

PMATH352W18 Complex Analysis - Class Notes

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January 3, 2018

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Chapter 1

Lecture 1 - Jan 3, 2018

1.1 Complex Numbers and Their Properties

Definition 1.1.1 (Complex Number, Complex Plane)

A **complex number** is a vector in \mathbb{R}^2 . The **complex plane**, denoted by \mathbb{C} , is a set of complex numbers,

$$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

In \mathbb{C} , we usually write

$$\begin{aligned} 0 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & 1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ i &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} & x &= \begin{pmatrix} x \\ 0 \end{pmatrix} \\ iy &= \begin{pmatrix} 0 \\ y \end{pmatrix} \end{aligned}$$

where $x, y \in \mathbb{R}$. Consequently, we have that

$$x + iy = x + yi = \begin{pmatrix} x \\ y \end{pmatrix}$$

If for $x, y \in \mathbb{R}$, $z = x + iy$, then x is called the real part of z and y is called the imaginary part of z , and we write

$$\operatorname{Re}(z) = x \quad \operatorname{Im}(z) = y.$$

Definition 1.1.2 (Sum and Product)

We define the sum of two complex numbers to be the usual vector sum, i.e.

$$(a + ib) + (c + id) = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a + c \\ b + d \end{pmatrix} = (a + c) + i(b + d)$$

where $a, b, c, d \in \mathbb{R}$.

We define the product of two complex numbers by setting $i^2 = -1$, and by requiring the product to be commutative, associative, and distributive over the sum. In this setup, we have that

$$\begin{aligned} (a + ib)(c + id) &= ac + iad + ibc + i^2bd \\ &= (ac - bd) + i(ad + bc) \end{aligned} \tag{1.1}$$

Example 1.1.1

Let $z = 2 + i, w = 1 + 3i$. Find $z + w$ and zw .

$$\begin{aligned} z + w &= (2 + i) + (1 + 3i) \\ &= 3 + 4i \end{aligned}$$

$$\begin{aligned} zw &= (2 + i)(1 + 3i) \\ &= (2 - 3) + i(6 + 1) \quad \text{By Equation (1.1)} \\ &= -1 + 7i \end{aligned}$$

Example 1.1.2

Show that every non-zero complex number has a multiplicative inverse, z^{-1} , and find a formula for this inverse.

Let $z = a + ib$ where $a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$. Then

$$\begin{aligned}
 & z(x + iy) = 1 \\
 \iff & (ax - by) + i(ay + bx) = 1 \\
 \iff & \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \iff & \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \iff & \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \iff & \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a \\ -b \end{pmatrix} \\
 \iff & x + iy = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2}
 \end{aligned}$$

Therefore, we have that the formula for the inverse is

$$(a + ib)^{-1} = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2} \quad (1.2)$$

Notation

For $z, w \in \mathbb{C}$, we write

$$\begin{aligned}
 -z &= -1z & w - z &= w + (-z) \\
 \frac{1}{z} &= z^{-1} & \frac{w}{z} &= wz^{-1}
 \end{aligned}$$

Example 1.1.3

Find $\frac{(4-i)-(1-2i)}{1+2i}$.

$$\begin{aligned}
 \frac{(4-i)-(1-2i)}{1+2i} &= \frac{3+i}{1+2i} \\
 &= (3+i)\left(\frac{1}{5} - i\frac{2}{5}\right) \\
 &= 1 - i
 \end{aligned}$$

Note

The set of complex numbers is a **field** under the operations of addition and multiplication. This means that $\forall u, v, w \in \mathbb{C}$,

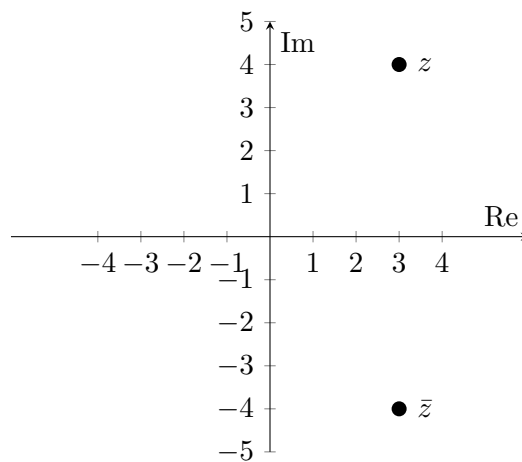
$$\begin{array}{ll}
u + v = v + u & uv = vu \\
(u + v) + w = u + (v + w) & (uv)w = u(vw) \\
0 + u = u & 1u = u \\
u + (-u) = 0 & uu^{-1} = 1, \quad u \neq 0 \\
u(v + w) = uv + uw &
\end{array}$$

