PMATH352W18 Complex Analysis - Class Notes

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Chapter 1

Lecture 1 - Jan 3, 2018

1.1 Complex Numbers and Their Properties

Definition 1.1.1 (Complex Number, Complex Plane)
A ***complex number*** *is a vector in* $\mathbb{R}^2$*. The* ***complex plane****, denoted by* $\mathbb{C}$*, is a set of complex numbers,*

$$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

In $\mathbb{C}$*, we usually write*

$$\begin{aligned} 0 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & 1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ i &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} & x &= \begin{pmatrix} x \\ 0 \end{pmatrix} \\ iy &= \begin{pmatrix} 0 \\ y \end{pmatrix} \end{aligned}$$

where $x, y \in \mathbb{R}$*. Consequently, we have that*

$$x + iy = x + yi = \begin{pmatrix} x \\ y \end{pmatrix}$$

If for $x, y \in \mathbb{R}$*,* $z = x + iy$*, then* x *is aclled the real part of* z *and* y *is called the imaginary part of* z*, and we write*

$$\mathrm{Re}(z) = x \quad \mathrm{Im}(z) = y.$$

Definition 1.1.2 (Sum and Product)
We define the sum of two complex numbers to be the usual vector sum, i.e.

$$\begin{aligned}(a+ib)+(c+id) &= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \begin{pmatrix} a+c \\ b+d \end{pmatrix} \\ &= (a+c)+i(b+d)\end{aligned}$$

where $a, b, c, d \in \mathbb{R}$.

We define the product of two complex numbers by setting $i^2 = -1$, *and by requiring the product to be commutative, associative, and distributive over the sum. In this setup, we have that*

$$\begin{aligned}(a+ib)(c+id) &= ac + iad + ibc + i^2 bd \\ &= (ac - bd) + i(ad + bc) \qquad (1.1)\end{aligned}$$

Example 1.1.1
Let $z = 2 + i, w = 1 + 3i$. *Find* $z + w$ *and* zw.

$$\begin{aligned}z + w &= (2+i) + (1+3i) \\ &= 3 + 4i\end{aligned}$$

$$\begin{aligned}zw &= (2+i)(1+3i) \\ &= (2-3) + i(6+1) \quad \text{By Equation (1.1)} \\ &= -1 + 7i\end{aligned}$$

Example 1.1.2
Show that every non-zero complex number has a multiplicative inverse, z^{-1}, *and find a formula for this inverse.*

Let $z = a + ib$ where $a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$. Then

$$\begin{aligned}
&z(x+iy) = 1 \\
\iff &(ax - by) + i(ay + bx) = 1 \\
\iff &\begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\iff &\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\iff &\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\iff &\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a \\ -b \end{pmatrix} \\
\iff &x + iy = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}
\end{aligned}$$

Therefore, we have that the formula for the inverse is

$$(a + ib)^{-1} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2} \tag{1.2}$$

Notation

For $z, w \in \mathbb{C}$, we write

$$\begin{matrix} -z = -1z & w - z = w + (-z) \\ \frac{1}{z} = z^{-1} & \frac{w}{z} = wz^{-1} \end{matrix}$$

Example 1.1.3

Find $\frac{(4-i)-(1-2i)}{1+2i}$.

$$\begin{aligned}
\frac{(4-i)-(1-2i)}{1+2i} &= \frac{3+i}{1+2i} \\
&= (3+i)(\frac{1}{5} - i\frac{2}{5}) \\
&= 1 - i
\end{aligned}$$

