PMATH433/733 - Model Theory and Set Theory

Classnotes for Fall 2018

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Foreword

This course has a ratio of about 1:3 for naive set theory to model theory.

1 Lecture 1 Sep 06th

1.1 Introduction to Set Theory

In this course, we shall focus only on **practical set theory**, which is more commonly knowned as **naive set theory**. In practical set theory, we look at set theory as a language of mathematics. Some of the examples of which we look into in this flavour of set theory are (transfinite) induction and recursion, and the measuring of the sizes of sets.

Another approach to set theory, one that is deemed required in order to learn set theory is a more formal way, is to look at set theory as the foundations of mathematics. Such an approach is more axiomatic, rigorous, and grounding as compared to practical set theory. This course will try to work around going into these topics, as they can take a life of their own, and within the context of this course, the topics that will be explored using this approach are not required.

1.2 Ordinals

1.2.1 Zermelo-Fraenkel Axioms

We use the natural numbers, i.e.

$$0, 1, 2, 3, 4, \ldots$$

to **count** finite sets. There are two related meanings attached to the word “count” here:

- enumeration; and
- measuring (of sizes)

In order to facilitate the introduction to certain axioms that we shall need, let our current goal be to develop an infinitary generalization of the natural numbers, so as to be able to enumerate and measure arbitrary sets.

We have that

$$\begin{aligned} \text{enumeration} &\to \text{ordinals} \\ \text{measuring} &\to \text{cardinals} \end{aligned}$$

where $\to$ represents "leads to" here.

To CONSTRUCT the natural numbers, we require 3 basic notions that shall remain undefined but understood:

- a set;
- membership, denoted by $\in$; and
- equality.

One such construction is

$$\begin{aligned} &0 := \emptyset, \text{ the empty set} \\ &1 := \{0\} = \{\emptyset\}, \text{ the set whose only member is } 0 \\ &2 := \{0, 1\} = \{\emptyset, \{\emptyset\}\}, \text{ the set whose only members are } 0 \text{ and } 1. \end{aligned}$$

Definition 1 (Successor)

*Given a natural number n, the **successor** of n is the natural number next to n, which can be obtained by*

$$S(n) := n \cup \{n\}.$$

We can use the definition of a successor to construct the rest of the natural numbers.

Example 1.2.1

Just to verify to ourselves that the definition indeed works, observe that

$$S(1) = 2 = \{\emptyset, \{\emptyset\}\} = \{\emptyset\} \cup \{\{\emptyset\}\}.$$

So to construct the natural number 3, we see that

$$\begin{aligned} S(2) = 3 = \{0, 1, 2\} = 2 \cup \{2\} &= \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} \\ &= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \end{aligned}$$

Looking at these, we start wondering to ourselves: how do we know that $\emptyset$ exists in the first place? How do we know that we can use $\cup$ and what does it even mean? Now it is meaningless if we cannot take that $\emptyset$ always exists, nor is it meaningful if we cannot take the $\cup$ of sets. And so, to allow us to continue, or even start with these notions, we require axioms.

Axiom 1 (Empty Set Axiom)

There exists a set, denoted by $\emptyset$, with no members.

With this axiom, we can indeed construct 0. To get 1 from 0, we have that 1 is a set whose only member is zero, and so if we take a member from 1, that member must be 0.

Axiom 2 (Pairset Axiom)

Given set x, y, there exists a set, denoted by $\{x, y\}$, whose only members are x and y. In other words,

$$t \in \{x, y\} \leftrightarrow (t = x \lor t = y)$$

Now note that in Axiom 2, if $x = y$, then the set $\{x, y\}$ has only x as its member. For example, we realize that $1 = \{0, 0\} = \{0\}$. But why exactly does this equality make sense? What exactly does "realize" mean?

Axiom 3 (Axiom of Extension)

Given sets x, y, $x = y$ if and only if x and y have the same members.

Now, using the above 3 axioms, we are guaranteed that

$$\begin{aligned}
0 &= \emptyset \text{ exists by the Empty Set Axiom} \\
1 &= \{\emptyset\} \text{ exists by the Pairset Axiom} \\
2 &= \{\emptyset, \{\emptyset\}\} \text{ exists by the Pairset Axiom}
\end{aligned}$$

Now we've constructed 3 to be the set whose only members are $0, 1$, and 2. So far, within our axioms, there is no such thing as $\{0, 1, 2\}$, which is what our 3 is supposed to be. We now require the following axiom:

Axiom 4 (Union Set Axiom)

Given a set x, there exists a set denoted by $\cup x$, whose members are precisely the members of the members of x, i.e.

$$t \in \cup x \leftrightarrow (t \in y \text{ for some } y \in x)$$

So, by this axiom, we have that given any n, $S(n) = \cup\{n, \{n\}\}$, or in other words,

$$t \in S(n) \leftrightarrow t \in n \vee t = n.$$

With all of the above axioms, we can iteratively construct each and every natural number in a rigorous manner. However, our goal is to construct infinitely many of them.

It is tempting to simply take the infinitude of natural numbers simply as an axiom, i.e.

> *There exists a set whose members are precisely the natural numbers.*

There is a certain rule to which we set down axioms, and that is, axioms must be expressable in a "finitary" manner, i.e. they must be expressible using first-order logic.

