PMATH352W18 Complex Analysis - Class Notes

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Chapter 1

Lecture 1 Jan 3 2018

1.1 Complex Numbers and Their Properties

Definition 1.1.1 (Complex Number, Complex Plane)
*A **complex number** is a vector in $\mathbb{R}^2$. The **complex plane**, denoted by $\mathbb{C}$, is a set of complex numbers,*

$$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

In $\mathbb{C}$, we usually write

$$\begin{aligned} 0 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & 1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ i &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} & x &= \begin{pmatrix} x \\ 0 \end{pmatrix} \\ iy &= \begin{pmatrix} 0 \\ y \end{pmatrix} \end{aligned}$$

where $x, y \in \mathbb{R}$. Consequently, we have that

$$x + iy = x + yi = \begin{pmatrix} x \\ y \end{pmatrix}$$

*If for $x, y \in \mathbb{R}$, $z = x + iy$, then x is called the **real part** of z and y is called the **imaginary part** of z, and we write*

$$\mathrm{Re}(z) = x \quad \mathrm{Im}(z) = y.$$

Note

- *It is easy to see how $\mathbb{R}$ is a subset of $\mathbb{C}$.*

- *Complex Numbers of the form* $\binom{0}{y}$ *where* $y \in \mathbb{R}$ *are called* ***purely imaginary numbers***.
- *Certain authors may prefer to denote* $i = \binom{0}{1}$.

Definition 1.1.2 (Sum and Product)
We define the sum of two complex numbers to be the usual vector sum, i.e.

$$\begin{aligned}(a+ib)+(c+id) &= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \begin{pmatrix} a+c \\ b+d \end{pmatrix} \\ &= (a+c)+i(b+d)\end{aligned}$$

where $a, b, c, d \in \mathbb{R}$.

We define the product of two complex numbers by setting $i^2 = -1$*, and by requiring the product to be* ***commutative, associative, and distributive*** *over the sum. In this setup, we have that*

$$\begin{aligned}(a+ib)(c+id) &= ac + iad + ibc + i^2 bd \\ &= (ac - bd) + i(ad + bc) \qquad (1.1)\end{aligned}$$

Note
It is interesting to note that ***any complex number times zero is zero****, just like what we have with real numbers.*

$$\forall z = x + iy \in \mathbb{C} \; x, y \in \mathbb{R} \;\; 0 \in \mathbb{C}$$
$$z \cdot 0 = (x+iy)(0+i0) = 0 + i0 = 0$$

Example 1.1.1
Let $z = 2 + i, w = 1 + 3i$*. Find* $z + w$ *and* zw.

$$\begin{aligned}z + w &= (2+i)+(1+3i) \\ &= 3 + 4i\end{aligned}$$

$$\begin{aligned}zw &= (2+i)(1+3i) \\ &= (2-3) + i(6+1) \quad \text{By Equation (1.1)} \\ &= -1 + 7i\end{aligned}$$

Example 1.1.2
Show that every non-zero complex number has a ***multiplicative inverse***, z^{-1}, *and find a formula for this inverse.*

Let $z = a + ib$ *where* $a, b \in \mathbb{R}$ *with* $a^2 + b^2 \neq 0$. *Then*

$$\begin{aligned} & z(x+iy) = 1 \\ \iff & (ax - by) + i(ay + bx) = 1 \\ \iff & \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \iff & \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \iff & \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \iff & \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2+b^2} \begin{pmatrix} a \\ -b \end{pmatrix} \\ \iff & x + iy = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2} \end{aligned}$$

Therefore, we have that the formula for the inverse is

$$(a+ib)^{-1} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2} \tag{1.2}$$

Notation
For $z, w \in \mathbb{C}$, *we write*

$$\begin{array}{ll} -z = -1z & w - z = w + (-z) \\ \frac{1}{z} = z^{-1} & \frac{w}{z} = wz^{-1} \end{array}$$

Example 1.1.3
Find $\frac{(4-i)-(1-2i)}{1+2i}$.

$$\begin{aligned} \frac{(4-i)-(1-2i)}{1+2i} &= \frac{3+i}{1+2i} \\ &= (3+i)(\frac{1}{5} - i\frac{2}{5}) \\ &= 1 - i \end{aligned}$$

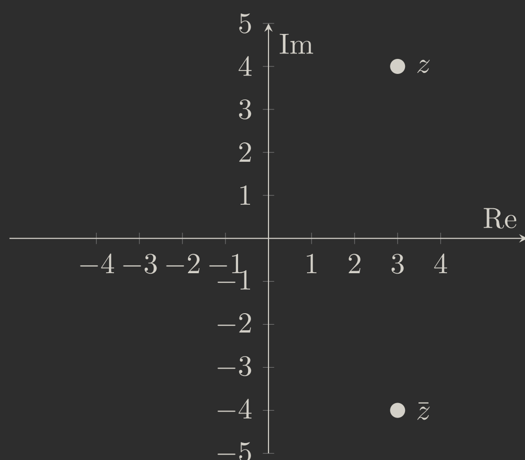
Note
The set of complex numbers is a ***field*** *under the operations of addition and multiplication. This means that* $\forall u, v, w \in \mathbb{C}$,

$$
\begin{aligned}
u + v &= v + u & uv &= vu \\
(u + v) + w &= u + (v + w) & (uv)w &= u(vw) \\
0 + u &= u & 1u &= u \\
u + (-u) &= 0 & uu^{-1} &= 1,\ u \neq 0 \\
u(v + w) &= uv + uw
\end{aligned}
$$

Since the distributive law holds for complex numbers, note that the ***binomial expansion works*** *for* $(w+z)^n$ *where* $w, z \in \mathbb{C}$ *and* $n \in \mathbb{N}$. *(I did not verify if this is still true for when* $n \in \mathbb{R}$.*)*

Definition 1.1.3 (Conjugate)
If $z = x + iy$ *where* $x, y \in \mathbb{R}$, *then the* ***conjugate of z*** *is given by* $\bar{z} = x - iy$

Example 1.1.4
Let $z = 3 + 4i$. *Then the* $\bar{z} = 3 - 4i$. *Represented in the complex plane, we have the following:*



We observe that on the complex plane, the conjugate of a complex number is simply its reflection on the real axis.

Definition 1.1.4 (Modulus)
We define the ***modulus*** *(length, magnitude) of* $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$, *to be*

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}. \tag{1.3}$$

Note
Note that this definition is consistent with the notion of the absolute value in real numbers when z is a real number, since if $y = 0$, $|z| = |x + i0| = \sqrt{x^2} = \pm x$.

Note
For $z, w \in \mathbb{C}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} \bar{\bar{z}} &= z & z + \bar{z} &= 2\,\mathrm{Re}(z) & z - \bar{z} &= 2i\,\mathrm{Im}(z) \\ z\bar{z} &= |z|^2 & |z| &= |\bar{z}| & \overline{z \pm w} &= \bar{z} \pm \overline{w} \\ \overline{zw} &= \bar{z}\overline{w} & |zw| &= |z|\,|w| & \bar{z}^n &= \overline{z^n} \end{aligned}$$

but note that $|z + w| \neq |z| + |w|$.

Also, note that the last equation is a generalization of the ***highlighted equation***.

Note
While inequalities such as $z_1 < z_2$, where $z_1, z_2 \in \mathbb{C}$, are meaningless unless if both of them are real, $|z_1| < |z_2|$ means that the point z_1 in the complex plane is closer to the origin than the point z_2.

Proposition 1.1.1 (Basic Inequalities)

1. $|\mathrm{Re}(z)| \leq |z|$

2. $|\mathrm{Im}(z)| \leq |z|$

3. $|z + w| \leq |z| + |w|$ *Triangle Inequality*

4. $|z + w| \geq \big|\, |z| - |w| \,\big|$ *Inverse Triangle Inequality*

Proof
Note that $|z|^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$ and that we can express $|x| = \sqrt{x^2}$ for any $x \in \mathbb{R}$. 1 and 2 immediately follows from that.

To prove 3, we have that

$$\begin{aligned} |z + w|^2 &= (z + w)(\bar{z} + \overline{w}) \\ &= |z|^2 + |w|^2 + (w\bar{z} + \overline{w}z) \\ &= |z|^2 + |w|^2 + 2\,\mathrm{Re}(w\bar{z}) \\ &\leq |z|^2 + |w|^2 + 2\,|w\bar{z}| \quad \text{by 1} \\ &= |z|^2 + |w|^2 + 2\,|wz| \quad \text{since } |w\bar{z}| = |w|\,|\bar{z}| \text{ and } |z| = |\bar{z}| \\ &= (|z| + |w|)^2 \end{aligned}$$

To prove 4, note that

$$|z| \ = |z + w - w| \leq |z + w| + |w| \tag{1.4}$$
$$|w| = |w + z - z| \ \leq |z + w| + |z| \tag{1.5}$$

Observe that

$$\textit{Equation (1.4)} \implies |z| - |w| \leq |z + w|$$
$$\textit{Equation (1.5)} \implies |w| - |z| \leq |z + w|$$

Thus, we have that

$$|z + w| \geq |\ |z| - |w|\ |$$

as required. □

Item 3 in Proposition 1.1.1 can be generalized by the means of mathematical induction to sums involving any finite number of terms, as:

$$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n| \tag{1.6}$$

where $n \in \mathbb{N} \setminus \{0, 1\}$.

To note the induction proof, when $n = 2$, Equation (1.6) is just Item 3. If Equation (1.6) is true for when $n = m$ where $m \in \mathbb{N} \setminus \{0, 1\}$, $n = m + 1$ is also true since by Item 3,

$$\begin{aligned} |(z_1 + z_2 + \ldots + z_m) + z_{m+1}| &\leq |z_1 + z_2 + \ldots + z_m| + |z_{m+1}| \\ &\leq (|z_1| + |z_2| + \ldots + |z_m|) + |z_{m+1}|. \end{aligned}$$

The distance between two points $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}, x_1, x_2, y_1, y_2 \in \mathbb{R}$ is $|z_1 - z_2|$, since $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 (y_1 - y_2)^2}$ is our usual notion of the Euclidean distance of two points on a plane.

Also, note that

$$z_1 - z_2 = z_1 + (-z_2)$$

and thus if we apply our knowledge of vector representation, $z_1 - z_2$ is the directed line segment from the point z_2 to z_1.

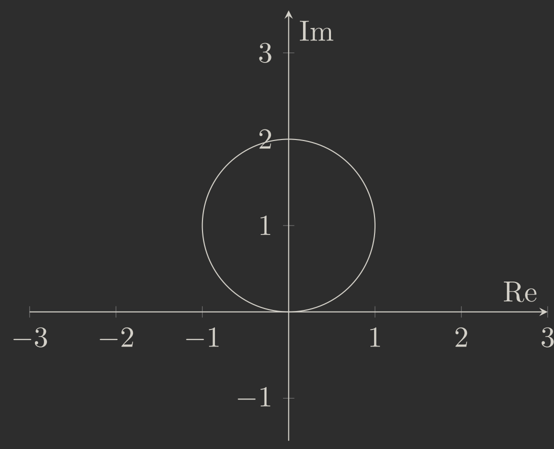
With the notion of a "distance" set on the complex plane, we can now explore upon points lying on a circle with a center z_0 and radius R, which satisfies the equation

$$|z - z_0| = R.$$

We may simply refer to this set of points as the circle $|z - z_0| = R$.

