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Chapter 1

Lecture 1 Jan 3 2018

1.1 Complex Numbers and Their Properties

Definition 1.1.1 (Complex Number, Complex Plane)

A **complex number** is a vector in \mathbb{R}^2 . The **complex plane**, denoted by \mathbb{C} , is a set of complex numbers,

$$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

In \mathbb{C} , we usually write

$$\begin{aligned} 0 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & 1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ i &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} & x &= \begin{pmatrix} x \\ 0 \end{pmatrix} \\ iy &= \begin{pmatrix} 0 \\ y \end{pmatrix} \end{aligned}$$

where $x, y \in \mathbb{R}$. Consequently, we have that

$$x + iy = x + yi = \begin{pmatrix} x \\ y \end{pmatrix}$$

If for $x, y \in \mathbb{R}$, $z = x + iy$, then x is called the **real part** of z and y is called the **imaginary part** of z , and we write

$$\operatorname{Re}(z) = x \quad \operatorname{Im}(z) = y.$$

Note

- It is easy to see how \mathbb{R} is a subset of \mathbb{C} .

- Complex Numbers of the form $\begin{pmatrix} 0 \\ y \end{pmatrix}$ where $y \in \mathbb{R}$ are called **purely imaginary numbers**.
- Certain authors may prefer to denote $i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Definition 1.1.2 (Sum and Product)

We define the sum of two complex numbers to be the usual vector sum, i.e.

$$\begin{aligned} (a + ib) + (c + id) &= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \begin{pmatrix} a + c \\ b + d \end{pmatrix} \\ &= (a + c) + i(b + d) \end{aligned}$$

where $a, b, c, d \in \mathbb{R}$.

We define the product of two complex numbers by setting $i^2 = -1$, and by requiring the product to be **commutative, associative, and distributive** over the sum. In this setup, we have that

$$\begin{aligned} (a + ib)(c + id) &= ac + iad + ibc + i^2bd \\ &= (ac - bd) + i(ad + bc) \end{aligned} \tag{1.1}$$

Note

It is interesting to note that **any complex number times zero is zero**, just like what we have with real numbers.

$$\begin{aligned} \forall z = x + iy \in \mathbb{C} \quad x, y \in \mathbb{R} \quad 0 \in \mathbb{C} \\ z \cdot 0 = (x + iy)(0 + i0) = 0 + i0 = 0 \end{aligned}$$

Example 1.1.1

Let $z = 2 + i, w = 1 + 3i$. Find $z + w$ and zw .

$$\begin{aligned} z + w &= (2 + i) + (1 + 3i) \\ &= 3 + 4i \end{aligned}$$

$$\begin{aligned} zw &= (2 + i)(1 + 3i) \\ &= (2 - 3) + i(6 + 1) \quad \text{By Equation (1.1)} \\ &= -1 + 7i \end{aligned}$$

Example 1.1.2

Show that every non-zero complex number has a **multiplicative inverse**, z^{-1} , and find a formula for this inverse.

Let $z = a + ib$ where $a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$. Then

$$\begin{aligned}
 z(x + iy) &= 1 \\
 \iff (ax - by) + i(ay + bx) &= 1 \\
 \iff \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \iff \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \iff \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \iff \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{a^2 + b^2} \begin{pmatrix} a \\ -b \end{pmatrix} \\
 \iff x + iy &= \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2}
 \end{aligned}$$

Therefore, we have that the formula for the inverse is

$$(a + ib)^{-1} = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2} \quad (1.2)$$

Notation

For $z, w \in \mathbb{C}$, we write

$$\begin{aligned}
 -z &= -1z & w - z &= w + (-z) \\
 \frac{1}{z} &= z^{-1} & \frac{w}{z} &= wz^{-1}
 \end{aligned}$$

Example 1.1.3

Find $\frac{(4-i)-(1-2i)}{1+2i}$.

$$\begin{aligned}
 \frac{(4-i)-(1-2i)}{1+2i} &= \frac{3+i}{1+2i} \\
 &= (3+i)\left(\frac{1}{5} - i\frac{2}{5}\right) \\
 &= 1 - i
 \end{aligned}$$

Note

The set of complex numbers is a **field** under the operations of addition and multiplication. This means that $\forall u, v, w \in \mathbb{C}$,

$$\begin{array}{ll}
u + v = v + u & uv = vu \\
(u + v) + w = u + (v + w) & (uv)w = u(vw) \\
0 + u = u & 1u = u \\
u + (-u) = 0 & uu^{-1} = 1, \quad u \neq 0 \\
u(v + w) = uv + uw &
\end{array}$$

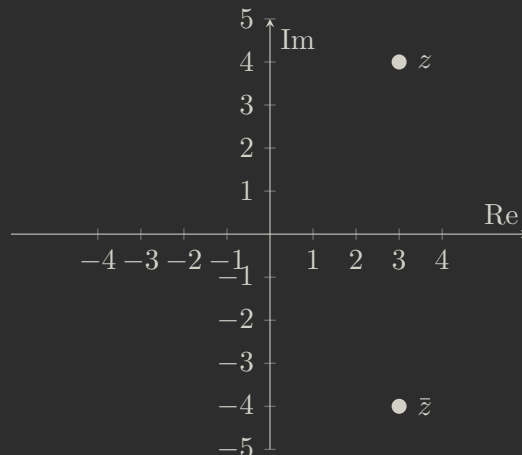
Since the distributive law holds for complex numbers, note that the **binomial expansion works** for $(w + z)^n$ where $w, z \in \mathbb{C}$ and $n \in \mathbb{N}$. (I did not verify if this is still true for when $n \in \mathbb{R}$.)

Definition 1.1.3 (Conjugate)

If $z = x + iy$ where $x, y \in \mathbb{R}$, then the **conjugate of z** is given by $\bar{z} = x - iy$

Example 1.1.4

Let $z = 3 + 4i$. Then the $\bar{z} = 3 - 4i$. Represented in the complex plane, we have the following:



We observe that on the complex plane, the conjugate of a complex number is simply its reflection on the real axis.

