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# Chapter 1

## Lecture 1 Jan 3 2018

### 1.1 Complex Numbers and Their Properties

#### Definition 1.1.1 (Complex Number, Complex Plane)

A **complex number** is a vector in  $\mathbb{R}^2$ . The **complex plane**, denoted by  $\mathbb{C}$ , is a set of complex numbers,

$$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

In  $\mathbb{C}$ , we usually write

$$\begin{aligned} 0 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & 1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ i &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} & x &= \begin{pmatrix} x \\ 0 \end{pmatrix} \\ iy &= \begin{pmatrix} 0 \\ y \end{pmatrix} \end{aligned}$$

where  $x, y \in \mathbb{R}$ . Consequently, we have that

$$x + iy = x + yi = \begin{pmatrix} x \\ y \end{pmatrix}$$

If for  $x, y \in \mathbb{R}$ ,  $z = x + iy$ , then  $x$  is called the **real part** of  $z$  and  $y$  is called the **imaginary part** of  $z$ , and we write

$$\operatorname{Re}(z) = x \quad \operatorname{Im}(z) = y.$$

#### Note

- It is easy to see how  $\mathbb{R}$  is a subset of  $\mathbb{C}$ .



- Complex Numbers of the form  $\begin{pmatrix} 0 \\ y \end{pmatrix}$  where  $y \in \mathbb{R}$  are called **purely imaginary numbers**.
- Certain authors may prefer to denote  $i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ .

**Definition 1.1.2 (Sum and Product)**

We define the sum of two complex numbers to be the usual vector sum, i.e.

$$\begin{aligned} (a + ib) + (c + id) &= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \begin{pmatrix} a + c \\ b + d \end{pmatrix} \\ &= (a + c) + i(b + d) \end{aligned}$$

where  $a, b, c, d \in \mathbb{R}$ .

We define the product of two complex numbers by setting  $i^2 = -1$ , and by requiring the product to be **commutative, associative, and distributive** over the sum. In this setup, we have that

$$\begin{aligned} (a + ib)(c + id) &= ac + iad + ibc + i^2bd \\ &= (ac - bd) + i(ad + bc) \end{aligned} \tag{1.1}$$

**Note**

It is interesting to note that **any complex number times zero is zero**, just like what we have with real numbers.

$$\begin{aligned} \forall z = x + iy \in \mathbb{C} \quad x, y \in \mathbb{R} \quad 0 \in \mathbb{C} \\ z \cdot 0 = (x + iy)(0 + i0) = 0 + i0 = 0 \end{aligned}$$

**Example 1.1.1**

Let  $z = 2 + i, w = 1 + 3i$ . Find  $z + w$  and  $zw$ .

$$\begin{aligned} z + w &= (2 + i) + (1 + 3i) \\ &= 3 + 4i \end{aligned}$$

$$\begin{aligned} zw &= (2 + i)(1 + 3i) \\ &= (2 - 3) + i(6 + 1) \quad \text{By Equation (1.1)} \\ &= -1 + 7i \end{aligned}$$

**Example 1.1.2**

Show that every non-zero complex number has a **multiplicative inverse**,  $z^{-1}$ , and find a formula for this inverse.

Let  $z = a + ib$  where  $a, b \in \mathbb{R}$  with  $a^2 + b^2 \neq 0$ . Then

$$\begin{aligned}
 z(x + iy) &= 1 \\
 \iff (ax - by) + i(ay + bx) &= 1 \\
 \iff \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \iff \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \iff \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \iff \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{a^2 + b^2} \begin{pmatrix} a \\ -b \end{pmatrix} \\
 \iff x + iy &= \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2}
 \end{aligned}$$

Therefore, we have that the formula for the inverse is

$$(a + ib)^{-1} = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2} \quad (1.2)$$

**Notation**

For  $z, w \in \mathbb{C}$ , we write

$$\begin{aligned}
 -z &= -1z & w - z &= w + (-z) \\
 \frac{1}{z} &= z^{-1} & \frac{w}{z} &= wz^{-1}
 \end{aligned}$$

**Example 1.1.3**

Find  $\frac{(4-i)-(1-2i)}{1+2i}$ .

$$\begin{aligned}
 \frac{(4-i)-(1-2i)}{1+2i} &= \frac{3+i}{1+2i} \\
 &= (3+i)\left(\frac{1}{5} - i\frac{2}{5}\right) \\
 &= 1 - i
 \end{aligned}$$

**Note**

The set of complex numbers is a **field** under the operations of addition and multiplication. This means that  $\forall u, v, w \in \mathbb{C}$ ,

$$\begin{array}{ll}
u + v = v + u & uv = vu \\
(u + v) + w = u + (v + w) & (uv)w = u(vw) \\
0 + u = u & 1u = u \\
u + (-u) = 0 & uu^{-1} = 1, \quad u \neq 0 \\
u(v + w) = uv + uw &
\end{array}$$

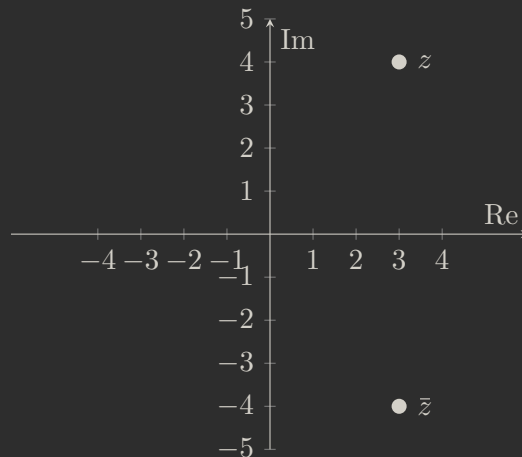
Since the distributive law holds for complex numbers, note that the **binomial expansion works** for  $(w + z)^n$  where  $w, z \in \mathbb{C}$  and  $n \in \mathbb{N}$ . (I did not verify if this is still true for when  $n \in \mathbb{R}$ .)

### Definition 1.1.3 (Conjugate)

If  $z = x + iy$  where  $x, y \in \mathbb{R}$ , then the **conjugate of  $z$**  is given by  $\bar{z} = x - iy$

### Example 1.1.4

Let  $z = 3 + 4i$ . Then the  $\bar{z} = 3 - 4i$ . Represented in the complex plane, we have the following:



We observe that on the complex plane, the conjugate of a complex number is simply its reflection on the real axis.

### Definition 1.1.4 (Modulus)

We define the **modulus** (length, magnitude) of  $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$ , to be

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}. \quad (1.3)$$

### Note

Note that this definition is consistent with the notion of the absolute value in real numbers when  $z$  is a real number, since if  $y = 0$ ,  $|z| = |x + i0| = \sqrt{x^2} = \pm x$ .

**Note**

For  $z, w \in \mathbb{C}$  and  $n \in \mathbb{N}$ , we have

$$\begin{array}{lll} \bar{\bar{z}} = z & z + \bar{z} = 2 \operatorname{Re}(z) & z - \bar{z} = 2i \operatorname{Im}(z) \\ z\bar{z} = |z|^2 & |z| = |\bar{z}| & \overline{z \pm w} = \bar{z} \pm \bar{w} \\ \overline{zw} = \bar{z}\bar{w} & |zw| = |z| |w| & \bar{z}^n = \overline{z^n} \end{array}$$

but note that  $|z + w| \neq |z| + |w|$ .

Also, note that the last equation is a generalization of the **highlighted equation**.

**Note**

While inequalities such as  $z_1 < z_2$ , where  $z_1, z_2 \in \mathbb{C}$ , are meaningless unless if both of them are real,  $|z_1| < |z_2|$  means that the point  $z_1$  in the complex plane is closer to the origin than the point  $z_2$ .

**Proposition 1.1.1 (Basic Inequalities)**

1.  $|\operatorname{Re}(z)| \leq |z|$
2.  $|\operatorname{Im}(z)| \leq |z|$
3.  $|z + w| \leq |z| + |w|$      *Triangle Inequality*
4.  $|z + w| \geq ||z| - |w||$      *Inverse Triangle Inequality*

**Proof**

Note that  $|z|^2 = \operatorname{Re}(z)^2 + \operatorname{Im}(z)^2$  and that we can express  $|x| = \sqrt{x^2}$  for any  $x \in \mathbb{R}$ . 1 and 2 immediately follows from that.

To prove 3, we have that

$$\begin{aligned} |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) \\ &= |z|^2 + |w|^2 + (w\bar{z} + \bar{w}z) \\ &= |z|^2 + |w|^2 + 2 \operatorname{Re}(w\bar{z}) \\ &\leq |z|^2 + |w|^2 + 2 |w\bar{z}| \quad \text{by 1} \\ &= |z|^2 + |w|^2 + 2 |wz| \quad \text{since } |w\bar{z}| = |w| |\bar{z}| \text{ and } |z| = |\bar{z}| \\ &= (|z| + |w|)^2 \end{aligned}$$

To prove 4, note that

$$|z| = |z + w - w| \leq |z + w| + |w| \quad (1.4)$$

$$|w| = |w + z - z| \leq |z + w| + |z| \quad (1.5)$$

Observe that

$$\text{Equation (1.4)} \implies |z| - |w| \leq |z + w|$$

$$\text{Equation (1.5)} \implies |w| - |z| \leq |z + w|$$

Thus, we have that

$$|z + w| \geq ||z| - |w||$$

as required.  $\square$

Item 3 in Proposition 1.1.1 can be generalized by the means of mathematical induction to sums involving any finite number of terms, as:

$$|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n| \quad (1.6)$$

where  $n \in \mathbb{N} \setminus \{0, 1\}$ .

To note the induction proof, when  $n = 2$ , Equation (1.6) is just Item 3. If Equation (1.6) is true for when  $n = m$  where  $m \in \mathbb{N} \setminus \{0, 1\}$ ,  $n = m + 1$  is also true since by Item 3,

$$\begin{aligned} |(z_1 + z_2 + \dots + z_m) + z_{m+1}| &\leq |z_1 + z_2 + \dots + z_m| + |z_{m+1}| \\ &\leq (|z_1| + |z_2| + \dots + |z_m|) + |z_{m+1}|. \end{aligned}$$

The distance between two points  $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}, x_1, x_2, y_1, y_2 \in \mathbb{R}$  is  $|z_1 - z_2|$ , since  $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$  is our usual notion of the Euclidean distance of two points on a plane.

Also, note that

$$z_1 - z_2 = z_1 + (-z_2)$$

and thus if we apply our knowledge of vector representation,  $z_1 - z_2$  is the directed line segment from the point  $z_2$  to  $z_1$ .

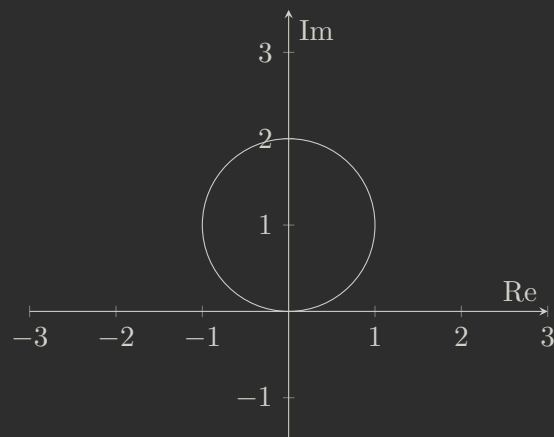
With the notion of a “distance” set on the complex plane, we can now explore upon points lying on a circle with a center  $z_0$  and radius  $R$ , which satisfies the equation

$$|z - z_0| = R.$$

We may simply refer to this set of points as the circle  $|z - z_0| = R$ .

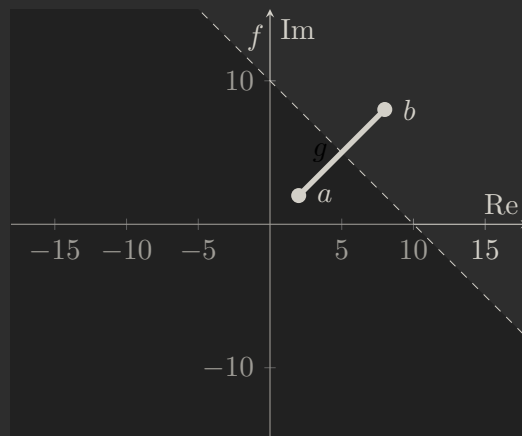
**Example 1.1.5**

We may describe a set  $\{z \in \mathbb{C} : |z - i| = 1\}$  as follows:



Let  $a, b \in \mathbb{C}$  describe the set  $\{z \in \mathbb{C} : |z - a| < |z - b|\}$ .

Suppose the following coordinates for  $a$  and  $b$  are arbitrary,



In the above,  $g$  is the line segment that connects the points  $a$  and  $b$  on the complex plane, while  $f$  is the perpendicular bisector of the line segment  $g$ . The area described by the set  $\{z \in \mathbb{C} : |z - a| < |z - b|\}$  is the shaded area which is below  $f$ .

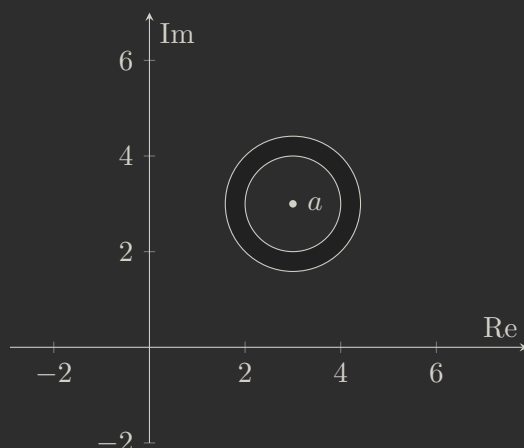
## Chapter 2

### Lecture 2 Jan 5th 2018

#### 2.1 Complex Numbers and Their Properties (Continued)

**Example 2.1.1**

Let  $a \in \mathbb{C}$ . Describe the set  $\{z \in \mathbb{C} : 1 < |z - a| < 2\}$ .



**Example 2.1.2**

Show that every non-zero complex number has exactly two complex square roots, and find a formula for the square roots.

Let  $z = x + iy \in \mathbb{C}$ ,  $x, y \in \mathbb{R}$ , and let  $w = u + iv$ ,  $u, v \in \mathbb{R}$ . Then





**Remark**

Let  $z \in \mathbb{C}$ . The notation  $\sqrt{z}$  may represent either one of the square roots of  $z$  or both of the square roots, i.e. **it is possible that  $\sqrt{z}$  represents a set.**

**Exercise 2.1.1**

Is it always okay for complex numbers such that  $\sqrt{zw} = \sqrt{z}\sqrt{w}$ , for  $z, w \in \mathbb{C}$ ?

No. For example, consider  $z = w = -1$ . Then we have

$$\sqrt{zw} = \sqrt{1} = \pm 1$$

while

$$\sqrt{z}\sqrt{w} = i \cdot i = -1$$

and thus

$$\sqrt{zw} \neq \sqrt{z}\sqrt{w}.$$

**Example 2.1.3**

Find the values of  $\sqrt{3-4i}$ .

By Example 2.1.2,

$$\begin{aligned} \sqrt{3-4i} &= \pm \left( \sqrt{\frac{3+\sqrt{9+16}}{2}} - i\sqrt{\frac{-3+\sqrt{9+16}}{2}} \right) \\ &= \pm(2-i) \end{aligned}$$

**Remark**

The quadratic formula holds for complex polynomials, i.e.

$$\forall a, b, c \in \mathbb{C} \quad a \neq 0 \quad \forall z \in \mathbb{C} \quad az^2 + bz + c = 0,$$

the solution for  $z$  is given by

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (2.3)$$

The following is a short proof.



