{Re}(z_n) - \mathrm{Re}(z)| &= |\mathrm{Re}(z_n - z)| \\
&\leq |z_n - z| < \varepsilon \\
|\mathrm{Im}(z_n) - \mathrm{Im}(z)| &= |\mathrm{Im}(z_n - z)| \\
&\leq |z_n - z| < \varepsilon
\end{aligned}$$

For the other direction,

$$\forall \frac{\varepsilon}{2} > 0 \ \exists N_0 \in \mathbb{N} \ \forall n > N_0 \ \ |\mathrm{Re}(z_n) - \mathrm{Re}(z)| < \frac{\varepsilon}{2}$$
$$\forall \frac{\varepsilon}{2} > 0 \ \exists N_1 \in \mathbb{N} \ \forall n > N_1 \ \ |\mathrm{Im}(z_n) - \mathrm{Im}(z)| < \frac{\varepsilon}{2}.$$

Therefore,

$$\begin{aligned} |z_n - z| &= |\mathrm{Re}(z_n) + \mathrm{Im}(z_n) - \mathrm{Re}(z) - \mathrm{Im}(z)| \\ &\leq |\mathrm{Re}(z_n) - \mathrm{Re}(z)| + |\mathrm{Im}(z_n) - \mathrm{Im}(z)| \\ &\leq \varepsilon \end{aligned}$$

□

5.1.2 Continuity

Definition 5.1.3 (Continuity)
$\forall \Omega \subseteq \mathbb{C}$, let $f : \Omega \to \mathbb{C}$. We say that f is ***continuous*** *at $z_0 \in \Omega$ if*

1. $\forall \{z_n\}_{n \in \mathbb{N}}$
   $z_n \to z_0 \implies f(z_n) \to f(z_0)$
2. $\forall \varepsilon > 0 \ \exists \delta > 0$
   $|z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon$

Remark

1. *f is continuous on Ω if it is continuous on every point in Ω.*
2. *We may* ***split*** *f into its feal and imaginary parts, i.e.*

$$f(z) = f(x, y) = u(x, y) + iv(x, y) \tag{5.5}$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$.

Example 5.1.3
Let $f : \mathbb{C} \to \mathbb{C}$ and for $z \in \mathbb{C}, f(z) = \frac{\bar{z}}{z}$. To split f into real and imaginary parts:

$$\begin{aligned} f(z) &= \frac{\bar{z}}{z} \\ &= (x + iy)\left(\frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}\right) \\ &= \frac{x^2 - y^2}{x^2 + y^2} + i\frac{(-2xy)}{x^2 + y^2} \end{aligned}$$

and we get

$$u(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$
$$v(x, y) = -\frac{2xy}{x^2 + y^2}$$

Chapter 6

Lecture 6 Jan 15th 2018

6.1 Continuity (Continued)

Exercise 6.1.1
Let $f : \Omega \to \mathbb{C}$. Prove that $f(z)$ is continuous at $z_0 = x_0 + iy_0 \in \mathbb{C} \iff$ functions $u, v : \mathbb{R}^2 \to \mathbb{R}$, such that $f(z) = u(x, y) + iv(x, y)$ are both continuous at (x_0, y_0).

Solution
We shall first prove the forward direction. Suppose that $f(z)$ is continuous at $z_0 = x_0+iy_0 \in \mathbb{C}$. By Definition 5.1.3, $\forall \{z_n\}_{n\in\mathbb{N}} \subseteq \Omega$, $z_n \to z_0 \implies f(z_n) \to f(z_0)$. By Exercise 5.1.1,

$$\begin{aligned} z_n \to z_0 &\iff \operatorname{Re} z_n \to \operatorname{Re} z_0 \wedge \operatorname{Im} z_n \to \operatorname{Im} z_0 \\ &\iff x_n \to x_0 \wedge y_n \to y_0 \end{aligned} \tag{6.1}$$

where $z_n = x_n + iy_n$ for $x_n, y_n \in \mathbb{R}$.

Similarly so, and by Equation (5.5),

$$f(z_n) + f(z_0) \iff u(x_n, y_n) \to u(x_0, y_0) \wedge v(x_n, y_n) \to v(x_0, y_0) \tag{6.2}$$

Putting together Equation (6.1) and Equation (6.2), we get

$$(x_n, y_n) \to (x_0, y_0) \implies u(x_n, y_n) \to u(x_0, y_0) \wedge v(x_n, y_n) \to v(x_0, y_0)$$

as desired.

The proof of the other direction is simply a reversed process of the above. $\square$

6.2 Differentiability

Definition 6.2.1 (Neighbourhood)
For $z_0 \in \mathbb{C}, r \in \mathbb{R}$, let

$$D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}. \tag{6.3}$$

On the complex plane, this is seen as a open disk centered around the point z_0 with radius r, as shown below.

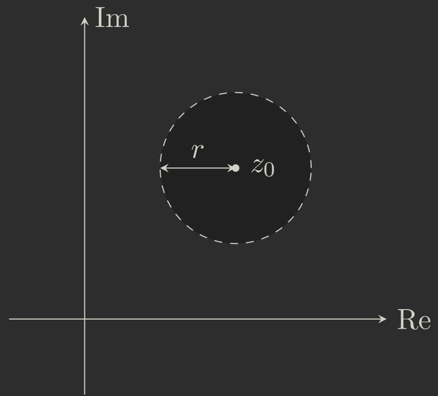


Figure 6.1: Open disk centered around z_0 with radius r

This open disk is called a ***neighbourhood*** *of z_0.*

Definition 6.2.2 (Differentiable/Holomorphic)
Let $f(z)$ be defined in a neighbourhood of $z_0 \in \mathbb{C}$. We say f is ***differentiable/holomorphic*** *at z_0 if for some $h \in \mathbb{C}$,*

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{6.4}$$

exists. If such a limit exists, we denote the limit by $f'(z_0)$.

Remark
$h \in \mathbb{C} : h$ need not necessarily be real. In this sense, h approaches 0 from ***any direction*** *around $0 \in \mathbb{C}$.*

Example 6.2.1
For $z \in \mathbb{C} \setminus \{0\}$, let $f(z) = \frac{1}{z}$. Let $z_0 \in \mathbb{C} \setminus \{0\}$. Note that

$$\lim_{h \to 0} \frac{\frac{1}{z_0 + h} - \frac{1}{z_0}}{h} = \lim_{h \to 0} \frac{1}{h} \left[\frac{-h}{(z_0 + h) z_0} \right] = -\frac{1}{z_0^2}$$

Thus f is holomorphic at any $z \in \mathbb{C} \setminus \{0\}$, and hence $f'(z) = -\frac{1}{z}$.

Example 6.2.2
For $z \in \mathbb{C}$, let $f(z) = \bar{z}$. Let $z_0 \in \mathbb{C}$. Notice that

$$\lim_{h \to 0} \frac{\overline{z_0 + h} - \bar{z}}{h} = \lim_{h \to 0} \frac{\bar{h}}{h}.$$

From Example 5.1.2, we know that such a limit does not exist. Thus f is not holomorphic on any $z \in \mathbb{C}$.

Exercise 6.2.1 (Holomorphic Functions Properties)
If f, g are holomorphic at $z \in \mathbb{C}$, prove that

1. *$f + g$ is holomorphic and $(f+g)' = f' + g'$.*
2. *fg is holomorphic and $(fg)' = f'g + fg'$.*
3. *if $g(z) \neq 0$, $\frac{f}{g}$ is holomorphic and $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$.*

Solution

1. *For $f + g$,*

$$\begin{aligned} &\lim_{h \to 0} \frac{f(z+h) + g(z+h) - f(z) - g(z)}{h} \\ =&\lim_{h \to 0} \left[\frac{f(z+h) - f(z)}{h} + \frac{g(z+h) - g(z)}{g} \right] \\ =&f'(z) + g'(z) \end{aligned}$$

Thus $(f+g)' = f' + g'$.

2. *For fg,*

$$\begin{aligned} &\lim_{h \to 0} \frac{f(z+h)g(z+h) - f(z)g(z)}{h} \\ =&\lim_{h \to 0} \frac{f(z+h)g(z+h) + f(z)g(z+h) - f(z)g(z+h) - f(z)g(z)}{h} \\ =&\lim_{h \to 0} \left[\frac{f(z+h) - f(z)}{h} g(z+h) + f(z) \frac{g(z+h) - g(z)}{h} \right] \\ =&f'(z)g(z) + f(z)g'(z) \end{aligned}$$

Therefore, $(fg)' = f'g + fg'$.

3. *When* $\forall z \in \mathbb{C}\ g(z) \neq 0$, *for* $\frac{f}{g}$,

$$\begin{aligned}
&\lim_{h\to 0} \frac{\frac{f(z+h)}{g(z+h)} - \frac{f(z)}{g(z)}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{f(z+h)g(z) - f(z)g(z+h)}{g(z+h)g(z)}\right] \\
&= \lim_{h\to 0} \frac{1}{g(z+h)g(z)}\left[\frac{f(z+h)g(z) + f(z)g(z) - f(z)g(z) - f(z)g(z+h)}{g}\right] \\
&= \lim_{h\to 0} \frac{1}{g(z+h)g(z)}\left[\frac{[f(z+h) - f(z)]g(z) - f(z)[g(z+h) - g(z)]}{h}\right] \\
&= \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}
\end{aligned}$$

Hence, $\frac{f}{g} = \frac{f'g - fg'}{g^2}$

Note
If we look at the example above from the perspective of f *being treated as a real-valued function, i.e.* $f(z) = u(x, y) + iv(x, y)$ *where* $u, v : \mathbb{R}^2 \to \mathbb{R}$ *and* $z = x + iy$, *observe that* $\forall (x, y) \in \mathbb{R}^2, (x, y) \mapsto (x, -y)$, *which we see that* u *and* v *are partially differentiable in* $\mathbb{R}^2$.

We will now look into this "discrepancy".

6.2.1 Cauchy-Riemann Equations

Consider the following function taken from Equation (6.4),

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{6.5}$$

While h may approach $0 \in \mathbb{C}$ from infinitely many sides on the complex plane, we will consider 2 cases.

Case 1: $h \to 0$ *via the real axis*

In this case, $h = x + i(0)$ and $x \to 0 \in \mathbb{R}$. Then Equation (6.5) gives

$$\begin{aligned}
f'(z_0) &= \lim_{x\to 0} \frac{u(x_0 + x, y_0) + iv(x_0 + x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{x} \\
&= \lim_{x\to 0} \left[\frac{u(x_0 + x, y_0) - u(x_0, y_0)}{x} + i\frac{v(x_0 + x, y_0) - v(x_0, y_0)}{x}\right] \\
&= \frac{\partial u}{\partial x}\bigg|_{(x_0,y_0)} + i\frac{\partial v}{\partial x}\bigg|_{(x_0,y_0)} \qquad (6.6)
\end{aligned}$$

Case 2: $h \to 0$ via the imaginary axis

In this case, $h = 0 + iy$ and $y \to 0 \in \mathbb{R}$. In a similar fashion, Equation (6.5) becomes

$$\begin{aligned} f'(z_0) &= \lim_{y \to 0} \left[\frac{u(x_0, y_0 + y) - u(x_0, y_0)}{iy} + \frac{v(x_0, y_0 + y) - v(x_0, y_0)}{y} \right] \\ &= \frac{1}{i} \cdot \frac{\partial u}{\partial y}\bigg|_{(x_0, y_0)} + \frac{\partial v}{\partial y}\bigg|_{(x_0, y_0)} \end{aligned} \tag{6.7}$$

Note that since $f'(z_0)$ exists, the real and imaginary part of Equation (6.6) and Equation (6.7) must equate. Also note that $\frac{1}{i} = -i$. With that, we obtain the following theorem.

Theorem 6.2.1 (Cauchy-Riemann Equations)
If $f(z)$ is holomorphic at $z_0 = x_0 + iy_0 \in \mathbb{C}$ where $x_0, y_0 \in \mathbb{R}$, then, at (x_0, y_0),

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{6.8}$$

Chapter 7

Lecture 7 Jan 17th 2018

7.1 Differentiability (Continued)

7.1.1 Cauchy-Riemann Equations (Continued)

It is natural to wonder if the **converse** of Theorem 6.2.1 is true. We present the following example.

Example 7.1.1
Let

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \textit{if } z \neq 0 \\ 0 & \textit{if } z = 0 \end{cases}$$

Check if

1. *f is holomorphic at* 0.
2. *Theorem 6.2.1 holds at* $(0, 0)$.

Proof

1. *Observe that by letting $h = x_h + iy_h$ where $x_h, y_h \in \mathbb{R}$,*

$$\lim_{h \to 0} \frac{\frac{\overline{0+h}^2}{0+h} - 0}{h} = \lim_{h \to 0} \frac{\bar{h}^2}{h} = \lim_{x_h + iy_h \to 0} \left(\frac{x_h - iy_h}{x_h + iy_h} \right)^2$$

Consider $y_h = kx_h$, for $k \in \mathbb{R} \setminus \{0\}$. Then

$$\lim_{x_h \to 0} \left(\frac{x_h - ikx_h}{x_h + ikx_h} \right)^2 = \left(\frac{1 - ik}{1 + ik} \right)^2,$$

where we see that the limit depends on the value of k. Therefore, the limit DNE. Hence f is not holomorphic at 0.

2. *Let $z = x + iy$ for $x, y \in \mathbb{R}$. Then*

$$\frac{\bar{z}^2}{z} = \frac{(x-iy)^2}{x+iy} = \frac{(x-iy)^3}{x^2+y^2} = \frac{x^3-3xy^2}{x^2+y^2} + i\frac{(-3x^2y+y^3)}{x^2+y^2}$$

Therefore, we obtain

$$u(x,y) = \begin{cases} \frac{x^3-3xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$
$$v(x,y) = \begin{cases} \frac{y^3-3x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Observe that

$$\frac{\partial u}{\partial x}\bigg|_{(0,0)} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = 1$$
$$\frac{\partial v}{\partial y}\bigg|_{(0,0)} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = 1$$
$$\text{and}$$
$$\frac{\partial u}{\partial y}\bigg|_{(0,0)} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = 0$$
$$\frac{\partial v}{\partial x}\bigg|_{(0,0)} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = 0$$

satisfies Equation (6.8).

This illustrates that the converse of Theorem 6.2.1 is not true. We will, however, show that the converse will be true given an extra condition.

Theorem 7.1.1 (Conditional Converse of CRE)
Let $z_0 = x_0 + iy_0 \in \Omega \subseteq \mathbb{C}, x_0, y_0 \in \mathbb{R}$, and $u, v : \mathbb{R}^2 \to \mathbb{R}, f = u + iv : \Omega \to \mathbb{C}$. If

1. *the partials of u, v exist in a neighbourhood of (x_0, y_0),*
2. *the partials of u, v are continuous at (x_0, y_0), and*
3. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ *and* $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ *at (x_0, y_0),*

then f is holomorphic at z_0.

A proof of the theorem is in page 36 of Newman and Bak (recommended text of PMATH352W18). I may include the proof whenever I am free.

7.1.2 Power Series

Definition 7.1.1 (Power Series)
A ***power series*** *in* $\mathbb{C}$ *is an infinite series of the form*

$$\sum_{n \in \mathbb{N}} c_n z^n, \tag{7.1}$$

where each $c_n \in \mathbb{C}$ *is the coefficient of* z *of the* n*-th power.*

In this subsection, we are interested to see if Equation (7.1) converges.

Recall the notion of convergence in series from $\mathbb{R}$. Equation (7.1) converges if the sequence of partial sums $\{S_N\}$ converges as $N \to \infty$, where

$$S_N := \sum_{n=0}^{N} c_n z^n$$

In other words, using the same definition of S_N,

$$\begin{gathered} \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \setminus \{0\} \quad \forall n > N \\ |S_n - L| < \varepsilon \end{gathered}$$

where $L \in \mathbb{C}$ is the limit that the sequence converges to.

