PMATH352W18 Complex Analysis - Class Notes

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Chapter 1

Lecture 1 - Jan 3, 2018

1.1 Complex Numbers and Their Properties

Definition 1.1.1 (Complex Number, Complex Plane)
*A **complex number** is a vector in $\mathbb{R}^2$. The **complex plane**, denoted by $\mathbb{C}$, is a set of complex numbers,*

$$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

In $\mathbb{C}$, we usually write

$$0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad 1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$i = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad x = \begin{pmatrix} x \\ 0 \end{pmatrix}$$
$$iy = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

where $x, y \in \mathbb{R}$. Consequently, we have that

$$x + iy = x + yi = \begin{pmatrix} x \\ y \end{pmatrix}$$

If for $x, y \in \mathbb{R}$, $z = x + iy$, then x is aclled the real part of z and y is called the imaginary part of z, and we write

$$\mathrm{Re}(z) = x \quad \mathrm{Im}(z) = y.$$

Note

- *It is easy to see how $\mathbb{R}$ is a subset of $\mathbb{C}$.*

- *Complex Numbers of the form* $\left(\begin{smallmatrix}0\\y\end{smallmatrix}\right)$ *where* $y \in \mathbb{R}$ *are called purely imaginary numbers.*
- *Certain authors may prefer to denote* $i = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$.

Definition 1.1.2 (Sum and Product)
We define the sum of two complex numbers to be the usual vector sum, i.e.

$$\begin{aligned}(a+ib)+(c+id) &= \begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}\\ &= \begin{pmatrix}a+c\\b+d\end{pmatrix}\\ &= (a+c)+i(b+d)\end{aligned}$$

where $a, b, c, d \in \mathbb{R}$.

We define the product of two complex numbers by setting $i^2 = -1$, *and by requiring the product to be commutative, associative, and distributive over the sum. In this setup, we have that*

$$\begin{aligned}(a+ib)(c+id) &= ac + iad + ibc + i^2bd\\ &= (ac-bd)+i(ad+bc) \qquad (1.1)\end{aligned}$$

Note
It is interesting to note that any complex number times zero is zero, just like what we have with real numbers.

$$\forall z = x+iy \in \mathbb{C}\ x, y \in \mathbb{R}\ \ 0 \in \mathbb{C}$$
$$z \cdot 0 = (x+iy)(0+i0) = 0+i0 = 0$$

Example 1.1.1
Let $z = 2+i, w = 1+3i$. *Find* $z+w$ *and* zw.

$$\begin{aligned}z+w &= (2+i)+(1+3i)\\ &= 3+4i\end{aligned}$$

$$\begin{aligned}zw &= (2+i)(1+3i)\\ &= (2-3)+i(6+1) \quad \text{By Equation (1.1)}\\ &= -1+7i\end{aligned}$$

Example 1.1.2
Show that every non-zero complex number has a multiplicative inverse, z^{-1}, *and find a formula for this inverse.*

Let $z = a + ib$ where $a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$. Then

$$\begin{aligned}
& z(x+iy) = 1 \\
\iff & (ax - by) + i(ay + bx) = 1 \\
\iff & \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\iff & \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\iff & \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\iff & \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2+b^2} \begin{pmatrix} a \\ -b \end{pmatrix} \\
\iff & x + iy = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}
\end{aligned}$$

Therefore, we have that the formula for the inverse is

$$(a+ib)^{-1} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2} \tag{1.2}$$

Notation
For $z, w \in \mathbb{C}$, we write

$$\begin{array}{ll} -z = -1z & w - z = w + (-z) \\ \frac{1}{z} = z^{-1} & \frac{w}{z} = wz^{-1} \end{array}$$

Example 1.1.3
Find $\frac{(4-i)-(1-2i)}{1+2i}$.

$$\begin{aligned}
\frac{(4-i)-(1-2i)}{1+2i} &= \frac{3+i}{1+2i} \\
&= (3+i)(\frac{1}{5} - i\frac{2}{5}) \\
&= 1 - i
\end{aligned}$$

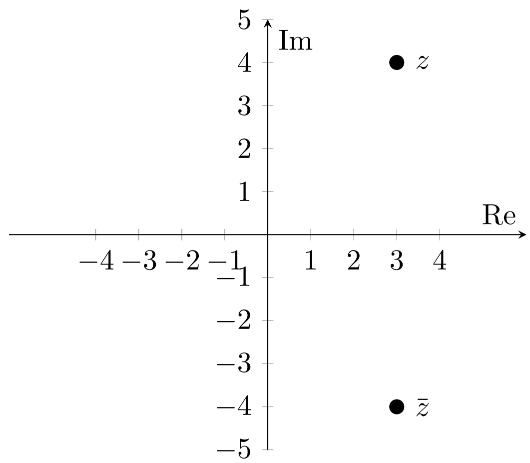
Note
The set of complex numbers is a ***field*** *under the operations of additiona and multiplication. This means that $\forall u, v, w \in \mathbb{C}$,*

$$\begin{aligned} u + v &= v + u & uv &= vu \\ (u + v) + w &= u + (v + w) & (uv)w &= u(vw) \\ 0 + u &= u & 1u &= u \\ u + (-u) &= 0 & uu^{-1} &= 1,\ u \neq 0 \\ u(v + w) &= uv + uw & & \end{aligned}$$