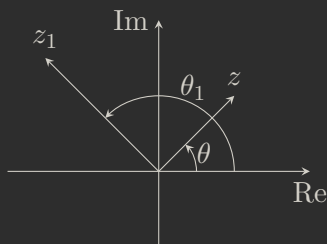
# Chapter 3

## Lecture 3 Jan 8th 2018

### 3.1 Complex Numbers and Their Properties (Continued 2)

#### Definition 3.1.1 (Argument of a Complex Number)

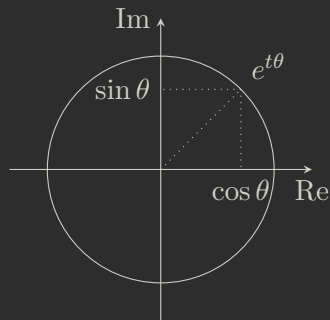
Let  $z \in \mathbb{C} \setminus \{0\}$ . The **argument** (or the angle) of  $z$ , denoted by  $\arg z$ ,  $\text{Arg } z$ , or simply  $\theta = \theta(z)$ , is the angle modulo  $2\pi$  (i.e.  $0 \leq \theta < 2\pi$ ) between the vector defining  $z$  and the positive real axis (in the counterclockwise direction).



#### Notation

Let  $e^{i\theta} := \cos \theta + i \sin \theta$ . Note that this definition, called **Euler's formula**, can be derived by extending the Taylor expansion of  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  for when  $x \in \mathbb{C}$  (the sum of the real parts of the expansion is the Taylor expansion of cosine while the imaginary part for sine).

Now  $e^{i\theta}$  is on the unit circle.

**Remark**

If  $z = 0$ , the coordinate  $\theta$  is undefined, and so it is implied that  $z \neq 0$  whenever we use the polar form.

**Example 3.1.1**

Some examples of  $\theta \in [0, 2\pi)$ :

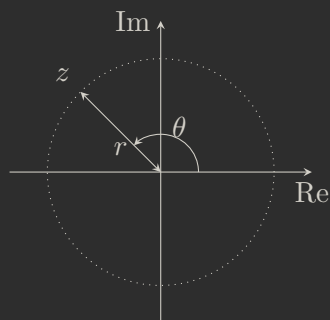
$$\begin{aligned} e^{i\frac{\pi}{4}} &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\frac{\pi}{2}} &= i \\ e^{i\frac{3\pi}{4}} &= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\pi} + 1 &= 0 \end{aligned}$$

**Remark**

$$\forall k \in \mathbb{Z} \quad \forall \theta \in \mathbb{R} \quad e^{i\theta} = e^{i(\theta+2\pi k)}$$

**Remark**

The complex number  $re^{i\theta}$ , where  $r > 0, \theta \in [0, 2\pi)$ , represents the complex number with modulus  $r$  and argument  $\theta$ .



Therefore,  $\forall z \in \mathbb{C}$ , we can express

$$z := |z| e^{i \operatorname{Arg} z}. \quad (3.1)$$





The  $n$ th roots of  $z$  is described by the set

$$\left\{ r^{\frac{1}{n}} e^{i\left(\frac{\theta+2\pi k}{n}\right)} : k = 0, 1, \dots, n-1 \right\} \quad (3.8)$$

**Proof**

$$\begin{aligned} s^n = r &\iff s = r^{\frac{1}{n}} \\ e^{in\theta} = e^{i\tau} &\iff \theta = \frac{\tau + 2\pi k}{n} \end{aligned}$$

Therefore, the set that describes the  $n$ th roots of  $z$  is

$$\left\{ w = r^{\frac{1}{n}} e^{i\left(\frac{\theta+2\pi k}{n}\right)} : k = 0, 1, \dots, n-1 \right\}$$

**Remark (nth Roots of Unity)**

The ***nth roots of unity*** is a direct consequence of *Proposition 3.1.1* where we solve for the equation  $z^n = 1$  for any  $z \in \mathbb{C}, n \in \mathbb{Z}$ .

The set that describes the  $n$ th roots of unity is

$$\left\{ e^{i\theta} : \theta = \frac{2\pi k}{n}, k = 0, 1, \dots, n-1 \right\} \quad (3.9)$$

It is easy to see how the  $n$ th roots of unity **partitions the unit circle into  $n$  parts**.

### Example 3.1.3

Find the cubic roots of  $-2 + 2i$ .

Let  $z = -2 + 2i$ . Note that  $|z| = 2\sqrt{2}$  and  $\text{Arg } z = \frac{3\pi}{4}$ .

Therefore, in polar form,  $z = 2\sqrt{2}e^{i\frac{3\pi}{4}}$ .

Let  $w = re^{i\theta}$ , where  $\theta \in [0, 2\pi)$ , and  $w^3 = z$ . Then

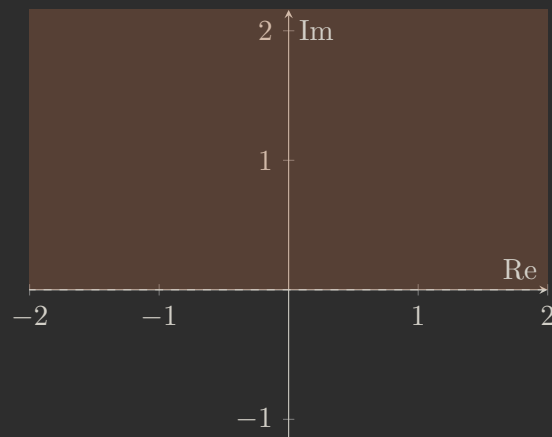
$$\begin{aligned} r &= (2\sqrt{2})^{\frac{1}{3}} \\ \theta &= \frac{\frac{3\pi}{4} + 2\pi k}{3}, \quad k = 0, 1, 2 \end{aligned}$$

The set that describes the cubic root of  $-2 + 2i$  is thus

$$\left\{ (2\sqrt{2})^{\frac{1}{3}} e^{i\theta} : \theta = \frac{\frac{3\pi}{4} + 2\pi k}{3}, k = 0, 1, 2 \right\}$$

**Example 3.1.4**

Describe the set  $\{z \in \mathbb{C} : |\operatorname{Arg} z - \frac{\pi}{2}| < \frac{\pi}{2}\}$ . (Note:  $\operatorname{Arg} z \in [0, 2\pi)$ )

**Exercise 3.1.1**

Solve

1.  $z^4 = -1$

$$\text{Let } z = re^{i\theta}$$

$$r = |-1| = 1 \quad \theta = \frac{\pi + 2\pi k}{4} = \frac{(2k+1)\pi}{4}, \quad k = 0, 1, 2, 3$$

2.  $z^4 = -1 + \sqrt{3}i$

$$\text{Let } z = re^{i\theta}$$

$$r = \left| -1 + \sqrt{3}i \right| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\theta = \frac{\frac{2\pi}{3} + 2\pi k}{4} = \frac{(2k + \frac{2}{3})\pi}{4}, \quad k = 0, 1, 2, 3$$



# Chapter 4

## Lecture 4 Jan 10th 2018

### 4.1 Examples for $n$ th Roots of Unity

Recall that the  $n$ th roots of unity are given by  $e^{i\frac{2\pi k}{n}}, k = 0, 1, \dots, n-1$ .

#### Exercise 4.1.1

Let  $z$  be any  $n$ th root of unity other than 1. Show that

$$z^{n-1} + z^{n-2} + \dots + z + 1 = 0 \quad (4.1)$$

#### Proof

By the Sum of Finite Geometric Terms,

$$z^{n-1} + z^{n-2} + \dots + z + 1 = \frac{1 - z^n}{1 - z}.$$

Since  $z^n = 1$ , RHS is thus zero, which in turn completes the proof.

As an aside, if we wish to remove the restriction that  $z$  can also be 1, we may consider that

$$z^n - 1 = (z - 1)(1 + z + \dots + z^{n-1})$$

Since  $z^n = 1$ , LHS is zero. Then either  $z = 1$  or  $(1 + z + \dots + z^{n-1}) = 0$ .

#### Exercise 4.1.2

Consider the  $n-1$  diagonals of a regular  $n$ -gon, inscribed in a circle of radius 1, obtained by connecting one vertex on the  $n$ -gon to all its other vertices.

For example, if we are given  $n = 6$ , we obtain the following diagram.





therefore we obtain

$$\begin{aligned}
 2^{3n} + (1 + \alpha)^{3n} + (1 + \alpha^2)^{3n} &= 3 \sum_{j=0}^n \binom{3n}{3j} \\
 \frac{1}{3} [2^{3n} + (1 + \alpha)^{3n} + (1 + \alpha^2)^{3n}] &= \sum_{j=0}^n \binom{3n}{3j} \\
 \frac{1}{3} [2^{3n} + (-\alpha^2)^{3n} + (-\alpha)^{3n}] &= \sum_{j=0}^n \binom{3n}{3j} \quad \text{since } 1 + \alpha + \alpha^2 = 0 \\
 \frac{1}{3} [2^{3n} + (-1)^n + (-1)^n] &= \sum_{j=0}^n \binom{3n}{3j} \quad \text{since } \alpha^3 = 1 \\
 \frac{2^{3n} + 2(-1)^n}{3} &= \sum_{j=0}^n \binom{3n}{3j}
 \end{aligned}$$

as required.

#### Exercise 4.1.4

Note that we can define  $\text{Arg } z$  in any interval of length  $2\pi$ , i.e. it is not necessary that  $\text{Arg } z \in [0, 2\pi)$ .

For example, if we restrict  $\text{Arg } z \in [-\pi, \pi]$ , then we can write

$$\text{Arg} \left( -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) = -\frac{3\pi}{4}$$

Let  $z$  be on the unit circle and  $\text{Arg } z \in [-\pi, \pi]$ . Suppose that  $z \notin \mathbb{R}$ , i.e.  $z \neq 1, z \neq -1$ . Show that

$$\text{Arg} \left( \frac{z-1}{z+1} \right) = \begin{cases} \frac{\pi}{2} & \text{Im } z > 0 \\ -\frac{\pi}{2} & \text{Im } z < 0 \end{cases}$$

#### Proof

Note that  $\forall w_1, w_2 \in \mathbb{C}$ , where  $\text{Arg } w_1 = \tau_1, \text{Arg } w_2 = \tau_2$  for  $\tau_1, \tau_2$  in the same  $2\pi$ -interval,

$$\text{Arg} \frac{w_1}{w_2} = \frac{e^{i\tau_1}}{e^{i\tau_2}} \equiv e^{i(\tau_1 - \tau_2)} = \text{Arg } w_1 - \text{Arg } w_2 \quad (4.7)$$

in modulo  $2\pi$ .

Suppose  $\text{Im } z > 0$ . Let  $\theta_1 = \text{Arg}(z-1)$  and  $\theta_2 = \text{Arg}(z+1)$ . Consider Figure 4.3. Note that since both  $\theta_1, \theta_2 \in [0, \pi]$ , we have that  $\theta_1 - \theta_2 \in [-\pi, \pi]$ , and thus Equation (4.7) holds

true without the need of the condition of being in modulo  $2\pi$ . We observe that

$$\begin{aligned}\frac{\pi}{2} &= \theta_2 + \pi - \theta_1 \\ \theta_1 - \theta_2 &= \frac{\pi}{2}\end{aligned}$$

as desired.

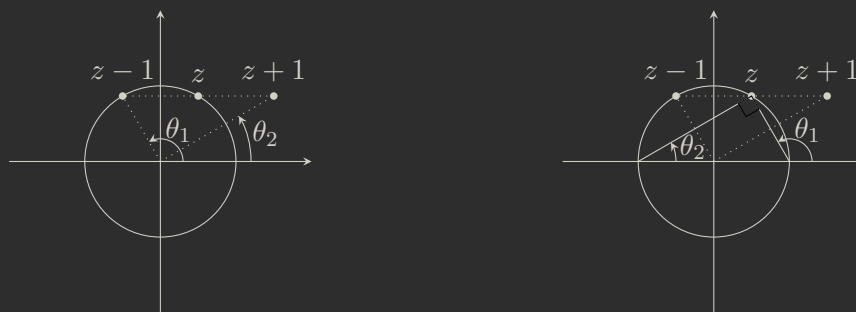


Figure 4.3: (Right) Depicted question, (Left) Translated Angles

Similarly, we can obtain  $\theta_1 - \theta_2 = -\frac{\pi}{2}$  for when  $\text{Im } z < 0$ . This completes the proof.

#### Exercise 4.1.5

Let  $f(z) = e^z$  for  $z \in \mathbb{C}$ . Let  $A = \{z = x + iy \in \mathbb{C} : x \leq 1, y \in [0, \pi]\}$ . Describe the image of  $f(A)$ .

#### Solution

Firstly, note that

$$\begin{aligned}e^z &= e^{x+iy} \\ e^x &\in (0, e] \\ y &\in [0, \pi]\end{aligned}$$



# Chapter 5

## Lecture 5 Jan 12 2018

### 5.1 Complex Functions

#### 5.1.1 Limits

##### Definition 5.1.1 (Convergence)

A sequence of complex numbers  $z_1, z_2, z_3, \dots$  **converges** to  $z \in \mathbb{C}$  if

$$\lim_{n \rightarrow \infty} |z_n - z| = 0 \quad (5.1)$$

or we may say

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \quad |z_n - z| < \varepsilon \quad (5.2)$$

##### Note

If  $\{z_n\}_{n \in \mathbb{N}}$  converges to  $z$ , we may write  $\lim_{n \rightarrow \infty} z_n = z$  or  $z_n \rightarrow z$  (as  $n \rightarrow \infty$ ).

##### Example 5.1.1

For  $|z| > 1$ , does  $\{\frac{1}{z^n}\}_{n=1}^{\infty}$  converge? Explain.

##### Solution

We claim that the limit is 0. Since  $|z| > 1$ , we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{1}{z^n} - 0 \right| &= \lim_{n \rightarrow \infty} \left| \frac{1}{z} \right|^n \\ &= 0 \end{aligned}$$

Another way to prove this, since  $|z| > 1 \implies 0 < \left|\frac{1}{z}\right| < 1$ ,

$$\begin{aligned} \forall \varepsilon = \left|\frac{1}{z}\right| > 0 \\ \left|\frac{1}{z^n} - 0\right| = \left|\frac{1}{z}\right|^n < \left|\frac{1}{z}\right| = \varepsilon \end{aligned}$$

**Definition 5.1.2 (Convergence for Complex Functions)**

$\forall \Omega \subseteq \mathbb{C}$ , let  $f : \Omega \rightarrow \mathbb{C}$ . We say that

$$\lim_{z \rightarrow z_0} f(z) = L \quad (5.3)$$

for some  $L \in \mathbb{C}$  if for every sequence  $\{z_n\}_n \subseteq \Omega$  (not including  $z_0$  if it is in  $\Omega$ ), we have that

$$z_n \rightarrow z_0 \implies f(z_n) \rightarrow L \quad (5.4)$$

Note that  $L$  need not be in  $\Omega$ .

**Example 5.1.2**

Let  $f(z) = \frac{z}{z}, z \in \mathbb{C} \setminus \{0\}$ . Find  $\lim_{z \rightarrow 0} f(z)$ .

**Solution**

Suppose  $z = x \in \mathbb{R} \setminus \{0\}$ . Then  $f(z) = f(x) = \frac{x}{x} = 1$ .

Suppose  $z = iy, y \in \mathbb{R} \setminus \{0\}$ . Then  $f(z) = f(iy) = \frac{-iy}{iy} = -1$ .

Therefore, the limit  $\lim_{z \rightarrow 0} f(z)$  does not exist.

**Exercise 5.1.1**

Show that  $z_n \rightarrow z \iff \text{Re}(z_n) \rightarrow \text{Re}(z) \wedge \text{Im}(z_n) \rightarrow \text{Im}(z)$ .

