PMATH351 - Real Analysis

Classnotes for Fall 2018

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1 Lecture 1 Sep 06th

1.1 Course Logistics

No content is covered in today's lecture so this chapter will cover some of the important logistical highlights that were mentioned in class.

- Assignments are designed to help students understand the content.
- Due to shortage of manpower, not all assignment questions will be graded; however, students are encouraged to attempt all of the questions.
- To further motivate students to work on ungraded questions, the midterm and final exam will likely recycle some of the assignment questions.
- There are no required text, but the professor has prepared course notes for reading. The course note are self-contained.
- The approach of the class will be more interactive than most math courses.
- Due to the size of the class, students are encouraged to utilize Waterloo Learn for questions, so that similar questions by multiple students can be addressed at the same time.

1.2 Preview into the Introduction

How do we compare the size of two sets?

- If the sets are finite, this is a relatively easy task.
- If the sets are infinite, we will have to rely on functions.

- Injective functions tell us that the **domain is of size that is lesser than or equal to the codomain**.
- Surjective functions tell us that the **codomain is of size that is lesser than or equal to the domain**.
- So does a bijective function tell us that the domain and codomain have the same size? Yes, although this is not as intuitive as it looks, as it relies on **Cantor-Schröder-Bernstein Theorem**.

Now, given two arbitrary sets, are we guaranteed to always be able to compare their sizes? It would be very tempting to immediately say yes, but to do that, one would have to agree on the **Axiom of Choice**. Fortunately, within the realm of this course, the Axiom of Choice is taken for granted.

2 Lecture 2 Sep 10th

2.1 Basic Set Theory

We shall use the following notations for some of the common set of numbers that we are already familiar with:

- $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \ldots\}$;
- $\mathbb{Z}$ denotes the set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$;
- $\mathbb{Q}$ denotes the set of rational numbers $\left\{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N}\right\}$; and
- $\mathbb{R}$ denotes the set of real numbers.

We shall start with having certain basic properties of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$.

We will use the notation $A \subset B$ and $A \subseteq B$ interchangably to mean that A is a subset of B with the possibility that $A = B$. When we wish to explicitly emphasize this possibility, we shall use $A \subseteq B$. When we wish to explicitly state that A is a **proper subset** of B, we will either specify that $A \neq B$ or simply $A \subsetneq B$.

Definition 1 (Universal Set)

A universal set, which we shall generally give the label X, is a set that contains all the mathematical objects that we are interested in.

This is a hand-wavy definition, but it is not in the interest of this course to further explore on this topic.

With a universal set in place, we can have the following definitions:

Definition 2 (Union)

Let X be a set. If $\{A_\alpha\}_{\alpha \in I}$ such that $A_\alpha \subset X$, then the ***union*** *for all A_α is defined as*

$$\bigcup_{\alpha \in I} A_\alpha := \{x \in X \mid \exists \alpha \in I, x \in A_\alpha\}.$$

Definition 3 (Intersection)

Let X be a set. If $\{A_\alpha\}_{\alpha \in I}$ such that $A_\alpha \subset X$, then the ***intersection*** *for all A_α is defined as*

$$\bigcap_{\alpha \in I} A_\alpha := \{x \in X \mid \forall \alpha \in I, x \in A_\alpha\}.$$

Definition 4 (Set Difference)

Let X be a set and $A, B \subseteq X$. The ***set difference*** *of A from B is defined as*

$$A \setminus B := \{x \in X \mid x \in A, x \notin B\}.$$

On a similar notion:

Definition 5 (Symmetric Difference)

Let X be a set and $A, B \subseteq X$. The ***symmetric difference*** *of A and B is defined as*

$$A \Delta B := \{x \in X \mid (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)\}.$$

In words, for an element in the symmetric difference of two sets, the element is either in A or B but not both. We can also think of the symmetric difference as

$$(A \cup B) \setminus (A \cap B)$$

or

$$(A \setminus B) \cup (B \setminus A).$$

We can also talk about the non-members of a set:

Definition 6 (Set Complement)

Let X be a set and $A \subset X$. The set of all non-members of A is called the ***complement*** *of A, which we denote as*

$$A^c := \{x \in X \mid x \notin A\}.$$

Note

Note that

$$(A^c)^c = \{x \in X \mid x \notin A^c\} = \{x \in X \mid x \in A\} = A.$$

Now taking a step away from that, we define the following:

Definition 7 (Empty Set)

*An **empty set**, denoted by $\emptyset$, is a set that contains nothing.*

Note

The empty set is set to be a subset of all sets.

