

# PMATH351 - Real Analysis (Class Notes)

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# Chapter 1

## Lecture 1: Sep 8, 2017

### 1.1 Logistics

Course Website: <http://www.math.uwaterloo.ca/~nspronk/math351/math351.html>

### 1.2 Brief Introduction to the Course

#### 1.2.1 Set Theory (Naive, for Real Analysis)

Sets whose existence that we shall take for granted:

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

$$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}, \gcd(m, n) = 1\}$$

#### **Definition 1.2.1 (Inclusion)**

*Given two sets  $A$ ,  $B$ , write*

$$A \subseteq B, \quad A \subset B \text{ or } B \supseteq A, \quad \text{etc.} \tag{1.1}$$

*for “ $B$  contains  $A$ ”, i.e.  $\forall x \in A \implies x \in B$ . We shall write*

$$A \subsetneq B \text{ if } A \subset B \wedge A \neq B \tag{1.2}$$



**Definition 1.2.2 (Power Set)**

Let  $X$  be a set. Let

$$\mathcal{P}(X) := \{A : A \subseteq X\} \quad (1.3)$$

Note that if  $X = \{1, \dots, n\}$ , notice that  $\mathcal{P}(X)$  has  $2^n$  elements.

**Definition 1.2.3 (Unions and Intersections)**

Let  $A, B \in \mathcal{P}(X)$  where  $X$  is the universe, and  $\{A_i\}_{i \in I} \subseteq \mathcal{P}(X)$  where  $I \neq \emptyset$ .

$$\begin{aligned} A \cup B &= \{x \in X : x \in A \vee x \in B\} & \bigcup_{i \in I} A &= \{x \in X : x \in A \text{ for some } u \in I\} \\ A \cap B &= \{x \in X : x \in A \wedge x \in B\} & \bigcap_{i \in I} A &= \{x \in X : x \in A \forall i \in I\} \end{aligned}$$

If we do not have  $A, B$  in a common universe, we let the "external union" be

$$A \sqcup B = \{x : x \in A \vee x \in B\} \quad (1.4)$$

**Example 1.2.1**

Suppose  $I \neq \emptyset$ . What is the meaning of

$$\bigcup_{i \in I} A_i, \quad \bigcap_{i \in I} A_i? \quad (1.5)$$

**Definition 1.2.4 (Difference Set)**

If  $A, B \in \mathcal{P}(X)$ . Let

$$A \setminus B = \{x \in X : x \in A \wedge x \notin B\} \quad (1.6)$$

In particular

$$X \setminus B = \{x \in X : x \notin B\} \text{ (complement)} \quad (1.7)$$

**Proposition 1.2.1 (De Morgan's Laws)**

If  $X$  is a set, with  $\{A_i\} \in \mathcal{P}(X)$ , then

$$X \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in I} (X \setminus A_i), \quad X \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (X \setminus A_i) \quad (1.8)$$

The proof is straightforward and should be done in two lines.

**Definition 1.2.5 (Product Sets)**

Let  $A, B$  be sets.

$$A \times B = \{(a, b) : a \in A, b \in B\} \quad (\text{ordered pairs}) \quad (1.9)$$

**Definition 1.2.6 (Function)**

$f \subseteq A \times B$  is called a function if

$$\forall a \in A \quad \exists! b = f(a) \in B \quad (1.10)$$

so that  $(a, b) \in f$ .

In practice, we write  $f : A \rightarrow B$  and the ordered pairs are all denoted  $(a, f(a))$ .

If  $X_1, \dots, X_n$  are sets, where  $n \in \mathbb{N}$ , then

$$X_1 \times \dots \times X_n = \prod_{j=1}^n X_j = \{(x_1, \dots, x_n) : x_j \in X_j \forall j \in \{1, \dots, n\}\} \quad (1.11)$$

is called the  $n$ -tuples of  $X$ .

If  $\{X_i\}_{i \in I, I \neq \emptyset}$ , is a (or an infinite) family of sets

$$\prod_{i \in I} X_i = \{(x_i)_{i \in I} : x_i \in X_i \forall i \in I\} \quad (1.12)$$

**Axiom 1.2.1 (Axiom of Choice)**

Given any non empty collection of nonempty sets  $\{A_i\}_{i \in I}$ , we have  $\prod_{i \in I} A_i \neq \emptyset$ .

**Remark (B. Russell)**

1.  $\forall n \in \mathbb{N}$ , let  $S_n = \{l_n, r_n\}$  be a pair of shoes. Surely,  $\prod_{i \in I} S_n \neq \emptyset$ .

2.  $\forall n \in \mathbb{N}$ , let  $T_n = \{s_n, s'_n\}$  be a pair of socks. Why do we expect  $\prod_{i \in I} T_n \neq \emptyset$ ?

**Proposition 1.2.2 (AC')**

The AC is equivalent to  $(AC')$  given any nonempty set  $A$ ,

$$\exists f : \mathcal{P}(A) \setminus \{\emptyset\} \rightarrow A \quad \forall B \in \mathcal{P}(A) \setminus \{\emptyset\} \quad f(B) \in B \quad (1.13)$$

**Proof**

$$(AC) \implies (AC')$$

We assume there is

$$(x_B)_{B \in \mathcal{P}(A) \setminus \{\emptyset\}} \in \prod_{B \in \mathcal{P}(A) \setminus \{\emptyset\}} B \quad (1.14)$$

(which is nonempty by assumption).

Then we simply have to let  $f(B) = x_B$  for each  $B$ .

$(AC') \implies (AC)$

Given a non-empty collection of nonempty sets  $\{A_i\}_{i \in I}$ , let

$$A = \bigsqcup_{i \in I} A_i \quad (\text{external product}) \quad (1.15)$$

We have a choice function  $f : \mathcal{P}(A) \setminus \{\emptyset\} \rightarrow A$ ,  $f(B) \in B$  for each  $B$ . Then

$$(f(A_i))_{i \in I} \in \prod_{i \in I} A_i \quad (1.16)$$

### 1.3 Relations, Ordering and Zorn

#### Definition 1.3.1 (Relation)

Let  $X$  be a nonempty set. A relation on  $X$  is any subset

$$R \subseteq X \times X \quad (1.17)$$

We write  $xRy$  provided that  $(x, y) \in R$ .

#### Example 1.3.1

1. A function  $f \subseteq X \times X$  is a relation.
2. In  $\mathbb{N} \times \mathbb{N}$ , consider

$$mRn \iff \exists p \in \{0\} \cup \mathbb{N} \quad n = m + p \quad (1.18)$$

We write  $m \leq n \iff mRn$ .

3. On  $\mathbb{Z}$ ,  $m \leq n \iff n - m \in \{0\} \cup \mathbb{N}$ .
4. On  $\mathbb{Q}$ ,  $\frac{m}{n} \leq \frac{\mu}{\nu} \iff m\nu \leq \mu n$  in  $(\mathbb{Z}, \leq)$ .
5. On  $\mathcal{P}(X)$ , we have relations

$$A \subseteq B$$

$$A \supseteq B$$

# Chapter 2

## Lecture 2: Sep 11, 2017

### 2.1 More on Relations

#### Definition 2.1.1 (More on Relations)

A relation  $R$  on  $X$  is

1. **Symmetric** if  $xRy \implies yRx$ .
2. **Reflexive** if  $\forall x \in X \ xRx$
3. **Transitive** if  $xRy \wedge yRz \implies xRz$
4. **Anti-Symmetric** if  $xRy \wedge yRx \implies x = y \in X$

(i), (ii) and (iii) makes up the **Equivalence Relation**. We usually use notations like  $\sim, \approx$ .