(Hint:  $|\text{Re}(z)|, |\text{Im}(z)| \leq |z| \leq |\text{Re}(z)| + |\text{Im}(z)|$ )

**Solution**

Suppose  $z_n \rightarrow z$ . Then  $\forall \varepsilon_0 > 0 \exists N \in \mathbb{N} \forall n > N |z_n - z| < \varepsilon$ . Note once and for all that

$$\begin{aligned} \text{Re}(z_n - z) &= \text{Re}(z_n) - \text{Re}(z) \\ \text{Im}(z_n - z) &= \text{Im}(z_n) - \text{Im}(z). \end{aligned}$$

Thus

$$\begin{aligned} |\text{Re}(z_n) - \text{Re}(z)| &= |\text{Re}(z_n - z)| \\ &\leq |z_n - z| < \varepsilon \\ |\text{Im}(z_n) - \text{Im}(z)| &= |\text{Im}(z_n - z)| \\ &\leq |z_n - z| < \varepsilon \end{aligned}$$



For the other direction,

$$\begin{aligned}\forall \frac{\varepsilon}{2} > 0 \quad \exists N_0 \in \mathbb{N} \quad \forall n > N_0 \quad |\operatorname{Re}(z_n) - \operatorname{Re}(z)| < \frac{\varepsilon}{2} \\ \forall \frac{\varepsilon}{2} > 0 \quad \exists N_1 \in \mathbb{N} \quad \forall n > N_1 \quad |\operatorname{Im}(z_n) - \operatorname{Im}(z)| < \frac{\varepsilon}{2}.\end{aligned}$$

Therefore,

$$\begin{aligned}|z_n - z| &= |\operatorname{Re}(z_n) + i\operatorname{Im}(z_n) - \operatorname{Re}(z) - i\operatorname{Im}(z)| \\ &\leq |\operatorname{Re}(z_n) - \operatorname{Re}(z)| + |\operatorname{Im}(z_n) - \operatorname{Im}(z)| \\ &\leq \varepsilon\end{aligned}$$

□

### 5.1.2 Continuity

#### Definition 5.1.3 (Continuity)

$\forall \Omega \subseteq \mathbb{C}$ , let  $f : \Omega \rightarrow \mathbb{C}$ . We say that  $f$  is **continuous** at  $z_0 \in \Omega$  if

1.  $\forall \{z_n\}_{n \in \mathbb{N}} \quad z_n \rightarrow z_0 \implies f(z_n) \rightarrow f(z_0)$
2.  $\forall \varepsilon > 0 \quad \exists \delta > 0 \quad |z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon$

#### Remark

1.  $f$  is continuous on  $\Omega$  if it is continuous on every point in  $\Omega$ .
2. We may **split**  $f$  into its real and imaginary parts, i.e.

$$f(z) = f(x, y) = u(x, y) + iv(x, y) \tag{5.5}$$

where  $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$ .

#### Example 5.1.3

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  and for  $z \in \mathbb{C}$ ,  $f(z) = \frac{\bar{z}}{z}$ . To split  $f$  into real and imaginary parts:

$$\begin{aligned}f(z) &= \frac{\bar{z}}{z} \\ &= (x + iy) \left( \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2} \right) \\ &= \frac{x^2 - y^2}{x^2 + y^2} + i \frac{(-2xy)}{x^2 + y^2}\end{aligned}$$



# Chapter 6

## Lecture 6 Jan 15th 2018

### 6.1 Continuity (Continued)

#### Exercise 6.1.1

Let  $f : \Omega \rightarrow \mathbb{C}$ . Prove that  $f(z)$  is continuous at  $z_0 = x_0 + iy_0 \in \mathbb{C} \iff$  functions  $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$ , such that  $f(z) = u(x, y) + iv(x, y)$  are both continuous at  $(x_0, y_0)$ .

#### Solution

We shall first prove the forward direction. Suppose that  $f(z)$  is continuous at  $z_0 = x_0 + iy_0 \in \mathbb{C}$ . By Definition 5.1.3,  $\forall \{z_n\}_{n \in \mathbb{N}} \subseteq \Omega$ ,  $z_n \rightarrow z_0 \implies f(z_n) \rightarrow f(z_0)$ . By Exercise 5.1.1,

$$\begin{aligned} z_n \rightarrow z_0 &\iff \operatorname{Re} z_n \rightarrow \operatorname{Re} z_0 \wedge \operatorname{Im} z_n \rightarrow \operatorname{Im} z_0 \\ &\iff x_n \rightarrow x_0 \wedge y_n \rightarrow y_0 \end{aligned} \tag{6.1}$$

where  $z_n = x_n + iy_n$  for  $x_n, y_n \in \mathbb{R}$ .

Similarly so, and by Equation (5.5),

$$f(z_n) \rightarrow f(z_0) \iff u(x_n, y_n) \rightarrow u(x_0, y_0) \wedge v(x_n, y_n) \rightarrow v(x_0, y_0) \tag{6.2}$$

Putting together Equation (6.1) and Equation (6.2), we get

$$(x_n, y_n) \rightarrow (x_0, y_0) \implies u(x_n, y_n) \rightarrow u(x_0, y_0) \wedge v(x_n, y_n) \rightarrow v(x_0, y_0)$$

as desired.

The proof of the other direction is simply a reversed process of the above. □

## 6.2 Differentiability

### Definition 6.2.1 (Neighbourhood)

For  $z_0 \in \mathbb{C}$ ,  $r \in \mathbb{R}$ , let

$$D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}. \quad (6.3)$$

On the complex plane, this is seen as a open disk centered around the point  $z_0$  with radius  $r$ , as shown below.

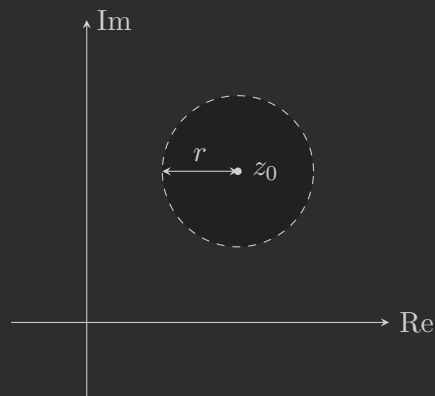


Figure 6.1: Open disk centered around  $z_0$  with radius  $r$

This open disk is called a **neighbourhood** of  $z_0$ .

### Definition 6.2.2 (Differentiable/Holomorphic)

Let  $f(z)$  be defined in a neighbourhood of  $z_0 \in \mathbb{C}$ . We say  $f$  is **differentiable/holomorphic** at  $z_0$  if for some  $h \in \mathbb{C}$ ,

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad (6.4)$$

exists. If such a limit exists, we denote the limit by  $f'(z_0)$ .

### Remark

$h \in \mathbb{C}$  :  $h$  need not necessarily be real. In this sense,  $h$  approaches 0 from **any direction** around 0  $\in \mathbb{C}$ .

### Example 6.2.1

For  $z \in \mathbb{C} \setminus \{0\}$ , let  $f(z) = \frac{1}{z}$ . Let  $z_0 \in \mathbb{C} \setminus \{0\}$ . Note that

$$\lim_{h \rightarrow 0} \frac{\frac{1}{z_0+h} - \frac{1}{z_0}}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{-h}{(z_0 + h)z_0} \right] = -\frac{1}{z_0^2}$$

Thus  $f$  is holomorphic at any  $z \in \mathbb{C} \setminus \{0\}$ , and hence  $f'(z) = -\frac{1}{z}$ .

### Example 6.2.2

For  $z \in \mathbb{C}$ , let  $f(z) = \bar{z}$ . Let  $z_0 \in \mathbb{C}$ . Notice that

$$\lim_{h \rightarrow 0} \frac{\overline{z_0 + h} - \bar{z}}{h} = \lim_{h \rightarrow 0} \frac{\bar{h}}{h}.$$

From [Example 5.1.2](#), we know that such a limit does not exist. Thus  $f$  is not holomorphic on any  $z \in \mathbb{C}$ .

### Exercise 6.2.1 (Holomorphic Functions Properties)

If  $f, g$  are holomorphic at  $z \in \mathbb{C}$ , prove that

1.  $f + g$  is holomorphic and  $(f + g)' = f' + g'$ .
2.  $fg$  is holomorphic and  $(fg)' = f'g + fg'$ .
3. if  $g(z) \neq 0$ ,  $\frac{f}{g}$  is holomorphic and  $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$ .

#### Solution

1. For  $f + g$ ,

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(z+h) + g(z+h) - f(z) - g(z)}{h} \\ &= \lim_{h \rightarrow 0} \left[ \frac{f(z+h) - f(z)}{h} + \frac{g(z+h) - g(z)}{h} \right] \\ &= f'(z) + g'(z) \end{aligned}$$

Thus  $(f + g)' = f' + g'$ .

2. For  $fg$ ,

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(z+h)g(z+h) - f(z)g(z)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(z+h)g(z+h) + f(z)g(z+h) - f(z)g(z+h) - f(z)g(z)}{h} \\ &= \lim_{h \rightarrow 0} \left[ \frac{f(z+h) - f(z)}{h} g(z+h) + f(z) \frac{g(z+h) - g(z)}{h} \right] \\ &= f'(z)g(z) + f(z)g'(z) \end{aligned}$$

Therefore,  $(fg)' = f'g + fg'$ .



*Case 2:  $h \rightarrow 0$  via the imaginary axis*

In this case,  $h = 0 + iy$  and  $y \rightarrow 0 \in \mathbb{R}$ . In a similar fashion, Equation (6.5) becomes

$$\begin{aligned} f'(z_0) &= \lim_{y \rightarrow 0} \left[ \frac{u(x_0, y_0 + y) - u(x_0, y_0)}{iy} + \frac{v(x_0, y_0 + y) - v(x_0, y_0)}{y} \right] \\ &= \frac{1}{i} \cdot \frac{\partial u}{\partial y} \Big|_{(x_0, y_0)} + \frac{\partial v}{\partial y} \Big|_{(x_0, y_0)} \end{aligned} \quad (6.7)$$

Note that since  $f'(z_0)$  exists, the real and imaginary part of Equation (6.6) and Equation (6.7) must equate. Also note that  $\frac{1}{i} = -i$ . With that, we obtain the following theorem.

**Theorem 6.2.1 (Cauchy-Riemann Equations)**

*If  $f(z)$  is holomorphic at  $z_0 = x_0 + iy_0 \in \mathbb{C}$  where  $x_0, y_0 \in \mathbb{R}$ , then, at  $(x_0, y_0)$ ,*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \quad (6.8)$$

# Chapter 7

## Lecture 7 Jan 17 2018

### 7.1 Differentiability (Continued)

#### 7.1.1 Cauchy-Riemann Equations (Continued)

It is natural to wonder if the **converse** of Theorem 6.2.1 is true. We present the following example.

**Example 7.1.1**

Let

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Check if

1.  $f$  is holomorphic at 0.
2. Theorem 6.2.1 holds at  $(0,0)$ .

**Proof**

1. Observe that by letting  $h = x_h + iy_h$  where  $x_h, y_h \in \mathbb{R}$ ,

$$\lim_{h \rightarrow 0} \frac{\overline{0+h}^2 - 0}{0+h} = \lim_{h \rightarrow 0} \frac{\bar{h}^2}{h} = \lim_{x_h + iy_h \rightarrow 0} \left( \frac{x_h - iy_h}{x_h + iy_h} \right)^2$$

Consider  $y_h = kx_h$ , for  $k \in \mathbb{R} \setminus \{0\}$ . Then

$$\lim_{x_h \rightarrow 0} \left( \frac{x_h - ikx_h}{x_h + ikx_h} \right)^2 = \left( \frac{1 - ik}{1 + ik} \right)^2,$$



where we see that the limit depends on the value of  $k$ . Therefore, the limit DNE. Hence  $f$  is not holomorphic at 0.

2. Let  $z = x + iy$  for  $x, y \in \mathbb{R}$ . Then

$$\frac{\bar{z}^2}{z} = \frac{(x - iy)^2}{x + iy} = \frac{(x - iy)^3}{x^2 + y^2} = \frac{x^3 - 3xy^2}{x^2 + y^2} + i \frac{(-3x^2y + y^3)}{x^2 + y^2}$$

Therefore, we obtain

$$u(x, y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$v(x, y) = \begin{cases} \frac{y^3 - 3x^2y}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Observe that

$$\left. \frac{\partial u}{\partial x} \right|_{(0,0)} = \lim_{x \rightarrow 0} \frac{u(x, 0) - u(0, 0)}{x} = 1$$

$$\left. \frac{\partial v}{\partial y} \right|_{(0,0)} = \lim_{y \rightarrow 0} \frac{v(0, y) - v(0, 0)}{y} = 1$$

and

$$\left. \frac{\partial u}{\partial y} \right|_{(0,0)} = \lim_{y \rightarrow 0} \frac{u(0, y) - u(0, 0)}{y} = 0$$

$$\left. \frac{\partial v}{\partial x} \right|_{(0,0)} = \lim_{x \rightarrow 0} \frac{v(x, 0) - v(0, 0)}{x} = 0$$

satisfies Equation (6.8).

This illustrates that the converse of Theorem 6.2.1 is not true. We will, however, show that the converse will be true given an extra condition.

### Theorem 7.1.1 (Conditional Converse of CRE)

Let  $z_0 = x_0 + iy_0 \in \Omega \subseteq \mathbb{C}$ ,  $x_0, y_0 \in \mathbb{R}$ , and  $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $f = u + iv : \Omega \rightarrow \mathbb{C}$ . If

1. the partials of  $u, v$  exist in a neighbourhood of  $(x_0, y_0)$ ,
2. the partials of  $u, v$  are continuous at  $(x_0, y_0)$ , and
3.  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$  and  $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$  at  $(x_0, y_0)$ ,

then  $f$  is holomorphic at  $z_0$ .

A proof of the theorem is in page 36 of Newman and Bak (recommended text of PMATH352W18). I may include the proof whenever I am free.

### 7.1.2 Power Series

#### Definition 7.1.1 (Power Series)

A **power series** in  $\mathbb{C}$  is an infinite series of the form

$$\sum_{n \in \mathbb{N}} c_n z^n, \quad (7.1)$$

where each  $c_n \in \mathbb{C}$  is the coefficient of  $z$  of the  $n$ -th power.

In this subsection, we are interested to see if Equation (7.1) converges.

Recall the notion of convergence in series from  $\mathbb{R}$ . Equation (7.1) converges if the sequence of partial sums  $\{S_N\}$  converges as  $N \rightarrow \infty$ , where

$$S_N := \sum_{n=0}^N c_n z^n$$

In other words, using the same definition of  $S_N$ ,

$$\begin{aligned} \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \setminus \{0\} \quad \forall n > N \\ |S_n - L| < \varepsilon \end{aligned}$$

where  $L \in \mathbb{C}$  is the limit that the sequence converges to.

We also know that Equation (7.1) converges absolutely if  $\sum_{n=0}^{\infty} |c_n| |z|^n$  converges. This is a stronger statement (i.e. absolute convergence  $\implies$  convergence)

$$\because \left| \sum_{n=0}^N c_n z^n \right| \leq \sum_{n=0}^N |c_n| |z|^n \quad \text{for each } N \in \mathbb{N}$$

#### Example 7.1.2

$\sum_{n=0}^{\infty} z^n$  converges absolutely for  $|z| < 1$ .

Note that the partial sum of a geometric series is

$$\sum_{n=0}^N r^n = \frac{1 - r^{N+1}}{1 - r}$$

and so the limit as  $N \rightarrow \infty$  exists if  $|r| < 1$ , and hence we see that

$$\sum_{n=0}^{\infty} r^n \rightarrow \frac{1}{1 - r}$$

if  $|r| < 1$  as  $N \rightarrow \infty$ .