Since the distributive law holds for complex numbers, note that the binomial expansion works for $(w+z)^n$ *where* $w, z \in \mathbb{C}$ *and* $n \in \mathbb{N}$. *(I did not verify if this is still true for when* $n \in \mathbb{R}$.*)*

Definition 1.1.3 (Conjugate)
If $z = x + iy$ *where* $x, y \in \mathbb{R}$*, then the* ***conjugate of z*** *is given by* $\bar{z} = x - iy$

Example 1.1.4
Let $z = 3 + 4i$. *Then the* $\bar{z} = 3 - 4i$. *Represented in the complex plane, we have the following:*



We observe that on the complex plane, the conjugate of a complex number is simply its reflection on the real axis.

Definition 1.1.4 (Modulus)
We define the ***modulus*** *(length, magnitude) of* $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$*, to be*

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}. \tag{1.3}$$

Note
Note that this definition is consistent with the notion of the absolute value in real numbers when z *is a real number, since if* $y = 0$, $|z| = |x + i0| = \sqrt{x^2} = \pm x$.

Note
For $z, w \in \mathbb{R}$, we have

$$\begin{aligned} \bar{\bar{z}} &= z & z + \bar{z} &= 2\,\mathrm{Re}(z) & z - \bar{z} &= 2i\,\mathrm{Im}(z) \\ z\bar{z} &= |z|^2 & |z| &= |\bar{z}| & \overline{z \pm w} &= \bar{z} \pm \overline{w} \\ \overline{zw} &= \bar{z} - \overline{w} & |zw| &= |z|\,|w| \end{aligned}$$

but note that $|z + w| \neq |z| + |w|$.

Note
While inequalities such as $z_1 < z_2$, where $z_1, z_2 \in \mathbb{C}$, are meaningless unless if both of them are real, $|z_1| < |z_2|$ means that the point z_1 in the complex plane is closer to the origin than the point z_2.

Proposition 1.1.1 (Basic Inequalities)

1. $|\mathrm{Re}(z)| \leq |z|$
2. $|\mathrm{Im}(z)| \leq |z|$
3. $|z + w| \leq |z| + |w|$ *Triangle Inequality*
4. $|z + w| \geq |\,|z| - |w|\,|$ *Inverse Triangle Inequality*

Proof
Note that $|z|^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$ and that we can express $|x| = \sqrt{x^2}$ for any $x \in \mathbb{R}$. 1 and 2 immediately follows from that.

To prove 3, we have that

$$\begin{aligned} |z + w|^2 &= (z + w)(\bar{z} + \overline{w}) \\ &= |z|^2 + |w|^2 + (w\bar{z} + \overline{w}z) \\ &= |z|^2 + |w|^2 + 2\,\mathrm{Re}(w\bar{z}) \\ &\leq |z|^2 + |w|^2 + 2\,|w\bar{z}| \quad \textit{by 1} \\ &= |z|^2 + |w|^2 + 2\,|wz| \quad \textit{since } |w\bar{z}| = |w|\,|\bar{z}| \textit{ and } |z| = |\bar{z}| \\ &= (|z| + |w|)^2 \end{aligned}$$

To prove 4, note that

$$|z| = |z + w - w| \leq |z + w| + |w| \tag{1.4}$$
$$|w| = |w + z - z| \leq |z + w| + |z| \tag{1.5}$$

Observe that

$$\begin{aligned}\text{Equation (1.4)} &\implies |z| - |w| \leq |z+w| \\ \text{Equation (1.5)} &\implies |w| - |z| \leq |z+w|\end{aligned}$$

Thus, we have that

$$|z+w| \geq \big|\, |z| - |w| \,\big|$$

as required. □

Item 3 in Proposition 1.1.1 can be generalized by the means of mathematical induction to sums involving any finite number of terms, as:

$$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n| \tag{1.6}$$

where $n \in \mathbb{N} \setminus \{0, 1\}$.

To note the induction proof, when $n = 2$, Equation (1.6) is just Item 3. If Equation (1.6) is true for when $n = m$ where $m \in \mathbb{N} \setminus \{0, 1\}$, $n = m + 1$ is also true since by Item 3,

$$\begin{aligned}|(z_1 + z_2 + \ldots + z_m) + z_{m+1}| &\leq |z_1 + z_2 + \ldots + z_m| + |z_{m+1}| \\ &\leq (|z_1| + |z_2| + \ldots + |z_m|) + |z_{m+1}| \,.\end{aligned}$$

The distance between two points $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}, x_1, x_2, y_1, y_2 \in \mathbb{R}$ is $|z_1 - z_2|$, since $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 (y_1 - y_2)^2}$ is our usual notion of the Euclidean distance of two points on a plane.