(ii), (iii) and (iv) makes up the **Partial Order** definition. We usually use notations like  $\leq, \geq$

In Example 1.3.1, (ii), (iii), (iv) and (v) are all partial orders. In (i),  $f$  is an equivalence relation only if  $f$  is an identity function.

#### Definition 2.1.2 (Total Order)

A total order is a partial order where for  $x, y$  we have at least one of

$$x \leq y \quad \text{or} \quad y \leq x \tag{2.1}$$

holds.

Notice that in Example 1.3.1, (ii), (iii) and (iv) are total orders. However, (v) is not if  $X$  has at least two elements.

If  $\sim$  is an equivalence relation on  $X$ , then we denote the equivalence class by  $[x] = \{y \in X : y \sim x\}$

**Example 2.1.1**

On  $\mathbb{Z} \times \mathbb{N}$ , let  $(m, n) \sim (\mu, v)$  if  $m\nu = \mu n$  in  $\mathbb{Z}$ . Then equivalence classes  $[(m, n)]$  are elements of  $\mathbb{Q}$ . Generally,

$$\frac{m}{n} = [(m, n)] \quad (2.2)$$

## 2.2 Construction of the Real Numbers

We provide a sketch of Cantor's construction:

**Notation:** On  $\mathbb{Q}$ , define  $|\frac{m}{n}| = \begin{cases} \frac{m}{n} & m > 0 \\ -\frac{m}{n} & m < 0 \end{cases}, n \in \mathbb{Z}$

We have the usual properties (triangle inequalities): for  $p, q \in \mathbb{Q}$

$$|p + q| \leq |p| + |q| \quad (2.3)$$

$$||p| - |q|| \leq |p - q| \quad (2.4)$$

Let  $\mathbb{Q}_+ = \{q \in \mathbb{Q} : q > 0\}$

$$X = \{(q_n) = (q_n)_{n=1}^\infty \in \mathbb{Q}^\mathbb{N} : \forall \epsilon \in \mathbb{Q}_+ \exists n_\epsilon \in \mathbb{N} \forall n, m \geq n_\epsilon |q_n - q_m| < \epsilon\}$$

( $X$  is set of Cauchy sequences of rationals)

On  $X$  we define

$$(q_n) \sim (r_n) \text{ if } \forall \epsilon \in \mathbb{Q} \exists n_\epsilon \in \mathbb{N} |q_n - r_n| < \epsilon \text{ whenever } n \geq n_\epsilon \quad (2.5)$$

(tails become closer together)

Then  $\sim$  is an equivalence relation (verify yourselves).

We let

$$\mathbb{R} = \{[(q_n)] : (q_n) \in X\} \quad (2.6)$$

**Note**

$\mathbb{R}$  is a field.

$$(q_n) \sim (s_n), (r_n) \sim (t_n) \implies (q_n + r_n) \sim (s_n + t_n), (q_n r_n) \sim (s_n t_n) \quad (2.7)$$

(Check! To check for multiplication, observe that elements of  $X$  form bounded sets in  $\mathbb{Q}$ ).

$(r_n) \not\sim (0, 0, \dots) \implies r_n = 0$  for at most finitely many  $n$

$\implies$  define

$$t_n = \begin{cases} 1 & \text{if } r_n = 0 \\ \frac{1}{r_n} & \text{otherwise} \end{cases}$$

$$\implies (r_n)(t_n) \sim (1, 1, 1, \dots)$$

We can define multiplication, addition, etc. on  $\mathbb{R}$  and it follows that  $\mathbb{R}$  is a field.

**Note (Properties)**

1.  $\mathbb{Q}$  is a subfield:

$$\mathbb{Q} \hookrightarrow \mathbb{R}, \quad q \mapsto [(q, q, \dots)] \quad (2.8)$$

(eq. class of const. seq.)

2. Total order: On  $X$  let  $(q_n) \leq (r_n)$  if

$$\forall \epsilon \in \mathbb{Q}_+ \exists n_\epsilon \in \mathbb{N} \forall n \geq n_\epsilon \quad q_n \leq r_n + \epsilon \quad (2.9)$$

(Eq.  $(1 - \frac{1}{n}) \leq (1, 1, \dots)$ )

Then  $(q_n) \leq (r_n), (q_n) \sim (s_n), (r_n) \sim (t_n) \implies (s_n) \leq (t_n)$  (check)

Hence, let

$[(q_n)] \leq [(r_n)]$  if  $(q_n) \leq (r_n)$ .

3. Density of  $\mathbb{Q}$ : (HW 1)

If  $[(q_n)] < [(r_n)]$  then there is  $q$  in  $\mathbb{Q}$  s.t.

$$[(q_n)] < [(q, q, \dots)] < [(r_n)] \quad (2.10)$$

4. Absolute value:  $|[(q_n)]| = [|q_n|]$

This is the usual absolute value (check)

## 2.3 Dyadic representation of $\mathbb{R}$

Like the density of  $\mathbb{Q} \in \mathbb{R}$ , we can show that for  $[(q_n)] \in \mathbb{R}$  there is  $q$  in  $\mathbb{Q}$  s.t.  $[(q_n)] \leq [(q, q, \dots)]$  (HW 1).

Let  $X = [(q_n)] \in \mathbb{R}$ . Suppose  $x \geq 0$ . Then there is unique  $m \in \mathbb{N}$  s.t.

$$[(m, m, \dots)] \leq x < [(m+1, m+1, \dots)] \quad (2.11)$$

Call  $m = \lfloor x \rfloor$ .

Define

$$x_1 = \begin{cases} 0 & \text{if } x - \lfloor x \rfloor < \frac{1}{2} = [(\frac{1}{2})] \\ 1 & \text{if } x - \lfloor x \rfloor \geq \frac{1}{2} \end{cases} \quad (2.12)$$

$$\vdots \quad (2.13)$$

$$x_{n+1} = \begin{cases} 0 & \text{if } x - (\lfloor x \rfloor - \sum_{k=1}^n \frac{x_k}{2^k}) < \frac{1}{2^{n+1}} \\ 1 & \text{if } x - (\lfloor x \rfloor - \sum_{k=1}^n \frac{x_k}{2^k}) \geq \frac{1}{2^{n+1}} \end{cases} \quad (2.14)$$

Then, check that

$$x \sim \left( \lfloor x \rfloor + \sum_{k=1}^{\infty} \frac{x_k}{2^k} \right)_{n=1}^{\infty} \quad (2.15)$$

Write  $x = \lfloor x \rfloor . x_1 x_2 x_3 \dots$

Similarly, we have decimal (base 10) or ternary representation (base 3).

