PMATH450 — Lesbesgue Integration and Fourier Analysis

Classnotes for Spring 2019

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Preface

The pre-requisite to this course is Real Analysis. We will use a lot of the concepts introduced in Real Analysis, at times without explicitly stating it. Refer to notes on PMATH351.

This course is spiritually broken into 2 pieces:

- Lesbesgue Integration; and
- Fourier Analysis,

which is as the name of the course.

In this set of notes, we use a special topic environment called **culture** to discuss interesting contents related to the course, but will not be throughly studied and not tested on exams.

1 Lecture 1 May 07th 2019

Since many of our results work for both $\mathbb{C}$ and $\mathbb{R}$, we shall use $\mathbb{K}$ throughout this course to represent either $\mathbb{C}$ or $\mathbb{R}$.

1.1 Riemannian Integration

Definition 1 (Norm and Semi-Norm)

Let V be a vector space over $\mathbb{K}$. We define a ***semi-norm*** *on V as a function*

$$\nu : V \to \mathbb{R}$$

that satisfies

1. *(**Positive Semi-Definite**) $v(x) \geq 0$ for all $x \in V$;*
2. *$\nu(\kappa x) = |\kappa| \, \nu(x)$ for any $\kappa \in \mathbb{K}$ and $x \in V$; and*
3. *(**Triangle Inequality**) $\nu(x+y) \leq \nu(x) + \nu(y)$ for all $x, y \in V$.*

If $\nu(x) = 0 \implies x = 0$, then we say that ν is a ***norm****. In this case, we usually write $\|\cdot\|$ to denote the norm, instead of ν.*

Note 1.1.1

- *We sometimes call a semi-norm a* ***pseudo-length****.*

Remark 1.1.1

Notice that we wrote $\nu(x) = 0 \implies x = 0$ *instead of* $\nu(x) = 0 \iff x = 0$. *This is because if* $z = 0 \in V$, *then*

$$v(z) = v(0z) = 0.$$

Exercise 1.1.1

Show that if ν *is a semi-norm on a vector space* V, *then* $\forall x, y \in V$,

$$|\nu(x) - \nu(y)| \leq \nu(x - y).$$

Proof

Notice that by condition (2) and (3), we have

$$\nu(x-y) \leq \nu(x) + \nu(-y) = \nu(x) - \nu(y),$$

and

$$\nu(x-y) = -\nu(y-x) \geq -(\nu(y) - \nu(x)) = \nu(x) - \nu(y).$$

It follows that indeed

$$|\nu(x) - \nu(y)| \leq \nu(x-y). \qquad \square$$

Example 1.1.1

The absolute value $|\cdot|$ is a norm on $\mathbb{K}$.

Example 1.1.2 (*p*-norms)

Consider $N \geq 1$ an integer. We define a family of norms on

$$\mathbb{K}^N = \underbrace{K \times K \times \ldots \times K}_{N \text{ times}}.$$

1-norm

$$\left\| (x_n)_{n=1}^N \right\|_1 := \sum_{n=1}^N |x_n|.$$

Infinity-norm, ∞-norm

$$\left\| (x_n)_{n=1}^N \right\|_\infty := \max_{1 \leq n \leq N} |x_n| .$$

Euclidean-norm, 2-norm

$$\left\| (x_n)_{n=1}^N \right\|_2 := \left(\sum_{n=1}^N |x_n|^2 \right)^{\frac{1}{2}}$$

It is relatively easy to check that the above norms are indeed norms, except for the 2-form. In particular, the **triangle inequality** is not as easy to show [1].

[1] See Minkowski's Inequality.

Less obviously so, but true nonetheless, we can define the following p-norms on $\mathbb{K}^N$:

$$\left\| (x_n)_{n=1}^N \right\|_p := \left(\sum_{n=1}^N |x_n|^p \right)^{\frac{1}{p}},$$

for $1 \leq p < \infty$.

