PMATH467 — Algebraic Geometry

Classnotes for Winter 2019

by

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Preface

The basic goal of the course is to be able to find **algebraic invariants**, which we shall use to classify topological spaces up to homeomorphism.

Other questions that we shall also look into include a uniqueness problem about manifolds; in particular, how many manifolds exist for a given invariant up to homeomorphism? We shall see that for a **2-manifold**, the only such manifold is the 2-dimensional sphere S^2. For a 4-manifold, it is the 4-dimensional sphere S^4. In fact, for any other n-manifold for $n > 4$, the unique manifold is the respective n-sphere. The problem is trickier with the 3-manifold, and it is known as the Poincaré Conjecture, solved in 2003 by Russian Mathematician Grigori Perelman. Indeed, the said manifold is homeomorphic to the 3-sphere.

For this course, you are expected to be familiar with notions from **real analysis**, such as topology, and concepts from **group theory**. Due to the structure of which this course is designed, each lecture may be much longer than reality, as I am also making heavy references to the recommended text that is Lee's Introduction to Topological Manifolds [1].

[1] Lee, J. M. (2000). *Graduate Texts in Mathematics: Introduction to Topological Manifolds*. Springer

The following topics shall be covered:

1. Point-Set Topology
2. Introduction to Topological Manifolds
3. Simplicial complexes & Introduction to Homology
4. Fundamental Groups & Covering Spaces
5. Classification of Surfaces

Feb 12th I have decided to delve deeper into the recommended text as the organization of the course demands. As so, changes might be made to earlier lectures as I go further down, so as to introduce the definitions and provide propositions at a timing deemed appropriate. To keep track of the changes, please look for **PMATH467** among the commits on `https://gitlab.com/japorized/TeX_notes/commits/master` using the provided filter. If you are unfamiliar with version controlling and writing in LaTeX, once you have found the lecture that you wish to compare, expand the **diff** for `classnotes.tex` if it is collapsed.

Basic Logistics for the Course

I shall leave this here for my own notes, in case something happens to my hard copy.

- OH: (Tue) 1630 - 1800, (Fri) 1245 - 1320
- OR: MC 6457
- EM: aaleyasin

Part I

Point-Set Topology

1 Lecture 1 Jan 07th

We will not be too rigorous in this part.

1.1 Euclidean Space

For any $(x_1, \ldots, x_m) \in \mathbb{R}^m$, we can measure its distance from the origin 0 using either

- $\|x\|_\infty = \max\{|x_i|\}$ (the supremum-norm);
- $\|x\|_2 = \sqrt{\sum (x_j)^2}$ (the 2-norm); or
- $\|x\|_p = \left(\sum |x_j|^p\right)^{\frac{1}{p}}$ (the p-norm),

where we may define a “distance” by

$$d_p(x, y) = \|x - y\|_p \,.$$

Definition 1 (Metric)

*Let X be an arbitrary space. A function $d : X \times X \to \mathbb{R}$ is called a **metric** if it satisfies*

1. *(symmetry) $d(x, y) = d(y, x)$ for any $x, y \in X$;*
2. *(positive definiteness) $d(x, y) \geq 0$ for any $x, y \in X$, and $d(x, y) = 0 \iff x = y$; and*
3. *(triangle inequality) $\forall x, y, z \in X$*

$$d(x, y) \leq d(x, z) + d(y, z).$$

Definition 2 (Open and Closed Sets)

Given a space X with a metric d, and $r > 0$, the set

$$B(x, r) := \{w \in X \mid d(x, w) < r\}$$

is called the ***open ball*** *of radius r centered at x. An* ***open set*** *A is such that $\forall a \in A$, $\exists r > 0$ such that*

$$B(a, r) \subseteq A.$$

We say that a set is ***closed*** *if its complement is open.*

Definition 3 (Continuous Map)

A function

$$f : (X, d_1) \to (Y, d_2)$$

is said to be ***continuous*** *if the preimage of an open set in Y is open in X.*

See notes on Real Analysis for why we defined a continuous map in such a way.

Warning

This definition does not imply that a continuous map f maps open sets to open sets.

Definition 4 (Open and Closed Maps)

A mapping $f : X \to Y$ is said to be ***open*** *if for all open $U \subset X$, $f(U)$ is open. We say that f is a* ***closed map*** *if for all closed $F \subset X$, $f(F)$ is closed.*

Exercise 1.1.2

Contruct a function on $[0, 1]$ which assumes all values between its maximum and minimum, but is not continuous.

Exercise 1.1.1

Suppose $f : X \to Y$ is a bijective continuous map. Then TFAE.

1. *f is a homeomorphism.*
2. *f is open.*
3. *f is closed.*

Solution
Consider the piecewise function

$$f(x) = \begin{cases} x & 0 \leq x < \frac{1}{2} \\ x - \frac{1}{2} & x \geq \frac{1}{2}. \end{cases}$$

It is clear that the maximum and minimum are $\frac{1}{2}$ and 0 respectively, and f assumes all values between 0 and $\frac{1}{2}$. However, a piecewise function is not continuous.

Definition 5 (Homeomorphism)

*A function f is a **homeomorphism** if it is a bijection and both f and f^{-1} are continuous.*

Example 1.1.1

The function

$$g : [0, 2\pi) \to \mathbb{R}^2 \text{ given by } \theta \mapsto (\cos\theta, \sin\theta)$$

is not homeomorphic, since if we consider an alternating series that converges to 0 on the unit circle on $\mathbb{R}^2$, we have that the preimage of the series does not converge and f^{-1} is in fact discontinuous.

Now, we want to talk about topologies without referring to a metric.

Definition 6 (Topology)

*Let X be a space. We say that the set $\mathcal{T} \subseteq \mathcal{P}(X)$ is a **topology** if*

1. $X, \emptyset \in \mathcal{T}$;

2. *if* $\{x_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ *for an arbitrary index set* A, *then*

$$\bigcup_{\alpha \in A} x_\alpha \in \mathcal{T}; \text{ and}$$

3. *If* $\{x_\beta\}_{\beta \in B} \subset \mathcal{T}$ *for some finite index set* B*, then*

$$\bigcap_{\beta \in B} x_\beta \in \mathcal{T}.$$

2 Lecture 2 Jan 09th

2.1 Euclidean Space (Continued)

In the last lecture, from metric topology, we generalized the notion to a more abstract one that is based solely on open sets.

Example 2.1.1

Let X be a set. The following two are uninteresting examples of topologies:

1. The **trivial topology** $\mathcal{T} = \{\emptyset, X\}$.
2. The **discrete topology** $\mathcal{T} = \mathcal{P}(X)$.

We shall now continue with looking at more concepts that we shall need down the road.

Definition 7 (Closure of a Set)

Let A be a set. Its ***closure****, denoted as $\overline{A}$, is defined as*

$$\overline{A} = \bigcap_{C \supset A}^{C\text{: closed}} C.$$

It is the smallest closed set that contains A.

Note 2.1.1

In metric topology, one typically defines the closure of a set by taking the union of A and its limit points.

Definition 8 (Interior of a Set)

Let A be a set. Its ***interior****, denoted either as* Int (A), A° *or* $\mathrm{Int}(A)$, *is defined as*

$$\mathrm{Int}(A) = \bigcup_{G \subseteq A}^{G\text{: open}} G.$$

Definition 9 (Boundary of a Set)

Let A be a set. Its ***boundary****, denoted as ∂A, is defined as*

$$\partial A = \overline{A} \setminus \mathrm{Int}(A).$$

Exercise 2.1.1

Let A be a set. Prove that ∂A is closed.

Proof

Notice that

$$(\partial A)^C = (\overline{A} \setminus \mathrm{Int}(A))^C = X \setminus \overline{A} \cup \mathrm{Int}(A) = X \cap \overline{A}^C \cup \mathrm{Int}(A)$$

which is open. □

Exercise 2.1.2

Let A be a set. Show that

$$\partial(\partial A) = \partial A.$$

Proof

First, notice that $\mathrm{Int}(\partial A) = \emptyset$. Since ∂A is closed, $\overline{\partial A} = \partial A$. Then

$$\partial(\partial A) = \overline{\partial A} \setminus \mathrm{Int}(\partial A) = \partial A \setminus \emptyset = \partial A$$ □

Example 2.1.2

We know that $\mathbb{Q} \subseteq \mathbb{R}$, and $\overline{\mathbb{Q}} = \mathbb{R}$. We say that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 10 (Dense)

We say that a subset A of a set X is dense if

$$\overline{A} = X.$$

Example 2.1.3

From the last example, we have that $\operatorname{Int}(\mathbb{Q}) = \emptyset$.

Definition 11 (Limit Point)

We say that $p \in X \supseteq A$ is a limit point of A if any neighbourhood of p has a nontrivial intersection with A.

Example 2.1.4 (A Topologist's Circle)

Consider the function

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

on the interval $\left[-\frac{1}{2\pi}, \frac{1}{2\pi} \right]$. Extend the function on both ends such that we obtain Figure 2.1 (See also: Desmos).

The limit points of the graph includes all the points on the straight line from $(0, -1)$ to $(0, 1)$, including the endpoints. This is the case because for any of the points on this line, for any neighbourhood around the point, the neighbourhood intersects the graph f infinitely many times.

Going back to continuity, given a function f, how do we know if f^{-1} maps an open set to an open set?

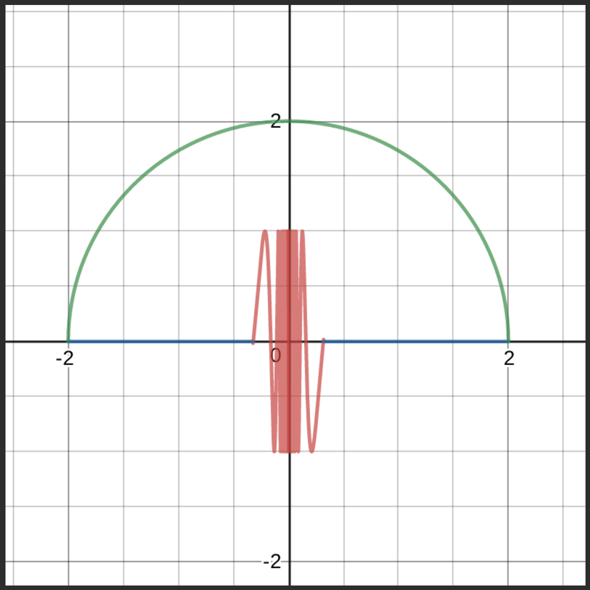

Figure 2.1: A Topologist's Circle

We can actually reduce the problem to only looking at open balls. But why are we allowed to do that?