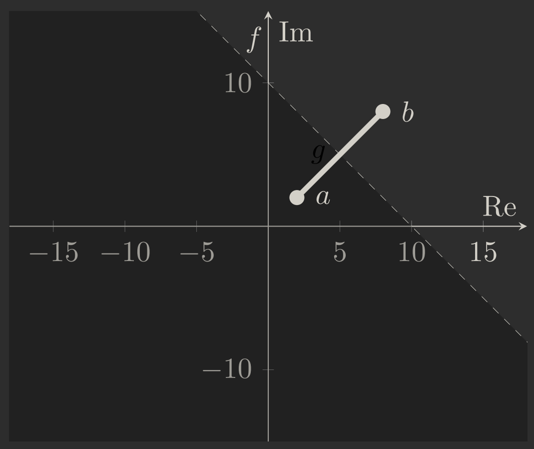
Example 1.1.5

We may describe a set $\{z \in \mathbb{C} : |z - i| = 1\}$ *as follows:*



Let $a, b \in \mathbb{C}$ *describe the set* $\{z \in \mathbb{C} : |z - a| < |z - b|\}$.

Suppose the following coordinates for a *and* b *are arbitrary,*



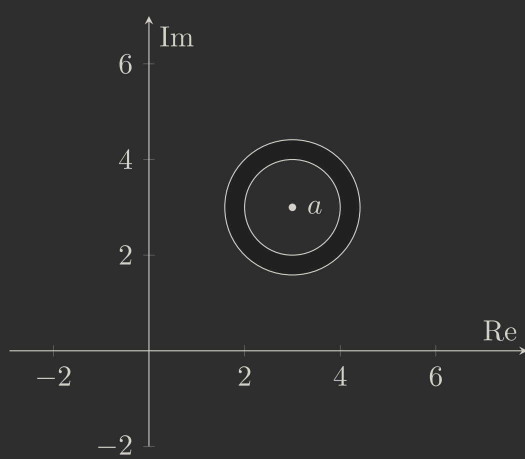
In the above, g *is the line segment that connects the points* a *and* b *on the complex plane, while* f *is the perpendicular bisector of the line segment* g*. The area described by the set* $\{z \in \mathbb{C} : |z - a| < |z - b|\}$ *is the shaded area which is below* f.

Chapter 2

Lecture 2 Jan 5th 2018

2.1 Complex Numbers and Their Properties (Continued)

Example 2.1.1
Let $a \in \mathbb{C}$. Describe the set $\{z \in \mathbb{C} : 1 < |z - a| < 2\}$.



Example 2.1.2
Show that every non-zero complex number has exactly two complex square roots, and find a formula for the square roots.

Let $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$, and let $w = u + iv, u, v \in \mathbb{R}$. Then

$$\begin{aligned} w^2 = z &\iff (u+iv)^2 = x+iy \\ &\iff (u^2-v^2)+i(2uv) = x+iy \\ &\iff x = u^2+v^2 \quad \textit{and} \qquad (2.1) \\ &\quad\;\; y = 2uv \qquad (2.2) \end{aligned}$$

Square both sides of Equation (2.2), and thus we have $y^2 = 4u^2v^2$.

Multiply Equation (2.1) by $4u^2$*, and we get*

$$\begin{aligned} 4u^2x &= 4u^4 - 4u^2v^2 = 4u^4 - y^2 \\ \iff 0 &= 4u^4 - 4u^2x - y^2 \\ \iff u^2 &= \frac{4x \pm \sqrt{16x^2+16y^2}}{8} \\ &= \frac{x \pm \sqrt{x^2+y^2}}{2} \end{aligned}$$

Suppose $y \neq 0$*. Note that* $x < \sqrt{x^2+y^2}$*. Thus* $u^2 = \frac{x+\sqrt{x^2+y^2}}{2} \implies u = \left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}$.

Similarly, we can get

$$v = \pm\left(\frac{-x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}$$

Note that all four choices of signs satisfy Equation (2.1). If $y > 0$*, then* u *and* v *are either both positive or both negative by Equation (2.2).*

Suppose $y = 0$*. Then we have*

$$w^2 = z = x$$

Therefore, we get

$$w = \begin{cases} \pm\left[\left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}} + i\left(\frac{-x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}\right] & y > 0 \\ \pm\left[\left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}} - i\left(\frac{-x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}\right] & y < 0 \\ \pm\sqrt{x} & y = 0, x > 0 \\ \pm i\sqrt{x} & y = 0, x < 0 \end{cases}$$

Remark
Let $z \in \mathbb{C}$. The notation $\sqrt{z}$ may represent either one of the square roots of z or both of the square roots, i.e. ***it is possible that $\sqrt{z}$ represents a set****.*

Exercise 2.1.1
Is it always okay for complex numbers such that $\sqrt{zw} = \sqrt{z}\sqrt{w}$, for $z, w \in \mathbb{C}$?

No. For example, consider $z = w = -1$. Then we have

$$\sqrt{zw} = \sqrt{1} = \pm 1$$

while

$$\sqrt{z}\sqrt{w} = i \cdot i = -1$$

and thus

$$\sqrt{zw} \neq \sqrt{z}\sqrt{w}.$$

Example 2.1.3
Find the values of $\sqrt{3-4i}$.

By Example 2.1.2,

$$\begin{aligned}\sqrt{3-4i} &= \pm\left(\sqrt{\frac{3+\sqrt{9+16}}{2}} - i\sqrt{\frac{-3+\sqrt{9+16}}{2}}\right) \\ &= \pm(2-i)\end{aligned}$$

Remark
The quadratic formula holds for complex polynomials, i.e.

$$\forall a, b, c \in \mathbb{C} \quad a \neq 0 \quad \forall z \in \mathbb{C}\ az^2 + bz + c = 0,$$

the solution for z is given by

$$z_{1,2} = \frac{-b + \sqrt{b^2 - 4ac}}{b} \tag{2.3}$$

The following is a short proof.

Proof

$$\begin{aligned}
az^2 + bz + c = 0 &\iff z^2 + \frac{b}{a}z + \frac{c}{a} = 0 \\
&\iff z^2 + \frac{b}{a}z + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \\
&\iff \left(z + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2} \\
&\iff z = \frac{-b + \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

(Personal Note: where did the $-$ for the supposed $\pm$ go? Or should it really be $\pm$?)

Example 2.1.4
Solve $iz^2 - (2+3i)z + 5(1+i) = 0$.

$$\begin{aligned}
z &= \frac{2 + 3i + \sqrt{(2+3i)^2 - 4i[5(1+i)]}}{2i} \\
&= \frac{2 + 3i + \sqrt{-5 + 12i - 20i + 20}}{2i} \\
&= \frac{2 + 3i + \sqrt{15 + 8i}}{2i}
\end{aligned}$$

Note that by Example 2.1.2,

$$\begin{aligned}
\sqrt{15 - 8i} &= \pm\left[\sqrt{\frac{15 + \sqrt{225 + 64}}{2}} - i\sqrt{\frac{-15 + \sqrt{225 + 64}}{2}}\right] \\
&= \pm\left[\sqrt{\frac{15 + 17}{2}} - i\sqrt{\frac{-15 + 17}{2}}\right] \\
&= \pm(4 - i)
\end{aligned}$$

Thus we have

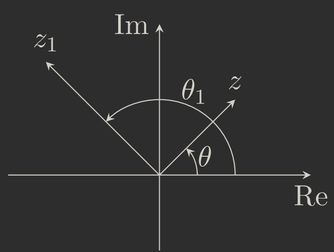
$$\begin{aligned}
z &= \frac{2 + 3i + \sqrt{15 + 8i}}{2i} \\
&= \frac{2 + 3i \pm (4 - i)}{2i} \\
&= (6 + 2i)\left(-\frac{1}{2}i\right) \;\; \textit{or} \; (-2 + 4i)\left(-\frac{1}{2}i\right) \qquad \textit{by Example } 1.1.2 \\
&= (1 - 3i) \;\; \textit{or} \; (2 + i)
\end{aligned}$$

Chapter 3

Lecture 3 Jan 8th 2018

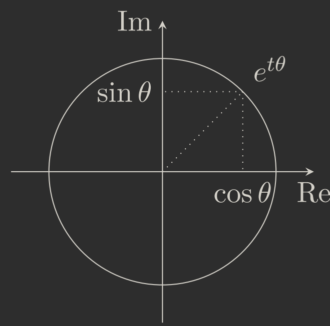
3.1 Complex Numbers and Their Properties (Continued 2)

Definition 3.1.1 (Argument of a Complex Number)
Let $z \in \mathbb{C} \setminus \{0\}$. The ***argument*** *(or the angle) of z, denoted by $\arg z$, $\operatorname{Arg} z$, or simply $\theta = \theta(z)$, is the angle modulo 2π (i.e. $0 \leq \theta < 2\pi$) between the vector defining z and the positive real axis (in the counterclockwise direction).*



Notation
Let $e^{i\theta} := \cos\theta + i \sin\theta$. Note that this definition, called ***Euler's formula****, can be derived by the extending the Taylor expansion of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for when $x \in \mathbb{C}$ (the sum of the real parts of the expansion is the Taylor expansion of cosine while the imaginary part for sine).*

Now $e^{i\theta}$ is on the unit circle.



Remark

If $z = 0$, the coordinate θ is undefined, and so it is implied that $z \neq 0$ whenever we use the polar form.

Example 3.1.1

Some examples of $\theta \in [0, 2\pi)$:

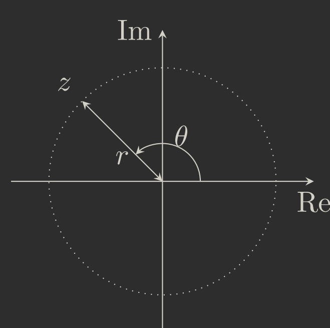
$$\begin{aligned} e^{i\frac{\pi}{4}} &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\frac{\pi}{2}} &= i \\ e^{i\frac{3\pi}{4}} &= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\pi} + 1 &= 0 \end{aligned}$$

Remark

$$\forall k \in \mathbb{Z} \ \ \forall \theta \in \mathbb{R} \ \ e^{i\theta} = e^{i(\theta + 2\pi k)}$$

Remark

The complex number $re^{i\theta}$, where $r > 0, \theta \in [0, 2\pi)$, represents the complex number with modulus r and argument θ.



