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# **14** Lecture 14 Jun 01 2018

## **14.1** *Isomorphism Theorems (Continued 2)*

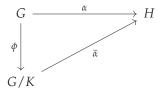
### **14.1.1** *Isomorphism Theorems (Continued)*

#### Note (Recall)

In First Isomorphism Theorem 38, we had that for a group homomorphism  $\alpha: G \to H$  where G and H are groups,

$$G_{\ker \alpha} \cong \operatorname{im} \alpha$$

Now let  $\alpha: G \to H$  be a group homomorphism,  $K = \ker \alpha$ ,  $\phi: G \to G/K$  be the coset map, and  $\bar{\alpha}$  be as defined in the proof of First Isomorphism Theorem 38. We then have the following commutative diagram to illustrate the relationship between the three groups.



A natural question to ask after seeing the relationship is: Is  $\bar{\alpha}\phi = \alpha$ ? If it is, is the definition of  $\bar{\alpha}$  unique? The answer is: **YES!** on both accounts.

#### **Proof**

Let  $g \in G$ . Then

$$\bar{\alpha}\phi(g) = \bar{\alpha}(\phi(g)) = \bar{\alpha}(Kg) = \alpha(g)$$

Suppose  $\alpha = \beta \phi$  where  $\beta : {}^G\!/_K \to H$ . Then

$$\beta(Kg) \stackrel{(1)}{=} \beta(\phi(g)) = \beta\phi(g) = \alpha(g) = \bar{\alpha}(Kg)$$

where (1) is because  $\phi$  is surjective by Proposition 35. Therefore, we observe that  $\beta = \bar{\alpha}$  for any  $Kg \in {}^{G}/_{K}$ . This proves that  $\bar{\alpha}$  is the unique homomorphism such that  ${}^{G}/_{K} \to H$  satisfying  $\alpha = \bar{\alpha}\phi$ .

With that, we have the following proposition.

#### **Proposition 39**

Let  $\alpha: G \to H$  be a group homomorphism, where G and H are groups. Let  $K = \ker \alpha$ . Then  $\alpha$  factors uniquely as  $\alpha = \bar{\alpha}\phi <$  where  $\phi: G \to {}^G/_K$  is the coset map and  $\bar{\alpha}: GK \to H$  is defined by

$$\bar{\alpha}(Kg) = \alpha(g).$$

*Note that*  $\phi$  *is surjective and*  $\bar{\alpha}$  *is injective.* 

In such a scenario, we also say that  $\alpha$  factors through  $\phi$ .<sup>1</sup>

<sup>1</sup> Reference for the terminology: https://math.stackexchange. com/questions/68941/ terminology-a-homomorphism-factors.

#### Example 14.1.1

Let  $G = \langle g \rangle$  be a cyclic group. Consider  $\alpha : \mathbb{Z} \to G$ , defined as

$$\forall k \in \mathbb{Z} \quad \alpha(k) = g^k,$$

which is a group homomorphism. By definition,  $\alpha$  is surjective. Note that

$$\ker \alpha = \{k \in \mathbb{Z} : g^k = 1\}.$$

We have, therefore, two cases to consider.

#### • *G* is an infinite group

This would imply that  $\ker \alpha = \{0\}$  since only  $g^0 = 1$ . Then by First Isomorphism Theorem 38, we have that

$$\mathbb{Z}_{\ker \alpha} \cong G$$

Note that<sup>2</sup>

$$\mathbb{Z}_{\ker \alpha} = \{(\ker \alpha)k : k \in \mathbb{Z}\} = \{0 + k : k \in \mathbb{Z}\} = \mathbb{Z}.$$

<sup>2</sup> We are assuming that the group  $\mathbb{Z}$  here works under the operation of addition, otherwise, if we employ multiplication, then  $\mathbb{Z}$  would not be a group and  $\alpha$  would not be a group homomorphism.

*Therefore* 

$$\mathbb{Z}\cong G$$

• *G* is a finite group

Suppose that  $|G| = o(g) = n \in \mathbb{N}$ , which is valid by Corollary 24. Then

$$\ker \alpha = n\mathbb{Z}$$

Then by the First Isomorphism Theorem 38, we have

$$\mathbb{Z}_{n\mathbb{Z}} \cong G$$
.

Observe that

$$\mathbb{Z}_{n\mathbb{Z}} = \{n\mathbb{Z} + k : k \in \mathbb{Z}\} = \mathbb{Z}_n$$

since the set in the middle is the definition of the set of integers modulo n.3 Therefore,

$$\mathbb{Z}_n \cong G$$

$$\mathbb{Z} \cong G \text{ or } \mathbb{Z}_{o(g)} \cong G$$

<sup>3</sup> This is why we often see texts from various authors using  $\mathbb{Z}/_{n\mathbb{Z}}$  to represent the set of integers modulo n.

Theorem 40 (Second Isomorphism Theorem)

Let H and K be the subgroups of a group G with  $K \triangleleft G$ . Then

- HK is a subgroup of G;
- *K* ⊲ *HK*;
- $H \cap K \triangleleft H$ ; and
- $HK/K \cong H/H \cap K$

#### **Proof**

Since  $K \triangleleft G$ , by Lemma 29 and Proposition 30, we have that HK = KHis a subgroup of G. Consequently, we have  $K \triangleleft HK$ , since K is clearly a subgroup of HK and  $K \triangleleft G$ , and so  $\forall x \in HK \subseteq G$  we have that gK = Kg.

Consider  $\alpha: H \to {HK}/_K$ , defined by<sup>4</sup>

 $^4$  Note that  $Kh \in {HK}/_{K}$  since  $h \in H \subseteq$ HK.