Also, note that

$$z_1 - z_2 = z_1 + (-z_2)$$

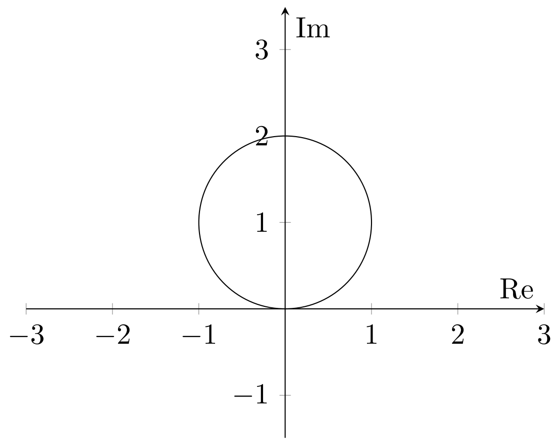
and thus if we apply our knowledge of vector representation, $z_1 - z_2$ is the directed line segment from the point z_2 to z_1.

With the notion of a "distance" set on the complex plane, we can now explore upon points lying on a circle with a center z_0 and radius R, which satisfies the equation

$$|z - z_0| = R.$$

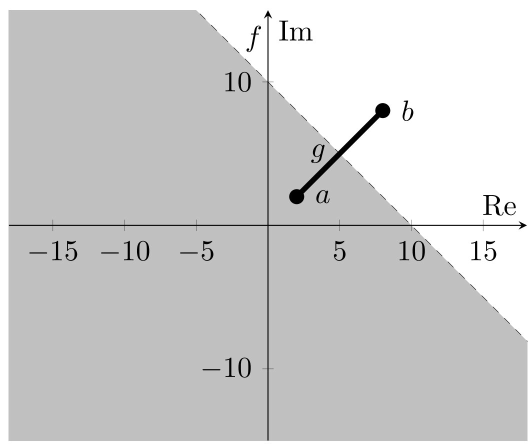
We may simply refer to this set of points as the circle $|z - z_0| = R$.

Example 1.1.5
We may describe a set $\{z \in \mathbb{C} : |z - i| = 1\}$ *as follows:*



Let $a, b \in \mathbb{C}$ describe the set $\{z \in \mathbb{C} : |z - a| < |z - b|\}$.

Suppose the following coordinates for a and b are arbitrary,



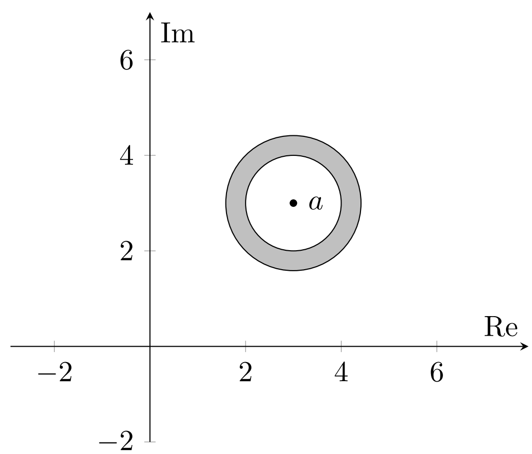
In the above, g is the line segment that connects the points a and b on the complex plane, while f is the perpendicular bisector of the line segment g. The area described by the set $\{z \in \mathbb{C} : |z - a| < |z - b|\}$ is the shaded area which is below f.

Chapter 2

Lecture 2 - Jan 5th, 2018

2.1 Complex Numbers and Their Properties (Continued)

Example 2.1.1
Let $a \in \mathbb{C}$. Describe the set $\{z \in \mathbb{C} : 1 < |z - a| < 2\}$.



Example 2.1.2
Show that every non-zero complex number has exactly two complex square roots, and find a formula for the square roots.

Let $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$, and let $w = u + iv, u, v \in \mathbb{R}$. Then

$$\begin{aligned} w^2 = z &\iff (u+iv)^2 = x+iy \\ &\iff (u^2-v^2)+i(2uv) = x+iy \\ &\iff x = u^2+v^2 \quad \textit{and} \qquad (2.1) \\ &\quad\;\; y = 2uv \qquad (2.2) \end{aligned}$$

Square both sides of Equation (2.2), and thus we have $y^2 = 4u^2v^2$.

Multiply Equation (2.1) by $4u^2$*, and we get*

$$\begin{aligned} 4u^2x &= 4u^4 - 4u^2v^2 = 4u^4 - y^2 \\ \iff 0 &= 4u^4 - 4u^2x - y^2 \\ \iff u^2 &= \frac{4x \pm \sqrt{16x^2+16y^2}}{8} \\ &= \frac{x \pm \sqrt{x^2+y^2}}{2} \end{aligned}$$

Suppose $y \neq 0$*. Note that* $x < \sqrt{x^2+y^2}$*. Thus* $u^2 = \frac{x+\sqrt{x^2+y^2}}{2} \implies u = \left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}$.