Definition 8 (Power Set)

Let X be a set. The power set of X is the set that contains all subsets of X, i.e.

$$\mathcal{P}(X) := \{A \mid A \subset X\}.$$

Note

A power set is always non-empty, since $\emptyset \in \mathcal{P}(\emptyset)$, and since $\emptyset \subset X$ for any set X, we have $\emptyset \in \mathcal{P}(X)$.

Example 2.1.1

Let $X = \{1, 2, \ldots, n\}$. There are several ways we can show that the size of $\mathcal{P}(X)$ is 2^n. One of the methods is by using a characteristic function that *maps from A to $\{0,1\}$, defined by*

$$\begin{aligned} \mathcal{X}_A &: A \to \{0,1\} \\ \mathcal{X}_A(x) &= \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}. \end{aligned}$$

Using this function, each element in X have 2 states: one being in the subset, and the other being not in the subset, which are represented by 1 and 0 respectively. It is then clear that there are 2^n of such configurations.

Theorem 1 (De Morgan's Laws)

Let X be a set. Given $\{A_\alpha\}_{\alpha \in I} \subset \mathcal{P}(X)$, we have

1. $\left(\bigcup_{\alpha \in I} A_\alpha \right)^c = \bigcap_{\alpha \in I} A_\alpha^c$; *and*
2. $\left(\bigcap_{\alpha \in I} A_\alpha \right)^c = \bigcup_{\alpha \in I} A_\alpha^c$.

Proof

1. *Note that*

$$\begin{aligned} x \in \left(\bigcup_{\alpha \in I} A_\alpha \right)^c &\iff \nexists \alpha \in I \;\; x \in A_\alpha \\ &\iff \forall \alpha \in I \;\; x \notin A_\alpha \\ &\iff \forall \alpha \in I \;\; x \in A_\alpha^c \text{ by set complementation} \\ &\iff x \in \bigcap_{\alpha \in I} A_\alpha^c. \end{aligned}$$

2. *Observe that, by part 1,*

$$\left(\bigcap_{\alpha \in I} A_\alpha \right)^c = \left(\left(\bigcup_{\alpha \in I} A_\alpha^c \right)^c \right)^c = \bigcup_{\alpha \in I} A_\alpha^c.$$

$\square$

Example 2.1.2

Suppose $I = \emptyset$. Then what is $\bigcup_{\alpha \in \emptyset} A_\alpha$? It is sensible to think that all we are left with is simply a union of empty sets, and so

$$\bigcup_{\alpha \in \emptyset} A_\alpha = \emptyset. \tag{2.1}$$

And what about $\bigcap_{\alpha \in \emptyset} A_\alpha$? By Theorem 1, it is quite clear from Equation (2.1) that

$$\bigcap_{\alpha \in \emptyset} A_\alpha = X.$$

2.2 Products of Sets

Definition 9 (Product of Sets)

*Given 2 sets X and Y, the **product** of X and Y is given by*

$$X \times Y := \{(x, y) \mid x \in X, y \in Y\}.$$

*We often refer to elements of $X \times Y$ as **tuples**.*

Note

Now if

$$X = \{x_1, x_2, ..., x_n\},$$
$$Y = \{y_1, y_2, ..., y_m\},$$

then

$$X \times Y = \{(x_i, y_j) \mid i = 1, 2, ..., n, \; j = 1, 2, ..., m\}$$

and so the size of $X \times Y$ is mn.

Consequently, we can think of tuples as two elements being in some "relation".

Definition 10 (Relation)

*A **relation** on sets X and Y is a subset R of the product $X \times Y$. We write*

$$xRy \quad \text{if } (x, y) \in R \subset X \times Y.$$

We call

- $\{x \in X \mid \exists y \in Y, (x, y) \in R\}$ *as the* ***domain*** *of* R*; and*
- $\{y \in Y \mid \exists x \in X, (x, y) \in R\}$ *as the* ***range*** *of* R.

In relation to that, functions are, essentially, relations.

Definition 11 (Function)

A ***function*** *from* X *to* Y *is a relation* R *such that*

$$\forall x \in X\ \exists! y \in Y\ (x, y) \in R.$$

Suppose $X_1, X_2, ..., X_n$ are non-empty[1] sets. We can define

$$X_1 \times X_2 \times \ldots \times X_n = \prod_{i=1}^{n} X_i := \{(x_1, x_2, ..., x_n) \mid x_i \in X_i\}.$$

[1] We are typically only interested in non-empty sets, since empty sets usually lead us to vacuous truths, which are not interesting.