Definition 2 (Definite Condition)

*We define a **definite condition** as follows:*

- *$x \in y$ and $x = y$ are definite conditions, where x and y are both indeterminants, standing for sets, or are sets themselves;*
- *if P and Q are definite conditions, then so are*
  - *not P, denoted as $\neg P$;*
  - *P and Q, denoted as $P \wedge Q$;*

- *P or Q, denoted as $P \vee Q$;*
- *for all x, P, denoted as $\forall x P$; and*
- *there exists x, P, denoted as $\exists x P$.*

Example 1.2.2

$$x \in 1, 0 \in 2, 2 \in 0$$

are all definite conditions. Note, however, that $2 \in 0$ is false.

Note

*"If P then Q", which is also written as $P \to Q$, is also a definite condition since it is "**equivalent**"[1] to the statement $\neg P \vee Q$.*

Consequently, $P if and only if Q$, which is also expressed as $P \leftrightarrow Q$, can be written as

$$(\neg P \vee Q) \wedge (\neg Q \vee P)$$

[1] We have yet to define what equivalent statements are but we shall take this for granted for now.

Now, with this definition, and first-order logic notations in mind, we can write:

- Empty Set Axiom: $\exists x \, \forall t \, \neg(t \in x)$
- Pairset Axiom: $\forall x \, \forall y \, \exists p \, \forall t \, (t \in p \leftrightarrow ((t = x) \vee (t = y)))$
- Union Set Axiom: $\forall x \, \exists z \, \forall t \, ((t \in z) \leftrightarrow (\exists y \, ((y \in x) \wedge (t \in y))))$

Exercise 1.2.1
Write Axiom 3 in first-order logic notation.

Solution

$$\forall x \, \forall y$$
$$(x = y) \leftrightarrow (\forall t \, ((t \in x) \leftrightarrow (t \in y)))$$

Note that the statement that we proposed as an axiom for the set of natural numbers in page 14 is not definite, although that itself is not obvious.

For example, we may try to write

$$\exists x \, (\forall t \, ((t \in x) \leftrightarrow ((t = 0) \vee (t = 1) \vee (t = 2) \vee \ldots)))$$

and then notice that we do not have the notion of "..." within the "tools" that we are allowed to use.

2 Lecture 2 Sep 11th

2.1 Ordinals (Continued)

2.1.1 Zermelo-Fraenkel Axioms (Continued)

We stopped at the discussion about allowing for an infinite set, so that we can construct our set of infinite natural numbers. The idea here is to take *the smallest set that contains* 0 *and is preserved by the successor function*[1]

[1] Q: Why the smallest set?

Axiom 5 (Infinity Axiom)

There exists a set I *that contains* 0 *and is preserved by the successor function. We may express this as*

$$\exists I((0 \in I) \wedge \forall x(x \in I \to S(x) \in I))$$

where we have defined that $S(x) \in I$ *means*

$$\exists y(\forall t(t \in y \leftrightarrow (t \in x) \vee (t = x)) \wedge (y \in I))$$

We call I *the* ***successor set****.*

Since we want the smallest of such successor sets, we can try taking the intersection of all successor sets. But before we can do that, we require more axiomatic statements.

Definition 3 (Subsets)

$x \subseteq y$ means that every element of x is an element of y, i.e.

$$\forall t((t \in x) \to (t \in y))$$

With a definition of a subset, we can define the Powerset Axiom.

Axiom 6 (Powerset Axiom)

Given a set x, there exists a set $\mathcal{P}(x)$ that contains all subsets of x, i.e.

$$\forall t((t \subseteq \mathcal{P}(x)) \leftrightarrow (t \subseteq x))$$

We also require the following axiom.

Axiom 7 ((Bounded) Separation Axiom)

Given a set x and a definition condition P, there exists a set whose elements are precisely the members of x that satisfies P, i.e.

$$\forall x \, \exists y \, \forall t \, ((t \in y) \leftrightarrow \forall y \, ((t \in x) \land P(t)))$$

where

$$y = \{z \in x \mid P(z)\}.$$

There are two important aspects to the Bounded Separation Axiom:

- it is bounded by the set x; and
- P is a definite condition.

Exercise 2.1.1 (Set Intersection)

Prove that given a non-empty set x, there exists a set $\cap x$ satisfying

$$\forall t \, ((t \in \cap x) \leftrightarrow \forall y \, ((y \in x) \to (t \in y)))$$

Proof

to be solved

Definition 4 (Natural Numbers)

Let I be a successor set. The set of natural numbers is[2]

$$\omega := \cap\{J \subseteq I : J \text{ is a successor set }\}$$

[2] We can also write $J \subseteq I$ as $J \in \mathcal{P}(I)$ and invoke the Bounded Separation Axiom.

Note

J being a successor set can be expressed by the definite condition

$$(0 \in J) \wedge \forall x\, (x \in J \to S(x) \in J),$$

so we can write the definite condition in the above definition by

$$\omega := \cap\{J \subseteq I : (0 \in J) \vee \forall x\, (x \in J \to S(x) \in J)\}$$

Exercise 2.1.2

Show that the definition of ω does not actually depend on I, i.e. if given I_1 and I_2 such that we have

$$\omega_1 = \cap\{J \subseteq I_1 : J \text{ is a successor set }\}$$
$$\omega_2 = \cap\{J \subseteq I_2 : J \text{ is a successor set }\}$$

we have

$$\omega_1 = \omega_2.$$

Proof

to be solved

Another useful axiom that we will use later is the following:

Axiom 8 (Replacement Axiom)

Suppose P is a binary definite condition[3] *such that for every set x, there is a unique y satisfying $P(x, y)$. Given a set A, there is a set B such that $t \in B$ if and only if there is an $a \in A$ with $P(a, t)$.*

[3] A **binary definite condition** has only two variables.