Therefore, $\forall z \in \mathbb{C}$, we can express

$$z := |z|\, e^{i\,\mathrm{Arg}\, z}. \tag{3.1}$$

With that, we now have two representations of a complex number:

- **Cartesian representation**: $z = x + iy$ where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$
- **Polar representation**: $z = re^{i\theta}$ where $r = |z|$ and $\theta = \operatorname{Arg} z \in [0, 2\pi)$

To convert between the two representations, we have the following equations:

Polar $\to$ Cartesian:

$$x = r\cos\theta \quad y = r\sin\theta \tag{3.2}$$

Cartesian $\to$ Polar:

$$\begin{gathered} r = |z| \\ x \neq 0 \implies \tan\theta = \frac{y}{x} \\ x = 0 \implies \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \end{gathered} \tag{3.3}$$

On another note,

$$z = re^{i\theta} \implies \bar{z} = re^{-i\theta}$$

and

$$z \neq 0 \implies \frac{1}{z} = \frac{1}{r}e^{-i\theta} \tag{3.4}$$

Remark

$$\begin{gathered} \forall r_1, r_2 \in \mathbb{R}\ \forall \theta_1, \theta_2 \in [0, 2\pi) \\ z_1 := r_1 e^{i\theta_1} \quad z_2 := r_2 e^{i\theta_2} \end{gathered}$$

Then

$$z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

Note that $e^{ix}e^{iy} = e^{i(x+y)}$ *is true for all* $x, y \in \mathbb{R}$ *since*

$$\begin{aligned} e^{ix}e^{iy} &= (\cos x + i\sin x)(\cos y + i\sin y) \\ &= (\cos x\cos y - \sin x\sin y) + i(\cos x\sin y + \cos y\sin x) \\ &= \cos(x+y) + i\sin(x+y) \\ &= e^{i(x+y)}. \end{aligned}$$

Generalizing the above, we get that

$$\forall n \in \mathbb{Z} \;\; z = (re^{in}) = r^n e^{in\theta} \tag{3.5}$$

which is commonly known as ***deMoivre's Law****. Note that by simply generalizing the above, all we have is that $n \in \mathbb{Z}^+$. But by Equation (3.4), we can have that for $n \in \mathbb{Z}^-$, let $m = -n$, and thus*

$$z^n = \left[\frac{1}{r}e^{i(-\theta)}\right]^m = \left(\frac{1}{r}\right)^m e^{im(-\theta)} = \left(\frac{1}{r}\right)^{-n} e^{i(-n)(-\theta)} = r^n e^{i\theta}$$

This proves that deMoivre's Law also holds for when $n \in \mathbb{Z}^-$.

Observe that if $r = 1$, Equation (3.5) becomes

$$(e^{i\theta})^n = e^{in\theta} \quad \textit{for all } n \in \mathbb{Z} \setminus \{0\} \tag{3.6}$$

When written in the form

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad (n \in \mathbb{Z} \setminus \{0\}) \tag{3.7}$$

this is known as ***deMoivre's formula****.*

Example 3.1.2
Equation (3.7) with $n = 2$ tells us that

$$(\cos\theta + i\sin\theta)^n = \cos 2\theta + i\sin 2\theta$$

or we can express the equation as

$$\cos^2\theta - \sin^2\theta + i2\sin\theta\cos\theta = \cos 2\theta + i\sin 2\theta$$

Equating real and imaginary parts, we have the familiar double angle trigonometric identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

3.1.1 Roots of Complex Numbers

Proposition 3.1.1 (nth Roots of a Complex Number)

$$\begin{gathered}
\forall z = re^{i\theta} \in \mathbb{C} \;\; r = |z| \in \mathbb{R} \;\; \theta \in [0, 2\pi) \\
\exists w = se^{i\tau} \in \mathbb{C} \;\; s \in \mathbb{R} \;\; \tau \in [0, 2\pi) \\
\forall n \in \mathbb{Z} \\
w^n = \left(se^{i\tau}\right)^n = z = re^{i\theta}
\end{gathered}$$

The nth roots of z is described by the set

$$\left\{ r^{\frac{1}{n}} e^{i\left(\frac{\theta+2\pi k}{n}\right)} : k = 0, 1, ..., n-1 \right\} \tag{3.8}$$

Proof

$$s^n = r \iff s = r^{\frac{1}{n}}$$
$$e^{in\theta} = e^{i\tau} \iff \theta = \frac{\tau + 2\pi k}{n}$$

Therefore, the set that describes the nth roots of z is

$$\left\{ w = r^{\frac{1}{n}} e^{i\left(\frac{\theta+2\pi k}{n}\right)} : k = 0, 1, ..., n-1 \right\}$$

Remark (nth Roots of Unity)
*The **nth roots of unity** is a direct consequence of Proposition 3.1.1 where we solve for the equation $z^n = 1$ for any $z \in \mathbb{C}, n \in \mathbb{Z}$.*

The set that describes the nth roots of unity is

$$\left\{ e^{i\theta} : \theta = \frac{2\pi k}{n}, k = 0, 1, ..., n-1 \right\} \tag{3.9}$$

It is easy to see how the nth roots of unity **partitions the unit circle into n parts**.

Example 3.1.3
Find the cubic roots of $-2 + 2i$.

Let $z = -2 + 2i$. Note that $|z| = 2\sqrt{2}$ and $\operatorname{Arg} z = \frac{3\pi}{4}$.

Therefore, in polar form, $z = 2\sqrt{2}e^{i\frac{3\pi}{4}}$.

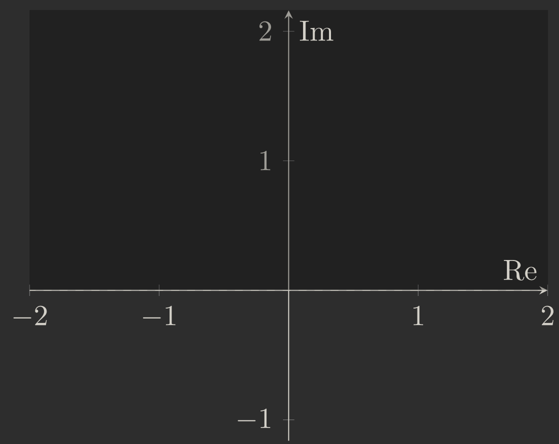
Let $w = re^{i\theta}$, where $\theta \in [0, 2\pi)$, and $w^3 = z$. Then

$$r = (2\sqrt{2})^{\frac{1}{3}}$$
$$\theta = \frac{\frac{3\pi}{4} + 2\pi k}{3}, \quad k = 0, 1, 2$$

The set that describes the cubic root of $-2 + 2i$ is thus

$$\left\{ (2\sqrt{2})^{\frac{1}{3}} e^{i\theta} : \theta = \frac{\frac{3\pi}{4} + 2\pi k}{3}, k = 0, 1, 2 \right\}$$

Example 3.1.4
Describe the set $\{z \in \mathbb{C} : \left|\operatorname{Arg} z - \frac{\pi}{2}\right| < \frac{\pi}{2}\}$*. (Note:* $\operatorname{Arg} z \in [0, 2\pi)$*)*



Exercise 3.1.1
Solve

1. $z^4 = -1$

$$\text{Let } z = re^{i\theta}$$
$$r = |-1| = 1 \quad \theta = \frac{\pi + 2\pi k}{4} = \frac{(2k+1)\pi}{4}, \quad k = 0, 1, 2, 3$$

2. $z^4 = -1 + \sqrt{3}i$

$$\text{Let } z = re^{i\theta}$$
$$r = \left|-1 + \sqrt{3}i\right| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$
$$\theta = \frac{\frac{2\pi}{3} + 2\pi k}{4} = \frac{(2k + \frac{2}{3})\pi}{4}, \quad k = 0, 1, 2, 3$$

Chapter 4

Lecture 4 Jan 10th 2018

4.1 Examples for nth Roots of Unity

Recall that the nth roots of unity are given by $e^{i\frac{2\pi k}{n}}, k = 0, 1, ..., n-1$.

Exercise 4.1.1
Let z be any nth root of unity other than 1. Show that

$$z^{n-1} + z^{n-2} + \ldots + z + 1 = 0 \tag{4.1}$$

Proof
By the Sum of Finite Geometric Terms,

$$z^{n-1} + z^{n-2} + \ldots + z + 1 = \frac{1 - z^n}{1 - z}.$$

Since $z^n = 1$, RHS is thus zero, which in turn completes the proof.

As an aside, if we wish to remove the restriction that z can also be 1, we may consider that

$$z^n - 1 = (z - 1)(1 + z + \ldots + z^{n-1})$$

Since $z^n = 1$, LHS is zero. Then either $z = 1$ or $(1 + z + \ldots + z^{n-1}) = 0$.

Exercise 4.1.2
Consider the $n - 1$ diagonals of a regular n-gon, inscribed in a circle of radius 1, obtained by connecting one vertex on the n-gon to all its other vertices.

For example, if we are given $n = 6$, we obtain the following diagram.

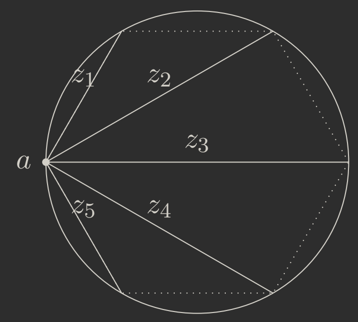


Figure 4.1: $n = 6$, where a is an arbitrary vertex on the hexagon

Show that the product of the lengths of these diagonals is equal to n.

Proof

Note that Figure 4.1 can be translated into Figure 4.2.

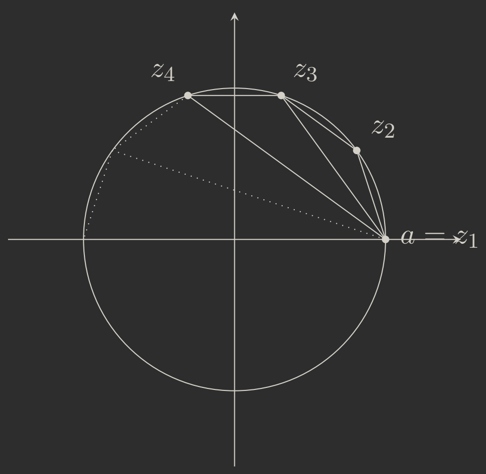


Figure 4.2: A regular n-gon with the roots of unity on its vertices

Thus the equation that we wish to prove becomes

$$|1 - z_2|\,|1 - z_3| \ldots |1 - z_n| = n \tag{4.2}$$

Note that $z_2, ..., z_n$ are the nth roots of unity other than 1.

Let z be a variable and consider the polynomial

$$P(z) := 1 + z + z^2 + \ldots + z^{n-1} \tag{4.3}$$

Since the roots of $P(z)$ are the nth roots of unity other than 1, we can factorize Equation (4.3) into

$$P(z) = (z - z_2)(z - z_3)\dots(z - z_n)$$

Now let $z = 1$ and take the modulus of $P(z)$, and we get Equation (4.2).

Exercise 4.1.3
Let $n \in \mathbb{N}$. Show that $\sum_{j=0}^{n} \binom{3n}{3j} = \frac{2^{3n}+2(-1)^n}{3}$.

Proof
Let $\alpha = e^{i\frac{2\pi}{3}}$. Then α is a cubic root of unity, i.e. $\alpha^3 = 1$, and from Exercise 4.1.1, $1 + \alpha + \alpha^2 = 0$.

