Foreword

Usage

- Notes are presented in two columns: main notes on the left, and sidenotes on the right. Main notes will have a larger margin.
- The following is the color code for the notes:

Blue	Definitions
Red	Important points
Yellow	Points to watch out for / comment for incompletion
Green	External definitions, theorems, etc.
Light Blue	Regular highlighting
Brown	Secondary highlighting

- The following is the color code for boxes, that begin and end with a line of the same color:

Blue	Definitions
Red	Warning
Yellow	Notes, remarks, etc.
Brown	Proofs
Magenta	Theorems, Propositions, Lemmas, etc.

- Hyperlinks are underlined in magenta. If your PDF reader supports it, you can follow the links to either be redirected to an external website, or a theorem, definition, etc., in the same document. Note that this is only reliable if you have the full set of notes as a single document, which you can find on:
`https://japorized.github.io/TeX_notes`

11 Lecture 11 May 25th 2018

The following theorem is useful for A2. The proof is not provided in this lecture, but expect the corollary to be restated and proven in a later lecture.

Corollary

Let G be a finite group and $H, K \triangleleft G$, $H - K = \{1\}$ and $|H|\,|K| = |G|$. Then $G \cong H \times K$.

11.1 Normal Subgroup (Continued 2)

11.1.1 Normal Subgroup (Continued)

Note (Recall)

Recall the definition of a normal subgroup as in Definition 23. Let H be a subgroup of G. If $gH = Hg$ for all $g \in G$, then $H \triangleleft G$.

Proposition 27 (Normality Test)

Let H be a subgroup of a group G. The following are equivalent:

1. $H \triangleleft G$
2. $\forall g \in G \;\; gHg^{-1} \subseteq H$
3. $\forall g \in G \;\; gHg^{-1} = H$

Note

*Note that item 3 is indeed a stronger statement that item 2. But since the statements are equivalent, while using the **Normality Test**, if we can show that item 2 is true, item 3 is automatically true.*

Proof

(1) $\implies$ (2):

$$\begin{aligned} x \in gHg^{-1} &\implies \exists h \in H \;\; x = ghg^{-1} \\ &\implies \exists h_1 \in H \;\; gh = h_1 g \qquad \because gh \in gH = Hg \\ &\implies x = ghg^{-1} = h_1 g g^{-1} = h_1 \in H \\ &\implies gHg^{-1} \subseteq H \end{aligned}$$

(2) $\implies$ (3):

$$\begin{aligned} (2) &\implies \forall g \in G \quad gHg^{-1} \subseteq H \\ &\implies \exists g^{-1} \in G \quad g^{-1}Hg \subseteq H \\ &\implies H \subseteq gHg^{-1} \\ &\overset{(2)}{\implies} gHg^{-1} = H \end{aligned}$$

(3) $\implies$ (1):

$$\begin{aligned} (3) &\implies \forall g \in G \quad gHg^{-1} = H \\ &\implies \forall x \in gH \quad xg^{-1} \in gHg^{-1} = H \\ &\implies x \in Hg \qquad \because gg^{-1} = 1 \\ &\implies gH \subseteq Hg \end{aligned}$$

Using a similar argument, we would have $Hg \subseteq Hg$. And so $gH = Hg$ as required. □

Example 11.1.1

Let $G = GL_n(\mathbb{R})$ and $H = SL_n(\mathbb{R})$.[1] *For $A \in G$ and $B \in H$ we have*

$$\det ABA^{-1} = \det A \det B \det A^{-1} = \det A(1)\frac{1}{\det A} = 1.$$

Thus $\forall A \in G$, $ABA^{-1} \in H$. By Proposition 27, $H \triangleleft G$, i.e. $SL_n(\mathbb{R}) \triangleleft GL_n(\mathbb{R})$.[2]

[1] Recall Definition 8 and Definition 11.

[2] **Note**

The normality is true for any field, not just $\mathbb{R}$.

Proposition 28 (Subgroup of Index 2 is Normal)

$$\forall H \text{ subgroup of } G \wedge [G : H] = 2 \implies H \triangleleft G$$

Proof

Let $a \in G$.

$$\begin{aligned} a \in H &\implies aH = H = Ha \\ a \notin H &\implies G = H \cup Ha \implies Ha = G \setminus H \quad \because \text{Proposition } 22 \\ a \notin H &\implies G = H \cup aH \implies aH = G \setminus H \quad \because \text{Proposition } 22 \end{aligned}$$

That implies that $aH = Ha$ for any $a \in G$. Hence, by Proposition 27, $H \triangleleft G$. □

Example 11.1.2

*Let A_n be the **Alternating Group** contained by S_n.[3] By Proposition 28, since $[S_n : A_n] = 2$ because $S_n = A_n \cup O_n$ and O_n is a coset of A_n, we have that*

$$A_n \triangleleft S_n.$$

[3] Recall the definition of alternating group from Theorem 11 and S_n from Definition 4

Example 11.1.3

Let

$$D_{2n} = \{1, a, a^2, ..., a^{n-1}, b, ba, ba^2, ..., ba^{n-1}\}$$

*be the **Dihedral Group** of order $2n$. Since $[D_{2n} : \langle\ a\ \rangle] = 2$,[4] we have that*

$$\langle\ a\ \rangle \triangleleft D_{2n} \quad \because \text{Proposition } 27.$$

[4] The coset of $\langle\ a\ \rangle$ is $b\langle\ a\ \rangle$.