However, if  $|z| = 1$ , the power series diverges.

Another note that we shall point out is that if Equation (7.1) converges absolutely for some  $z_0 \in \mathbb{C}$ , then it converges absolutely for any  $z$  where  $|z| < |z_0|$ .

These notions, in turn, begs the question of **what is the largest possible  $|z_0|$  for the series to converge absolutely.**

# Chapter 8

## Lecture 8 Jan 19 2018

### 8.1 Power Series (Continued)

#### 8.1.1 Radius of Convergence

##### Theorem 8.1.1 (Convergence in the Radius of Convergence)

For any power series  $\sum_{n \in \mathbb{N}} c_n z^n$ ,  $\exists 0 \leq R < \infty$ , such that

1.  $|z| < R \implies$  series converges absolutely.
2.  $|z| > R \implies$  series diverges.

Moreover,  $R$  is given by **Hadamard's Formula**:

$$\frac{1}{R} := \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}} \quad (8.1)$$

##### Remark

1.  $R$  is called the **radius of convergence** of the series.  $\{z \in \mathbb{C} : |z| < R\}$  is called the disk of convergence of the series.
2. Recall the definition of the **limit supremum**

$$\limsup_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} \left( \sup_{m \geq n} a_m \right) \quad (8.2)$$

which we may colloquially say as the “highest peak ‘reached’ by  $a_n$ ’s as  $n \rightarrow \infty$ ”

**Proposition 8.1.1 (A Property of limsup)**

$$\begin{aligned} \forall \{a_n\}_{n \in \mathbb{N}} \quad L := \limsup_{n \rightarrow \infty} a_n &\implies \\ \forall \varepsilon > 0 \quad \exists N > 0 \quad \forall n > N & \\ L - \varepsilon < a_n < L + \varepsilon & \end{aligned}$$

(Proof to be included)

**Proof (Theorem 8.1.1)**

Let  $L := \frac{1}{R} = \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}$ . Clearly,  $L \geq 0$ .

1. Suppose  $|z| < R$ .  $\exists \varepsilon > 0, r := |z|(L + \varepsilon)$  such that  $0 < r < 1$ . By Proposition 8.1.1,  $\exists N \in \mathbb{N}, \forall n > N, |c_n|^{\frac{1}{n}} < L + \varepsilon$ .

Now since  $L = \frac{1}{R}$ ,

$$\sum_{n=N}^{\infty} |c_n| |z|^n = \sum_{n=N}^{\infty} (|c_n|^{\frac{1}{n}} |z|)^n < \sum_{n=N}^{\infty} r^n$$

and since  $0 < r < 1$ , the final summation converges (as it is a geometric sum). Thus by comparison test,  $\sum_{n=N}^{\infty} |c_n| |z|^n$  converges.

We may also proceed with noticing that the partial sum of  $\sum_{n=N}^{\infty} |c_n| |z|^n$  is **bounded and monotonic**, which shows that the series converges.

2. Suppose  $|z| > R$ .  $\exists \varepsilon > 0, r := |z|(L - \varepsilon)$  such that  $r > 1$ . By Proposition 8.1.1,  $\exists N \in \mathbb{N}, \forall n > N, |c_n|^{\frac{1}{n}} > L - \varepsilon$ . Then analogous to the proof above,

$$\sum_{n=N}^{\infty} |c_n| |z|^n = \sum_{n=N}^{\infty} (|c_n|^{\frac{1}{n}} |z|)^n > \sum_{n=N}^{\infty} r^n$$

where the final summation diverges, and thus implying that  $\sum_{n=N}^{\infty} |c_n| |z|^n$  diverges.

**Theorem 8.1.2 (Power function, holomorphic function, region of convergence)**

Suppose  $f(z) = \sum_{n \in \mathbb{N}} c_n z^n$  has a radius of convergence  $R \in \mathbb{R}$ . Then  $f'(z)$  exists and equals

$$\sum_{n=1}^{\infty} n c_n z^{n-1}$$

throughout  $|z| < R$ .

Moreover,  $f'$  has the **same radius of convergence** as  $f$ .





# Chapter 9

## Lecture 9 Jan 22 2018

### 9.1 Power Series (Continued 2)

#### 9.1.1 Radius of Convergence (Continued)

##### Example 9.1.1

Let  $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ . To find the radius of convergence, we use Hadamard's Formula:

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} \left( \frac{1}{n} \right)^{\frac{1}{n}} = 1 \quad \because \lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$$

Therefore  $R = 1$ . Thus, by *Theorem 8.1.1*,  $f$  converges absolutely when  $|z| < 1$  and diverges when  $|z| > 1$ . As for the boundary, i.e.  $|z| = 1$ , consider the following two cases:

1. If  $z = 1$ , then  $f(1) = \sum_{n=1}^{\infty} \frac{1}{n}$  is a **harmonic series**, and hence  $f$  diverges.
2. If  $z = i$ , then

$$\begin{aligned} f(i) &= \sum_{n=1}^{\infty} \frac{i^n}{n} \\ &= i - \frac{1}{2} + \frac{-i}{3} + \frac{1}{4} + \frac{i}{5} - \frac{1}{6} \\ &= \left( -\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \dots \right) + i \left( 1 - \frac{1}{3} + \frac{1}{5} + \dots \right). \end{aligned}$$

Observe that both the real and imaginary parts are alternating series where the absolute values of each term is decreasing, which, by the **alternating series test**, converge. Thus in this case,  $f$  converges.



Therefore, we observe that **both convergence and divergence may occur** on the boundary, depending on the value of  $z$ .

### Note

We may not always exchange the position of  $\lim$  and  $\sum_{a=1}^b$  when we consider an infinite sum (i.e.  $b = \infty$ ). Here's an example why this is true. Consider the function  $f(x) = \sum_{n=1}^{\infty} (x^n - x^{n-1})$  for  $|x| < 1$ . Is

$$\lim_{x \rightarrow 1} \sum_{n=1}^{\infty} (x^n - x^{n-1}) = \sum_{n=1}^{\infty} \lim_{x \rightarrow 1} (x^n - x^{n+1})$$

true?

Clearly, RHS is 0. For LHS, note that

$$\begin{aligned} f(x) &= \lim_{N \rightarrow \infty} \sum_{n=1}^N (x^n - x^{n+1}) \\ &= \lim_{N \rightarrow \infty} (x - x^2 + x^2 - x^3 + \dots + x^N - x^{N+1}) \\ &= \lim_{N \rightarrow \infty} (x - x^{N+1}) = x. \end{aligned}$$

So,

$$LHS = \lim_{x \rightarrow 1} x = 1$$

And we see that  $RHS \neq LHS$ .

### Definition 9.1.1 (Entire Function)

A function  $f$  is said to be **entire** if  $f$  is holomorphic in **the entire complex plane**.

### Exercise 9.1.1

Define  $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ . Show that

1. the radius of convergence of this series is  $\infty$ , and hence that  $e^z$  is an entire function.  
(Hint: Use **Stirling's formula**:  $n! \sim (\frac{n}{e})^n \sqrt{2\pi n}$ )
2.  $(e^z)' = e^z$

### Solution

1. Using Stirling's formula, note that we have

$$e^z = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2\pi n}} \left(\frac{ez}{n}\right)^n$$



# Chapter 10

## Lecture 10 Jan 24 2018

### 10.1 Power Series (Continued 3)

#### 10.1.1 Radius of Convergence (Continued 2)

A power series is infinitely  $\mathbb{C}$ -differentiable in its radius of convergence. All its derivatives are also power series, obtained by term-wise differentiation.

E.g.

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \quad \text{then} \quad f^{(2)}(z) = \sum_{n=0}^{\infty} n(n-1)c_n z^{n-2}$$

In general, we may have  $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ , which is a power series centered at  $z_0 \in \mathbb{C}$ . Then, as before, the radius of convergence of this power series is given by

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}$$

So instead of having the disc of convergence centered around 0, we now have one that is centered around  $z_0$ .

#### **Corollary 10.1.1 (Corollary of Theorem 8.1.2)**

*From Theorem 8.1.2, we have shown that*

$f(z)$  has a power series expansion at  $z_0$   
 (i.e.  $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$  in some  
 neighbourhood of  $z_0$ ) with radius of  
 convergence  $R > 0$ 
 $\implies$ 
 $f$  is holomorphic at  $z_0$

The converse of the statement above is true, i.e.

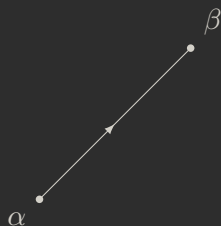
$f$  is holomorphic at  $z_0$ 
 $\implies$ 
 $f(z)$  has a power series expansion at  $z_0$   
 (i.e.  $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$  in some  
 neighbourhood of  $z_0$ ) with radius of  
 convergence  $R > 0$

This converse, however, is not possible to be proven given the current tools on our belt. And so we now have to venture into integrals in  $\mathbb{C}$ .

## 10.2 Integration in $\mathbb{C}$

### 10.2.1 Curves and Paths

Before we begin with the definition of a curve in  $\mathbb{C}$ , let us consider how a straight line should be described as a vector-valued function in the complex plane. For instance, if we have two points  $\alpha, \beta \in \mathbb{C}$ , and we want to describe the straight line connecting the two.



Let  $\gamma$  be the function that describes this line. We may then define  $\gamma : [0, 1] \rightarrow \mathbb{C}$  to be either

$$\gamma(t) = \alpha + (\beta - \alpha)t \quad \text{or} \quad \gamma = \alpha(1 - t) + \beta t.$$

We would then have the following mapping:

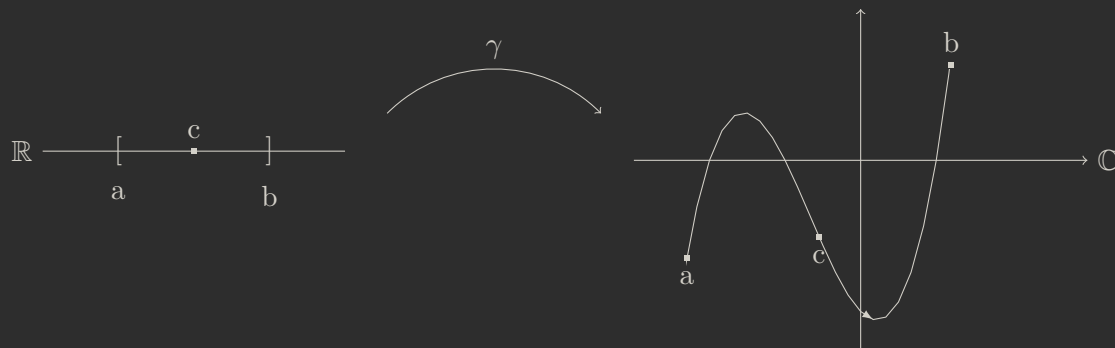


Figure 10.1: Mapping from  $\mathbb{R} \rightarrow \mathbb{C}$  with  $\gamma$ , which is called **the curve  $\gamma$**

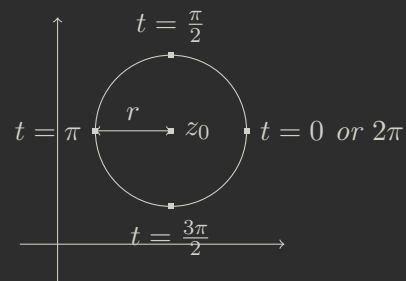
**Definition 10.2.1 (Curves in  $\mathbb{C}$ )**

A curve in  $\mathbb{C}$  is a continuous function,  $\gamma(t) : [a, b] \rightarrow \mathbb{C}$ , where  $a, b \in \mathbb{R}$ . The image of  $\gamma$  in  $\mathbb{C}$  is called  $\gamma^*$ .

**Example 10.2.1**

Let  $z_0 \in \mathbb{C}, r > 0$ .

1. Let  $\gamma : [0, 2\pi] \rightarrow \mathbb{C}$ , such that  $\gamma(t) = z_0 + re^{it}$ .
  2. Let  $\gamma' : [0, 1] \rightarrow \mathbb{C}$ , such that  $\gamma'(t) = z_0 + re^{2\pi it}$ .
- The two functions above describe a circle centered at  $z_0$  with radius  $r$ , anticlockwise-oriented.



We say that  $\gamma$  and  $\gamma'$  are equivalent parameterizations for the same oriented path.

**Definition 10.2.2 (Equivalent Parameterization)**

Let  $\gamma_1 : [a, b] \rightarrow \mathbb{C}, \gamma_2 : [c, d] \rightarrow \mathbb{C}$  where  $a, b, c, d \in \mathbb{R}$  describe the path  $\gamma^*$ . The two **parameterizations are said to be equivalent** if  $\exists h : [a, b] \rightarrow [c, d]$  that is a bijection and a continuous function such that

$$\gamma_1(t) = \gamma_2(h(t))$$

where  $t \in [a, b]$ .

**Note**

We will not look at functions like the Weierstrass function in this course.

**Definition 10.2.3 (Smooth Curve)**

Let  $\gamma : [a, b] \rightarrow \mathbb{C}$ ,  $a, b \in \mathbb{C}$ .  $\gamma$  is said to be smooth if its derivative  $\gamma'$  exists and is continuous on  $[a, b]$  and  $\forall t \in [a, b], \gamma'(t) \neq 0$ .

**Definition 10.2.4 (Piecewise Smooth)**

Let  $\gamma : [a, b] \rightarrow \mathbb{C}$ .  $\gamma$  is said to be piecewise smooth if it is smooth on  $[a, b]$  except on finitely many points in  $[a, b]$ .

**Remark**

Piecewise smooth curves shall be called paths.

**10.2.2 Integral****Definition 10.2.5 (Contour)**

Given a path  $\gamma : [a, b] \rightarrow \mathbb{C}$  and  $f : \mathbb{C} \rightarrow \mathbb{C}$ , a function continuous on  $\gamma$ . We define the integral  $f$  along  $\gamma$ , called a **contour**, as

$$\int_{\gamma} f(z) dz := \int_a^b f(\gamma(t)) \gamma'(t) dt \quad (10.1)$$

where we let  $z = \gamma(t)$  and hence  $dz = \gamma'(t)dt$ .

**Remark**

1. Suppose  $g$  is a complex-valued function, then

$$\int_a^b g(t) dt = \int_a^b \operatorname{Re}(g(t)) dt + i \int_a^b \operatorname{Im}(g(t)) dt$$

2. The integral of  $f$  along  $\gamma$  can be shown to be independent of the chosen parameterization for  $\gamma^*$ .

**Proof**

Let  $a, b, c, d \in \mathbb{R}$ ,  $\gamma_1 : [a, b] \rightarrow \mathbb{C}$ ,  $\gamma_2 : [c, d] \rightarrow \mathbb{C}$  describe the same path  $\gamma^*$ . By Definition 10.2.2, define a bijection  $h : [a, b] \rightarrow [c, d]$  that is a continuous function such that  $t \mapsto \tau$ , so that

$$\gamma_1(t) = \gamma_2(h(t)) = \gamma(\tau).$$

Note that

$$\begin{aligned}\gamma_1'(t) &= h'(t)\gamma_2'(h(t)) \text{ and} \\ h(t) = \tau &\implies h'(t)dt = d\tau.\end{aligned}$$

Now since  $h$  is a bijection, we claim that  $h(a) = c$  while  $h(b) = d$ .

We know that  $h$  cannot be a constant function. Suppose  $h$  is an increasing function, then since  $a \leq b$  and  $c \leq d$ , it is clear that  $h(a) = c$  and  $h(b) = d$ . Similarly, if  $h$  is a decreasing function, then  $h(a) = d$  and  $h(b) = c$ . But this is a contradiction to our supposition that  $\gamma_1$  and  $\gamma_2$  describe the same orientation. Thus  $h$  must be an increasing function, and hence we have  $h(a) = c$  and  $h(b) = d$ .

