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1. A total ordering on $\mathbb{C}$ is a relation $\succ$ between complex numbers that satisfies *all* the following conditions:

   (C1) For any two complex numbers z, w, one and only one of the following is true: $z \succ w, w \succ z$ or $z = w$.

   (C2) For all $z_1, z_2, z_3 \in \mathbb{C}$, the relation $z_1 \succ z_2$ implies $z_1 + z_3 \succ z_1 + z_3$.

   (C3) For all $z_1, z_2, z_3 \in \mathbb{C}$, if $z_3 \succ 0$, then the relation $z_1 \succ z_2$ implies $z_1 z_3 \succ z_2 z_3$.

   Show that it is impossible to define a total ordering on $\mathbb{C}$. [Hint: Assume a relation between i and 0]

   Proof
   We know that $i \neq 0$.

   Suppose $i \succ 0$. Then by (C2),

$$i \succ 0 \quad \Longrightarrow \quad i - i = 0 \succ -i = 0 - i$$

   But by (C3),

$$i \succ 0 \quad \Longrightarrow \quad i^2 = -1 \succ 0 \quad \Longrightarrow \quad (-1)i = -i \succ 0.$$

   Thus $i \not\succ 0$. Suppose $0 \succ i$. By (C2),

$$0 \succ i \quad \Longrightarrow \quad 0 - i = -i \succ 0 = i - i$$

   But by (C3),

$$-i \succ 0 \quad \Longrightarrow \quad (-i)(-i) = -1 \succ 0 \quad \Longrightarrow \quad (-1)(-i) = i \succ 0.$$

   Then $0 \not\succ i$. Hence, it is impossible to define a total ordering on $\mathbb{C}$. □

2. Let w be a complex number with $0 < |w| < 1$. Show that the set of all $z \in \mathbb{C}$ with $|z - w| < |1 - \overline{w}z|$ is the disc $\{z \in \mathbb{C} : |z| < 1\}$.

   Proof
   Let $w = u + iv$ and $z = x + iy$ where $u, v, x, y \in \mathbb{R}$. Note that

$$\overline{w}z = (u - iv)(x + iy) = ux + vy + i(uy - vx).$$

Thus

$$\begin{aligned}|z-w| &< |1-\overline{w}z| \\ (x-u)^2+(y-v)^2 &< (1-ux-vy)^2+(uy-vx)^2 \\ (x^2+y^2)[1-(u^2+v^2)]+u^2+v^2 &< 1 \\ x^2+y^2 &< 1\end{aligned}$$

where in the second line, we may disregard the square roots since all the terms are squares and are, therefore, positive. This completes the proof. □

3. Let $P(z)$ be a polynomial with real coefficients. Show that the complex roots of P appear in conjugate pairs.

Proof

Let

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = \sum_{k=0}^{n} a_k z^k$$

and $w \in \mathbb{C}$ be a complex root of P. Thus

$$\sum_{k=0}^{n} a_k w^k = 0 \tag{1}$$

Now

$$P(\overline{w}) = \sum_{k=0}^{n} a_k \overline{w}^k = \sum_{k=0}^{n} a_k \overline{w^k} = \sum_{k=0}^{n} \overline{a_k w^k} = \overline{\sum_{k=0}^{n} a_k w^k}$$

by the properties of complex numbers. By Equation (1), we obtain that

$$P(\overline{w}) = \overline{\sum_{k=0}^{n} a_k w^k} = \bar{0} = 0,$$

showing that the conjugate of w is also a root of P. □

4. Suppose $f, g : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ are holomorphic at $z_0 \in \Omega$. Show that fg is holomorphic and at z_0 and $(fg)' = f'g + fg'$ at z_0.

Proof

Consider the limit

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(z_0+h)g(z_0+h) - f(z_0)g(z_0)}{h} \\
=&\lim_{h\to 0} \frac{f(z_0+h)g(z_0+h) + f(z_0)g(z_0+h) - f(z_0)g(z_0+h) - f(z_0)g(z_0)}{h} \\
=&\lim_{h\to 0} \left[\frac{f(z_0+h) - f(z_0)}{h} g(z_0+h) + f(z_0)\frac{g(z_0+h) - g(z_0)}{h}\right] \\
=&f'(z_0)g(z_0) + f(z_0)g'(z_0)
\end{aligned}$$

Thus, we have that fg is holomorphic, and that $(fg)' = f'g + fg'$ at z_0. □

5. A function f is said to be entire if f is holomorphic in the entire complex plane. Consider a polynomial in z of degree $n \geq 1$:

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0, \quad (a_i \in \mathbb{C}, a_n \neq 0)$$

Show that

(a) $P(z)$ is an entire function and $P'(z) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \ldots + a_1$.

(b) P cannot take only imaginary values.

Proof

(a) We will prove this statement inductively. $\forall n \in \mathbb{N}\backslash\{0\}$, let $Q(n)$ be the statement: $\forall z \in \mathbb{C}, P(z)$ is an entire function and $P'(z) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \ldots + a_1$.

When $n = 1, P(z) = a_1 z + a_0$, then

$$\lim_{h\to 0} \frac{a_1(z+h) + a_0 - a_1 z - a_0}{h} = \lim_{h\to 0} \frac{a_1 h}{h} = a_1.$$

Thus $Q(1)$ is true. Let $k \in \mathbb{N} \setminus \{0\}$, and suppose that $Q(k)$ is true, i.e. we have

$$\begin{aligned}
P'(z) &= \lim_{h\to 0} \frac{\sum_{i=0}^{k} a_i \left(\sum_{j=0}^{i} \binom{i}{j} z^{i-j}h^j\right) - \sum_{i=0}^{k} a_i z^i}{h} \\
&= \lim_{h\to 0} \frac{\sum_{i=0}^{k} a_i \left[\sum_{j=0}^{i} \left(\binom{i}{j} z^{i-j}h^j - z^i\right)\right]}{h} \\
&= ka_k z^{k-1} + (k-1)a_{k-1}z^{k-2} + \ldots + a_1
\end{aligned} \tag{2}$$

When $n = k + 1$,

$$
\begin{aligned}
&\lim_{h\to 0} \frac{\sum_{i=0}^{k+1} a_i \left(\sum_{j=0}^{i} \binom{i}{j} z^{i-j} h^j\right) - \sum_{i=0}^{k+1} a_i z^i}{h} \\
=&\lim_{h\to 0} \frac{a_{k+1}\left[\sum_{j=0}^{k+1}\left(\binom{k+1}{j} z^{k+1-j} h^j - z^{k+1}\right)\right]}{h} \\
&\quad + \lim_{h\to 0} \frac{\sum_{i=0}^{k} a_i \left[\sum_{j=0}^{i}\left(\binom{i}{j} z^{i-j} h^j - z^i\right)\right]}{h} \\
=&\lim_{h\to 0} \frac{a_{k+1}\left[z^{k+1} + \binom{k+1}{1} z^k h + \binom{k+1}{2} z^{k-1} h^2 + \ldots + h^{k+1} - z^{k+1}\right]}{h} \\
&\quad + \text{Equation (2)} \\
=&(k+1)a_{k+1} z^k + \text{Equation (2)}
\end{aligned}
$$

Thus $Q(k+1)$ *is true, and hence by mathematical induction,* $\forall n \in \mathbb{N} \setminus \{0\}, P(z)$ *is an entire function.*

(b)