We also know that Equation (7.1) converges absolutely if $\sum_{n=0}^{\infty} |c_n| \, |z|^n$ converges. This is a stronger statement (i.e. absolute convergence $\implies$ convergence)

$$\because \left| \sum_{n=0}^{N} c_n z^n \right| \leq \sum_{n=0}^{N} |c_n| \, |z|^n \quad \text{for each } N \in \mathbb{N}$$

Example 7.1.2
$\sum_{n=0}^{\infty} z^n$ *converges absolutely for* $|z| < 1$.

Note that the partial sum of a geometric series is

$$\sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r}$$

and so the limit as $N \to \infty$ *exists if* $|r| < 1$*, and hence we see that*

$$\sum_{n=0}^{N} r^n \to \frac{1}{1 - r}$$

if $|r| < 1$ *as* $N \to \infty$.

However, if $|z| = 1$*, the power series diverges.*

Another note that we shall point out is that if Equation (7.1) converges absolutely for some $z_0 \in \mathbb{C}$, then it converges absolutely for any z where $|z| < |z_0|$.

These notions, in turn, begs the question of **what is the largest possible $|z_0|$ for the series to converge absolutely**.

Chapter 8

Lecture 8 Jan 19 2018

8.1 Power Series (Continued)

8.1.1 Radius of Convergence

Theorem 8.1.1 (Convergence in the Radius of Convergence)
For any power series $\sum_{n \in \mathbb{N}} c_n z^n$, $\exists 0 \leq R < \infty$, such that

1. $|z| < R \implies$ *series converges absolutely.*
2. $|z| > R \implies$ *series diverges.*

*Moreover, R is given by **Hadamard's Formula**:*

$$\frac{1}{R} := \limsup_{n \to \infty} |c_n|^{\frac{1}{n}} \tag{8.1}$$

Remark

1. *R is called the **radius of convergence** of the series. $\{z \in \mathbb{C} : |z| < R\}$ is called the disk of convergence of the series.*
2. *Recall the definition of the **limit supremum***

$$\limsup_{n \to \infty} a_n := \lim_{n \to \infty} \left(\sup_{m \geq n} a_m \right) \tag{8.2}$$

which we may colloquially say as the "highest peak 'reached' by a_n's as $n \to \infty$"

Proposition 8.1.1 (A Property of limsup)

$$\forall \{a_n\}_{n\in\mathbb{N}} \quad L := \limsup_{n\to\infty} a_n \implies$$
$$\forall \varepsilon > 0 \quad \exists N > 0 \quad \forall n > N$$
$$L - \varepsilon < a_n < L + \varepsilon$$

(Proof to be included)

Proof (Theorem 8.1.1)
Let $L := \frac{1}{R} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$. *Clearly,* $L \geq 0$.

1. *Suppose* $|z| < R$. $\exists \varepsilon > 0, r := |z|(L+\varepsilon)$ *such that* $0 < r < 1$. *By Proposition 8.1.1,* $\exists N \in \mathbb{N}, \forall n > N, |c_n|^{\frac{1}{n}} < L + \varepsilon$.

   Now since $L = \frac{1}{R}$,

   $$\sum_{n=N}^{\infty} |c_n| |z|^n = \sum_{n=N}^{\infty} (|c_n|^{\frac{1}{n}} |z|)^n < \sum_{n=N}^{\infty} r^n$$

   and since $0 < r < 1$, *the final summation converges (as it is a geometric sum). Thus by comparison test,* $\sum_{n=N}^{\infty} |c_n| |z|^n$ *converges.*

   We may also proceed with noticing that the partial sum of $\sum_{n=N}^{\infty} |c_n| |z|^n$ *is* ***bounded and monotonic****, which shows that the series converges.*

2. *Suppose* $|z| > R$. $\exists \varepsilon > 0, r := |z|(L-\varepsilon)$ *such that* $r > 1$. *By Proposition 8.1.1,* $\exists N \in \mathbb{N}, \forall n > N, |c_n|^{\frac{1}{n}} > L - \varepsilon$. *Then analogous to the proof above,*

   $$\sum_{n=N}^{\infty} |c_n| |z|^n = \sum_{n=N}^{\infty} (|c_n|^{\frac{1}{n}} |z|)^n > \sum_{n=N}^{\infty} r^n$$

   where the final summation diverges, and thus implying that $\sum_{n=N}^{\infty} |c_n| |z|^n$ *diverges.*

Theorem 8.1.2 (Power function, holomorphic function, region of convergence)
Suppose $f(z) = \sum_{n\in\mathbb{N}} c_n z^n$ *has a radius of convergence* $R \in \mathbb{R}$. *Then* $f'(z)$ *exists and equals*

$$\sum_{n=1}^{\infty} n c_n z^{n-1}$$

throughout $|z| < R$.

Moreover, f' *has the* ***same radius of convergence*** *as* f.

Proof
Note that f' has the same radius of convergence as f since

$$\limsup_{n\to\infty} |nc_n|^{\frac{1}{n}} = \limsup_{n\to\infty} |n|^{\frac{1}{n}} |c_n|^{\frac{1}{n}} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$$

where note that $\lim_{n\to\infty} |n|^{\frac{1}{n}} = 1$.

Let $|z_0| \leq r < R$ and $g(z_0) := \sum_{n=1}^{\infty} nc_n z_0^{n-1}$.

WTS

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = g(z_0)$$

OR

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \tag{$\dagger$}$$
$$|h| < \delta \implies \left| \frac{f(z_0+h) - f(z_0)}{h} - g(z_0) \right| < \varepsilon$$

(Note that we would want $\delta > 0$ such that $|z_0 + h| \leq r < R$)

WTP($\dagger$). *$\forall \varepsilon > 0$, choose $\delta > 0$. Suppose $|z_0 + h| < |z_0 + \delta| \leq r < R$. Write*

$$f(z) = \sum_{n=0}^{N} c_n z^n + \sum_{n=N}^{\infty} c_n z^n$$

and let $S_N(z)$ and $E_N(z)$ be the first and second terms respectively. Then we have that

$$S_N'(z) = \sum_{n=1}^{N} nc_n z^{n-1}$$

Consider

$$\begin{aligned} &\left| \frac{S_N(z_0+h) - S_N(z_0)}{h} + \frac{E_N(z_0+h) - E_N(z_0)}{h} - g(z_0) + S_N'(z_0) - S_N'(z_0) \right| \\ \leq &\left| \frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0) \right| + \left| \frac{E_N(z_0+h) - E_N(z_0)}{h} \right| + \left| S_N'(z_0) - g(z_0) \right| \end{aligned} \tag{8.3}$$

For the second term, since $a^2 - b^2 = (a-b)(a^{n-1} + a^{n-2}b + \ldots ab^{n-2} + b^{n-1})$ and $|z_0|, |z_0 + h| < r \leq R$, we obtain $(z_0+h)^n - z_0^n \leq hnr^{n-1}$. Thus

$$\left| \frac{E_N(z_0+h) - E_N(z_0)}{h} \right| = \left| \frac{1}{h} \sum_{n=N+1}^{\infty} c_n [(z_0+h)^n - z_0^n] \right| \leq \sum_{n=N+1}^{\infty} nc_n r^{n-1}.$$

Note that

$$\sum_{n=1}^{\infty} nc_n r^{n-1} = g(r), \tag{8.4}$$

and since $r < R$, *Equation* (8.4) *converges absolutely by Theorem 8.1.1, thus the tail* $\sum_{n=N+1}^{\infty} nc_n r^{n-1}$ *converges absolutely. Therefore, by comparison, we can pick* $\frac{\varepsilon}{3} > 0$ *so that* $\exists N_1 \in \mathbb{N}, \forall n > N_1$

$$\left| \frac{E_N(z_0+h) - E_N(z_0)}{h} \right| < \frac{\varepsilon}{3}.$$

For the third term in Equation (8.3), *we observe that by definition,* $S_N'(z_0) = \sum_{n=1}^{N} nc_n z^{n-1}$. *Since*

$$\lim_{N\to\infty} S_N'(z_0) = \lim_{N\to\infty} \sum_{n=1}^{N} nc_n z^{n-1} = \sum_{n=1}^{\infty} nc_n z^{n-1} = g(z_0)$$

we know that we can pick $\frac{\varepsilon}{3} > 0, \exists N_2 \in \mathbb{N}, \forall n > N_2$, *we have*

$$\left| S_N'(z_0) - g(z_0) \right| < \frac{\varepsilon}{3}$$

.

For the first term in Equation (8.3), *note that* $(S_N(z_0))' = S_N'(z_0)$. *Let* $\frac{\varepsilon}{3} > 0, \exists \delta > 0, \exists N > \max\{N_1, N_2\}, \forall n > N$, *since* $|h| < \delta$, *we have that*

$$\left| \frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0) \right| < \frac{\varepsilon}{3}$$

This completes the proof. □

Chapter 9

Lecture 9 Jan 22nd 2018

9.1 Power Series (Continued 2)

9.1.1 Radius of Convergence (Continued)

Example 9.1.1
Let $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$. To find the radius of convergence, we use Hadamard's Formula:

$$\frac{1}{R} = \limsup_{n \to \infty} \left(\frac{1}{n} \right)^{\frac{1}{n}} = 1 \qquad \because \lim_{n \to \infty} n^{\frac{1}{n}} = 1$$

Therefore $R = 1$. Thus, by Theorem 8.1.1, f converges absolutely when $|z| < 1$ and diverges when $|z| > 1$. As for the boundary, i.e. $|z| = 1$, consider the following two cases:

1. *If $z = 1$, then $f(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ is a* ***harmonic series****, and hence f diverges.*
2. *If $z = i$, then*

$$\begin{aligned} f(i) &= \sum_{n=1}^{\infty} \frac{i^n}{n} \\ &= i - \frac{1}{2} + \frac{-i}{3} + \frac{1}{4} + \frac{i}{5} - \frac{1}{6} \\ &= \left(-\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \ldots \right) + i \left(1 - \frac{1}{3} + \frac{1}{5} + \ldots \right). \end{aligned}$$

   Observe that both the real and imaginary parts are alternating series where the absolute values of each term is decreasing, which, by the ***alternating series test****, converge. Thus in this case, f converges.*

Therefore, we observe that ***both convergence and divergence may occur*** *on the boundary, depending on the value of z.*

Note
We may not always exchange the position of $\lim$ *and* $\sum_{a=1}^{b}$ *when we consider an infinite sum (i.e. $b = \infty$). Here's an example why this is true. Consider the function $f(x) = \sum_{n=1}^{\infty}(x^n - x^{n-1})$ for $|x| < 1$. Is*

$$\lim_{x \to 1} \sum_{n=1}^{\infty}(x^n - x^{n-1}) = \sum_{n=1}^{\infty} \lim_{x \to 1}(x^n - x^{n+1})$$

true?

Clearly, RHS is 0. For LHS, note that

$$\begin{aligned} f(x) &= \lim_{N \to \infty} \sum_{n=1}^{N}(x^n - x^{n+1}) \\ &= \lim_{N \to \infty} (x - x^2 + x^2 - x^3 + \ldots + x^N - x^{N+1}) \\ &= \lim_{N \to \infty} (x - x^{N+1}) = x. \end{aligned}$$

So,

$$LHS = \lim_{x \to 1} x = 1$$

And we see that RHS $\neq$ LHS.

Definition 9.1.1 (Entire Function)
A function f is said to be ***entire*** *if f is holomorphic in* ***the entire complex plane****.*

Exercise 9.1.1
Define $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Show that

1. *the radius of convergence of this series is ∞, and hence that e^z is an entire function. (Hint: Use* ***Stirling's formula****: $n! \sim (\frac{n}{e})^n \sqrt{2\pi n}$)*
2. $(e^z)' = e^z$

Solution

1. *Using Stirling's formula, note that we have*

$$e^z = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2\pi n}} \left(\frac{ez}{n} \right)^n$$

To find R, we have

$$\begin{aligned}
\frac{1}{R} &= \limsup_{n\to\infty} \left| \frac{1}{\sqrt{2\pi n}} \left(\frac{e}{n}\right)^n \right|^{\frac{1}{n}} \\
&= \limsup_{n\to\infty} \left| \frac{e}{n} \right| \limsup_{n\to\infty} \left| \frac{1}{\sqrt{2\pi n}} \right|^{\frac{1}{n}} \\
&= 0
\end{aligned}$$

since $\limsup_{n\to\infty} \frac{e}{n} = 0$ *and* $\limsup_{n\to\infty} \left| \frac{1}{\sqrt{2\pi n}} \right|^{\frac{1}{n}} = 1$. *Thus* $R = \infty$. *By Theorem 8.1.1,* e^z *is an entire function.*

2. *Note that*

$$\begin{aligned}
\lim_{h\to 0} \frac{e^{z+h} - e^z}{h} &= e^z \lim_{h\to 0} \frac{e^h - 1}{h} \\
&= e^z \lim_{h\to 0} \frac{1 + h + \frac{h^2}{2} + \frac{h^3}{3} + \ldots - 1}{h} \\
&= e^z
\end{aligned}$$

Thus $(e^z)' = e^z$ *as required.*

Chapter 10

Lecture 10 Jan 24th 2018

10.1 Power Series (Continued 3)

10.1.1 Radius of Convergence (Continued 2)

A power series is infinitely $\mathbb{C}$-differentiable in its radius of convergence. All its derivatives are also power series, obtained by term-wise differentiation.

E.g.

$$f(z) - \sum_{n=0}^{\infty} c_n z^n \quad \text{then} \quad f^{(2)}(z) = \sum_{n=0}^{\infty} n(n-1) c_n z^{n-2}$$

In general, we may have $\sum_{n=0}^{\infty} c_n (z - z_0)^n$, which is a power series centered at $z_0 \in \mathbb{C}$. Then, as before, the radius of convergence of this power series is given by

$$\frac{1}{R} = \limsup_{n \to \infty} |c_n|^{\frac{1}{n}}$$

So instead of having the disc of convergence centered around 0, we now have one that is centered around z_0.

Corollary 10.1.1 (Corollary of Theorem 8.1.2)
From Theorem 8.1.2, we have shown that

$f(z)$ has a power series expansion at z_0 (i.e. $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ in some neighbourhood of z_0) with radius of convergence $R > 0$ $\implies$ *f is holomorphic at z_0*

The converse of the statement above is true, i.e.

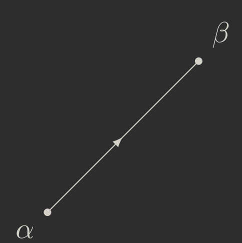
f is holomorphic at z_0 $\implies$ $f(z)$ has a power series expansion at z_0 (i.e. $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ in some neighbourhood of z_0) with radius of convergence $R > 0$

This converse, however, is not possible to be proven given the current tools on our belt. And so we now have to venture into integrals in $\mathbb{C}$.

10.2 Integration in $\mathbb{C}$

10.2.1 Curves and Paths

Before we begin with the definition of a curve in $\mathbb{C}$, let us consider how a straight line should be described as a vector-valued function in the complex plane. For instance, if we have two points $\alpha, \beta \in \mathbb{C}$, and we want to describe the straight line connecting the two.



