1 Complex Numbers and Their Properties

Complex Plane as a Set

$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$

Real and Imaginary Part of a Complex Number

$\forall z = x + iy \in \mathbb{C} \; x, y \in \mathbb{R}$
$\mathrm{Re}(z) = x \; \mathrm{Im}(z) = y$

Product of Complex Numbers

$\forall z = a + ib, w = c + id \in \mathbb{C} \; a, b, c, d \in \mathbb{R}$
$zw = (ac - bd) + i(ad + bc)$

Inverse of a Complex Number

$\forall z = a + ib \in \mathbb{C} \; a, b \in \mathbb{R}$
$\exists z^{-1} = \frac{a}{aa^2+b^2} - i\frac{b}{a^2+b^2} \in \mathbb{C}$

Conjugate of a Complex Number

$\forall z = a + ib \in \mathbb{C} \; a, b \in \mathbb{R}$
$\exists \overline{z} = a - ib \in \mathbb{C}$

Modulus of a Complex Number

$\forall z = x = iy \in \mathbb{C} \; x, y \in \mathbb{R}$
$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}$

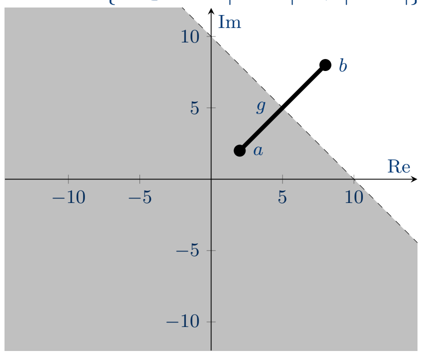
Basic Inequalities of Complex Numbers

$\forall z, w \in \mathbb{C}$,

1. $|\mathrm{Re}(z)| \leq |z|$
2. $|\mathrm{Im}(z)| \leq |z|$
3. $|z + w| \leq |z| + |w|$
4. $|z + w| \geq \big|\, |z| + |w| \,\big|$

Region of a set of Complex Numbers

Describe $\{z \in \mathbb{C} : |z - a| < |z - b|\}$.



Every complex number has exactly 2 roots

$\forall z = x = iy \in \mathbb{C} \; x, y \in \mathbb{R}$
$\exists w_{1,2} = u + iv \in \mathbb{C} \; u, v \in \mathbb{R}$

$$w = \begin{cases} \pm\left[\left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}} + i\left(\frac{-x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}\right] & y > 0 \\ \pm\left[\left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}} - i\left(\frac{-x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}\right] & y < 0 \\ \pm\sqrt{x} & y = 0, x > 0 \\ \pm i\sqrt{x} & y = 0, x < 0 \end{cases}$$

Quadratic Formula for Complex Numbers

$\forall a, b, c \in \mathbb{C} \; a \neq 0 \; az^2 + bz + c = 0$

$$z = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$