Consider

$$\begin{aligned}(1+1)^{3n} =& \binom{3n}{0} + \binom{3n}{1} + \binom{3n}{2} + \binom{3n}{3} + \binom{3n}{4} \\ &+ \binom{3n}{5} + \binom{3n}{6} + \dots + \binom{3n}{3n}\end{aligned} \tag{4.4}$$

$$\begin{aligned}(1+\alpha)^{3n} =& \binom{3n}{0} + \binom{3n}{1}\alpha + \binom{3n}{2}\alpha^2 + \binom{3n}{3} + \binom{3n}{4}\alpha \\ &+ \binom{3n}{5}\alpha^2 + \binom{3n}{6} + \dots + \binom{3n}{3n}\end{aligned} \tag{4.5}$$

$$\begin{aligned}(1+\alpha^2)^{3n} =& \binom{3n}{0} + \binom{3n}{1}\alpha^2 + \binom{3n}{2}\alpha + \binom{3n}{3} + \binom{3n}{4}\alpha^2 \\ &+ \binom{3n}{5}\alpha + \binom{3n}{6} + \dots + \binom{3n}{3n}\end{aligned} \tag{4.6}$$

Adding Equation (4.4), Equation (4.5) and Equation (4.6), we observe that the terms with coefficients $\binom{3n}{k}$ where k is not a multiple of 3 sums to 0 as given by $1 + \alpha + \alpha^2 = 0$, and *therefore we obtain*

$$\begin{aligned}
2^{3n} + (1+\alpha)^{3n} + (1+\alpha^2)^{3n} &= 3\sum_{j=0}^{n}\binom{3n}{3j} \\
\frac{1}{3}\left[2^{3n} + (1+\alpha)^{3n} + (1+\alpha^2)^{3n}\right] &= \sum_{j=0}^{n}\binom{3n}{3j} \\
\frac{1}{3}\left[2^{3n} + (-\alpha^2)^{3n} + (-\alpha)^{3n}\right] &= \sum_{j=0}^{n}\binom{3n}{3j} \quad \textit{since } 1+\alpha+\alpha^2 = 0 \\
\frac{1}{3}\left[2^{3n} + (-1)^{n} + (-1)^{n}\right] &= \sum_{j=0}^{n}\binom{3n}{3j} \quad \textit{since } \alpha^3 = 1 \\
\frac{2^{3n} + 2(-1)^n}{3} &= \sum_{j=0}^{n}\binom{3n}{3j}
\end{aligned}$$

as required.

Exercise 4.1.4

Note that we can define $\operatorname{Arg} z$ *in any interval of length* 2π*, i.e. it is not necessary that* $\operatorname{Arg} z \in [0, 2\pi)$.

For example, if we restrict $\operatorname{Arg} z \in [-\pi, \pi]$*, then we can write*

$$\operatorname{Arg}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -\frac{3\pi}{4}$$

Let z *be on the unit circle and* $\operatorname{Arg} z \in [-\pi, \pi]$*. Suppose that* $z \notin \mathbb{R}$*, i.e.* $z \neq 1, z \neq -1$*. Show that*

$$\operatorname{Arg}\left(\frac{z-1}{z+1}\right) = \begin{cases} \frac{\pi}{2} & \operatorname{Im} z > 0 \\ -\frac{\pi}{2} & \operatorname{Im} z < 0 \end{cases}$$

Proof

Note that $\forall w_1, w_2 \in \mathbb{C}$*, where* $\operatorname{Arg} w_1 = \tau_1, \operatorname{Arg} w_2 = \tau_2$ *for* τ_1, τ_2 *in the same* 2π*-interval,*

$$\operatorname{Arg}\frac{w_1}{w_2} = \frac{e^{i\tau_1}}{e^{i\tau_2}} \equiv e^{i(\tau_1 - \tau_2)} = \operatorname{Arg} w_1 - \operatorname{Arg} w_2 \tag{4.7}$$

in modulo 2π.

Suppose $\operatorname{Im} z > 0$*. Let* $\theta_1 = \operatorname{Arg}(z-1)$ *and* $\theta_2 = \operatorname{Arg}(z+1)$*. Consider Figure 4.3. Note that since both* $\theta_1, \theta_2 \in [0, \pi]$*, we have that* $\theta_1 - \theta_2 \in [-\pi, \pi]$*, and thus Equation (4.7) holds* *true without the need of the condition of being in modulo 2π. We observe that*

$$\frac{\pi}{2} = \theta_2 + \pi - \theta_1$$
$$\theta_1 - \theta_2 = \frac{\pi}{2}$$

as desired.

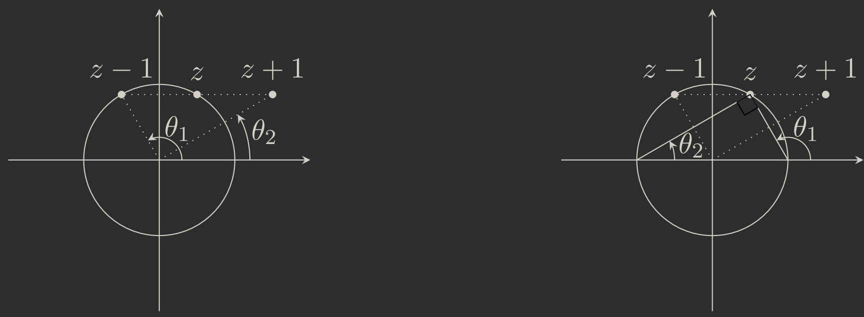


Figure 4.3: (Right) Depicted question, (Left) Translated Angles

Similarly, we can obtain $\theta_1 - \theta_2 = -\frac{\pi}{2}$ for when $\operatorname{Im} z < 0$. This completes the proof.

Exercise 4.1.5
Let $f(z) = e^z$ for $z \in \mathbb{C}$. Let $A = \{z = x + iy \in \mathbb{C} : x \leq 1, y \in [0, \pi]\}$. Describe the image of $f(A)$.

Solution
Firstly, note that

$$e^z = e^{x+iy}$$
$$e^x \in (0, e]$$
$$y \in [0, \pi]$$

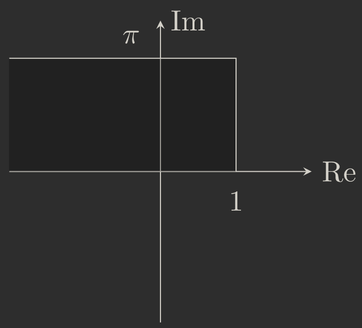


Figure 4.4: (Right) Domain of $f(A)$, (Left) Image of $f(A)$

It is clear that the image will be in on the positive side of the imaginary-axis. Also, since $e^x \in (0, e]$*, we get the right graph represented in Figure 4.4. The image of* $f(A)$ *is described in the left image of Figure 4.4.*

Chapter 5

Lecture 5 Jan 12 2018

5.1 Complex Functions

5.1.1 Limits

Definition 5.1.1 (Convergence)
A sequence of complex numbers $z_1, z_2, z_3, \ldots$ ***converges*** *to $z \in \mathbb{C}$ if*

$$\lim_{n\to\infty} |z_n - z| = 0 \tag{5.1}$$

or we may say

$$\forall \epsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n > N \quad |z_n - z| < \epsilon \tag{5.2}$$

Note
If $\{z_n\}_{n\in\mathbb{N}}$ converges to z, we may write $\lim_{n\to\infty} z_n = z$ or $z_n \to z$ (as $n \to \infty$).

Example 5.1.1
For $|z| > 1$, does $\{\frac{1}{z^n}\}_{n=1}^{\infty}$ converge? Explain.

Solution
We claim that the limit is 0. Since $|z| > 1$, we have that

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{1}{z^n} - 0\right| &= \lim_{n\to\infty}\left|\frac{1}{z}\right|^n \\ &= 0\end{aligned}$$

Another way to prove this, since $|z| > 1 \implies 0 < \left|\frac{1}{z}\right| < 1$,

$$\forall \epsilon = \left|\frac{1}{z}\right| > 0$$
$$\left|\frac{1}{z^n} - 0\right| = \left|\frac{1}{z}\right|^n < \left|\frac{1}{z}\right| = \epsilon$$

Definition 5.1.2 (Convergence for Complex Functions)
$\forall \Omega \subseteq \mathbb{C}$, *let* $f : \Omega \to \mathbb{C}$. *We say that*

$$\lim_{z \to z_0} f(z) = L \tag{5.3}$$

for some $L \in \mathbb{C}$ *if for every sequence* $\{z_n\}_n \subseteq \Omega$ *(not including* z_0 *if it is in* Ω*), we have that*

$$z_n \to z_0 \implies f(z_n) \to L \tag{5.4}$$

Note that L *need not be in* Ω.

Example 5.1.2
Let $f(z) = \frac{\bar{z}}{z}, z \in \mathbb{C} \setminus \{0\}$. *Find* $\lim_{z \to 0} f(z)$.

Solution
Suppose $z = x \in \mathbb{R} \setminus \{0\}$. *Then* $f(z) = f(x) = \frac{x}{x} = 1$.

Suppose $z = iy, y \in \mathbb{R} \setminus \{0\}$. *Then* $f(z) = f(iy) = \frac{-iy}{iy} = -1$.

Therefore, the limit $\lim_{z \to 0} f(z)$ *does not exist.*

Exercise 5.1.1
Show that $z_n \to z \iff \mathrm{Re}(z_n) \to \mathrm{Re}(z) \wedge \mathrm{Im}(z_n) \to \mathrm{Im}(z)$.
(Hint: $|\mathrm{Re}(z)|, |\mathrm{Im}(z)| \leq |z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$*)*

Solution
Suppose $z_n \to z$. *Then* $\forall \epsilon_0 > 0\ \exists N \in \mathbb{N}\ \forall n > N\ |z_n - z| < \epsilon$. *Note once and for all that*

$$\mathrm{Re}(z_n - z) = \mathrm{Re}(z_n) - \mathrm{Re}(z)$$
$$\mathrm{Im}(z_n - z) = \mathrm{Im}(z_n) - \mathrm{Im}(z).$$

Thus

$$\begin{aligned} |\mathrm{Re}(z_n) - \mathrm{Re}(z)| &= |\mathrm{Re}(z_n - z)| \\ &\leq |z_n - z| < \epsilon \\ |\mathrm{Im}(z_n) - \mathrm{Im}(z)| &= |\mathrm{Im}(z_n - z)| \\ &\leq |z_n - z| < \epsilon \end{aligned}$$

For the other direction,

$$\forall \frac{\epsilon}{2} > 0 \ \exists N_0 \in \mathbb{N} \ \forall n > N_0 \quad |\mathrm{Re}(z_n) - \mathrm{Re}(z)| < \frac{\epsilon}{2}$$
$$\forall \frac{\epsilon}{2} > 0 \ \exists N_1 \in \mathbb{N} \ \forall n > N_1 \quad |\mathrm{Im}(z_n) - \mathrm{Im}(z)| < \frac{\epsilon}{2}.$$

Therefore,

$$\begin{aligned} |z_n - z| &= |\mathrm{Re}(z_n) + \mathrm{Im}(z_n) - \mathrm{Re}(z) - \mathrm{Im}(z)| \\ &\leq |\mathrm{Re}(z_n) - \mathrm{Re}(z)| + |\mathrm{Im}(z_n) - \mathrm{Im}(z)| \\ &\leq \epsilon \end{aligned}$$

□

5.1.2 Continuity

Definition 5.1.3 (Continuity)
*$\forall \Omega \subseteq \mathbb{C}$, let $f : \Omega \to \mathbb{C}$. We say that f is **continuous** at $z_0 \in \Omega$ if*

1. $\forall \{z_n\}_{n \in \mathbb{N}}$
   $z_n \to z_0 \implies f(z_n) \to f(z_0)$
2. $\forall \epsilon > 0 \ \exists \delta > 0$
   $|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$

Remark

1. *f is continuous on Ω if it is continuous on every point in Ω.*
2. *We may **split** f into its feal and imaginary parts, i.e.*

$$f(z) = f(x, y) = u(x, y) + iv(x, y) \tag{5.5}$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$.