Let γ be the function that describes this line. We may then define $\gamma : [0, 1] \to \mathbb{C}$ to be either

$$\gamma(t) = \alpha + (\beta - \alpha)t \quad \text{or} \quad \gamma = \alpha(1 - t) + \beta t.$$

We would then have the following mapping:

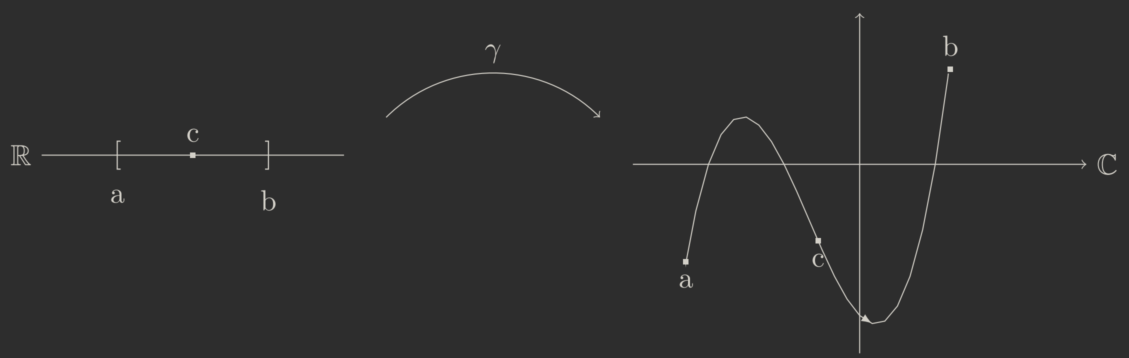


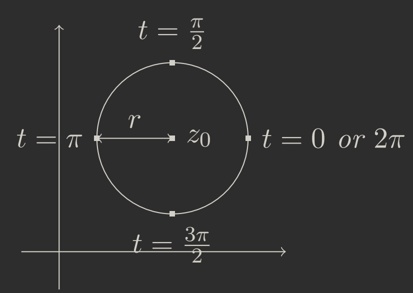
Figure 10.1: Mapping from $\mathbb{R} \to \mathbb{C}$ with γ, which is called **the curve γ**

Definition 10.2.1 (Curves in $\mathbb{C}$)
A curve in $\mathbb{C}$ is a continuous function, $\gamma(t) : [a, b] \to \mathbb{C}$, where $a, b \in \mathbb{R}$. The image of γ in $\mathbb{C}$ is called γ^*.

Example 10.2.1
Let $z_0 \in \mathbb{C}, r > 0$.

1. *Let $\gamma : [0, 2\pi] \to \mathbb{C}$, such that $\gamma(t) = z_0 + re^{it}$.*
2. *Let $\gamma' : [0, 1] \to \mathbb{C}$, such that $\gamma'(t) = z_0 + re^{2\pi it}$.*

The two functions above describe a circle centered at z_0 with radius r, anticlockwise-oriented.



We say that γ and γ' are equivalent parameterizations for the same oriented path.

Definition 10.2.2 (Equivalent Parameterization)
*Let $\gamma_1 : [a, b] \to \mathbb{C}, \gamma_2 : [c, d] \to \mathbb{C}$ where $a, b, c, d \in \mathbb{C}$ describe the path γ^*. The two **paramaterization are said to be equivalent** if $\exists h : [a, b] \to [c, d]$ that is a bijection and a continuous function such that*

$$\gamma_1(t) = \gamma_2(h(t))$$

where $t \in [a, b]$.

Note

We will not look at functions like the Weierstrass function in this course.

Definition 10.2.3 (Smooth Curve)

Let $\gamma : [a, b] \to \mathbb{C}, a, b, \in \mathbb{C}$*.* γ *is said to be smooth if its derivative* γ' *exists and is continuous on* $[a, b]$ *and* $\forall t \in [a, b], \gamma'(t) \neq 0$*.*

Definition 10.2.4 (Piecewise Smooth)

Let $\gamma : [a, b] \to \mathbb{C}$*.* γ *is said to be piecewise smooth if it is smooth on* $[a, b]$ *except on finitely many points in* $[a, b]$*.*

Remark

Piecewise smooth curves shall be called paths.

10.2.2 Integral

Definition 10.2.5 (Contour)

Given a path $\gamma : [a, b] \to \mathbb{C}$ *and* $f : \mathbb{C} \to \mathbb{C}$*, a function continuous on* γ*. We define the integral* f *along* γ*, called a* ***contour****, as*

$$\int_\gamma f(z)\, dz := \int_a^b f(\gamma(t))\gamma'(t)dt \tag{10.1}$$

where we let $z = \gamma(t)$ *and hence* $dz = \gamma'(t)dt$*.*

Remark

1. *Suppose* g *is a complex-valued function, then*

$$\int_a^b g(t)dt = \int_a^b \mathrm{Re}(g(t))dt + i \int_a^b \mathrm{Im}(g(t))dt$$

2. *The integral of* f *along* γ *can be shown to be independent of the chosen parameterization for* γ^**.*

   Proof

   Let $a, b, c, d \in \mathbb{R}, \gamma_1 : [a, b] \to \mathbb{C}, \gamma_2 : [c, d] \to \mathbb{C}$ *describe the same path* γ^**. By Definition 10.2.2, define a bijection* $h : [a, b] \to [c, d]$ *that is a continuous function such that* $t \mapsto \tau$*, so that*

$$\gamma_1(t) = \gamma_2(h(t)) = \gamma(\tau).$$

Note that

$$\gamma_1'(t) = h'(t)\gamma_2'(h(t)) \text{ and}$$
$$h(t) = \tau \implies h'(t)dt = d\tau.$$

Now since h is a bijection, we claim that $h(a) = c$ while $h(b) = d$.

We know that h cannot be a constant function. Suppose h is an increasing function, then since $a \leq b$ and $c \leq d$, it is clear that $h(a) = c$ and $h(b) = d$. Similarly, if h is a decreasing function, then $h(a) = d$ and $h(b) = c$. But this is a contradiction to our supposition that γ_1 and γ_2 describe the same orientation. Thus h must be an increasing function, and hence we have $h(a) = c$ and $h(b) = d$.

(This can be more rigorous but that is an easy proof, and we may use perhaps the Approximation Property of $\mathbb{R}$ to that end, which is a fun exercise that shall not be included within these covers.)

Now

$$\begin{aligned}
\int_{\gamma_1} f(z)\,dz &= \int_a^b f\big(\gamma_1(t)\big)\gamma_1'(t)dt \\
&= \int_a^b f\Big(\gamma_2\big(h(t)\big)\Big)h'(t)\gamma_2'(h(t))dt \\
&= \int_c^d f\big(\gamma_2(\tau)\big)\gamma_2'(\tau)d\tau \\
&= \int_{\gamma_2} f(z)\,dz
\end{aligned}$$

This completes the proof. □

Chapter 11

Lecture 11 Jan 26th 2018

11.1 Integration in $\mathbb{C}$ (Continued)

11.1.1 Integral (Continued)

Note (Recall)
Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth curve. For a function f that is continuous on γ, we defined

$$\begin{aligned} \int_\gamma f(z)\, dz &= \int_a^b f\big(\gamma(t)\big)\gamma'(t)dt \\ &= \int_a^b \mathrm{Re}\Big(f\big(\gamma(t)\big)\gamma'(t)\Big)dt + i\int_a^b \mathrm{Im}\Big(f\big(\gamma(t)\big)\gamma'(t)\Big)dt \end{aligned}$$

and have

$$\begin{gathered} \gamma'(t) = u'(t) + iv'(t) \\ \textit{if } \gamma(t) = u(t) + iv(t) \end{gathered}$$

Example 11.1.1
Let $f(z) = f(x + iy) = x^2 + y^2$ be continuous along $\gamma : [0, 1] \to \mathbb{C}$ $t \mapsto t + it$. Evaluate $\int_\gamma f(z)\, dz$.

Solution

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_0^1 f(t+it)(1+i)dt \\ &= (1+i)^2 \int_0^1 t^2 dt \\ &= (1+i)^2 \cdot \frac{1}{3}t^3\Big|_0^1 \\ &= \frac{2i}{3}\end{aligned}$$

Example 11.1.2
$\forall n \in \mathbb{Z}$, evaluate $\int_\gamma z^n\,dz$ that is continue on the path γ that describes any circle centered at origin oriented anticlockwise.

Solution
Let $R \in \mathbb{R}$, and define

$$\gamma : [0,1] \to \mathbb{C} \;\; t \mapsto Re^{2\pi it}$$
$$\gamma'(t) = 2R\pi i e^{2\pi it} = 2\pi i\gamma(t)$$

Then

$$\begin{aligned}\int_\gamma z^n\,dz &= \int_0^1 R^n e^{2\pi int} \cdot 2\pi i \cdot Re^{2\pi it} dt \\ &= 2\pi i R^{n+1} \int_0^1 e^{2\pi i(n+1)t} dt \\ &= \begin{cases} \frac{R^{n+1}}{n+1} e^{2\pi i(n+1)t}\Big|_0^1 & \text{if } n \in \mathbb{Z}\setminus\{-1\} \\ 2\pi it\Big|_0^1 & \text{if } n = -1 \end{cases} \\ &= \begin{cases} \frac{R^{n+1}}{n+1}\left(e^{2\pi i(n+1)} - 1\right) & \text{if } n \in \mathbb{Z}\setminus\{-1\} \\ 2\pi i & \text{if } n = -1 \end{cases} \quad \because e^{2\pi ki} \equiv 1 \mod 2\pi \\ &= \begin{cases} 0 & \text{if } n \in \mathbb{Z}\setminus\{-1\} \\ 2\pi i & \text{if } n = -1 \end{cases}\end{aligned}$$

Note that our final answer does not depend on R, the radius of the circle.

Proposition 11.1.1 (Properties of integrals in $\mathbb{C}$)

1. ***(Linearity)*** *Let $\alpha, \beta \in \mathbb{C}$. $\int_\gamma (\alpha f(z) + \beta g(z))\,dz = \alpha \int_\gamma f(z)\,dz + \beta \int_\gamma g(z)\,dz$.*

2. (a) *For any complex-valued function g, and $b \geq a$,*

$$\left| \int_a^b g(t)dt \right| \leq \int_a^b |g(t)|\, dt$$

(b) *For any function $f(z)$ that is continuous on a path $\gamma : [a, b] \to \mathbb{C}$,*

$$\left| \int_\gamma f(z)\, dz \right| \leq \sup_{z \in \gamma} |f(z)| \cdot \underbrace{\int_a^b \left| \gamma'(t) \right| dt}_{\text{length of the path}}$$

3. *If γ^- is the path $\gamma : [a, b] \to \mathbb{C}$ with a reversed direction, then*

$$\int_{\gamma^-} f(z)\, dz = -\int_\gamma f(z)\, dz$$

Proof

1.

$$\begin{aligned}
LHS &= \int_\gamma \alpha f(z) + \beta g(z)\, dz \\
&= \int_\gamma \alpha \Big(\mathrm{Re}(f(z)) + i\, \mathrm{Im}(f(z)) \Big) + \beta \Big(\mathrm{Re}(g(z)) + i\, \mathrm{Im}(g(z)) \Big)\, dz \\
&= \begin{aligned} &\alpha \int_a^b \Big(\mathrm{Re}(f(z)) + i\, \mathrm{Im}(f(z)) \Big) dt \\ &\quad + \beta \int_a^b \Big(\mathrm{Re}(g(z)) + i\, \mathrm{Im}(g(z)) \Big) dt \end{aligned} \qquad (\dagger) \\
&= \alpha \int_a^b f(z)\, dz + \beta \int_a^b g(z)\, dz
\end{aligned}$$

where ($\dagger$) *is because of Item 1 from an earlier remark and Linearity of Integrals in* $\mathbb{R}$. □

2. (a) *Let $R \in \mathbb{R}$. Suppose $\int_a^b g(t)dt = Re^{i\theta}$. Then*

$$\begin{aligned}
LHS = R &= \int_a^b \underbrace{e^{-i\theta} g(t)}_{u(t)+iv(t)}\, dt \\
&= \int_a^b u(t)dt + i \int_a^b v(t)dt
\end{aligned}$$

Since $R \in \mathbb{R}$, *we have* $\int_a^b v(t)dt = 0$, *so*

$$\begin{aligned}
R &= \int_a^b u(t)dt \\
&\leq \int_a^b |u(t)|\,dt \quad \because a \leq b \wedge u(t) \leq |u(t)| \in \mathbb{R} \\
&\leq \int_a^b |u(t) + iv(t)|\,dt \\
&= \int_a^b \left| e^{-i\theta} g(t) \right| dt \\
&= \int_a^b |g(t)|\,dt = RHS
\end{aligned}$$

□

(b)

$$\begin{aligned}
LHS &= \left| \int_\gamma f(z)\,dz \right| \\
&= \left| \int_a^b f\big(\gamma(t)\big)\gamma'(t)dt \right| \\
&\leq \int_a^b \left| f\big(\gamma(t)\big)\gamma'(t) \right| dt \quad \text{by Item 2a} \\
&\leq \int_a^b \sup_{z\in\gamma} |f(z)| \left|\gamma'(t)\right| dt \quad \text{since } |f(z)| \leq \sup_{z\in\gamma} |f(z)| \\
&= \sup_{z\in\gamma} |f(z)| \cdot \int_a^b \left|\gamma'(t)\right| dt = RHS
\end{aligned}$$

□

3. *Let* $\gamma^- : [b,a] \to \mathbb{C}$ *such that* $\gamma^- = \gamma(b-t+a)$. *Let* $k = b-t+a$ *so that* $dk = -dt$, *and* $t = b \implies k = a \wedge t = a \implies k = b$. *Note that* $k \in [a,b]$ *with this definition.*

Then

$$\begin{aligned}
LHS &= \int_{\gamma^-} f(z)\, dz \\
&= \int_b^a f\big(\gamma^-(t)\big)\gamma^{-\prime}(t)dt \\
&= \int_b^a f\big(\gamma(b-t+a)\big)\gamma'(b-t+a)dt \\
&= -\int_a^b f\big(\gamma(k)\big)\gamma'(k)dk \\
&= -\int_\gamma f(z)\, dz = RHS
\end{aligned}$$

as required. □

We are now in a position to generalize the **Fundamental Theorem of Calclus** for $\mathbb{C}$.

Chapter 12

Lecture 12 Jan 29th 2018

12.1 Integration in $\mathbb{C}$ (Continued 2)

12.1.1 Fundamental Theorem of Calculus

To simplify statements from hereon, we shall use the following notations.

Notation
Let $\Omega \subseteq \mathbb{C}$ be an open set in $\mathbb{C}$. We denote $f \in H(\Omega) \iff f$ is holomorphic on Ω.

Theorem 12.1.1 (Fundamental Theorem of Calculus)
Let $\gamma : [a, b] \to \mathbb{C}$ be a path inside an open set $\Omega \subseteq \mathbb{C}$. Suppose $f(z)$ is continuous on γ, and has an antiderivative $F \in \Omega$. Then

$$\int_\gamma f(z)\, dz = F\big(\gamma(b)\big) - F\big(\gamma(a)\big) \tag{12.1}$$

Proof
Let $G = F \circ \gamma$ and suppose γ is a smooth function. Since γ is smooth, γ' exists and is continuous on $[a, b]$ and $\gamma'(t) \neq 0$ for all $t \in [a, b]$, and since f is continuous on $[a, b]$, $G(t) = F'(\gamma(t))\gamma'(t)$ is continuous as well.

Now

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b f\big(\gamma(t)\big)\gamma'(t)dt \\ &= \int_a^b F'\big(\gamma(t)\big)\gamma'(t)dt \\ &= \int_a^b G'(t)dt \\ &= G(b) - G(a) \quad \textit{by applying FTC in } \mathbb{R} \textit{ to real and imaginary parts} \\ &= F(\gamma(b)) - F(\gamma(a))\end{aligned}$$

If γ is piecewise smooth, then we can simply apply the above to each of the smooth paths separately and sum up all of the integrals. □

Definition 12.1.1 (Closed Path)
*A path $\gamma : [a, b] \to \mathbb{C}$ is said to be **closed** if $\gamma(a) = \gamma(b)$.*

Corollary 12.1.1 (Corollary of FTC)
If $F \in H(\Omega), \Omega \subseteq \mathbb{C}$ (hence F' is continuous on Ω), then

$$\int_\gamma F'(z)\,dz = 0$$

on any closed path γ on Ω.