Definition 1.1.3 (Conjugate)

If $z = x + iy$ where $x, y \in \mathbb{R}$, then the **conjugate of z** is given by $\bar{z} = x - iy$

Example 1.1.4

Let $z = 3 + 4i$. Then the $\bar{z} = 3 - 4i$. Represented in the complex plane, we have the following:

**Definition 1.1.4 (Modulus)**

We define the **modulus** (length, magnitude) of $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$, to be

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}. \quad (1.3)$$

Note

For $z, w \in \mathbb{R}$, we have

$$\begin{array}{lll}
\bar{\bar{z}} = z & z + \bar{z} = 2 \operatorname{Re}(z) & z - \bar{z} = 2i \operatorname{Im}(z) \\
z\bar{z} = |z|^2 & |z| = |\bar{z}| & \overline{z \pm w} = \bar{z} \pm \bar{w} \\
\overline{z\bar{w}} = \bar{z} - \bar{w} & |zw| = |z| |w| &
\end{array}$$

but note that $|z + w| \neq |z| + |w|$.

Proposition 1.1.1 (Basic Inequalities)

$$1. |\operatorname{Re}(z)| \leq |z|$$

$$2. |\operatorname{Im}(z)| \leq |z|$$

$$3. |z + w| \leq |z| + |w| \quad \text{Triangle Inequality}$$

$$4. |z + w| \geq ||z| - |w|| \quad \text{Inverse Triangle Inequality}$$

Proof

Note that $|z|^2 = \operatorname{Re}(z)^2 + \operatorname{Im}(z)^2$ and that we can express $|x| = \sqrt{x^2}$ for any $x \in \mathbb{R}$. 1 and 2 immediately follows from that.

To prove 3, we have that

$$\begin{aligned} |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) \\ &= |z|^2 + |w|^2 + (w\bar{z} + \bar{w}z) \\ &= |z|^2 + |w|^2 + 2\operatorname{Re}(w\bar{z}) \\ &\leq |z|^2 + |w|^2 + 2|w\bar{z}| \quad \text{by 1} \\ &= |z|^2 + |w|^2 + 2|wz| \quad \text{since } |w\bar{z}| = |w||\bar{z}| \text{ and } |z| = |\bar{z}| \\ &= (|z| + |w|)^2 \end{aligned}$$

To prove 4, note that

$$|z| = |z + w - w| \leq |z + w| + |w| \quad (1.4)$$

$$|w| = |w + z - z| \leq |z + w| + |z| \quad (1.5)$$

Observe that

$$\text{Equation (1.4)} \implies |z| - |w| \leq |z + w|$$

$$\text{Equation (1.5)} \implies |w| - |z| \leq |z + w|$$

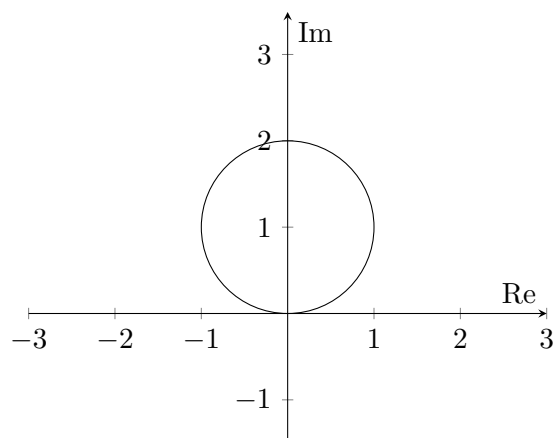
Thus, we have that

$$|z + w| \geq ||z| - |w||$$

as required. □

Example 1.1.5

We may describe a set $\{z \in \mathbb{C} : |z - i| = 1\}$ as follows:



Let $a, b \in \mathbb{C}$ describe the set $\{z \in \mathbb{C} : |z - a| < |z - b|\}$.