Let H and K be subgroups of a group G. Recall an earlier discussion: $H \cap K$ is the largest subgroup contained in both H and K.

What is the "smallest" subgroup that contains both H and K? Since $H \cap K$ is the largest, it makes sense to think about $H \cup K$. However,

$$H \cup K \text{ is a subgroup of } G \iff H \subseteq K \veebar K \subseteq H$$

While we know that $H \cup K$ can indeed be such a subgroup, the price of the restriction is too high, since it is overly restrictive.

A more "useful" construction turns out to be the **product** of the subgroups.

Definition 24 (Product of Groups)

$$HK := \{hk : h \in H, k \in K\}$$

However, HK is not necessarily a subgroup. For example, for $h_1k_1, h_2k_2 \in HK$, it is not necessary that $h_1k_1h_2k_2 \in HK$, since k_1h_2 is not necessarily equal to h_2k_1.

Lemma 29 (Product of Groups as a Subgroup)
Let H and K be subgroups of G. The following are equivalent:

1. *HK is a subgroup of G*
2. *$HK = KH$* [5]
3. *KH is a subgroup of G*

[5] If one of H or K is normal, then the lemma immediately kicks in.

Proof
It suffices to prove $(1) \iff (2)$, since $(1) \iff (3)$ simply through exchanging H and K.

$(1) \implies (2)$: Let $kh \in KH$ such that $k \in K$ and $h \in H$. Their inverses are $k^{-1} \in K$ and $h^{-1} \in H$, since K and H are groups. Note that

$$kh = (h^{-1}k^{-1})^{-1} \in HK \quad \because HK \text{ is a subgroup of } G.$$

Therefore $kh \in HK$, which implies $KH \subseteq HK$. By a similar argument, we can arrive at $HK \subseteq KH$ and so $HK = KH$.

$(2) \implies (1)$: Note that $1 = 1 \cdot 1 \in HK$.Forallhk $\in HK$, $(hk)^{-1} = k^{-1}h^{-1} \in KH = HK$. For $h_1k_1, h_2k_2 \in HK$, note that $k_1h_2 \in KH = HK$, so there exists $hk \in HK$ such that $k_1h_2 = hk$. Therefore,

$$h_1k_1h_2k_2 = h_1hkk_2 \in HK.$$

By the Subgroup Test, HK is a subgroup of G. $\square$

Proposition 30 (Product of Normal Subgroups is Normal)
Let H and K be subgroups of G.

1. $H \triangleleft G \lor K \triangleleft G \implies HK = KH$ *is a subgroup of* G
2. $H, K \triangleleft G \implies HK = KH \triangleleft G$

Proof
1. *Without loss of generality, suppose* $H \triangleleft G$. *Then*

$$HK = \bigcup_{k \in K} Hk = \bigcup_{k \in K} kH = KH \tag{11.1}$$

By Lemma 29, $HK = KH$ *is a subgroup of* G.

2. *Suppose* $H, K \triangleleft G$. *Then*

$$\forall g \in G \;\; \forall hk \in HK \quad g^{-1}(hk)g = (g^{-1}hg)(g^{-1}kg) \in HK$$

Thus $gHKg^{-1} \subseteq HK$. *Thus by Proposition 27, we have that* $HK \triangleleft G$.

□

Note
Note that Equation (11.1) is a weaker statement than the regular normality that we have defined, since it only requires all elements of K *to work instead of the entire* G.

With that, we define the following notion:

Definition 25 (Normalizer)
Let H *be a subgroup of* G. *The* ***normalizer of H****, denoted by* $N_G(H)$, *is defined to be*

$$N_G(H) := \{g \in G : gH = Hg\}$$

Note

By the above definition, we immediately see that $H \triangleleft G \iff N_G(H) = G$ *by Equation* (11.1)*. Observe that since we only needed* $kH = Hk$ *in Equation* (11.1) *for all* $k \in K$*, we have that* $k \in N_G(H)$*.*

We shall now provide the following corollary which shall be proven in the next lecture.

Corollary

Let H *and* K *be subgroups of a group* G*.*

$$K \subseteq N_G(H) \vee H \subseteq N_G(K) \implies HK = KH \text{ is a subgroup of } G$$