1 Complex Numbers and Their Properties

Complex Plane as a Set

$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$

Real and Imaginary Part

$\forall z = x + iy \in \mathbb{C}\ x, y \in \mathbb{R}$
$\mathrm{Re}(z) = x\ \ \mathrm{Im}(z) = y$

Product

$\forall z = a + ib, w = c + id \in \mathbb{C}\ a, b, c, d \in \mathbb{R}$
$zw = (ac - bd) + i(ad + bc)$

Inverse of a Complex Number

$\forall z = a + ib \in \mathbb{C}\ a, b \in \mathbb{R}$
$\exists z^{-1} = \frac{a}{aa^2 + b^2} - i\frac{b}{a^2 + b^2} \in \mathbb{C}$

Conjugate

$\forall z = a + ib \in \mathbb{C}\ a, b \in \mathbb{R}$
$\exists \bar{z} = a - ib \in \mathbb{C}$

Modulus

$\forall z = x = iy \in \mathbb{C}\ x, y \in \mathbb{R}$
$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}$

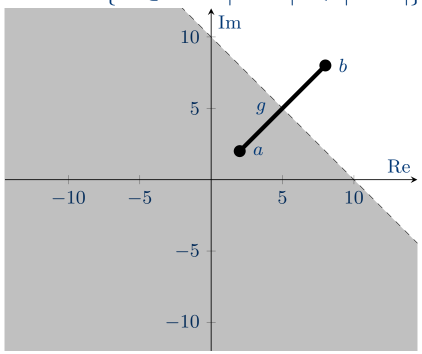
Basic Inequalities

$\forall z, w \in \mathbb{C}$,

1. $|\mathrm{Re}(z)| \leq |z|$
2. $|\mathrm{Im}(z)| \leq |z|$
3. $|z + w| \leq |z| + |w|$
4. $|z + w| \geq \big|\, |z| + |w| \,\big|$

Region of a set of Complex Numbers

Describe $\{z \in \mathbb{C} : |z - a| < |z - b|\}$.



Every complex number has exactly 2 roots

$\forall z = x = iy \in \mathbb{C}\ x, y \in \mathbb{R}$
$\exists w_{1,2} = u + iv \in \mathbb{C}\ u, v \in \mathbb{R}$

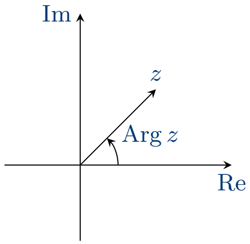
$$w = \begin{cases} \pm\left[\left(\frac{x + \sqrt{x^2 + y^2}}{2}\right)^{\frac{1}{2}} + i\left(\frac{-x + \sqrt{x^2 + y^2}}{2}\right)^{\frac{1}{2}}\right] & y > 0 \\ \pm\left[\left(\frac{x + \sqrt{x^2 + y^2}}{2}\right)^{\frac{1}{2}} - i\left(\frac{-x + \sqrt{x^2 + y^2}}{2}\right)^{\frac{1}{2}}\right] & y < 0 \\ \pm\sqrt{x} & y = 0, x > 0 \\ \pm i\sqrt{x} & y = 0, x < 0 \end{cases}$$

Quadratic Formula

$\forall a, b, c \in \mathbb{C}\ a \neq 0\ az^2 + bz + c = 0$

$$z = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Argument



Polar Form

$\forall z \in \mathbb{C}\ \exists r, \theta \in \mathbb{R}\ \theta \in [0, 2\pi)$
$z = re^{i\theta}$

Polar to Cartesian

$x = r\cos\theta \quad y = r\sin\theta$

Cartesian to Polar

$r = |z| \quad \tan\theta = \frac{x}{y}$

Conjugate in Polar Form

$z = re^{i\theta} \iff \bar{z} = re^{-i\theta}$

Inverse in Polar Form

$z = re^{i\theta} \wedge z \neq 0$
$\implies z^{-1} = \frac{1}{r}e^{-i\theta}$

Product in Polar Form

- $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$
- $\forall n \in \mathbb{Z}\ (re^{in}) = r^n e^{in\theta}$

nth Roots of a Complex Number

$\left\{ r^{\frac{1}{n}} e^{i\left(\frac{\theta + 2\pi k}{n}\right)} : k = 0, 1, ..., n - 1 \right\}$

nth Roots of Unity

$\left\{ e^{i\left(\frac{2\pi k}{n}\right)} : k = 0, 1, ..., n - 1 \right\}$

2 Complex Functions

2.1 Convergence

$\forall \{z_n\}_{n \in \mathbb{N}} \subseteq \mathbb{C} \ \wedge\ z \in \mathbb{Z}$
$(n \to \infty \implies z_n \to z) \iff \lim_{n \to \infty} |z_n - z| = 0$
May also write as $\lim_{n \to \infty} z_n = z$

2.2 Convergence for Complex Functions

$\forall \Omega \subseteq \mathbb{C}\ \forall f : \Omega \to \mathbb{C}\ z_0 \in \mathbb{C}\ \exists L \in \mathbb{C}\ \forall \{z_n\}_{n \in \mathbb{N}} \subseteq \Omega \setminus \{z_0\}$
$(z_n \to z_0 \implies f(z_0) \to L) \implies \lim_{z \to z_0} f(z) = L$

2.3 Continuity

$\forall f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$
f is continuous on $z_0 \implies$

1. $\forall \{z_0\}_{n \in \mathbb{N}}\ z_n \to z_0 \implies f(z_n) \to f(z_0)$
2. $\forall z \in \Omega\ \forall \epsilon > 0\ \exists \delta > 0\ |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$

2.4 Real and Imaginary Parts of a Function

$f(z) = u(x, y) + iv(x, y)$

3 Differentiation

3.1 Neighbourhood

$\forall z_0 \in \mathbb{C}\ r \in \mathbb{R}\ D(z_0, r) := \{z in\mathbb{C} : |z - z_0| < r\}$ is the neighbourhood of radius r around z_0.

