1 Complex Numbers and Their Properties

Complex Plane as a Set

$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$

Real and Imaginary Part

$\forall z = x + iy \in \mathbb{C}\ x, y \in \mathbb{R}$
$\mathrm{Re}(z) = x\ \mathrm{Im}(z) = y$

Product

$\forall z = a + ib, w = c + id \in \mathbb{C}\ a, b, c, d \in \mathbb{R}$
$zw = (ac - bd) + i(ad + bc)$

Inverse of a Complex Number

$\forall z = a + ib \in \mathbb{C}\ a, b \in \mathbb{R}$
$\exists z^{-1} = \frac{a}{aa^2+b^2} - i\frac{b}{a^2+b^2} \in \mathbb{C}$

Conjugate

$\forall z = a + ib \in \mathbb{C}\ a, b \in \mathbb{R}$
$\exists \bar{z} = a - ib \in \mathbb{C}$

Modulus

$\forall z = x = iy \in \mathbb{C}\ x, y \in \mathbb{R}$
$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}$

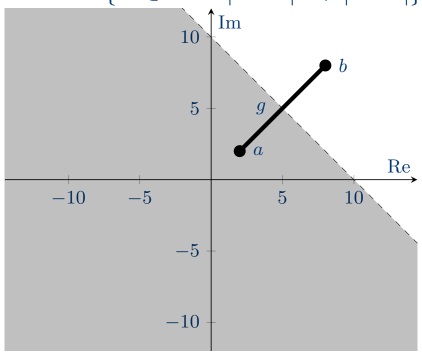
Basic Inequalities

$\forall z, w \in \mathbb{C}$,

1. $|\mathrm{Re}(z)| \le |z|$
2. $|\mathrm{Im}(z)| \le |z|$
3. $|z + w| \le |z| + |w|$
4. $|z + w| \ge \big|\, |z| + |w| \,\big|$

Region of a set of Complex Numbers

Describe $\{z \in \mathbb{C} : |z - a| < |z - b|\}$.



Every complex number has exactly 2 roots

$\forall z = x = iy \in \mathbb{C}\ x, y \in \mathbb{R}$
$\exists w_{1,2} = u + iv \in \mathbb{C}\ u, v \in \mathbb{R}$

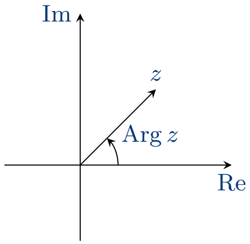
$$w = \begin{cases} \pm\left[\left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}} + i\left(\frac{-x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}\right] & y > 0 \\ \pm\left[\left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}} - i\left(\frac{-x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}\right] & y < 0 \\ \pm\sqrt{x} & y = 0, x > 0 \\ \pm i\sqrt{x} & y = 0, x < 0 \end{cases}$$

Quadratic Formula

$\forall a, b, c \in \mathbb{C}\ a \neq 0\ az^2 + bz + c = 0$

$$z = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Argument



Polar Form

$\forall z \in \mathbb{C}\ \exists r, \theta \in \mathbb{R}\ \theta \in [0, 2\pi)$
$z = re^{i\theta}$

Polar to Cartesian

$x = r\cos\theta \quad y = r\sin\theta$

Cartesian to Polar

$r = |z| \quad \tan\theta = \frac{x}{y}$

Conjugate in Polar Form

$z = re^{i\theta} \iff \bar{z} = re^{-i\theta}$

Inverse in Polar Form

$z = re^{i\theta} \wedge z \neq 0$
$\implies z^{-1} = \frac{1}{r}e^{-i\theta}$

Product in Polar Form

- $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$
- $\forall n \in \mathbb{Z}\ (re^{in}) = r^n e^{in\theta}$

nth Roots of a Complex Number

$\left\{ r^{\frac{1}{n}} e^{i\left(\frac{\theta + 2\pi k}{n}\right)} : k = 0, 1, \ldots, n-1 \right\}$

nth Roots of Unity

$\left\{ e^{i\left(\frac{2\pi k}{n}\right)} : k = 0, 1, \ldots, n-1 \right\}$

2 Complex Functions

2.1 Convergence

$\forall \{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C} \wedge z \in \mathbb{Z}$
$(n \to \infty \implies z_n \to z) \iff \lim_{n\to\infty} |z_n - z| = 0$
May also write as $\lim_{n\to\infty} z_n = z$

2.2 Convergence for Complex Functions

$\forall \Omega \subseteq \mathbb{C}\ \forall f : \Omega \to \mathbb{C}\ z_0 \in \mathbb{C}\ \exists L \in \mathbb{C}\ \forall \{z_n\}_{n\in\mathbb{N}} \subseteq \Omega \setminus \{z_0\}$
$(z_n \to z_0 \implies f(z_0) \to L) \implies \lim_{z\to z_0} f(z) = L$

2.3 Continuity

$\forall f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$
f is continuous on $z_0 \implies$

1. $\forall \{z_0\}_{n\in\mathbb{N}}\ z_n \to z_0 \implies f(z_n) \to f(z_0)$
2. $\forall z \in \Omega\ \forall \varepsilon > 0\ \exists \delta > 0\ |z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon$

2.4 Real and Imaginary Parts of a Function

$f(z) = u(x, y) + iv(x, y)$

3 Differentiation

3.1 Neighbourhood

$\forall z_0 \in \mathbb{C}\ r \in \mathbb{R}\ D(z_0, r) := \{zin\mathbb{C} : |z - z_0| < r\}$ is the neighbourhood of radius r around z_0.