Definition 12 (Basis of a Topology)

Given a topology $\mathcal{T}$, we say that $\mathcal{B} = \{B_\alpha\}_{\alpha \in I}$ is a **basis** *if $\forall T \in \mathcal{T}$, there exists $J \subset I$ such that*

$$T = \bigcup_{\alpha \in J} B_\alpha.$$

Note that while the definition is similar to that of a cover, we are now "covering" over sets and not points.

Example 2.1.5

Let $\mathcal{T}$ be the Euclidean topology on $\mathbb{R}$. Then we can take

$$\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}, \, a \leq b\}.$$

Note that $\mathcal{B}$ is **uncountable**. We can, in fact, have [1]

$$\mathcal{B}_1 = \{(a, b) \mid a, b \in \mathbb{Q}, a \leq b\},$$

which is countable, as a basis for $\mathbb{R}$. Furthermore, we can consider the set

$$\mathcal{B}_2 = \left\{ (a, b) \,\middle|\, a \leq b, \, a = \frac{m}{2^p}, \, b = \frac{n}{2^q}, \, m, n, p, q \in \mathbb{Z} \right\},$$

[1] Recall from PMATH 351 that we can write $\mathbb{R}$ as a disjoint union of open intervals with rational endpoints.

which is also a countable basis for $\mathbb{R}$. Notice that

$$\mathcal{B}_2 \subseteq \mathcal{B}_1 \subseteq \mathcal{B}.$$

Example 2.1.6

In $\mathbb{R}^2$, we can do a similar construction of $\mathcal{B}$, $\mathcal{B}_1$, and $\mathcal{B}_2$ as in the last example and use them as a basis for $\mathbb{R}^2$. In particular, we would have

$$\mathcal{B} = \{(a_1, b_1) \times (a_2, b_2) \mid a_1, a_2, b_1, b_2 \in \mathbb{R}\}.$$

This is called a **dyadic partitioning** of $\mathbb{R}^2$.

Example 2.1.7

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be two topological spaces. Then the Cartesian product $X_1 \times X_2$ has topology induced from $\mathcal{T}_1$ and $\mathcal{T}_2$ by taking the set

$$\mathcal{B} = \{\beta_1 \times \beta_2 \mid \beta_1 \in \mathcal{T}_1,\ \beta_2 \in \mathcal{T}_2\}$$

as the basis.

Exercise 2.1.3

Prove that

1. *β_1 and β_2 can be taken to be elements of bases $\mathcal{B}_1 \subset \mathcal{T}_1$ and $\mathcal{B}_2 \subset \mathcal{T}_2$, respectively.*
2. *the product topology on $\mathbb{R}^2$ is the same as the Euclidean topology.*

3 Lecture 3 Jan 11th

3.1 Euclidean Space (Continued 2)

The student is recommended to do a quick review for the first 3 chapters of the recommended text.

Let $\tilde{X}$ be a metric space, and $p, q \in \tilde{X}$ with $p \neq q$. Then we have that $d(p, q) = r > 0$.

Then we must have that

$$B\left(p, \frac{r}{3}\right) \cap B\left(q, \frac{r}{3}\right) = \emptyset.$$

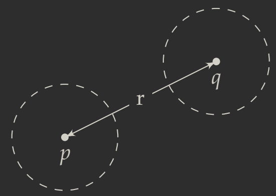

Figure 3.1: Idea of separation

Exercise 3.1.1

Prove that the above claim is true. (Use the triangle inequality)

Proof

Suppose $\exists x \in B\left(p, \frac{r}{3}\right) \cap B\left(q, \frac{r}{3}\right)$. Then

$$d(p, x) + d(q, x) < \frac{2r}{3} < r = d(p, q),$$

which violates the triangle inequality. □

We observe here that the two open sets (or balls) "separate" p and q.

Definition 13 (Hausdorff / T_2)

*Let X be a topological space. X is said to be **Hausdorff** or T_2 iff any 2 distinct points can be separated by disjoint open sets.*

Note 3.1.1

1. *The Hausdorff space (or* T_2 *space) is an important space; we can only define a metric on spaces that are* T_2*.*
2. *A space is called* T_1 *is for any* $p, q \in X$ *with* $p \neq q$*,* $\exists U \ni p$ *open such that* $q \notin U$ *and* $\exists V \ni q$ *open such that* $p \notin v$*. It is worth noting that a* T_2 *space is also* T_1*.*

Example 3.1.1 (The Discrete Topology)

Suppose X is a metric space. For any $x \in X$, we have that $\{x\}$ is open. Thus for any $x_1, x_2 \in X$, if $x_1 \neq x_2$, then the open sets $\{x_1\}$ and $\{x_2\}$ separates x_1 and x_2.

This is true as we can define the following metric on the space: let $d : X \times X \to \mathbb{R}$ such that

$$d(x_1, x_2) = \begin{cases} 0 & x_1 = x_2 \\ 1 & x_1 \neq x_2 \end{cases}$$

This topology is called a **discrete topology**, and it is a metric space.

Let X be a metric space and $A \subseteq X$. Then there is a metric induced by X on A, and this in turn induces a topology on A.

More generally, if $A \subset X$ where X is some arbitrary topological space, then a set $U \subseteq A$ is open iff $U = A \cap V$ for some $V \subseteq X$ that is open. In other words, a subset U of A is said to be open iff we can find an open set V in X such that the intersection of A and V gives us U.

Exercise 3.1.2

Prove that the construction above gives us a topology.

Proof

Let $A \subseteq X$. We shall show that τ_A is a topological space induced by the topological space τ of X. It is clear that $\emptyset \in \tau_A$, since it is open in X, and so $A \cap \emptyset = \emptyset$. Since X is open, we have $A \cap X = A$, and so $A \in \tau_A$.

Now if $\{U_\alpha\}_{\alpha \in I} \subseteq \tau_A$, then $\exists V_\alpha \subseteq X$ such that $U_\alpha = A \cap V_\alpha$. Then

$$\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} A \cap V_\alpha = A \cap \bigcup_{\alpha \in I} V_\alpha, \qquad \square$$

and $\bigcup_{\alpha \in I} V_\alpha$ is open in X by the properties of open sets. Thus $\bigcup_{\alpha \in I} U_\alpha \in \tau_A$.

If $\{U_i\}_{i=1}^n \subset \tau_A$, then again, by the properties of open sets, finite intersection of open sets is open, and so $\bigcap_{i=1}^n U_i \in \tau_A$.

Note 3.1.2

We can say the same can be said about closed sets of A.

Example 3.1.2

Let $A \subseteq X$ and consider the function

$$\iota_A : A \to X \text{ given by } x \mapsto x,$$

which is the inclusion map.

Then ι_A is continuous when the topology on A is chosen to be the induced subspace topology. This is rather clear; notice that the inverse of the inclusion map brings open sets to open sets.

Let Y be an arbitrary topological space. Then let

$$\begin{tikzcd} & X \\ Y \arrow[ur, "\iota_A \circ f"] \arrow[r, "f"] & A \arrow[u, "\iota_A"'] \end{tikzcd}$$

Figure 3.2: Composition of a function and the inclusion map

where f is continuous. Then $\iota_A \circ f$ is continuous.

The converse is also true: if $\iota_A \circ f$ is continuous, then f is continuous. However, we will not prove this. This property is known as the **characteristic property of the subspace topology**.

Theorem 1 (Characteristic Property of the Subspace Topology)
Suppose X is a topological space and $S \subseteq X$ is a subspace. For any topological space T, a map $f : Y \to S$ is continuous iff the composite map $\iota_S \circ f : Y \to X$ is continuous.

$$\begin{tikzcd} & X \\ Y \arrow[ur, "\iota_S \circ f"] \arrow[r, "f"'] & S \arrow[u, hook, "\iota_S"'] \end{tikzcd}$$

Figure 3.3: Characteristic Property of the Subspace Topology

Lemma 2 (Restriction of a Continuous Map is Continuous)

Let $X \xrightarrow{f} Y$ be continuous, and $A \subseteq X$ [1]. Then

[1] Here, A is equipped with the subspace topology

$$f \restriction_A : A \to Y$$

is also continuous.

Proposition 3 (Other Properties of the Subspace Topology)

Suppose S is a subspace of the topological space X.

1. *If $R \subseteq S$ is a subspace of S, then R is a subspace of X; i.e. the subspace topologies that R inherits from S and from X agree.*
2. *If $\mathcal{B}$ is a basis for the topology of X, then*

   $$\mathcal{B}_S = \{B \cap S : B \in \mathcal{B}\}$$

   is a basis for the topology of S.
3. *If $\{p_i\}$ is a sequence of points in S and $p \in S$, then p_i converges to p in S iff p_i converges to p in X.*
4. *Every subspace of a Hausdorff space is Hausdorff.*

3.2 Connected Spaces

Consider the real line $\mathbb{R}$, and consider two disjoint intervals on $\mathbb{R}$.

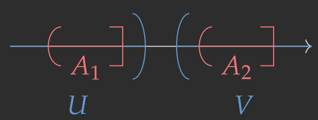

Figure 3.4: Motivation for Connectedness

Observe that we may find two open subsets U and V of $\mathbb{R}$ such that $A_1 \subseteq U$ and $A_2 \subseteq V$, which effectively separates the two intervals on the space $\mathbb{R}$.

Definition 14 (Disconnectedness)

A space X is said to be ***disconnected*** *iff X can be written as a disjoint union*

$$X = A_1 \coprod A_2$$

where $A_1, A_2 \subseteq X$, $A_1 = A_2^C$, that they are both non-empty and open [2].

[2] It goes without saying that the two sets are also simultaneously closed.

Definition 15 (Connctedness)

A space X is said to be ***connected*** *if it is not disconnected.*

Note 3.2.1

By the above definitions, we have that X is connected iff for any partition $X = A \coprod A^C$ with A being open, either A is $\emptyset$ or A is X.

Example 3.2.1

The space $\mathbb{R} \setminus \{0\}$ is disconnected; our disjoint sets are $(-\infty, 0)$ and $(0, \infty)$.

However, $\mathbb{R}^2 \setminus \{0\}$ is connected, but it is not easy to describe why.

Definition 16 (Path)

A ***path*** *in a space X from p to q (both in X) is a continuous map $f : [0,1] \to X$ such that $f(0) = p$ and $f(1) = q$. We say that X is* ***path connected*** *if $\forall p, q \in X$, there is a path in X from p to q.*

Lemma 4 (Path Connectedness implies Connectedness)

If a space X is path connected, then it is connected.

Theorem 5 (From Connected Space to Connected Space)

If $X \xrightarrow{f} Y$ is continuous and X is connected, then $\mathrm{Img}(f)$ is connected.