# Chapter 3

## Lecture 3: Sep 13, 2017

### 3.1 Last Time

#### Definition 3.1.1 (Partial Order)

A partial order is a relation  $\leq$  on  $X$  which is

- reflexive
- transitive
- anti-symmetric

We write  $(X, \leq)$  as a “partially ordered set” or a poset.

### 3.2 Bounds and Completeness

#### Definition 3.2.1 (Upper Bound, Supremum)

Let  $(X, \leq)$  be a partially ordered set (aka poset). Given  $A \subset X$ ,

- an upper bound is any  $u \in X$  s.t.  $\forall x \in A, x \leq u$
- a supremum (aka least upper bound) is an upper bound  $s$  s.t.  $s \leq u$  for any upper bound  $u$ .

#### Note

1. A supremum need not exist.

For example, in  $(\mathbb{Q}, \leq)$ ,

- $\mathbb{N}$  is not bounded above



- $A = \{q \in \mathbb{Q} : q^2 \leq 2\}$  is bounded above (e.g. 2 is an upper bound) but admits no supremum.
- 2. If a supremum exists, then it is unique (appeal to the anti-symmetry property of  $\leq$ ), so we write  $s = \sup A$ .

**Definition 3.2.2 (Complete)**

We say that  $(X, \leq)$  is complete if any set  $A \subset X$  which admits an upper bound has a supremum,  $\sup A$ .

**Example 3.2.1**

1.  $X \neq \emptyset$ , consider  $(\mathcal{P}(X), \subseteq)$ . Given  $A = \{A_i\}_{i \in I} \subseteq \mathcal{P}(X)$ , we have  $\sup A = \bigcup_{i \in I} A_i$ , so  $\mathcal{P}(X), \subseteq$  is complete.
2.  $(\mathbb{R}, \leq)$  is complete.

(Sketch proof) Suppose  $\emptyset \neq A \subset \mathbb{R}$  is bounded above. Based on (HW1), we can find  $q_0, r_0 \in \mathbb{Q} [\mathbb{Q} \hookrightarrow \mathbb{R}, q \mapsto [(q, q, \dots)]]$  s.t.

- $q_0$  is not an upper bound for  $A$
- $r_0$  is an upper bound for  $A$

Inductively, define for  $n \in \{0\} \cup \mathbb{N}$ ,  $(q_{n+1}, r_{n+1}) \in \mathbb{Q}^2$ .

$$(q_{n+1}, r_{n+1}) = \begin{cases} (q_n, \frac{1}{2}(q_n + r_n)) & \frac{1}{2}(q_n + r_n) \text{ is an upper bound for } A \\ (\frac{1}{2}(q_n + r_n), r_n) & \text{otherwise} \end{cases} \quad (3.1)$$

Fact (check):  $[(q_n)_{n=1}^\infty] = [(r_n)_{n=1}^\infty]$  and is  $\sup A$ .

**Definition 3.2.3 (Maximum)**

Further, we call  $m \in A (A \subset X, (X, \leq))$  poset a maximum of  $A$  if

- $m = \sup A$
- $m \in A$

**Definition 3.2.4 (Lower Bound, Infimum, Minimum)**

We have symmetric definition for lower bounds, infimums (greatest lower bound) and minimums.

Note: The infimum of  $A$  is unique if it exists, denoted as  $\inf A$

**Proposition 3.2.1 (Infimum of a subset of a space)**

If  $(X, \leq)$  is a complete partially ordered space, then any  $A \subseteq X$  which is bounded below, admits an infimum.

**Proof**

Let  $L = \{x \in X : \forall a \in A \ x \leq a\}$ . Notice that  $L \neq \emptyset$  (by assumption on  $A$ ). Also,  $L$  is bounded above, since any element of  $A$  is an upper bound.

Then  $\sup L = \inf A$ .

**3.3 Chains and Zorn's Lemma****Definition 3.3.1 (Chain)**

Let  $(X, \leq)$  be a poset. A chain is any subset  $C \subseteq X$  s.t.  $(C, \leq)$  is totally ordered.

(Note: Strictly, we should have  $(C, \leq \upharpoonright_{C \times C})$ ).

**Definition 3.3.2 (Maximal)**

We say an element  $m \in X$  is maximal if we have that  $\forall x \in X \ m \leq x \implies x = m$ .

**Axiom 3.3.1 (Zorn's Lemma)**

Suppose in a poset  $(X, \leq)$  every chain  $C \subseteq X$  admits an upper bound, i.e.

$$\exists u \in X \ \forall x \in C \ x \leq u \tag{3.2}$$

Then  $(X, \leq)$  admits a maximal element.

**Definition 3.3.3 (Linearly Independent, Spanning, Basis)**

Let  $V$  be a vector space over a field  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{Q}$ ). A subset  $L \subseteq V$  is **linearly independent** (aka **lin. ind.**) if for each finite  $\{v_1, \dots, v_n\} \subseteq L$ ,

$$\forall \alpha_n \in \mathbb{K} \ 0 = \sum_{i=1}^n \alpha_i v_i \implies \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

A subset  $S \subset V$  is **spanning** if for each  $v \in V$  there are finite  $\{v_1, \dots, v_n\} \subseteq S, \{\alpha_1, \dots, \alpha_n\} \subseteq \mathbb{K}$  s.t.

$$v = \sum_{i=1}^n \alpha_i v_i$$

A **basis** is a set  $B \subset V$  which is both linearly independent and spanning.

**Theorem 3.3.1 (Vector space over  $\mathbb{K}$  has a basis)**

A vector space  $V$  over  $\mathbb{K}$  always admits a basis.

**Proof**

Let  $\mathcal{L} = \{L \subset V : L \text{ is linearly independent}\}$ . We note that  $(\mathcal{L}, \subseteq)$  is a poset.

Furthermore,  $\{\{v\} : v \in V \setminus \{0\}\} \subseteq \mathcal{L}$ . So  $\mathcal{L} \neq \emptyset$ .

Let  $\mathcal{C} = \{L_i\}_{i \in I}$  be a chain in  $\mathcal{L}$ , and consider  $L = \bigcup_{i \in I} L_i$ . If  $\{v_1, \dots, v_n\} \subseteq L$ , we have  $v_k \in L_{i_k}$  for some  $k \in [0, n]$ , and since  $\mathcal{C}$  is a chain, we may relate so  $L_{i_1} \subseteq L_{i_2} \subseteq \dots \subseteq L_{i_k}$ . Thus  $\{v_1, \dots, v_n\} \subseteq L_{i_n}$  and is lin. ind. It follows  $L$  is lin. ind. Hence, [Axiom 3.3.1](#) tells us that  $\mathcal{L}$  admits a maximal element  $B$ .