Similarly, we can get

$$v = \pm\left(\frac{-x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}$$

Note that all four choices of signs satisfy Equation (2.1). If $y > 0$*, then* u *and* v *are either both positive or both negative by Equation (2.2).*

Suppose $y = 0$*. Then we have*

$$w^2 = z = x$$

Therefore, we get

$$w = \begin{cases} \pm\left[\left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}} + i\left(\frac{-x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}\right] & y > 0 \\ \pm\left[\left(\frac{x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}} - i\left(\frac{-x+\sqrt{x^2+y^2}}{2}\right)^{\frac{1}{2}}\right] & y < 0 \\ \pm\sqrt{x} & y = 0, x > 0 \\ \pm i\sqrt{x} & y = 0, x < 0 \end{cases}$$

Remark
Let $z \in \mathbb{C}$. The notation $\sqrt{z}$ may represent either one of the square roots of z or both of the square roots, i.e. it is possible that $\sqrt{z}$ represents a set.

Exercise 2.1.1
Is it always okay for complex numbers such that $\sqrt{zw} = \sqrt{z}\sqrt{w}$, for $z, w \in \mathbb{C}$?

No. For example, consider $z = w = -1$. Then we have

$$\sqrt{zw} = \sqrt{1} = \pm 1$$

while

$$\sqrt{z}\sqrt{w} = i \cdot i = -1$$

and thus

$$\sqrt{zw} \neq \sqrt{z}\sqrt{w}.$$

Example 2.1.3
Find the values of $\sqrt{3-4i}$.

By Example 2.1.2,

$$\begin{aligned}\sqrt{3-4i} &= \pm\left(\sqrt{\frac{3+\sqrt{9+16}}{2}} - i\sqrt{\frac{-3+\sqrt{9+16}}{2}}\right) \\ &= \pm(2-i)\end{aligned}$$

Remark
The quadratic formula holds for complex polynomials, i.e.

$$\forall a, b, c \in \mathbb{C} \quad a \neq 0 \quad \forall z \in \mathbb{C}\; az^2 + bz + c = 0,$$

the solution for z is given by

$$z_{1,2} = \frac{-b + \sqrt{b^2 - 4ac}}{b} \tag{2.3}$$

The following is a short proof.

Proof

$$
\begin{aligned}
az^2 + bz + c = 0 &\iff z^2 + \frac{b}{a}z + \frac{c}{a} = 0 \\
&\iff z^2 + \frac{b}{a}z + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \\
&\iff \left(z + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2} \\
&\iff z = \frac{-b + \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

(Personal Note: where did the $-$ for the supposed $\pm$ go? Or should it really be $\pm$?)

Example 2.1.4
Solve $iz^2 - (2+3i)z + 5(1+i) = 0$.

$$
\begin{aligned}
z &= \frac{2 + 3i + \sqrt{(2+3i)^2 - 4i[5(1+i)]}}{2i} \\
&= \frac{2 + 3i + \sqrt{-5 + 12i - 20i + 20}}{2i} \\
&= \frac{2 + 3i + \sqrt{15 + 8i}}{2i}
\end{aligned}
$$

Note that by Example 2.1.2,

$$
\begin{aligned}
\sqrt{15 - 8i} &= \pm\left[\sqrt{\frac{15 + \sqrt{225 + 64}}{2}} - i\sqrt{\frac{-15 + \sqrt{225 + 64}}{2}}\right] \\
&= \pm\left[\sqrt{\frac{15 + 17}{2}} - i\sqrt{\frac{-15 + 17}{2}}\right] \\
&= \pm(4 - i)
\end{aligned}
$$

Thus we have

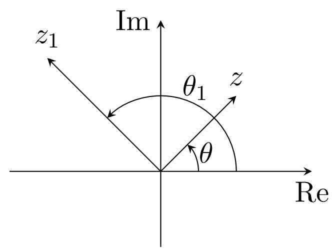
$$
\begin{aligned}
z &= \frac{2 + 3i + \sqrt{15 + 8i}}{2i} \\
&= \frac{2 + 3i \pm (4 - i)}{2i} \\
&= (6 + 2i)\left(-\frac{1}{2}i\right) \;\; \textit{or} \;\; (-2 + 4i)\left(-\frac{1}{2}i\right) \quad \textit{by Example } 1.1.2 \\
&= (1 - 3i) \;\; \textit{or} \;\; (2 + i)
\end{aligned}
$$