*(This can be more rigorous but that is an easy proof, and we may use perhaps the Approximation Property of  $\mathbb{R}$  to that end, which is a fun exercise that shall not be included within these covers.)*

Now

$$\begin{aligned}\int_{\gamma_1} f(z) dz &= \int_a^b f(\gamma_1(t))\gamma_1'(t)dt \\ &= \int_a^b f(\gamma_2(h(t)))h'(t)\gamma_2'(h(t))dt \\ &= \int_c^d f(\gamma_2(\tau))\gamma_2'(\tau)d\tau \\ &= \int_{\gamma_2} f(z) dz\end{aligned}$$

This completes the proof. □

# Chapter 11

## Lecture 11 Jan 26 2018

### 11.1 Integration in $\mathbb{C}$ (Continued)

#### 11.1.1 Integral (Continued)

**Note (Recall)**

Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a piecewise smooth curve. For a function  $f$  that is continuous on  $\gamma$ , we defined

$$\begin{aligned}\int_{\gamma} f(z) dz &= \int_a^b f(\gamma(t)) \gamma'(t) dt \\ &= \int_a^b \operatorname{Re} \left( f(\gamma(t)) \gamma'(t) \right) dt + i \int_a^b \operatorname{Im} \left( f(\gamma(t)) \gamma'(t) \right) dt\end{aligned}$$

and have

$$\begin{aligned}\gamma'(t) &= u'(t) + iv'(t) \\ \text{if } \gamma(t) &= u(t) + iv(t)\end{aligned}$$

**Example 11.1.1**

Let  $f(z) = f(x + iy) = x^2 + y^2$  be continuous along  $\gamma : [0, 1] \rightarrow \mathbb{C} \ t \mapsto t + it$ . Evaluate  $\int_{\gamma} f(z) dz$ .



**Solution**

$$\begin{aligned}
 \int_{\gamma} f(z) dz &= \int_0^1 f(t+it)(1+i)dt \\
 &= (1+i)^2 \int_0^1 t^2 dt \\
 &= (1+i)^2 \cdot \frac{1}{3} t^3 \Big|_0^1 \\
 &= \frac{2i}{3}
 \end{aligned}$$

**Example 11.1.2**

$\forall n \in \mathbb{Z}$ , evaluate  $\int_{\gamma} z^n dz$  that is continue on the path  $\gamma$  that describes any circle centered at origin oriented anticlockwise.

**Solution**

Let  $R \in \mathbb{R}$ , and define

$$\begin{aligned}
 \gamma : [0, 1] &\rightarrow \mathbb{C} \quad t \mapsto Re^{2\pi it} \\
 \gamma'(t) &= 2R\pi ie^{2\pi it} = 2\pi i\gamma(t)
 \end{aligned}$$

Then

$$\begin{aligned}
 \int_{\gamma} z^n dz &= \int_0^1 R^n e^{2\pi int} \cdot 2\pi i \cdot Re^{2\pi it} dt \\
 &= 2\pi i R^{n+1} \int_0^1 e^{2\pi i(n+1)t} dt \\
 &= \begin{cases} \frac{R^{n+1}}{n+1} e^{2\pi i(n+1)t} \Big|_0^1 & \text{if } n \in \mathbb{Z} \setminus \{-1\} \\ 2\pi i t \Big|_0^1 & \text{if } n = -1 \end{cases} \\
 &= \begin{cases} \frac{R^{n+1}}{n+1} (e^{2\pi i(n+1)} - 1) & \text{if } n \in \mathbb{Z} \setminus \{-1\} \\ 2\pi i & \text{if } n = -1 \end{cases} \quad \because e^{2\pi ki} \equiv 1 \pmod{2\pi} \\
 &= \begin{cases} 0 & \text{if } n \in \mathbb{Z} \setminus \{-1\} \\ 2\pi i & \text{if } n = -1 \end{cases}
 \end{aligned}$$

Note that our final answer does not depend on  $R$ , the radius of the circle.

**Proposition 11.1.1 (Properties of integrals in  $\mathbb{C}$ )**

1. **(Linearity)** Let  $\alpha, \beta \in \mathbb{C}$ .  $\int_{\gamma} (\alpha f(z) + \beta g(z)) dz = \alpha \int_{\gamma} f(z) dz + \beta \int_{\gamma} g(z) dz$ .







# Chapter 12

## Lecture 12 Jan 29th 2018

### 12.1 Integration in $\mathbb{C}$ (Continued 2)

#### 12.1.1 Fundamental Theorem of Calculus

To simplify statements from hereon, we shall use the following notations.

##### **Notation**

Let  $\Omega \subseteq \mathbb{C}$  be an open set in  $\mathbb{C}$ . We denote  $f \in H(\Omega) \iff f$  is holomorphic on  $\Omega$ .

##### **Theorem 12.1.1 (Fundamental Theorem of Calculus)**

Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a path inside an open set  $\Omega \subseteq \mathbb{C}$ . Suppose  $f(z)$  is continuous on  $\gamma$ , and has an antiderivative  $F \in \Omega$ . Then

$$\int_{\gamma} f(z) dz = F(\gamma(b)) - F(\gamma(a)) \quad (12.1)$$

##### **Proof**

Let  $G = F \circ \gamma$  and suppose  $\gamma$  is a smooth function. Since  $\gamma$  is smooth,  $\gamma'$  exists and is continuous on  $[a, b]$  and  $\gamma'(t) \neq 0$  for all  $t \in [a, b]$ , and since  $f$  is continuous on  $[a, b]$ ,  $G(t) = F'(\gamma(t))\gamma'(t)$  is continuous as well.

Now

$$\begin{aligned}
 \int_{\gamma} f(z) dz &= \int_a^b f(\gamma(t)) \gamma'(t) dt \\
 &= \int_a^b F'(\gamma(t)) \gamma'(t) dt \\
 &= \int_a^b G'(t) dt \\
 &= G(b) - G(a) \quad \text{by applying FTC in } \mathbb{R} \text{ to real and imaginary parts} \\
 &= F(\gamma(b)) - F(\gamma(a))
 \end{aligned}$$

If  $\gamma$  is piecewise smooth, then we can simply apply the above to each of the smooth paths separately and sum up all of the integrals.  $\square$

### Definition 12.1.1 (Closed Path)

A path  $\gamma : [a, b] \rightarrow \mathbb{C}$  is said to be **closed** if  $\gamma(a) = \gamma(b)$ .

### Corollary 12.1.1 (Corollary of FTC)

If  $F \in H(\Omega)$ ,  $\Omega \subseteq \mathbb{C}$  (hence  $F'$  is continuous on  $\Omega$ ), then

$$\int_{\gamma} F'(z) dz = 0$$

on any closed path  $\gamma$  on  $\Omega$ .

### Proof

A closed path  $\gamma : [a, b] \rightarrow \mathbb{C}$  has  $\gamma(a) = \gamma(b)$ . By Theorem 12.1.1,  $\int_{\gamma} F'(z) dz = F(\gamma(b)) - F(\gamma(a)) = 0$  as required.  $\square$

### Example 12.1.1

Take  $f(z) = z^n$  where  $n \in \mathbb{Z} \setminus \{-1\}$  as in Example 11.1.2. Then  $f$  is continuous on  $\mathbb{C} \setminus \{0\}$  (**not sure why this would be problematic when we've already excluded -1 for  $n$** ). Then  $f = F'$  for  $F(z) = \frac{z^{n+1}}{n+1}$  and  $F \in H(\mathbb{C} \setminus \{0\})$ . Therefore by Corollary 12.1.1,  $\int_{\gamma} z^n dz = 0$  for any closed path  $\gamma$  not passing through 0.

If we do include  $-1$  for  $n$ , note that  $F'$  would not be continuous on 0, and thus the corollary would not apply. We have also shown in the earlier example that  $\int_{\gamma} \frac{1}{z} dz = 2\pi i$ .

### Note (Recall)

The **interior** of a set  $\Omega$  is defined as  $\{z \in \Omega : \forall \varepsilon > 0 \ B(z, \varepsilon) \subseteq \Omega\}$ , and denoted as  $\Omega^0$ .

**Theorem 12.1.2 (Goursat's Theorem / Cauchy's Theorem for a triangle)**

Let  $\Omega \subseteq \mathbb{C}$  be an open set. Suppose  $\Delta \subseteq \Omega$  is a closed triangle whose interior is also contained in  $\Omega$ . Let  $f \in H(\Omega)$ . Then

$$\int_{\Delta} f(z) dz = 0$$

This theorem holds more meaning than the presented statement, as it implies that, essentially, given any two points connected by two different paths in an open set in  $\mathbb{C}$ , and a function that is holomorphic over the two paths, the **two path integrals of the function will yield the same result!**

**Proof**

Let  $\Delta_1^{(1)}, \Delta_2^{(1)}, \Delta_3^{(1)}, \Delta_4^{(1)}$  be smaller triangles by bisecting each side of  $\Delta$ .  $\forall i \in \{1, 2, 3, 4\}$ , orient  $\Delta_i^{(1)}$  anticlockwise. Then we have

$$J := \int_{\Delta} f(z) dz = \sum_{i=1}^4 \int_{\Delta_i^{(1)}} f(z) dz \quad (12.2)$$

Note that there must at least one of the  $\Delta_i^{(1)}$  such that  $\left| \int_{\Delta_i^{(1)}} f(z) dz \right| \geq \frac{|J|}{4}$ , since  $\forall i \in \{1, 2, 3, 4\}$ ,  $\left| \int_{\Delta_i^{(1)}} f(z) dz \right| < \frac{|J|}{4}$  would contradict Equation (12.2). Without loss of generality, let  $\Delta_1^{(1)}$  be the largest triangle of the four.

Now note that each of the perimeter of  $\Delta_i^{(1)}$  is half of the perimeter of  $\Delta$ . Let  $\ell(x)$  be the perimeter of  $x$ . Continue with taking bisectors of  $\Delta_1^{(1)}, \Delta_1^{(2)}, \dots$  such that

$$\Delta \supseteq \Delta_1^{(1)} \supseteq \Delta_1^{(2)} \supseteq \dots,$$

then we have that for each  $j \in \mathbb{N} \setminus \{0\}$ ,  $\Delta_i^{(j)}$  is such that

$$\left| \int_{\Delta_i^{(j)}} f(z) dz \right| \geq \frac{|J|}{4^j}$$

and  $\ell(\Delta_i^{(j)}) = \frac{1}{2^j} \ell(\Delta)$ . By the **Nested Rectangle Theorem from Real Analysis**,  $\exists z_0 \in \mathbb{C}$  such that  $z_0 \in \Delta_i^{(j)}$  for all  $j \in \mathbb{N} \setminus \{0\}$  that is a limit point. Since  $z_0 \in \Omega \wedge f \in H(\Omega)$ , we have that

$$\begin{aligned} \forall z \in \Omega \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \\ 0 < |z - z_0| < \delta \implies \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \end{aligned}$$





# Tutorial Jan 31 2018

## Note

Consider the power series  $\sum_{n \geq 0} a_n(z - z_0)^n$  and let  $\frac{1}{R} := \limsup_{n \rightarrow \infty} \sup \sqrt[n]{|a_n|} \in [0, \infty)$ .

- If  $|z - z_0| < R$ ,  $\sum_{n \geq 0} a_n(z - z_0)^n$  converges absolutely.
- If  $|z - z_0| > R$ ,  $\sum_{n \geq 0} a_n(z - z_0)^n$  diverges.
- If  $0 < r < R$ , then  $\sum_{n \geq 0} a_n(z - z_0)^n$  converges uniformly on  $\{z : |z - z_0| < r\}$ .

## 12.2 Practice Problems

1. Parameterize the semicircle  $|z - 4 - 5i| = 3$  clockwise, starting from  $z = 4 + 8i$  to  $z = 4 + 2i$ .

### Solution

Let  $\gamma : [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow \mathbb{C}$  such that  $\gamma(t) = 3e^{-it} + 4 + 5i$ . Note that  $\gamma$  parameterizes the given semicircle:

$$\gamma\left(-\frac{\pi}{2}\right) = 4 + 8i$$

$$\gamma(0) = 7 + 5i$$

$$\gamma\left(\frac{\pi}{2}\right) = 4 + 2i$$

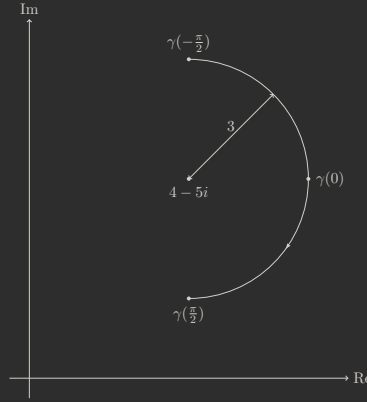


Figure 12.1: Semicircle  $|z - 4 - 5i| = 3$  oriented clockwise, parameterized by  $\gamma$

2. If the power series  $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$  centered at  $z_0$  has a non-zero radius of convergence, then show that

$$c_m = \frac{f^{(m)}(z_0)}{m!}$$

for any  $m \in \mathbb{Z}, m \geq 0$ , where  $f^{(m)}(z_0)$  denotes the  $m$ th derivative of  $f$  at  $z_0$ .

**Solution**

Since  $f(z)$  is a power series and the radius of convergence  $R \neq 0$ , by [Theorem 8.1.2](#),  $f(z)$  is  $\mathbb{C}$ -differentiable and each derivative has the same radius of convergence. By induction, it can be shown that

$$f^{(m)}(z) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} c_n (z - z_0)^{n-m}$$

Evaluating  $f^{(m)}$  at  $z_0$ , we have

$$\begin{aligned} f^{(m)}(z_0) &= \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} c_n (z_0 - z_0)^{n-m} \\ &= m! c_m \end{aligned}$$

where all terms above  $m$  are 0. Then we obtain

$$c_m = \frac{f^{(m)}(z_0)}{m!}$$

as desired. □

3. Let  $\gamma$  be the arc of the unit circle centered at the origin in the first quadrant oriented clockwise (from  $i$  to 1). Evaluate the integral

$$\int_{\gamma} \bar{z}^2 dz$$

by parameterizing the curve.

**Solution**

Consider the parameterization  $\gamma : [-\frac{\pi}{2}, 0] \rightarrow \mathbb{C}$  given by  $\gamma(t) = e^{-it}$ . Note that  $\overline{e^{-it}} = e^{it}$ . Then

$$\begin{aligned} \int_{\gamma} \bar{z}^2 dz &= \int_{-\frac{\pi}{2}}^0 e^{2it} \cdot (-ie^{-it}) dt \\ &= -i \int_{-\frac{\pi}{2}}^0 e^{it} dt \\ &= -e^{it} \Big|_{-\frac{\pi}{2}}^0 \\ &= -1 - i \end{aligned}$$

□

4. Evaluate the above integral by finding an antiderivative. (Hint: Use  $(\frac{z\bar{z}}{z})^2$ )

**Solution**

Note that  $z\bar{z} = |z|^2$ , so on the circle, we have  $\bar{z} = \frac{1}{z}$ . Thus the integral is equivalent to

$$\int_{\gamma} \frac{1}{z^2} dz$$

Note that the antiderivative of  $\frac{1}{z^2}$  is  $-\frac{1}{z}$ . Thus by Theorem 12.1.1,

$$\int_{\gamma} \bar{z}^2 dz = \int_{\gamma} \frac{1}{z^2} dz = F(\gamma(0)) - F\left(\gamma\left(-\frac{\pi}{2}\right)\right) = -\frac{1}{e^{-i(0)}} + \frac{1}{e^{-i(-\pi/2)}} = -1 - i$$

5. Let  $\{c_n\}_{n=0}^{\infty}$  be a sequence of positive real numbers such that

$$L = \lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n}$$

exists. Then show that

$$\lim_{n \rightarrow \infty} c_n^{\frac{1}{n}} = L$$

This shows that, when applicable, the **ratio test** can be used instead of the root test to calculate the radius of convergence of a power series.