3.2 Differentiation/Holomorphic

Let $z_0 \in \mathbb{C}\ r \in \mathbb{R}\ \exists D(z_0, r) \subseteq \mathbb{R}$. $\forall f : D(z_0, r) \to \mathbb{C}\ \forall h \in \mathbb{C}$
$\exists \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \implies f$ is differentiable/holomorphic $\wedge\ f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$

3.3 Properties of Holomorphic Functions

f, g are holomorphic at $z \in \mathbb{C} \implies$

1. $(f + g)' = f' + g'$
2. $(fg)' = f'g + fg'$
3. $(g \neq 0 \implies (\frac{f}{g})' = \frac{f'g - fg'}{g^2})$

3.4 Cauchy-Riemann Equations

$\forall z_0 = x_0 + iy_0 \in \mathbb{C}\ x_o, y_o \in \mathbb{R}$. $f(z)$ is holomorphic at $z_0 \implies$ at (x_0, y_0)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

3.5 Conditional Converse of CRE

Let $z_0 = z_0 + iy_0 \in \nleqslant \subseteq \mathbb{R}\ x_0, y_0 \in \mathbb{R}\ u, v : \mathbb{R}^{\nleqslant} \to \mathbb{R}\ f = u + iv : \Omega \to \mathbb{C}$.

1. partials of u, v exist in nbd of $x_0, y_0)$
2. partials of u, v are cont' at (x_0, y_0)
3. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\implies f$ is holo at z_0.

3.6 Power Series

Infinite series of the form $\sum_{n \in \mathbb{N}} c_n z^n$

3.7 Convergence for Power Series

We will usually aim for absolute convergence, for

$$\left| \sum_{n=0}^{N} c_n z^n \right| \leq \sum_{n=0}^{N} |c_n|\, |z|^n$$

3.8 Hadamard's Formula

$\frac{1}{R} := \limsup_{n \to \infty} |c_n|^{\frac{1}{n}}$.

3.9 Limit Supremum

$$\limsup_{n \to \infty} a_n := \lim_{n \to \infty} \sup_{m \geq n} a_m$$

3.10 limsup Property

$\forall \{a_n\}_{n \in \mathbb{N}} L := \limsup_{n \to \infty} a_n \implies \forall \epsilon > 0\ \exists N > 0\ \forall n > N$
$|a_0 - L| < \epsilon$

3.11 Radius of Convergence

$\forall \sum_{n \in \mathbb{N}} c_n z^n\ \exists 0 \leq R < \infty$

1. $|z| < R \implies$ absolute convergence
2. $|z| > R \implies$ divergence

3.12 Power Function and its Holomorphic Function share the same Region of Convergence

$f(z) = \sum_{n \in \mathbb{N}} c_n z^n$ had a rad of conv $R \in \mathbb{R} \implies \forall \{z : |z| < R\}$

$$f'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1}$$

rad of conv of f' is R.

3.13 Entire Function

f is said to be entire if f is holomorphic in the entire $\mathbb{C}$.

4 Integration

4.1 Curves

A curve in $\mathbb{C}$ is a cont' fn $\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{C}$. Image of γ is called γ^*.

4.2 Equivalent Parameterization

Let $\gamma_1 : [a, b] \subseteq \mathbb{R} \to \mathbb{C}\ \gamma_2 : [c, d] \subseteq \mathbb{R} \to \mathbb{C}$ desc path γ^*. γ_1, γ_2 are equiv if $\exists h : [a, b] \to [c, d]$, bijective and cont', s.t. $\forall t \in \mathrm{Dom}(h)\ \gamma_1(t) = \gamma_2(h(t))$.

4.3 Smooth Curve

γ is smooth if $\exists \gamma'$ is cont' on $\mathrm{Dom}(\gamma) \wedge \forall t \in \mathrm{Dom}(\gamma)\ \gamma'(t) \neq 0$.

4.4 Piecewise Smooth Curve

γ is piecewise smooth if γ is smooth on $\mathrm{Dom}(\gamma)$ except on finitely many pts.

4.5 Integral over path

Let $\gamma : [a, b] \to \mathbb{C} \wedge f : \mathbb{C} \to \mathbb{C}$ con' on γ. Integral f along γ is

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Integral over a curve γ^* is independent of the path chosen.

4.6 Splitting Integral into Real and Imaginary Parts

$$\int_a^b g(t)dt = \int_a^b \mathrm{Re}(g(t))dt + i\int_a^b \mathrm{Im}(g(t))dt$$

4.7 Integral Properties

1. (Linearity) $\int_\gamma (\alpha f + \beta g) = \alpha \int_\gamma f + \beta \int_\gamma g$
2. (a) $\left| \int_a^b g \right| \leq \int_a^b |g|$
   (b) $\left| \int_\gamma f dz \right| \leq \sup_{z \in \Omega} f(z) \cdot \int_a^b |\gamma'(t)|\, dt$
3. γ^- is in opposite orientation of $\gamma \implies \int_{\gamma^-} f = -\int_\gamma f$

4.8 Fundamental Theorem of Calculus

Let $(\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{C}) \in \Omega \subseteq \mathbb{C}$. f cont' on γ $\exists F' = f$ holo on $\Omega \implies \int_\gamma f = F(\gamma(b)) - F(\gamma(a))$