Proof
A closed path $\gamma : [a, b] \to \mathbb{C}$ has $\gamma(a) = \gamma(b)$. By Theorem 12.1.1, $\int_\gamma F'(z)\,dz = F(\gamma(b)) - F(\gamma(a)) = 0$ as required. □

Example 12.1.1
*Take $f(z) = z^n$ where $n \in \mathbb{Z} \setminus \{-1\}$ as in Example 11.1.2. Then f is continuous on $\mathbb{C} \setminus \{0\}$ **(not sure why this would be problematic when we've already excluded -1 for n)**. Then $f = F'$ for $F(z) = \frac{z^{n+1}}{n+1}$ and $F \in H(\mathbb{C} \setminus \{0\})$. Therefore by Corollary 12.1.1, $\int_\gamma z^n\,dz = 0$ for any closed path γ not passing through 0.*

If we do include -1 for n, note that F' would not be continuous on 0, and thus the corollary would not apply. We have also shown in the earlier example that $\int_\gamma \frac{1}{z}\,dz = 2\pi i$.

Note (Recall)
*The **interior** of a set Ω is defined as $\{z \in \Omega : \forall \varepsilon > 0\ B(z, \varepsilon) \subseteq \Omega\}$, and denoted as Ω^0.*

Theorem 12.1.2 (Goursat's Theorem / Cauchy's Theorem for a triangle)
Let $\Omega \subseteq \mathbb{C}$ be an open set. Suppose $\Delta \subseteq \Omega$ is a closed triangle whose interior is also contained in Ω. Let $f \in H(\Omega)$. Then

$$\int_\Delta f(z)\, dz = 0$$

This theorem holds more meaning than the presented statement, as it implies that, essentially, given any two points connected by two different paths in an open set in $\mathbb{C}$, and a function that is holomorphic over the two paths, the **two path integrals of the function will yield the same result**!

Proof
Let $\Delta_1^{(1)}, \Delta_2^{(1)}, \Delta_3^{(1)}, \Delta_4^{(1)}$ be smaller triangles by bisecting each side of Δ. $\forall i \in \{1, 2, 3, 4\}$, orient $\Delta_i^{(1)}$ anticlockwise. Then we have

$$J := \int_\Delta f(z)\, dz = \sum_{i=1}^{4} \int_{\Delta_i^{(1)}} f(z)\, dz \tag{12.2}$$

Note that there must at least one of the $\Delta_i^{(1)}$ such that $\left| \int_{\Delta_i^{(1)}} \right| \geq \frac{|J|}{4}$, since $\forall i \in \{1, 2, 3, 4\}$, $\left| \int_{\Delta_i^{(1)}} \right| < \frac{|J|}{4}$ would contradict Equation (12.2). Without loss of generality, let $\Delta_1^{(1)}$ be the largest triangle of the four.

Now note that each of the perimeter of $\Delta_i^{(1)}$ is half of the perimeter of Δ. Let $\ell(x)$ be the perimeter of x. Continue with taking bisectors of $\Delta_1^{(1)}, \Delta_1^{(2)}, \ldots$ such that

$$\Delta \supseteq \Delta_1^{(1)} \supseteq \Delta_1^{(2)} \supseteq \ldots,$$

then we have that for each $j \in \mathbb{N} \setminus \{0\}$, $\Delta_i^{(j)}$ is such that

$$\left| \int_{\Delta_i^{(j)}} f(z)\, dz \right| \geq \frac{|J|}{4^j}$$

*and $\ell(\Delta_i^{(j)}) = \frac{1}{2^j} \ell(\Delta)$. By the **Nested Rectangle Theorem from Real Analysis**, $\exists z_0 \in \mathbb{C}$ such that $z_0 \in \Delta_i^{(j)}$ for all $j \in \mathbb{N} \setminus \{0\}$ that is a limit point. Since $z_0 \in \Omega \ \wedge \ f \in H(\Omega)$, we have that*

$$\forall z \in \Omega \ \ \forall \varepsilon > 0 \ \ \exists \delta > 0$$
$$0 < |z - z_0| < \delta \implies \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

Consider $D(z_0, \delta), \exists n \in \mathbb{N} \setminus \{0\}, \Delta_1^{(n)} \subseteq D(z_0, \delta)$. *Consider*

$$\int_{\Delta_1^{(n)}} f(z)\,dz = \int_{\Delta_1^{(n)}} \big(f(z) - f(z_0) - f'(z_0)(z - z_0)\big)\,dz,$$

where we note that $\int_{\Delta_1^{(n)}} f(z_0)\,dz = \int_{\Delta_1^{(n)}} f'(z_0)(z - z_0)\,dz = 0$ *by Corollary 12.1.1.*

By Item 2a in Proposition 11.1.1,

$$\begin{aligned} \left| \int_{\Delta_1^{(n)}} f(z)\,dz \right| &\leq \int_{\Delta_1^{(n)}} \big| f(z) - f(z_0) - f'(z_0)(z - z_0) \big|\,dz \\ &\leq \int_{\Delta_1^{(n)}} \varepsilon\,|z - z_0|\,dz \leq \varepsilon \ell(\Delta_1^{(n)}) \int_{\Delta_1^{(n)}} dz = \varepsilon \ell(\Delta_1^{(n)})^2 \end{aligned}$$

where we note that $|z - z_0| \leq \ell(\Delta_1^{(n)}$ *since* $z \in \Delta_1^{(n)}$. *This implies that*

$$\frac{|J|}{4^n} \leq \varepsilon \ell(\Delta_1^{(n)})^2 \leq \varepsilon \frac{\ell(\Delta)^2}{4^n} = |J| \leq \varepsilon$$

and therefore, $|J| = 0$*!*

Tutorial Jan 31 2018

Note
Consider the power series $\sum_{n\geq 0} a_n(z-z_0)^n$ *and let* $\frac{1}{R} := \limsup_{n\to\infty} \sup \sqrt[n]{|a_n|} \in [0,\infty)$.

- *If* $|z-z_0| < R$, $\sum_{n\geq 0} a_n(z-z_0)^n$ *converges absolutely.*
- *If* $|z-z_0| > R, \sum_{n\geq 0} a_n(z-z_0)^n$ *diverges.*
- *If* $0 < r < R$, *then* $\sum_{n\geq 0} a_0(z-z_0)^n$ *converges uniformly on* $\{z : |z-z_0| < r\}$.

12.2 Practice Problems

1. Parameterize the semicircle $|z-4-5i| = 3$ clockwise, starting from $z = 4+8i$ to $z = 4+2i$.

   Solution
   Let $\gamma : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{C}$ *such that* $\gamma(t) = 3e^{-it} + 4 + 5i$. *Note that* γ *parameterizes the given semicircle:*

$$\begin{aligned} \gamma\left(-\frac{\pi}{2}\right) &= 4+8i \\ \gamma(0) &= 7+5i \\ \gamma\left(\frac{\pi}{2}\right) &= 4+2i \end{aligned}$$

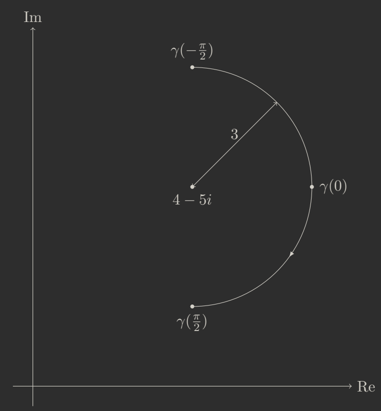


Figure 12.1: Semicircle $|z-4-5i|=3$ oriented clockwise, parameterized by γ

2. If the power series $f(z)=\sum_{n=0}^{\infty} c_n(z-z_0)^n$ centered at z_0 has a non-zero radius of convergence, then show that

$$c_m = \frac{f^{(m)}(z_0)}{m!}$$

for any $m \in \mathbb{Z}, m \geq 0$, where $f^{(m)}(z_0)$ denotes the mth derivative of f at z_0.

Solution

Since $f(z)$ is a power series and the radius of convergence $R \neq 0$, by Theorem 8.1.2, $f(z)$ is $\mathbb{C}$-differentiable and each derivative has the same radius of convergence. By induction, it can be shown that

$$f^{(m)}(z) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} c_n (z-z_0)^{n-m}$$

Evaluating $f^{(m)}$ at z_0, we have

$$\begin{aligned} f^{(m)}(z_0) &= \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} c_n (z_0-z_0)^{n-m} \\ &= m! c_m \end{aligned}$$

where all terms above m are 0. Then we obtain

$$c_m = \frac{f^{(m)}(z_0)}{m!}$$

as desired. □

3. Let γ be the arc of the unit circle centered at the origin in the first quadrant oriented clockwise (from i to 1). Evaluate the integral

$$\int_\gamma \bar{z}^2\, dz$$

by parameterizing the curve.

Solution
Consider the parameterization $\gamma : [-\frac{\pi}{2}, 0] \to \mathbb{C}$ given by $\gamma(t) = e^{-it}$. Note that $\overline{e^{-it}} = e^{it}$. Then

$$\begin{aligned}\int_\gamma \bar{z}^2\, dz &= \int_{-\frac{\pi}{2}}^{0} e^{2it} \cdot \left(-ie^{-it}\right) dt \\ &= -i \int_{-\frac{\pi}{2}}^{0} e^{it} dt \\ &= -e^{it}\Big|_{\frac{\pi}{2}}^{0} \\ &= -1 - i\end{aligned}$$

□

4. Evaluate the above integral by finding an antiderivative. (Hint: Use $\left(\frac{z\bar{z}}{z}\right)^2$)

Solution
Note that $z\bar{z} = |z|^2$, so on the circle, we have $\bar{z} = \frac{1}{z}$. Thus the integral is equivalent to

$$\int_\gamma \frac{1}{z^2}\, dz$$

Note that the antiderivative of $\frac{1}{z^2}$ is $-\frac{1}{z}$. Thus by Theorem 12.1.1,

$$\int_\gamma \bar{z}^2\, dz = \int_\gamma \frac{1}{z^2} = F(\gamma(0)) - F\left(\gamma\left(-\frac{\pi}{2}\right)\right) = -\frac{1}{e^{-i(0)}} + \frac{1}{e^{-i(-\pi/2)}} = -1 - i$$

5. Let $\{c_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers such that

$$L = \lim_{n\to\infty} \frac{c_{n+1}}{c_n}$$

exists. Then show that

$$\lim_{n\to\infty} c_n^{\frac{1}{n}} = L$$

This shows that, when applicable, the **ratio test** can be used instead of the root test to calculate the radius of convergence of a power series.

Solution
Suppose that

$$L = \lim_{n\to\infty} \frac{c_{n+1}}{c_n}$$

exists. By definition, we have

$$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n > N$$
$$\left| \frac{c_n}{c_{n-1}} - L \right| < \varepsilon$$

Thus for $n \geq N$,

$$\begin{aligned} c_n^{\frac{1}{n}} &= \left(\frac{c_n}{c_{n-1}} \cdot \frac{c_{n-1}}{c_{n-2}} \dots \frac{c_N}{c_{N-1}} \cdot c_{N-1} \right)^{\frac{1}{n}} \\ &= \left(\frac{c_n}{c_{n-1}} \right)^{\frac{1}{n}} \left(\frac{c_{n-1}}{c_{n-2}} \right)^{\frac{1}{n}} \dots \left(\frac{c_N}{c_{N-1}} \right)^{\frac{1}{n}} c_{N-1}^{\frac{1}{n}} \end{aligned}$$

Now

$$(L-\varepsilon)^{\frac{1}{n}} (L-\varepsilon)^{\frac{1}{n}} \dots (L-\varepsilon)^{\frac{1}{n}} c_{N-1}^{\frac{1}{n}} \leq c_n^{\frac{1}{n}} \leq (L+\varepsilon)^{\frac{1}{n}} (L+\varepsilon)^{\frac{1}{n}} \dots (L+\varepsilon)^{\frac{1}{n}} c_{N-1}^{\frac{1}{n}}$$
$$(L-\varepsilon)^{\frac{n-N+1}{n}} c_{N-1}^{\frac{1}{n}} \leq c_n^{\frac{1}{n}} \leq (L+\varepsilon)^{\frac{n-N+1}{n}} c_{N-1}^{\frac{1}{n}}$$

Note that

$$\lim_{n\to\infty} (L-\varepsilon)^{\frac{n-N+1}{n}} c_{N-1}^{\frac{1}{n}} = L - \varepsilon$$
$$\lim_{n\to\infty} (L+\varepsilon)^{\frac{n-N+1}{n}} c_{N-1}^{\frac{1}{n}} = L + \varepsilon$$

Thus we have

$$L - \varepsilon \leq c_n^{\frac{1}{n}} \leq L + \varepsilon$$
$$\left| c_n^{\frac{1}{n}} - L \right| \leq \varepsilon$$

as desired. □

6. Find the radius of convergence of
   (a) $\sum_{n=0}^{\infty} \frac{n^n z^n}{n!}$
   (b) $\sum_{n=0}^{\infty} z^{2^n}$

Solution

(a) *By Stirling's Approximation, i.e.* $n! \sim (\frac{n}{e})^n \sqrt{2\pi n}$, *we have that Hadamard's formula is*

$$\begin{aligned}\frac{1}{R} &= \limsup_{n\to\infty} \left| \frac{n^n}{n!} \right|^{\frac{1}{n}} \\ &= \limsup_{n\to\infty} \left| \frac{n^n}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} \right|^{\frac{1}{n}} \\ &= \limsup_{n\to\infty} \left| \frac{e^n}{\sqrt{2\pi n}} \right|^{\frac{1}{n}} \\ &= e \limsup_{n\to\infty} \left| \frac{1}{\sqrt{2\pi n}} \right|^{\frac{1}{n}} = e\end{aligned}$$

Therefore, $R = \frac{1}{e}$.

(b) ***no solution yet: current problem, not being able to express the sum as a power series, in turn failing to get*** c_n ***which is needed for*** $\frac{1}{R}$.

7. Show that for any path $\gamma : [a, b] \to \mathbb{C}$ and $f(z)$ continuous on γ, we have

$$\left| \int_\gamma f(z)\, dz \right| \leq \sup_{z\in\gamma} |f(z)| \int_a^b \left|\gamma'(t)\right| dt$$

Solution

$$\begin{aligned} LHS &= \left| \int_\gamma f(z)\, dz \right| \\ &= \left| \int_a^b f\big(\gamma(t)\big)\gamma'(t) dt \right| \quad \textit{by definition} \\ &\leq \int_a^b \left| f\big(\gamma(t)\big)\gamma'(t) \right| dt \quad \textit{by Item 2a of Proposition } 11.1.1 \\ &\leq \int_a^b \sup_{z\in\gamma} |f(z)| \left|\gamma'(t)\right| dt \quad \textit{since } |f(z)| \leq \sup_{z\in\gamma} |f(z)| \\ &= \sup_{z\in\gamma} |f(z)| \cdot \int_a^b \left|\gamma'(t)\right| dt = RHS \end{aligned}$$

Chapter 13

Lecture 13 Feb 9th 2018

13.1 Cauchy's Integral Formula

Definition 13.1.1 (Convex Set)
A set $S \subseteq \mathbb{C}$ is called a ***convex set*** *if the line segment joining any pair of points in S lies entirely in S.*

Theorem 13.1.1 (Cauchy's Theorem for Convex Set)
Let $\Omega \subseteq \mathbb{C}$ be a convex open set, and $f \in H(\Omega)$. Then

1. *$f = F'$ for some $F \in H(\Omega)$.*
2. *$\int_\gamma f(z)\, dz = 0$ for any closed path $\gamma \in \Omega$.*

Proof
Note that it is sufficient to prove 1 since 1 $\implies$ 2 by Theorem 12.1.1.