3.2 Differentiation/Holomorphic

Let $z_0 \in \mathbb{C}\ r \in \mathbb{R}\ \exists D(z_0, r) \subseteq \mathbb{R}$. $\forall f : D(z_0, r) \to \mathbb{C}\ \forall h \in \mathbb{C}$
$\exists \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} \implies f$ is differentiable/holomorphic $\wedge\ f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$

3.3 Properties of Holomorphic Functions

f, g are holomorphic at $z \in \mathbb{C} \implies$

1. $(f + g)' = f' + g'$
2. $(fg)' = f'g + fg'$
3. $(g \neq 0 \implies (\frac{f}{g})' = \frac{f'g - fg'}{g^2})$

3.4 Cauchy-Riemann Equations

$\forall z_0 = x_0 + iy_0 \in \mathbb{C}\ x_o, y_o \in \mathbb{R}$. $f(z)$ is holomorphic at $z_0 \implies$ at (x_0, y_0)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

3.5 Conditional Converse of CRE

Let $z_0 = z_0 + iy_0 \in \not\leq \subseteq \mathbb{R}\ x_0, y_0 \in \mathbb{R}\ u, v : \mathbb{R}^{\not\leq} \to \mathbb{R}\ f = u + iv : \Omega \to \mathbb{C}$.

1. partials of u, v exist in nbd of $x_0, y_0)$
2. partials of u, v are cont' at (x_0, y_0)
3. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\implies f$ is holo at z_0.

3.6 Power Series

Infinite series of the form $\sum_{n\in\mathbb{N}} c_n z^n$

3.7 Convergence for Power Series

We will usually aim for absolute convergence, for

$$\left| \sum_{n=0}^{N} c_n z^n \right| \le \sum_{n=0}^{N} |c_n| \, |z|^n$$

3.8 Hadamard's Formula

$\frac{1}{R} := \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$.

3.9 Limit Supremum

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \sup_{m\ge n} a_m$$

3.10 limsup Property

$\forall \{a_n\}_{n\in\mathbb{N}} L := \limsup_{n\to\infty} a_n \implies \forall \varepsilon > 0\ \exists N > 0\ \forall n > N\ |a_0 - L| < \varepsilon$

3.11 Radius of Convergence

$\forall \sum_{n\in\mathbb{N}} c_n z^n\ \exists 0 \le R < \infty$

1. $|z| < R \implies$ absolute convergence
2. $|z| > R \implies$ divergence

3.12 Power Function and its Holomorphic Function share the same Region of Convergence

$f(z) = \sum_{n\in\mathbb{N}} c_n z^n$ had a rad of conv $R \in \mathbb{R} \implies \forall \{z : |z| < R\}$

$$f'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1}$$

rad of conv of f' is R.

3.13 Entire Function

f is said to be entire if f is holomorphic in the entire $\mathbb{C}$.

4 Integration

4.1 Curves

A curve in $\mathbb{C}$ is a cont' fn $\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{C}$. Image of γ is called γ^*.

4.2 Equivalent Parameterization

Let $\gamma_1 : [a, b] \subseteq \mathbb{R} \to \mathbb{C}\ \gamma_2 : [c, d] \subseteq \mathbb{R} \to \mathbb{C}$ desc path γ^*. γ_1, γ_2 are equiv if $\exists h : [a, b] \to [c, d]$, bijective and cont', s.t. $\forall t \in \mathrm{Dom}(h)\ \gamma_1(t) = \gamma_2(h(t))$.

4.3 Smooth Curve

γ is smooth if $\exists \gamma'$ is cont' on $\mathrm{Dom}(\gamma) \wedge \forall t \in \mathrm{Dom}(\gamma)\ \gamma'(t) \neq 0$.

4.4 Piecewise Smooth Curve

γ is piecewise smooth if γ is smooth on $\mathrm{Dom}(\gamma)$ except on finitely many pts.

4.5 Integral over path

Let $\gamma : [a, b] \to \mathbb{C} \wedge f : \mathbb{C} \to \mathbb{C}$ con' on γ. Integral f along γ is

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Integral over a curve γ^* is independent of the path chosen.