4 Lecture 4 Jan 14th

4.1 Connected Spaces (Continued)

Definition 17 (Locally Connected)

*We say that X is **locally connected** at x if for every open set V containing x there exists a connected, open set U with $x \in U \subseteq V$. We say that the space X is **locally connected** if it is locally connected $\forall x \in X$.*

Example 4.1.1

The space S generated by the function $\sin \frac{1}{x}$ with 0 at $x = 0$, on the $\mathbb{R}^2$ is not locally connected: consider $(0, y) \in S, y \neq 0$. Then any small open ball at this point will contain infinitely many line segments from S. This cannot be connected, as each one of these constitutes a component, within the neighborhood.

Definition 18 (Connected Component)

*The maximal connected subsets of any topological space X are called **connected components** of the space. The components form a partition of the space.*

4.2 Compactness

Definition 19 (Sequential Compactness)

For $A \subseteq X$, where X is a topological space, if $\{x_i\}_{i \in I} \subseteq A$, arbitrary sequence in A, has a convergent subsequence, we say that A is ***sequentially compact****.*

Definition 20 (Compactness)

We say that a topological space X is ***compact*** *if every* ***open cover*** *of X has a finite* ***subcover****.*

Lemma 6 (Compactness implies Sequential Compactness)

Compactness implies sequential compactness.

Example 4.2.1

$[0, 1)$ is not compact: consider the open cover $\left\{ \left[0, 1 - \frac{1}{n}\right) \right\}_{n \in \mathbb{N}}$, which contains $[0, 1)$ as $n \to \infty$. But whenever n is finite, we have $1 - \frac{1}{n} < 1$, and so any finite collection of the $\left[0, \frac{1}{n}\right)$ is not a cover of $[0, 1)$.

Theorem 7 (Continuous Maps map Compact Sets to Compact Images)

Let $f : X \to Y$ be continuous, where X is compact. Then $f(X)$ is compact.

Proof

Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $f(X)$. Since f is continuous, we have that $f^{-1}(U_\alpha)$ is open for each $\alpha \in I$. Since f is bijective between the image set and its domain, we have that $\{f^{-1}(U_\alpha)\}_{\alpha \in I}$ is an open cover of X. Since X is compact, this cover has a finite subcover, say $\{f^{-1}(U_i)\}_{i=1}^{n}$. Thus

$$X = \bigcup_{i=1}^{n} f^{-1}(U_i).$$

Thus

$$f(X) = f\left(\bigcup_{i=1}^{n} f^{-1}(U_i)\right) = \bigcup_{i=1}^{n} U_i.$$

Hence $\{U_\alpha\}_{\alpha \in I}$ has a finite subcover and so $f(X)$ is compact. □

Corollary 8 (Homeomorphic Maps map Compact Sets to Compact Sets)

Let $X \xrightarrow{f} Y$ be a homeomorphism. Then X is compact iff Y is compact.

Note 4.2.1

Compactness is a topological property.

Lemma 9 (Properties of Compact Sets)

1. *A closed subset of a compact space is compact.*
2. *A compact subset of a topological space is closed provided that the space is Hausdorff.*
3. *In a metric space, a compact set is bounded.*
4. *Finite (Cartesian) product of compact sets is compact.*

The proof for the first item is simple: consider an open cover of the closed subset, and union them with the complement of the closed subset. This covers the entire space, and so it has a finite subcover. We just need to then remove that complement set, and that would be a finite subcover for the closed subset.

Example 4.2.2

The subset $[-a, b]$, $a, b \in \mathbb{R}$, is compact.

Example 4.2.3

$[0, 1]^{\mathbb{N}}$ is not compact: the space is equivalent to ℓ_∞.

Theorem 10 (Heine-Borel)

Let $X \subseteq \mathbb{R}^n$. Then X is compact iff X is closed and bounded.

Proof

$(\Longrightarrow)$ We say that compactness impliess boundedness. Also, since $\mathbb{R}^n$ is Hausdorff, X is closed.[1].

$(\Longleftarrow)$ X is bounded implies that $X \subseteq [-R, R]^n$ with R sufficiently large. Since X is closed, and $[-R, R]^n$ is compact, X is necessarily compact by Lemma 9. $\square$

[1] Both from Lemma 9

Theorem 11 (Bolzano-Weierstrass)

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Exercise 4.2.1

Prove Theorem 11 as an exercise.

We shall start the next part this lecture.

4.3 Manifolds

Definition 21 (Locally Homeomorphic)

A space is said to be ***locally homeomorphic*** *to $\mathbb{R}^n$ provided that $\forall x \in X, \exists U \ni x$ open such that U is homeomorphic to $\mathbb{R}^n$.*

Definition 22 (Manifold)

An n-dimensional ***manifold*** *is a* ***second countable***[2]*,* ***Hausdorff*** *topological space that is* ***locally homeomorphic*** *to $\mathbb{R}^n$.*

[2] a topological space is said to be **second countable** if its basis is countable. Note that for a second countable set X, every open cover of X has a countable subcover (see pg 32 of Lee (2000)).

Note 4.3.1

We may also call the last criterion of being a manifold, that is, to be locally homeomorphic to $\mathbb{R}^n$, as to be ***locally Euclidean*** *of dimension n.*

Note 4.3.2

One can give an equivalent definition of locally homeomorphic by requiring that U be homeomorphic to an open ball $B^n \subseteq \mathbb{R}^n$. Notice that B^n is homeomorphic to $\mathbb{R}^n$ [3]

[3] By scaling, really.

The following is a quick fact about second countable spaces, which will be helpful when we start creating subspaces from manifolds.

Proposition 12 (Subspaces of Second Countable Spaces are Second Countable)

Every subspace of a second countable space is second countable.

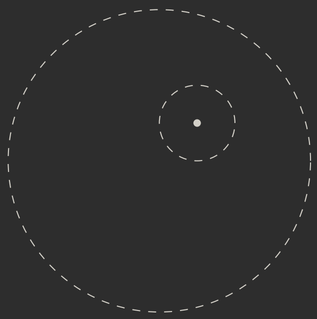

Figure 4.1: Open ball in an open set in $\mathbb{R}^2$

Figure 4.2: Open ball on a point on the boundary of a closed set

Example 4.3.1

Let $B^n = B^n(0,1) \subseteq \mathbb{R}^n$. Then B^n is homeomorphic to $\mathbb{R}^n$.

Example 4.3.2

Now consider the closed ball $\overline{B}^n = \overline{B}^n(0,1) \subset \mathbb{R}^n$. This is actually not a manifold, but we are not yet there to prove this. This sort of a structure motivates us to the next definition.

Definition 23 (Manifold with Boundary)

An n-dimensional space that is second countable and Hausdorff, such that $\forall x \in X$, there exists a neighbourhood either homeomorphic to $\overline{B}^n \subseteq \mathbb{R}^n$ or $B^n \cap \mathbb{H}^n$.

Note 4.3.3

Note that $\mathbb{H}^n$ is defined as

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) : x_n \geq 0\}.$$

For instance, $\mathbb{H}^2$ has the following graph:

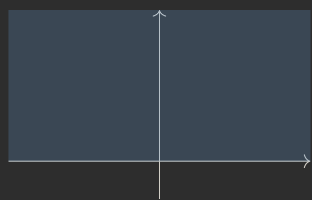

Figure 4.3: Graph of $\mathbb{H}^2$

Part II

Introduction to Topological Manifolds

5 Lecture 5 Jan 16th

5.1 Manifolds (Continued)

Definition 24 (Interior Point)

*A point $x \in M$ is called an **interior point** if there is a local homeomorphism*

$$\phi : \mathcal{U} \to \mathbb{B} \subseteq \mathbb{R}^n,$$

where $\mathcal{U}$ is open.

In the last lecture we asked ourselves the following: how do we know if the idea of 'being on a boundary' is a well-defined notion? In particular, how do well tell the difference between the following two graphs, mathematically?

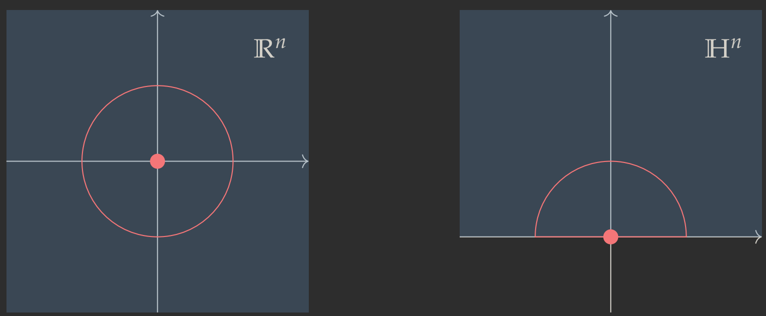

Figure 5.1: How do we tell the difference between the two graphs?

Definition 25 (Boundary Point)

A point x is on the boundary of M, denoted as $x \in \partial M$, if there exists

$\mathcal{U} \ni x$ that is open, and a homeomorphism

$$\psi : \mathcal{U} \to \mathbb{B}_{0,1} \cap \mathbb{H}^n.$$

Note 5.1.1

Note that the definition of a boundary and interior is different from our earlier definitions for the same terminologies. A manifold with boundary will always have an empty boundary, as a subset, despite the fact that its boundary as a manifold may not be empty.

Note 5.1.2

*The ϕ in Definition 24 and ψ in Definition 25 are called **local charts**.*

Also, our definitions do not rule out, e.g.

$$\phi_2 : \mathcal{U} \to \mathbb{B}^2 \subseteq \mathbb{R}^2$$
$$\phi_5 : \mathcal{U} \to \mathbb{B}^5 \subseteq \mathbb{R}^5.$$

Definition 26 (Coordinate Chart)

*If M is locally homeomorphic to $\mathbb{R}^n$, a homeomorphism from an open subset $U \subset M$ to an open subset of $\mathbb{R}^n$ is called a **coordinate chart** (or simply a **chart**).*

We shall later on prove that a point cannot simultaneously be a boundary point and an interior point. This property is called **the invariance of the boundary**. [1]

[1] This also means that $\mathrm{Int}(M) \cap \partial M = \emptyset$. Also, by Definition 22, $\mathrm{Int}(M) \cup \partial M = M$.

Note 5.1.3

$\mathrm{Int}(M)$ *is open. Thus, we can use the same definition about open sets as before using ϕ.*

Note 5.1.4

In contrast, ∂M is closed; thanks to the invariance of the boundary we have that $\partial M = M \setminus \mathrm{Int}(M)$ and $\mathrm{Int}(M)$ is open, and so $(\partial M)^C = \mathrm{Int}(M)$.

We shall also prove the following theorem later on:

Theorem (Invariance of the Dimension)
The n in $\mathbb{R}^n$ is well-defined.