Let $a \in \Omega$, and let $[a, z]$ denote the straight line from a to z. Since Ω is a convex set, $[a, z]$ is in Ω. Define $F(z)$[1] $= \int_{[a,z]} f(z)\, dz$[2].

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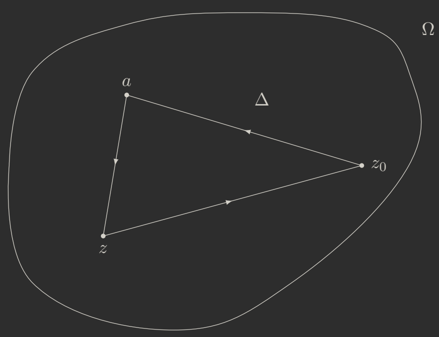
[1]It can be verified that F is continuous.
[2]This is a keys step: defining an "antiderivative" as how we would expect it to be.



WTS $F \in H(\Omega), F'(z_0) = f(z_0)$ *for any* $z_0 \in \Omega$.
Now by Theorem 12.1.2,

$$\begin{aligned} 0 &= \int_\Delta f(z)\,dz \\ &= \int_{[a,z]} f(z)\,dz + \int_{[z,z_0]} f(z)\,dz + \int_{[z_0,a]} f(z)\,dz \\ &= F(z) + \int_{[z,z_0]} f(z)\,dz + (-F(z_0)) \end{aligned}$$

This implies that

$$F(z) - F(z_0) = \int_{[z_0,z]} f(z)\,dz.$$

Divide both sides by $z - z_0$, *then*

$$\begin{aligned} \frac{F(z)-F(z_0)}{z-z_0} - f(z_0) &= \frac{1}{z-z_0}\int_{[z_0,z]} f(z)\,dz - f(z_0) \\ &= \frac{1}{z-z_0}\int_{[z_0,z]} f(z) - f(z_0)\,dz \quad \textit{since } \int_{[z_0,z]} dz = z - z_0 \end{aligned}$$

Since $f \in H(\Omega)$ *and is hence continuous, we have that*

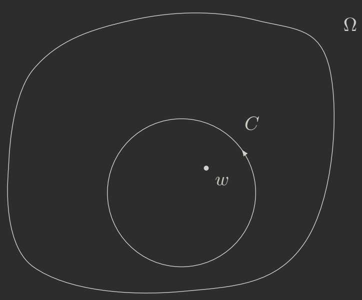
$$\forall \varepsilon > 0\ \exists \delta > 0$$
$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon$$

which in turn implies that

$$\left| \frac{F(z)-F(z_0)}{z-z_0} - f(z_0) \right| = \left| \frac{1}{z-z_0}\int_{[z_0,z]} [f(z) - f(z_0)]\,dz \right| \le \frac{1}{|z-z_0|}\left| \int_{[z_0,z]} \varepsilon\ dz \right| = \varepsilon$$

Hence, by first principle, $F'(z_0) = f(z_0)$. □

Theorem 13.1.2 (Cauchy's Integral Formula 1)
Let $\Omega \subseteq \mathbb{C}$ *be a convex open set, and* C *be a closed circle path in* Ω. *If* $w \in \Omega \setminus \partial C$, *where* ∂C *is the* ***boundary of*** C, *and* $f \in H(\Omega)$, *then*



$$f(w)\,\mathrm{Ind}_C(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dz$$

where

$$\mathrm{Ind}_C(w) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-w}$$

denotes the number of times the countour C *winds around the point* w.

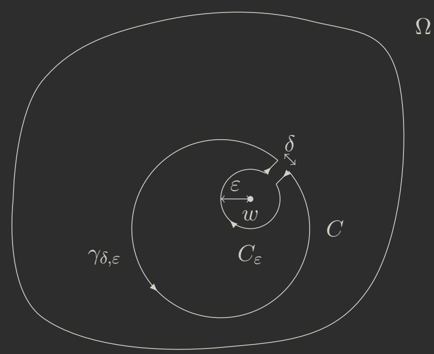
is called the ***index of w with respect to C****, or the* ***winding number*** *of C around w.*

Proof

Let $w \in \Omega \setminus \partial C$. Define

$$g(w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w \end{cases}$$

By the construction of g, g is continuous on Ω, and $g \in H(\Omega \setminus \{w\})$.
We need to *construct a convex set $\Omega' \subseteq \Omega$ that contains $\gamma_{\delta,\varepsilon}$ such that $g \in H(\Omega')$.*



We now follow a similar argument as in the proof for Theorem 13.1.1. Let $\varepsilon > 0$ such that $\exists \delta > 0$, so that we can define the "keyhole" $\gamma_{\delta,\varepsilon}$ which omits w. Consider $D(w, \varepsilon)$, call the image of the border of $D(w, \varepsilon)$ as C_ε, let δ be the width of the "corridor", and the two paths that are the "sides of the corridor" be called $C_{\delta_1}, C_{\delta_2}$ respectively. Define $G(z) = \int_{[a,z]} g(z)\,dz$, where a and z are in the interior of C but not in the interior of C_ε. Then if we define a set Ω' such that it contains the interior of $\gamma_{\delta,\varepsilon}$, we have that Ω' is a convex open set, and $G \in H(\Omega')$. By Theorem 13.1.1, $G' = g$.

Also from Theorem 13.1.1, we have that $\int_{\gamma_{\delta,\varepsilon}} g(z)\,dz = 0$ for any $\varepsilon, \delta > 0$. As $\delta \to 0^+$, we have that the integrals over C_{δ_1} and C_{δ_2} cancel out. Hence, we are left with

$$\int_C g(z)\,dz + \int_{C_\varepsilon} g(z)\,dz = 0$$

Let's put our focus on the smaller circle, C_ε. Now as $\varepsilon \to 0^+$, $\frac{f(z)-f(w)}{z-w} \to 0$, and thus

$$\int_{C_\varepsilon} g(z)\,dz = \int_{C_\varepsilon} \frac{f(z)-f(w)}{z-w}\,dz \to 0$$

Therefore,

$$\int_C g(z)\,dz = 0$$

which implies, in the limit, that

$$\int_C \frac{f(z)}{z-w}\,dz = \int_C \frac{f(w)}{z-w}\,dz = f(w)\int_C \frac{dz}{z-w}$$

We now require $\int_C \frac{dz}{z-w} = 2\pi i$, but we shall prove for a more general case as a lemma.

Chapter 14

Lecture 14 Feb 12 2018

14.1 Cauchy's Integral Formula (Continued)

Lemma 14.1.1
(Lemma and proof from Newman & Bak on Complex Analysis, 3rd Ed.)

Suppose $a \in C_\rho^0$ such that $\exists \alpha \in C_\rho$ that is the center of the circle C_ρ, where ρ is the radius of C_ρ, and hence $|a - \alpha| < \rho$. Then

$$\int_{C_\rho} \frac{dz}{z-a} = 2\pi i$$

Proof
Let $z \equiv \alpha + \rho e^{i\theta}$, then $dz = i\rho e^{i\theta} d\theta$. Thus

$$\int_{C_\rho} \frac{dz}{z-\alpha} = \int_0^{2\pi} \frac{i\rho e^{i\theta}}{\rho e^{i\theta}} d\theta = 2\pi i$$

while

$$\int_{C_\rho} \frac{dz}{(z-\alpha)^{k+1}} = 0 \quad \textit{for } k = 1, 2, 3, \ldots . \tag{14.1}$$

The Equation (14.1) follows not only from a direct evaluation of the integral

$$\int_{C_\rho} \frac{dz}{(z-\alpha)^{k+1}} = \int_0^{2\pi} \frac{i\rho e^{i\theta}}{(\rho e^{i\theta})^{k+1}} d\theta = \frac{i}{\rho^k} \int_0^{2\pi} e^{-ik\theta} d\theta = 0$$

but also the fact that $\frac{1}{(z-\alpha)^{k+1}}$ is the derivative of $-\frac{1}{k(z-\alpha)^k}$, which can be verified to be holomorphic on C_ρ, which simply makes Equation (14.1) true by Theorem 12.1.1.

To evaluate $\int_{C_\rho} \frac{dz}{z-a}$, *write*

$$\begin{aligned}\frac{1}{z-a} &= \frac{1}{(z-\alpha)-(a-\alpha)} = \frac{1}{(z-\alpha)[1-\frac{a-\alpha}{z-\alpha}]} \\ &= \frac{1}{z-\alpha}\cdot\frac{1}{1-\omega}\end{aligned}$$

where

$$\omega = \frac{a-\alpha}{z-\alpha} \text{ has fixed modulus } \frac{|a-\alpha|}{\rho} < 1 \text{ throughout } C_\rho \tag{14.2}$$

By Equation (14.2) and by the ***Infinite Geometric Sum*** *that* $\frac{1}{1-\omega} = 1 + \omega + \omega^2 + \ldots$, *we get*

$$\begin{aligned}\frac{1}{z-a} &= \frac{1}{z-\alpha}\left[1 + \frac{a-\alpha}{z-\alpha} + \frac{(a-\alpha)^2}{(z-\alpha)^2} + \ldots\right] \\ &= \frac{1}{z-\alpha} + \frac{a-\alpha}{(z-\alpha)^2} + \frac{(a-\alpha)^2}{(z-\alpha)^3} + \ldots.\end{aligned}$$

Since the convergence is uniform throughout C_ρ,

$$\int_{C_\rho} \frac{1}{z-a}\,dz = \int_{C_\rho} \frac{1}{z-\alpha}\,dz + \sum_{k=1}^{\infty}\int_{C_\rho} \frac{(a-\alpha)^k}{(z-\alpha)^{k+1}}\,dz = 2\pi i$$

□

We may now continue with completing the previous proof.

Proof (Continued - Theorem 13.1.2)
Lemma 14.1.1 completes the part where we required $\int_C \frac{dz}{z-w} = 2\pi i$.

We now have

$$f(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dz$$

Now note that if we further generalize the number of times the contour C_ρ *made around* a, *where in this case* C_ρ *is a closed path instead of a simple circle in* Ω, *in Lemma 14.1.1, we would get* $\int_{C_\rho} \frac{dz}{z-a} = 2k\pi i$ *where* k *would represent that number.*

In this case, we would get

$$f(w)k = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dz$$

where $k = \mathrm{Ind}_C(w) = \frac{1}{2\pi i}\int_C \frac{dz}{z-w}$ *which represents the number of times the contour* C *winds around* w. □

Remark
As noted, Theorem 13.1.2 holds for any closed path $\gamma \in \Omega$ instead of a simple circle C. If $w \in \Omega \setminus \gamma^*$, we get

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz = f(w)\,\mathrm{Ind}_\gamma(w)$$

Proposition 14.1.1 (Holomorphic Functions can be expressed as Power series)
Let $\Omega \subseteq \mathbb{C}$ be an open set, $f \in H(\Omega)$. Then f can be expressed as a power series.

Proof
$\forall w \in \Omega, \exists C \subseteq \Omega$ that is a closed circle path with $w \in C^0$. By Theorem 13.1.2, and since C is a circle, i.e. the contour winds around w only once, we have

$$f(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dz.$$

Let $w_0 \in \Omega$ be the center of C. Then $\forall z \in \partial C, 0 < |w-w_0| < |z-w_0|$[1]. This implies that

$$\begin{aligned}
&0 < \frac{|w-w_0|}{|z-w_0|} < 1 \\
&\implies \sum_{n=0}^{\infty}\left(\frac{w-w_0}{z-w_0}\right)^n = \frac{1}{1-\frac{w-w_0}{z-w_0}} = \frac{z-w_0}{z-w} \quad \text{by the Infinite Geometric Sum} \\
&\implies \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w_0}\frac{z-w_0}{z-w}\,dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w_0}\sum_{n=0}^{\infty}\left(\frac{w-w_0}{z-w_0}\right)^n dz
\end{aligned}$$

Note that each of the terms in the integrand of the last expression are absolutely convergent, thus by ***Fubini's Theorem****, we can interchange the summation and integral sign to get*

$$f(w) = \sum_{n=0}^{\infty}\underbrace{\left[\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-w_0)^{n+1}}\,dz\right]}_{a_n}(w-w_0)^n$$

which is a power series centered at w_0 with coefficient a_n.

Note (Recall)
Consider the power series $f(w) = \sum_{n=0}^{\infty} a_n(w-w_0)^n$. Recall Item 2 from Section 12.2 that

$$a_n = \frac{f^{(n)}(w_0)}{n!}$$

[1] This is the key step

Applying this to Proposition 14.1.1, we get

$$\frac{f^{(n)}(w_0)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-w_0)^{n+1}}\,dz$$

which holds for any $w_0 \in \Omega$ by having $C \subseteq \Omega$ centered at w_0.

Theorem 14.1.1 (Cauchy's Integral Formula 2)
Let $\Omega \subseteq \mathbb{C}$ be open, $f \in H(\Omega)$. Then

1. *$\forall w \in \Omega$, f has a power series expansion at w.*
2. *f is differentiable infinitely many times in Ω.*
3. *$\forall C \subseteq \Omega$ that is a closed circle oriented anticlockwise, we have that $\forall w \in C^0$,*

$$f^{(n)}(w) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-w)^{n+1}}\,dz \tag{14.3}$$

Remark
Item 3 is the actual Cauchy's Integral Formula in the theorem.

Proof
We have shown 1 from Proposition 14.1.1 and 2 from Theorem 8.1.2. It remains to prove 3, which we shall prove by induction.

When $n = 0$, it is simply Theorem 13.1.2. Suppose f has up to $n-1$ complex derivatives and that

$$f^{(n-1)}(w) = \frac{(n-1)!}{2\pi i}\int_C \frac{f(z)}{(z-w)^n}\,dz.$$

Consider $h > 0$, the difference of the quotient for $f^{(n-1)}$ is

$$\frac{f^{(n-1)}(w-h) - f^{(n-1)}(w)}{h} = \frac{(n-1)!}{2\pi i}\int_C f(z)\frac{1}{h}\left[\frac{1}{z-w-h} - \frac{1}{z-w}\right]dz \tag{14.4}$$

Note that

$$A^n - B^n = (A-B)(A^{n-1} + A^{n-2}B + \ldots + AB^{n-2} + B^{n-1})$$

Let $A = \frac{1}{z-w-h}$, $B = \frac{1}{z-w}$[2]*, then the term in square brackets in Equation (14.4) becomes*

$$\frac{h}{(z-w-h)(z-w)}\left[A^{n-1} + A^{n-2}B + \ldots + AB^{n-2} + B^{n-1}\right]$$

[2] Key step

Thus as $h \to 0$, we have

$$f^{(n)} = \frac{(n-1)!}{2\pi i} \int_C f(z) \left[\frac{1}{(z-w)^2} \right] \left[\frac{n}{(z-w)^{n-1}} \right] dz = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-w)^{n+1}} \, dz$$

which completes the induction proof and proves 3. □

Corollary 14.1.1 (Taylor Expansion of Entire Functions)
If f is an entire function, then $\forall z_0 \in \mathbb{C}$, we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots$$

which is a ***Taylor Expansion*** *of f around z_0.*

Proof
By Proposition 14.1.1, we have that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^{n+1}} \, dw \right] (z - z_0)^n \\ &= \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_0} \, dw + \left[\frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^2} \right] (z - z_0) \\ &\quad + \left[\frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^3} \, dw \right] (z - z_0)^2 + \dots \\ &\quad + \left[\frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^{k+1}} \, dw \right] (z - z_0)^k + \dots \end{aligned} \tag{14.5}$$

Now by Theorem 14.1.1, we have

$$\begin{aligned} f(z_0) = f^{(0)}(z_0) &= \frac{0!}{2\pi i} \int_C \frac{f(w)}{w - z_0} \, dw = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_0} \, dw \\ f^{(1)}(z_0) &= \frac{1!}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^2} \, dw \\ f^{(2)}(z_0) &= \frac{2!}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^3} \, dw \\ &\vdots \\ f^{(k)}(z_0) &= \frac{k!}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^{k+1}} \, dw \\ &\vdots \end{aligned}$$

Thus Equation (14.5) becomes

$$f(z) = f(z_0) + f^{(1)}(z_0)(z - z_0) + \frac{f^{(2)}(z_0)}{2!}(z - z_0)^2 + \ldots + \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k + \ldots$$

as required. □

Chapter 15

Lecture 15 Feb 14th 2018

15.1 Cauchy's Integral Formula (Continued 1)

At this point, it is important that we provide the following definition:

Definition 15.1.1 (Analytic Functions)
*We say that f is **analytic** in Ω if f has a power series expansion at every $z \in \Omega$.*

Remark

1. *We have proven, in the previous lecture, that Holomorphicity $\implies$ Analyticity*
2. *Should we have defined, in Theorem 14.1.1, that the closed circle orients clockwise, then we would have a negative equation for Equation (14.3).*

15.1.1 Applications of Cauchy's Integral Formula

Exercise 15.1.1

1. ***(Cauchy's Inequality)***[1] *Prove that $\forall z_0 \in \mathbb{C}\ \forall R > 0 \in \mathbb{R}\ \forall f \in H(C = D(z_0, R))$*

$$f^{(n)}(z_0) \leq \frac{n!}{R^n} \cdot \sup_{z \in \mathbb{C}} |f(z)|$$

Proof
From Equation (14.3), we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

[1] In a sense, this inequality implies that as we take higher derivatives, the value of the derivatives become smaller.