Definition 1.1.4 (Modulus)

We define the **modulus** (length, magnitude) of $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$, to be

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}. \quad (1.3)$$

Note

Note that this definition is consistent with the notion of the absolute value in real numbers when z is a real number, since if $y = 0$, $|z| = |x + i0| = \sqrt{x^2} = \pm x$.

Note

For $z, w \in \mathbb{C}$ and $n \in \mathbb{N}$, we have

$$\begin{array}{lll} \bar{\bar{z}} = z & z + \bar{z} = 2 \operatorname{Re}(z) & z - \bar{z} = 2i \operatorname{Im}(z) \\ z\bar{z} = |z|^2 & |z| = |\bar{z}| & \overline{z \pm w} = \bar{z} \pm \bar{w} \\ \overline{zw} = \bar{z}\bar{w} & |zw| = |z| |w| & \bar{z}^n = \overline{z^n} \end{array}$$

but note that $|z + w| \neq |z| + |w|$.

Also, note that the last equation is a generalization of the **highlighted equation**.

Note

While inequalities such as $z_1 < z_2$, where $z_1, z_2 \in \mathbb{C}$, are meaningless unless if both of them are real, $|z_1| < |z_2|$ means that the point z_1 in the complex plane is closer to the origin than the point z_2 .

Proposition 1.1.1 (Basic Inequalities)

1. $|\operatorname{Re}(z)| \leq |z|$
2. $|\operatorname{Im}(z)| \leq |z|$
3. $|z + w| \leq |z| + |w|$ *Triangle Inequality*
4. $|z + w| \geq ||z| - |w||$ *Inverse Triangle Inequality*

Proof

Note that $|z|^2 = \operatorname{Re}(z)^2 + \operatorname{Im}(z)^2$ and that we can express $|x| = \sqrt{x^2}$ for any $x \in \mathbb{R}$. 1 and 2 immediately follows from that.

To prove 3, we have that

$$\begin{aligned} |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) \\ &= |z|^2 + |w|^2 + (w\bar{z} + \bar{w}z) \\ &= |z|^2 + |w|^2 + 2 \operatorname{Re}(w\bar{z}) \\ &\leq |z|^2 + |w|^2 + 2 |w\bar{z}| \quad \text{by 1} \\ &= |z|^2 + |w|^2 + 2 |wz| \quad \text{since } |w\bar{z}| = |w| |\bar{z}| \text{ and } |z| = |\bar{z}| \\ &= (|z| + |w|)^2 \end{aligned}$$

To prove 4, note that

$$|z| = |z + w - w| \leq |z + w| + |w| \quad (1.4)$$

$$|w| = |w + z - z| \leq |z + w| + |z| \quad (1.5)$$

Observe that

$$\text{Equation (1.4)} \implies |z| - |w| \leq |z + w|$$

$$\text{Equation (1.5)} \implies |w| - |z| \leq |z + w|$$

Thus, we have that

$$|z + w| \geq ||z| - |w||$$

as required. \square

Item 3 in Proposition 1.1.1 can be generalized by the means of mathematical induction to sums involving any finite number of terms, as:

$$|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n| \quad (1.6)$$

where $n \in \mathbb{N} \setminus \{0, 1\}$.

To note the induction proof, when $n = 2$, Equation (1.6) is just Item 3. If Equation (1.6) is true for when $n = m$ where $m \in \mathbb{N} \setminus \{0, 1\}$, $n = m + 1$ is also true since by Item 3,

$$\begin{aligned} |(z_1 + z_2 + \dots + z_m) + z_{m+1}| &\leq |z_1 + z_2 + \dots + z_m| + |z_{m+1}| \\ &\leq (|z_1| + |z_2| + \dots + |z_m|) + |z_{m+1}|. \end{aligned}$$

The distance between two points $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}, x_1, x_2, y_1, y_2 \in \mathbb{R}$ is $|z_1 - z_2|$, since $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ is our usual notion of the Euclidean distance of two points on a plane.

Also, note that

$$z_1 - z_2 = z_1 + (-z_2)$$

and thus if we apply our knowledge of vector representation, $z_1 - z_2$ is the directed line segment from the point z_2 to z_1 .

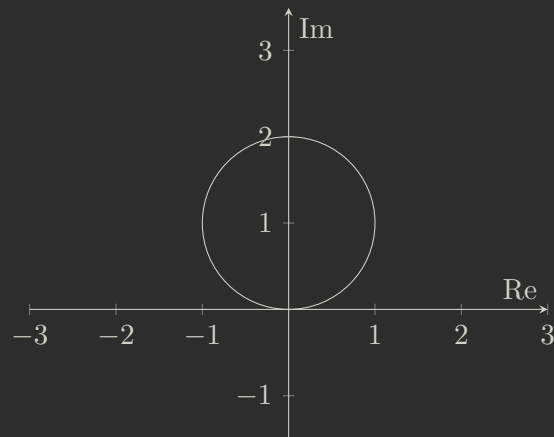
With the notion of a “distance” set on the complex plane, we can now explore upon points lying on a circle with a center z_0 and radius R , which satisfies the equation

$$|z - z_0| = R.$$

We may simply refer to this set of points as the circle $|z - z_0| = R$.

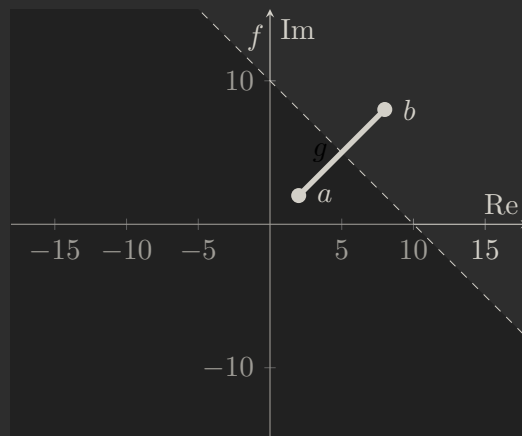
Example 1.1.5

We may describe a set $\{z \in \mathbb{C} : |z - i| = 1\}$ as follows:



Let $a, b \in \mathbb{C}$ describe the set $\{z \in \mathbb{C} : |z - a| < |z - b|\}$.