Note

The slogan for the Replacement Axiom is:

> *The image of a set under a definite operation exists.*

These eight axioms, along with another ninth axiom called the Regularity Axiom[4], constitutes the **Zermelo-Fraenkel Set Theory**.

Note that all axioms, save the Extensionality, assert the existence of sets.

[4] We shall not discuss too much about this. According to the lecture and the lecture notes, the Regularity Axiom states that every set has a minimal element. On Wikipedia, the axiom states that every set has an element that does not intersect with the set itself.

2.1.2 Classes

There are times where we are interested in a collection of sets that do not form a set themselves.

Example 2.1.1 (Russell's Paradox)

There is no set containing all sets.

Proof

Suppose such a set exists, and call it U. Now consider the set

$$R := \{x \in U : x \notin x\},$$

which exists by Bounded Separation. Observe that

$$\begin{aligned} R \in R &\implies R \notin R \quad \lightning \\ \implies R \notin R &\implies R \in R \quad \lightning \end{aligned}$$

Thus such a set U cannot exist.

To talk about such collections, that may or may not be sets, we define *classes*.

Definition 5 (Class)

A class is any collection of sets defined by definite property, i.e. given any difinite condition P,

$$[\![\ z \mid P(z) \ ]\!]$$

is the class of all sets satisfying P.

Here, instead of Bounded Separation, we have what is called **unbounded separation**.

Note

We shall use $[\![\ \ ]\!]$ rather than $\{\ \ \}$ to emphasize that we are talking about classes, i.e. we may be talking about non-sets.

Example 2.1.2

$$\mathrm{Set} := [\![\ z \mid z = z\ ]\!]$$

is the universal class of all sets.

Note

- *Every set is a class.*

Proof

Suppose x is a set. We may write

$$x = [\![\ z \mid z \in x\ ]\!].$$

$\square$

- *Some classes are not sets; these are called **proper classes**. E.g. the universal class of all sets, and*

$$\mathit{Russell} := [\![\ z \mid z \notin z\ ]\!].$$

3 Lecture 3 Sep 13th

3.1 Ordinals (Continued 2)

3.1.1 Cartesian Products and Function

Definition 6 (Ordered Pairs)

*Given sets x, y, an **ordered pair** of x and y is defined as*[1]

$$(x, y) = \{\{x\}, \{x, y\}\}$$

[1] This invokes the Pairset Axiom thrice. Why did we not define an ordered pair as

$$(x, y) = \{\{x\}, \{y\}\}$$

instead?

Note

Note that we must have

$$((x, y) = (x', y')) \iff (x = x' \land y = y').$$

Proof

The ($\Longleftarrow$) direction is clear by Extensionality. For the other direction, we shall break it into 2 cases:

***Case 1**: $x = y$. Then $\{x, y\} = \{x\}$ by Extensionality, and so*

$$(x, y) = \{\{x\}\}$$

Therefore, we have that

$$\{\{x\}\} = (x, y) = (x', y') = \{\{x'\}, \{x', y'\}\}$$

So we have

$$\{x\} = \{x'\} \implies x = x'$$

and

$$\{x\} = \{x', y'\} \implies y' = x = y.$$

Thus we have

$$x = x' \wedge y = y'.$$

Case 2*: Suppose $x \neq y$ and $x' \neq y'$ [2] We have*

[2] If any of them are equal, Case 1 would apply.

$$\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$$

Then

$$\{x\} = \{x'\} \vee \{x\} = \{x', y'\}$$

The latter leads to a contradiction, since it would imply

$$x' = x = y'.$$

Thus $x = x'$. Also, we have

$$\{x, y\} = \{x'\} \vee \{x, y\} = \{x', y'\}$$

Now the former leads to a contradiction since it would imply that

$$x = x' = y.$$

Now since $x = x'$, it must be that $y = y'$, otherwise $y = x' = x$ would contradict our assumption. Therefore, we have that

$$x = x' \wedge y = y'.$$

□

With ordered pairs, we can build Cartesian products:

Definition 7 (Cartesian Product)

Given classes X and Y, the ***Cartesian Product*** *of X and Y is defined as*

$$X \times Y := [\![\; z : z = (x, y), x \in X, y \in Y \;]\!]$$

Note

We can express this definition using definite conditions;

$$\forall x, y\Big((x \in X) \wedge (y \in Y) \wedge \Big(\exists a, b(\forall t(t \in a \leftrightarrow t = x)) \wedge \forall t(t \in b \leftrightarrow (t = x) \vee (t = y))\Big) \wedge \forall t\Big(t \in z \leftrightarrow ((t = a) \vee (t = b))\Big)\Big)$$

Note

- *A Cartesian product is a class.*
- *If A is a set and B is a class, and $B \subseteq A$, then B is also a set. This is easy to show: observe that by Extentionality,*

  $$B = \{a \in A \mid a \in B\}.$$

  By Bounded Separation Axiom, B is a set[3]*.*

[3] This statement can be rephrased as: subclasses of a set are subsets.

Consequently, Cartesian products of sets are sets themselves; if X and Y are sets, we want to show that $X \times Y$ is a set so it is sufficient to show that it is contained in one. Recall that

$$(x, y) = \{\{x\}, \{x, y\}\}$$

and $\{x, y\} \subset X \cup Y$ which means $\{x, y\} \in \mathcal{P}(X)$, and we observe that $\{x\} \in \mathcal{P}(X \cup Y)$. So $(x, y) \in \mathcal{P}(X \cup Y)$. Therefore, $X \times Y \subset \mathcal{P}(\mathcal{P}(X \cup Y))$, and we show to ourselves that $X \times Y$ is indeed a set.