Chapter 3

Lecture 3 - Jan 8th, 2018

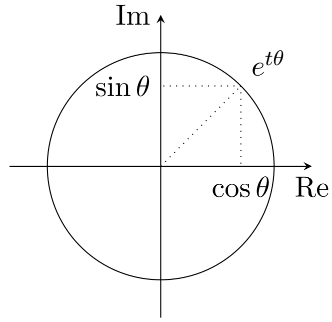
3.1 Complex Numbers and Their Properties (Continued 2)

Definition 3.1.1 (Argument of a Complex Number)
*Let $z \in \mathbb{C} \setminus \{0\}$. The **argument** (or the angle) of z, denoted by $\arg z$, $\operatorname{Arg} z$, or simply $\theta = \theta(z)$, is the angle modulo 2π (i.e. $0 \leq \theta < 2\pi$) between the vector defining z and the positive real axis (in the counterclockwise direction).*



Notation
Let $e^{i\theta} := \cos\theta + i\sin\theta$. Note that this definition can be derived by the extending the Taylor expansion of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for when $x \in \mathbb{C}$ (the sum of the real parts of the expansion is the Taylor expansion of cosine while the imaginary part for sine).

Now $e^{i\theta}$ is on the unit circle.



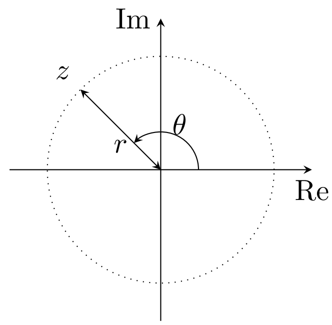
Example 3.1.1
Some examples of $\theta \in [0, 2\pi)$:

$$\begin{aligned} e^{i\frac{\pi}{4}} &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\frac{\pi}{2}} &= i \\ e^{i\frac{3\pi}{4}} &= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\pi} + 1 &= 0 \end{aligned}$$

Remark

$$\forall k \in \mathbb{Z}\ \ \forall \theta \in \mathbb{R}\ \ e^{i\theta} = e^{i(\theta + 2\pi k)}$$

Remark
The complex number $re^{i\theta}$, where $r > 0, \theta \in [0, 2\pi)$, represents the complex number with modulus r and argument θ.



Therefore, $\forall z \in \mathbb{C}$, we can express

$$z := |z|\, e^{i \operatorname{Arg} z}. \tag{3.1}$$

With that, we now have two representations of a complex number:

- Cartesian representation: $z = x + iy$ where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$
- Polar representation: $z = re^{i\theta}$ where $r = |z|$ and $\theta = \operatorname{Arg} z \in [0, 2\pi)$

To convert between the two representations, we have the following equations:

Polar $\to$ Cartesian:

$$x = r\cos\theta \quad y = r\sin\theta \tag{3.2}$$

Cartesian $\to$ Polar:

$$\begin{aligned} r &= |z| \\ x \neq 0 &\implies \tan\theta = \frac{y}{x} \\ x = 0 &\implies \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \end{aligned} \tag{3.3}$$

On another note,

$$z = re^{i\theta} \implies \bar{z} = re^{-i\theta}$$

and

$$z \neq 0 \implies \frac{1}{z} = \frac{1}{r}e^{-i\theta}$$

Remark

$$\begin{gathered} \forall r_1, r_2 \in \mathbb{R}\ \forall \theta_1, \theta_2 \in [0, 2\pi) \\ z_1 := r_1 e^{i\theta_1} \quad z_2 := r_2 e^{i\theta_2} \end{gathered}$$

Then

$$z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

Note that $e^{ix}e^{iy} = e^{i(x+y)}$ *is true for all* $x, y \in \mathbb{R}$ *since*

$$\begin{aligned} e^{ix}e^{iy} &= (\cos x + i\sin x)(\cos y + i\sin y) \\ &= (\cos x\cos y - \sin x\sin y) + i(\cos x\sin y + \cos y\sin x) \\ &= \cos(x+y) + i\sin(x+y) \\ &= e^{i(x+y)}. \end{aligned}$$

Generalizing the above, we get that

$$\forall n \in \mathbb{Z}\ \ (re^{in}) = r^n e^{in\theta}$$

*which is commonly known as **deMoivre's Law**.*

Proposition 3.1.1 (nth Roots of a Complex Number)

$$\forall z = re^{i\theta} \in \mathbb{C} \;\; r = |z| \in \mathbb{R} \;\; \theta \in [0, 2\pi)$$
$$\exists w = se^{i\tau} \in \mathbb{C} \;\; s \in \mathbb{R} \;\; \tau \in [0, 2\pi)$$
$$\forall n \in \mathbb{Z}$$
$$w^n = \left(se^{i\tau}\right)^n = z = re^{i\theta}$$

The nth roots of z is described by the set

$$\left\{ r^{\frac{1}{n}} e^{i\left(\frac{\theta+2\pi k}{n}\right)} : k = 0, 1, ..., n-1 \right\} \tag{3.4}$$

Proof

$$s^n = r \iff s = r^{\frac{1}{n}}$$
$$e^{in\theta} = e^{i\tau} \iff \theta = \frac{\tau + 2\pi k}{n}$$

Therefore, the set that describes the nth roots of z is

$$\left\{ w = r^{\frac{1}{n}} e^{i\left(\frac{\theta+2\pi k}{n}\right)} : k = 0, 1, ..., n-1 \right\}$$

Remark (nth Roots of Unity)
The nth roots of unity is a direct consequence of Proposition 3.1.1 where we solve for the equation $z^n = 1$ for any $z \in \mathbb{C}, n \in \mathbb{Z}$.