Example 5.1.1

Consider the equation

$$x^2 - y^2 - z^2 = 0. \tag{5.1}$$

Note that we may write

$$x = \pm\sqrt{y^2 + z^2},$$

and so the graph generated by Equation (5.1) is as shown in Figure 5.2.

However, this is not a manifold: if we assume that a ball arounnd the origin is homeomorphic to $\mathbb{R}^2$, then by removing the point at the origin in the cone, the cone becomes two disconnected components, but the ball in $\mathbb{R}^2$ homeomorphic to the aforementioned ball is still connected.

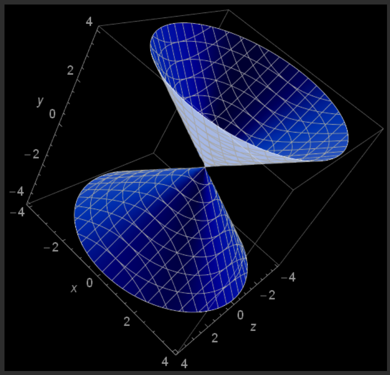

Figure 5.2: A 3D cone in $\mathbb{R}^3$, from WolframAlpha

Note 5.1.5

An open subset of a manifold is a manifold, by restriction.

5.1.1 The 1-Sphere S^1

From Example 2.1.5, we have

- $[0,1) \simeq [0,\infty)$ is a manifold with boundary;
- $(0,1) \simeq \mathbb{R}$ is a manifold; and
- $[0,1]$ is a manifold with boundary.

Example 5.1.2 (S^1 is a manifold)

Consider the function $f : [0,2\pi) \to e^{i\theta}$. The image of f is Consider

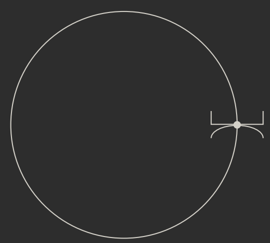

Figure 5.3: S^1 as a manifold

the following two functions $(0,2\pi) \to \mathbb{C}^2$ by

$$\theta_1 \to e^{i\theta_1} \text{ and } \theta_2 \to e^{i\theta_2+\pi},$$

which gives us the graphs:

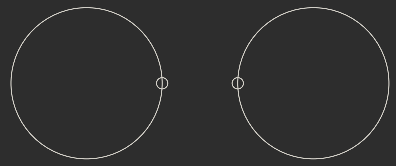

Figure 5.4: Basis for S^1

respectively. Note that the image of both these functions are not compact. Regardless, this gives us a basis for S^1, which we notice is countable, Hausdorff, and locally homeomorphic to $\mathbb{R}^2$.

Theorem 13 (1-Dimensional Manifolds Determined by Its Compactness)

Let M be a connected component of a 1-dimensional manifold. Then either

1. *M is compact, in which case if it is*
   - *without a boundary, then M is homeomorphic to S^1.*
   - *with a boundary, then M is homeomorphic to $[0,1]$.*

2. *M is not compact, in which case if it is*
   - *without a boundary, then M is homeomorphic to* $[0,1)$.
   - *with a boundary, then M is homeomorphic to* $(0,1)$.

6 Lecture 6 Jan 18th

6.1 Manifolds (Continued 2)

6.1.1 The 1-Sphere S^1 (Continued)

Set theoretic view of S^1 We showed that S^1 is a manifold. We can, in fact, set theoretically, look at S^1 as $A = [0, 1]$ glued at the endpoints, i.e. we **identify** the points 0 and 1 as ‘the same’, and label this notion as $0 \sim 1$.

Topological view of S^1 Topologically, for $0 \sim 1$ in A, we can construct an open set around the point such that the open set is properly contained in A.

Figure 6.1: Topological representation of A

But how can we describe this notion mathematically so?

CONSIDER the real line as follows:

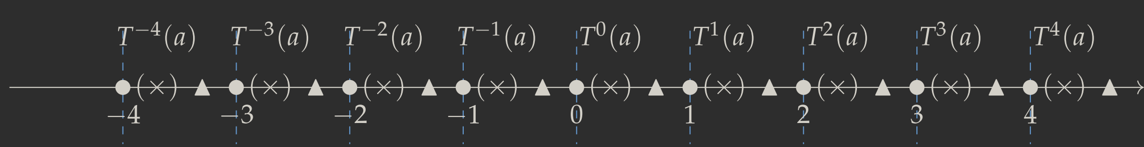

Figure 6.2: Breaking down the real line into parts

Let’s define $T : \mathbb{R} \to \mathbb{R}$ such that $x \mapsto x + 1$. Clearly so, T is bijective and, in particular, has an inverse $x \mapsto x - 1$.

Notice that within each interval $[x, x + 1]$, we can find a $\times$ and $\blacktriangle$ at the same distance from x. Also, notice that we can use the same radius for $\times$ such that the open ball around $\times$ sits in $[x, x + 1]$ for each $x \in \mathbb{Z}$.

Thus, instead of studying the entire real line at once, we can reduce our attention only to $[0,1]$, and simply scale the interval with a 'scalar multiplication' to get to wherever we want on the real line.

Now let

$$G := \left\{ T^k \;\middle|\; k \in \mathbb{Z} \right\},$$

which is evidently a **group**. Furthermore, every element in G is a homeomorphism to $\mathbb{R}$. Let G **act on** $\mathbb{R}$, and for $a \in \mathbb{R}$, consider the **orbit** of a, which is denoted as

$$G \cdot a := \left\{ T^k(a) \;\middle|\; k \in \mathbb{Z} \right\}.$$

Then

$$S^1 \simeq \text{ the space of all orbits of } G \text{ acting on } \mathbb{R} =: \mathbb{R}/G,$$

where $\simeq$ represents homeomorphism. [1] Also, notice that here, G is effectively $\mathbb{Z}$.

[1] $\mathbb{R}/G$ is the **quotient space** of $\mathbb{R}$ over G.

This realization implies the existence of some topology on S^1.

Thus far, we have seen that we may look at S^1

- set theoretically: as $A = [0,1]$ with glued endpoints; and
- topologically: as $\mathbb{R} \setminus \mathbb{Z}$.

S^1 as a topological group Since $\mathbb{C} \simeq \mathbb{R}^2$, we may think of S^1 as a sphere on the complex plane. We see that this 'group' takes on the

$$S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} = \left\{ e^{i\theta} \mid \theta \in \mathbb{R} \right\}$$

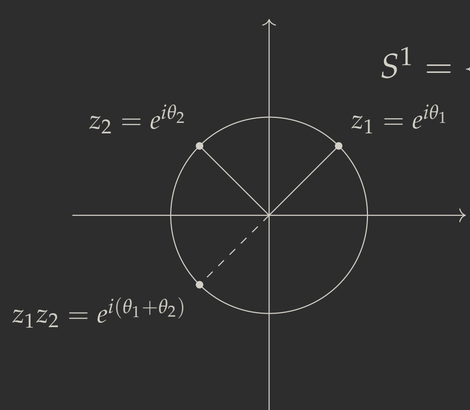

Figure 6.3: S^1 on the complex plane

operation of adding the indices of the exponents. Thus $G = (S^1, \cdot)$. Notice that G is indeed a group equipped with said operation, and for each $z_1 \in G$, there exists $z_1^{-1} = \frac{1}{z_1} \in G$ such that $z_1 \cdot \frac{1}{z_1} = 1$.

Furthermore, the function

$$\iota : S^1 \to S^1 \text{ given by } z \mapsto \frac{1}{z}$$
$$\text{which is } e^{i\theta} \mapsto e^{-i\theta}$$

is continuous.

Also, the function

$$P : S^1 \times S^1 \to S^1 \text{ given by } (z_1, z_2) \mapsto z_1 z_2$$

is continuous.

Definition 27 (Topological Group)

If G is a group, and functions ι and P as defined above, if both ι and P are continuous, then we say that G is a ***topological group****.*

S^1 as a moduli space

Definition 28 (Moduli Space)

A ***moduli space*** *is the space of all lines passing through the origin.*

On $\mathbb{R}^2$, we have

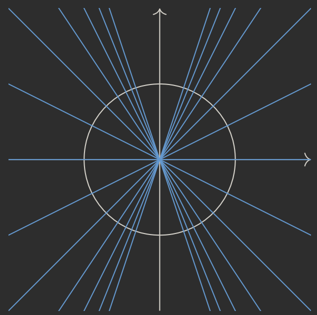

Figure 6.4: The moduli space on $\mathbb{R}^2$

First, how can we understand 'closeness' in a moduli space? We can actually look at the difference in the radians of each line, or really just $x/360$ and compare the x's.

Also, notice that each line passes through S^1 twice. We can indeed avoid this problem by shifting S^1 to one side, as shown in Figure 6.5.

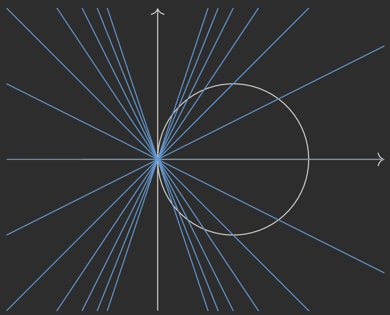

Figure 6.5: Shifted S^1 for the moduli space

Now each line intersects S^1 at the origin and another point on S^1, and this intersection is in fact unique.

6.1.1.1 The space of $S^1 \times S^1$

Observe that the product of two manifolds is a manifold [2].

[2] This is called a **product space**.

Consider using the set theoretical viewpoint, with credits to **Felix Klein**, the following figure By joining the sides with $>$, where we

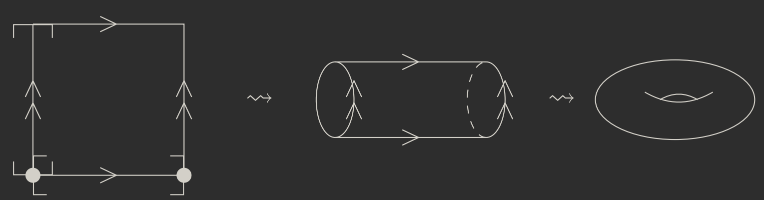

Figure 6.6: $S^1 \times S^1$ becomes a cylinder by identifying the edges

identify the endpoints, we can go from the figure introduced by Klein to a cylinder. Then by identifying the sides with $>>$, we get a **torus**.