Now if $X_i = X_j = X$ for all $i, j = 1, 2, ..., n$, we write

$$\prod_{i=1}^{n} X_i = \prod_{i=1}^{n} X = X^n.$$

And now comes the problem: given a collection $\{X_\alpha\}_{\alpha \in I}$ of non-empty sets[2], what do we mean by

$$\prod_{\alpha \in I} X_\alpha ?$$

[2] i.e. we now talk about arbitrary $\alpha \in I$.

To motivate for what comes next, consider

$$\prod_{i=1}^{n} X_i = X_1 \times \ldots \times X_n = \{(x_1, ..., x_n) \mid x_i \in X_i\}.$$

Choose $(x_1, ..., x_n) \in \prod_{i=1}^{n} X_i$. This induces a function

$$f_{(x_1, ..., x_n)} : \{1, ..., n\} \to \bigcup_{i=1}^{n} X_i$$

with

$$\begin{aligned} f(1) &= x_1 \in X_1 \\ f(2) &= x_2 \in X_2 \\ &\vdots \\ f(n) &= x_n \in X_n \end{aligned}$$

Now assume for a more general f such that

$$f : \{1, ..., n\} \to \bigcup_{i=1}^{n} X_i$$

is defined by

$$f(i) \in X_i.$$

Then, we have

$$(f(1), f(2), ..., f(n)) \in \prod_{i=1}^{n} X_i,$$

which leads us to the following notion:

Definition 12 (Choice Function)

Given a collection $\{X_\alpha\}_{\alpha \in I}$ of non-empty sets, let

$$\prod_{\alpha \in I} X_\alpha = \left\{ f : I \to \bigcup_{\alpha \in I} X_\alpha \right\}$$

*such that $f(\alpha) \in X_\alpha$. Such an f is called a **choice function**.*

And so we may ask a similar question as before: if each X_α is non-empty, is $\prod_{\alpha \in I} X_\alpha$ non-empty? Turns out this is not as easy to show. In fact, it is essentially impossible to show, because this is exactly the **Axiom of Choice**.

3 Lecture 3 Sep 12th

3.1 *Axiom of Choice*

RECALL our final question of last lecture: If $\{X_\alpha\}_{\alpha \in I}$ is a non-empty collection of non-empty sets, is

$$\prod_{\alpha \in I} X_\alpha \neq \emptyset \ ?$$

Turns out this is widely known (in the world of mathematics) as the **Axiom of Choice**.

Axiom 2 (Zermelo's Axiom of Choice)

If $\{X_\alpha\}_{\alpha \in I}$ is a non-empty collection of non-empty sets, then

$$\prod_{\alpha \in I} X_\alpha \neq \emptyset.$$

An equivalent statement of the above axiom is:

Axiom 3 (Zermelo's Axiom of Choice v2)

$X \neq \emptyset \implies$

$$\exists f : \mathcal{P}(X) \setminus \{\emptyset\} \to X \ \forall A \in \mathcal{P}(X) \setminus \{\emptyset\} \ f(A) \in A$$

where f is the choice function.

Exercise 3.1.1

Prove that Axiom 2 and Axiom 3 are equivalent.

Proof

From Axiom 2 to Axiom 3:

Since $X \neq \emptyset$, we have that $\mathcal{P}(X) \setminus \{\emptyset\}$ is a non-empty collection of non-empty sets. Therefore,

$$\prod_{A \in \mathcal{P}(X) \setminus \{\emptyset\}} A \neq \emptyset.$$

So we know that

$$\exists (x_A)_{A \in \mathcal{P}(X) \setminus \{\emptyset\}} \in \prod_{A \in \mathcal{P}(X) \setminus \{\emptyset\}} A.$$

We then simply need to choose the choice function $f : \mathcal{P}(X) \setminus \{\emptyset\} \to X$ such that

$$f(A) = x_A \in A.$$

From Axiom 3 to Axiom 2:

Let $X_\alpha \in \mathcal{P}(X)$ for $\alpha \in I$, where I is some index set. We know that not all $X_\alpha = \emptyset$ since $X \neq \emptyset$. Choose $J \subseteq I$ such that $\{X_\alpha\}_{\alpha \in J}$ is a non-empty collection of non-empty sets. Let $f : \mathcal{P}(X) \setminus \{\emptyset\}$ be any choice function. By Axiom 3,

$$\forall X_\alpha \in \mathcal{P}(X) \setminus \{\emptyset\} \quad f(X_\alpha) \in X_\alpha.$$

Therefore,

$$(f(X_\alpha))_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha.$$

$\square$

3.2 *Relations*

Now, it is in our interest to start talking about comparisons or relations between the mathematical objects that we have defined.