4.6 Integral Properties

1. (Linearity) $\int_\gamma (\alpha f + \beta g) = \alpha \int_\gamma f + \beta \int_\gamma g$
2. (a) $\left| \int_a^b g \right| \le \int_a^b |g|$
   (b) $\left| \int_\gamma f dz \right| \le \sup_{z\in\Omega} f(z) \cdot \int_a^b |\gamma'(t)| \, dt$
3. γ^- is in opposite orientation of $\gamma \implies \int_{\gamma^-} f = -\int_\gamma f$

4.7 Fundamental Theorem of Calculus

Let $(\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{C}) \in \Omega \subseteq \mathbb{C}$. f cont' on $\gamma\ \exists F' = f$ holo on $\Omega \implies \int_\gamma f = F(\gamma(b)) - F(\gamma(a))$

4.8 Corollary of FTC

If $F \in H(\Omega)$, $\Omega \subseteq \mathbb{C}$, $\gamma \subseteq \Omega$ that is a closed path, then

$$\int_\gamma F'(z) \, dz = 0$$

4.9 Goursat's Theorem

Let $\Omega \subseteq \mathbb{C}$ be open. Sps $\Delta \subseteq \Omega$ is a closed triangle, and $\Delta^0 \subseteq \Omega$, and let $f \in H(\Omega)$. Then

$$\int_\Delta f(z) \, dz = 0$$

4.10 Convex Set

A set $S \subseteq \mathbb{C}$ is a convex set if the line segment joining any pair of points in S lies entirely in S.

4.11 Cauchy's Theorem for Convex Set

Let $\Omega \subseteq \mathbb{C}$ be a convex open set, and $f \in H(\Omega)$. Then

1. $f = F'$ for some $F \in H(\Omega)$
2. $\int_\gamma f(z) \, dz = 0$ for any closed path $\gamma \in \Omega$

4.12 Cauchy's Integral Formula 1

Let $\Omega \subseteq \mathbb{C}$ be a convex open set, and C be a closed circle path in Ω. If $w \in \Omega \backslash \partial C$, and $f \in H(\Omega)$, then

$$f(w) \, \mathrm{Ind}_C(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - w} \, dz$$

where

$$\mathrm{Ind}_C(w) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - w}$$

4.13 Holomorphic Functions as Power Series

Let $\Omega \subseteq \mathbb{C}$ be an open set, $f \in H(\Omega)$. Then f can be expressed as a power series.

4.14 Cauchy's Integral Formula 2

Let $\Omega \subseteq \mathbb{C}$ be open, $f \in H(\Omega)$. Then

1. $\forall w \in \Omega$, f has a power series expansion at w
2. f is differentiable infinitely many times in Ω
3. $\forall C \subseteq \Omega$ that is a closed circle oriented anticlockwise, $\forall w \in C^0$,

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - w)^{n+1}} \, dz$$

4.15 Taylor Expansion of Entire Functions

If f is entire, then $\forall z_0 \in \mathbb{C}$,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z)^n$$

4.16 Analytic Functions

f is analytic in Ω if f has a power series expansion $\forall z \in \Omega$.

4.17 Analyticity & Holomorphicity

It is an iff relation

4.18 Cauchy's Inequality

$\forall z_0 \in \mathbb{C}\ \forall R > 0 \in \mathbb{R}\ \forall f \in H(C = D(z_0, R))$

$$f^{(n)}(z_0) \leq \frac{n!}{R^n} \cdot \sup_{z \in C} |f(z)|$$

4.19 Liouville's Theorem

A bounded entire function $f : \mathbb{C} \to \mathbb{C}$ is a constant.

4.20 Parseveal's Theorem

$\Omega \subseteq \mathbb{C}$ be open, $f \in H(\Omega)$, $\overline{D(z_0, R)} \subseteq \Omega \implies$
$\forall z \in \overline{D(z_0, R)},\ f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \implies$
$\forall z \in \overline{D(z_0, R)}\ f(z_0 + re^{i\theta}) = \sum_{n=0}^{\infty} c_n (re^{i\theta})^n$

4.21 Parseval's Indentity

Same setup as above,

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 = re^{i\theta}) \right|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}$$

4.22 Principle of Analytic Continuation

$\Omega \subseteq \mathbb{C}$ open & connected, $f \in H(\Omega)$. $Z(f) := \{a \in \Omega : f(a) = 0\}$. Then either $Z(f) = \Omega$ or $Z(f)$ has no limit point (i.e. points where $f = 0$ are isolated)

4.23 Maximum Modulus Principle

$\Omega \subseteq \mathbb{C} f \in H(\Omega)\, \exists r > 0\, D_{z_0} = \overline{D(z_0, r)} \subseteq \Omega \implies$
$|f(z_0)| \leq \max_{z \in \partial D_{z_0}} |f(z)|$ and
$|f(z_0)| = \max_{z \in \partial D_{z_0}} \iff f$ is constant on Ω

4.24 Fundamental Theorem of Algebra

$\forall P(z) \in \mathbb{C}[z] \deg P(z) = n \in \mathbb{N}\, \exists \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{C} \wedge \exists A \in \mathbb{C}$

$$P(z) = A(z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_n)$$

4.25 Uniqueness of a Function

$\Omega \subseteq \mathbb{C}$ open & connected, $\forall f, g \in H(\Omega)$
For any $\Omega' \subseteq \Omega$, $\forall z \in \Omega'\ f(z) = g(z) \implies$
$\forall z \in \Omega\ f(z) = g(z)$