WTP  $B$  is spanning. Suppose  $B$  is not spanning. Then there is  $v_o \in V$  which cannot be written as a linear combination of finitely many vectors from  $B$ . Consider

$$0 = \alpha_0 v_0 + \sum_{i=1}^n \alpha_i v_i \quad (3.3)$$

for  $\{v_1, \dots, v_n\} \subseteq B$ , and  $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ . If we can have  $\alpha_n \neq 0$ , then

$$v_0 = \sum_{i=1}^n \left( -\frac{\alpha_i}{\alpha_n} v_i \right) \quad (3.4)$$

which contradicts our assumption on  $v_o$ . Hence  $\alpha_n = 0$ , and thus  $0 = \sum_{i=1}^n \alpha_i v_i$ , so  $\alpha_1 = \dots = \alpha_n = 0$ , as well. Hence  $B \cup \{v_o\} \in \mathcal{L}$ . But  $B \subseteq B \cup \{v_o\}$ , contradicting maximality.

#### Remark

An easy modification of the proof shows that any  $L \in \mathcal{L}$  is a subset of a basis.

# Chapter 4

## Lecture 4: Sep 15, 2017

### 4.1 Logistics

#### Office Hours

- today: 1430 - 1520
- Wed, next week: 1430 - 1630

### 4.2 Cardinal arithmetic

#### Definition 4.2.1 (Injection, Surjection, Bijection)

Given nonempty sets  $X, Y$ , a function  $f : X \rightarrow Y$  is called a(n)

- **injection**  $x_1 \neq x_2 \in X \implies f(x_1) \neq f(x_2)$
- **surjection**  $\forall y \in Y \exists x \in X f(x) = y$
- **bijection** if it is both an injection and a surjection (aka invertible)

Of course, if  $f : X \rightarrow Y$  is a bijection then we can define  $f^{-1} : Y \rightarrow X$  by  $f^{-1}(f(x)) = x$ .

We write  $X \sim Y$  if there exists a bijection  $f : X \rightarrow Y$ .

Sometimes, we write

$$X \underset{f}{\sim} Y$$

#### Note ( $\sim$ as an equivalence relation)

- (reflexivity)  $X \underset{id}{\sim} X$  ( $id : X \rightarrow X$  is the identity function)

- (symmetry)  $X \underset{f}{\sim} Y \implies Y \underset{f^{-1}}{\sim} X$
- (transitivity)  $X \underset{f}{\sim} Y \wedge Y \underset{g}{\sim} Z \implies X \underset{gf}{\sim} Z$

Hence  $\sim$  is an equivalence relation on any given family of sets. We let  $|X|$  denote the equivalence class. We call this cardinality of  $X$ .

Note:  $|\emptyset| = 0$ ,  $|\{1, \dots, n\}| = n \in \mathbb{N}$

### Example 4.2.1

1.

$$\mathbb{N} \sim \mathbb{Z} \quad \because f(n) = \begin{cases} \frac{n}{2} & n \text{ is even} \\ \frac{1}{n}(1-n) & n \text{ is odd} \end{cases}$$

2.

$$\mathbb{R} \underset{f}{\sim} (-1, 1) \quad \because f(x) = \frac{x}{|x| + 1}$$

Exercise: exhibit  $f^{-1}$

$$\text{Answer: } f^{-1}(x) = \frac{x}{1-|x|}$$

3.  $a < b \in \mathbb{R}$   $(0, 1) \underset{g}{\sim} (a, b)$ ,  $g(x) = a + x(b - a)$

### Note (Notation)

$$\aleph_0 = |\mathbb{N}| \text{ ("aleph-naught")} \quad c = |\mathbb{R}| \text{ ("continuum")}$$

### Note (Arithmetic)

Let  $A, B$  be sets.

$$\begin{aligned} |A| + |B| &= |A \sqcup B| \\ |A||B| &= |A \times B| \\ |A|^{|B|} &= |A^B| \quad (B \neq \emptyset, A^B = \{f : B \rightarrow A \mid f \text{ is a function}\}) \end{aligned}$$

### Note (Properties)

- (commutativity)

$$\begin{aligned} |A| + |B| &= |B| + |A| \\ |A||B| &= |B||A| \end{aligned}$$

- (distributivity)

$$\begin{aligned} |A|(|B| + |C|) &= |A||B| + |A||C| \\ (A \times (B \sqcup C) &\sim (A \times B) \sqcup (A \times C)) \end{aligned}$$

- (exponential laws)

$$\begin{aligned} (B \neq \emptyset \neq C) \\ (1) \quad |A|^{|B|+|C|} &= |A|^{|B|}|A|^{|C|} \quad (2) \quad |A|^{|B||C|} = \left(|A|^{|B|}\right)^{|C|} \end{aligned}$$

$$\begin{aligned} (1) \quad (A^{B \sqcup C} &\sim A^B \times A^C \text{ via } \phi \mapsto (\phi|_B, \phi|_C)) \\ (2) \quad A^{B \times C} &\sim (A^B)^C \text{ via } \phi \mapsto (\phi(b, \cdot) : C \rightarrow A)_{b \in B} \end{aligned}$$

**Definition 4.2.2 (Precedence)**

For sets  $A, B$ , define

$$A \leq B \text{ if there is an injection } f : A \rightarrow B$$

We sometimes write the above as  $A \underset{f}{\leq} B$ .

- (reflexivity)  $A \leq A$
- (transitivity)  $A \leq B, B \leq C \implies A \leq C$

We are one property short of making  $\leq$  as an order relation.

**Note**

It seems reasonable to write  $|A| \leq |B|$ , in this case, our question is: Is  $\leq$  in cardinal numbers anti-symmetric?

**Theorem 4.2.1 (Cantor-Bernstein-Schröder)**

If, for non-empty set  $A, B$ , we have

$$A \leq B \wedge B \leq A \implies A \sim B \quad (4.1)$$

i.e.

$$|A| \leq |B| \wedge |B| \leq |A| \implies |A| = |B| \quad (4.2)$$

**Proof**

Our assumption is that we have injections

$$A \underset{\phi}{\leq} B, \quad B \underset{\psi}{\leq} A \quad (4.3)$$

To avoid triviality, let us suppose that neither  $\phi$  or  $\psi$  is surjective. Thus

$$\phi(A) \subsetneq B \quad \psi \circ \phi(A) \subsetneq \psi(B) \subsetneq A \quad (4.4)$$

Let  $A_0 = A$ ,  $A_1 = \psi(B)$ ,  $A_2 = \psi \circ \phi(A)$  and we inductively define

$$A_{n+1} = g(A_n) \text{ where } g = \psi \circ \phi \quad (4.5)$$

Then  $A_2 \subsetneq A_1 \subsetneq A_0$ , so by applying injection  $g$ ,

$$\begin{aligned} A_4 &\subsetneq A_3 \subsetneq A_2 \\ &\vdots \\ A_{n+1} &\subsetneq A_n \subsetneq A_{n-1} \end{aligned}$$

Hence, we may decompose

$$\begin{aligned} A &= A_0 = (A_0 \setminus A_1) \cup A_1 \\ &= (A_0 \setminus A_1) \cup (A_1 \setminus A_2) \cup A_2 \\ &\vdots \\ &= \bigcup_{n=1}^{\infty} (A_{n-1} \setminus A_n) \cup A_{\infty} \end{aligned}$$

where  $A_{\infty} = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=2}^{\infty} A_n$ , we likewise observe

$$A_i = \bigcup_{n=2}^{\infty} (A_{n-1} \setminus A_n) \cup A_{\infty}$$

Using definitions of the sets  $A_n$  ( $n \geq 2$ ) we have

$$g(A_{n-1} \setminus A_n) = A_{n+1} \setminus A_{n+2}$$

Define

$$h : A_0 \rightarrow A_1 \quad h(x) = \begin{cases} g(x) & x \in A_{n-1} \setminus A_n \text{ is odd} \\ x & \text{otherwise} \end{cases} \quad (4.6)$$

Then  $h$  is a bijection.