Now on a Cartesian plane, observe that

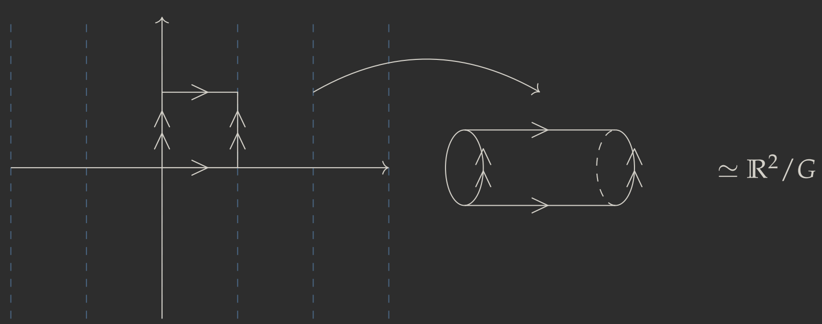

Figure 6.7: Klein's figure on a Cartesian plane to a cylinder

where we define $G = \left\{ T^k \;\middle|\; k \in \mathbb{Z} \right\}$ as before, for $T((x,y)) = (x+1, y)$. Now on a similar note, define $R : \mathbb{R} \to \mathbb{R}$ by $R((x,y)) = (x, y+1)$. Then let $G_2 = \left\{ R^k \;\middle|\; k \in \mathbb{Z} \right\}$. Thus

$$\mathbb{R}^2/G \oplus G_2 \simeq \mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z}.$$

7 Lecture 7 Jan 21st

7.1 Manifolds (Continued 3)

7.1.1 2-dimensional Manifolds

Example 7.1.1

S^2 is a 2-dimensional manifold (w/o boundary).

Note that we may 'cover' this sphere by the function $f_- : \mathbb{D} \subseteq \mathbb{R}^2 \to \mathbb{R}$, where $\mathbb{D}$ is the unit disc in $\mathbb{R}^2$, and f_- is given by

$$(x, y) \mapsto -\sqrt{1 - (x^2 + y^2)}.$$

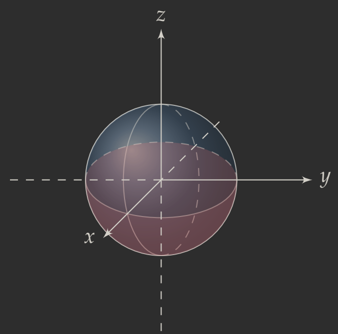

Figure 7.1: The 2-sphere S^2

We see that f_- is a chart for the lower hemisphere for Figure 7.1, shaded red. We can indeed cover the entire 2-sphere with similar hemispheres in different orientations: consider

$$\begin{aligned} f_+(x, y) &= \sqrt{1 - x^2 - y^2} \\ g_+(x, z) &= \sqrt{1 - x^2 - z^2} \\ g_-(x, z) &= -\sqrt{1 - x^2 - z^2} \\ h_+(y, z) &= \sqrt{1 - y^2 - z^2} \\ h_-(y, z) &= -\sqrt{1 - y^2 - z^2}, \end{aligned}$$

which represent the upper hemisphere, eastern hemisphere, western hemisphere, front hemisphere, and back hemisphere, respectively.

We have the following general observation: let $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ be a real-valued continuous function. Then $\Gamma(f)$ is **parameterized** using f, i.e. f constructs the chart $\Gamma(f)$.

We have not explicitly defined what $\Gamma(f)$ is, but it is recorded in PMATH 733, but we shall provide a short definition here for reference: the **graph** $\Gamma(f)$ of the function f is defined as

$$\Gamma(f) := \{(x_1, \ldots, x_n, y) \mid f(x_1, \ldots, x_n) = y\}.$$

It is interesting to note that $\Gamma(f)$ is homeomorphic to $\mathcal{U}$. Consider the function

$$\Phi : \mathcal{U} \to \Gamma(F) \text{ given by } \Phi(x_1, \ldots, x_n) = (x_1, \ldots, x_n, f(x_1, \ldots, x_n)).$$

It is clear that Φ is continuous since each of its components are continuous. However, it is not as easy to find a continuous map to go from $\Gamma(f)$ to $\mathcal{U}$.

Definition 29 (Projection)

We define function $\pi_{n+1} : \Gamma(f) \to \mathcal{U}$ by

$$\pi_{n+1}(x_1, \ldots, x_n, y) = (x_1, \ldots, x_n),$$

*which is called a **projection**.*

Note 7.1.1

1. *In A1, we showed that π_{n+1} is continuous.*
2. *Furthermore, π_{n+1} is injective.*
3. *Thus, we observe that*

$$\pi_{n+1} \circ \Phi(x_1, \ldots, x_n) = (x_1, \ldots, x_n) = id_{n+1}(x_1, \ldots, x_n).$$

Example 7.1.2

Consider the following figure:

We define the **stereographic projection** by

$$\Sigma(p) = p'.$$

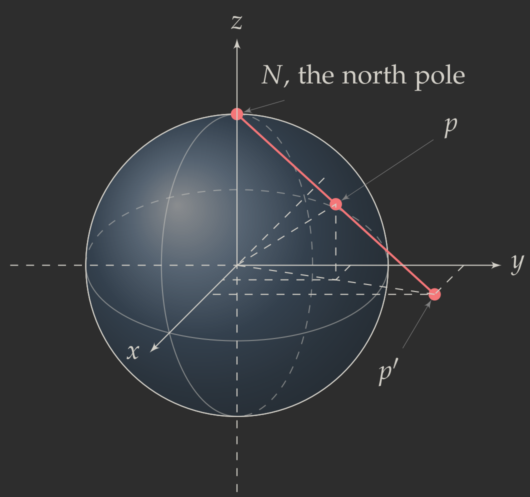

Figure 7.2: Stereographic Projection

7.1.2 n-spheres

We can now extend the notion we saw in Example 7.1.1 to higher dimensional manifolds. In particular, we want to be able to 'glue' the boundaries of the hemispheres, as shown in Figure 7.3.

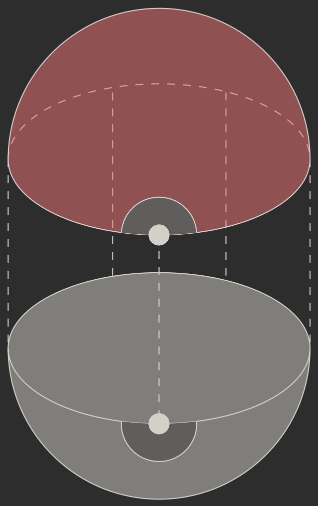

Figure 7.3: Glueing two hemispheres I was definitely not trying to make a Pokeball.

In a more mathematical sense, we are identifying the lower and upper boundaries of the upper and lower hemisphere, respectively, i.e. we identify

$$S^n = \overline{\mathbb{B}}^n \times \{0\} \cup \overline{\mathbb{B}}^n \times \{1\},$$

where the $\{0\}$ and $\{1\}$ represent the upper and lower hemispheres, respectively. We may also write this as

$$x \in \partial\overline{\mathbb{B}}^n : (x, 0) \sim (x, 1).$$

This calls for the notion of an equivalence class. Recall that an **equivalence relation** on a set A is a relation between elements such that

1. (reflexity) $x \sim x$;
2. (symmetry) $x \sim y \iff y \sim x$; and
3. (transitivity) $x \sim y \wedge y \sim z \implies x \sim z$,

where $\sim$ is our equivalence relation.

Equivalence relations give rise to the notion of **equivalence classes**, where we define an equivalence class as

$$[\beta] := \{\alpha \in A \mid \beta \sim \alpha\}.$$

In this course, we shall sometimes denote equivalence classes in the form of A_β. Notice that

$$A = \coprod_{\beta \in B} A_\beta,$$

where $B \subseteq A$.

This begs the question: do the set of equivalence classes retain the topology of the space?

Here, we look into **quotient topology**. Let A^* be the space of equivalence classes, and let's assume that A is endowed with a topology. Consider the function

$$\pi : A \to A^* \text{ given by } a \mapsto [a]_\sim,$$

where $\sim$ is the equivalence relation, and $[a]_\sim$ is the equivalence class that a belongs to[1].

[1] Note that the function is well-defined as the equivalence classes are disjoint.

$$\begin{tikzcd} A \arrow[d, "\pi"'] \arrow[dr, "g"] & \\ A^* \arrow[r, "\tilde{g}"'] & Z \end{tikzcd}$$

Figure 7.4: Relationship of A, A^* and Z

Consider an arbitrary space Z that is also endowed with a topology, such that $\exists g : A \to Z$ that is continuous, such that g is constant on each of the equivalence classes of A, that is, if $a \sim b$, then $g(a) = g(b)$.

Then $\tilde{g}$ is a well-defined function induced on A^*: we have that

$$\tilde{g}(\pi(a)) = g(a).$$

We want to endow A^* with a topology such that $\tilde{g}$ will be continuous. However,

- if A^* is too 'fine', then π may not be continuous; and
- if A^* is too 'course', then $\tilde{g}$ may not be continuous.

We need to strike a balance in the fineness of the topology of A^* to make sure that both π and $\tilde{g}$ are continuous.

Definition 30 (Strongly Continuous)

Let $\mathcal{V} \subseteq A^*$. $\mathcal{V}$ is open iff $\exists \mathcal{U} \subseteq A$ such that $\pi(\mathcal{U}) = \mathcal{V}$. We say that π is ***strongly continuous****.*

Warning

π may not be an open map!

8 Lecture 8 Jan 23rd

8.1 Quotient Spaces

With the topology that we last introduced for A^*, g is continuous iff $\tilde{f}$ is continuous. However, this is a rather cumbersome way to construct a quotient space. In particular, how do we know what equivalence class should we choose?

This way of constructing a quotient space is intuitive, ring theoretically.

Exercise 8.1.1
Let $\pi : X \to Y$ be any map. For a subset $U \subseteq X$, show that TFAE.

1. *U is saturated.*
2. *$U = \pi^{-1}(q(U))$.*
3. *U is a union of fibres.*
4. *If $x \in U$, then every point $x' \in X$ such that $q(x) = q(x')$ is also in U.*

Definition 31 (Saturated)

We say that a subset $S \subsetneq A$ is ***saturated*** *(wrt π) provided that it is has a* ***non-empty intersection*** *with a fibre $\pi^{-1}(\{q\})$, where $q \in A^*$, then $S \supseteq \pi^{-1}(\{q\})$.*

Note 8.1.1

This definition is equivalent to saying that $\pi^{-1} \circ \pi(S) = S$.

With this definition, we may restate the definition of a quotient map in a clearer way.

Definition 32 (Quotient Map)

A map $g : X \to Y$ is called a ***quotient map*** *if it sends saturated open subsets of X to open subsets of Y.*

Example 8.1.1

Let $g : X \to Y$ be surjective. We can define an equivalence relation on X by setting $x_1 \sim x_2$ iff $g(x_1) = g(x_2)$, i.e. x_1 and x_2 belong to the same fibre.