Culture

*Consider $V = \mathbb{M}_n(\mathbb{C})$, [2] where $n \in \mathbb{N}$ is fixed. For $T \in \mathbb{M}_n(\mathbb{C})$, we define the **singular numbers** of T to be*

$$s_1(T) \geq s_2(T) \geq \ldots \geq s_n(T) \geq 0,$$

where $\sigma(T^*T) = \{s_1(T)^2, s_2(T)^2, \ldots, s_n(T)^2\}$, including multiplicity. Then we can define

$$\|T\|_p := \left(\sum_{i=1}^n s_i(T)^p \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$, which is called the p-norm of T on $\mathbb{M}_n(\mathbb{C})$.

[2] Note that $\mathbb{M}_n(\mathbb{C})$ is the set of $n \times n$ matrices over $\mathbb{C}$.

Example 1.1.3

Let

$$V = \mathcal{C}([0,1], \mathbb{K}) = \{ f : [0,1] \to \mathbb{K} \mid f \text{ is continuous } \}.$$

Then
$$\|f\|_{\text{sup}} := \sup\{|f(x)| \mid x \in [0,1]\}$$
[3] defines a norm on $\mathcal{C}([0,1], \mathbb{K})$.

[3] Some authors use $\|f\|_\infty$, but we will have the notation $\|[f]\|_\infty$ later on, and so we shall use $\|f\|_{\text{sup}}$ for clarity.

A sequence $(f_n)_{n=1)^\infty}$ in V converges in this norm to some $f \in V$, i.e.
$$\lim_{n\to\infty} \|f_n - f\|_{\text{sup}} = 0,$$
which means that $(f_n)_{n=1}^\infty$ converges uniformly to f on $[0,1]$.

Definition 2 (Normed Linear Space)

*A **normed linear space (NLS)** is a pair $(V, \|\cdot\|)$ where V is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is a norm on V.*

Definition 3 (Metric)

*Given an NLS $(V, \|\cdot\|)$, we can define a **metric** d on V (called the **metric induced by the norm**) as follows:*
$$d : V \times V \to \mathbb{R} \quad d(x,y) = \|x - y\|,$$
such that

- *$d(x,y) \geq 0$ for all $x, y \in V$ and $d(x,y) = 0 \iff x = y$;*
- *$d(x,y) = d(y,x)$; and*
- *$d(x,y) \leq d(x,z) + d(y,z)$.*

Note 1.1.2

Norms are all metrics, and so any space that has a norm will induce a metric on the space.

Definition 4 (Banach Space)

We say that an NLS $(V, \|\cdot\|)$ is ***complete*** *or is a* ***Banach Space*** *if the corresponding (V, d), where d is the metric induced by the norm, is complete*[4].

[4] Completeness of a metric space is such that any of its Cauchy sequences converges in the space.

Example 1.1.4

$(\mathcal{C}([0,1], \mathbb{K}), \|\cdot\|_{\sup})$ is a Banach space.

Example 1.1.5

We can define a 1-norm $\|\cdot\|_1$ on $\mathcal{C}([0,1], \mathbb{K})$ via

$$\|f\|_1 := \int_0^1 |f|.$$

Then $(\mathcal{C}([0,1], \mathbb{K}), \|\cdot\|_1)$ is an NLS.

Exercise 1.1.2

Show that $(\mathcal{C}([0,1], \mathbb{K}), \|\cdot\|_1)$ is not complete, which will then give us an example of a ***normed linear space that is not Banach****.*

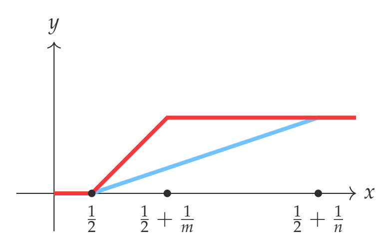

Figure 1.1: Sequence of functions $(f_n)_{n=1}^{\infty}$. We show for two indices $n < m$.