Thus  $A = A_0 \underset{h}{\sim} A_1 - \phi(B)$ ,  $B \underset{\phi}{\sim} \psi(B)$  so we conclude that  $A \sim B$ . □

**Example 4.2.2**

1. Let  $a < b \in \mathbb{R}$ . Then

$$\begin{aligned} [a, b] &\leq \mathbb{R} \quad \text{obvious} \\ \mathbb{R} &\sim (-1, 1) \sim (0, 1) \sim (a, b) \leq [a, b] \end{aligned}$$

i.e.  $[a, b] \leq \mathbb{R}$  and  $\mathbb{R} \leq [a, b]$  so  $\mathbb{R} \sim [a, b]$ .



# Chapter 5

## Lecture 5: Sep 18, 2017

### 5.1 Continuing CBS with examples

#### Example 5.1.1

2.  $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$ , i.e.  $|\mathcal{P}(\mathbb{N})| = c$

$$\mathcal{P}(\mathbb{N}) \sim \{0, 1\}^{\mathbb{N}} \text{ via } A \mapsto \chi(A) \quad (5.1)$$

where

$$\chi_A(n) = \begin{cases} 1 & n \in A \\ 0 & n \notin A \end{cases} \quad (5.2)$$

is the “characteristic indicator”.

$$\{0, 1\}^{\mathbb{N}} \leq [0, 1) \text{ via } (x_k)_{k=1}^{\infty} \mapsto \sum_{k=1}^{\infty} \frac{x_k}{3^k} = 0.x_1x_2x_3\ldots \text{ is the ternary rep'n} \quad (5.3)$$

which is injective.

Claim  $[0, 1) \leq \{0, 1\}^{\mathbb{N}}$ ,  $0.x_1x_2x_3\ldots = \sum_{k=1}^{\infty} \frac{x_k}{2^k} \mapsto (x_k)_{k=1}^{\infty}$  which is the binary rep'n.  
Note that this representation doesn't allow  $0.1111\ldots = 1$  (see [Lecture 2](#)).

$$\mathcal{P}(\mathbb{N}) \sim \{0, 1\}^{\mathbb{N}} \leq [0, 1) \leq \{0, 1\}^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N})$$

Thus by [Theorem 4.2.1](#),

$$|\mathcal{P}(\mathbb{N})| = |[0, 1)| = c = |\mathbb{R}| \quad (5.4)$$

$$3. \mathbb{N} \sim \mathbb{Q} \sim \mathbb{N}^2$$

- $\mathbb{N} \leq \mathbb{Q}$  (obvious)

- $\mathbb{Q} \leq \mathbb{Z} \times \mathbb{N}$ , which we pick  $\frac{m}{n} \mapsto (m, n)$  with  $\gcd(m, n) = 1$  where  $m \in \mathbb{Z}, n \in \mathbb{N}$ .
- $\mathbb{Z} \times \mathbb{N} \sim \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$  as  $\mathbb{Z} \sim \mathbb{N}$
- $\mathbb{N}^2 \sim \mathbb{N}$  via  $(m, n) \mapsto 2^m 3^n$

Therefore

$$\mathbb{N} \leq \mathbb{Q} \leq \mathbb{Z} \times \mathbb{N} \sim \mathbb{N}^2 \leq \mathbb{N} \quad (5.5)$$

Thus by [Theorem 4.2.1](#),  $\mathbb{N} \sim \mathbb{Q} \sim \mathbb{N}^2$ .

### Note (Notation)

We say that a set  $A$  is

- **countable** if  $A \leq \mathbb{N}$ , i.e.  $|A| \leq \aleph_0$
- **denumerable** if  $A \sim \mathbb{N}$ , i.e.  $|A| = \aleph_0$

## 5.2 Comparison Theorem

### Proposition 5.2.1 (Surjectivity)

Suppose  $X$  and  $Y$  are non-empty sets and there is a surjection  $g : X \rightarrow Y$ . Then  $Y \leq X$ .

#### Proof

Let  $f : \mathcal{P}(X) \setminus \{\emptyset\} \rightarrow X$  be a choice function (by [Axiom 1.2.1 AC](#)). For each  $y \in Y$ , we have  $g^{-1}(\{y\}) = \{x \in X : g(x) = y\} \neq \emptyset$ , as  $g$  is surjective. Define  $h : Y \rightarrow X$  be given by  $h(y) = f(g^{-1}(\{y\}))$  and  $h$  is injective, as if  $y_1 \neq y_2$ ,  $\{y_1\} \cap \{y_2\} = \emptyset$ , so we see that

$$g^{-1}(\{y_1\}) \cap g^{-1}(\{y_2\}) = \emptyset \quad (5.6)$$

too. □

### Theorem 5.2.1 (Comparison Theorem)

Let  $X$  and  $Y$  be sets. Then either  $X \leq Y$  or  $Y \leq X$ .

#### Proof

If  $X = \emptyset$  then  $X \leq Y$ ; likewise if  $Y = \emptyset$ . Hence, assume  $X \neq \emptyset \neq Y$ . Let

$$\Delta = \{(A, f) : A \in \mathcal{P}(X) \setminus \{\emptyset\}, f \in Y^A \text{ is an injection}\} \quad (5.7)$$

We observe that  $\Delta \neq \emptyset$ . If  $x \in X, y \in Y$ , then  $(\{x\}, x \mapsto y) \in \Delta$ .

On  $\Delta$  let

$$(A, f) \leq (B, g) \iff \begin{matrix} A \subseteq B \subseteq X \\ g|_A = f \end{matrix} \quad (5.8)$$

Notice that  $\leq$  is reflexive, anti-symmetric, and transitive. Thus  $\leq$  is a partial order on  $\Delta$ .

Let  $\Gamma = \{(A_i, f_i)\}_{i \in I}$  be a chain in  $(\Delta, \leq)$ . We let

$$A = \bigcup_{i \in I} A_i \quad (5.9)$$

and  $f \in Y^A$  be given by  $f(x) = f_i(x)$  provided  $x \in A_i$ .

Notice that  $f$  is well-defined. Say  $x \in A_i$  and  $x \in A_j$ , then since  $\Gamma$  is a chain, without loss of generality,  $A_i \subseteq A_j$ , and  $f_j|_{A_i} = f_i$ .

Furthermore, if  $x_1 \neq x_2 \in A$ , then  $x_1 \in A_{i_1}, x_2 \in A_{i_2}$ , and we may suppose  $A_{i_1} \subseteq A_{i_2}$ . Then  $f(x_1) = f_{i_1}(x_1) = f_{i_2}(x_1) \neq f_{i_2}(x_2) = f(x_2)$ .