Suppose the following coordinates for a and b are arbitrary,



In the above, g is the line segment that connects the points a and b on the complex plane, while f is the perpendicular bisector of the line segment g . The area described by the set $\{z \in \mathbb{C} : |z - a| < |z - b|\}$ is the shaded area which is below f .

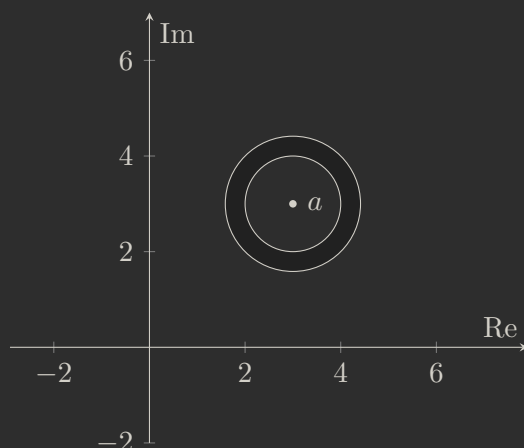
Chapter 2

Lecture 2 Jan 5th 2018

2.1 Complex Numbers and Their Properties (Continued)

Example 2.1.1

Let $a \in \mathbb{C}$. Describe the set $\{z \in \mathbb{C} : 1 < |z - a| < 2\}$.



Example 2.1.2

Show that every non-zero complex number has exactly two complex square roots, and find a formula for the square roots.

Let $z = x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$, and let $w = u + iv$, $u, v \in \mathbb{R}$. Then

Remark

Let $z \in \mathbb{C}$. The notation \sqrt{z} may represent either one of the square roots of z or both of the square roots, i.e. **it is possible that \sqrt{z} represents a set.**

Exercise 2.1.1

Is it always okay for complex numbers such that $\sqrt{zw} = \sqrt{z}\sqrt{w}$, for $z, w \in \mathbb{C}$?

No. For example, consider $z = w = -1$. Then we have

$$\sqrt{zw} = \sqrt{1} = \pm 1$$

while

$$\sqrt{z}\sqrt{w} = i \cdot i = -1$$

and thus

$$\sqrt{zw} \neq \sqrt{z}\sqrt{w}.$$

Example 2.1.3

Find the values of $\sqrt{3-4i}$.

By Example 2.1.2,

$$\begin{aligned} \sqrt{3-4i} &= \pm \left(\sqrt{\frac{3+\sqrt{9+16}}{2}} - i\sqrt{\frac{-3+\sqrt{9+16}}{2}} \right) \\ &= \pm(2-i) \end{aligned}$$

Remark

The quadratic formula holds for complex polynomials, i.e.

$$\forall a, b, c \in \mathbb{C} \quad a \neq 0 \quad \forall z \in \mathbb{C} \quad az^2 + bz + c = 0,$$

the solution for z is given by

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (2.3)$$

The following is a short proof.

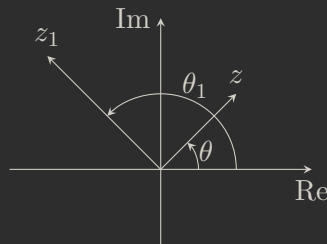
Chapter 3

Lecture 3 Jan 8th 2018

3.1 Complex Numbers and Their Properties (Continued 2)

Definition 3.1.1 (Argument of a Complex Number)

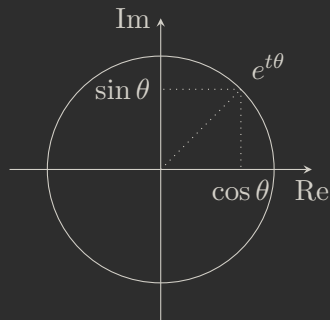
Let $z \in \mathbb{C} \setminus \{0\}$. The **argument** (or the angle) of z , denoted by $\arg z$, $\text{Arg } z$, or simply $\theta = \theta(z)$, is the angle modulo 2π (i.e. $0 \leq \theta < 2\pi$) between the vector defining z and the positive real axis (in the counterclockwise direction).



Notation

Let $e^{i\theta} := \cos \theta + i \sin \theta$. Note that this definition, called **Euler's formula**, can be derived by extending the Taylor expansion of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for when $x \in \mathbb{C}$ (the sum of the real parts of the expansion is the Taylor expansion of cosine while the imaginary part for sine).

Now $e^{i\theta}$ is on the unit circle.

**Remark**

If $z = 0$, the coordinate θ is undefined, and so it is implied that $z \neq 0$ whenever we use the polar form.

Example 3.1.1

Some examples of $\theta \in [0, 2\pi)$:

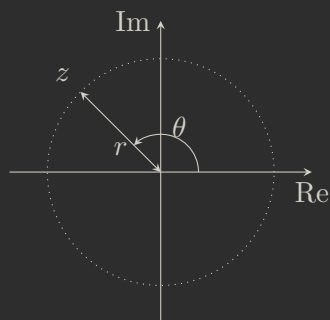
$$\begin{aligned} e^{i\frac{\pi}{4}} &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\frac{\pi}{2}} &= i \\ e^{i\frac{3\pi}{4}} &= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\pi} + 1 &= 0 \end{aligned}$$

Remark

$$\forall k \in \mathbb{Z} \quad \forall \theta \in \mathbb{R} \quad e^{i\theta} = e^{i(\theta+2\pi k)}$$

Remark

The complex number $re^{i\theta}$, where $r > 0, \theta \in [0, 2\pi)$, represents the complex number with modulus r and argument θ .



Therefore, $\forall z \in \mathbb{C}$, we can express

$$z := |z| e^{i \operatorname{Arg} z}. \quad (3.1)$$

The n th roots of z is described by the set

$$\left\{ r^{\frac{1}{n}} e^{i\left(\frac{\theta+2\pi k}{n}\right)} : k = 0, 1, \dots, n-1 \right\} \quad (3.8)$$

Proof

$$\begin{aligned} s^n = r &\iff s = r^{\frac{1}{n}} \\ e^{in\theta} = e^{i\tau} &\iff \theta = \frac{\tau + 2\pi k}{n} \end{aligned}$$

Therefore, the set that describes the n th roots of z is

$$\left\{ w = r^{\frac{1}{n}} e^{i\left(\frac{\theta+2\pi k}{n}\right)} : k = 0, 1, \dots, n-1 \right\}$$

Remark (nth Roots of Unity)

The ***nth roots of unity*** is a direct consequence of *Proposition 3.1.1* where we solve for the equation $z^n = 1$ for any $z \in \mathbb{C}, n \in \mathbb{Z}$.