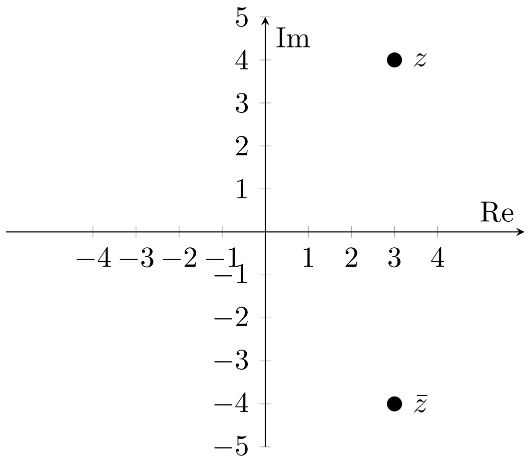
Note

*The set of complex numbers is a **field** under the operations of additiona and multiplication. This means that $\forall u, v, w \in \mathbb{C}$,*

$$
\begin{array}{rl}
u + v = v + u & uv = vu \\
(u + v) + w = u + (v + w) & (uv)w = u(vw) \\
0 + u = u & 1u = u \\
u + (-u) = 0 & uu^{-1} = 1,\ u \neq 0 \\
u(v + w) = uv + uw &
\end{array}
$$

Definition 1.1.3 (Conjugate)
*If $z = x + iy$ where $x, y \in \mathbb{R}$, then the **conjugate of z** is given by $\bar{z} = x - iy$*

Example 1.1.4
Let $z = 3 + 4i$. Then the $\bar{z} = 3 - 4i$. Represented in the complex plane, we have the following:



Definition 1.1.4 (Modulus)
*We define the **modulus** (length, magnitude) of $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$, to be*

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}. \tag{1.3}$$

Note
For $z, w \in \mathbb{R}$, we have

$$
\begin{array}{ccc}
\bar{\bar{z}} = z & z + \bar{z} = 2\,\mathrm{Re}(z) & z - \bar{z} = 2i\,\mathrm{Im}(z) \\
z\bar{z} = |z|^2 & |z| = |\bar{z}| & \overline{z \pm w} = \bar{z} \pm \bar{w} \\
\overline{zw} = \bar{z} - \bar{w} & |zw| = |z|\,|w| &
\end{array}
$$

but note that $|z + w| \neq |z| + |w|$.

Proposition 1.1.1 (Basic Inequalities)

1. $|\mathrm{Re}(z)| \leq |z|$

2. $|\mathrm{Im}(z)| \leq |z|$

3. $|z+w| \leq |z| + |w|$ *Triangle Inequality*

4. $|z+w| \geq \big|\, |z| - |w| \,\big|$ *Inverse Triangle Inequality*

Proof
Note that $|z|^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$ *and that we can express* $|x| = \sqrt{x^2}$ *for any* $x \in \mathbb{R}$. *1 and 2 immediately follows from that.*

To prove 3, we have that

$$\begin{aligned}
|z+w|^2 &= (z+w)(\bar{z}+\bar{w}) \\
&= |z|^2 + |w|^2 + (w\bar{z} + \bar{w}z) \\
&= |z|^2 + |w|^2 + 2\,\mathrm{Re}(w\bar{z}) \\
&\leq |z|^2 + |w|^2 + 2\,|w\bar{z}| \quad \textit{by 1} \\
&= |z|^2 + |w|^2 + 2\,|wz| \quad \textit{since } |w\bar{z}| = |w|\,|\bar{z}| \textit{ and } |z| = |\bar{z}| \\
&= (|z| + |w|)^2
\end{aligned}$$

To prove 4, note that

$$|z| = |z+w-w| \leq |z+w| + |w| \tag{1.4}$$
$$|w| = |w+z-z| \leq |z+w| + |z| \tag{1.5}$$

Observe that

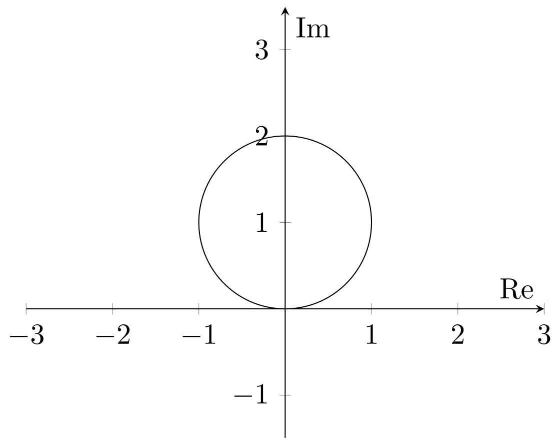
$$\begin{aligned}
\textit{Equation (1.4)} &\implies |z| - |w| \leq |z+w| \\
\textit{Equation (1.5)} &\implies |w| - |z| \leq |z+w|
\end{aligned}$$