Definition 8 (Definite Operation)

Given classes X and Y, a ***definite operation*** *$f : X \to Y$ is a subclass $\Gamma(f) \subseteq X \times Y$ such that*

$$\forall x \in X\ \exists! y \in Y\ (x, y) \in \Gamma(f).$$

Note

We write $f(x) = y$ to mean $(x, y) \in \Gamma(f)$. We also refer to $\Gamma(f)$ as the ***graph of f****.*

Example 3.1.1

The successor function $S : \mathsf{Set} \to \mathsf{Set}$ is a definite operation such that

$$S(x) = x \cup \{x\}$$

This is true since is can be expressed as

$$\forall t(t \in y \leftrightarrow (t \in x \vee t = x)).$$

To show that S is a definite operation, we need to show that S is a definite condition.

Note

If X and Y are sets and f is a definite operation, then $\Gamma(f) \subseteq X \times Y$ is a set. In such a case, we call f a function.

Definition 9 (Functions)

A function is a definite operation $f : X \to Y$ where X and Y are both sets.

We can now restate the Replacement Axiom.

Axiom 9 (Replacement Axiom (Restated))

If $f : X \to Y$ is a definite operation, and $A \subseteq X$ is a set, then $\exists B \subseteq Y$ that is a set such that $t \in B$ if and only if $t = f(a)$ for some $a \in A$.

3.1.2 The Natural Numbers

Theorem 10 (Induction Principle)

Suppose $J \subseteq \omega$, $0 \in J$ and whenever $n \in J$, $S(n) \in J$. Then $J = \omega$.

Proof

By assumption, J is a successor set, therefore $\omega \subseteq J$ by definition. Thus, sinec $J \subseteq \omega$, we have $J = \omega$. $\square$

Lemma 11 (Properties of the Natural Numbers)

Suppose $n \in \omega$. We have

1. $n \subseteq \omega$;
2. $\forall m \in n \quad m \subseteq n$;
3. $n \notin n$;
4. $n = 0 \veebar 0 \in n$; *and*
5. $y \in n \implies S(y) \in n \veebar S(y) = n$.

Proof

1. *Let*[4]
$$J := \{n \in \omega : n \subseteq \omega\} \subseteq \omega.$$
Note that $\emptyset \subseteq \omega$ and so $0 \subseteq \omega$. By membership, $0 \in J$.

Suppose $m \in J$. Consider $S(m) = m \cup \{m\}$. Since $J \subseteq \omega$, $m \in \omega$. Since $m \in \omega$, $\{m\} \subseteq \omega$. Therefore $S(m) = m \cup \{m\} \subseteq \omega$, and so $S(m) \in J$. So J is a successor set. And thus by Induction Principle, $J = \omega$.

2. *Let*
$$J := \{n \in \omega : \forall m \in n, m \subseteq n\}.$$
It is vacuously true that $0 \in J$ since $\emptyset$ is a subset of every $n \in J$. Suppose $n \in J$. Then $\forall m \in n$, we have $m \subseteq n$. Consider $S(n) = n \cup \{n\}$. Note that $n \in S(n)$ and $n \subseteq S(n)$. For $x \in S(n)$ such that $x \neq n$, we must have that $x \in n$. By assumption, $x \subseteq n \subseteq S(n)$. Therefore, $S(n) \in J$, and so J is a successor set. By the Induction Principle, $J = \omega$.

[4] We construct this J and show that it is a successor set. Note that if $J = \omega$, our proof is complete.

3. *Let*
$$J := \{n \in \omega : n \notin n\}.$$
We have $0 = \emptyset \notin \emptyset$. *So* $0 \in J$.

Let $n \in J$. *Consider* $S(n) = n \cup \{n\}$. *In particular, note that* $n \in S(n)$. *Suppose, for contradiction, that* $S(n) \in S(n)$. *Then* $S(n) = n$ *or* $S(n) \in n$.

$S(n) = n \implies n \in S(n) = n \lightning n \notin n$.

$S(n) \in n \implies S(n) \subseteq n$ *by part 2* $\implies n \in n \lightning n \notin n$.

Thus $S(n) \notin S(n)$ *and so* $S(n) \in J$. *So* J *is a successor set, and so by the Induction Principle,* $J = \omega$.

4. *It suffices to show that*
$$\omega = \{0\} \cup \{n \in \omega : 0 \in n\}.$$
Let $J =$ *RHS. We have that* $0 \in J$. *Suppose* $n \in J$ *such that* $n \neq 0$. *Then* $0 \in n$. *Since* $n \subseteq S(n) = n \cup \{n\}$, *we have that* $0 \in S(n)$. *Therefore,* $S(n) \in J$. *So* J *is a successor set, and so by the Induction Principle,* $J = \omega$ *as required.*

5. *Let*
$$J := \{n \in \omega : y \in n \implies S(y) \in n \veebar S(y) = n\}.$$
$0 \in J$ *is vacuously true, since there are no* $y \in 0$. *Suppose* $n \in J$. *Let* $y \in S(n) = n \cup \{n\}$. *We have two choices: either* $y \in n$ *or* $y = n$. *If* $y \in n$, *then* $S(y) \in n \veebar S(y) = n$, *since* $n \in J$. *We have that*

$S(y) \in n \subseteq S(n)$ *in which case we are done; and*

$(sy) \subseteq n \in S(n)$.