Example 8.1.2

Let $\overline{\mathbb{B}}^2$ be the closed unit disk in $\mathbb{R}^2$, and let $\sim$ be the equivalence relation on $\overline{\mathbb{B}}^2$ defined by $(x, y) \sim (-x, y)$ for all $(x, y) \in \partial\overline{\mathbb{B}}^2$ (See Figure 8.1).

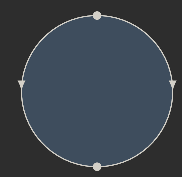

Figure 8.1: A quotient of $\overline{\mathbb{B}}^2$

We can think of this space as one that is obtained from $\mathbb{B}^2$ by "pasting" the left half of the boundary to the right half. It is not difficult to imagine this transformation and notice that we can 'continuously morph' $\mathbb{B}^2$ under this equivalence relation into S^2. We shall prove much later on that this is indeed the case.

The above process is also called 'collapsing $\partial\mathbb{B}^2$ to a point'.

Example 8.1.3

Define an equivalence relation on the square $I \times I$ by setting $(x, 0) \sim (x, 1)$ for all $x \in I$, and $(0, y) \sim (1, y)$ for all $y \in I$ (See Figure 8.2).

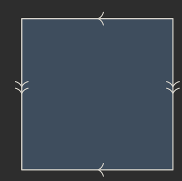

Figure 8.2: A quotient of $I \times I$

Example 8.1.4

Define $\mathbb{P}^n$, the **real projective space of dimension n**, to be the set of 1-dimensional linear subpaces (lines through the origin) in $\mathbb{R}^{n+1}$. There exists a map $q : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, defined by sending a point x to its span. We apply the quotient topology with respect to q on $\mathbb{P}^2$. The 2-dimensional projective space $\mathbb{P}^2$ is usually called the **projective plane**.

Example 8.1.5

Let X be a topological space, the quotient $(X \times I)/(X \times \{0\})$ obtained from the "cylinder" $X \times I$ by collapsing one end to a point is called the **cone** on X. For instance, if $X = S^1$ and $I = (0, 1)$, then taking the quotient $(S^1 \times (0, 1)/(S^1 \times \{0\})$ is exactly the process shown in Figure 8.3.

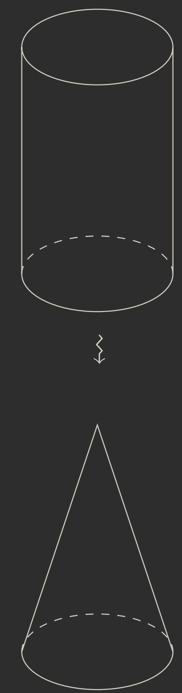

Figure 8.3: Cone of S^1

Example 8.1.6 (Wedge sum of Manifolds)

Let M_1 and M_2 be two manifolds. We define the **wedge sum** of M_1 and M_2 as

$$M_1 \vee M_2 := (M_1 \cup M_2)/p_1 \sim p_2,$$

where $p_1 \in M_1$ and $p_2 \in M_2$. The wedge sum is also sometimes called the **one-point union**.

For instance, the wedge sum of $\mathbb{R} \wedge \mathbb{R}$ is homeomorphic to the union of the x-axis and y-axis on a Cartesian plane (cf Figure 8.4), and the wedge sum $S^1 \wedge S^1$ is homeomorphic to the **figure-eight space**, which is made up by the union of two circles of radius 1 centered at $(0, 1)$ and $(0, -1)$ in the plane (cf Figure 8.5).

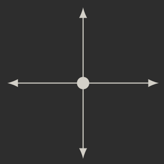

Figure 8.4: Wedge sum of two lines.

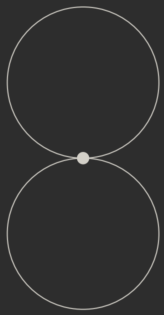

Figure 8.5: Wedge sum of two circles.

Warning

Unlike subspaces and product spaces, quotient spaces do not behave well wrt most topological properties. In particular, none of the definiting properties of manifolds are necessarily inherited by quotient spaces.

For example, a quotient space can be

- *locally Euclidean and second countable, but not Hausdorff;*
- *Hausdorff and second countable, but not locally Euclidean; or*
- *not even first countable.*

The following is a proposition that gives us some peace of mind when working within certain spaces.

Proposition 14 (Locally Euclidean Quotient Space of a Second Countable Space is Second Countable)

Suppose M is a second countable space and N is a quotient space of M. If N is locally Euclidean, then it is second countable.

Note 8.1.2

Thus if the original space is, say, a manifold, then for any of its quotient spaces, we only need to check that the quotient space is both Hausdorff and locally Euclidean.

Proof

Let $q : M \to N$ be the quotient map, and suppose N is locally Euclidean. Let $\mathcal{U}$ be a cover of N. Then the set $\{q^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of M, which therefore has a countable subcover. Let $\mathcal{U}' \subseteq \mathcal{U}$ be the countable subset such that $\{q^{-1}(U) : U \in \mathcal{U}'\}$ covers M. Then $\mathcal{U}'$ is a countable subcover of N. $\square$

Exercise 8.1.2

Consider the function $g : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$. Verify that g is a quotient map.

Example 8.1.7

The map $g : S^1 \to S^1 \subseteq \mathbb{R}^2$ as given by the above is indeed a quotient map. Thus we observe that S^1 is a quotient space of itself.

8.1.1 *Characteristic Property and Uniqueness of Quotient Spaces*

Theorem 15 (Characteristic Property of the Quotient Topology)

Suppose X and Y are two topological spaces and $\pi : X \to Y$ is a quotient map. For any topological space Z, a map $f : Y \to Z$ is continuous iff the composite map $f \circ \pi$ is continuous (cf Figure 8.6).

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f \circ \pi"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

Figure 8.6: Characteristic property of the quotient topology.

Proof

Observe that for any open $U \subseteq Z$, $f^{-1}(U)$ is open in Y iff

$$\pi^{-1}(f^{-1}(U)) = (f \circ \pi)^{-1}(U)$$

is open in X. Our result follows immediately from this observation. $\square$

Theorem 16 (Uniqueness of the Quotient Topology)

Given a topological space X, a set Y and a surjective map $\pi : X \to Y$, the quotient topology is the only topology on Y for which the characterisitc *property holds.*

Theorem 17 (Descends to the Quotient)

Suppose $\pi : X \to Y$ is a quotient map, Z a topological space, and $f : X \to Z$ is any continuous map that is constant on the fibres of π [1]*. Then there exists a unique continuous map $\tilde{f} : Y \to Z$ such that $f = \tilde{f} \circ \pi$ (cf Figure 8.7).*

Exercise 8.1.3
Prove Theorem 16.

[1] This means that if $\pi(x) = \pi(x')$, then $f(x) = f(x')$.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ Y \arrow[r, dashed, "\tilde{f}"'] & Z \end{tikzcd}$$

Figure 8.7: Descends to the Quotient

Proof

Since π is surjective, $\forall y \in Y$, $\exists x \in X$ such that $\pi(x) = y$. Then consider $\tilde{f} : Y \to Z$ given by $\tilde{f}(y) = f(x)$ for any x that we discover from the last statement. By the hypothesis on f, $\tilde{f}$ is guaranteed to be unique and well-defined. It follows from Theorem 15 that $\tilde{f}$ is continuous. □

The following theorem, which is a consequence of Theorem 15, gives to us that quotient spaces are unique up to homeomorphism by the identifications made by their quotient maps.

Theorem 18 (Uniqueness of Quotient Spaces)

Suppose $\pi_1 : X \to Y_1$ and $\pi_2 : X \to Y_2$ are quotient maps that make the same identifications, i.e.

$$\pi_1(x) = \pi_1(x') \iff \pi_2(x) = \pi_2(x').$$

Then there exists a unique homeomorphism $\phi : Y_1 \to Y_2$ such that $\pi \circ \pi_1 = \pi_2$.

The proof is relatively straightforward so I will only jot down the gist of the proof and provide commutative diagrams for visualization of the relationships between these spaces.

Proof (Sketch)
Observe Figure 8.8.

$$\begin{tikzcd} X \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Y_1 \arrow[r, dashed, "\tilde{\pi}_2"'] & Y_2 \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ Y_2 \arrow[r, dashed, "\tilde{\pi}_1"'] & Y_1 \end{tikzcd}$$

Figure 8.8: Relationships of the quotient spaces

Then

$$\tilde{\pi}_1 \circ (\tilde{\pi}_2 \circ \pi_1) = \tilde{\pi}_1 \circ \pi_2 = \pi_1. \qquad \square$$

Thus, we have Figure 8.9.

$$\begin{tikzcd} X \arrow[d, "\pi_1"'] \arrow[dr, "\pi_1"] & \\ Y_1 \arrow[r, dashed, "\tilde{\pi}_1 \circ \tilde{\pi}_2"'] & Y_1 \end{tikzcd}$$

Figure 8.9: Consequence of the relationship between the quotient spaces.

Then by Theorem 16, the map is indeed unique, and it follows from the identity map that the two maps are equal. Similarly, $\tilde{\pi}_2 \circ \tilde{\pi}_1$ is the identity on Y_2.

Then $\phi = \tilde{\pi}_2$ is the required homeomorphism, and it is unique by Theorem 16.

9 Lecture 9 Jan 25th

9.1 Topological Embeddings

Definition 33 (Topological Embedding)

An injective continuous map $g : S \to Y$ is called a ***topological embedding*** *(or just an* ***embedding****) if it is a homeomorphism onto its image.*

Note 9.1.1

In other words, $g : S \to Y$ is called an embedding if it is a homeomorphism between S and $g(S)$.

Example 9.1.1

Consider the function $f : \mathbb{R} \to \mathbb{R}^3$ given by

$$x \mapsto (\cos x, \sin x, x).$$

We know that f is both bijective and continuous (since each of its components are continuous). Note that

$$f \circ \pi_3 \restriction_{f(\mathbb{R})} = \mathrm{id}_{\mathbb{R}} .$$

It follows that f is an embedding.

Example 9.1.2

The map $f : x \mapsto e^{ix}$ from $[0, 2\pi)$ to $\mathbb{C}$ is continuous and injective, but not a homeomorphism (problem lies on the endpoints). So f is not an embedding, despite being continuous and injective.

However, the restriction of f to any proper subinterval is an embedding, as is the interval $(0, 2\pi)$.

As we've seen above, a continuous injective map is not necessarily an embedding. The following proposition provides us with two sufficient but not necessary conditions to ensure that a continuous injective map is an embedding.