So  $f$  is an injection. Thus  $(A, f) \in \Delta$  and is an upper bound for  $\Gamma$ .

Thus there is a maximal element  $(M, g) \in \Delta$ , by [Axiom 3.3.1](#) Zorn's Lemma.

1. Case 1:  $M = X$ . Then  $X = M \leq_g Y$ .
2. Case 2:  $M \subsetneq X$ . We wish to see that  $g$  is surjective.

Suppose not, i.e.  $\exists y_0 \in Y \setminus g(M)$ . Since  $M \subsetneq X$ ,  $\exists x_0 \in X \setminus M$ . Define  $h : M \cup \{x_0\} \rightarrow Y$  by

$$h(x) = \begin{cases} g(x) & x \in M \\ y_0 & x = x_0 \end{cases} \quad (5.10)$$

which is injective.

Then  $(M \cup \{x_0\}, h) \in \Delta$ , and  $(M, g) \leq (M \cup \{x_0\}, h)$ , contradicting the maximality of  $(M, g) \in \Delta$ . Thus  $g$  is surjective as desired.

Therefore,  $Y \leq_{g^{-1}} X$ . □

### Proposition 5.2.2 (Alternative Definitions of an Infinite Set)

Let  $A$  be a set. Then TFAE:

1.  $n \leq |A|$  for all  $n \in \mathbb{N}$ .
2.  $\aleph_0 \leq |A|$ , i.e.  $A$  is infinite
3.  $\exists B \subsetneq A$  s.t.  $|B| = |A|$ .
4.  $1 + |A| = |A|$  (Hilbert hotel)
5.  $\aleph_0 + |A| = |A|$

# Chapter 6

## Lecture 6: Sep 20, 2017

### 6.1 Continuing ordinal arithmetic

#### Proof

1.  $1 \implies 2$

We have that for each  $n \in \mathbb{N}$  there is an injection  $\phi_n : \{1, \dots, n\} \rightarrow A$ . Inductively, define  $f : \mathbb{N} \rightarrow A$  by

$$f(1) = \phi_1(1)$$

$\vdots$

$$f(n+1) = \phi_{n+1}(k) \quad \text{where } k = \min\{j \in \{1, \dots, n+1\} : \phi_{n+1}(j) \notin \{f(1), \dots, f(n)\}\}$$

The  $f$  is injective by construction, i.e.  $\mathbb{N} \underset{f}{\preceq} A$  or  $\aleph_0 \leq |A|$

2.  $2 \implies 3$

We have  $\mathbb{N} \underset{f}{\preceq} A$ . Let  $B = A \setminus \{f(1)\}$ .

Define  $g : A \rightarrow B$  by

$$g(x) = \begin{cases} f(n+1) & x = f(n), n \in \mathbb{N} \\ x & \text{otherwise} \end{cases} \tag{6.1}$$

Then  $A \underset{g}{\sim} B$ , i.e.  $|A| = |B|$ .

3.  $3 \implies 4$

We suppose that there is  $x_0 \in A \setminus B$  and  $B \sim A$ . Thus,

$$A \sim B \leq B \cup \{x_0\} \leq A \quad (6.2)$$

Then by **Theorem 4.2.1**,  $A \sim B$  and furthermore  $A \sim B \cup \{x_0\} \sim A \sqcup \{1\}$ , i.e.  $|A| = |A| + 1$ .

4.  $4 \implies 5$

We have  $\{1\} \sqcup A \sim A$ . Then  $\phi(A) \subsetneq A$ . Thus  $\phi \circ \phi(A) \subsetneq \phi(A) \subsetneq A$ , and by induction

$$\underbrace{\phi^{\circ n}}_{\phi \text{ composed with itself } n \text{ times}}(A) \subsetneq \phi^{\circ(n-1)}(A) \subsetneq \dots \subsetneq A \quad (6.3)$$

Hence  $|A| \geq |A \setminus \phi^{\circ n}(A)| \geq n$  (at each stage above, we gain at least one point).

5.  $2 \implies 5$

We have  $\mathbb{N} \leq_f A$ . Let

$$g : \mathbb{N} \sqcup A \rightarrow A, \quad g(x) = \begin{cases} f(2n) & x = n, n \in \mathbb{N} \\ f(2n+1) & x = f(n) \in A, n \in \mathbb{N} \\ x & \text{otherwise} \end{cases} \quad (6.4)$$

6.  $5 \implies 2$

$$\aleph_0 \leq \aleph_0 + |A| \underset{\text{by assumption}}{=} |A|.$$

### Note

Any set satisfying 1 to 5 of the above is called infinite.

### Corollary 6.1.1 (A set is either finite or denumerable)

If  $A \in \mathcal{P}(\mathbb{N})$ , then either  $A$  is finite or denumerable.

### Proof

Either  $n \leq |A|$  for all  $n \in \mathbb{N}$ , or  $|A| < n$  for some  $n \in \mathbb{N}$ .

### Theorem 6.1.1 (Cantor)

For any set  $X$

$$|X| \leq |\mathcal{P}(X)|, \text{ i.e. } X \leq \mathcal{P}(X) \wedge X \not\sim \mathcal{P}(X) \quad (6.5)$$

### Proof

If  $X = \emptyset$ ,  $0 = |\emptyset| \leq 1 = |\{\emptyset\}|$ .

If  $X \neq \emptyset$ , then  $x \mapsto \{x\} : X \rightarrow \mathcal{P}(X)$  shows  $X \leq \mathcal{P}(X)$ .

Now suppose  $X \neq \emptyset$ ,  $f : X \rightarrow \mathcal{P}(X)$ . We will show that  $f$  cannot be surjective. Let

$$E = \{x \in X : x \notin f(x)\} \quad (6.6)$$

i.e.  $E$  is a set that is not in the range of  $f$ .

If we had  $E \subseteq f(X)$ , i.e.  $E = f(x)$  for some  $x \in X$ , then either

- $x \in E$ , i.e.  $x \notin f(x)$ , which means that  $E \neq f(x)$ , or
- $x \notin E = f(x)$ , so  $x \in E$ .

These contradictions show that  $E \not\subseteq f(X)$ .

Hence there is no surjection  $f : X \rightarrow \mathcal{P}(X)$ .

### Example 6.1.1

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = c$$

### Theorem 6.1.2 (Cantor's Continuum Hypothesis)

This is no set  $A$  such that

$$\aleph_0 < |A| < c \quad (6.7)$$

### Remark

This theorem has recently been proven (about a month ago from Sep 20, 2017). This theorem is independent of ordinary set theory.

### Theorem 6.1.3 (Generalized Continuum Hypothesis)

Given an infinite set  $C$ , there is no set  $A$  such that

$$|C| < |A| < |\mathcal{P}(C)| \quad (6.8)$$

### Theorem 6.1.4 (Cantor's Paradox)

There is no "set" of all sets.

Suppose there was a universal set  $\mathcal{U}$ , i.e. any set  $A \subseteq \mathcal{U}$ . But then,

$$|\mathcal{U}| < |\mathcal{P}(\mathcal{U})|, \text{ so } \mathcal{P}(\mathcal{U}) \not\subseteq \mathcal{U} \quad (6.9)$$

so  $\mathcal{U}$  cannot exist.