The set that describes the nth roots of unity is

$$\left\{ e^{i\theta} : \theta = \frac{2\pi k}{n}, k = 0, 1, ..., n-1 \right\} \tag{3.5}$$

It is easy to see how the nth roots of unity partitions the unit circle into n parts.

Example 3.1.2
Find the cubic roots of $-2 + 2i$.

Let $z = -2 + 2i$. Note that $|z| = 2\sqrt{2}$ and $\operatorname{Arg} z = \frac{3\pi}{4}$.

Therefore, in polar form, $z = 2\sqrt{2}e^{i\frac{3\pi}{4}}$.

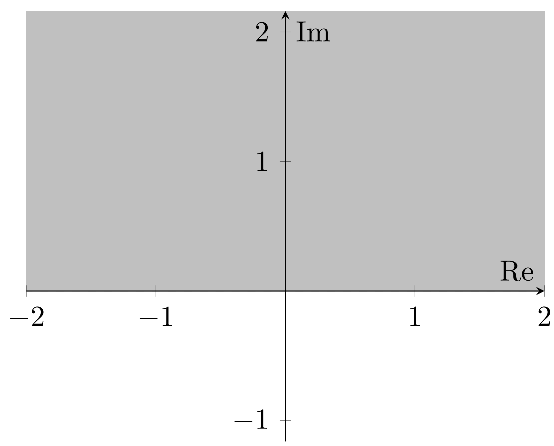
Let $w = re^{i\theta}$*, where* $\theta \in [0, 2\pi)$*, and* $w^3 = z$*. Then*

$$r = (2\sqrt{2})^{\frac{1}{3}}$$
$$\theta = \frac{\frac{3\pi}{4} + 2\pi k}{3}, \;\; k = 0, 1, 2$$

The set that describes the cubic root of $-2 + 2i$ *is thus*

$$\left\{ (2\sqrt{2})^{\frac{1}{3}} e^{i\theta} : \theta = \frac{\frac{3\pi}{4} + 2\pi k}{3}, k = 0, 1, 2 \right\}$$

Example 3.1.3
Describe the set $\{z \in \mathbb{C} : \left|\operatorname{Arg} z - \frac{\pi}{2}\right| < \frac{\pi}{2}\}$*. (Note:* $\operatorname{Arg} z \in [0, 2\pi)$*)*



Exercise 3.1.1
Solve

1. $z^4 = -1$

$$\text{Let } z = re^{i\theta}$$
$$r = |-1| = 1 \quad \theta = \frac{\pi + 2\pi k}{4} = \frac{(2k+1)\pi}{4}, \;\; k = 0, 1, 2, 3$$

2. $z^4 = -1 + \sqrt{3}i$

$$\text{Let } z = re^{i\theta}$$
$$r = \left|-1 + \sqrt{3}i\right| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$
$$\theta = \frac{\frac{2\pi}{3} + 2\pi k}{4} = \frac{(2k + \frac{2}{3})\pi}{4}, \quad k = 0, 1, 2, 3$$

Chapter 4

Lecture 4 - Jan 10th, 2018

4.1 Examples for nth Roots of Unity

Recall that the nth roots of unity are given by $e^{i\frac{2\pi k}{n}}, k = 0, 1, ..., n-1$.

Exercise 4.1.1
Let z be any nth root of unity other than 1. Show that

$$z^{n-1} + z^{n-2} + \ldots + z + 1 = 0 \tag{4.1}$$

Proof
By the Sum of Finite Geometric Terms,

$$z^{n-1} + z^{n-2} + \ldots + z + 1 = \frac{1 - z^n}{1 - z}.$$

Since $z^n = 1$, RHS is thus zero, which in turn completes the proof.

As an aside, if we wish to remove the restriction that z can also be 1, we may consider that

$$z^n - 1 = (z - 1)(1 + z + \ldots + z^{n-1})$$

Since $z^n = 1$, LHS is zero. Then either $z = 1$ or $(1 + z + \ldots + z^{n-1}) = 0$.

Exercise 4.1.2
Consider the $n-1$ diagonals of a regular n-gon, inscribed in a circle of radius 1, obtained by connecting one vertex on the n-gon to all its other vertices.

For example, if we are given $n = 6$, we obtain the following diagram.

Show that the product of the lengths of these diagonals is equal to n.