Parameterize C with $\gamma : [0, 2\pi] \to \mathbb{C}$, where $t \mapsto z_0 + Re^{it}$. Then

$$\begin{aligned}
f^{(n)}(z_0) &= \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{(Re^{it})^{n+1}} Rie^{it} \, dt \\
\left| f^{(n)}(z_0) \right| &\leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{\left| f(z_0 + Re^{it}) \right|}{R^n} \, dt \qquad \because \left| Re^{it} \right| = R \\
&\leq \frac{n!}{2\pi R^n} \sup_{z \in C} |f(z)| \int_0^{2\pi} dt \\
&= \frac{n!}{R^n} \sup_{z \in C} |f(z)|
\end{aligned}$$

This completes the proof. □

2. ***(Liouville's Theorem)*** *A bounded entire function $f : \mathbb{C} \to \mathbb{C}$ is a constant*[2] [3].

   Proof

   Since f is entire, we may take R, in Item 1, to be any large value. Let M be the bound of f, i.e. $\exists M \in \mathbb{C}, \forall z_0 \in \mathbb{C}, \left| f^{(n)}(z_0) \right| \leq \frac{n!}{R^n} \sup_{z \in \mathbb{C}} |f(z)| = \frac{n!}{R^n} \sup_{z \in \mathbb{C}} M$. Let $n = 1$, then $|f'(z_0)| = \frac{M}{R}$. Thus we observe that $R \to \infty \implies f(z_0) \to 0$ for any $z_0 \in \mathbb{C}$. By A2Q5(a), f is a constant.

3. ***(Parseval's Theorem)*** *Let $\Omega \subseteq \mathbb{C}$ be open, $f \in H(\Omega)$, $\overline{D(z_0, R)} \subseteq \Omega$. Then $\forall z \in \overline{D(z_0, R)}$, $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$, which in turn implies that* [4]

$$\forall z \in \overline{D(z_0, R)} \quad f(z_0 + re^{i\theta}) = \sum_{n=0}^{\infty} c_n (re^{i\theta})^n \tag{$\dagger$}$$

[2] The theorem is not true in $\mathbb{R}$, since $\sin x$ is a bounded function differentiable everywhere, but is not a constant.

[3] The theorem also implies that "trigonometry" in $\mathbb{C}$ is unbounded, whatever the definition of "trigonometry" may be.

[4] This is why the L^2-norm is perserved, as seen in AMATH231.

Consider (the L^2norm)

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\left|f(z_0+re^{i\theta})\right|^2 d\theta \\
=&\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{n=0}^{\infty}c_n(re^{i\theta})^n\right|^2 d\theta \\
=&\frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{n=0}^{\infty}c_n r^n e^{in\theta}\right]\left[\sum_{m=0}^{\infty}\overline{c_m}r^m e^{-in\theta}\right] d\theta \\
=&\frac{1}{2\pi}\int_0^{2\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c_n\overline{c_m}r^{n+m}e^{i(n-m)\theta}\,d\theta
\end{aligned}
$$

Since the series are absolutely convergent, use may use Fubini's Theorem, and thus

$$
\begin{aligned}
=&\frac{1}{2\pi}\sum_{n,m=0}^{\infty}c_n\overline{c_m}r^{n+m}\int_0^{2\pi}e^{i(n-m)\theta}\,d\theta \\
=&\begin{cases}\frac{1}{2\pi}\sum_{n,m=0}^{\infty}c_n\overline{c_m}r^{n+m}2\pi & \text{if } n=m \\ \frac{1}{2\pi}\sum_{n,m=0}^{\infty}c_n\overline{c_m}r^{n+m}\frac{e^{i(n-m)\theta}}{i(n-m)}\Big|_0^{2\pi}=0 & \text{if } n\neq m\end{cases} \\
=&\sum_{n=0}^{\infty}|c_n|^2 r^{2n} \quad \text{if } n=m
\end{aligned}
$$

Therefore, we have what is known as ***Parseval's Identity****:*

$$
\frac{1}{2\pi}\int_0^{2\pi}\left|f(z_0+re^{i\theta})\right|^2 d\theta=\sum_{n=0}^{\infty}|c_n|^2 r^{2n} \tag{15.1}
$$

Parseval's Theorem states that:

L^2-norm of LHS in Equation (15.1) = L^2-norm of RHS of Equation (†)

Before going into the next application, please see Lemma 15.1.1.

4. ***(Maximum Modulus Principle)*** *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and $f \in H(\Omega)$. Then*

$$
\sup_{z\in\Omega}|f(z)|=\max_{z\in\partial\Omega}|f(z)|.
$$

This implies that f cannot attain its maximum value in Ω^0.

Proof
Suppose not, i.e. $\exists z_0 \in \Omega^0$, $\forall z \in \Omega$ *such that* $|f(z_0)| = \max_{z\in\Omega} |f(z)| \geq |f(z)|$

$$\implies \exists r > 0 \quad \overline{D(z_0, r)} \subseteq \Omega$$
$$\implies \forall z \in \overline{D(z_0, r)} \quad f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

Note that $c_0 = \frac{f^{(0)}(z_0)}{0!} = f(z_0)$. *By Item 3,*

$$\begin{aligned}
\sum_{n=0}^{\infty} |c_n|^2 r^{2n} &= \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + re^{i\theta}) \right|^2 d\theta \\
\implies f(z_0)^2 + \sum_{n=1}^{\infty} |c_n|^2 r^{2n} &= \frac{1}{2\pi} \left| f(z_0 + re^{i\theta}) \right|^2 d\theta \\
&\leq \frac{1}{2\pi} |f(z_0)|^2 (2\pi) \quad \because f(z_0) = \max_{z\in\Omega} f(z) \\
\implies f(z_0)^2 + \sum_{n=1}^{\infty} |c_n|^2 r^{2n} &\leq |f(z_0)|^2 \\
\implies \sum_{n=1}^{\infty} |c_n|^2 r^{2n} &\leq 0 \\
\implies c_1, c_2, \ldots &= 0 \\
&\implies f \text{ is a constant in } \overline{D(z_0, r)} \\
&\implies f \text{ is a constant in } \Omega \text{ by Lemma 15.1.1}
\end{aligned}$$

which is a contradiction. □

Lemma 15.1.1 (Principle of Analytic Continuation)
Let $\Omega \subseteq \mathbb{C}$ *be open and connected, and* $f \in H(\Omega)$. *Let* $Z(f) = \{a in \Omega : f(a) = 0\}$. *Then either*

- $Z(f) = \Omega$, *i.e.* $\forall z \in \Omega$, $f(z) = 0$; *or*
- $Z(f)$ *has no limit point, i.e. points where* $f = 0$ *are isolated*

This is a powerful result, since if we can find a small region for where f is 0 in Ω, then f would be 0 in the entirety of Ω. If not, then f is only 0 at isolated points, i.e. points where $f = 0$ are all apart from each other.

Chapter 16

Lecture 16 Feb 16th 2018

16.1 Cauchy's Integral Formula (Continued 3)

16.1.1 Applications of Cauchy's Integral Formula (Continued)

Exercise 15.1.1 (Continued)

We shall restate the Item 4 in the following manner.

4. **Maximum Modulus Principle (MMP)** Let $\Omega \subseteq \mathbb{C}$, $f \in H(\Omega)$, $D_{z_0} = \overline{D(z_0, r)} \subseteq \Omega$. Then $|f(z_0)| \leq \max_{z \in \partial D_{z_0}} |f(z)|$ with

$$|f(z_0)| = \max_{z \in \partial D_{z_0}} |f(z)| \iff f \text{ is a constant on } \Omega$$

Remark

(a) This implies that for a non-constant analytic function f, $\forall z \in \Omega^0$, $f(z) \neq \max_{w \in \Omega} f(w)$.

(b) Since a global maximum is also a local maximum, we observe that for any smaller region $\Omega_0 \subseteq \Omega$, f cannot attain its maximum value for any point in Ω_0^0. This is a stronger statement than the our previous statement about the MMP.

Proof

Suppose for $\lightning$ that f has a maximum in Ω^0, say at z_0. Hence $\exists r > 0, D_{z_0} = \overline{D(z_0, r)}$ where

$$|f(z_0)| \geq \max_{z \in D_{z_0}} |f(z)|$$

On D_{z_0}, we have

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \tag{16.1}$$

Note that $c_0 = f(z_0)$. By Item 3, on D_{z_0},

$$\begin{aligned}
\sum_{n=0}^{\infty} |c_n|^2 r^{2n} &= \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + re^{i\theta}) \right|^2 d\theta \quad \text{by Equation (15.1)} \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|^2 \, d\theta \quad \text{by Equation (16.1)} \\
&= |f(z_0)|^2 .
\end{aligned}$$

Then we have

$$\begin{aligned}
|c_0|^2 + \sum_{n=1}^{\infty} |c_n|^2 r^{2n} &= |f(z_0)|^2 \\
|f(z_0)|^2 + \sum_{n=1}^{\infty} |c_n|^2 r^{2n} &= |f(z_0)|^2 \\
\sum_{n=1}^{\infty} |c_n|^2 r^{2n} &= 0
\end{aligned}$$

which $\implies$ $c_1 = c_2 = \ldots = 0$. Thus $\forall z \in D_{z_0}$, $\underline{f(z) \equiv c_0}$ mod 2π. Then by Lemma 15.1.1, since $f(z_0) - c_0$, as $f(z) - c_0)$ contains $D(z - 0, r)$, we see that $f(z) - c_0 \equiv 0$ in Ω, which implies the equality of Item 4. □

5. **Fundamental Theorem of Algebra (FTA)** Any polynomial $P(z) \in \mathbb{C}[z]$ of degree greater than 1 has precisely n roots in $\mathbb{C}$, given by $\alpha_1, \alpha_2, ..., \alpha_n$. $P(z)$ can be factored as $P(z) = A(z - \alpha_1) \ldots (z - \alpha_n)$ for some $A \in \mathbb{C}$.

   Proof
   We may write $P(z) = A(z^n + a_{n-1} z^{n-1} + \ldots a_1 z + a_0)$, which then

   $$\frac{P(z)}{z^n} = A \left(1 + \frac{a_{n+1}}{z} + \ldots \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right)$$

   which then, by the Reverse Triangle Inequality,

   $$\implies \left| \frac{P(z)}{z^n} \right| \geq |A| \left[1 - \frac{|a_{n-1}|}{|z|} - \ldots - \frac{|a_1|}{|z^{n-1}|} - \frac{|a_0|}{|z^n|} \right] \tag{16.2}$$

So as $|z| \to \infty$, $\left|\frac{P(z)}{z^n}\right| \to |A|$, *from Equation (16.2). Since* $|z| \to \infty$, $\exists R > 0$, $\forall |z| > R$, *then* $\forall \theta \in [0, 2\pi]$,

$$\left|P(Re^{i\theta})\right| = |P(z)| \geq \frac{|A|}{2} |z|^n \geq \frac{|A|}{2} R^n$$

Taking R *to be even larger if necessary, we can get*

$$\left|P(Re^{i\theta})\right| \geq |P(0)| \tag{†}$$

Suppose, for contradiction, $P(z)$ *has no root in* $\mathbb{C}$*. Then* $g(z) = \frac{1}{P(z)}$ *is an entire function. By Equation (†), we have that* $\left|g(Re^{i\theta})\right| \leq |g(0)|$ *for all* $\theta \in [0, 2\pi]$*. But this contradicts Item 4 unless if* $g(z)$ *is constant on* $\mathbb{C}$*, which in turn implies that* P *is a constant, but that contradicts that* P *has degree greater than* 1*.*

$\therefore$ $P(z)$ *has to have a zero in* $\mathbb{C}$*, say* α_1*. This implies that*

$$P(z) = A(z - \alpha_1) P_1(z)$$

where $P_1(z) \in \mathbb{C}[z]$*. By repeatedly taking the above steps, inductively so, for* $P_1, P_2, \ldots$*, the proof is completed.* $\square$

Chapter 17

Lecture 17 Feb 26th 2018

17.1 Analytic Continuity

We shall restate the important lemma that we have been using in the last two lectures, and proceed to prove this lemma.

Lemma 17.1.1 (Principle of Analytic Continuity)
Let $\Omega \subseteq \mathbb{C}$ be open and connected, and $f \in H(\Omega)$. Let $Z(f) = \{a in \Omega : f(a) = 0\}$. Then either

- *$Z(f) = \Omega$, i.e. $\forall z \in \Omega$, $f(z) = 0$; or*
- *$Z(f)$ has no limit point, i.e. points where $f = 0$ are isolated*

Proof
Let $z_0 \in Z(f)^*$.

Step 1: *Show that $z_0 \in Z(f)^0$, i.e. f is identically 0 on some $\overline{D(z_0, r)} \subseteq \Omega$ for $r > 0$.*

On $\overline{D(z_0, r)}$, $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$. Suppose f is not identically 0 on $\overline{D(z_0, r)}$. Then $\exists m \in \mathbb{N}$, $c_m \neq 0$, $\forall j < m$, $c_j = 0$, i.e. $f(z) = c_m (z - z_0)^m + c_{m+1}(z - z_0)^{m+1} + \ldots$.

Define, in Ω,

$$g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^m} & z \in \Omega \setminus \{z_0\} \\ c_m & z = z_0 \end{cases}$$

Clearly, $g \in H(\Omega \setminus \{z_0\})$. But on $\overline{D(z_0, r)}$,

$$g(z) = c_m + c_{m+1}(z - z_0) + c_{m+2}(z - z_0)^2 + \ldots$$

which implies $g \in H(\Omega)$. Now $g(z_0) = c_m \neq 0$, so there exists a neighbourhood U_{z_0} of z_0, such that $g \neq 0$ on U_{z_0}.

$\forall a \neq z_0 \in Z(f)$, we have that $g(a) = 0$ by defintion of $Z(f)$, which implies that $a \notin U_{z_0}$, which contradicts that $z_0 \in Z(f)^*$. This implies $f \equiv 0$ in $\overline{D(z_0, r)}$.

Step 2: *$Z(f)^0$ is both open and closed.*

Note that

$$Z(f)^0 := \left\{ a \in Z(f) \,:\, \exists r > 0,\, \overline{D(a,r)} \subseteq Z(f) \right\}$$

is open by definition.