The set that describes the n th roots of unity is

$$\left\{ e^{i\theta} : \theta = \frac{2\pi k}{n}, k = 0, 1, \dots, n-1 \right\} \quad (3.9)$$

It is easy to see how the n th roots of unity **partitions the unit circle into n parts**.

Example 3.1.3

Find the cubic roots of $-2 + 2i$.

Let $z = -2 + 2i$. Note that $|z| = 2\sqrt{2}$ and $\text{Arg } z = \frac{3\pi}{4}$.

Therefore, in polar form, $z = 2\sqrt{2}e^{i\frac{3\pi}{4}}$.

Let $w = re^{i\theta}$, where $\theta \in [0, 2\pi)$, and $w^3 = z$. Then

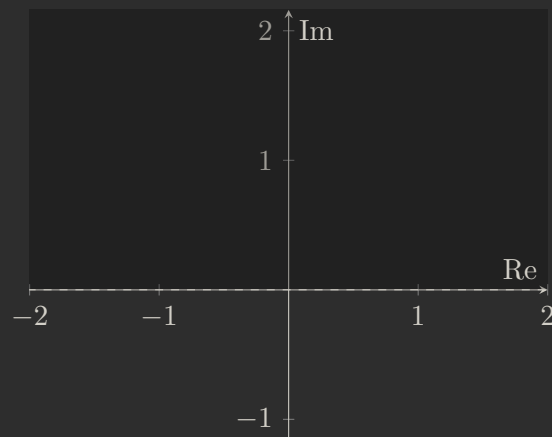
$$\begin{aligned} r &= (2\sqrt{2})^{\frac{1}{3}} \\ \theta &= \frac{\frac{3\pi}{4} + 2\pi k}{3}, \quad k = 0, 1, 2 \end{aligned}$$

The set that describes the cubic root of $-2 + 2i$ is thus

$$\left\{ (2\sqrt{2})^{\frac{1}{3}} e^{i\theta} : \theta = \frac{\frac{3\pi}{4} + 2\pi k}{3}, k = 0, 1, 2 \right\}$$

Example 3.1.4

Describe the set $\{z \in \mathbb{C} : |\operatorname{Arg} z - \frac{\pi}{2}| < \frac{\pi}{2}\}$. (Note: $\operatorname{Arg} z \in [0, 2\pi)$)

**Exercise 3.1.1**

Solve

1. $z^4 = -1$

$$\text{Let } z = re^{i\theta}$$

$$r = |-1| = 1 \quad \theta = \frac{\pi + 2\pi k}{4} = \frac{(2k+1)\pi}{4}, \quad k = 0, 1, 2, 3$$

2. $z^4 = -1 + \sqrt{3}i$

$$\text{Let } z = re^{i\theta}$$

$$r = \left| -1 + \sqrt{3}i \right| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\theta = \frac{\frac{2\pi}{3} + 2\pi k}{4} = \frac{(2k + \frac{2}{3})\pi}{4}, \quad k = 0, 1, 2, 3$$

Chapter 4

Lecture 4 Jan 10th 2018

4.1 Examples for n th Roots of Unity

Recall that the n th roots of unity are given by $e^{i\frac{2\pi k}{n}}, k = 0, 1, \dots, n-1$.

Exercise 4.1.1

Let z be any n th root of unity other than 1. Show that

$$z^{n-1} + z^{n-2} + \dots + z + 1 = 0 \quad (4.1)$$

Proof

By the Sum of Finite Geometric Terms,

$$z^{n-1} + z^{n-2} + \dots + z + 1 = \frac{1 - z^n}{1 - z}.$$

Since $z^n = 1$, RHS is thus zero, which in turn completes the proof.

As an aside, if we wish to remove the restriction that z can also be 1, we may consider that

$$z^n - 1 = (z - 1)(1 + z + \dots + z^{n-1})$$

Since $z^n = 1$, LHS is zero. Then either $z = 1$ or $(1 + z + \dots + z^{n-1}) = 0$.

Exercise 4.1.2

Consider the $n-1$ diagonals of a regular n -gon, inscribed in a circle of radius 1, obtained by connecting one vertex on the n -gon to all its other vertices.

For example, if we are given $n = 6$, we obtain the following diagram.

therefore we obtain

$$\begin{aligned}
 2^{3n} + (1 + \alpha)^{3n} + (1 + \alpha^2)^{3n} &= 3 \sum_{j=0}^n \binom{3n}{3j} \\
 \frac{1}{3} [2^{3n} + (1 + \alpha)^{3n} + (1 + \alpha^2)^{3n}] &= \sum_{j=0}^n \binom{3n}{3j} \\
 \frac{1}{3} [2^{3n} + (-\alpha^2)^{3n} + (-\alpha)^{3n}] &= \sum_{j=0}^n \binom{3n}{3j} \quad \text{since } 1 + \alpha + \alpha^2 = 0 \\
 \frac{1}{3} [2^{3n} + (-1)^n + (-1)^n] &= \sum_{j=0}^n \binom{3n}{3j} \quad \text{since } \alpha^3 = 1 \\
 \frac{2^{3n} + 2(-1)^n}{3} &= \sum_{j=0}^n \binom{3n}{3j}
 \end{aligned}$$

as required.

Exercise 4.1.4

Note that we can define $\text{Arg } z$ in any interval of length 2π , i.e. it is not necessary that $\text{Arg } z \in [0, 2\pi)$.

