Foreword

Usage

- Notes are presented in two columns: main notes on the left, and sidenotes on the right. Main notes will have a larger margin.
- The following is the color code for the notes:

Blue	Definitions
Red	Important points
Yellow	Points to watch out for / comment for incompletion
Green	External definitions, theorems, etc.
Light Blue	Regular highlighting
Brown	Secondary highlighting

- The following is the color code for boxes, that begin and end with a line of the same color:

Blue	Definitions
Red	Warning
Yellow	Notes, remarks, etc.
Brown	Proofs
Magenta	Theorems, Propositions, Lemmas, etc.

- Hyperlinks are underlined in magenta. If your PDF reader supports it, you can follow the links to either be redirected to an external website, or a theorem, definition, etc., in the same document. Note that this is only reliable if you have the full set of notes as a single document, which you can find on: `https://japorized.github.io/TeX_notes`

2 Lecture 2 May 04th 2018

2.1 Introduction (Continued)

2.1.1 Permutations

Definition 1 (Injectivity)
Let $f : X \to Y$ be a function. We say that f is ***injective*** *(or* ***one-to-one****) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.*

Definition 2 (Surjectivity)
Let $f : X \to Y$ be a function. We say that f is ***surjective*** *(or* ***onto****) if $\forall y \in Y \;\; \exists x \in X \;\; f(x) = y$.*

Definition 3 (Bijectivity)
Let $f : X \to Y$ be a function. We say that f is ***bijective*** *if it is both* ***injective*** and ***surjective***.

Definition 4 (Permutations)
Given a non-empty set L, a permutation of L is a bijection from L to L. The set of all permutations of L is denoted by S_L.

Example 2.1.1
Consider the set $L = \{1, 2, 3\}$, which has the following 6 different permuta-

tions:

$$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}\quad\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}\quad\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}\quad\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}\quad\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$$

Note

$$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$$

indicates the bijection $\sigma : \{1,2,3\} \to \{1,2,3\}$ *with* $\sigma(1) = 1$, $\sigma(2) = 3$ *and* $\sigma(3) = 2$.

For $n \in \mathbb{N}$, we denote $S_n := S_{\{1,2,\ldots,n\}}$, the set of all permutations of $\{1,2,\ldots,n\}$. Example 2.1.1 shows the elements of the set S_3.

Definition 5 (Order)

The ***order*** *of a set* A*, denoted by* $|A|$*, is the cardinality of the set.*

Example 2.1.2

We have seen that the order of S_3, $|S_3|$ *is* $6 = 3!$.

Proposition 1

$|S_n| = n!$

Proof

$\forall \sigma \in S_n$*, there are* n *choices for* $\sigma(1)$, $n-1$ *choices for* $\sigma(2)$, ..., 2 *choices for* $\sigma(n-1)$*, and finally* 1 *choice for* $\sigma(n)$. $\square$

Do elements of S_n *share the same properties as what we've seen in the numbers?* Given $\sigma, \tau \in S_n$, we can **compose** the 2 together to get a third element in S_n, namely $\sigma\tau$ (wlog), where $\sigma\tau : \{1,\ldots,n\} \to \{1,\ldots,n\}$ is given by $\forall x \in \{1,\ldots,n\}$, $x \mapsto \sigma(\tau(x))$.

It is important to note that $\because$ σ, τ are **both bijective**, $\sigma\tau$ is also bijective. Thus, together with the fact that $\sigma\tau : \{1,\ldots,n\} \to \{1,\ldots,n\}$, we have that $\sigma\tau \in S_n$ by definition of S_n.

$\therefore \forall \sigma, \tau \in S_n$, $\sigma\tau, \tau\sigma \in S_n$, but $\sigma\tau \neq \tau\sigma$ in general. The following is an example of the stated case:

Example 2.1.3
Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \text{ and } \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}.$$

Compute $\sigma\tau$ and $\tau\sigma$ to show that they are not equal.

Solution

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \text{ but } \tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

Perhaps what is interesting is the question of: **when does commutativity occur?** One such case is when σ and τ have support sets that are disjoint[1].

[1] This is proven in A1

On the other hand, the associative property holds[2], i.e.

$$\forall \sigma, \tau, \mu \in S_n \;\; \sigma(\tau\mu) = (\sigma\tau)\mu$$

[2]

Exercise 2.1.1
Prove this as an exercise.