### Axiom 6.1.1 (Well-Ordering)

Given a non-empty set  $X$ , a **well-order** is a partial order on  $X$  such that any  $\emptyset \neq A \subseteq X$  admits a minimal element, i.e.

$$\exists m_A \in A \forall a \in A \ m_A \leq a \quad (6.10)$$

**Remark**

Well-order VS total order:  $x, y \in X$  consider  $A = \{x, y\}$ .

**Example 6.1.2**

1.  $(\mathbb{N}, \leq)$  is well-ordered (principle of mathematical induction).
2.  $\mathbb{N}^2$ . Let us consider two well-orders.

(pyramid)  $(m, n) \leq (\mu, \nu) \iff$

$$\begin{cases} \text{either } m + n < \mu + \nu \\ m + n = \mu + \nu \text{ and } m \leq \mu \end{cases} \quad (6.11)$$

(lexicographic)  $(m, n) \leq_l (\mu, \nu) \iff$

$$\begin{cases} \text{either } m < \mu \text{ or} \\ m = \mu \text{ and } n \leq \nu \end{cases} \quad (6.12)$$

Notice that  $(m, n) \leq_l (\mu, \nu) \iff 2m - \frac{1}{n} \leq 2\mu - \frac{2}{\nu} \in (\mathbb{Q}, \leq)$

# Chapter 7

## Lecture 7: Sep 22, 2017

### 7.1 Metric Spaces

#### Note

We can use  $\mathbb{R}$  in any reasonable manner.

#### Definition 7.1.1 (Metric and Metric Space)

Let  $X$  be a nonempty set. A metric  $d : X \times X \rightarrow \mathbb{R}$  is a function which satisfies, for  $x, y, z \in X$

- **(non-negativity)**  $d(x, y) \geq 0$
- **(non-degeneracy)**  $d(x, y) = 0 \iff x = y$
- **(symmetry)**  $d(x, y) = d(y, x)$
- **(triangle inequality)**  $d(x, z) \leq d(x, y) + d(y, z)$

We often call the pair  $(X, d)$  a metric space.

#### Example 7.1.1

1. On  $\mathbb{R}$ ,  $d(x, y) = |x - y|$
2. Let  $X \neq \emptyset$  any set. Define the “discrete” metric

$$d : X \times X \rightarrow \{0, 1\} \subseteq \mathbb{R}, \quad d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases} \quad (7.1)$$

Note that non-degeneracy and symmetry are obvious. The triangle inequality is sat-



isfied since

$$\begin{aligned} \text{Case: } x \neq y \neq z \neq x \\ 1 = d(x, z) \leq 2 = d(x, y) + d(y, z) \end{aligned}$$

3. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be strictly increasing. Let

$$d_f : \mathbb{R}^2 \rightarrow [0, \infty) \quad d_f(x, y) = |f(x) - f(y)| \quad (7.2)$$

Exercise: check for its properties.

$$\text{E.g. } f(x) = \frac{x}{|x|+1}.$$

4. (French railroad metric) Suppose we have a set  $X \neq \emptyset$ , and a function  $f : X \rightarrow [0, \infty)$  which satisfies  $f^{-1}(\{0\}) = \{p_0\}$ . Notice that  $f(x) > 0$  if  $x \in X \setminus \{p_0\}$ .

$$d_f : X \times X \rightarrow [0, \infty) \quad d_f(x, y) = \begin{cases} 0 & x = y \\ f(x) + f(y) & x \neq y \end{cases} \quad (7.3)$$

Easy exercise: This is a metric.

### Definition 7.1.2 (Norm, Normed Vector Space)

Let  $V$  be a vector space over  $\mathbb{R}$ . A **norm** is a function  $\|\cdot\| : V \rightarrow \mathbb{R}$  which satisfies, for  $x, y \in V$ ,  $\alpha \in \mathbb{R}$

1. (**non-negativity**)  $\|x\| \geq 0$
2. (**non-degeneracy**)  $\|x\| = 0 \iff x = 0$
3. ( **$\|\cdot\|$ -homogeneity**)  $\|\alpha x\| = |\alpha| \|x\|$
4. (**subadditivity**)  $\|x + y\| \leq \|x\| + \|y\|$

We call the pair  $(V, \|\cdot\|)$  a **normed vector space**.

### Note

If  $(V, \|\cdot\|)$  is a normed vector space, then

$$d : V \times V \rightarrow [0, \infty) \quad d(x, y) = \|x - y\| \quad (7.4)$$

is always a metric on  $V$ . Everything is easy to check; subadditivity of  $\|\cdot\| \implies$  triangle inequality of  $d$ .

### Example 7.1.2

1.  $(\mathbb{R}, |\cdot|)$  is a normed vector space.

2. On  $\mathbb{R}^n$ , for  $x = (x_1, \dots, x_n)$

$$\|x\|_2 = \sqrt{x_1^2 + \dots + x_n^2} \quad (7.5)$$

This is the Euclidean norm.

Consider, also

$$\begin{aligned} \|x\|_1 &= |x_1| + \dots + |x_n| \\ \|x\|_\infty &= \max\{|x_1|, \dots, |x_n|\} \end{aligned}$$

**Note**

non-degeneracy and  $|\cdot|$ -homogeneity are obvious for  $\|\cdot\|_1$ ,  $\|\cdot\|_\infty$

Let us consider subadditivity

$$\begin{aligned} \|x + y\|_1 &= |x_1 + y_1| + \dots + |x_n + y_n| \\ &\leq |x_1| + |y_1| + \dots + |x_n| + |y_n| \\ &= |x_1| + \dots + |x_n| + |y_1| + \dots + |y_n| \\ &= \|x\|_1 + \|y\|_1 \end{aligned}$$

$$\begin{aligned} \|x + y\|_\infty &= \max\{|x_i + y_i| : i = 1, \dots, n\} \\ &= \max\{|x_i| + |y_i| : i = 1, \dots, n\} \\ &= \max\{|x_i| + |y_j| : i, j = 1, \dots, n\} \\ &= \max\{|x_i| : i = 1, \dots, n\} + \max\{|y_j| : j = 1, \dots, n\} \\ &= \|x\|_\infty + \|y\|_\infty \end{aligned}$$

Now for  $1 < p < \infty$  consider

$$x^p = \begin{cases} e^{p \log x} & x > 0 \\ 0 & x = 0 \end{cases} \quad (7.6)$$

$$\|x\|_p = (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}}$$

**Remark (Cauchy-Bunyakovsky-Schwartz)**

$$|x \cdot y| \leq \|x\|_2 \|y\|_2$$

**Lemma 7.1.1** ( $\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$ )

Let  $\alpha, \beta \leq 0 \in \mathbb{R}$ ,  $1 < p < \infty$  and  $q$  is chosen such that  $\frac{1}{p} + \frac{1}{q} = 1$  (i.e.  $q = \frac{p}{p-1}$ ) then

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q} \quad (7.7)$$

with the equality when  $\alpha^p = \beta^q$ .