WTP *$[Z(f)^0]^* \subseteq [Z(f)]^*$.*

*From **Step 1**, we know that $[Z(f)^0]^* \subseteq Z(f)^0$. Thus $Z(f)^0$ contains its limit points and is hence closed by definition.*

Step 3: *$Z(f) = \emptyset$ or Ω.*

$$\begin{aligned} &\Omega \text{ is connected} \\ &\implies \Omega = Z(f)^0 \sqcup \left(Z(f)^0\right)^c \\ &\implies \left(Z(f)^0\right)^c \text{ is open and closed by } \textbf{Step 2} \end{aligned}$$

A connected set cannot be expressed as a disjoint union of non-trivial open sets. Therefore, either $Z(f)^0 = \emptyset$ or $Z(f)^0 = \Omega$.

$$\begin{aligned} Z(f)^0 = \emptyset &\implies Z(f)^* = \emptyset \text{ by } \textbf{Step 1} \implies Z(f) = \emptyset \\ Z(f)^0 = \Omega &\implies Z(f) = \Omega \text{ by } \textbf{Step 1} \end{aligned}$$

□

Corollary 17.1.1 (Uniqueness of a Function)
Let $\Omega \subseteq \mathbb{C}$ be open and connected. $\forall f, g \in H(\Omega)$ with $f(z) = g(z)$ for $z \in \Omega_1 \subseteq \Omega$ where Ω_1 has limit points. Then $\forall z \in \Omega$, $f(z) = g(z)$.

Proof
Apply Lemma 15.1.1 to the function $f - g$.

Remark

1. *In $\mathbb{C}$, we cannot have two functions sharing a region of points in their images. (But this is possible in $\mathbb{R}$)*

2. *Suppose $f \in H(\Omega)$, $\Omega \in \mathbb{C}$ is open and connected, $F \in H(\Omega')$ with $\Omega \subseteq \Omega'$. If f, F agree on Ω, then F is called an analytic continuation of f in Ω' (i.e. F 'extends' f in Ω'). Lemma 15.1.1 states that F is uniquely determined by f, i.e. there is a unique way to analytically 'continue' f.*

17.2 Morera's Theorem

Remark (Recall)
From Cauchy's Theorem, we know that $\forall f \in H(\Omega) \implies \forall \gamma \in \Omega \int f = 0$. We used Goursat's Theorem, i.e. $\forall \Delta \in \Omega \int_\Delta f = 0$ to proof this, and in the process we constructed an antiderivative. Now, our question is, is the converse of the said Cauchy's Theorem true?

Unfortunately for us, that is not true (**example needed**). But a "partial" converse exists.

Theorem 17.2.1 (Morera's Theorem)
Let f be continuous on $\Omega \subseteq \mathbb{C}$, which is an open set, and $\forall \Delta \in \Omega$, $\int_\Delta f = 0$, where Δ is a triangular path. Then $f \in H(\Omega)$.

Proof
Use the same construction as in Cauchy's Theorem for Convex Sets to get an antiderivative F for f, where $F \in H(\Omega)$, i.e.

$$F(z) := \int_{[a,z]} f(z)\, dz$$

Then $F'(z) = f(z)$, which in turn implies that $f \in H(\Omega)$ since F is $\mathbb{C}$-differentiable on Ω by Theorem 14.1.1.

Chapter 18

Lecture 18 Feb 28th 2018

18.1 Winding Numbers

Recall Cauchy's Integral Formula. We claimed that

$$\mathrm{Ind}_C(w) = \begin{cases} 1 & w \in C^0 \\ 0 & w \notin C \end{cases}$$

We will now formally define this index.

Definition 18.1.1 (Winding Numbers)
Let $\gamma : [a, b] \to \mathbb{C}$ be a closed and oriented anti-clockwise, and γ^* be the image of γ in $\mathbb{C}$. Let $\Omega = \mathbb{C} \setminus \gamma^*$. $\forall w \in \Omega$, define the index of w with respect to γ as

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z - w}$$

in which shall be called the winding number of γ around w.

Theorem 18.1.1 (Winding Number Theorem)
We shall use notation as the definition above. $\mathrm{Ind}_\gamma(w)$ is

1. *always an integer;*
2. *constant on any connected component of Ω; and*
3. *zero on the unbounded component of Ω.*

Note

γ is compact in $\mathbb{C}$ (since it creates a ring from $[a, b]$ under γ). So for some disc D, $\gamma^* \subseteq D$. Let $\Omega \supset \mathbb{C} \setminus D$, where we note that the contained set is connected and unbounded. Then Ω contains one unbounded component, while other components of Ω are inside D. Therefore, we know that components in D are bounded.

Proof

1. *By definition,*

$$\begin{aligned}\operatorname{Ind}_\gamma(w) &= \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-w} \\ &= \frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)\,dt}{\gamma(t)-w}\end{aligned}$$

WTS $\operatorname{Ind}_\gamma(w) \in \mathbb{Z} \equiv \int_a^b \frac{\gamma'(t)\,dt}{\gamma(t)-w} \in 2\pi i\mathbb{Z}$.

Note that $z \in 2\pi i\mathbb{Z} \iff e^z = 1$. Thus it suffices to show that

$$e^{\int_a^b \frac{\gamma'(t)\,dt}{\gamma(t)-w}} = 1$$

Idea: *Think of* $\exp\left(\int_a^u \frac{\gamma'(t)\,dt}{\gamma(t)-w}\right)$ *as a function of u, call it $\phi(u)$. Then we just need to show that $\phi(b) = 1$. We know that $\phi(a) = \exp\left(\int_a^a \dots\right) = 1$. This motivates us to find the derivative of ϕ.*

Define ϕ accordingly, and then since $(e^{f(u)})' = e^{f(u)} \cdot f'(u)$,

$$\begin{aligned}\phi'(u) &= \phi(u) \cdot \frac{d}{du}\int_a^u \frac{\gamma'(t)\,dt}{\gamma(t)-w} \\ \textit{by FTC} \implies \frac{\phi'(u)}{\phi(u)} &= \frac{\gamma'(u)}{\gamma(u)-w} \\ &\implies \phi'(u)\left(\gamma(u)-w\right) - \gamma'(u)\phi(u) = 0 \\ &\implies \frac{d}{du}\left(\frac{\phi(u)}{\gamma(u)-w}\right) = 0 \quad \textit{by quotient rule} \\ &\implies \frac{\phi(b)}{\gamma(b)-w} = \frac{\phi(a)}{\gamma(a)-w} \;\; \textit{since } \frac{\phi(u)}{\gamma(u)-w} \textit{ is a constant function of } u \\ &\implies \phi(b) = \phi(a) = 1 \;\; \because \gamma \textit{ is closed.}\end{aligned}$$

We will prove that $\operatorname{Ind}_\gamma(w)$ *is continuous.*

$$\forall w \in \Omega\ \forall z \in \gamma^*\ \ \exists M > 0\ \ |w - z| > M$$

$$\forall \varepsilon > 0\ \exists \delta = \frac{M^2 \pi \varepsilon}{\int_\gamma dz} > 0\ \forall w_0 \in \Omega$$

$$|w - w_0| < \delta \wedge |w_0 - z| > \frac{M}{2}$$

then

$$\begin{aligned}
|\operatorname{Ind}_\gamma(w) - \operatorname{Ind}_\gamma(w_0)| &= \left| \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - w} - \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - w_0} \right| \\
&= \frac{1}{2\pi} \left| \int_\gamma \frac{w - w_0}{(z - w)(z - w_0)}\, dz \right| \\
&\leq \frac{1}{2\pi} \int_\gamma \left| \frac{w - w_0}{(z - w)(z - w_0)} \right| dz \\
&< \frac{1}{2\pi} \delta \int_\gamma \left| \frac{2}{M \cdot M} \right| dz \\
&= \frac{1}{M^2 \pi} \delta \int_\gamma dz = \varepsilon
\end{aligned}$$

2. *Also* $\operatorname{Ind}_\gamma(w)$ *takes only integer values, thus it must be constant on each open connected component*[1] *(**why?**).*

3. *Note that*

$$|\operatorname{Ind}_\gamma(w)| = \frac{1}{2\pi} \left| \int_a^b \frac{\gamma'(t)\, dt}{\gamma(t) - w} \right|$$

Let w *be in the unbounded component in the complement of* γ *such that* $|w| \to \infty$. *Then* $\forall t \in [a, b]$, $\exists M > 0$ *such that*

$$\frac{1}{|\gamma(t) - w|} \leq \frac{1}{M}$$

which implies that

$$|\operatorname{Ind}_\gamma(w)| \leq \frac{1}{2\pi} \frac{1}{M} \cdot \underbrace{\int_a^b |\gamma'(t)|\, dt}_{\substack{\text{is a fixed constant} \\ \text{as } \gamma \text{ is a fixed path}}}$$

$$\implies (|w| \to \infty \implies M \to \infty \implies |\operatorname{Ind}_\gamma(w)| \to 0)$$

Then by parts 1 and 2, the proof is completed. □

[1] We may invoke Lemma 15.1.1 but it is, to an extent, unnecessary for such a powerful statement.

Remark

Note that by 2, we have that $\forall w \in C^0$,

$$\frac{1}{2\pi i}\int_C \frac{dz}{z-w} = \frac{1}{2\pi i}\int_C \frac{dz}{z-z_0} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{Rie^{i\theta}}{Re^{i\theta}}\,d\theta = 1$$

where z_0 *is the center of the circle path* C*.*

Chapter 19

Lecture 19 Mar 2nd 2018

19.1 Singularities

Exercise 19.1.1
Let $C : [0, 2\pi] \to \mathbb{C}$ *such that* $\forall t \in [0, 2\pi]$, $t \to e^{it}$. *Suppose* $f \in H(\Omega)$, *then by Cauchy*

$$\int_C f(z)\, dz = 0$$

Let $f(z) = \frac{1}{z}$, *then* $\int_C \frac{1}{z}\, dz = 2\pi i \operatorname{Ind}_C(0) = 2\pi i$ *when it is "supposed" to be* 0 *by the argument above. Then in this case,* $f \notin H(\Omega)$. *In fact,* f *is undefined at* 0.

The example above introduces us to the study of such exceptional points.

Definition 19.1.1 ((Isolated) Singularity)
$\forall a \in \mathbb{C}$, $\exists r > 0$, $\exists D = D(a, r)$.

$$f \in H(D \setminus \{a\}) \;\wedge\; f(a) \text{ is undefined} \iff$$

f has a(n) ***point/isolated singularity*** *at* $z = a$.

Example 19.1.1

1. *Given* $f \in H(\mathbb{C} \setminus \{0\})$, *define* $f(z) = \frac{e^z - 1}{z}$. *Clearly,* z *is a singularity. Consider the function* $(e^z - 1) \in H(\mathbb{C})$. *Then we have that the function has a power series expansion around* $z = 0$. *So* $\forall z \in \mathbb{C}$,

$$e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

And for $z \neq 0$, we have

$$\frac{e^z - 1}{z} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots \tag{19.1}$$

This motivates us to define

$$g(z) = \begin{cases} \frac{e^z - 1}{z} & z \in \mathbb{C} \setminus \{0\} \\ 1 & z = 0 \end{cases}$$

Clearly then $g \in H(\mathbb{C})$, where in $\mathbb{C} \setminus \{0\}$ its holomorphicity is given by f, and in a neighbourhood of 0, from Equation (19.1). Therefore, y assigning f the value of 1 at $z = 0$, we can make f "entire".

We call such a point z as a ***removable singularity*** *for f.*

2. *Given $f \in H(\mathbb{C} \setminus \{0\})$, define $f(z) = \frac{1}{z}$. Is the singularity at 0 removable?*

   Suppose $\exists g \in H(\mathbb{C})$ such that

$$\forall z \in \mathbb{C} \setminus \{0\} \quad g(z) = f(z) \tag{19.2}$$

   $\therefore \exists r > 0 \; \forall z \in D(0, r)$

$$g(z) = c_0 + c_1 z + c_2 z^2 + \dots \tag{19.3}$$

   Consider the function $zg(z)$. By Equation (19.2),

$$\forall z \in \mathbb{C} \setminus \{0\} \quad zg(z) = 1$$

   *By Equation (19.3), $z = 0 \implies zg(z) = 0$. But this cannot happen since $zg(z) \in H(\mathbb{C})$ (**if we pick an open ball of, say, $\frac{1}{2}$ around 0, then there are no points in the entirety of $\mathbb{C}$ that is close to 0**). Therefore $z = 0$ is not a removable singularity for f.*

Definition 19.1.2 (Removable Singularity, Pole, Essential Singularity)
Let f have a singularity at $z_0 \in \mathbb{C}$.

1. $\exists r > 0 \;\; \forall z \in D = D(z_0, r) \;\; \exists g(z) \in H(D) \;\; \forall z \in D \setminus \{z_0\} \;\; g(z) = f(z)$
   $\implies$ *f has a* ***removable singularity*** at z_0[1].

2. $\exists r > 0 \;\; \forall z \in D = D*(z_0, r) \;\; \exists A, B \in H(D) \;\; A(z_0) \neq 0 \wedge B(z_0) = 0 \;\; f(z) = \frac{A(z)}{B(z)}$
   $\implies$ *f has a* ***pole*** at z_0 *(a non-removable singularity)*[2]

3. *f has a singularity at z_0 which is neither removeable nor a pole $\implies$ f has an* ***essential singularity*** at z_0.

[1] For the laymen, "the value of f at z_0 can be corrected or defined to make it holomorphic in its designated region."

[2] For the laymen, "the singularity of f comes from a zero of its denominator."

Example 19.1.2

To show an example of an essential singularity, consider the function $f(z) = e^{\frac{1}{z}}$. If we attempt to do a "Taylor expansion" on the function (which is invalid at $z = 0$), we have

$$f(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$$

The point 0 for f is said to be a "pole of infinite order" (this shall be defined later on)

While **removable singularities** are nice to have, they are not as interesting to us. On the other hand, we are more interested in their non-removable counterpart, the **poles**. This motivates the study of zeros of holomorphic functions.

Theorem 19.1.1 (Theorem 9)

Let $\Omega \subseteq \mathbb{C}$ be open and connected. Suppose that $f \in H(\Omega)$ with $f \not\equiv 0$ on Ω and that f has a zero at $z_0 \in \Omega$. Then

$$\begin{gathered} \exists r > 0 \;\; \forall z \in D = D(z_0, r) \;\; \exists g \in H(D) \;\; g(z_0) \neq 0 \;\; \exists! n \in \mathbb{N} \\ f(z) = (z - z_0)^n \cdot g(z) \end{gathered} \tag{19.4}$$

Proof

By Analytic Continuation, zeros of f are isolated since $f \not\equiv 0$. So $\exists r > 0$ such that $\exists D = D(z_0, r)$, in which $\forall z \in D \setminus \{z_0\}$, $f(z) \neq 0$.

Since $f \in H(\Omega)$, $\forall z \in D$,

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k$$

As $f \not\equiv 0$ in D, $\exists n \in \mathbb{N} \setminus \{\nvdash\}$ that is the smallest such that $c_n \neq 0$[3].

$$\begin{aligned} \therefore f(z) &= c_n (z - z_0)^n + c_{n+1} (z - z_0)^{n+1} + \dots \\ &= (z - z_0)^n \underbrace{[c_n + c_{n+1}(z - z_0) + \dots]}_{\text{call this } g(z)} \end{aligned}$$

Note that $g(z_0) \neq 0$ since $c_n \neq 0$. Thus $g(z) \in H(D)$ since it has the same radius of convergence as f.