Thus, we have that

$$|z+w| \geq \big|\, |z| - |w| \,\big|$$

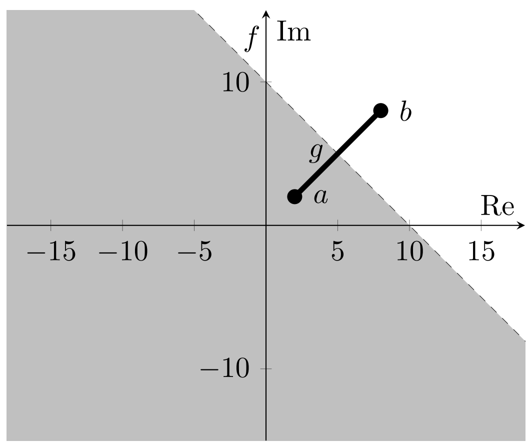
as required. □

Example 1.1.5
We may describe a set $\{z \in \mathbb{C} : |z - i| = 1\}$ *as follows:*



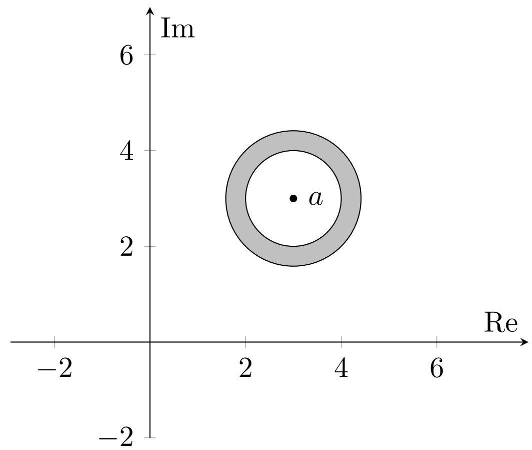
Let $a, b \in \mathbb{C}$ describe the set $\{z \in \mathbb{C} : |z-a| < |z-b|\}$.

Suppose the following coordinates for a and b are arbitrary,



In the above, g is the line segment that connects the points a and b on the complex plane, while f is the perpendicular bisector of the line segment g. The area described by the set $\{z \in \mathbb{C} : |z-a| < |z-b|\}$ is the shaded area which is below f.

Example 1.1.6
Let $a \in \mathbb{C}$. Describe the set $\{z \in \mathbb{C} : 1 < |z-a| < 2\}$.



Example 1.1.7
Show that every non-zero complex number has exactly two complex square roots, and find a formula for the square roots.

Let $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$*, and let* $w = u + iv, u, v \in \mathbb{R}$*. Then*

$$\begin{aligned} w^2 = z &\iff (u+iv)^2 = x + iy \\ &\iff (u^2 - v^2) + i(2uv) = x + iy \\ &\iff x = u^2 + v^2 \quad \textit{and} \qquad (1.6) \\ &\quad\;\;\;\; y = 2uv \qquad (1.7) \end{aligned}$$

Square both sides of Equation (1.7), and thus we have $y^2 = 4u^2v^2$*.*

Multiply Equation (1.6) by $4u^2$*, and we get*

$$\begin{aligned} 4u^2x &= 4u^4 - 4u^2v^2 = 4u^4 - y^2 \\ \iff 0 &= 4u^4 - 4u^2x - y^2 \\ \iff u^2 &= \frac{4x \pm \sqrt{16x^2 + 16y^2}}{8} \\ &= \frac{x \pm \sqrt{x^2 + y^2}}{2} \end{aligned}$$

Suppose $y \neq 0$*. Note that* $x < \sqrt{x^2 + y^2}$*. Thus* $u^2 = \frac{x + \sqrt{x^2+y^2}}{2} \implies u = \left(\frac{x+\sqrt{x^2+y^2}}{2} \right)^{\frac{1}{2}}$*.*

Similarly, we can get

$$v = \pm \left(\frac{-x + \sqrt{x^2 + y^2}}{2} \right)^{\frac{1}{2}}$$

Note that all four choices of signs satisfy Equation (1.6). If $y > 0$, then u and v are either both positive or both negative by Equation (1.7).