Example 5.1.3
Let $f : \mathbb{C} \to \mathbb{C}$ and for $z \in \mathbb{C}, f(z) = \frac{\bar{z}}{z}$. To split f into real and imaginary parts:

$$\begin{aligned} f(z) &= \frac{\bar{z}}{z} \\ &= (x + iy)\left(\frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}\right) \\ &= \frac{x^2 - y^2}{x^2 + y^2} + i\frac{(-2xy)}{x^2 + y^2} \end{aligned}$$

and we get

$$u(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

$$v(x,y) = -\frac{2xy}{x^2 + y^2}$$

Chapter 6

Lecture 6 Jan 15th 2018

6.1 Continuity (Continued)

Exercise 6.1.1
Let $f : \Omega \to \mathbb{C}$. *Prove that* $f(z)$ *is continuous at* $z_0 = x_0 + iy_0 \in \mathbb{C} \iff$ *functions* $u, v : \mathbb{R}^2 \to \mathbb{R}$, *such that* $f(z) = u(x,y) + iv(x,y)$ *are both continuous at* (x_0, y_0).

Solution
We shall first prove the forward direction. Suppose that $f(z)$ *is continuous at* $z_0 = x_0 + iy_0 \in \mathbb{C}$. *By Definition 5.1.3,* $\forall \{z_n\}_{n \in \mathbb{N}} \subseteq \Omega$, $z_n \to z_0 \implies f(z_n) \to f(z_0)$. *By Exercise 5.1.1,*

$$\begin{aligned} z_n \to z_0 &\iff \operatorname{Re} z_n \to \operatorname{Re} z_0 \wedge \operatorname{Im} z_n \to \operatorname{Im} z_0 \\ &\iff x_n \to x_0 \wedge y_n \to y_0 \end{aligned} \tag{6.1}$$

where $z_n = x_n + iy_n$ *for* $x_n, y_n \in \mathbb{R}$.

Similarly so, and by Equation (5.5),

$$f(z_n) + f(z_0) \iff u(x_n, y_n) \to u(x_0, y_0) \wedge v(x_n, y_n) \to v(x_0, y_0) \tag{6.2}$$

Putting together Equation (6.1) and Equation (6.2), we get

$$(x_n, y_n) \to (x_0, y_0) \implies u(x_n, y_n) \to u(x_0, y_0) \wedge v(x_n, y_n) \to v(x_0, y_0)$$

as desired.

The proof of the other direction is simply a reversed process of the above. $\square$

6.2 Differentiability

Definition 6.2.1 (Neighbourhood)
For $z_0 \in \mathbb{C}, r \in \mathbb{R}$, let

$$D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}. \tag{6.3}$$

On the complex plane, this is seen as a open disk centered around the point z_0 with radius r, as shown below.

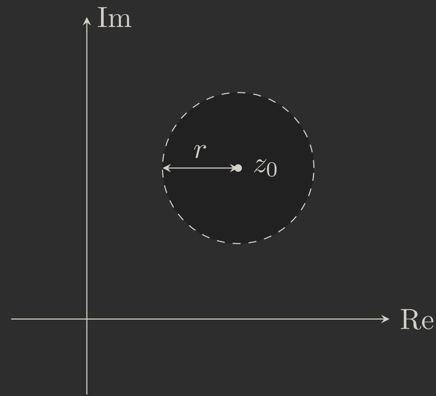


Figure 6.1: Open disk centered around z_0 with radius r

This open disk is called a ***neighbourhood*** *of z_0.*

Definition 6.2.2 (Differentiable/Holomorphic)
Let $f(z)$ be defined in a neighbourhood of $z_0 \in \mathbb{C}$. We say f is ***differentiable/holomorphic*** *at z_0 if for some $h \in \mathbb{C}$,*

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{6.4}$$

exists. If such a limit exists, we denote the limit by $f'(z_0)$.

Remark
$h \in \mathbb{C}$: h need not necessarily be real. In this sense, h approaches 0 from ***any direction*** *around $0 \in \mathbb{C}$.*

Example 6.2.1
For $z \in \mathbb{C} \setminus \{0\}$, let $f(z) = \frac{1}{z}$. Let $z_0 \in \mathbb{C} \setminus \{0\}$. Note that

$$\lim_{h \to 0} \frac{\frac{1}{z_0 + h} - \frac{1}{z_0}}{h} = \lim_{h \to 0} \frac{1}{h} \left[\frac{-h}{(z_0 + h) z_0} \right] = -\frac{1}{z_0^2}$$

Thus f is holomorphic at any $z \in \mathbb{C} \setminus \{0\}$, and hence $f'(z) = -\frac{1}{z}$.

Example 6.2.2
For $z \in \mathbb{C}$, let $f(z) = \bar{z}$. Let $z_0 \in \mathbb{C}$. Notice that

$$\lim_{h \to 0} \frac{\overline{z_0 + h} - \bar{z}}{h} = \lim_{h \to 0} \frac{\bar{h}}{h}.$$

From Example 5.1.2, we know that such a limit does not exist. Thus f is not holomorphic on any $z \in \mathbb{C}$.

Exercise 6.2.1 (Holomorphic Functions Properties)
If f, g are holomorphic at $z \in \mathbb{C}$, prove that

1. *$f + g$ is holomorphic and $(f + g)' = f' + g'$.*
2. *fg is holomorphic and $(fg)' = f'g + fg'$.*
3. *if $g(z) \neq 0$, $\frac{f}{g}$ is holomorphic and $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$.*

Solution

1. *For $f + g$,*

$$\begin{aligned}
&\lim_{h \to 0} \frac{f(z+h) + g(z+h) - f(z) - g(z)}{h} \\
=&\lim_{h \to 0} \left[\frac{f(z+h) - f(z)}{h} + \frac{g(z+h) - g(z)}{g} \right] \\
=&f'(z) + g'(z)
\end{aligned}$$

Thus $(f + g)' = f' + g'$.

2. *For fg,*

$$\begin{aligned}
&\lim_{h \to 0} \frac{f(z+h)g(z+h) - f(z)g(z)}{h} \\
=&\lim_{h \to 0} \frac{f(z+h)g(z+h) + f(z)g(z+h) - f(z)g(z+h) - f(z)g(z)}{h} \\
=&\lim_{h \to 0} \left[\frac{f(z+h) - f(z)}{h} g(z+h) + f(z) \frac{g(z+h) - g(z)}{h} \right] \\
=&f'(z)g(z) + f(z)g'(z)
\end{aligned}$$

Therefore, $(fg)' = f'g + fg'$.

3. *When $\forall z \in \mathbb{C}$ $g(z) \neq 0$, for $\frac{f}{g}$,*

$$\begin{aligned}
&\lim_{h\to 0} \frac{\frac{f(z+h)}{g(z+h)} - \frac{f(z)}{g(z)}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{f(z+h)g(z) - f(z)g(z+h)}{g(z+h)g(z)}\right] \\
&= \lim_{h\to 0} \frac{1}{g(z+h)g(z)}\left[\frac{f(z+h)g(z) + f(z)g(z) - f(z)g(z) - f(z)g(z+h)}{g}\right] \\
&= \lim_{h\to 0} \frac{1}{g(z+h)g(z)}\left[\frac{[f(z+h) - f(z)]g(z) - f(z)[g(z+h) - g(z)]}{h}\right] \\
&= \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}
\end{aligned}$$

Hence, $\frac{f}{g} = \frac{f'g - fg'}{g^2}$

Note

If we look at the example above from the perspective of f being treated as a real-valued function, i.e. $f(z) = u(x,y) + iv(x,y)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$ and $z = x + iy$, observe that $\forall (x,y) \in \mathbb{R}^2, (x,y) \mapsto (x,-y)$, which we see that u and v are partially differentiable in $\mathbb{R}^2$.

We will now look into this "discrepancy".

6.2.1 Cauchy-Riemann Equations

Consider the following function taken from Equation (6.4),

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{6.5}$$

While h may approach $0 \in \mathbb{C}$ from infinitely many sides on the complex plane, we will consider 2 cases.

Case 1: $h \to 0$ via the real axis

In this case, $h = x + i(0)$ and $x \to 0 \in \mathbb{R}$. Then Equation (6.5) gives

$$\begin{aligned}
f'(z_0) &= \lim_{x\to 0} \frac{u(x_0 + x, y_0) + iv(x_0 + x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{x} \\
&= \lim_{x\to 0}\left[\frac{u(x_0 + x, y_0) - u(x_0, y_0)}{x} + i\frac{v(x_0 + x, y_0) - v(x_0, y_0)}{x}\right] \\
&= \frac{\partial u}{\partial x}\Big|_{(x_0,y_0)} + i\frac{\partial v}{\partial x}\Big|_{(x_0,y_0)} \qquad (6.6)
\end{aligned}$$

Case 2: $h \to 0$ via the imaginary axis

In this case, $h = 0 + iy$ and $y \to 0 \in \mathbb{R}$. In a similar fashion, Equation (6.5) becomes

$$\begin{aligned} f'(z_0) &= \lim_{y \to 0} \left[\frac{u(x_0, y_0 + y) - u(x_0, y_0)}{iy} + \frac{v(x_0, y_0 + y) - v(x_0, y_0)}{y} \right] \\ &= \frac{1}{i} \cdot \frac{\partial u}{\partial y}\Big|_{(x_0, y_0)} + \frac{\partial v}{\partial y}\Big|_{(x_0, y_0)} \end{aligned} \tag{6.7}$$

Note that since $f'(z_0)$ exists, the real and imaginary part of Equation (6.6) and Equation (6.7) must equate. Also note that $\frac{1}{i} = -i$. With that, we obtain the following theorem.

Theorem 6.2.1 (Cauchy-Riemann Equations)
If $f(z)$ is holomorphic at $z_0 = x_0 + iy_0 \in \mathbb{C}$ where $x_0, y_0 \in \mathbb{R}$, then, at (x_0, y_0),

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{6.8}$$

Chapter 7

Lecture 7 Jan 17 2018

7.1 Differentiability (Continued)

7.1.1 Cauchy-Riemann Equations (Continued)

It is natural to wonder if the **converse** of Theorem 6.2.1 is true. We present the following example.