Proof

Consider the sequence $(f_n)_{n=1}^{\infty}$ of continuous functions given by

$$f_n(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ n\left(x + \frac{1}{2}\right) & \frac{1}{2} \leq x \leq \frac{1}{2} + \frac{1}{n} \\ 1 & \text{otherwise} \end{cases}$$

Note that the sequence $(f_n)_{n=1}^{\infty}$ is indeed **Cauchy**: let $\varepsilon > 0$ and $|n - m| < \frac{\varepsilon}{\left|x - \frac{1}{2}\right|}$, and then we have

$$\begin{aligned} |f_n(x) - f_m(x)| &= \left| n\left(x - \frac{1}{2}\right) - m\left(x - \frac{1}{2}\right) \right| \\ &= \left| (n - m)\left(x - \frac{1}{2}\right) \right| = |n - m| \left| x - \frac{1}{2} \right| < \varepsilon. \end{aligned}$$

However, it is clear that the sequence $(f_n)_{n=1}^{\infty}$ converges to the piecewise function (in particular, a non-continuous function)

$$f(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}.$$ $\square$

Example 1.1.6

If $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ are NLS's, and if $T : \mathfrak{X} \to \mathfrak{Y}$ is a linear map, we define the **operator norm** of T to be

$$\|T\| := \sup\{\|T(x)\|_{\mathfrak{Y}} \mid \|x\|_{\mathfrak{X}} \leq 1\}.$$

We set

$$B(\mathfrak{X}, \mathfrak{Y}) := \{T : \mathfrak{X} \to \mathfrak{Y} \mid T \text{ is linear }, \|T\| < \infty\}.$$

Note that for any such linear map T, $\|T\| < \infty \iff T$ is continuous. Thus $B(\mathfrak{X}, \mathfrak{Y})$ is the set of all continuous functions from $\mathfrak{X}$ into $\mathfrak{Y}$.

Then $(B(\mathfrak{X}, \mathfrak{Y}), \|\cdot\|)$ is an NLS.

It is likely that we have seen this in Real Analysis.

Exercise 1.1.3

Show that $(B(\mathfrak{X}, \mathfrak{Y}), \|\cdot\|)$ is complete iff $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ is complete.

Note 1.1.3

*One example of the last example is when $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}}) = (\mathbb{K}, |\cdot|)$. In this case, $B(\mathfrak{X}, \mathbb{K})$ is known as the **dual space** of $\mathfrak{X}$, or simple the **dual** of $\mathfrak{X}$.*

We are interested in integrating over Banach spaces.

Definition 5 (Partition of a Set)

*Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a, b] \to \mathfrak{X}$ a function, where $a < b \in \mathbb{R}$. A **partition** P of $[a, b]$ is a finite set*

$$P = \{a = p_0 < p_1 < \ldots < p_N = b\}$$

for some $N \geq 1$. The set of all partitions of $[a, b]$ is denoted by $\mathcal{P}[a, b]$.

Definition 6 (Test Values)

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a, b] \to \mathfrak{X}$ a function, where $a < b \in \mathbb{R}$. Let $P \in \mathcal{P}[a, b]$. A set

$$P^* := \{p_k^*\}_{k=1}^N$$

satisfying

$$p_{k-1} \leq p_k^* \leq p_k, \text{ for } 1 \leq k \leq n$$

*is called a set of **test values** for P.*

Definition 7 (Riemann Sum)

*Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a, b] \to \mathfrak{X}$ a function, where $a < b \in \mathbb{R}$. Let $P \in \mathcal{P}[a, b]$ and P^* its corresponding set of test values. We define the **Riemann sum** as*

$$S(f, P, P^*) = \sum_{k=1}^N f(p_k^*)(p_k - p_{k-1}).$$

Remark 1.1.2

1. *Note that because Definition 5, $p_k - p_{k-1} > 0$.*
2. *When $(\mathfrak{X}, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, then this is the usual Riemann sum from first-year calculus.*
3. *In general, note that*

$$\frac{1}{b-a} S(f, P, P^*) = \sum_{k=1}^N \lambda_k f(p_k^*),$$

where $0 < \lambda_k = \frac{p_k - p_{k-1}}{b-a} < 1$ and [5]

$$\sum_{k=1}^N \lambda_k = 1.$$

[5] via the fact that the λ_k's form a telescoping sum

So $\frac{1}{b-a}S(f,P,P^*)$ is an averaging of f over $[a,b]$. We call $\frac{1}{b-a}S(f,P,P^*)$ the convex combination of the $f(p_k^*)$'s.