For example, if we restrict $\text{Arg } z \in [-\pi, \pi]$, then we can write

$$\text{Arg} \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) = -\frac{3\pi}{4}$$

Let z be on the unit circle and $\text{Arg } z \in [-\pi, \pi]$. Suppose that $z \notin \mathbb{R}$, i.e. $z \neq 1, z \neq -1$. Show that

$$\text{Arg} \left(\frac{z-1}{z+1} \right) = \begin{cases} \frac{\pi}{2} & \text{Im } z > 0 \\ -\frac{\pi}{2} & \text{Im } z < 0 \end{cases}$$

Proof

Note that $\forall w_1, w_2 \in \mathbb{C}$, where $\text{Arg } w_1 = \tau_1, \text{Arg } w_2 = \tau_2$ for τ_1, τ_2 in the same 2π -interval,

$$\text{Arg} \frac{w_1}{w_2} = \frac{e^{i\tau_1}}{e^{i\tau_2}} \equiv e^{i(\tau_1 - \tau_2)} = \text{Arg } w_1 - \text{Arg } w_2 \quad (4.7)$$

in modulo 2π .

Suppose $\text{Im } z > 0$. Let $\theta_1 = \text{Arg}(z-1)$ and $\theta_2 = \text{Arg}(z+1)$. Consider Figure 4.3. Note that since both $\theta_1, \theta_2 \in [0, \pi]$, we have that $\theta_1 - \theta_2 \in [-\pi, \pi]$, and thus Equation (4.7) holds

true without the need of the condition of being in modulo 2π . We observe that

$$\begin{aligned}\frac{\pi}{2} &= \theta_2 + \pi - \theta_1 \\ \theta_1 - \theta_2 &= \frac{\pi}{2}\end{aligned}$$

as desired.

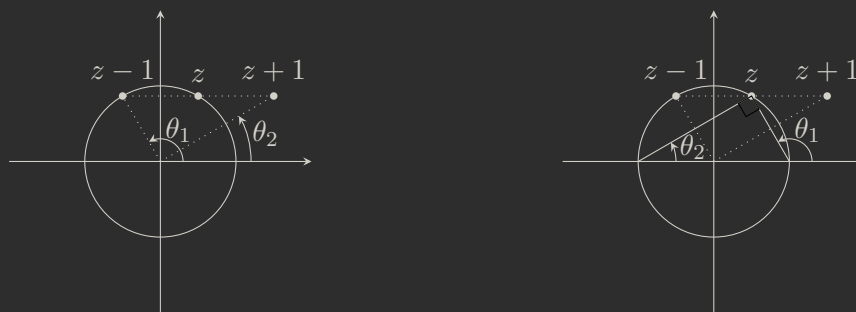


Figure 4.3: (Right) Depicted question, (Left) Translated Angles

Similarly, we can obtain $\theta_1 - \theta_2 = -\frac{\pi}{2}$ for when $\text{Im } z < 0$. This completes the proof.

Exercise 4.1.5

Let $f(z) = e^z$ for $z \in \mathbb{C}$. Let $A = \{z = x + iy \in \mathbb{C} : x \leq 1, y \in [0, \pi]\}$. Describe the image of $f(A)$.

Solution

Firstly, note that

$$\begin{aligned}e^z &= e^{x+iy} \\ e^x &\in (0, e] \\ y &\in [0, \pi]\end{aligned}$$

Chapter 5

Lecture 5 Jan 12 2018

5.1 Complex Functions

5.1.1 Limits

Definition 5.1.1 (Convergence)

A sequence of complex numbers z_1, z_2, z_3, \dots **converges** to $z \in \mathbb{C}$ if

$$\lim_{n \rightarrow \infty} |z_n - z| = 0 \quad (5.1)$$

or we may say

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \quad |z_n - z| < \epsilon \quad (5.2)$$

Note

If $\{z_n\}_{n \in \mathbb{N}}$ converges to z , we may write $\lim_{n \rightarrow \infty} z_n = z$ or $z_n \rightarrow z$ (as $n \rightarrow \infty$).

Example 5.1.1

For $|z| > 1$, does $\{\frac{1}{z^n}\}_{n=1}^{\infty}$ converge? Explain.

Solution

We claim that the limit is 0. Since $|z| > 1$, we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{1}{z^n} - 0 \right| &= \lim_{n \rightarrow \infty} \left| \frac{1}{z} \right|^n \\ &= 0 \end{aligned}$$

Another way to prove this, since $|z| > 1 \implies 0 < \left|\frac{1}{z}\right| < 1$,

$$\forall \epsilon = \left|\frac{1}{z}\right| > 0$$

$$\left|\frac{1}{z^n} - 0\right| = \left|\frac{1}{z}\right|^n < \left|\frac{1}{z}\right| = \epsilon$$

Definition 5.1.2 (Convergence for Complex Functions)

$\forall \Omega \subseteq \mathbb{C}$, let $f : \Omega \rightarrow \mathbb{C}$. We say that

$$\lim_{z \rightarrow z_0} f(z) = L \quad (5.3)$$

for some $L \in \mathbb{C}$ if for every sequence $\{z_n\}_n \subseteq \Omega$ (not including z_0 if it is in Ω), we have that

$$z_n \rightarrow z_0 \implies f(z_n) \rightarrow L \quad (5.4)$$

Note that L need not be in Ω .

Example 5.1.2

Let $f(z) = \frac{z}{z}, z \in \mathbb{C} \setminus \{0\}$. Find $\lim_{z \rightarrow 0} f(z)$.

Solution

Suppose $z = x \in \mathbb{R} \setminus \{0\}$. Then $f(z) = f(x) = \frac{x}{x} = 1$.

Suppose $z = iy, y \in \mathbb{R} \setminus \{0\}$. Then $f(z) = f(iy) = \frac{-iy}{iy} = -1$.

Therefore, the limit $\lim_{z \rightarrow 0} f(z)$ does not exist.

Exercise 5.1.1

Show that $z_n \rightarrow z \iff \text{Re}(z_n) \rightarrow \text{Re}(z) \wedge \text{Im}(z_n) \rightarrow \text{Im}(z)$.