Example 7.1.1
Let

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Check if

1. *f is holomorphic at 0.*
2. *Theorem 6.2.1 holds at $(0,0)$.*

Proof

1. *Observe that by letting $h = x_h + iy_h$ where $x_h, y_h \in \mathbb{R}$,*

$$\lim_{h \to 0} \frac{\frac{\overline{0+h}^2}{0+h} - 0}{h} = \lim_{h \to 0} \frac{\bar{h}^2}{h} = \lim_{x_h + iy_h \to 0} \left(\frac{x_h - iy_h}{x_h + iy_h} \right)^2$$

Consider $y_h = kx_h$, for $k \in \mathbb{R} \setminus \{0\}$. Then

$$\lim_{x_h \to 0} \left(\frac{x_h - ikx_h}{x_h + ikx_h} \right)^2 = \left(\frac{1 - ik}{1 + ik} \right)^2,$$

where we see that the limit depends on the value of k. Therefore, the limit DNE. Hence f is not holomorphic at 0.

2. *Let $z = x + iy$ for $x, y \in \mathbb{R}$. Then*

$$\frac{\bar{z}^2}{z} = \frac{(x-iy)^2}{x+iy} = \frac{(x-iy)^3}{x^2+y^2} = \frac{x^3 - 3xy^2}{x^2+y^2} + i\frac{(-3x^2y + y^3)}{x^2+y^2}$$

Therefore, we obtain

$$u(x,y) = \begin{cases} \frac{x^3-3xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$
$$v(x,y) = \begin{cases} \frac{y^3-3x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Observe that

$$\frac{\partial u}{\partial x}\bigg|_{(0,0)} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = 1$$
$$\frac{\partial v}{\partial y}\bigg|_{(0,0)} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = 1$$
$$\text{and}$$
$$\frac{\partial u}{\partial y}\bigg|_{(0,0)} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = 0$$
$$\frac{\partial v}{\partial x}\bigg|_{(0,0)} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = 0$$

satisfies Equation (6.8).

This illustrates that the converse of Theorem 6.2.1 is not true. We will, however, show that the converse will be true given an extra condition.

Theorem 7.1.1 (Conditional Converse of CRE)
Let $z_0 = x_0 + iy_0 \in \Omega \subseteq \mathbb{C}, x_0, y_0 \in \mathbb{R}$, and $u, v : \mathbb{R}^2 \to \mathbb{R}, f = u + iv : \Omega \to \mathbb{C}$. If

1. *the partials of u, v exist in a neighbourhood of (x_0, y_0),*
2. *the partials of u, v are continuous at (x_0, y_0), and*
3. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ *and* $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ *at (x_0, y_0),*

then f is holomorphic at z_0.

A proof of the theorem is in page 36 of Newman and Bak (recommended text of PMATH352W18). I may include the proof whenever I am free.

7.1.2 Power Series

Definition 7.1.1 (Power Series)
A ***power series*** *in* $\mathbb{C}$ *is an infinite series of the form*

$$\sum_{n \in \mathbb{N}} c_n z^n, \tag{7.1}$$

where each $c_n \in \mathbb{C}$ *is the coefficient of* z *of the* n*-th power.*

In this subsection, we are interested to see if Equation (7.1) converges.

Recall the notion of convergence in series from $\mathbb{R}$. Equation (7.1) converges if the sequence of partial sums $\{S_N\}$ converges as $N \to \infty$, where

$$S_N := \sum_{n=0}^{N} c_n z^n$$

In other words, using the same definition of S_N,

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \setminus \{0\} \quad \forall n > N$$
$$|S_n - L| < \epsilon$$

where $L \in \mathbb{C}$ is the limit that the sequence converges to.

We also know that Equation (7.1) converges absolutely if $\sum_{n=0}^{\infty} |c_n| \, |z|^n$ converges. This is a stronger statement (i.e. absolute convergence $\implies$ convergence)

$$\because \left| \sum_{n=0}^{N} c_n z^n \right| \leq \sum_{n=0}^{N} |c_n| \, |z|^n \quad \text{for each } N \in \mathbb{N}$$

Example 7.1.2
$\sum_{n=0}^{\infty} z^n$ *converges absolutely for* $|z| < 1$.

Note that the partial sum of a geometric series is

$$\sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r}$$

and so the limit as $N \to \infty$ *exists if* $|r| < 1$*, and hence we see that*

$$\sum_{n=0}^{N} r^n \to \frac{1}{1 - r}$$

if $|r| < 1$ *as* $N \to \infty$.

However, if $|z| = 1$*, the power series diverges.*

Another note that we shall point out is that if Equation (7.1) converges absolutely for some $z_0 \in \mathbb{C}$, then it converges absolutely for any z where $|z| < |z_0|$.

These notions, in turn, begs the question of **what is the largest possible $|z_0|$ for the series to converge absolutely**.

Chapter 8

Lecture 8 Jan 19 2018

8.1 Power Series (Continued)

8.1.1 Radius of Convergence

Theorem 8.1.1 (Convergence in the Radius of Convergence)
For any power series $\sum_{n\in\mathbb{N}} c_n z^n$, $\exists 0 \leq R < \infty$, such that

1. $|z| < R \implies$ *series converges absolutely.*
2. $|z| > R \implies$ *series diverges.*

Moreover, R is given by ***Hadamard's Formula****:*

$$\frac{1}{R} := \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} \tag{8.1}$$

Remark

1. *R is called the* ***radius of convergence*** *of the series. $\{z \in \mathbb{C} : |z| < R\}$ is called the disk of convergence of the series.*
2. *Recall the definition of the* ***limit supremum***

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \left(\sup_{m\geq n} a_m\right) \tag{8.2}$$

which we may colloquially say as the "highest peak 'reached' by a_n's as $n \to \infty$"

Proposition 8.1.1 (A Property of limsup)

$$\forall \{a_n\}_{n\in\mathbb{N}} \quad L := \limsup_{n\to\infty} a_n \implies$$
$$\forall \epsilon > 0 \quad \exists N > 0 \quad \forall n > N$$
$$L - \epsilon < a_n < L + \epsilon$$

(Proof to be included)

Proof (Theorem 8.1.1)
Let $L := \frac{1}{R} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$. *Clearly,* $L \geq 0$.

1. *Suppose* $|z| < R$. $\exists \epsilon > 0, r := |z|(L+\epsilon)$ *such that* $0 < r < 1$. *By Proposition 8.1.1,* $\exists N \in \mathbb{N}, \forall n > N, |c_n|^{\frac{1}{n}} < L + \epsilon$.

   Now since $L = \frac{1}{R}$,

   $$\sum_{n=N}^{\infty} |c_n| |z|^n = \sum_{n=N}^{\infty} (|c_n|^{\frac{1}{n}} |z|)^n < \sum_{n=N}^{\infty} r^n$$

   and since $0 < r < 1$, *the final summation converges (as it is a geometric sum). Thus by comparison test,* $\sum_{n=N}^{\infty} |c_n| |z|^n$ *converges.*

   We may also proceed with noticing that the partial sum of $\sum_{n=N}^{\infty} |c_n| |z|^n$ *is* ***bounded and monotonic****, which shows that the series converges.*

2. *Suppose* $|z| > R$. $\exists \epsilon > 0, r := |z|(L-\epsilon)$ *such that* $r > 1$. *By Proposition 8.1.1,* $\exists N \in \mathbb{N}, \forall n > N, |c_n|^{\frac{1}{n}} > L - \epsilon$. *Then analogous to the proof above,*

   $$\sum_{n=N}^{\infty} |c_n| |z|^n = \sum_{n=N}^{\infty} (|c_n|^{\frac{1}{n}} |z|)^n > \sum_{n=N}^{\infty} r^n$$

   where the final summation diverges, and thus implying that $\sum_{n=N}^{\infty} |c_n| |z|^n$ *diverges.*

Theorem 8.1.2 (Power function, holomorphic function, region of convergence)
Suppose $f(z) = \sum_{n\in\mathbb{N}} c_n z^n$ *has a radius of convergence* $R \in \mathbb{R}$. *Then* $f'(z)$ *exists and equals*

$$\sum_{n=1}^{\infty} n c_n z^{n-1}$$

throughout $|z| < R$.

Moreover, f' *has the* ***same radius of convergence*** *as* f.

Proof

Note that f' has the same radius of convergence as f since

$$\limsup_{n\to\infty} |nc_n|^{\frac{1}{n}} = \limsup_{n\to\infty} |n|^{\frac{1}{n}} |c_n|^{\frac{1}{n}} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$$

where note that $\lim_{n\to\infty} |n|^{\frac{1}{n}} = 1$.

Let $|z_0| \le r < R$ and $g(z_0) := \sum_{n=1}^{\infty} nc_n z_0^{n-1}$.

WTS

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = g(z_0)$$

OR

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \tag{$\dagger$}$$
$$|h| < \delta \implies \left| \frac{f(z_0+h) - f(z_0)}{h} - g(z_0) \right| < \epsilon$$

(Note that we would want $\delta > 0$ such that $|z_0 + h| \le r < R$)

WTP($\dagger$). *$\forall \epsilon > 0$, choose $\delta > 0$. Suppose $|z_0 + h| < |z_0 + \delta| \le r < R$. Write*

$$f(z) = \sum_{n=0}^{N} c_n z^n + \sum_{n=N}^{\infty} c_n z^n$$

and let $S_N(z)$ and $E_N(z)$ be the first and second terms respectively. Then we have that

$$S_N'(z) = \sum_{n=1}^{N} nc_n z^{n-1}$$

Consider

$$\begin{aligned} &\left| \frac{S_N(z_0+h) - S_N(z_0)}{h} + \frac{E_N(z_0+h) - E_N(z_0)}{h} - g(z_0) + S_N'(z_0) - S_N'(z_0) \right| \\ \le &\left| \frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0) \right| + \left| \frac{E_N(z_0+h) - E_N(z_0)}{h} \right| + \left| S_N'(z_0) - g(z_0) \right| \end{aligned} \tag{8.3}$$

For the second term, since $a^2 - b^2 = (a-b)(a^{n-1} + a^{n-2}b + \ldots ab^{n-2} + b^{n-1})$ and $|z_0|, |z_0 + h| < r \le R$, we obtain $(z_0+h)^n - z_0^n \le hnr^{n-1}$. Thus

$$\left| \frac{E_N(z_0+h) - E_N(z_0)}{h} \right| = \left| \frac{1}{h} \sum_{n=N+1}^{\infty} c_n [(z_0+h)^n - z_0^n] \right| \le \sum_{n=N+1}^{\infty} nc_n r^{n-1}.$$

Note that

$$\sum_{n=1}^{\infty} nc_n r^{n-1} = g(r), \tag{8.4}$$

and since $r < R$, *Equation* (8.4) *converges absolutely by Theorem 8.1.1, thus the tail* $\sum_{n=N+1}^{\infty} nc_n r^{n-1}$ *converges absolutely. Therefore, by comparison, we can pick* $\frac{\epsilon}{3} > 0$ *so that* $\exists N_1 \in \mathbb{N}, \forall n > N_1$

$$\left| \frac{E_N(z_0+h) - E_N(z_0)}{h} \right| < \frac{\epsilon}{3}.$$

For the third term in Equation (8.3), *we observe that by definition,* $S'_N(z_0) = \sum_{n=1}^{N} nc_n z^{n-1}$. *Since*

$$\lim_{N\to\infty} S'_N(z_0) = \lim_{N\to\infty} \sum_{n=1}^{N} nc_n z^{n-1} = \sum_{n=1}^{\infty} nc_n z^{n-1} = g(z_0)$$

we know that we can pick $\frac{\epsilon}{3} > 0, \exists N_2 \in \mathbb{N}, \forall n > N_2$, *we have*

$$\left| S'_N(z_0) - g(z_0) \right| < \frac{\epsilon}{3}$$

.