Example 1.1.7 (Silly example)

Let $(\mathfrak{X} = \mathcal{C}([-\pi,\pi],\mathbb{K}), \|\cdot\|_{\sup})$. Let

$$f : [0,1] \to \mathfrak{X} \text{ such that } x \mapsto e^{2\pi x}\sin 7\theta + \cos x \cos(12\theta),$$

where $\theta \in [-\pi,\pi]$. Now if we consider the partition

$$P = \left\{-\pi, \frac{1}{10}, \frac{1}{2}, \pi\right\}$$

and its corresponding test value

$$P^* = \left\{0, \frac{1}{3}, 2\right\},$$

then

$$\begin{aligned}
S(f,P,P^*) &= f(0)\left(\frac{1}{10}+\pi\right) + f\left(\frac{1}{3}\right)\left(\frac{1}{2}-\frac{1}{10}\right) + f(2)\left(\pi - \frac{1}{2}\right) \\
&= (\sin 7\theta + \cos 12\theta)\left(\pi + \frac{1}{10}\right) \\
&\quad + \left(e^{\frac{2\pi}{3}}\sin 7\theta + \cos\frac{1}{3}\cos 12\theta\right)\left(\frac{2}{5}\right) \\
&\quad + (e^{4\pi}\sin 7\theta + \cos 2\cos 12\theta)\left(\pi - \frac{1}{2}\right)
\end{aligned}$$

Definition 8 (Refinement of a Partition)

Let $a < b \in \mathbb{R}$, and $P \in \mathcal{P}[a,b]$. We say Q is a refinement of P is $Q \in \mathcal{P}[a,b]$ and $P \subseteq Q$.

Note 1.1.4

In simpler words, Q is a "finer" partition that is based on P.

Definition 9 (Riemann Integrable)

Let $a < b \in \mathbb{R}$, $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a,b] \to \mathfrak{X}$ be a function. We say that f is ***Riemann integrable*** *over $[a,b]$ if $\exists x_0 \in \mathfrak{X}$ such that*

$$\forall \varepsilon > 0 \quad \exists P \in \mathcal{P}[a,b],$$

such that if Q is any refinement of P, and Q^* is any set of test values of Q, then

$$\|x_0 - S(f, Q, Q^*)\|_{\mathfrak{X}} < \varepsilon.$$

In this case, we write

$$\int_a^b f = x_0.$$

Proposition 1 (Uniqueness of the Riemann Integral)

If f is Riemann integrable over $[a,b]$, then the value of $\int_a^b f$ is unique.

Proof

Suppose not, i.e.

$$\int_a^b f = x_0 \text{ and } \int_a^b f = y_0$$

for some $x_0 \neq y_0$. Then, let

$$\varepsilon = \frac{\|x_0 - y_0\|}{2},$$

which is > 0 since $\|x_0 - y_0\| > 0$. Let $P_{x_0}, P_{y_0} \in \mathcal{P}[a,b]$ be partitions corresponding to x_0 and y_0 as in the definition of Riemann integrability.

Then, let $R = P_{x_0} \cup P_{y_0}$, so that R is a **common refinement** of P_{x_0} and P_{y_0}. If Q is any refinement of R, then Q is also a common refinement of P_{x_0} and P_{y_0}. Then for any test values Q^* of Q, we have

$$\begin{aligned} 2\varepsilon &= \|x_0 - y_0\| \\ &\leq \|x_0 - S(f, Q, Q^*)\| + \|S(f, Q, Q^*) - y_0\| < \varepsilon + \varepsilon = 2\varepsilon, \end{aligned}$$

which is a contradiction.

Thus $x_0 = y_0$ as required. $\square$

Theorem 2 (Cauchy Criterion of Riemann Integrability)

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space, $a < b \in \mathbb{R}$ and $f : [a, b] \to \mathfrak{X}$ be a function. TFAE:

1. *f is Riemann integrable over $[a, b]$;*
2. *$\forall \varepsilon > 0$, $R \in \mathcal{P}[a, b]$, if P, Q is any refinement of R, and P^* (respectively Q^*) is any test values of P (respectively Q), then*

$$\|S(f, P, P^*) - S(f, Q, Q^*)\|_{\mathfrak{X}} < \varepsilon.$$

Proof

$\implies$ This is a rather straightforward proof. Suppose $P, Q \in \mathcal{P}[a, b]$ is some refinement of the given partition $R \in \mathcal{P}[a, b]$, and P^*, Q^* any test values for P, Q, respectively. Then by assumption and Proposition 1, $\exists x_0 \in \mathfrak{X}$ such that

$$\|x_0 - S(f, P, P^*)\|_{\mathfrak{X}} < \frac{\varepsilon}{2} \text{ and } \|x_0 - S(f, Q, Q^*)\|_{\mathfrak{X}} < \frac{\varepsilon}{2}.$$

It follows that

$$\begin{aligned} &\|S(f, P, P^*) - S(f, Q, Q^*)\|_{\mathfrak{X}} \\ &\leq \|x_0 - S(f, P, P^*)\|_{\mathfrak{X}} + \|x_0 - S(f, Q, Q^*)\|_{\mathfrak{X}} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

$\impliedby$ By hypothesis, wma $\varepsilon = \frac{1}{n}$ for some $n \geq 1$, such that if P, Q are any refinements of the partition $R_n \in \mathcal{P}[a, b]$, and P^*, Q^* are the respective arbitrary test values, then

$$\|S(f, P, P^*) - S(f, Q, Q^*)\|_{\mathfrak{X}} < \frac{1}{n}$$

Now for each $n \geq 1$, define

$$W_n := \bigcup_{k=1}^{n} R_k \in \mathcal{P}[a,b],$$

so that W_n is a common refinement for $R_1, R_2, \ldots, R_n$. For each $n \geq 1$, let W_n^* be an arbitrary set of test values for W_n. For simplicity, let us write

$$x_n = S(f,\, W_n,\, W_n^*),\ \text{for each } n \geq 1.$$

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[6] Note that it would be nice if for the finer and finer partitions that we have constructed, i.e. the W_n's, give us a convergent sequence of Riemann sums, since it makes sense that this convergence will give us the final value that we want.

Claim: $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence If $n_1 \geq n_2 > N \in \mathbb{N}$, then

$$\|x_{n_1} - x_{n_2}\|_{\mathfrak{X}} = \left\| S(f, W_{n_1}, W_{n_1}^*) - S(f, W_{n_2}, W_{n_2}^*) \right\| < \frac{1}{N}$$

by our assumption, since W_{n_1}, W_{n_2} are refinements of R_N. Then by picking $N = \frac{1}{\varepsilon}$ for any $\varepsilon > 0$, we have that $(x_n)_{n=1}^{\infty}$ is indeed a Cauchy sequence in $\mathfrak{X}$.

Since $\mathfrak{X}$ is a Banach space, it is complete, and so $\exists x_0 := \lim\limits_{n \to \infty} x_n \in \mathfrak{X}$. It remains to show that, indeed,

$$x_0 = \int_a^b f.$$

Let $\varepsilon > 0$, and choose $N \geq 1$ such that

- $\frac{1}{N} < \frac{\varepsilon}{2}$; and
- $k \geq N$ implies that $\|x_k - x_0\| < \frac{\varepsilon}{2}$.

Then suppose that V is any refinement of W_N, and V^* is an arbitrary set of test values of V. Then we have

$$\begin{aligned} \|x_0 - S(f, V, V^*)\|_{\mathfrak{X}} &\leq \|x_0 - x_N\|_{\mathfrak{X}} + \|x_N - S(f, V, V^*)\|_{\mathfrak{X}} \\ &< \frac{\varepsilon}{2} + \|S(f, W_N, W_N^*) - S(f, V, V^*)\|_{\mathfrak{X}} \\ &< \frac{\varepsilon}{2} + \frac{1}{N} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

It follows that

$$\int_a^b f = x_0,$$

as desired. $\square$

In first-year calculus, all continuous functions over $\mathbb{R}$ are integrable. A similar result holds in Banach spaces as well. In the next lecture, we shall prove the following theorem.

Theorem (Continuous Functions are Riemann Integrable)
Let $(\mathfrak{X}, \|\cdot\|)$ *be a Banach space and* $a < b \in \mathbb{R}$. *If* $f : [a, b] \to \mathfrak{X}$ *is continuous, then* f *is Riemann integrable over* $[a, b]$.

Bibliography

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