(Hint: $|\text{Re}(z)|, |\text{Im}(z)| \leq |z| \leq |\text{Re}(z)| + |\text{Im}(z)|$)

Solution

Suppose $z_n \rightarrow z$. Then $\forall \epsilon_0 > 0 \exists N \in \mathbb{N} \forall n > N |z_n - z| < \epsilon$. Note once and for all that

$$\text{Re}(z_n - z) = \text{Re}(z_n) - \text{Re}(z)$$

$$\text{Im}(z_n - z) = \text{Im}(z_n) - \text{Im}(z).$$

Thus

$$|\text{Re}(z_n) - \text{Re}(z)| = |\text{Re}(z_n - z)|$$

$$\leq |z_n - z| < \epsilon$$

$$|\text{Im}(z_n) - \text{Im}(z)| = |\text{Im}(z_n - z)|$$

$$\leq |z_n - z| < \epsilon$$

For the other direction,

$$\begin{aligned}\forall \frac{\epsilon}{2} > 0 \quad \exists N_0 \in \mathbb{N} \quad \forall n > N_0 \quad |\operatorname{Re}(z_n) - \operatorname{Re}(z)| < \frac{\epsilon}{2} \\ \forall \frac{\epsilon}{2} > 0 \quad \exists N_1 \in \mathbb{N} \quad \forall n > N_1 \quad |\operatorname{Im}(z_n) - \operatorname{Im}(z)| < \frac{\epsilon}{2}.\end{aligned}$$

Therefore,

$$\begin{aligned}|z_n - z| &= |\operatorname{Re}(z_n) + i\operatorname{Im}(z_n) - \operatorname{Re}(z) - i\operatorname{Im}(z)| \\ &\leq |\operatorname{Re}(z_n) - \operatorname{Re}(z)| + |\operatorname{Im}(z_n) - \operatorname{Im}(z)| \\ &\leq \epsilon\end{aligned}$$

□

5.1.2 Continuity

Definition 5.1.3 (Continuity)

$\forall \Omega \subseteq \mathbb{C}$, let $f : \Omega \rightarrow \mathbb{C}$. We say that f is **continuous** at $z_0 \in \Omega$ if

1. $\forall \{z_n\}_{n \in \mathbb{N}} \quad z_n \rightarrow z_0 \implies f(z_n) \rightarrow f(z_0)$
2. $\forall \epsilon > 0 \quad \exists \delta > 0 \quad |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$

Remark

1. f is continuous on Ω if it is continuous on every point in Ω .
2. We may **split** f into its real and imaginary parts, i.e.

$$f(z) = f(x, y) = u(x, y) + iv(x, y) \tag{5.5}$$

where $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Example 5.1.3

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ and for $z \in \mathbb{C}$, $f(z) = \frac{\bar{z}}{z}$. To split f into real and imaginary parts:

$$\begin{aligned}f(z) &= \frac{\bar{z}}{z} \\ &= (x + iy) \left(\frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2} \right) \\ &= \frac{x^2 - y^2}{x^2 + y^2} + i \frac{(-2xy)}{x^2 + y^2}\end{aligned}$$

Chapter 6

Lecture 6 Jan 15th 2018

6.1 Continuity (Continued)

Exercise 6.1.1

Let $f : \Omega \rightarrow \mathbb{C}$. Prove that $f(z)$ is continuous at $z_0 = x_0 + iy_0 \in \mathbb{C} \iff$ functions $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$, such that $f(z) = u(x, y) + iv(x, y)$ are both continuous at (x_0, y_0) .

Solution

We shall first prove the forward direction. Suppose that $f(z)$ is continuous at $z_0 = x_0 + iy_0 \in \mathbb{C}$. By Definition 5.1.3, $\forall \{z_n\}_{n \in \mathbb{N}} \subseteq \Omega$, $z_n \rightarrow z_0 \implies f(z_n) \rightarrow f(z_0)$. By Exercise 5.1.1,

$$\begin{aligned} z_n \rightarrow z_0 &\iff \operatorname{Re} z_n \rightarrow \operatorname{Re} z_0 \wedge \operatorname{Im} z_n \rightarrow \operatorname{Im} z_0 \\ &\iff x_n \rightarrow x_0 \wedge y_n \rightarrow y_0 \end{aligned} \tag{6.1}$$

where $z_n = x_n + iy_n$ for $x_n, y_n \in \mathbb{R}$.

Similarly so, and by Equation (5.5),

$$f(z_n) \rightarrow f(z_0) \iff u(x_n, y_n) \rightarrow u(x_0, y_0) \wedge v(x_n, y_n) \rightarrow v(x_0, y_0) \tag{6.2}$$

Putting together Equation (6.1) and Equation (6.2), we get

$$(x_n, y_n) \rightarrow (x_0, y_0) \implies u(x_n, y_n) \rightarrow u(x_0, y_0) \wedge v(x_n, y_n) \rightarrow v(x_0, y_0)$$

as desired.

The proof of the other direction is simply a reversed process of the above. □

6.2 Differentiability

Definition 6.2.1 (Neighbourhood)

For $z_0 \in \mathbb{C}$, $r \in \mathbb{R}$, let

$$D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}. \quad (6.3)$$

On the complex plane, this is seen as a open disk centered around the point z_0 with radius r , as shown below.

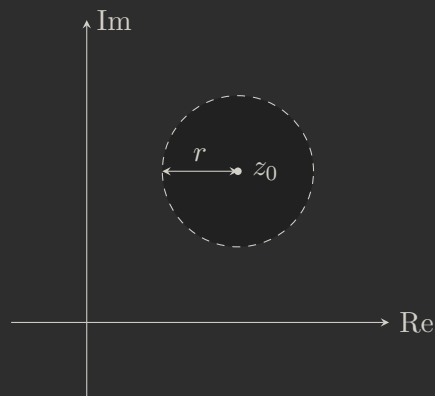


Figure 6.1: Open disk centered around z_0 with radius r

This open disk is called a **neighbourhood** of z_0 .

Definition 6.2.2 (Differentiable/Holomorphic)

Let $f(z)$ be defined in a neighbourhood of $z_0 \in \mathbb{C}$. We say f is **differentiable/holomorphic** at z_0 if for some $h \in \mathbb{C}$,

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad (6.4)$$

exists. If such a limit exists, we denote the limit by $f'(z_0)$.

Remark

$h \in \mathbb{C}$: h need not necessarily be real. In this sense, h approaches 0 from **any direction** around 0 $\in \mathbb{C}$.