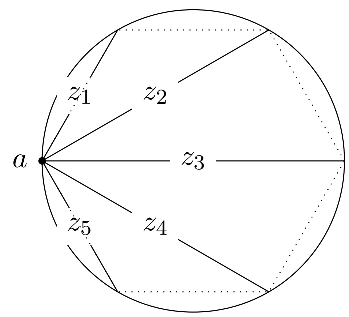


Figure 4.1: $n = 6$, where a is an arbitrary vertex on the hexagon

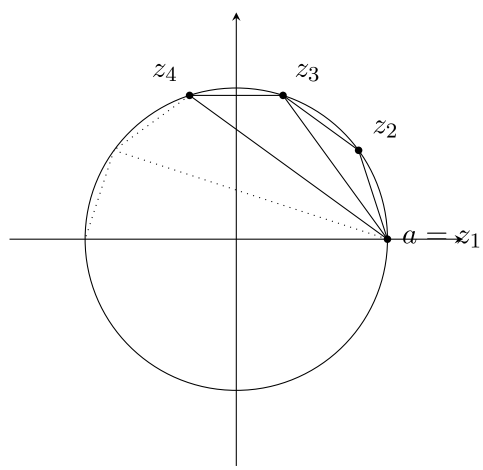


Figure 4.2: A regular n-gon with the roots of unity on its vertices

Proof

Note that Figure 4.1 can be translated into Figure 4.2.

Thus the equation that we wish to prove becomes

$$|1 - z_2|\,|1 - z_3| \ldots |1 - z_n| = n \tag{4.2}$$

Note that $z_2, ..., z_n$ are the nth roots of unity other than 1.

Let z be a variable and consider the polynomial

$$P(z) := 1 + z + z^2 + \ldots + z^{n-1} \tag{4.3}$$

Since the roots of $P(z)$ are the nth roots of unity other than 1, we can factorize Equation (4.3) into

$$P(z) = (z - z_2)(z - z_3) \ldots (z - z_n)$$

Now let $z = 1$ and take the modulus of $P(z)$, and we get Equation (4.2).

Exercise 4.1.3
Let $n \in \mathbb{N}$. Show that $\sum_{j=0}^{n} \binom{3n}{3j} = \frac{2^{3n}+2(-1)^n}{3}$.

Proof
Let $\alpha = e^{i\frac{2\pi}{3}}$. Then α is a cubic root of unity, i.e. $\alpha^3 = 1$, and from Exercise 4.1.1, $1+\alpha+\alpha^2 = 0$.

Consider

$$\begin{aligned}(1+1)^{3n} =& \binom{3n}{0} + \binom{3n}{1} + \binom{3n}{2} + \binom{3n}{3} + \binom{3n}{4} \\ &+ \binom{3n}{5} + \binom{3n}{6} + \ldots + \binom{3n}{3n}\end{aligned} \tag{4.4}$$

$$\begin{aligned}(1+\alpha)^{3n} =& \binom{3n}{0} + \binom{3n}{1}\alpha + \binom{3n}{2}\alpha^2 + \binom{3n}{3} + \binom{3n}{4}\alpha \\ &+ \binom{3n}{5}\alpha^2 + \binom{3n}{6} + \ldots + \binom{3n}{3n}\end{aligned} \tag{4.5}$$

$$\begin{aligned}(1+\alpha^2)^{3n} =& \binom{3n}{0} + \binom{3n}{1}\alpha^2 + \binom{3n}{2}\alpha + \binom{3n}{3} + \binom{3n}{4}\alpha^2 \\ &+ \binom{3n}{5}\alpha + \binom{3n}{6} + \ldots + \binom{3n}{3n}\end{aligned} \tag{4.6}$$

Adding Equation (4.4), Equation (4.5) and Equation (4.6), we observe that the terms with coefficients $\binom{3n}{k}$ where k is not a multiple of 3 sums to 0 as given by $1+\alpha+\alpha^2 = 0$, and therefore we obtain

$$\begin{aligned}
2^{3n} + (1+\alpha)^{3n} + (1+\alpha^2)^{3n} &= 3\sum_{j=0}^{n}\binom{3n}{3j} \\
\frac{1}{3}\left[2^{3n} + (1+\alpha)^{3n} + (1+\alpha^2)^{3n}\right] &= \sum_{j=0}^{n}\binom{3n}{3j} \\
\frac{1}{3}\left[2^{3n} + (-\alpha^2)^{3n} + (-\alpha)^{3n}\right] &= \sum_{j=0}^{n}\binom{3n}{3j} \quad \textit{since } 1+\alpha+\alpha^2 = 0 \\
\frac{1}{3}\left[2^{3n} + (-1)^n + (-1)^n\right] &= \sum_{j=0}^{n}\binom{3n}{3j} \quad \textit{since } \alpha^3 = 1 \\
\frac{2^{3n}+2(-1)^n}{3} &= \sum_{j=0}^{n}\binom{3n}{3j}
\end{aligned}$$

as required.