Otherwise, if $y \notin n$, *then* $y = n$. *Then we simply have* $S(y) = S(n)$. *Thus* J *is a succesor set and so by the Induction Principle,* $J = \omega$.

$\square$

3.1.3 *Well-Orderings*

Definition 10 (Strict Partially Ordered Set)

*A **strict partially ordered set** (or **strict poset**[5]) is a set E together with $R \subseteq E^2 = E \times E$ such that*

1. *(**anti-reflexive**)* $\forall a \in E \quad (a, a) \notin R$;
2. *(**anti-symmetric**)* $\forall a, b \in E \quad (a, b) \in R \land (b, a) \in R \implies a = b$; *and*
3. *(**transitivity**)* $\forall a, b, c \in E \quad (a, b), (b, c) \in R \implies (a, c) \in R$.

[5] This is my unofficial terminology

Definition 11 (Strict Totally Ordered Set)

*A strict poset is **total** (or **linear**) if*

$$\forall a, b \in E \quad (a, b) \in R \veebar (b, a) \in R$$

Definition 12 (Well-Order)

*A strict linear order is **well-ordered** if*

$$\forall X \subseteq E (X \neq \emptyset) \quad \exists a \in X \quad \forall b \in X (b \neq a) \quad (a, b) \in R$$

*i.e. every nonempty subset of E has a **least element**.*

We shall prove the following next lecture.[6]

[6] Anti-reflexivity and Anti-symmetry were proven in this lecture, but I am moving it to the next for ease of reading.

Proposition (ω is Strictly Well-ordered)
$(\omega, \in)$ is a strict well-ordering.

4 Lecture 4 Sep 18th

4.1 Ordinals (Continued 3)

4.1.1 Well-Orderings (Continued)

Proposition 12 (ω is Strictly Well-Ordered)

$(\omega, \in)$ is a strict well-ordering.

Proof

By Lemma 11, we have that $\forall n \in \omega$, $n \notin n$. (***anti-reflexivity*** ✔).

$\forall n, m \in \omega$, suppose, for contradiction, that $n \in m$ and $m \in n$. Again, by Lemma 11, we have $n \subseteq m$ and $m \subseteq n$, which implies that $n = m$. Thus, we have $n \in m = n$ and $m \in n = m$, a contradiction to the fact that $n \notin n$ and $m \notin m$ (***anti-symmetry*** ✔).

$\forall x, y, z \in \omega$ such that $x \in y$ and $y \in z$, by Lemma 11, $y \in z \implies y \subseteq z \implies x \in z$ (***transitivity*** ✔).

To show totality of the relation, let $n \in \omega$. WTS for any $m \in \omega$, either

$$m \in n, \quad m = n, \text{ or } n \in m.$$

Let[1]

$$J = \underbrace{n}_{\in n} \cup \underbrace{\{n\}}_{=n} \cup \underbrace{\{m \in \omega : n \in m\}}_{>n}.$$

Case 1: $n = 0$. In this case, we have[2]

$$J = \emptyset \cup \{\emptyset\} \cup \{m \in \omega : 0 \in m\}$$

As a consequence of Lemma 11 (4), we have that $J = \omega$.

Lemma (Lemma 11)

Suppose $n \in \omega$. We have

1. $n \subseteq \omega$;
2. $\forall m \in n \quad m \subseteq n$;
3. $n \notin n$;
4. $n = 0 \veebar 0 \in n$; and
5. $y \in n \implies S(y) \in n \veebar S(y) = n$.

[1] We construct J such that J will contain all the possible cases, and use this fact to prove that $J = \omega$ so these 3 cases are the only scenarios that can happen.

[2] Note that $0 = \emptyset$.

Case 2: $n \neq 0$. *Again, by Lemma 11 (4), since $n \neq 0$, we must have $0 \in n \subseteq J$ and so $0 \in J$. Now suppose that $m \in J$.*

Case 2(a): $m \in n$. *Then by Lemma 11 (5), $S(m) \in n$ or $S(m) = n$.*
$S(m) \in n \implies S(m) \in J$
$S(m) \in n \implies S(m) \in J$

Case 2(b): $m = n$. *Then $S(m) = S(n) = n \cup \{n\}$. And so $n \in S(m)$, which implies $S(m) \in J$.*

Case 2(c): $n \in m$ *Then since $S(m) = m \cup \{m\}$, we have that $m \in m \subseteq S(m)$. Therefore $S(m) \in J$.*

*Therefore, J is a sucessor subset of ω. Thus by the Induction Principle, $J = \omega$. (**totality** ✔)*

To prove that $\in$ is a well-ordering, suppose $X \subseteq \omega$ is non-empty. Suppose, for contradiction, that X has no $\in$-least element. Now consider

$$J = \{n \in \omega : S(n) \cap X = \emptyset\}$$

Claim: J is a successor set.[3]

[3] Since we want to prove that $\in$ is a well-ordering, we can suppose that there is a non-empty subset of ω that is not empty, and has no $\in$-least element. The core idea here is that, by the construction of J, if $J = \omega$, then all elements of ω would be disjoint from X, forcing X to be the empty set.

By Lemma 11 (4), 0 is the $\in$-least element of ω. If $0 \in X$, then 0 would be $\in$-least in X, contradicting our supposition. Thus $0 \notin X$, And so

$$S(0) \cap X = (0 \cup \{0\}) \cap X = \{0\} \cap X = \emptyset$$

since $0 \notin X$. Thus $0 \in J$.