Warning
*Notice that by our definition, and by taking note on Exercise 1.1.1, an embedding is **only a homeomorphism between the domain and its image** under the mapping. It need not be a homeomorphism between the domain and the codomain. Thus, an embedding need not be open or closed.*

Proposition 19 (Sufficient Conditions to be an Embedding)

A continuous injective map that is either open or closed is an embedding.

Proof
Let $f : X \to Y$ be a continuous injective map between 2 topological spaces. Note that f is bijective from X to $f(X)$. It suffices to show that f is open or closed from X to $f(X)$ by Exercise 1.1.1.

By assumption, if $f : X \to Y$ is open, then for any open $A \subset X$, $f(A) \subset f(X)$ is open, simply by definition. It also follows that if f is closed, then for any closed $B \subset X$, $f(B)$ is closed in $f(X)$. $\square$

Example 9.1.3
The map $f : \left(0, \frac{1}{2}\right) \to S^1$ given by $x \to e^{2\pi i x}$ is an embedding but it is neither open nor closed.

Exercise 9.1.1
Give another example of a topological embedding that is neither open nor closed.

Proposition 20 (Surjective Embeddings are Homeomorphisms)

A surjective topological embedding is a homeomorphism.

Exercise 9.1.2
Prove Proposition 20.

Remark 9.1.1

We now have some proper tools to build various examples of manifolds as subspaces of Euclidean spaces. Recall Proposition 3 and Proposition 12, to show that a subspace of $\mathbb{R}^n$ is a manifold, we need only to verify that it is locally Euclidean.

Example 9.1.4

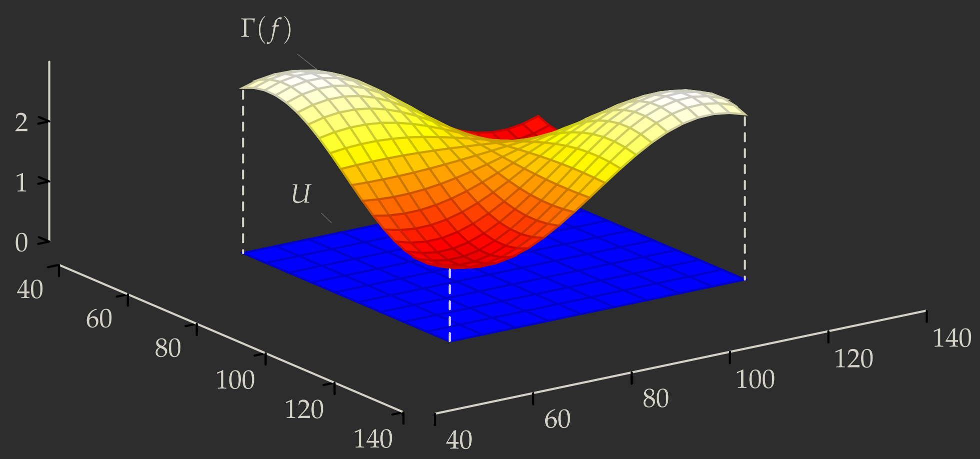

Figure 9.1: Graph of a continuous function

If $U \subseteq \mathbb{R}^n$ is an open subset and $f : U \to \mathbb{R}^k$ is any continuous map, the graph of f (cf Figure 9.1) is the subset $\Gamma(f) \subseteq \mathbb{R}^{n+k}$ defined by

$$\Gamma(f) = \{(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_k) : x \in U,\, y = f(x)\},$$

with the subspace topology inherited from $\mathbb{R}^{n+k}$. To verify that $\Gamma(f)$ is indeed a manifold, we need only to show that $\Gamma(f)$ is homeomorphic to U. Let $\phi_f : U \to \mathbb{R}^{n+k}$ be the continuous injective map

$$\phi_f(x) = (x, f(x)).$$

One can verify that ϕ_f is indeed a continuous bijection from U to $\Gamma(f)$, and for $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$, $\pi_{\Gamma(f)}$ is a continuous inverse for ϕ_f. It follows that ϕ_f is an embedding and $\Gamma(f)$ is homeomorphic to U.

Example 9.1.5 (*n*-spheres are Manifolds)

Recall that the n-sphere (or *unit*) n-sphere is the set S^n of unit vectors in $\mathbb{R}^n$. In low dimensions, spheres are easy to visualize:

- S^0 is the two-point discrete space $\{\pm 1\} \subseteq \mathbb{R}$;
- S^1 is the unit circle in $\mathbb{R}^2$; and
- S^2 is the unit spherical surface of radius 1 in $\mathbb{R}^3$.

Since we are working in $\mathbb{R}^n$, by Remark 9.1.1, it suffices for us to show that each of the S^n's are locally Euclidean. We shall show that each point has a neighbourhood in S^n that is the graph of a continuous function.

For each $i \in \{1, \ldots, n+1\}$, let

U_i^+ denote the open subset of $\mathbb{R}^{n+1}$ consisting of points with $x_i > 0$, and

U_i^- denote the open subset of $\mathbb{R}^{n+1}$ consisting of points with $x_i < 0$.

Then for any $x = (x_1, \ldots, x_n) \in S^n$, some coordinate x_i must be nonzero, and so the sets $U_1^\pm, \ldots, U_{n+1}^\pm$ cover S^n. Now for each $U_i^\pm$, we can solve for the equation $|x| = 1$, and find that $x \in S^n \cap U_i^\pm$ iff

$$x_i = \pm \sqrt{1 - \sum_{\substack{j=1 \\ j \neq i}}^{n+1} x_j^2}.$$

Since the square root is a continuous function, it follows that the intersection of S^n with $U_i^\pm$ is the graph of a continuous function. This intersection is therefore locally Euclidean, showing to us that S^n is indeed a manifold.

The following lemma allows us to, essentially, 'glue' surfaces to one another.

Lemma 21 (Glueing Lemma)

Let X and Y be topological spaces, and let $\{A_i\}$ be either an arbitrary open cover of X, or a finite closed cover of X. Suppose that we are given continuous maps $f_i : A_i \to Y$ that agree on overlaps, i.e.

$$f_i \restriction_{A_i \cap A_j} = f_k \restriction_{A_i \cap A_j} .$$

Then there exists a unique continuous map $f : X \to Y$ whose restriction to each A_i is equal to f_i.

With that, we can construct the following space.

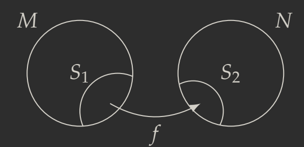

Figure 9.2: Glueing subsets of two disjoint spaces.

Definition 34 (Adjunction Space)

Consider 2 manifolds M and N that are of the same dimension, and let $S_1 \subseteq M$ and $S_2 \subseteq N$. Let $f : S_1 \to S_2$ be a homeomorphism (cf Figure 9.2). Then we define

$$M \cup_f N := M \coprod N \Big/ \left\{ \substack{a \sim f(a) \\ a \in S_1} \right.$$

*as the **adjunction space**, and is said to be formed by **attaching** Y **to** X **along** f. The map f is called the **attaching map**.*

Remark 9.1.2

By Lemma 21, there exists a continuous map between M and N, and so this allows us to know that this new structure $M \cup_f N$ is indeed a manifold.

Definition 35 (Double)

*If $M = N$, with the identity map $\mathrm{id} \restriction_{\partial M}$ as a homeomorphism between ∂M and ∂N, then we call $M \cup_{\mathrm{id}\restriction_{\partial M}} N$ the **double** of M.*

Lemma 22 (Attaching Manifolds along Their Boundaries)

Suppose M is an n-dimensional manifold with boundary. Then its double $M \cup_{\mathrm{id}\restriction_{\partial M}} M$ is a manifold without boundary. More generally, if M_1 and M_2 are manifolds with non-empty boundaries, then there are topological embeddings $e : M_1 \cup_h M_2$ and $f : M_1 \cup_h M_2$ whose images are closed subsets of $M_1 \cup_h M_2$ satisfying

$$e(M_1) \cup f(M_2) = M_1 \cup_h M_2$$
$$e(M_1) \cap f(M_2) = e(\partial M_1) = f(\partial M_2).$$

The core idea of the proof is illustrated in Figure 9.3.

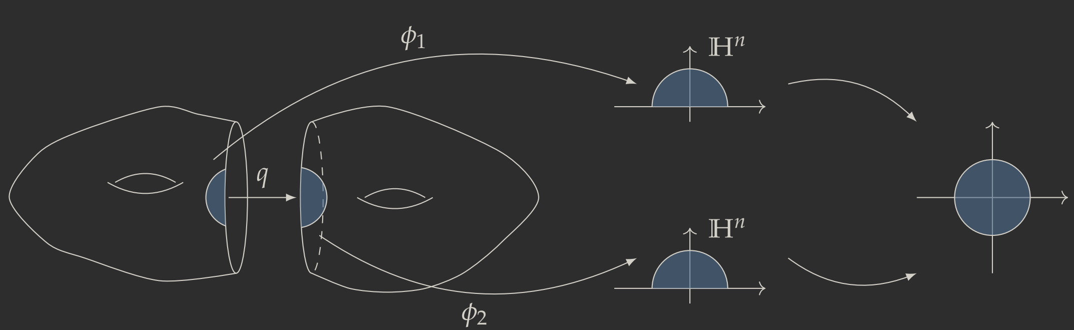

Figure 9.3: Attaching along the boundaries

Proof (Sketch)

Step 1 Find q.

Step 2 $q \restriction_{(} \operatorname{Int}(M_1))$ is an embedding.

Step 3 Define ϕ_1 and ϕ_2.

Step 4 Put the two together.

To END this lecture today, we recalled that we can look at the torus as a 2-dimensional rectangle with its sides properly identified. Now if we change the identification of one of the sides by swapping its orientation, we end up with what is known as a **Möbius band**.

The Möbius strip in Figure 9.4 is taken from TeX SE.

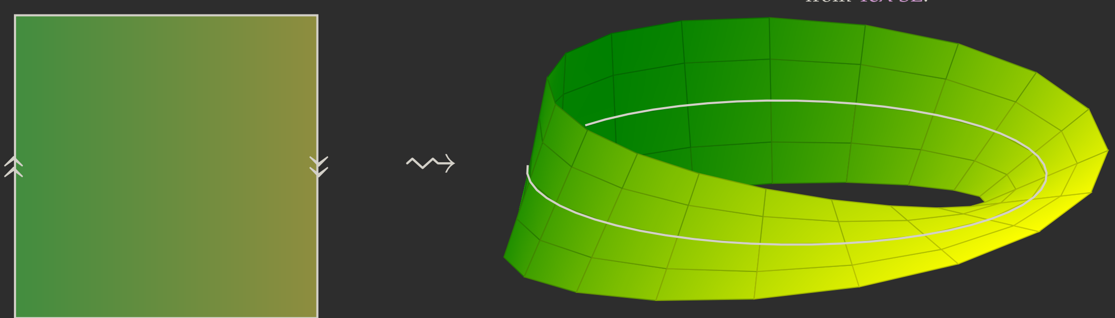

Figure 9.4: Möbius band

Following that, if we also swap the orientation of the other two sides, we get what is known as the **Klein Bottle**.