**Solution**

Suppose that

$$L = \lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n}$$

exists. By definition, we have

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N$$

$$\left| \frac{c_n}{c_{n-1}} - L \right| < \varepsilon$$

Thus for  $n \geq N$ ,

$$c_n^{\frac{1}{n}} = \left( \frac{c_n}{c_{n-1}} \cdot \frac{c_{n-1}}{c_{n-2}} \cdots \frac{c_N}{c_{N-1}} \cdot c_{N-1} \right)^{\frac{1}{n}}$$

$$= \left( \frac{c_n}{c_{n-1}} \right)^{\frac{1}{n}} \left( \frac{c_{n-1}}{c_{n-2}} \right)^{\frac{1}{n}} \cdots \left( \frac{c_N}{c_{N-1}} \right)^{\frac{1}{n}} c_{N-1}^{\frac{1}{n}}$$

Now

$$(L - \varepsilon)^{\frac{1}{n}} (L - \varepsilon)^{\frac{1}{n}} \cdots (L - \varepsilon)^{\frac{1}{n}} c_{N-1}^{\frac{1}{n}} \leq c_n^{\frac{1}{n}} \leq (L + \varepsilon)^{\frac{1}{n}} (L + \varepsilon)^{\frac{1}{n}} \cdots (L + \varepsilon)^{\frac{1}{n}} c_{N-1}^{\frac{1}{n}}$$

$$(L - \varepsilon)^{\frac{n-N+1}{n}} c_{N-1}^{\frac{1}{n}} \leq c_n^{\frac{1}{n}} \leq (L + \varepsilon)^{\frac{n-N+1}{n}} c_{N-1}^{\frac{1}{n}}$$

Note that

$$\lim_{n \rightarrow \infty} (L - \varepsilon)^{\frac{n-N+1}{n}} c_{N-1}^{\frac{1}{n}} = L - \varepsilon$$

$$\lim_{n \rightarrow \infty} (L + \varepsilon)^{\frac{n-N+1}{n}} c_{N-1}^{\frac{1}{n}} = L + \varepsilon$$

Thus we have

$$L - \varepsilon \leq c_n^{\frac{1}{n}} \leq L + \varepsilon$$

$$\left| c_n^{\frac{1}{n}} - L \right| \leq \varepsilon$$

as desired. □

6. Find the radius of convergence of

- (a)  $\sum_{n=0}^{\infty} \frac{n^n z^n}{n!}$
- (b)  $\sum_{n=0}^{\infty} z^{2^n}$

**Solution**

(a) By Stirling's Approximation, i.e.  $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ , we have that Hadamard's formula is

$$\begin{aligned} \frac{1}{R} &= \limsup_{n \rightarrow \infty} \left| \frac{n^n}{n!} \right|^{\frac{1}{n}} \\ &= \limsup_{n \rightarrow \infty} \left| \frac{n^n}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} \right|^{\frac{1}{n}} \\ &= \limsup_{n \rightarrow \infty} \left| \frac{e^n}{\sqrt{2\pi n}} \right|^{\frac{1}{n}} \\ &= e \limsup_{n \rightarrow \infty} \left| \frac{1}{\sqrt{2\pi n}} \right|^{\frac{1}{n}} = e \end{aligned}$$

Therefore,  $R = \frac{1}{e}$ .

(b) *no solution yet: current problem, not being able to express the sum as a power series, in turn failing to get  $c_n$  which is needed for  $\frac{1}{R}$ .*

7. Show that for any path  $\gamma : [a, b] \rightarrow \mathbb{C}$  and  $f(z)$  continuous on  $\gamma$ , we have

$$\left| \int_{\gamma} f(z) dz \right| \leq \sup_{z \in \gamma} |f(z)| \int_a^b |\gamma'(t)| dt$$

**Solution**

$$\begin{aligned} LHS &= \left| \int_{\gamma} f(z) dz \right| \\ &= \left| \int_a^b f(\gamma(t)) \gamma'(t) dt \right| \text{ by definition} \\ &\leq \int_a^b |f(\gamma(t)) \gamma'(t)| dt \text{ by Item 2a of Proposition 11.1.1} \\ &\leq \int_a^b \sup_{z \in \gamma} |f(z)| |\gamma'(t)| dt \text{ since } |f(z)| \leq \sup_{z \in \gamma} |f(z)| \\ &= \sup_{z \in \gamma} |f(z)| \cdot \int_a^b |\gamma'(t)| dt = RHS \end{aligned}$$

# Chapter 13

## Lecture 13 Feb 9th 2018

### 13.1 Cauchy's Integral Formula

#### Definition 13.1.1 (Convex Set)

A set  $S \subseteq \mathbb{C}$  is called a **convex set** if the line segment joining any pair of points in  $S$  lies entirely in  $S$ .

#### Theorem 13.1.1 (Cauchy's Theorem for Convex Set)

Let  $\Omega \subseteq \mathbb{C}$  be a convex open set, and  $f \in H(\Omega)$ . Then

1.  $f = F'$  for some  $F \in H(\Omega)$ .
2.  $\int_{\gamma} f(z) dz = 0$  for any closed path  $\gamma \in \Omega$ .

#### Proof

Note that it is sufficient to prove 1 since  $1 \implies 2$  by Theorem 12.1.1.

Let  $a \in \Omega$ , and let  $[a, z]$  denote the straight line from  $a$  to  $z$ . Since  $\Omega$  is a convex set,  $[a, z]$  is in  $\Omega$ . Define  $F(z)^1 = \int_{[a, z]} f(z) dz^2$ .

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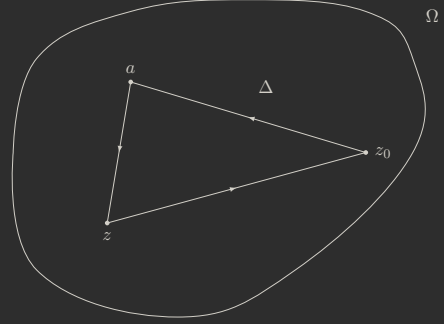
<sup>1</sup>It can be verified that  $F$  is continuous.

<sup>2</sup>This is a key step: defining an “antiderivative” as how we would expect it to be.

**WTS**  $F \in H(\Omega)$ ,  $F'(z_0) = f(z_0)$  for any  $z_0 \in \Omega$ .

Now by *Theorem 12.1.2*,

$$\begin{aligned} 0 &= \int_{\Delta} f(z) dz \\ &= \int_{[a,z]} f(z) dz + \int_{[z,z_0]} f(z) dz + \int_{[z_0,a]} f(z) dz \\ &= F(z) + \int_{[z,z_0]} f(z) dz + (-F(z_0)) \end{aligned}$$



This implies that

$$F(z) - F(z_0) = \int_{[z_0,z]} f(z) dz.$$

Divide both sides by  $z - z_0$ , then

$$\begin{aligned} \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) &= \frac{1}{z - z_0} \int_{[z_0,z]} f(z) dz - f(z_0) \\ &= \frac{1}{z - z_0} \int_{[z_0,z]} f(z) - f(z_0) dz \quad \text{since } \int_{[z_0,z]} dz = z - z_0 \end{aligned}$$

Since  $f \in H(\Omega)$  and is hence continuous, we have that

$$\begin{aligned} \forall \varepsilon > 0 \exists \delta > 0 \\ |z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon \end{aligned}$$

which in turn implies that

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| = \left| \frac{1}{z - z_0} \int_{[z_0,z]} [f(z) - f(z_0)] dz \right| \leq \frac{1}{|z - z_0|} \left| \int_{[z_0,z]} \varepsilon dz \right| = \varepsilon$$

Hence, by first principle,  $F'(z_0) = f(z_0)$ .  $\square$

### Theorem 13.1.2 (Cauchy's Integral Formula 1)

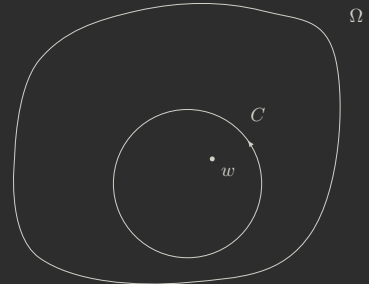
Let  $\Omega \subseteq \mathbb{C}$  be a convex open set, and  $C$  be a closed circle path in  $\Omega$ . If  $w \in \Omega \setminus \partial C$ , where  $\partial C$  is the **boundary of  $C$** , and  $f \in H(\Omega)$ , then

$$f(w) \text{Ind}_C(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - w} dz$$

where

$$\text{Ind}_C(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - w}$$

denotes the number of times the countour  $C$  winds around the point  $w$ .



is called the **index of  $w$  with respect to  $C$** , or the **winding number** of  $C$  around  $w$ .

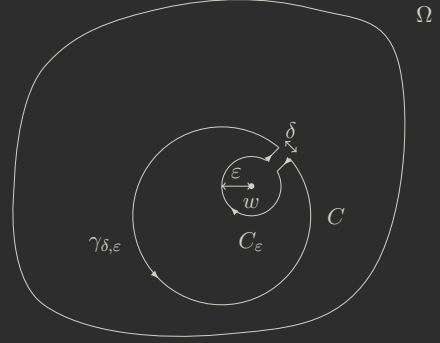
### Proof

Let  $w \in \Omega \setminus \partial C$ . Define

$$g(w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w \end{cases}$$

By the construction of  $g$ ,  $g$  is continuous on  $\Omega$ , and  $g \in H(\Omega \setminus \{w\})$ .

**We need to** construct a convex set  $\Omega' \subseteq \Omega$  that contains  $\gamma_{\delta,\varepsilon}$  such that  $g \in H(\Omega')$ .



We now follow a similar argument as in the proof for [Theorem 13.1.1](#). Let  $\varepsilon > 0$  such that  $\exists \delta > 0$ , so that we can define the “keyhole”  $\gamma_{\delta,\varepsilon}$  which omits  $w$ . Consider  $D(w,\varepsilon)$ , call the image of the border of  $D(w,\varepsilon)$  as  $C_\varepsilon$ , let  $\delta$  be the width of the “corridor”, and the two paths that are the “sides of the corridor” be called  $C_{\delta_1}, C_{\delta_2}$  respectively. Define  $G(z) = \int_{[a,z]} g(z) dz$ , where  $a$  and  $z$  are in the interior of  $C$  but not in the interior of  $C_\varepsilon$ . Then if we define a set  $\Omega'$  such that it contains the interior of  $\gamma_{\delta,\varepsilon}$ , we have that  $\Omega'$  is a convex open set, and  $G \in H(\Omega')$ . By [Theorem 13.1.1](#),  $G' = g$ .

Also from [Theorem 13.1.1](#), we have that  $\int_{\gamma_{\delta,\varepsilon}} g(z) dz = 0$  for any  $\varepsilon, \delta > 0$ . As  $\delta \rightarrow 0^+$ , we have that the integrals over  $C_{\delta_1}$  and  $C_{\delta_2}$  cancel out. Hence, we are left with

$$\int_C g(z) dz + \int_{C_\varepsilon} g(z) dz = 0$$

Let's put our focus on the smaller circle,  $C_\varepsilon$ . Now as  $\varepsilon \rightarrow 0^+$ ,  $\frac{f(z)-f(w)}{z-w} \rightarrow 0$ , and thus

$$\int_{C_\varepsilon} g(z) dz = \int_{C_\varepsilon} \frac{f(z) - f(w)}{z - w} dz \rightarrow 0$$

Therefore,

$$\int_C g(z) dz = 0$$

which implies, in the limit, that

$$\int_C \frac{f(z)}{z-w} dz = \int_C \frac{f(w)}{z-w} dz = f(w) \int_C \frac{dz}{z-w}$$

We now require  $\int_C \frac{dz}{z-w} = 2\pi i$ , but we shall prove for a more general case as a lemma.



# Chapter 14

## Lecture 14 Feb 12 2018

### 14.1 Cauchy's Integral Formula (Continued)

#### Lemma 14.1.1

*(Lemma and proof from Newman & Bak on Complex Analysis, 3rd Ed.)*

Suppose  $a \in C_\rho^0$  such that  $\exists \alpha \in C_\rho$  that is the center of the circle  $C_\rho$ , where  $\rho$  is the radius of  $C_\rho$ , and hence  $|a - \alpha| < \rho$ . Then

$$\int_{C_\rho} \frac{dz}{z - a} = 2\pi i$$

#### Proof

Let  $z \equiv \alpha + \rho e^{i\theta}$ , then  $dz = i\rho e^{i\theta} d\theta$ . Thus

$$\int_{C_\rho} \frac{dz}{z - \alpha} = \int_0^{2\pi} \frac{i\rho e^{i\theta}}{\rho e^{i\theta}} d\theta = 2\pi i$$

while

$$\int_{C_\rho} \frac{dz}{(z - \alpha)^{k+1}} = 0 \quad \text{for } k = 1, 2, 3, \dots \quad (14.1)$$

The Equation (14.1) follows not only from a direct evaluation of the integral

$$\int_{C_\rho} \frac{dz}{(z - \alpha)^{k+1}} = \int_0^{2\pi} \frac{i\rho e^{i\theta}}{(\rho e^{i\theta})^{k+1}} d\theta = \frac{i}{\rho^k} \int_0^{2\pi} e^{-ik\theta} d\theta = 0$$

but also the fact that  $\frac{1}{(z - \alpha)^{k+1}}$  is the derivative of  $-\frac{1}{k(z - \alpha)^k}$ , which can be verified to be holomorphic on  $C_\rho$ , which simply makes Equation (14.1) true by Theorem 12.1.1.

To evaluate  $\int_{C_\rho} \frac{dz}{z-a}$ , write

$$\begin{aligned} \frac{1}{z-a} &= \frac{1}{(z-\alpha) - (a-\alpha)} = \frac{1}{(z-\alpha)\left[1 - \frac{a-\alpha}{z-\alpha}\right]} \\ &= \frac{1}{z-\alpha} \cdot \frac{1}{1-\omega} \end{aligned}$$

where

$$\omega = \frac{a-\alpha}{z-\alpha} \text{ has fixed modulus } \frac{|a-\alpha|}{\rho} < 1 \text{ throughout } C_\rho \quad (14.2)$$

By Equation (14.2) and by the **Infinite Geometric Sum** that  $\frac{1}{1-\omega} = 1 + \omega + \omega^2 + \dots$ , we get

$$\begin{aligned} \frac{1}{z-a} &= \frac{1}{z-\alpha} \left[ 1 + \frac{a-\alpha}{z-\alpha} + \frac{(a-\alpha)^2}{(z-\alpha)^2} + \dots \right] \\ &= \frac{1}{z-\alpha} + \frac{a-\alpha}{(z-\alpha)^2} + \frac{(a-\alpha)^2}{(z-\alpha)^3} + \dots \end{aligned}$$

Since the convergence is uniform throughout  $C_\rho$ ,

$$\int_{C_\rho} \frac{1}{z-a} dz = \int_{C_\rho} \frac{1}{z-\alpha} dz + \sum_{k=1}^{\infty} \int_{C_\rho} \frac{(a-\alpha)^k}{(z-\alpha)^{k+1}} dz = 2\pi i$$

□

We may now continue with completing the previous proof.

### Proof (Continued - Theorem 13.1.2)

Lemma 14.1.1 completes the part where we required  $\int_C \frac{dz}{z-w} = 2\pi i$ .

