Personal Notes for An Introduction to Analysis William R. Wade

Johnson Ng

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Chapter 1

Differentiability on $\mathbb{R}$

1.1 The Derivative

Definition 1.1.1 (Differentiable)
A real function f is said to be differentiable at a point $a \in \mathbb{R}$ if and only if f is defined on some open interval I containing a and

$$f'(a) := \lim_{h \to \infty} \frac{f(a+h) - f(a)}{h} \tag{1.1}$$

exists. In this case, $f'(a)$ is called the derivative of f at a.

There are two characterizations of diffrentiability which we shall use to study derivatives. The first one which characterizes the derivatives in terms of the "chord function"

$$F(x) := \frac{f(x) - f(a)}{x - a} \quad x \neq a, \tag{1.2}$$

will be used to establish the Chain Rule.

Theorem 1.1.1
A real function f is differentiable at some point $a \in \mathbb{R} \iff \exists$ an open interval I and a function $F : I \to \mathbb{R}$ such that $a \in I$, f is defined on I, F is continuous on a, and

$$f(x) = F(x)(x - a) + f(a) \tag{1.3}$$

holds for all $x \in I$, in which case $F(a) = f'(a)$.

Proof

Notice that $\forall x \in I \setminus \{a\}$, 1.2 and 1.3 are equivalent. SPS f is differentiable at a. Then by definition, f is defined on some open interval I that contains a, and the limit in 1.1 exists. Define F on I by 1.2 if $x \neq a$ and $F(a) := f'(a)$. Then 1.3 holds $\forall x \in I$ and F is continuous on a by 1.2 since $f'(a)$ exists.

Conversely, SPS 1.3 holds. Then 1.2 holds $\forall x \in I \setminus \{a\}$. As $x \to a$, since F is continuous on a, we have that $F(a) = f'(a)$. Thus by definition, f is differentiable on a.