Example 6.2.1

For $z \in \mathbb{C} \setminus \{0\}$, let $f(z) = \frac{1}{z}$. Let $z_0 \in \mathbb{C} \setminus \{0\}$. Note that

$$\lim_{h \rightarrow 0} \frac{\frac{1}{z_0+h} - \frac{1}{z_0}}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-h}{(z_0 + h)z_0} \right] = -\frac{1}{z_0^2}$$

Thus f is holomorphic at any $z \in \mathbb{C} \setminus \{0\}$, and hence $f'(z) = -\frac{1}{z}$.

Example 6.2.2

For $z \in \mathbb{C}$, let $f(z) = \bar{z}$. Let $z_0 \in \mathbb{C}$. Notice that

$$\lim_{h \rightarrow 0} \frac{\overline{z_0 + h} - \bar{z}}{h} = \lim_{h \rightarrow 0} \frac{\bar{h}}{h}.$$

From [Example 5.1.2](#), we know that such a limit does not exist. Thus f is not holomorphic on any $z \in \mathbb{C}$.

Exercise 6.2.1 (Holomorphic Functions Properties)

If f, g are holomorphic at $z \in \mathbb{C}$, prove that

1. $f + g$ is holomorphic and $(f + g)' = f' + g'$.
2. fg is holomorphic and $(fg)' = f'g + fg'$.
3. if $g(z) \neq 0$, $\frac{f}{g}$ is holomorphic and $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$.

Solution

1. For $f + g$,

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(z+h) + g(z+h) - f(z) - g(z)}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{f(z+h) - f(z)}{h} + \frac{g(z+h) - g(z)}{h} \right] \\ &= f'(z) + g'(z) \end{aligned}$$

Thus $(f + g)' = f' + g'$.

2. For fg ,

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(z+h)g(z+h) - f(z)g(z)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(z+h)g(z+h) + f(z)g(z+h) - f(z)g(z+h) - f(z)g(z)}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{f(z+h) - f(z)}{h} g(z+h) + f(z) \frac{g(z+h) - g(z)}{h} \right] \\ &= f'(z)g(z) + f(z)g'(z) \end{aligned}$$

Therefore, $(fg)' = f'g + fg'$.

Case 2: $h \rightarrow 0$ via the imaginary axis

In this case, $h = 0 + iy$ and $y \rightarrow 0 \in \mathbb{R}$. In a similar fashion, Equation (6.5) becomes

$$\begin{aligned} f'(z_0) &= \lim_{y \rightarrow 0} \left[\frac{u(x_0, y_0 + y) - u(x_0, y_0)}{iy} + \frac{v(x_0, y_0 + y) - v(x_0, y_0)}{y} \right] \\ &= \frac{1}{i} \cdot \frac{\partial u}{\partial y} \Big|_{(x_0, y_0)} + \frac{\partial v}{\partial y} \Big|_{(x_0, y_0)} \end{aligned} \quad (6.7)$$

Note that since $f'(z_0)$ exists, the real and imaginary part of Equation (6.6) and Equation (6.7) must equate. Also note that $\frac{1}{i} = -i$. With that, we obtain the following theorem.

Theorem 6.2.1 (Cauchy-Riemann Equations)

If $f(z)$ is holomorphic at $z_0 = x_0 + iy_0 \in \mathbb{C}$ where $x_0, y_0 \in \mathbb{R}$, then, at (x_0, y_0) ,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \quad (6.8)$$

Chapter 7

Lecture 7 Jan 17 2018

7.1 Differentiability (Continued)

7.1.1 Cauchy-Riemann Equations (Continued)

It is natural to wonder if the **converse** of Theorem 6.2.1 is true. We present the following example.

Example 7.1.1

Let

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Check if

1. f is holomorphic at 0.
2. Theorem 6.2.1 holds at $(0,0)$.

Proof

1. Observe that by letting $h = x_h + iy_h$ where $x_h, y_h \in \mathbb{R}$,

$$\lim_{h \rightarrow 0} \frac{\overline{0+h}^2 - 0}{0+h} = \lim_{h \rightarrow 0} \frac{\bar{h}^2}{h} = \lim_{x_h + iy_h \rightarrow 0} \left(\frac{x_h - iy_h}{x_h + iy_h} \right)^2$$

Consider $y_h = kx_h$, for $k \in \mathbb{R} \setminus \{0\}$. Then

$$\lim_{x_h \rightarrow 0} \left(\frac{x_h - ikx_h}{x_h + ikx_h} \right)^2 = \left(\frac{1 - ik}{1 + ik} \right)^2,$$

where we see that the limit depends on the value of k . Therefore, the limit DNE. Hence f is not holomorphic at 0.

2. Let $z = x + iy$ for $x, y \in \mathbb{R}$. Then

$$\frac{\bar{z}^2}{z} = \frac{(x - iy)^2}{x + iy} = \frac{(x - iy)^3}{x^2 + y^2} = \frac{x^3 - 3xy^2}{x^2 + y^2} + i \frac{(-3x^2y + y^3)}{x^2 + y^2}$$

Therefore, we obtain

$$u(x, y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$v(x, y) = \begin{cases} \frac{y^3 - 3x^2y}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Observe that

$$\left. \frac{\partial u}{\partial x} \right|_{(0,0)} = \lim_{x \rightarrow 0} \frac{u(x, 0) - u(0, 0)}{x} = 1$$

$$\left. \frac{\partial v}{\partial y} \right|_{(0,0)} = \lim_{y \rightarrow 0} \frac{v(0, y) - v(0, 0)}{y} = 1$$

and

$$\left. \frac{\partial u}{\partial y} \right|_{(0,0)} = \lim_{y \rightarrow 0} \frac{u(0, y) - u(0, 0)}{y} = 0$$

$$\left. \frac{\partial v}{\partial x} \right|_{(0,0)} = \lim_{x \rightarrow 0} \frac{v(x, 0) - v(0, 0)}{x} = 0$$

satisfies Equation (6.8).

This illustrates that the converse of Theorem 6.2.1 is not true. We will, however, show that the converse will be true given an extra condition.