Suppose $n \in J$. By construction of J, we have $S(n) \cap X = \emptyset$. Observe that

$$S(S(n)) \cap X = (S(n) \cup \{S(n)\}) \cap X.$$

Now if RHS of the above is non-empty (aiming for contradiction), then we may have $S(n) \in X$. Then $S(n)$ would be the $\in$-least element in X, a contradiction. If $m \in S(n)$, we have that $m \notin X$ since $S(n) \cap X = \emptyset$. Thus $SS(n) \cap X = \emptyset$ and so $S(n) \in J$. Therefore, by the Induction Principle, $J = \omega$.

We observe that $\forall n \in \omega$,

$$\emptyset = S(n) \cap X) = (n \cup \{n\}) \cap X$$

$\implies$ *$n \notin X$, and so we must have $X = \emptyset$ (**well-ordered** ✔).* $\square$

Note

Given $n, m \in \omega$, we often write $n < m$ to mean $n \in m$.

Definition 13 (Ordinals)

*An **ordinal** is a set α satisfying:*

1. $x \in \alpha \implies x \subseteq \alpha$;
2. $(\alpha, \in)$ *is a strict well-ordering.*

Example 4.1.1

ω is an ordinal: $\forall n \in \omega$, by Lemma 11, $n \subseteq \omega$, and ω is proven to have a strict well-ordering under $\in$.

Example 4.1.2

Every natural number is an ordinal (finite ordinals): by Lemma 11 (2), the first property is satisfied; well-ordering follows from the property of ω.

Let Ord denote the class of all ordinals. We shall show later that Ord is a proper class.

Exercise 4.1.1

Verify that for a set to be an ordinal is a definite condition.

$$\forall t(t \in \mathrm{Ord} \leftrightarrow (\underbrace{\forall x(x \in t \to \forall a(a \in x \to a \in t))}_{x \in t \implies x \subseteq t} \wedge \underbrace{\forall s(s \subseteq t \wedge s \neq \emptyset \to \exists a(a \in s \to \forall b(b \in s \wedge b \neq a \to (a,b) \in (\in))))}_{(t, \in) \text{ is a strict well-ordering}}))$$

Lemma 13 (Proper Subsets of an Ordinal Are Its Elements)

If $\alpha, \beta \in \mathrm{Ord}$ and $\alpha \subsetneq \beta$, then $\alpha \in \beta$.

Proof

We shall prove that α is the least element in β that is not in α itself.[4]

Let $D := \beta \setminus \alpha = \{x \in \beta : x \notin \alpha\} \subset \beta$[5]*. Since $\alpha \subsetneq \beta$, $D \neq \emptyset$. Since*

[4] We shall construct a subset of $\beta \setminus \alpha$ and show that α is its element.

[5] Exists by Bounded Separation Axiom.

$\beta \in \mathrm{Ord}$, $(\beta, \in)$ has a strict well-ordering, and so D has a least element, d. Note that $d \in \beta$, and since $\beta \in \mathrm{Ord}$, $d \subseteq \beta$.

Claim: $\alpha = d$.[6] *WTS $\alpha \subseteq d$. $\forall x \in \alpha$, we have $x, d \in \beta$. Then since $(\beta, \in)$ is a strict well-ordering, we have either*

$$x < d, \quad x = d, \quad \text{or } d < x$$

Note that $x \neq d$, otherwise $x = d \in D = \beta \setminus \alpha$.

[7] *If $d < x$, then $d \in x$ (by our notation). Now since $\alpha \in \mathrm{Ord}$, $x < \alpha \implies x \in \subseteq$, and so $d \in \alpha$, which is yet another contradiction ($d \in D = \beta \setminus \alpha$).*

Thus we must have $x < d$, i.e. $x \in d$. So $\alpha \subseteq d$.

WTS $d \subseteq \alpha$. Suppose not. Then let $x \in d \setminus \alpha$. Then since $d \in D = \beta \setminus \alpha$, we have $x \in \beta \setminus \alpha$, which then contradicts the minimality of d. Therefore, $d = \alpha$ as required. □

[6] If $\alpha = d$, then α is the said least element.

[7] This is an errorneous proof.

Warning

$$d < x \wedge x < \alpha$$
$$\underset{\text{transitivity}}{\implies} d < \alpha \implies d \in \alpha$$

This argument is errorneous because we do not yet know if $\alpha \in \beta$.

Proposition 14 (Properties of Ordinals)

1. *Every member of an ordinal is an ordinal.*
2. $\alpha \in \mathrm{Ord} \implies \alpha \notin \alpha$.
3. $\alpha \in \mathrm{Ord} \implies S(\alpha) \in \mathrm{Ord}$.
4. $\alpha, \beta \in \mathrm{Ord} \implies \alpha \cap \beta \in \mathrm{Ord}$.
5. $\alpha, \beta \in \mathrm{Ord} \implies \alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha$.
6. *$E \subseteq \mathrm{Ord}$ a subset $\implies (E, \in)$ is a strict well-ordering.*[8]
7. *Ord is a proper class.*

Some of proofs of these properties are available in the course notes.

Exercise 4.1.2

Prove Item 3, Item 4, and Item 5 of Proposition 14.

[8] **I think** that such an E need not be an ordinal itself. For example, $E = \{1, 5, 10\} \subset \mathrm{Ord}$, but $4 \in 5$ and $4 \notin E$, and so $5 \in E$ but $5 \not\subseteq E$.