For the first term in Equation (8.3), *note that* $(S_N(z_0))' = S'_N(z_0)$. *Let* $\frac{\epsilon}{3} > 0, \exists \delta > 0, \exists N > \max\{N_1, N_2\}, \forall n > N$, *since* $|h| < \delta$, *we have that*

$$\left| \frac{S_N(z_0+h) - S_N(z_0)}{h} - S'_N(z_0) \right| < \frac{\epsilon}{3}$$

This completes the proof. □

Chapter 9

Lecture 9 Jan 22 2018

9.1 Power Series (Continued 2)

9.1.1 Radius of Convergence (Continued)

Example 9.1.1
Let $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$. To find the radius of convergence, we use Hadamard's Formula:

$$\frac{1}{R} = \limsup_{n \to \infty} \left(\frac{1}{n} \right)^{\frac{1}{n}} = 1 \qquad \because \lim_{n \to \infty} n^{\frac{1}{n}} = 1$$

Therefore $R = 1$. Thus, by Theorem 8.1.1, f converges absolutely when $|z| < 1$ and diverges when $|z| > 1$. As for the boundary, i.e. $|z| = 1$, consider the following two cases:

1. *If $z = 1$, then $f(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ is a **harmonic series**, and hence f diverges.*
2. *If $z = i$, then*

$$\begin{aligned} f(i) &= \sum_{n=1}^{\infty} \frac{i^n}{n} \\ &= i - \frac{1}{2} + \frac{-i}{3} + \frac{1}{4} + \frac{i}{5} - \frac{1}{6} \\ &= \left(-\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \ldots \right) + i \left(1 - \frac{1}{3} + \frac{1}{5} + \ldots \right). \end{aligned}$$

   *Observe that both the real and imaginary parts are alternating series where the absolute values of each term is decreasing, which, by the **alternating series test**, converge. Thus in this case, f converges.*

Therefore, we observe that ***both convergence and divergence may occur*** *on the boundary, depending on the value of z.*

Note
We may not always exchange the position of $\lim$ *and* $\sum_{a=1}^{b}$ *when we consider an infinite sum (i.e. $b = \infty$). Here's an example why this is true. Consider the function $f(x) = \sum_{n=1}^{\infty}(x^n - x^{n-1})$ for $|x| < 1$. Is*

$$\lim_{x\to 1}\sum_{n=1}^{\infty}(x^n - x^{n-1}) = \sum_{n=1}^{\infty}\lim_{x\to 1}(x^n x^{n-1})$$

true?

Clearly, RHS is 0. For LHS, note that

$$\begin{aligned} f(x) &= \lim_{N\to\infty}\sum_{n=1}^{N}(x^n - x^{n-1}) \\ &= \lim_{N\to\infty}(x - x^2 + x^2 - x^3 + \ldots + x^N x^{N+1}) \\ &= \lim_{N\to\infty}(x - x^{N+1}) = x. \end{aligned}$$

So,

$$LHS = \lim_{x\to 1} x = 1$$

And we see that RHS $\neq$ LHS.

Definition 9.1.1 (Entire Function)
A function f is said to be ***entire*** *if f is holomorphic in* ***the entire complex plane****.*

Exercise 9.1.1
Define $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$. Show that

1. *the radius of convergence of this series is ∞, and hence that e^z is an entire function. (Hint: Use* ***Stirling's formula****: $n! \sim (\frac{n}{e})^n\sqrt{2\pi n}$)*
2. *$(e^z)' = e^z$*

Solution

1. *Using Stirling's formula, note that we have*

$$e^z = \sum_{n=0}^{\infty}\frac{1}{\sqrt{2\pi n}}\left(\frac{ez}{n}\right)^n$$

To find R, we have

$$\begin{aligned}\frac{1}{R} &= \limsup_{n\to\infty} \left| \frac{1}{\sqrt{2\pi n}} \left(\frac{e}{n}\right) \right|^{\frac{1}{n}} \\ &= \limsup_{n\to\infty} \left| \frac{e}{n} \right| \limsup_{n\to\infty} \left| \frac{1}{\sqrt{2\pi n}} \right|^{\frac{1}{n}} \\ &= 0\end{aligned}$$

since $\limsup_{n\to\infty} \frac{e}{n} = 0$. *Thus* $R = \infty$. *By Theorem 8.1.1,* e^z *is an entire function.*

2. *Note that*

$$\begin{aligned}\lim_{h\to 0} \frac{e^{z+h} - e^z}{h} &= e^z \lim_{h\to 0} \frac{e^h - 1}{h} \\ &= e^z \lim_{h\to 0} \frac{1 + h + \frac{h^2}{2} + \frac{h^3}{3} + \ldots - 1}{h} \\ &= e^z\end{aligned}$$

Thus $(e^z)' = e^z$ *as required.*

Chapter 10

Lecture 10 Jan 24 2018

10.1 Power Series (Continued 3)

10.1.1 Radius of Convergence (Continued 2)

A power series is infinitely $\mathbb{C}$-differentiable in its radius of convergence. All its derivatives are also power series, obtained by term-wise differentiation.

E.g.

$$f(z) - \sum_{n=0}^{\infty} c_n z^n \quad \text{then} \quad f^{(2)}(z) = \sum_{n=0}^{\infty} n(n-1) c_n z^{n-2}$$

In general, we may have $\sum_{n=0}^{\infty} c_n (z - z_0)^n$, which is a power series centered at $z_0 \in \mathbb{C}$. Then, as before, the radius of convergence of this power series is given by

$$\frac{1}{R} = \limsup_{n \to \infty} |c_n|^{\frac{1}{n}}$$

So instead of having the disc of convergence centered around 0, we now have one that is centered around z_0.

Corollary 10.1.1 (Corollary of Theorem 8.1.2)
From Theorem 8.1.2, we have shown that

$f(z)$ has a power series expansion at z_0 (i.e. $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$ in some neighbourhood of z_0) with radius of convergence $R > 0$ $\implies$ *f is holomorphic at z_0*

The converse of the statement above is true, i.e.

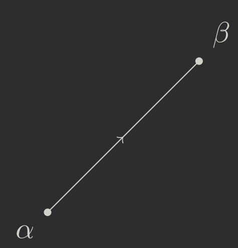
f is holomorphic at z_0 $\implies$ $f(z)$ has a power series expansion at z_0 (i.e. $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$ in some neighbourhood of z_0) with radius of convergence $R > 0$

This converse, however, is not possible to be proven given the current tools on our belt. And so we now have to venture into integrals in $\mathbb{C}$.

10.2 Integration in $\mathbb{C}$

10.2.1 Curves and Paths

Before we begin with the definition of a curve in $\mathbb{C}$, let us consider how a straight line should be described as a vector-valued function in the complex plane. For instance, if we have two points $\alpha, \beta \in \mathbb{C}$, and we want to describe the straight line connecting the two.



Let γ be the function that describes this line. We may then define $\gamma : [0,1] \to \mathbb{C}$ to be either

$$\gamma(t) = \alpha + (\beta - \alpha)t \quad \text{or} \quad \gamma = \alpha(1-t) + \beta t.$$

We would then have the following mapping:

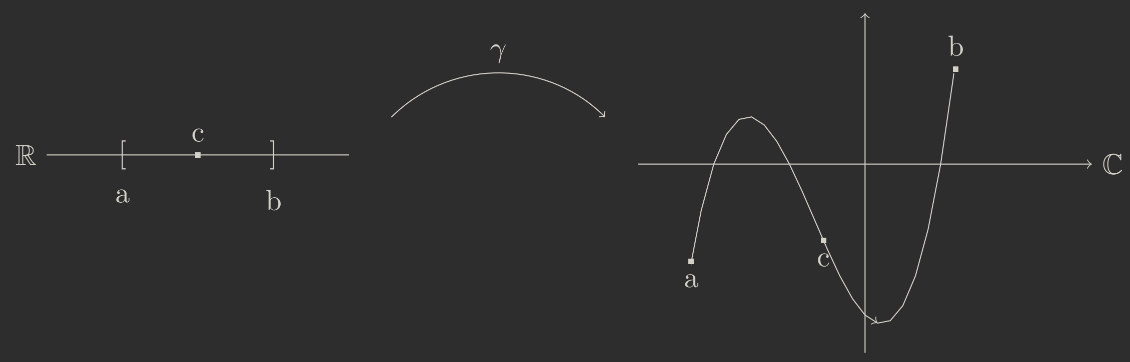


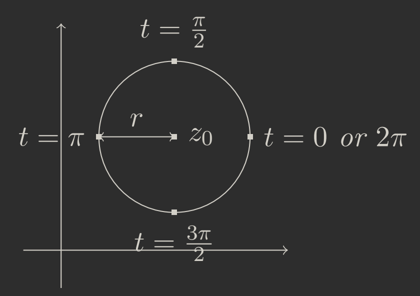
Figure 10.1: Mapping from $\mathbb{R} \to \mathbb{C}$ with γ, which is called **the curve γ**

Definition 10.2.1 (Curves in $\mathbb{C}$)
A curve in $\mathbb{C}$ is a continuous function, $\gamma(t) : [a, b] \to \mathbb{C}$, where $a, b \in \mathbb{R}$. The image of γ in $\mathbb{C}$ is called γ^*.

Example 10.2.1
Let $z_0 \in \mathbb{C}, r > 0$.

1. *Let $\gamma : [0, 2\pi] \to \mathbb{C}$, such that $\gamma(t) = z_0 + re^{it}$.*
2. *Let $\gamma' : [0, 1] \to \mathbb{C}$, such that $\gamma'(t) = z_0 + re^{2\pi it}$.*

The two functions above describe a circle centered at z_0 with radius r, anticlockwise-oriented.



We say that γ and γ' are equivalent parameterizations for the same oriented path.

Definition 10.2.2 (Equivalent Parameterization)
Let $\gamma_1 : [a, b] \to \mathbb{C}, \gamma_2 : [c, d] \to \mathbb{C}$ where $a, b, c, d \in \mathbb{C}$ describe the path γ^*. The two ***paramaterization are said to be equivalent*** *if $\exists h : [a, b] \to [c, d]$ that is a bijection and a continuous function such that*

$$\gamma_1(t) = \gamma_2(h(t))$$

where $t \in [a, b]$.

Note
We will not look at functions like the Weierstrass function in this course.

Definition 10.2.3 (Smooth Curve)
Let $\gamma : [a, b] \to \mathbb{C}, a, b, \in \mathbb{C}$. γ *is said to be smooth if its derivative* γ' *exists and is continuous on* $[a, b]$ *and* $\forall t \in [a, b], \gamma'(t) \neq 0$.

