List of Theorems

One reason that we are interested in the symmetric group is that they contain all finite groups.

Theorem (Cayley's Theorem)

If G is a finite group of order n, then G is isomorphic to a subgroup of S_n.

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15 Lecture 15 Jun 04 2018

15.1 Group Action

15.1.1 Cayley's Theorem

Theorem 42 (Cayley's Theorem)
If G is a finite group of order n, then G is isomorphic to a subgroup of S_n.

Proof
Since G is finite, let $G = \{g_1, g_2, ..., g_n\}$ and let S_G be the permutation group of G. By identifying g_i with i, where $1 \leq i \leq n$, we see that $S_G \cong S_n$ [1]*. Therefore, it suffices to find an injective homomorphism*[2] *$\sigma : G \to S_G$.*

Consider the function $\mu_a : G \to G$, where $a \in G$, such that $\mu_a(g) = ag$ for all $g \in G$. Clearly, μ_a is surjective. Suppose $\mu_a = \mu_b$, where $b \in G$. Then $a = \mu_a(1) = \mu_b(1) = b$. Thus μ_a is also injective. It follows that $\mu_a \in S_G$ by definition.

Now define the function $\sigma : G \to S_G$ such that $\sigma(a) = \mu_a$. Clearly, σ is injective, since $\sigma(a) = \sigma(b) \implies \mu_a = \mu_b$. Observe that $\sigma(ab) = \mu_{ab} = ab = \mu_a\mu_b$. Thus σ is a group homomorphism. Note that $\ker\sigma = \{1\}$, the trivial group. It follows from the First Isomorphism Theorem that $G \cong \operatorname{Im}\sigma \leq S_G \cong S_n$.[3] [4] □

Cayley's Theorem is, however, too strong at times. We can certainly find a smaller integer m such that G is contained in S_m. Consider the following example.

[1] S_G is the permutation group of G. We can think of S_G as a group of permutations that permutes the index of the elements of G. Since there are n indices, there are $n!$ ways to permute the indices, and so $|S_G| = n! = |S_n|$. Then we can certainly find some isomorphism from S_G to S_n, and so $S_G \cong S_n$.

[2] **Why do we need injectivity?** We need homomorphicity in order to invoke the First Isomorphism Theorem so that we can get $G \cong \operatorname{im}\sigma \leq S_G \cong S_n$.

[3] We shall use $H \leq G$ to denote that H is a subgroup of G from here on.

[4] This is a result from Proposition 36