We now have

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-w} dz$$

Now note that if we further generalize the number of times the contour  $C_\rho$  made around  $a$ , where in this case  $C_\rho$  is a closed path instead of a simple circle in  $\Omega$ , in Lemma 14.1.1, we would get  $\int_{C_\rho} \frac{dz}{z-a} = 2k\pi i$  where  $k$  would represent that number.

In this case, we would get

$$f(w)k = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-w} dz$$

where  $k = \text{Ind}_C(w) = \frac{1}{2\pi i} \int_C \frac{dz}{z-w}$  which represents the number of times the contour  $C$  winds around  $w$ . □

**Remark**

As noted, *Theorem 13.1.2* holds for any closed path  $\gamma \in \Omega$  instead of a simple circle  $C$ . If  $w \in \Omega \setminus \gamma^*$ , we get

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-w} dz = f(w) \text{Ind}_{\gamma}(w)$$

**Proposition 14.1.1 (Holomorphic Functions can be expressed as Power series)**

Let  $\Omega \subseteq \mathbb{C}$  be an open set,  $f \in H(\Omega)$ . Then  $f$  can be expressed as a power series.

**Proof**

$\forall w \in \Omega, \exists C \subseteq \Omega$  that is a closed circle path with  $w \in C^0$ . By *Theorem 13.1.2*, and since  $C$  is a circle, i.e. the contour winds around  $w$  only once, we have

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-w} dz.$$

Let  $w_0 \in \Omega$  be the center of  $C$ . Then  $\forall z \in \partial C, 0 < |w - w_0| < |z - w_0|^1$ . This implies that

$$\begin{aligned} 0 &< \frac{|w - w_0|}{|z - w_0|} < 1 \\ \implies \sum_{n=0}^{\infty} \left( \frac{w - w_0}{z - w_0} \right)^n &= \frac{1}{1 - \frac{w-w_0}{z-w_0}} = \frac{z - w_0}{z - w} \text{ by the Infinite Geometric Sum} \\ \implies \frac{1}{2\pi i} \int_C \frac{f(z)}{z - w} dz &= \frac{1}{2\pi i} \int_C \frac{f(z)}{z - w_0} \frac{z - w_0}{z - w} dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - w_0} \sum_{n=0}^{\infty} \left( \frac{w - w_0}{z - w_0} \right)^n dz \end{aligned}$$

Note that each of the terms in the integrand of the last expression are absolutely convergent, thus by **Fubini's Theorem**, we can interchange the summation and integral sign to get

$$f(w) = \sum_{n=0}^{\infty} \underbrace{\left[ \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - w_0)^{n+1}} dz \right]}_{a_n} (w - w_0)^n$$

which is a power series centered at  $w_0$  with coefficient  $a_n$ .

**Note (Recall)**

Consider the power series  $f(w) = \sum_{n=0}^{\infty} a_n(w - w_0)^n$ . Recall *Item 2* from *Section 12.2* that

$$a_n = \frac{f^{(n)}(w_0)}{n!}$$

---

<sup>1</sup>This is the key step

Applying this to *Proposition 14.1.1*, we get

$$\frac{f^{(n)}(w_0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - w_0)^{n+1}} dz$$

which holds for any  $w_0 \in \Omega$  by having  $C \subseteq \Omega$  centered at  $w_0$ .

**Theorem 14.1.1 (Cauchy's Integral Formula 2)**

Let  $\Omega \subseteq \mathbb{C}$  be open,  $f \in H(\Omega)$ . Then

1.  $\forall w \in \Omega$ ,  $f$  has a power series expansion at  $w$ .
2.  $f$  is differentiable infinitely many times in  $\Omega$ .
3.  $\forall C \subseteq \Omega$  that is a closed circle oriented anticlockwise, we have that  $\forall w \in C^0$ ,

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - w)^{n+1}} dz \quad (14.3)$$

**Remark**

*Item 3 is the actual Cauchy's Integral Formula in the theorem.*

**Proof**

We have shown 1 from *Proposition 14.1.1* and 2 from *Theorem 8.1.2*. It remains to prove 3, which we shall prove by induction.

When  $n = 0$ , it is simply *Theorem 13.1.2*. Suppose  $f$  has up to  $n - 1$  complex derivatives and that

$$f^{(n-1)}(w) = \frac{(n-1)!}{2\pi i} \int_C \frac{f(z)}{(z - w)^n} dz.$$

Consider  $h > 0$ , the difference of the quotient for  $f^{(n-1)}$  is

$$\frac{f^{(n-1)}(w - h) - f^{(n-1)}(w)}{h} = \frac{(n-1)!}{2\pi i} \int_C f(z) \frac{1}{h} \left[ \frac{1}{z - w - h} - \frac{1}{z - w} \right] dz \quad (14.4)$$

Note that

$$A^n - B^n = (A - B)(A^{n-1} + A^{n-2}B + \dots + AB^{n-2} + B^{n-1})$$

Let  $A = \frac{1}{z-w-h}$ ,  $B = \frac{1}{z-w}$ <sup>2</sup>, then the term in square brackets in *Equation (14.4)* becomes

$$\frac{h}{(z - w - h)(z - w)} [A^{n-1} + A^{n-2}B + \dots + AB^{n-2} + B^{n-1}]$$

---

<sup>2</sup>Key step

Thus as  $h \rightarrow 0$ , we have

$$f^{(n)} = \frac{(n-1)!}{2\pi i} \int_C f(z) \left[ \frac{1}{(z-w)^2} \right] \left[ \frac{n}{(z-w)^{n-1}} \right] dz = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-w)^{n+1}} dz$$

which completes the induction proof and proves 3.  $\square$

### Corollary 14.1.1 (Taylor Expansion of Entire Functions)

If  $f$  is an entire function, then  $\forall z_0 \in \mathbb{C}$ , we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots$$

which is a **Taylor Expansion** of  $f$  around  $z_0$ .

#### Proof

By Proposition 14.1.1, we have that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \left[ \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{n+1}} dw \right] (z - z_0)^n \\ &= \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_0} dw + \left[ \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^2} dw \right] (z - z_0) \\ &\quad + \left[ \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^3} dw \right] (z - z_0)^2 + \dots \\ &\quad + \left[ \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{k+1}} dw \right] (z - z_0)^k + \dots \end{aligned} \tag{14.5}$$

Now by Theorem 14.1.1, we have

$$\begin{aligned} f(z_0) &= f^{(0)}(z_0) = \frac{0!}{2\pi i} \int_C \frac{f(w)}{w - z_0} dw = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_0} dw \\ f^{(1)}(z_0) &= \frac{1!}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^2} dw \\ f^{(2)}(z_0) &= \frac{2!}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^3} dw \\ &\vdots \\ f^{(k)}(z_0) &= \frac{k!}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{k+1}} dw \\ &\vdots \end{aligned}$$



# Chapter 15

## Lecture 15 Feb 14 2018

### 15.1 Cauchy's Integral Formula (Continued 1)

At this point, it is important that we provide the following definition:

**Definition 15.1.1 (Analytic Functions)**

We say that  $f$  is *analytic* in  $\Omega$  if  $f$  has a power series expansion at every  $z \in \Omega$ .

**Remark**

1. We have proven, in the previous lecture, that Holomorphicity  $\implies$  Analyticity
2. Should we have defined, in *Theorem 14.1.1*, that the closed circle orients clockwise, then we would have a negative equation for *Equation (14.3)*.

#### 15.1.1 Applications of Cauchy's Integral Formula

**Exercise 15.1.1**

1. (*Cauchy's Inequality*)<sup>1</sup> Prove that  $\forall z_0 \in \mathbb{C} \forall R > 0 \in \mathbb{R} \forall f \in H(C = D(z_0, R))$

$$f^{(n)}(z_0) \leq \frac{n!}{R^n} \cdot \sup_{z \in \mathbb{C}} |f(z)|$$

**Proof**

From *Equation (14.3)*, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

---

<sup>1</sup>In a sense, this inequality implies that as we take higher derivatives, the value of the derivatives become smaller.

Parameterize  $C$  with  $\gamma : [0, 2\pi] \rightarrow \mathbb{C}$ , where  $t \mapsto z_0 + Re^{it}$ . Then

$$\begin{aligned} f^{(n)}(z_0) &= \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{(Re^{it})^{n+1}} Rie^{it} dt \\ |f^{(n)}(z_0)| &\leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{|f(z_0 + Re^{it})|}{R^n} dt \quad \because |Re^{it}| = R \\ &\leq \frac{n!}{2\pi R^n} \sup_{z \in C} |f(z)| \int_0^{2\pi} dt \\ &= \frac{n!}{R^n} \sup_{z \in C} |f(z)| \end{aligned}$$

This completes the proof. □

2. **(Liouville's Theorem)** A bounded entire function  $f : \mathbb{C} \rightarrow \mathbb{C}$  is a constant<sup>2 3</sup>.

**Proof**

Since  $f$  is entire, we may take  $R$ , in Item 1, to be any large value. Let  $M$  be the bound of  $f$ , i.e.  $\exists M \in \mathbb{C}, \forall z_0 \in \mathbb{C}, |f^{(n)}(z_0)| \leq \frac{n!}{R^n} \sup_{z \in \mathbb{C}} |f(z)| = \frac{n!}{R^n} \sup_{z \in \mathbb{C}} M$ . Let  $n = 1$ , then  $|f'(z_0)| = \frac{M}{R}$ . Thus we observe that  $R \rightarrow \infty \implies f'(z_0) \rightarrow 0$  for any  $z_0 \in \mathbb{C}$ . By A2Q5(a),  $f$  is a constant.

3. **(Parseval's Theorem)** Let  $\Omega \subseteq \mathbb{C}$  be open,  $f \in H(\Omega)$ ,  $\overline{D(z_0, R)} \subseteq \Omega$ . Then  $\forall z \in \overline{D(z_0, R)}, f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ , which in turn implies that<sup>4</sup>

$$\forall z \in \overline{D(z_0, R)} \quad f(z_0 + re^{i\theta}) = \sum_{n=0}^{\infty} c_n(re^{i\theta})^n \quad (\dagger)$$

---

<sup>2</sup>The theorem is not true in  $\mathbb{R}$ , since  $\sin x$  is a bounded function differentiable everywhere, but is not a constant.

<sup>3</sup>The theorem also implies that “trigonometry” in  $\mathbb{C}$  is unbounded, whatever the definition of “trigonometry” may be.

<sup>4</sup>This is why the  $L^2$ -norm is perserved, as seen in AMATH231.



Consider (the  $L^2$  norm)

$$\begin{aligned}
 & \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + re^{i\theta}) \right|^2 d\theta \\
 &= \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=0}^{\infty} c_n (re^{i\theta})^n \right|^2 d\theta \\
 &= \frac{1}{2\pi} \int_0^{2\pi} \left[ \sum_{n=0}^{\infty} c_n r^n e^{in\theta} \right] \left[ \sum_{m=0}^{\infty} \overline{c_m} r^m e^{-im\theta} \right] d\theta \\
 &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n \overline{c_m} r^{n+m} e^{i(n-m)\theta} d\theta
 \end{aligned}$$

Since the series are absolutely convergent, we may use Fubini's Theorem, and thus

$$\begin{aligned}
 &= \frac{1}{2\pi} \sum_{n,m=0}^{\infty} c_n \overline{c_m} r^{n+m} \int_0^{2\pi} e^{i(n-m)\theta} d\theta \\
 &= \begin{cases} \frac{1}{2\pi} \sum_{n,m=0}^{\infty} c_n \overline{c_m} r^{n+m} 2\pi & \text{if } n = m \\ \frac{1}{2\pi} \sum_{n,m=0}^{\infty} c_n \overline{c_m} r^{n+m} \frac{e^{i(n-m)\theta}}{i(n-m)} \Big|_0^{2\pi} = 0 & \text{if } n \neq m \end{cases} \\
 &= \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \quad \text{if } n = m
 \end{aligned}$$

Therefore, we have what is known as **Parseval's Identity**:

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + re^{i\theta}) \right|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \tag{15.1}$$

Parseval's Theorem states that:

$L^2$ -norm of LHS in Equation (15.1) =  $L^2$ -norm of RHS of Equation (†)

Before going into the next application, please see Lemma 15.1.1.

4. (**Maximum Modulus Principle**) Let  $\Omega \subseteq \mathbb{C}$  be open and connected, and  $f \in H(\Omega)$ . Then

$$\sup_{z \in \Omega} |f(z)| = \max_{z \in \partial\Omega} |f(z)|.$$

This implies that  $f$  cannot attain its maximum value in  $\Omega^0$ .

**Proof**

Suppose not, i.e.  $\exists z_0 \in \Omega^0, \forall z \in \Omega$  such that  $|f(z_0)| = \max_{z \in \Omega} |f(z)| \geq |f(z)|$

$$\implies \exists r > 0 \quad \overline{D(z_0, r)} \subseteq \Omega$$

$$\implies \forall z \in \overline{D(z_0, r)} \quad f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

Note that  $c_0 = \frac{f^{(0)}(z_0)}{0!} = f(z_0)$ . By *Item 3*,

$$\begin{aligned} \sum_{n=0}^{\infty} |c_n|^2 r^{2n} &= \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + re^{i\theta}) \right|^2 d\theta \\ \implies f(z_0)^2 + \sum_{n=1}^{\infty} |c_n|^2 r^{2n} &= \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + re^{i\theta}) \right|^2 d\theta \\ &\leq \frac{1}{2\pi} |f(z_0)|^2 (2\pi) \quad \because f(z_0) = \max_{z \in \Omega} |f(z)| \\ \implies f(z_0)^2 + \sum_{n=1}^{\infty} |c_n|^2 r^{2n} &\leq |f(z_0)|^2 \\ \implies \sum_{n=1}^{\infty} |c_n|^2 r^{2n} &\leq 0 \\ \implies c_1, c_2, \dots &= 0 \\ &\implies f \text{ is a constant in } \overline{D(z_0, r)} \\ &\implies f \text{ is a constant in } \Omega \text{ by Lemma 15.1.1} \end{aligned}$$

which is a contradiction. □

**Lemma 15.1.1 (Principle of Analytic Continuation)**

Let  $\Omega \subseteq \mathbb{C}$  be open and connected, and  $f \in H(\Omega)$ . Let  $Z(f) = \{a \in \Omega : f(a) = 0\}$ . Then either

- $Z(f) = \Omega$ , i.e.  $\forall z \in \Omega, f(z) = 0$ ; or
- $Z(f)$  has no limit point, i.e. points where  $f = 0$  are isolated

This is a powerful result, since if we can find a small region for where  $f$  is 0 in  $\Omega$ , then  $f$  would be 0 in the entirety of  $\Omega$ . If not, then  $f$  is only 0 at isolated points, i.e. points where  $f = 0$  are all apart from each other.

# Chapter 16

## Lecture 16 Feb 16th 2018

### 16.1 Cauchy's Integral Formula (Continued 3)

#### 16.1.1 Applications of Cauchy's Integral Formula (Continued)

##### Exercise 15.1.1 (Continued)

We shall restate the **Item 4** in the following manner.

4. **Maximum Modulus Principle (MMP)** Let  $\Omega \subseteq \mathbb{C}$ ,  $f \in H(\Omega)$ ,  $D_{z_0} = \overline{D(z_0, r)} \subseteq \Omega$ . Then  $|f(z_0)| \leq \max_{z \in \partial D_{z_0}} |f(z)|$  with

$$|f(z_0)| = \max_{z \in \partial D_{z_0}} |f(z)| \iff f \text{ is a constant on } \Omega$$

##### Remark

- (a) This implies that for a non-constant analytic function  $f$ ,  $\forall z \in \Omega^0$ ,  $f(z) \neq \max_{w \in \Omega} f(w)$ .
- (b) Since a global maximum is also a local maximum, we observe that for any smaller region  $\Omega_0 \subseteq \Omega$ ,  $f$  cannot attain its maximum value for any point in  $\Omega_0^0$ . This is a stronger statement than the our previous statement about the MMP.