The set S_n also has an identity element[3], namely

$$\varepsilon = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}$$

[3]

Exercise 2.1.2
Verify that the given identity element is indeed the identity, i.e.

$$\forall \sigma \in S_n \;\; \sigma\varepsilon = \sigma = \varepsilon\sigma.$$

Finally, $\forall \sigma \in S_n$, since σ is a bijection, we have that its inverse function, σ^-1 is also a bijection, and thus satisfies the requirements to be in S_n. We call $\sigma^{-1} \in S_n$ to be the **inverse permutation** of σ, such that

$$\forall x, y \in \{1, ..., n\} \quad \sigma^{-1}(x) = y \iff \sigma(y) = x.$$

It follows, immediately, that

$$\sigma\big(\sigma^{-1}(x)\big) = x \wedge \sigma^{-1}\big(\sigma(y)\big) = y.$$

$\therefore$ We have that

$$\sigma\sigma^{-1} = \varepsilon = \sigma^{-1}\sigma.$$

Example 2.1.4
Find the inverse of

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$$

Solution
By rearranging the image in ascending order, using them now as the object

and their respective objects as their image, construct

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}.$$

It can easily (although perhaps not so prettily) be shown that

$$\sigma\tau = \varepsilon = \tau\sigma.$$

With all the above, we have for ourselves the following proposition:

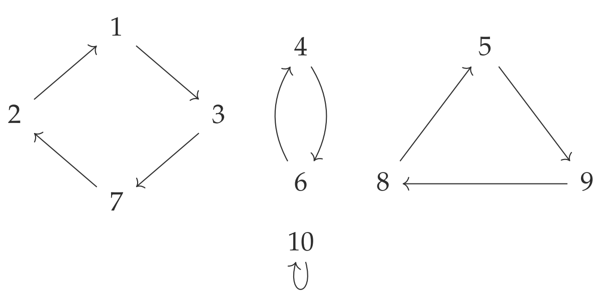
Proposition 2 (Properties of S_n)
We have

1. $\forall \sigma, \tau \in S_n \;\; \sigma\tau, \tau\sigma \in S_n$.
2. $\forall \sigma, \tau, \mu \in S_n \;\; \sigma(\tau\mu) = (\sigma\tau)\mu$.
3. $\exists \varepsilon \in S_n \;\; \forall \sigma \in S_n \;\; \sigma\varepsilon = \sigma = \varepsilon\sigma$.
4. $\forall \sigma \in S_n \;\; \exists ! \sigma^{-1} \in S_n \;\; \sigma\sigma^{-1} = \varepsilon = \sigma^{-1}\sigma$.

Consider

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 7 & 6 & 9 & 4 & 2 & 5 & 8 & 10 \end{pmatrix} \in S_{10}$$

If we represent the action of σ geometrically, we get



We observe that σ can be **decomposed** into one 4-cycle, $\begin{pmatrix} 1 & 3 & 7 & 2 \end{pmatrix}$, one 2-cycle, $\begin{pmatrix} 4 & 6 \end{pmatrix}$, one 3-cycle, $\begin{pmatrix} 5 & 9 & 8 \end{pmatrix}$, and one 1-cycle, $\begin{pmatrix} 10 \end{pmatrix}$.

Note that these cycles are (pairwise) **disjoint**, and we can write[4]

[4] We generally do not include the 1-cycle and assume that by excluding them, it is known that any number that is supposed to appear loops back to themselves.

$$\sigma = \begin{pmatrix} 1 & 3 & 7 & 2 \end{pmatrix} \begin{pmatrix} 4 & 6 \end{pmatrix} \begin{pmatrix} 5 & 9 & 8 \end{pmatrix}$$

Note that we may also write

$$\begin{aligned} \sigma &= \begin{pmatrix} 4 & 6 \end{pmatrix} \begin{pmatrix} 5 & 9 & 8 \end{pmatrix} \begin{pmatrix} 1 & 3 & 7 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 4 \end{pmatrix} \begin{pmatrix} 9 & 8 & 5 \end{pmatrix} \begin{pmatrix} 7 & 2 & 1 & 3 \end{pmatrix} \end{aligned}$$

It is interesting to note that the cycles can rotate their "elements" in a **cyclic** manner, i.e.

$$\begin{pmatrix} 1 & 3 & 7 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 2 & 1 & 3 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 7 & 3 \end{pmatrix}.$$

Although the decomposition of the cycle notation is not unique (i.e. you may rearrange them), each individual cycle is unique, and is proven below[5].

[5] See bonus question of A1. Proof will be included in the notes once the assignment is over.

Theorem 3 (Cycle Decomposition Theorem)
If $\sigma \in S_n$, $\sigma \neq \varepsilon$, then σ is a product of (one or more) disjoint cycles of length at least 2. This factorization is unique up to the order of the factors.

Note (Convention)
Every permutation in S_n can be regarded as a permutation of S_{n+1} by fixing the permutation of $n+1$. Therefore, we have that

$$S_1 \subseteq S_2 \subseteq \ldots \subseteq S_n \subseteq S_{n+1} \subseteq \ldots$$