Proof

1. *Suppose $x \in \beta \in \mathrm{Ord}$. WTS $x \in \mathrm{Ord}$, and we shall show that x satisfies Definition 13.*

   Since $\beta \in \mathrm{Ord}$, $x \in \beta \implies x \subseteq \beta$. Thus $(x, \in)$ is a strict well-ordering (through inheriting the property). So it suffices to show that $y \in xy \subseteq x$. So let $y \in x$, and let $t \in x$[9]*. Observe that*

[9] To show that $y \subseteq x$, we need to show that $\forall t \in y$, $t \in x$.

$$t \in y \implies t < y$$
$$y \in x \implies y < x$$

and $t, y, x \in \beta \in \mathrm{Ord}$. Therefore, by transitivity, we have $t < y < x \implies t \in x$.

2. *Suppose not, i.e. $\alpha \in \alpha$. Then $\alpha \subseteq \alpha \in \mathrm{Ord}$, and so $(\alpha, \in)$ is a strict well-ordering, i.e. $\alpha \notin \alpha$, a contradiction.*

6. *Suppose $A \subseteq E$ and $A \neq \emptyset$. Let $\alpha \in A$.*

   Case 1: $\alpha \cap A = \emptyset$. Then $\forall \beta \in \alpha \implies \beta \notin A$. Therefore α is $\in$-least in A.

   Case 2: $a \cap A \neq \emptyset$. Let $A' = \alpha \cap A \subseteq \alpha$. Since $\alpha \in A \subseteq E \subseteq \mathrm{Ord}$, we have $(\alpha, \in)$ is a strict well-ordering, and so A' has a strict well-ordering as well, and thus it must have a $\in$-least element, x. Then x is the $\in$-least element in A.

7. *If Ord is a set, then by Item 6, $(\mathrm{Ord}, \in)$ is a strict well-ordering. Also, by Item 1, every element of Ord is a subset of Ord. Therefore, Ord satisfies Definition 13, and so $\mathrm{Ord} \in \mathrm{Ord}$, which contradicts Item 2. Therefore $\mathrm{Ord} \notin \mathrm{Set}$.*

$\square$

5 Lecture 5 Sep 20th

5.1 Ordinals (Continued 4)

Note

If $A, B \in \mathrm{Ord}$, we will write $A < B$ to mean $A \in B$.

Proposition 15 (Properties of Ordinals 2)

1. *If $\alpha \in \mathrm{Ord}$, then $\alpha < S(\alpha)$, and there is nothing in between.*
2. *Let $E \subseteq \mathrm{Ord}$, where $E \neq \emptyset$ is a set, and $\sup E := \cup E$. Then $\sup E \in \mathrm{Ord}$, and it is a least upper bound for E.*[1]
3. *If $E \subseteq \mathrm{Ord}$ is a subset, then there is a least ordinal that is not in E.*

[1] I noted down from the lectures that this is "not necessarily strict", but I do not remember what it means now. **(Clarification required.)**
Perhaps this related to my question; can $E = \sup E$?

This is not necessarily true. If $E \notin \mathrm{Ord}$, then $E \neq \sup E$.

Proof

1. *Since $S(\alpha) = \alpha \cup \{\alpha\}$, $\alpha \in S(\alpha)$ and so $\alpha < S(\alpha)$.*

   It suffices to show that $\forall x < S(\alpha)$, we have $x \leq \alpha$. Let $x < S(\alpha)$, i.e. $x \in S(\alpha)$. So $x \in \alpha$ or $x = \alpha$, i.e. $x < \alpha$ or $x = \alpha$.

2. *By definition, $\forall x \in E \subseteq \mathrm{Ord}$, we have that $x \subseteq \mathrm{Ord}$. Since $\cup E \subseteq E$, we have that $\cup E \subseteq \mathrm{Ord}$ is a subset. Thus by Proposition 14 Item 6, $(\cup E, \in)$ is a strict well-ordering.*

   [2] *Suppose $\alpha \in \cup E$, then $\exists e \in E$ such that $\alpha \in e \subseteq E \subseteq \mathrm{Ord}$. So e is an ordinal and so $\alpha \subseteq e$.* [3] *$\forall x \in \alpha$, we have $x \in e \in E$, and so $x \in \cup E$ by definition. Therefore $\alpha \subseteq \cup E$.*

   And so, we have shown that $\cup E = \sup E \in \mathrm{Ord}$.

Exercise 5.1.1

Prove Proposition 15 Item 3.

Recommended strategy: $\alpha \in \mathrm{Ord}$ such that $E \subsetneq \alpha$ and take the least element of $\alpha \setminus E$ (which is non-empty). Prove that this least element is the least ordinal that is not in E.
You can take $\alpha = SS(\sup E)$. Verify that this works.

[2] This part shows that $\cup E$ is also an ordinal.
[3] Now we show that $\alpha \subseteq \cup E$.

Claim 1: $\sup E$ is an upper bound for E.

Suppose, for contradiction, that $\exists e \in E$ such that $\sup E < e$. Then since $\sup E$ and e are both ordinals, we have $\sup E \in e \in E$. Then by definition of $\cup$, we have that $\sup E \in \cup E = \sup E$, but by Proposition 14 Item 2, $\sup E \notin \sup E$, a contradiction.

Thus $\sup E$ is an upper bound as claimed.

Claim 2: $\sup E$ is the supremum (least upper bound).