##### Proof

Suppose for  $\nexists$  that  $f$  has a maximum in  $\Omega^0$ , say at  $z_0$ . Hence  $\exists r > 0$ ,  $D_{z_0} = \overline{D(z_0, r)}$  where

$$|f(z_0)| \geq \max_{z \in D_{z_0}} |f(z)|$$

On  $D_{z_0}$ , we have

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \quad (16.1)$$

Note that  $c_0 = f(z_0)$ . By *Item 3*, on  $D_{z_0}$ ,

$$\begin{aligned} \sum_{n=0}^{\infty} |c_n|^2 r^{2n} &= \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + re^{i\theta}) \right|^2 d\theta \quad \text{by Equation (15.1)} \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|^2 d\theta \quad \text{by Equation (16.1)} \\ &= |f(z_0)|^2. \end{aligned}$$

Then we have

$$\begin{aligned} |c_0|^2 + \sum_{n=1}^{\infty} |c_n|^2 r^{2n} &= |f(z_0)|^2 \\ |f(z_0)|^2 + \sum_{n=1}^{\infty} |c_n|^2 r^{2n} &= |f(z_0)|^2 \\ \sum_{n=1}^{\infty} |c_n|^2 r^{2n} &= 0 \end{aligned}$$

which  $\implies c_1 = c_2 = \dots = 0$ . Thus  $\forall z \in D_{z_0}$ ,  $f(z) \equiv c_0 \pmod{2\pi}$ . Then by *Lemma 15.1.1*, since  $f(z) - c_0$ , as  $f(z) - c_0$  contains  $\overline{D(z - 0, r)}$ , we see that  $f(z) - c_0 \equiv 0$  in  $\Omega$ , which implies the equality of *Item 4*.  $\square$

5. **Fundamental Theorem of Algebra (FTA)** Any polynomial  $P(z) \in \mathbb{C}[z]$  of degree greater than 1 has precisely  $n$  roots in  $\mathbb{C}$ , given by  $\alpha_1, \alpha_2, \dots, \alpha_n$ .  $P(z)$  can be factored as  $P(z) = A(z - \alpha_1) \dots (z - \alpha_n)$  for some  $A \in \mathbb{C}$ .

**Proof**

We may write  $P(z) = A(z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0)$ , which then

$$\frac{P(z)}{z^n} = A \left( 1 + \frac{a_{n-1}}{z} + \dots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right)$$

which then, by the Reverse Triangle Inequality,

$$\implies \left| \frac{P(z)}{z^n} \right| \geq |A| \left[ 1 - \frac{|a_{n-1}|}{|z|} - \dots - \frac{|a_1|}{|z^{n-1}|} - \frac{|a_0|}{|z^n|} \right] \quad (16.2)$$

So as  $|z| \rightarrow \infty$ ,  $\left| \frac{P(z)}{z^n} \right| \rightarrow |A|$ , from Equation (16.2). Since  $|z| \rightarrow \infty$ ,  $\exists R > 0$ ,  $\forall |z| > R$ , then  $\forall \theta \in [0, 2\pi]$ ,

$$\left| P(Re^{i\theta}) \right| = |P(z)| \geq \frac{|A|}{2} |z|^n \geq \frac{|A|}{2} R^n$$

Taking  $R$  to be even larger if necessary, we can get

$$\left| P(Re^{i\theta}) \right| \geq |P(0)| \quad (\dagger)$$

Suppose, for contradiction,  $P(z)$  has no root in  $\mathbb{C}$ . Then  $g(z) = \frac{1}{P(z)}$  is an entire function. By Equation  $(\dagger)$ , we have that  $|g(Re^{i\theta})| \leq |g(0)|$  for all  $\theta \in [0, 2\pi]$ . But this contradicts Item 4 unless if  $g(z)$  is constant on  $\mathbb{C}$ , which in turn implies that  $P$  is a constant, but that contradicts that  $P$  has degree greater than 1.

$\therefore P(z)$  has to have a zero in  $\mathbb{C}$ , say  $\alpha_1$ . This implies that

$$P(z) = A(z - \alpha_1)P_1(z)$$

where  $P_1(z) \in \mathbb{C}[z]$ . By repeatedly taking the above steps, inductively so, for  $P_1, P_2, \dots$ , the proof is completed.  $\square$

# Chapter 17

## Lecture 17 Feb 26th 2018

### 17.1 Analytic Continuity

We shall restate the important lemma that we have been using in the last two lectures, and proceed to prove this lemma.

**Lemma 17.1.1 (Principle of Analytic Continuity)**

Let  $\Omega \subseteq \mathbb{C}$  be open and connected, and  $f \in H(\Omega)$ . Let  $Z(f) = \{a \in \Omega : f(a) = 0\}$ . Then either

- $Z(f) = \Omega$ , i.e.  $\forall z \in \Omega, f(z) = 0$ ; or
- $Z(f)$  has no limit point, i.e. points where  $f = 0$  are isolated

**Proof**

Let  $z_0 \in Z(f)^*$ .

**Step 1:** Show that  $z_0 \in Z(f)^0$ , i.e.  $f$  is identically 0 on some  $\overline{D(z_0, r)} \subseteq \Omega$  for  $r > 0$ .

On  $\overline{D(z_0, r)}$ ,  $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ . Suppose  $f$  is not identically 0 on  $\overline{D(z_0, r)}$ . Then  $\exists m \in \mathbb{N}, c_m \neq 0, \forall j < m, c_j = 0$ , i.e.  $f(z) = c_m(z - z_0)^m + c_{m+1}(z - z_0)^{m+1} + \dots$

Define, in  $\Omega$ ,

$$g(z) = \begin{cases} \frac{f(z)}{(z - z_0)^m} & z \in \Omega \setminus \{z_0\} \\ c_m & z = z_0 \end{cases}$$

Clearly,  $g \in H(\Omega \setminus \{z_0\})$ . But on  $\overline{D(z_0, r)}$ ,

$$g(z) = c_m + c_{m+1}(z - z_0) + c_{m+2}(z - z_0)^2 + \dots$$

which implies  $g \in H(\Omega)$ . Now  $g(z_0) = c_m \neq 0$ , so there exists a neighbourhood  $U_{z_0}$  of  $z_0$ , such that  $g \neq 0$  on  $U_{z_0}$ .

$\forall a \neq z_0 \in Z(f)$ , we have that  $g(a) = 0$  by definition of  $Z(f)$ , which implies that  $a \notin U_{z_0}$ , which contradicts that  $z_0 \in Z(f)^*$ . This implies  $f \equiv 0$  in  $\overline{D(z_0, r)}$ .

**Step 2:**  $Z(f)^0$  is both open and closed.

Note that

$$Z(f)^0 := \left\{ a \in Z(f) : \exists r > 0, \overline{D(a, r)} \subseteq Z(f) \right\}$$

is open by definition.

**WTP**  $[Z(f)^0]^* \subseteq [Z(f)]^*$ .

From **Step 1**, we know that  $[Z(f)^0]^* \subseteq Z(f)^0$ . Thus  $Z(f)^0$  contains its limit points and is hence closed by definition.

**Step 3:**  $Z(f) = \emptyset$  or  $\Omega$ .

$\Omega$  is connected

$$\implies \Omega = Z(f)^0 \sqcup (Z(f)^0)^c$$

$$\implies (Z(f)^0)^c \text{ is open and closed by Step 2}$$

A connected set cannot be expressed as a disjoint union of non-trivial open sets. Therefore, either  $Z(f)^0 = \emptyset$  or  $Z(f)^0 = \Omega$ .

$$Z(f)^0 = \emptyset \implies Z(f)^* = \emptyset \text{ by Step 1} \implies Z(f) = \emptyset$$

$$Z(f)^0 = \Omega \implies Z(f) = \Omega \text{ by Step 1}$$

□

### Corollary 17.1.1 (Uniqueness of a Function)

Let  $\Omega \subseteq \mathbb{C}$  be open and connected.  $\forall f, g \in H(\Omega)$  with  $f(z) = g(z)$  for  $z \in \Omega_1 \subseteq \Omega$  where  $\Omega_1$  has limit points. Then  $\forall z \in \Omega$ ,  $f(z) = g(z)$ .

#### Proof

Apply Lemma 15.1.1 to the function  $f - g$ .

#### Remark

1. In  $\mathbb{C}$ , we cannot have two functions sharing a region of points in their images. (But this is possible in  $\mathbb{R}$ )

2. Suppose  $f \in H(\Omega)$ ,  $\Omega \subseteq \mathbb{C}$  is open and connected,  $F \in H(\Omega')$  with  $\Omega \subseteq \Omega'$ . If  $f, F$  agree on  $\Omega$ , then  $F$  is called an analytic continuation of  $f$  in  $\Omega'$  (i.e.  $F$  ‘extends’  $f$  in  $\Omega'$ ). Lemma 15.1.1 states that  $F$  is uniquely determined by  $f$ , i.e. there is a unique way to analytically ‘continue’  $f$ .

## 17.2 Morera’s Theorem

### Remark (Recall)

From Cauchy’s Theorem, we know that  $\forall f \in H(\Omega) \implies \forall \gamma \in \Omega \int \gamma f = 0$ . We used Goursat’s Theorem, i.e.  $\forall \Delta \in \Omega \int_{\Delta} f = 0$  to prove this, and in the process we constructed an antiderivative. Now, our question is, is the converse of the said Cauchy’s Theorem true?

Unfortunately for us, that is not true (**example needed**). But a “partial” converse exists.

### Theorem 17.2.1 (Morera’s Theorem)

Let  $f$  be continuous on  $\Omega \subseteq \mathbb{C}$ , which is an open set, and  $\forall \Delta \in \Omega, \int_{\Delta} f = 0$ , where  $\Delta$  is a triangular path. Then  $f \in H(\Omega)$ .

### Proof

Use the same construction as in Cauchy’s Theorem for Convex Sets to get an antiderivative  $F$  for  $f$ , where  $F \in H(\Omega)$ , i.e.

$$F(z) := \int_{[a,z]} f(z) dz$$

Then  $F'(z) = f(z)$ , which in turn implies that  $f \in H(\Omega)$  since  $F$  is  $\mathbb{C}$ -differentiable on  $\Omega$  by Theorem 14.1.1.



# Chapter 18

## Lecture 18 Feb 28th 2018

### 18.1 Winding Numbers

Recall Cauchy's Integral Formula. We claimed that

$$\text{Ind}_C(w) = \begin{cases} 1 & w \in C^0 \\ 0 & w \notin C \end{cases}$$

We will now formally define this index.

**Definition 18.1.1 (Winding Numbers)**

Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a closed and oriented anti-clockwise, and  $\gamma^*$  be the image of  $\gamma$  in  $\mathbb{C}$ . Let  $\Omega = \mathbb{C} \setminus \gamma^*$ .  $\forall w \in \Omega$ , define the index of  $w$  with respect to  $\gamma$  as

$$\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - w}$$

in which shall be called the winding number of  $\gamma$  around  $w$ .

**Theorem 18.1.1 (Winding Number Theorem)**

We shall use notation as the definition above.  $\text{Ind}_{\gamma}(w)$  is

1. always an integer;
2. constant on any connected component of  $\Omega$ ; and
3. zero on the unbounded component of  $\Omega$ .

**Note**

$\gamma$  is compact in  $\mathbb{C}$  (since it creates a ring from  $[a, b]$  under  $\gamma$ ). So for some disc  $D$ ,  $\gamma^* \subseteq D$ . Let  $\Omega \supset \mathbb{C} \setminus D$ , where we note that the contained set is connected and unbounded. Then  $\Omega$  contains one unbounded component, while other components of  $\Omega$  are inside  $D$ . Therefore, we know that components in  $D$  are bounded.

**Proof**

1. By definition,

$$\begin{aligned} \text{Ind}_\gamma(w) &= \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-w} \\ &= \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t) dt}{\gamma(t)-w} \end{aligned}$$

**WTS**  $\text{Ind}_\gamma(w) \in \mathbb{Z} \equiv \int_a^b \frac{\gamma'(t) dt}{\gamma(t)-w} \in 2\pi i \mathbb{Z}.$

Note that  $z \in 2\pi i \mathbb{Z} \iff e^z = 1$ . Thus it suffices to show that

$$e^{\int_a^b \frac{\gamma'(t) dt}{\gamma(t)-w}} = 1$$

**Idea:** Think of  $\exp\left(\int_a^u \frac{\gamma'(t) dt}{\gamma(t)-w}\right)$  as a function of  $u$ , call it  $\phi(u)$ . Then we just need to show that  $\phi(b) = 1$ . We know that  $\phi(a) = \exp\left(\int_a^a \dots\right) = 1$ . This motivates us to find the derivative of  $\phi$ .

Define  $\phi$  accordingly, and then since  $(e^{f(u)})' = e^{f(u)} \cdot f'(u)$ ,

$$\begin{aligned} \phi'(u) &= \phi(u) \cdot \frac{d}{du} \int_a^u \frac{\gamma'(t) dt}{\gamma(t)-w} \\ \text{by FTC} \implies \frac{\phi'(u)}{\phi(u)} &= \frac{\gamma'(u)}{\gamma(u)-w} \\ \implies \phi'(u)(\gamma(u)-w) - \gamma'(u)\phi(u) &= 0 \\ \implies \frac{d}{du} \left( \frac{\phi(u)}{\gamma(u)-w} \right) &= 0 \quad \text{by quotient rule} \\ \implies \frac{\phi(b)}{\gamma(b)-w} &= \frac{\phi(a)}{\gamma(a)-w} \quad \text{since } \frac{\phi(u)}{\gamma(u)-w} \text{ is a constant function of } u \\ \implies \phi(b) = \phi(a) = 1 &\quad \because \gamma \text{ is closed.} \end{aligned}$$

2. Note that  $\text{Ind}_\gamma(w)$  is continuous (**why?**) and takes only integer values, thus it must be constant on each open connected component<sup>1</sup> (**why?**).

---

<sup>1</sup>We may invoke Lemma 15.1.1 but it is, to an extent, unnecessary for such a powerful statement.

3. Note that

$$|\text{Ind}_\gamma(w)| = \frac{1}{2\pi} \left| \int_a^b \frac{\gamma'(t) dt}{\gamma(t) - w} \right|$$

Let  $w$  be in the unbounded component in the complement of  $\gamma$  such that  $|w| \rightarrow \infty$ .  
Then  $\forall t \in [a, b]$ ,  $\exists M > 0$  such that

$$\frac{1}{|\gamma(t) - w|} \leq \frac{1}{M}$$

which implies that

$$\begin{aligned} |\text{Ind}_\gamma(w)| &\leq \frac{1}{2\pi} \frac{1}{M} \cdot \underbrace{\int_a^b |\gamma'(t)| dt}_{\substack{\text{is a fixed constant} \\ \text{as } \gamma \text{ is a fixed path}}} \\ &\implies (|w| \rightarrow \infty \implies M \rightarrow \infty \implies |\text{Ind}_\gamma(w)| \rightarrow 0) \end{aligned}$$

Then by parts 1 and 2, the proof is completed. □

#### Remark

Note that by 2, we have that  $\forall w \in C^0$ ,

$$\frac{1}{2\pi i} \int_C \frac{dz}{z - w} = \frac{1}{2\pi i} \int_C \frac{dz}{z - z_0} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{Rie^{i\theta}}{Re^{i\theta}} d\theta = 1$$

where  $z_0$  is the center of the circle path  $C$ .