To prove uniqueness, suppose that we may write

$$f(z) = \sum_{k=0}^{\infty} (z - z_0)^n \cdot g(z) = (z - z_0)^m \cdot h(z)$$

[3] $n \neq 0$ since we have $f(z_0) = 0$ which implies $c_0 = 0$.

for some $m \in \mathbb{N}$ *and that* $h(z) \neq 0$*. If* $m > n$*, dividing both sides by* $(z - z_0)^n$*,*

$$g(z) = (z - z_0)^{m-n} h(z).$$

As $z \to z_0$*, we would have* $g(z_0) = 0$*, which is a contradiction. If* $m < n$*, we can perform a similar argument and have that as* $z \to z_0$*,* $h(z_0) = 0$*, also a contradiction. Therefore,* $m = n$ *and* $h = g$*.* $\square$

We say that f has a **zero of order** n at z_0 if Equation (19.4) holds.

Chapter 20

Lecture 20 Mar 5th 2018

20.1 Singularity (Continued)

Recall the definition of a **removable singularity** from Definition 19.1.2.

Theorem 20.1.1 (Theorem 10)
If $f \in H(\Omega \setminus \{z_0\})$ has an isolated singularity at z_0 and $\lim_{z \to z_0}(z - z_0)f(z) = 0$, then the singularity at z_0 is removable.

Proof
Since $f(z_0)$ is undefined, set

$$h(z) = \begin{cases} (z - z_0)^2 f(z) & \forall z \in \Omega \setminus \{z_0\} \\ 0 & z = z_0 \end{cases}$$

Clearly $h \in H(\Omega \setminus \{z_0\})$. At z_0,

$$\begin{aligned} \lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} &= \lim_{z \to z_0} \frac{(z - z_0)^2 f(z)}{z - z_0} \quad [1] \\ &= 0 \text{ by assumption} \end{aligned}$$

$\therefore$ $h'(z_0)$ exists and equals 0. Clearly then that $h \in H(\Omega)$. So $\exists r > 0$ such that $\exists D = D(z_0, r)$, so that $\forall z \in D$,

$$h(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots$$

[1] Goes to show that the definition of h is no foresight.

But $c_0 = h(z_0) = 0$ and $c_1 = h'(z_0) = 0$. Thus the power series can be written as

$$\begin{aligned} h(z) &= c_2(z - z_0)^2 + c_3(z - z_0)^3 + \ldots \\ &= (z - z_0)^2 \left[c_2 + c_3(z - z_0) + \ldots \right] \end{aligned}$$

Hence by the definition of h, $\forall z \in \Omega \setminus \{z_0\}$, $f(z) = c_2 + c_3(z - z_0) + \ldots$. Therefore, by redefining $f(z_0) = c_2$, we see that the singularity at z_0 is removable.

We may also complete the proof by defining a function g as, $\forall z \in \Omega$,

$$g(z) = \begin{cases} f(z) & z \neq z_0 \\ c_2 & z = z_0 \end{cases}$$

□

Recall Theorem 19.1.1

Let $\Omega \subseteq \mathbb{C}$ be open and connected, and $f \in H(\Omega)$ where $\forall z \in \Omega$, $f(z) \neq 0$.

$f(z_0) = 0 \implies$

$$\begin{gathered} \exists r > 0 \;\; \exists D = D(z_0, r) \;\; \forall z \in D \;\; \exists! n \in \mathbb{N} \\ \exists! g \in H(D) \;\; g(z_0) \neq 0 \\ f(z) = (z - z_0)^n g(z) \end{gathered}$$

Definition 20.1.1 (Zero of Order n & Simple Zero)
*By the above setting, we say that f has a **zero of order n** at z_0.*[2]

*If $n = 1$, we say that z_0 is a **simple zero**.*

Recall definition of a pole from Definition 19.1.2

Suppose f has an isolated singularity at z_0, and that there exists a neighbourhood D around z_0 where $A, B \in H(D)$, in which A and B are defined such that $\forall z \neq z_0 \in D$, $A(z_0) \neq 0 \wedge B(z_0) = 0$, so that we can let $f(z) = \frac{A(z)}{B(z)}$. Then f has a pole at z_0.

Theorem 20.1.2 (Theorem 9.1)
If f has a pole at $z_0 \in \Omega$, then in a neighbourhood of that point there exists a non-vanishing holomorphic function h and a unique positive integer n such that

$$f(z) = (z - z_0)^{-n} h(z)$$

[2] In laymen terms, "Rate at which the function vanishes at z_0. The greater n is, the greater the rate."

Stein & Shakarchi - Complex Analysis (pg. 74)

Proof
By Theorem 19.1.1, we have $\frac{1}{f(z)} = (z - z_0)^n g(z)$*, where* g *is holomorphic and non-vanishing in a neighbourhood of* z_0*, so the result follows with* $h(z) = \frac{1}{g(z)}$. □

Definition 20.1.2 (Pole of order n & Simple Pole)
With the above setting, we say that f *has a* ***pole of order*** n *at* z_0 *if the function* B *has a zero of order* n[3]

If $n = 1$*, then* z_0 *is a simple pole.*

Theorem 20.1.3 (Theorem 11)
Let f *have a pole of order* n *at* z_0*. Then* $\exists r > 0$, $\exists D = D(z_0, r)$*, such that* $\forall z \in D \setminus \{z_0\}$,

$$f(z) = \frac{c_{-n}}{(z - z_0)^n} + \frac{c_{-(n-1)}}{(z - z_0)^{n-1}} + \ldots + \frac{c_{-1}}{z - z_0} + G(z)$$

for some $G \in H(D)$.

Proof
By Theorem 20.1.2, write the holomorphic function h *as* $h(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$*, then*

$$f(z) = \frac{1}{(z - z_0)^n} \left[a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots \right].$$

The proof is complete by expanding the equation. □

Definition 20.1.3 (Principal Part)
In Theorem 20.1.3, the sum $\sum_{j=1}^{n} = \frac{c_{-j}}{(z - z_0)^j}$ *is called the* ***principal part*** *of* f *at the pole* z_0.

Definition 20.1.4 (Residue)
In Theorem 20.1.3, the coefficient c_{-1} *is called the* ***residue*** *of* f *at the pole* z_0*, denoted* $\mathrm{res}_{z_0} f$.

The **residue** shall be more carefully studied later on.

[3] In laymen terms, "Rate at which f 'grows' near z_0."

Chapter 21

Lecture 21 Mar 7th 2018

21.1 Singularity (Continued 2)

Theorem 21.1.1 (Casorati-Weierstrass)
Let $z_0 \in \Omega$ and $f \in H(\Omega \setminus \{z_0\})$. Suppose f has a singularity at z_0. Then one of the following occurs:

1. *f is a removable singularity at z_0;*
2. *$\exists m \in \mathbb{N}$, $\{c_j\}_{j=1}^m \subseteq \mathbb{C}$, $f(z) - \sum_{j=1}^m c_j(z - z_0)^- j$ has a removable singularity at z_0; or*
3. *$\forall r > 0$, $B(z_0, r) \subseteq$ such that $f(B^0(z_0, r))$ is dense in $\mathbb{C}$ (Note: $B^0(z_0, r)$ is the punctured ball)*

Proof
Suppose 3. does not hold, i.e. $f(B^0(z_0, r))$ is not dense in $\mathbb{C}$ for some $r > 0$. Then $\exists w \in \mathbb{C}$, $\exists \delta > 0$, such that

$$f\big(B^0(z_0, r)\big) \cap B(w, \delta) = \emptyset$$
$$\implies \forall z \in B^0(z_0, r) \quad |f(z) - w| > \delta$$

Consider $g(z) = \frac{1}{f(z)-w}$ for $z \in B^0(z_0, r)$, in which $g \in H(B^0(z_0, r))$. Then $|g(z)| \leq \frac{1}{\delta}$ for all $z \in B^0(z_0, r)$, which implies that

$$\lim_{z \to z_0} (z - z_0) g(z) = 0$$

By Theorem 20.1.1, g has a removable singularity at z_0, thus we can extend the function to a function $\tilde{g} \in H(B(z_0, r))$. From here, we try to construct a function that extends on f

onto the singularity z_0, say, $\tilde{f}$. The construction of $\tilde{g}$ satisfies the equation $\frac{1}{\tilde{g}(z)} + w = f(z)$ except, possibly, at z_0.

Case 1: *Suppose $\tilde{g}(z_0) \neq 0$.*

We can simply define

$$\frac{1}{\tilde{g}(z)} = \begin{cases} f(z) - w & z \in B^0(z_0, r) \\ \frac{1}{\tilde{g}(z_0)} & z = z_0 \end{cases}$$

Clearly then $\frac{1}{\tilde{g}} \in H(B^0(z_0, r))$. At z_0,

$$\tilde{g}(z_0) \neq 0 \implies \exists r_1 > 0 \; D = D(z_0, r_1) \; \forall z \in D \; \tilde{g}(z) \neq 0 \implies \frac{1}{\tilde{g}} \in H(D)$$

Therefore, $\frac{1}{\tilde{g}} \in H(B(z_0, r))$[1]

Since $\forall z \in B^0(z_0, r)$ $f(z) = \frac{1}{\tilde{g}(z)} + w$ by construction, we may define

$$\tilde{f}(z) = \frac{1}{\tilde{g}(z)} + w$$

such that $f(z_0)$ is defined as $\frac{1}{\tilde{g}(z_0)} + w$. By this construction, 1. holds.

Case 2: *$\tilde{g}(z_0) = 0$.*

$\because \tilde{g} \in H(B^0(z_0, r)) \wedge \tilde{g}(z_0) = 0 \wedge (\forall z \in B^0(z_0, r) \; \tilde{g}(z_0) \neq 0)$

By Theorem 19.1.1,

$$\begin{gathered} \exists! m \in \mathbb{N} \;\; \exists 0 < r_1 < r \;\; \exists g_1 \in H\big(B(z_0, r_1)\big) \\ \forall z \in B(z_0, r_1) \;\; \tilde{g}(z) = (z - z_0)^m g_1(z) \end{gathered} \tag{21.1}$$

*$\because g_1 \in H\big(B(z_0, r_1)\big) \wedge g(z_0) \neq 0$, we can repeat the argument as in **Case 1** for g_1 to get that $\frac{1}{g_1} \in H\big(B(z_0, r_1)\big)$, which implies*

$$\forall z \in B(z_0, r_1) \;\; \frac{1}{g_1(z)} = a_0 + a_1(z - z_0) + \ldots$$

By construction given by Equation (21.1), $\forall z \in B^0(z_0, r_1)$

$$\begin{aligned} f(z) - w &= \frac{1}{g_1(z)(z - z_0)^m} \\ &= \frac{a_0}{(z - z_0)^m} + \frac{a_1}{(z - z_0)^{m-1}} + \ldots + \frac{a_{m-1}}{z - z_0} + a_m + a_{m+1}(z - z_0) + \ldots \end{aligned}$$

[1] $\frac{1}{\tilde{g}}$ is the inverse of a non-zero holomorphic function.

$\therefore \forall z \in B^0(z_0, r_1)$,

$$f(z) - \frac{a_0}{(z-z_0)^m} - \ldots - \frac{a_{m-1}}{z-z_0} = w + a_m + a_{m+1}(z-z_0)\ldots$$

Thus we may define an "extended" function of f, $\tilde{f}$ to be $w + a_m$ at the singularity z_0. Therefore, $f(z) - \sum_{j=1}^{m} a_j(z-z_0)^{-j}$ has a removable singularity at z_0, i.e. 2. holds. $\square$

Chapter 22

Lecture 22 Mar 9th 2018

22.1 Singularity (Continued 3)

Corollary 22.1.1
If f has an essential singularity at z_0 and is holomorphic in some $B^0(z_0, r)$ where $r > 0$, then $f(B^0(z_0, r))$ is dense in $\mathbb{C}$.

Proof
Suppose not, i.e. 3. of Theorem 21.1.1 does not hold. Then either 1., which implies that z_0 is removable, or 2., which implies that z_0 is a pole, is true. This contradicts the assumption that z_0 is an essential singularity. $\square$

Remark
*There are a lot more that are actually true from Theorem 21.1.1! **Picard** showed that in any such punctured ball $B^0(z_0, r)$ around the essential singularity z_0, f takes on every complex value (except possibly one value) infinitely often.*

22.2 The Residue Theorem

Note (Recall)
If f has a pole at z_0, $f \in H(\Omega \setminus \{z_0\})$, then in some open neighbourhood D of z_0, we can write $\forall z \in D \setminus \{z_0\}$

$$f(z) = \underbrace{\frac{c_{-k}}{(z - z_0)^k} + \ldots + \frac{c_{-1}}{(z - z_0)}}_{\text{Principal Part}} + \underbrace{c_0 + c_1(z - z_0) + \ldots}_{G(z)} \tag{22.1}$$

with $G \in H(D)$.

Theorem 22.2.1 (Cauchy's Residue Theorem)
Let $\Omega \subseteq \mathbb{C}$ be open, $f \in H(\Omega \setminus \{z_0\})$ where $z_0 \in \Omega$ is a pole. If γ is a closed path in $\Omega \setminus \{z_0\}$ such that $\forall w \notin \Omega$, $\operatorname{Ind}_\gamma(w) = 0$. Then

$$\frac{1}{2\pi i} \int_\gamma f(z)\, d(z) = (\operatorname{res}_{z_0} f) \operatorname{Ind}_\gamma(z_0)$$

where $\operatorname{Ind}_\gamma(z_0) := \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0}\, dz$.

Proof
Using notation of Equation (21.1), define $g(z)$ such that

$$g(z) := \begin{cases} f(z) - \sum_{j=1}^{k} \frac{c_{-j}}{(z-z_0)^j} & z \in \Omega \setminus \{z_0\} \\ c_0 & z = z_0 \end{cases}$$

Clearly, $g \in H(\Omega \setminus \{z_0\})$, since $f(z)$ minus finitely many polynomials with non-zero denominators is still a holomorphic function. At z_0, with a neighbourhood D around the point, we have, from Equation (21.1),

$$g(z) = c_0 + c_1(z - z_0) + \dots$$

which $g(z_0)$ agrees with c_0 and for any point $z \in D \setminus \{z_0\}$, by definition of g using Equation (21.1). This implies that $g \in H(D) \implies g \in H(\Omega)$.

Thus, by Theorem 13.1.1,

$$\int_\gamma g(z)\, dz = 0$$

Then $\forall z \in \gamma$ and since $z_0 \notin \gamma$, we get

$$\int_\gamma f(z)\, dz = \int_\gamma \sum_{j=1}^{k} \frac{c_{-j}}{(z - z_0)^j}\, dz$$

Consider each term of RHS in turn. Note that for $m \geq 2$, since $\frac{-1}{(m-1)(z-z_0)^{m-1}}$ is the antiderivative of $\frac{1}{(z-z_0)^m}$,

$$\begin{aligned} \int_\gamma \frac{1}{(z - z_0)^m}\, dz &= F\big(\gamma(b)\big) - F\big(\gamma(a)\big) \quad \textit{by Theorem } 12.1.1 \\ &= 0 \quad \textit{since } \gamma \textit{ is closed.} \end{aligned}$$

If $m = 1$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}\,dz = \operatorname{Ind}_\gamma(z_0) \text{ by definition}$$

$$\begin{aligned}\therefore \frac{1}{2\pi i}\int_\gamma f(z)\,dz &= c_{-1}\operatorname{Ind}_\gamma(z_0)\\ &= (\operatorname{res}_{z_0} f)\operatorname{Ind}_\gamma(z_0)\end{aligned}$$

□

Definition 22.2.1 (Meromorphic Functions)
A function f is said to be meromorphic on Ω if $\exists \mathscr{A} \subseteq \Omega$ such that

1. $A^* = \emptyset$
2. $f \in H(\Omega \setminus \mathscr{A})$
3. $\forall z \in \mathscr{A} \quad f$ *has a pole of finite order on z.*

Remark
Holomorphicity $\subseteq$ Meromorphicity (let $A = \emptyset$)