$\forall \alpha < \sup E$, we have that $\alpha \in \sup E = \cup E$, and so $\exists e \in E$ such that $\alpha \in e$. Then $\alpha < e \in E$, i.e. α is not an upper bound of E.

Definition 14 (Successor Ordinal)

*The **successor ordinal** is an ordinal of the form $S(\alpha)$ for some $\alpha \in \mathrm{Ord}$.*

Definition 15 (Limit Ordinal)

*A **limit ordinal** is an ordinal that is not a successor.*

Example 5.1.1

0 and ω are both limit ordinals; 0 is vacuosly a limit ordinal, and ω is not a successor of any $\alpha \in$ Ord [4].

On the other hand, for $n \in \omega$ such that $n \neq 0$, $\exists \cup n \in \omega$ such that $S(\cup n) = n$ [5].

[4] Need a more careful proof, which I cannot do. The idea is to show that any such ordinal α will be an element of ω, and so will its successor $S(\alpha)$, and $\omega \notin \omega$.

[5] See A1.

Exercise 5.1.2

Prove that $S(\omega)$ is a successor ordinal.

Solution

We have that $\omega \in \mathrm{Ord}$, and so $S(\omega)$ is a successor ordinal.

5.1.1 *Transfinite Induction & Recursion*

Theorem 16 (Transfinite Induction Theorem v1)

Suppose P is a definite condition, with the property

$$\forall \alpha \in \mathrm{Ord} \wedge (\forall \beta < \alpha \; P(\beta)) \implies P(\alpha). \tag{5.1}$$

Then P is true of all ordinals.

Proof

$P(0)$ is vacuously true, since there are no elements that are less than 0. Suppose $P(\alpha)$ is false for some $\alpha \in \mathrm{Ord}$ such that $\alpha > 0$. By the Bounded Separation Axiom,

$$D := \{\beta \leq \alpha : \neg P(\beta)\}$$

is a set [6]. Note that $D \neq \emptyset$, since $\alpha \in D$. Since $\alpha \in \mathrm{Ord}$, we have $D \subseteq \alpha \subseteq \mathrm{Ord}$, and so $(D, \in)$ has a strict well-ordering. Let $\alpha_0 \in D$ be $\in$-least. Then $\forall \beta < \alpha_0$, we have that $\neg P(\beta)$, which contradicts the assumption Equation (5.1). Thus $P(\alpha)$ is true for all ordinals. □

[6] Note that $\beta \leq \alpha \iff \beta < \alpha \vee \beta = \alpha \iff \beta \in S(\alpha)$

Theorem 17 (Transfinite Induction Theorem v2)

Suppose P is a definite condition satisfying

1. *$P(0)$;*
2. *$\forall \beta \in \mathrm{Ord} \; P(\beta) \implies P(S(\beta))$; and*
3. *If $\alpha \in \mathrm{Ord}$ is a limit ordinal and $\forall \beta < \alpha$, $P(\beta)$, then $P(\alpha)$.*

Then P is true of all ordinals.

This statement strongly resembles the Induction Princple that we have learnt in the earlier years of university. In contrast, v1 resembles Strong Induction Principle. It can be shown that v1 $\iff$ v2. v1 $\implies$ v2 is proven in this lecture.

Exercise 5.1.3
Prove that Theorem 17 $\implies$ Theorem 16.

Proof

It suffices to show that P satisfies Equation (5.1), i.e. $\forall \alpha \in \mathrm{Ord}$, we want to prove that $\forall \beta < \alpha$, if $P(\beta)$, then $P(\alpha)$.

When $\alpha = 0$, we have $P(0)$ and so Equation (5.1) is satisfies. When $\alpha > 0$ is a limit ordinal, our assumption immediately satisfies Equation (5.1). Now suppose $\alpha > 0$ is a successor ordinal, and suppose that $\alpha = S(\gamma)$ for some $\gamma \in \mathrm{Ord}$. By the assumption in Equation (5.1), we have that

$$\forall \beta < \gamma \; P(\beta) \implies P(\gamma)$$

and so by condition (2), we have $P(S(\gamma))$ *since* $\gamma \in \mathrm{Ord}$. *Thus we have* $P(\alpha) = P(S(\gamma))$. ☐

We shall prove the following in the next lecture:

Theorem (Transfinite Recursion)

Let X *be a class of all definite operations whose domain is an ordinal. Given a definite operation*

$$G : X \to \mathrm{Set}$$

$\exists! F : \mathrm{Ord} \to \mathrm{Set}$, *a definite operation, such that* $F(\alpha = F(F\restriction_\alpha))$, *for all* $\alpha \in \mathrm{Ord}$.

We want to use Transfinite Recursion to construct definite operations on ordinals such that they have properties that we are familiar with (and hence desire).

Note (Notation - Restriction)

Let $H : U \to Y$ *be a definite operation on classes* U, Y, *and* $Z \subseteq U$ *a subclass.* $H\restriction_Z$ *is the definite operation*

$$H\restriction_Z : Z \to Y$$

obtained by restricting H *onto* Z.

Note

In the theorem, we stated that F *has its domain on* Ord. *We know that for* $\alpha \in \mathrm{Ord}$, $\alpha \subseteq \mathrm{Ord}$, *and so* $F\restriction_\alpha$ *makes sense; in particular,*

$$F\restriction_\alpha : \alpha \to \mathrm{Set}.$$

Note that $F\restriction_\alpha \in X$, *and so* $G(F\restriction_\alpha)$ *is valid and makes sense.*

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