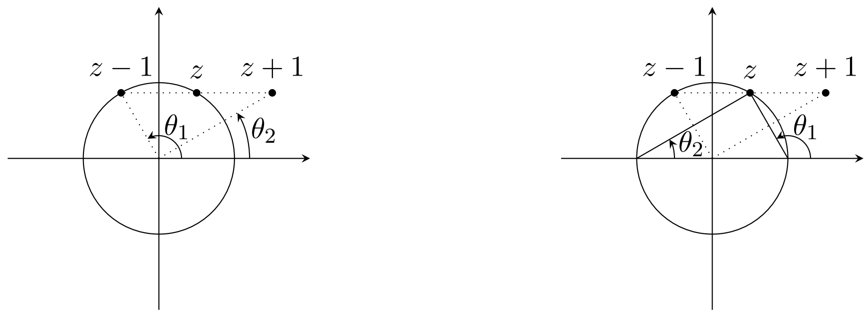


Figure 4.3: (Right) Depicted question, (Left) Translated Angles

Exercise 4.1.4

Note that we can define $\operatorname{Arg} z$ *in any interval of length* 2π*, i.e. it is not necessary that* $\operatorname{Arg} z \in [0, 2\pi)$*.*

For example, if we restrict $\operatorname{Arg} z \in [-\pi, \pi]$*, then we can write*

$$\operatorname{Arg}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right) = -\frac{3\pi}{4}$$

Let z *be on the unit circle and* $\operatorname{Arg} z \in [-\pi, \pi]$*. Suppose that* $z \notin \mathbb{R}$*, i.e.* $z \neq 1, z \neq -1$*. Show that*

$$\operatorname{Arg}\left(\frac{z-1}{z+1}\right) = \begin{cases} \frac{\pi}{2} & \operatorname{Im} z > 0 \\ -\frac{\pi}{2} & \operatorname{Im} z < 0 \end{cases}$$

Proof

Note that $\forall w_1, w_2 \in \mathbb{C}$*, where* $\operatorname{Arg} w_1 = \tau_1, \operatorname{Arg} w_2 = \tau_2$ *for* τ_1, τ_2 *in the same* 2π*-interval,*

$$\operatorname{Arg} \frac{w_1}{w_2} = \frac{e^{i\tau_1}}{e^{i\tau_2}} \equiv e^{i(\tau_1 - \tau_2)} = \operatorname{Arg} w_1 - \operatorname{Arg} w_2$$

in modulo 2π*.*

Suppose $\operatorname{Im} z > 0$*. Let* $\theta_1 = \operatorname{Arg}(z-1)$ *and* $\theta_2 = \operatorname{Arg}(z+1)$*. Consider Figure 4.3. We observe that*

$$\begin{aligned} \frac{\pi}{2} &= \theta_2 + \pi - \theta_1 \\ \theta_1 - \theta_2 &= \frac{\pi}{2} \end{aligned}$$

as desired.

Similarly, we can obtain $\theta_1 - \theta_2 = -\frac{\pi}{2}$ *for when* $\operatorname{Im} z < 0$*. This completes the proof.*

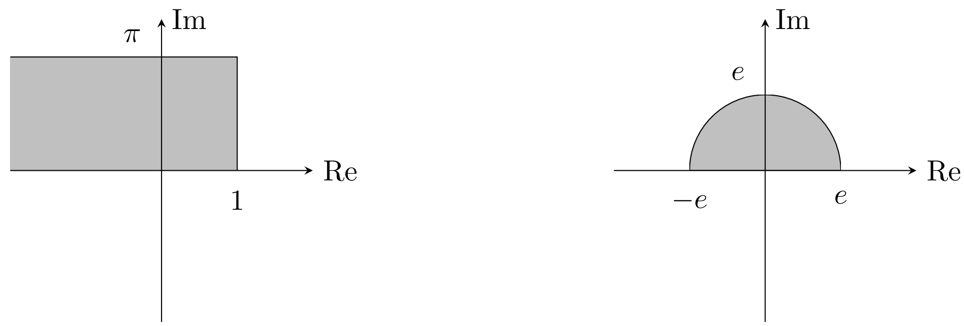


Figure 4.4: (Right) Domain of $f(A)$, (Left) Image of $f(A)$

Exercise 4.1.5
Let $f(z) = e^z$ for $z \in \mathbb{C}$. Let $A = \{z = x + iy \in \mathbb{C} : x \leq 1, y \in [0, \pi]\}$. Describe the image of $f(A)$.

Solution
Firstly, note that

$$\begin{aligned} e^z &= e^{x+iy} \\ e^x &\in (0, e] \\ y &\in [0, \pi] \end{aligned}$$

It is clear that the image will be in on the positive side of the imaginary-axis. Also, since $e^x \in (0, e]$, we get the right graph represented in Figure 4.4. The image of $f(A)$ is described in the left image of Figure 4.4.