Definition 10.2.4 (Piecewise Smooth)
Let $\gamma : [a, b] \to \mathbb{C}$. γ *is said to be piecewise smooth if it is smooth on* $[a, b]$ *except on finitely many points in* $[a, b]$.

Remark
Piecewise smooth curves shall be called paths.

10.2.2 Integral

Definition 10.2.5 (Integral of f over a path γ)
Given a path $\gamma : [a, b] \to \mathbb{C}$ *and* $f : \mathbb{C} \to \mathbb{C}$, *a function continuous on* γ. *We define the integral* f *along* γ *as*

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt \tag{10.1}$$

where we let $z = \gamma(t)$ *and hence* $dz = \gamma'(t)dt$.

Remark

1. *Suppose* g *is a complex-valued function, then*

$$\int_a^b g(t)dt = \int_a^b \mathrm{Re}(g(t))dt + i\int_a^b \mathrm{Im}(g(t))dt$$

2. *The integral of* f *along* γ *can be shown to be independent of the chosen parameterization for* γ^*.

   Proof
   Let $a, b, c, d \in \mathbb{R}, \gamma_1 : [a, b] \to \mathbb{C}, \gamma_2 : [c, d] \to \mathbb{C}$ *describe the same path* γ^*. *By Definition 10.2.2, define a bijection* $h : [a, b] \to [c, d]$ *that is a continuous function such that* $t \mapsto \tau$, *so that*

$$\gamma_1(t) = \gamma_2(h(t)) = \gamma(\tau).$$

   Note that

$$\begin{aligned} \gamma_1'(t) &= h'(t)\gamma_2'(h(t)) \text{ and} \\ h(t) = \tau &\implies h'(t)dt = d\tau. \end{aligned}$$

Now since h is a bijection, we claim that $h(a) = c$ while $h(b) = d$.

We know that h cannot be a constant function. Suppose h is an increasing function, then since $a \leq b$ and $c \leq d$, it is clear that $h(a) = c$ and $h(b) = d$. Similarly, if h is a decreasing function, then $h(a) = d$ and $h(b) = c$. But this is a contradiction to our supposition that γ_1 and γ_2 describe the same orientation. Thus h must be an increasing function, and hence we have $h(a) = c$ and $h(b) = d$.

(This can be more rigorous but that is an easy proof, and we may use perhaps the Approximation Property of $\mathbb{R}$ to that end, which is a fun exercise that shall not be included within these covers.)

Now

$$\begin{aligned}
\int_{\gamma_1} f(z)dz &= \int_a^b f\big(\gamma_1(t)\big)\gamma_1'(t)dt \\
&= \int_a^b f\Big(\gamma_2\big(h(t)\big)\Big)h'(t)\gamma_2'(h(t))dt \\
&= \int_c^d f\big(\gamma_2(\tau)\big)\gamma_2'(\tau)d\tau \\
&= \int_{\gamma_2} f(z)dz
\end{aligned}$$

This completes the proof. $\square$

Chapter 11

Lecture 11 Jan 26 2018

11.1 Integration in $\mathbb{C}$ (Continued)

11.1.1 Integral (Continued)

Note (Recall)
Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth curve. For a function f that is continuous on γ, we defined

$$\begin{aligned}\int_\gamma f(z)dz &= \int_a^b f\big(\gamma(t)\big)\gamma'(t)dt \\ &= \int_a^b \mathrm{Re}\Big(f\big(\gamma(t)\big)\gamma'(t)\Big)dt + i\int_a^b \mathrm{Im}\Big(f\big(\gamma(t)\big)\gamma'(t)\Big)dt\end{aligned}$$

and have

$$\begin{gathered}\gamma'(t) = u'(t) + iv'(t) \\ \text{if } \gamma(t) = u(t) + iv(t)\end{gathered}$$

Example 11.1.1
Let $f(z) = f(x + iy) = x^2 + y^2$ be continuous along $\gamma : [0, 1] \to \mathbb{C}$ $t \mapsto t + it$. Evaluate $\int_\gamma f(z)dz$.

Solution

$$\begin{aligned}\int_\gamma f(z)dz &= \int_0^1 f(t+it)(1+i)dt \\ &= (1+i)^2\int_0^1 t^2 dt \\ &= (1+i)^2 \cdot \frac{1}{3}t^3\Big|_0^1 \\ &= \frac{2i}{3}\end{aligned}$$

Example 11.1.2
$\forall n \in \mathbb{Z}$, evaluate $\int_\gamma z^n dz$ that is continue on the path γ that describes any circle centered at origin oriented anticlockwise.

Solution
Let $R \in \mathbb{R}$, and define

$$\gamma : [0,1] \to \mathbb{C} \;\; t \mapsto Re^{2\pi it}$$
$$\gamma'(t) = 2R\pi i e^{2\pi it} = 2\pi i\gamma(t)$$

Then

$$\begin{aligned}\int_\gamma z^n dz &= \int_0^1 R^n e^{2\pi int} \cdot 2\pi i \cdot Re^{2\pi it} dt \\ &= 2\pi i R^{n+1}\int_0^1 e^{2\pi i(n+1)t}dt \\ &= \begin{cases} \frac{R^{n+1}}{n+1} e^{2\pi i(n+1)t}\Big|_0^1 & \textit{if } n \in \mathbb{Z}\setminus\{-1\} \\ 2\pi it\Big|_0^1 & \textit{if } n = -1 \end{cases} \\ &= \begin{cases} \frac{R^{n+1}}{n+1}\left(e^{2\pi i(n+1)} - 1\right) & \textit{if } n \in \mathbb{Z}\setminus\{-1\} \\ 2\pi i & \textit{if } n = -1 \end{cases} \quad \because e^{2\pi ki} \equiv 1 \mod 2\pi \\ &= \begin{cases} 0 & \textit{if } n \in \mathbb{Z}\setminus\{-1\} \\ 2\pi i & \textit{if } n = -1 \end{cases}\end{aligned}$$

Note that our final answer does not depend on R, the radius of the circle.

Proposition 11.1.1 (Properties of integrals in $\mathbb{C}$)

1. ***(Linearity)*** *Let $\alpha, \beta \in \mathbb{C}$. $\int_\gamma (\alpha f(z) + \beta g(z))\, dz = \alpha\int_\gamma f(z)dz + \beta\int_\gamma g(z)dz$.*

2. (a) *For any complex-valued function g, and $b \geq a$,*

$$\left| \int_a^b g(t)dt \right| \leq \int_a^b |g(t)|\, dt$$

(b) *For any function $f(z)$ that is continuous on a path $\gamma : [a,b] \to \mathbb{C}$,*

$$\left| \int_\gamma f(z)dz \right| \leq \sup_{z \in \gamma} |f(z)| \cdot \underbrace{\int_a^b \left|\gamma'(t)\right| dt}_{\text{length of the path}}$$

3. *If γ^- is the path $\gamma : [a,b] \to \mathbb{C}$ with a reversed direction, then*

$$\int_{\gamma^-} f(z)dz = -\int_\gamma f(z)dz$$

Proof

1.

$$\begin{aligned}
LHS &= \int_\gamma \alpha f(z) + \beta g(z) dz \\
&= \int_\gamma \alpha\Big(\operatorname{Re}(f(z)) + i\operatorname{Im}(f(z))\Big) + \beta\Big(\operatorname{Re}(g(z)) + i\operatorname{Im}(g(z))\Big)dz \\
&= \begin{aligned} &\alpha \int_a^b \Big(\operatorname{Re}(f(z)) + i\operatorname{Im}(f(z))\Big)dt \\ &+ \beta \int_a^b \Big(\operatorname{Re}(g(z)) + i\operatorname{Im}(g(z))\Big)dt \end{aligned} \qquad (\dagger) \\
&= \alpha \int_a^b f(z)dz + \beta \int_a^b g(z)dz
\end{aligned}$$

where ($\dagger$) is because of Item 1 from an earlier remark and Linearity of Integrals in $\mathbb{R}$. □

2. (a) *Let $R \in \mathbb{R}$. Suppose $\int_a^b g(t)dt = Re^{i\theta}$. Then*

$$\begin{aligned}
LHS = R &= \int_a^b \underbrace{e^{-i\theta} g(t)}_{u(t)+iv(t)} dt \\
&= \int_a^b u(t)dt + i\int_a^b v(t)dt
\end{aligned}$$

Since $R \in \mathbb{R}$*, we have* $\int_a^b v(t)dt = 0$*, so*

$$\begin{aligned} R &= \int_a^b u(t)dt \\ &\leq \int_a^b |u(t)|\, dt \quad \because a \leq b \wedge u(t) \leq |u(t)| \in \mathbb{R} \\ &\leq \int_a^b |u(t) + iv(t)|\, dt \\ &= \int_a^b \left| e^{-i\theta} g(t) \right| dt \qquad \textbf{*(is this really correct?)} \\ &= \int_a^b |g(t)|\, dt = \mathit{RHS} \end{aligned}$$

□

(b)

$$\begin{aligned} \mathit{LHS} &= \left| \int_\gamma f(z)dz \right| \\ &= \left| \int_a^b f\big(\gamma(t)\big)\gamma'(t)dt \right| \\ &\leq \int_a^b \left| f\big(\gamma(t)\big)\gamma'(t) \right| dt \quad \text{by Item 2a} \\ &\leq \int_a^b \sup_{z\in\gamma} |f(z)| \left|\gamma'(t)\right| dt \quad \text{since } |f(z)| \leq \sup_{z\in\gamma} |f(z)| \\ &= \sup_{z\in\gamma} |f(z)| \cdot \int_a^b \left|\gamma'(t)\right| dt = \mathit{RHS} \end{aligned}$$

□

3. *Let* $\gamma^- : [b, a] \to \mathbb{C}$ *such that* $\gamma^- = \gamma(b - t + a)$*. Let* $k = b - t + a$ *so that* $dk = -dt$*, and* $t = b \implies k = a \wedge t = a \implies k = b$*. Note that* $k \in [a, b]$ *with this definition.*

Then

$$\begin{aligned}
LHS &= \int_{\gamma^-} f(z)dz \\
&= \int_b^a f\big(\gamma^-(t)\big)\gamma^{-\prime}(t)dt \\
&= \int_b^a f\big(\gamma(b-t+a)\big)\gamma'(b-t+a)dt \\
&= -\int_a^b f\big(\gamma(k)\big)\gamma'(k)dk \\
&= -\int_\gamma f(z)dz = RHS
\end{aligned}$$

as required. □

We are now in a position to generalize the **Fundamental Theorem of Calclus** for $\mathbb{C}$.

Theorem 11.1.1 (Fundamental Theorem of Calculus)
Let $\gamma : [a, b] \to \mathbb{C}$ be a path inside an open set $\Omega \subseteq \mathbb{C}$. Suppose $f(z)$ is continuous on γ, and has an antiderivative F which is holomorphic on Ω (i.e. $\forall z \in \Omega\ f(z) = F'(z)$ and F is holomorphic in Ω). Then

$$\int_\gamma f(z)dz = F\big(\gamma(b)\big) - F\big(\gamma(a)\big) \tag{11.1}$$