Foreword

Usage

- Notes are presented in two columns: main notes on the left, and sidenotes on the right. Main notes will have a larger margin.
- The following is the color code for the notes:

Blue	Definitions
Red	Important points
Yellow	Points to watch out for / comment for incompletion
Green	External definitions, theorems, etc.
Light Blue	Regular highlighting
Brown	Secondary highlighting

- The following is the color code for boxes, that begin and end with a line of the same color:

Blue	Definitions
Red	Warning
Yellow	Notes, remarks, etc.
Brown	Proofs
Magenta	Theorems, Propositions, Lemmas, etc.

- Hyperlinks are underlined in magenta. If your PDF reader supports it, you can follow the links to either be redirected to an external website, or a theorem, definition, etc., in the same document. Note that this is only reliable if you have the full set of notes as a single document, which you can find on: `https://japorized.github.io/TeX_notes`

16 Lecture 16 Jun 06 2018

16.1 Group Action (Continued)

16.1.1 Group Action (Continued)

Remark
Let G be a group acting on a set X. For $a, b \in G$, and $x, y \in X$, we have that

$$a \cdot x = b \cdot y \iff (b^{-1}a) \cdot x = y.$$

In particular, we have

$$a \cdot x = a \cdot y \iff x = y.$$

For $a \in G$, define $\sigma_a : X \to X$ by $\sigma_a(x) = a \cdot x$ for all $x \in X$. In A3, we will be showing that[1]:

1. $\sigma_a \in S_X$, the permutation group of X; and
2. The function $\Theta : G \to S_X$ given by $\Theta(a) = \sigma_a$ is a group homomorphism with
$$\ker \Theta = \{a \in G : a \cdot x = x,\ x \in X\}.$$

Note that the group homomorphism $\Theta : G \to S_X$ gives an **equivalent definition** of a **Group Action** of G on X. If $X = G$, $|G| = n$ and $\ker \Theta = \{1\}$ [2], then the map $\Theta : G \to S_G \cong S_n$ shows that G is isomorphic to a subgroup of S_n [3], which the equivalent statement of Cayley's Theorem.

[1] This will be added after the assignment.

[2] This is also called a **faithful group action**.

[3]

Exercise 16.1.1
Verify that G is indeed isomorphic to a subgroup of S_n using the given information and the equivalent definition of a group action.

Example 16.1.1

If G is a group, let G act on itself by $a \cdot x = a \cdot x \cdot a^{-1}$, for all $a, x \in G$. Note that the axioms of a group action is satisfied:

1. $1 \cdot x = 1 \cdot x \cdot 1^{-1} = x$; *and*
2. $a \cdot (b \cdot x) = a \cdot (b \cdot x \cdot b^{-1}) \cdot a = ab \cdot x \cdot (ab)^{-1} = (ab) \cdot x$.

In this case, we say that G ***acts on itself by conjugation****.*

Definition 29 (Orbit & Stabilizer)

Let G be a group acting on a set X, and $x \in X$. We denote by

$$G \cdot x = \{g \cdot x : \forall g \in G\}$$

the ***orbit*** *of X and*

$$S(x) = \{g \in G : g \cdot x = x\} \subseteq G$$

the ***stabilizer*** *of X.*

There is no standardized way of expressing the orbit and the stabilizer, i.e. the notation for orbit and stabilizers will be different across many references.

Proposition 45

Let G be a group acting on a set X an $x \in X$. Let $G \cdot x$ and $S(x)$ be the orbit and stabilizer of X respectively. Then

1. $S(x) \leq G$
2. *there is a bijection from $G \cdot x$ to $\{gS(x) : g \in G\}$ and thus $|G \cdot x| = [G : S(x)]$.*

Proof

1. *Since $1 \cdot x = x$, we have $1 \in S(x)$. If $g, h \in S(x)$, then*

$$gh \cdot x = g \cdot (h \cdot x) = g \cdot x = x$$

i.e. $S(x)$ is closed under "composition of group action". Also note that

$$g^{-1} \cdot x = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = 1 \cdot x = 1.$$

Thus the inverse of each element is also in $S(x)$. Therefore, by the ***Subgroup Test****, $S(x) \leq G$.*

2. *For the sake of simplicity, let us write $S = S(x)$. Consider the map*

$$\phi : G \cdot x \to \{gS(x) : g \in G\}$$

defined by $\phi(g \cdot x) = gS$ [4]*. To verify that the map is well-defined, note that*

$$\begin{aligned} g \cdot x = h \cdot x &\iff (h^{-1}g) \cdot x = x = 1 \cdot x \\ &\iff \phi(h^{-1}g \cdot x) = \phi(1 \cdot x) \\ &\iff h^{-1}gS = 1 \cdot S = S \\ &\iff gS = hS \end{aligned}$$

We also observe that ϕ is injective. It is also clear that ϕ is onto, and therefore we have that ϕ is a bijection. It follows that

$$|G \cdot x| = |\{gS : g \in G\}| = [G : S]$$

□

[4] We go with the most simplistic and rather naive kind of function here.

Theorem 46 (Orbit Decomposition Theorem)

Let G be a group acting on a non-empty finite set X. Let

$$X_f = \{x \in X : a \cdot x = x, \forall a \in G\}$$

(Note that $x \in X_f \iff |G \cdot x| = 1$)[5]

Let $G \cdot x_1$, $G \cdot x_2$, ..., $G \cdot x_n$ denote the distinct nonsingleton orbits (i.e. $|G \cdot x_i| > 1$ for all $1 \leq i \leq n$). Then

$$|X| = \left|X_f\right| + \sum_{i=1}^{n} [G : S(x_i)].$$

[5] Notice that

$$\begin{aligned} x \in X_f &\iff \forall a \in G \;\; a \cdot x = x \\ &\iff \forall g \cdot x \in G \cdot x \;\; g \cdot x = x \\ &\iff |G \cdot x| = 1 \end{aligned}$$

Proof

Note that for $a, b \in G$ and $x, y \in X$,

$$\begin{aligned} a \cdot x = b \cdot y &\overset{WLOG}{\iff} (b^{-1}a) \cdot x = y \\ &\iff y \in G \cdot x \\ &\overset{(1)}{\iff} G \cdot x = G \cdot y \end{aligned}$$

where (1) *is the conclusion after consider the other case where* $(a^{-1}b) \cdot y = x$.

Thus, we see that the two orbits are either disjoint or the same, but not both. It follows that the orbits form a disjoint union of X*. Since* $x \in X_f \iff |G \cdot x| = 1$*, the set* $X \setminus X_f$ *contains all nonsingleton orbits, which are disjoint. It follows that*

$$|X| = \left| X_f \right| + \sum_{i=1}^{n} |G \cdot x_i| \overset{(2)}{=} \left| X_f \right| + \sum_{i=1}^{n} [G : S(x_i)]$$

where (2) *is by Proposition 45.* □