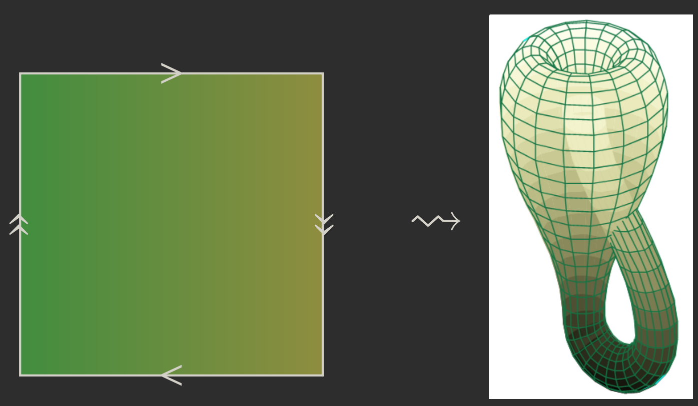

Figure 9.5: Klein Bottle

10 Lecture 10 Jan 28th

10.1 An Enpasse into Orientability

We talked about orientability in the last lecture. We shall now introduce a more concrete way of talking about orientability of a surface.

Definition 36 (Frame)

*A **frame** is a continuous pair of vectors v_1, v_2 defined on the entire surface such that v_1 and v_2 are linearly independent.*

Example 10.1.1

Here are two graphical examples of a frame:

- **Cylinder**

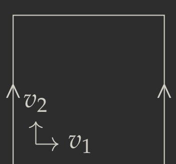

Figure 10.1: Frame on a Cylinder

Note that if we 'move' this frame towards the identified sides, the 'direction' of which the vectors 'point at' remains the same (cf. Figure 10.2).

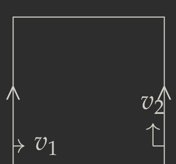

Figure 10.2: Frame on the identified sides of a cylinder

- **Möbius strip**

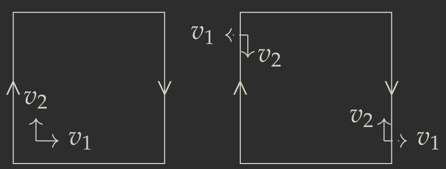

Figure 10.3: Frame on a Möbius strip

The two frames on the second diagram in Figure 10.3 are the same frame. Notice that v_2 is now 'pointing' in the 'opposite direction' simultaneously.

Note 10.1.1

The notion of having a 'side' for a Möbius strip makes sense when we embed it in $\mathbb{R}^3$.

Example 10.1.2

Recall Example 8.1.4, where we introduced the projective space. Since $S^n \simeq \mathbb{R}^{n+1}$, we can define a projective space with respect to n-spheres. Let us consider $n = 2$, and consider the group action $\mathcal{G}_2 = \{-1, 1\}$ given by

$$(-1)(x, y, z) = (-x, -y, -z) \text{ and } (1)(x, y, z) = (x, y, z).$$

Then

$$\mathbb{P}^2 = S^2 / \mathcal{G}_2 \simeq \overline{S^+}^2 / {}_{\substack{(x,y,z)\sim(-x,-y,-z) \\ (x,y,z)\in\partial S^+}},$$

where S^+ is the upper hemisphere.

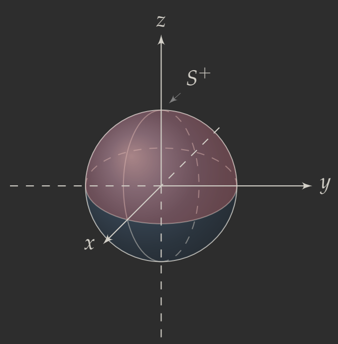

Figure 10.4: Projective Plane from S^2

In a very simple sense, we are 'compressing' the sphere by identifying the upper hemisphere with the lower hemisphere.

Note that because of this construction, $\mathbb{P}^2$ is not orientable: a point where the vector pointing at a direction parallel to y on the upper hemisphere is identified with a point whose corresponding vector points at the opposite direction of y.

Example 10.1.3

Consider the diagram presented in Figure 10.5. Now we have a more concrete reason to explain why the Klein Bottle is not embeddable

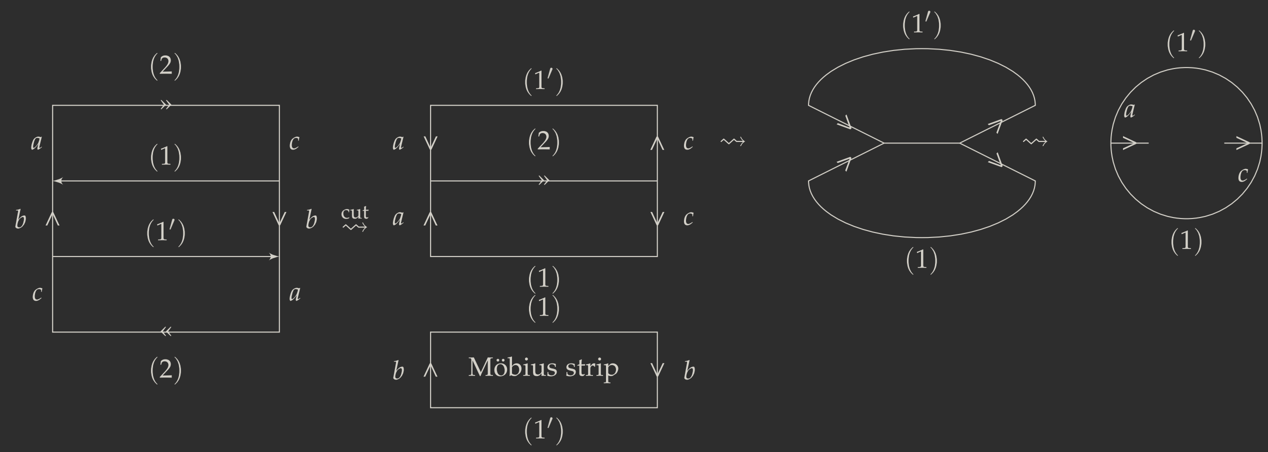

Figure 10.5: Klein Bottle as a Möbius Strip and a Disc

into $\mathbb{R}^3$, since the disc itself is not embeddable into $\mathbb{R}^3$.

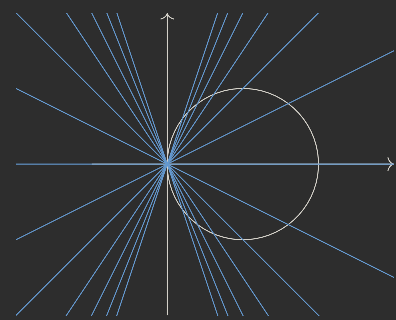

Figure 10.6: Shifted S^1 for the moduli space, as shown in Figure 6.5.

Recall Figure 6.5 where we saw how S^1 is homeomorphic to the space of all lines in $\mathbb{R}^2$ passing through the origin $(0, 0)$.

It follows, therefore, that $S^1 \simeq \mathbb{P}^1$.

> *What about the space of all lines in $\mathbb{R}^2$?*

Observing Figure 10.7, we see that since we can rotate γ at the origin, we may, in particular, rotate γ by $180°$ to get the opposite direction. It follows that this space of all lines in $\mathbb{R}^2$ is thus homeomorphic to $\mathfrak{M}$, a Möbius strip. Thus, the space of all lines in $\mathbb{R}^2$ is also not orientable.

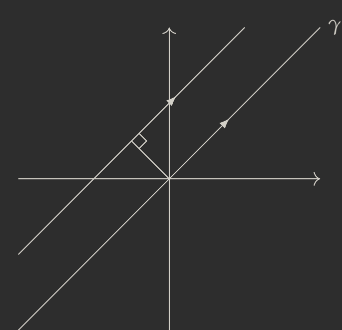

Figure 10.7: Space of all lines in $\mathbb{R}^2$

Following the same question as above, we may ask:

> *What is the space of all lines passing through $(0, 0, 0)$ in $\mathbb{R}^3$?*

It is not difficult to see that this space is homeomorphic to S^2, via the map $x \mapsto \frac{x}{|x|}$ (which is also a quotient map).

Just right before this, we observed, in Example 10.1.2 that

$$S^2 / x \sim (-x) \simeq \mathbb{P}^2,$$

where $x \sim (-x)$ via what is called the **antipedal map**.

> *What about the space of all lines in $\mathbb{R}^3$?*

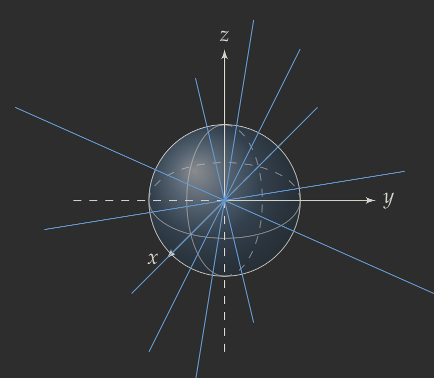

Figure 10.8: Space of all lines passing through $(0, 0, 0)$ in $\mathbb{R}^3$

Following a similar observation in Figure 10.7, we have the following:

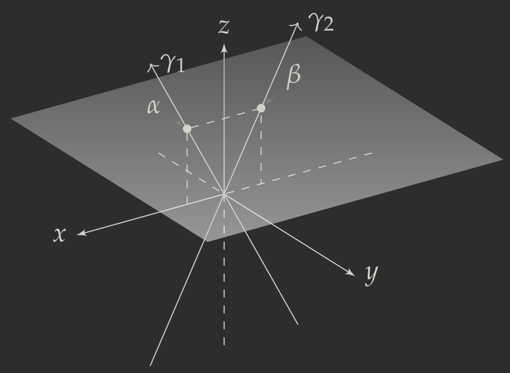

Figure 10.9: Space of all lines in $\mathbb{R}^3$

Observe that if we rotate γ_1 and γ_2 downwards, anchored at α and β respectively, we will eventually get that the two lines are the same line but pointing at different directions. We see that the space of all lines is also not orientable.

Bibliography

Lee, J. M. (2000). *Graduate Texts in Mathematics: Introduction to Topological Manifolds*. Springer.

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