Theorem 7.1.1 (Conditional Converse of CRE)

Let $z_0 = x_0 + iy_0 \in \Omega \subseteq \mathbb{C}$, $x_0, y_0 \in \mathbb{R}$, and $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f = u + iv : \Omega \rightarrow \mathbb{C}$. If

1. the partials of u, v exist in a neighbourhood of (x_0, y_0) ,
2. the partials of u, v are continuous at (x_0, y_0) , and
3. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ at (x_0, y_0) ,

then f is holomorphic at z_0 .

A proof of the theorem is in page 36 of Newman and Bak (recommended text of PMATH352W18). I may include the proof whenever I am free.

7.1.2 Power Series

Definition 7.1.1 (Power Series)

A **power series** in \mathbb{C} is an infinite series of the form

$$\sum_{n \in \mathbb{N}} c_n z^n, \quad (7.1)$$

where each $c_n \in \mathbb{C}$ is the coefficient of z of the n -th power.

In this subsection, we are interested to see if Equation (7.1) converges.

Recall the notion of convergence in series from \mathbb{R} . Equation (7.1) converges if the sequence of partial sums $\{S_N\}$ converges as $N \rightarrow \infty$, where

$$S_N := \sum_{n=0}^N c_n z^n$$

In other words, using the same definition of S_N ,

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \setminus \{0\} \quad \forall n > N \\ |S_n - L| < \epsilon$$

where $L \in \mathbb{C}$ is the limit that the sequence converges to.

We also know that Equation (7.1) converges absolutely if $\sum_{n=0}^{\infty} |c_n| |z|^n$ converges. This is a stronger statement (i.e. absolute convergence \implies convergence)

$$\because \left| \sum_{n=0}^N c_n z^n \right| \leq \sum_{n=0}^N |c_n| |z|^n \quad \text{for each } N \in \mathbb{N}$$

Example 7.1.2

$\sum_{n=0}^{\infty} z^n$ converges absolutely for $|z| < 1$.

Note that the partial sum of a geometric series is

$$\sum_{n=0}^N r^n = \frac{1 - r^{N+1}}{1 - r}$$

and so the limit as $N \rightarrow \infty$ exists if $|r| < 1$, and hence we see that

$$\sum_{n=0}^{\infty} r^n \rightarrow \frac{1}{1 - r}$$

if $|r| < 1$ as $N \rightarrow \infty$.

However, if $|z| = 1$, the power series diverges.

Another note that we shall point out is that if Equation (7.1) converges absolutely for some $z_0 \in \mathbb{C}$, then it converges absolutely for any z where $|z| < |z_0|$.

These notions, in turn, begs the question of **what is the largest possible $|z_0|$ for the series to converge absolutely.**

Chapter 8

Lecture 8 Jan 19 2018

8.1 Power Series (Continued)

Theorem 8.1.1 (Convergence in the Radius of Convergence)

For any power series $\sum_{n \in \mathbb{N}} c_n z^n$, $\exists 0 \leq R < \infty$, such that

1. $|z| < R \implies$ series converges absolutely.
2. $|z| > R \implies$ series diverges.

Moreover, R is given by **Hadamard's Formula**:

$$\frac{1}{R} := \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}} \quad (8.1)$$

Remark

1. R is called the **radius of convergence** of the series. $\{z \in \mathbb{C} : |z| < R\}$ is called the disk of convergence of the series.
2. Recall the definition of the **limit supremum**

$$\limsup_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} \left(\sup_{m \geq n} a_m \right) \quad (8.2)$$

which we may colloquially say as the “highest peak ‘reached’ by a_n ’s as $n \rightarrow \infty$ ”

Proposition 8.1.1 (A Property of limsup)

$$\begin{aligned} \forall \{a_n\}_{n \in \mathbb{N}} \quad L := \limsup_{n \rightarrow \infty} a_n &\implies \\ \forall \epsilon > 0 \quad \exists N > 0 \quad \forall n > N & \\ L - \epsilon < a_n < L + \epsilon & \end{aligned}$$

(Proof to be included)

Proof (Theorem 8.1.1)

Let $L := \frac{1}{R} = \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}$. Clearly, $L \geq 0$.

1. Suppose $|z| < R$. $\exists \epsilon > 0, r := |z|(L + \epsilon)$ such that $0 < r < 1$. By Proposition 8.1.1, $\exists N \in \mathbb{N}, \forall n > N, |c_n|^{\frac{1}{n}} < L + \epsilon$.

Now since $L = \frac{1}{R}$,

$$\sum_{n=N}^{\infty} |c_n| |z|^n = \sum_{n=N}^{\infty} (|c_n|^{\frac{1}{n}} |z|)^n < \sum_{n=N}^{\infty} r^n$$

and since $0 < r < 1$, the final summation converges (as it is a geometric sum). Thus by comparison test, $\sum_{n=N}^{\infty} |c_n| |z|^n$ converges.

We may also proceed with noticing that the partial sum of $\sum_{n=N}^{\infty} |c_n| |z|^n$ is **bounded and monotonic**, which shows that the series converges.

2. Suppose $|z| > R$. $\exists \epsilon > 0, r := |z|(L - \epsilon)$ such that $r > 1$. By Proposition 8.1.1, $\exists N \in \mathbb{N}, \forall n > N, |c_n|^{\frac{1}{n}} > L - \epsilon$. Then analogous to the proof above,

$$\sum_{n=N}^{\infty} |c_n| |z|^n = \sum_{n=N}^{\infty} (|c_n|^{\frac{1}{n}} |z|)^n > \sum_{n=N}^{\infty} r^n$$

where the final summation diverges, and thus implying that $\sum_{n=N}^{\infty} |c_n| |z|^n$ diverges.

Theorem 8.1.2 (Power function, holomorphic function, region of convergence)

Suppose $f(z) = \sum_{n \in \mathbb{N}} c_n z^n$ has a radius of convergence $R \in \mathbb{R}$. Then $f'(z)$ exists and equals

$$\sum_{n=1}^{\infty} n c_n z^{n-1}$$

throughout $|z| < R$.

Moreover, f' has the **same radius of convergence** as f .