Definition 13 (Relations)

A relation R on a set X is [1]

- *(**Reflexive**)* $\forall x \in X \;\; xRx$;
- *(**Symmetric**)* $\forall x, y \in X \;\; xRy \iff yRx$;
- *(**Anti-symmetric**)* $\forall x, y \in X \;\; xRy \land yRx \implies x = y$;
- *(**Transitive**)* $\forall x, y, z \in X \;\; xRy \land yRz \implies xRz$.

[1] We can look at this definition as $R \subseteq X \times X$. Under such a definition, we would have

- **(Reflexive)** $\forall x \in X \;\; (x, x) \in R$;
- **(Symmetric)** $\forall x, y \in X \;\; (x, y) \in R \iff (y, x) \in R$;
- **(Anti-symmetric)** $\forall x, y \in X \;\; (x, y), (y, x) \in R \implies x = y$;
- **(Transitive)** $\forall x, y, z \in X \;\; (x, y), (y, z) \in R \implies (x, z) \in R$.

Example 3.2.1

Let $X = \mathbb{R}$, and let $xRy \iff x \leq y$, where $\leq$ is the notion of "less than or equal to", which we shall assume that it has the meaning that we know. Observe that $\leq$ is:

- *reflexive:* $\forall x \in \mathbb{R} \;\; x \leq x$ *is true;*
- *anti-symmetric:* $\forall x, y \in \mathbb{R} \;\; x \leq y \land y \leq x \implies x = y$*; and*
- *transitive:* $\forall x, y, z \in \mathbb{R} \;\; x \leq y \land y \leq z \implies x \leq z$.

Example 3.2.2

Let $Y \neq \emptyset$, $X = \mathcal{P}(Y)$, with $ARB \iff A \subseteq B$. Observe that $\subseteq$ is:

- *reflexive:* $\forall A \in \mathcal{P}(Y) \;\; ARA \iff A \subseteq A$ *is true;*
- *anti-symmetric:* $\forall A, B \in \mathcal{P}(Y) \;\; ARB \land BRA \iff A \subseteq B \land B \subseteq A \implies A = B$;
- *transitive:* $\forall A, B, C \in \mathcal{P}(Y) \;\; ARB \land BRC \iff A \subseteq B \land B \subseteq C \implies A \subseteq C$.

Example 3.2.3

Let $Y \neq \emptyset$, $X = \mathcal{P}(Y)$, with $ARB \iff A \supseteq B$. Observe that $\supseteq$ is:

- *reflexive:* $\forall A \in \mathcal{P}(Y) \;\; ARA \iff A \subseteq A$;
- *anti-symmetric:* $\forall A, B \in \mathcal{P}(Y) \;\; ARB \land BRA \iff A \supseteq B \land B \supseteq A \implies A = B$;
- *transitive:* $\forall A, B, C \in \mathcal{P}(Y) \;\; ARB \land BRC \iff A \supseteq B \land B \supseteq C \implies A \supseteq C$.

All the above examples are also known as *partially ordered sets*.

Definition 14 (Partially Ordered Sets)

The set X with the relation R on X is called a ***partially ordered set*** *(or a* ***poset****) if R is*

- *reflexive;*
- *anti-symmetric; and*
- *transitive.*

We denote a poset by (X, R).

The "partial" in 'partially ordered" indicates that not every pair of elements need to be comparable, i.e. there may be pairs for which neither precedes the other (anti-symmetry).

Note

If (X, R) is a poset, then if $A \subseteq X$, and $R_1 = R \restriction_{A \times A}$, then (A, R_1) is also a poset.

Example 3.2.4

How many possible relations can we define on these sets to make them into posets?

1. $X = \emptyset$

   Solution
   We have that $R = \emptyset \times \emptyset$, and so the only relation we have is an empty relation. Then it is vacuously true that (X, R) a poset.