Suppose $y = 0$. Then we have

$$w^2 = z = x$$

Therefore, we get

$$w = \begin{cases} \pm \left[\left(\frac{x+\sqrt{x^2+y^2}}{2} \right)^{\frac{1}{2}} + i \left(\frac{-x+\sqrt{x^2+y^2}}{2} \right)^{\frac{1}{2}} \right] & y > 0 \\ \pm \left[\left(\frac{x+\sqrt{x^2+y^2}}{2} \right)^{\frac{1}{2}} - i \left(\frac{-x+\sqrt{x^2+y^2}}{2} \right)^{\frac{1}{2}} \right] & y < 0 \\ \pm\sqrt{x} & y = 0, x > 0 \\ \pm i\sqrt{x} & y = 0, x < 0 \end{cases}$$

Remark
Let $z \in \mathbb{C}$. The notation $\sqrt{z}$ may represent either one of the square roots of z or both of the square roots, i.e. it is possible that $\sqrt{z}$ represents a set.

Exercise 1.1.1
Is it always okay for complex numbers such that $\sqrt{zw} = \sqrt{z}\sqrt{w}$, for $z, w \in \mathbb{C}$?

No. For example, consider $z = w = -1$. Then we have

$$\sqrt{zw} = \sqrt{1} = \pm 1$$

while

$$\sqrt{z}\sqrt{w} = i \cdot i = -1$$

and thus

$$\sqrt{zw} \neq \sqrt{z}\sqrt{w}.$$

Example 1.1.8
Find the values of $\sqrt{3 - 4i}$.

By Example 1.1.7,

$$
\begin{aligned}
\sqrt{3-4i} &= \pm\left(\sqrt{\frac{3+\sqrt{9+16}}{2}} - i\sqrt{\frac{-3+\sqrt{9+16}}{2}}\right) \\
&= \pm(2-i)
\end{aligned}
$$

Remark
The quadratic formula holds for complex polynomials, i.e.

$$\forall a, b, c \in \mathbb{C} \quad a \neq 0 \quad \forall z \in \mathbb{C}\; az^2 + bz + c = 0,$$

the solution for z is given by

$$z_{1,2} = \frac{-b+\sqrt{b^2-4ac}}{b} \tag{1.8}$$

The following is a short proof.

Proof

$$
\begin{aligned}
az^2 + bz + c = 0 &\iff z^2 + \frac{b}{a}z + \frac{c}{a} = 0 \\
&\iff z^2 + \frac{b}{a}z + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \\
&\iff \left(z + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2-4ac}{4a^2} \\
&\iff z = \frac{-b+\sqrt{b^2-4ac}}{2a}
\end{aligned}
$$

(Personal Note: where did the $-$ for the supposed $\pm$ go? Or should it really be $\pm$?)

Example 1.1.9
Solve $iz^2 - (2+3i)z + 5(1-i) = 0$.

$$
\begin{aligned}
z &= \frac{2+3i+\sqrt{(2+3i)^2 - 4i[5(1-i)]}}{2i} \\
&= \frac{2+3i+\sqrt{-5+12i-20i-20}}{2i} \\
&= \frac{2+3i+\sqrt{-25-8i}}{2i}
\end{aligned}
$$

$$\sqrt{-25-8i} = \pm\left[\left(\frac{-25+\sqrt{625+64}}{2}\right)^{\frac{1}{2}} - i\left(\frac{25+\sqrt{625+64}}{2}\right)^{\frac{1}{2}}\right]$$

(Personal note: temporarily stuck, seeing that there's no "nice" solution)