2. $X = \{x\}$

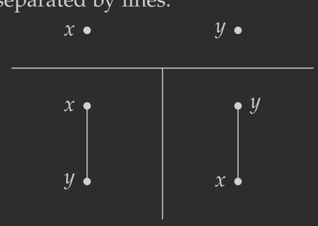
   Solution
   We have that $R = X \times X = \{(x, x)\}$. It it clear that (X, R) is a poset.

3. $X = \{x, y\}$

   Solution
   There are 3 possible relations:

   - *a relation where xRx and yRy;*
   - *a relation where xRy; or*
   - *a relation where yRx.*

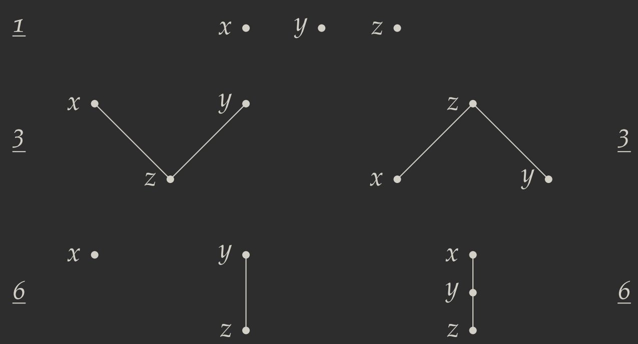
3 possibilities illustrated as graphs, separated by lines:



4. $X = \{x, y, z\}$

Solution

The following are all the possibilities represented by graphs, where the underlined numbers represent the number of ways we can rearrange the elements for unique relations:



Therefore, we see that there are a total of

$$1 + 3 + 3 + 6 + 6 = 19 \text{ relations.}$$

5. *X with 6 elements.*

Exercise 3.2.1

How many possible relations can we define on a set of 6 elements to the set into a poset?

Solution

to be added

Definition 15 (Totally Ordered Sets / Chains)

The set X with the relation R on X is called a ***totally ordered set*** *(or a* ***chain****) if (X, R) is a poset with the exception that, for any $x, y \in X$, either xRy or yRx but not both.*

Definition 16 (Bounds)

Let $(X, \leq)$ be a poset. Let $A \subset X$. We say $x_0 \in X$ is an ***upper bound*** *for A if*

$$\forall a \in A \quad a \leq x_0.$$

If A has an upper bound, we say that A is ***bounded above****. If A is* *bounded above, then x_0 is the* ***least upper bound*** *(or* ***supremum****) of A is for any $x_1 \in X$ that is an upper bound of A, we have*

$$x_0 \leq x_1.$$

We write $x_0 = \mathrm{lub}(A) = \sup(A)$. If $\sup(A) \in A$, then $\sup(A) = \max(A)$ is the ***maximum*** *of A.*

We can analogously define for:

$$\begin{aligned} \textit{upper bound } &\to \textit{ lower bound} \\ \textit{bounded above } &\to \textit{ bounded below} \\ \textit{least upper bound, } \mathrm{lub} &\to \textit{ greatest lower bound, } \mathrm{glb} \\ \textit{supremum, } \sup &\to \textit{ infimum, } \inf \\ \textit{maximum, } \max &\to \textit{ minimum, } \min \end{aligned}$$

Note

By ***anti-symmetry*** *of posets, we have that $\max, \sup, \min, \inf$ are all unique if they exists.*

Example 3.2.5 (Least Upper Bound Property of $\mathbb{R}$)

Let $X = \mathbb{R}$, and $\leq$ be the order that we have defined. Every bounded non-empty subset of X has a supremum.

Example 3.2.6

Let $Y \neq \emptyset$, and $X = \mathcal{P}(Y)$, and $\subseteq$ the ordering by inclusion. We know that Y is the maximum element of $(X, \subseteq)$. Then the collection $\{A_\alpha\}_{\alpha \in I} \subset \mathcal{P}(Y)$ is bounded above by Y, and we have that

$$\sup\left(\{A_\alpha\}_{\alpha \in I}\right) = \bigcup_{\alpha \in I} A_\alpha$$
$$\inf\left(\{A_\alpha\}_{\alpha \in I}\right) = \bigcap_{\alpha \in I} A_\alpha$$

Now if $Y = \emptyset$, we would end up having

$$\sup\left(\{A_\alpha\}_{\alpha \in I}\right) = \emptyset$$
$$\inf\left(\{A_\alpha\}_{\alpha \in I}\right) = X$$

But this makes sense, since the empty set would be the least of upper bounds, and since $X = \mathcal{P}(Y)$ would have to be the greatest of lower bounds.

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