**Proof**

Consider the graph of  $y = x^{p-1}$  (assume  $p \geq 2$ ). Then

$$\begin{aligned}\alpha\beta &\leq \int_0^\alpha x^{p-1} dx + \int_0^b y^{q-1} dy \\ &= \frac{\alpha^p}{p} + \frac{\beta^q}{q}\end{aligned}$$

Equality holds only if  $\beta = \alpha^{p-1} \implies \beta^{\frac{1}{p-1}} = \alpha \implies \beta^q = \alpha^p$

**Theorem 7.1.1 (Holder's Inequality)**

Let  $x, y \in \mathbb{R}^n$ ,  $1 < p < \infty$  and  $q$  be so  $\frac{1}{p} + \frac{1}{q} = 1$ , then

$$\left| \sum_{j=1}^n x_j y_j \right| \leq \sum_{j=1}^n |x_j| |y_j| \leq \left( \sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^n |y_j|^q \right)^{\frac{1}{q}} = \|x\|_p \|y\|_q \quad (7.8)$$

## Chapter 8

# Lecture 8: Sep 25, 2017

### 8.1 Logistics

Expect assignment 2 to be up tonight!

### 8.2 Continuing Normed Vector Space

#### Proof (Holder's Inequality)

$\|x\|_p \|y\|_q = 0 \implies (x = 0 \vee y = 0) \wedge$  the inequality is trivial. Let us assume  $\|x\|_p \|y\|_q \neq 0$ .  
For  $j = 1, \dots, n$

$$\alpha_j = \frac{|x_j|}{\|x\|_p}, \quad \beta_j = \frac{|y_j|}{\|y\|_q}$$

Then

$$\begin{aligned} \frac{1}{\|x\|_p \|y\|_q} \sum_{j=1}^n |x_j| |y_j| &= \sum_{j=1}^n \alpha_j \beta_j \stackrel{(1)}{\leq} \sum_{j=1}^n \left( \frac{\alpha_j^p}{p} + \frac{\beta_j^q}{q} \right) \\ &= \frac{1}{p} \sum_{j=1}^n \alpha_j^p + \frac{1}{q} \sum_{j=1}^n \beta_j^q \\ &= \frac{1}{p \|x\|_p^p} \sum_{j=1}^n |x_j|^p + \frac{1}{q \|y\|_q^q} \sum_{j=1}^n |y_j|^q \\ &= \frac{1}{p \|x\|_p^p} \|x\|_p^p + \frac{1}{q \|y\|_q^q} \|y\|_q^q = \frac{1}{p} + \frac{1}{q} \stackrel{(2)}{=} 1 \end{aligned}$$

where (1) is by [Lemma 7.1.1](#) and (2) is by choice of  $q$ .

Hence, we multiply by  $\|x\|_p \|y\|_q$  and see that

$$\sum_{j=1}^n |x_j| |y_j| \leq \|x\|_p \|y\|_q \quad (8.1)$$

□

### Theorem 8.2.1 (Minkowski's Inequality)

Let  $x, y \in \mathbb{R}^n$  and  $1 < p < \infty$ . Then

$$\|x + y\| \leq \|x\|_p + \|y\|_p \quad (8.2)$$

#### Proof

If  $x + y = 0$ , this is trivial, hence suppose  $x + y \neq 0$ . Compute

$$\begin{aligned} \|x + y\|_p^p &= \sum_{j=1}^n |x_j + y_j|^p = \sum_{j=1}^n |x_j + y_j| |x_j + y_j|^{p-1} \\ &= \sum_{j=1}^n (|x_j| + |y_j|) |x_j + y_j|^{p-1} \\ &= \sum_{j=1}^n |x_j| |x_j + y_j|^{p-1} + \sum_{j=1}^n |y_j| |x_j + y_j|^{p-1} \\ &\leq \left( \sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}} \\ &\quad + \left( \sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}} \\ &= (\|x\|_p + \|y\|_p) \left( \sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}} \end{aligned}$$

We have  $\frac{1}{p} + \frac{1}{q} = 1 \implies \frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p} \implies p = q(p-1)$ , and thus

$$\begin{aligned} \|x + y\|_p^p &\leq (\|x\|_p + \|y\|_p) \left( \sum_{j=1}^n |x_j + y_j|^p \right)^{\frac{1}{q}} \\ &= (\|x\|_p + \|y\|_p) \|x + y\|_p^{\frac{p}{q}} \end{aligned}$$

Now divide  $\|x + y\|_p^{\frac{p}{q}} \neq 0$ , we get

$$\|x + y\|_p = \|x + y\|_p^{p - \frac{p}{q}} \leq \|x\|_p + \|y\|_p \quad (\text{since } p - \frac{p}{q} = p(1 - \frac{1}{q}) = \frac{p}{p} = 1) \quad (8.3)$$

**Corollary 8.2.1** ( $\|\cdot\|_p$  is a norm)

Given  $1 < p < \infty$ ,  $\|\cdot\|_p$  is a norm on  $\mathbb{R}^n$ .

**Proof**

Clearly,  $\|\cdot\|_p$  is non-negative and non-degenerate. If  $\alpha \in \mathbb{R}, x \in \mathbb{R}^n$  then

$$\begin{aligned} \|\alpha x\|_p &= \left( \sum_{j=1}^n |\alpha x_j|_p^p \right)^{\frac{1}{p}} = \left( \sum_{j=1}^n |\alpha|^p |x_j|^p \right)^{\frac{1}{p}} \\ &= |\alpha| \left( \sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} = |\alpha| \|x\|_p \end{aligned}$$

Finally, subadditivity is provided by [Theorem 8.2.1](#).

### 8.3 $\ell_p$ -spaces

Consider  $\mathbb{R}^n = \{x = (x_k)_{k=1}^\infty : x_k \in \mathbb{R}\}$  which is a  $\mathbb{R}$ -vector space:

$$(x_k)_{k=1}^\infty + (y_k)_{k=1}^\infty = (x_k + y_k)_{k=1}^\infty, \quad \alpha(x_k)_{k=1}^\infty = 1^\infty = (\alpha x_k)_{k=1}^\infty = 1^\infty \quad (8.4)$$

We let, for  $1 \leq p < \infty$ ,

$$\bullet \ell_p = \{x = (x_k)_{k=1}^\infty \in \mathbb{R}^\mathbb{N} : \sum_{k=1}^\infty |x_k|^p = \lim_{n \rightarrow \infty} \sum_{k=1}^n |x_k|^p < \infty\}$$

and

$$\ell_\infty = \{x = (x_k)_{k=1}^\infty : \sup_{k \in \mathbb{N}} |x_k| < \infty\}$$

On  $\ell_p$  we define

$$\|x\|_p = \begin{cases} \left( \sum_{k=1}^\infty |x_k|^p \right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \sup_{k \in \mathbb{N}} |x_k| & p = \infty \end{cases} \quad (8.5)$$

**Theorem 8.3.1** ( $\ell_p$  is a  $\mathbb{R}$ -subspace)

Let  $1 \leq p < \infty$ . Then  $\ell_p$  is a  $\mathbb{R}$ -subspace of  $\mathbb{R}^\mathbb{N}$  and  $\|\cdot\|_p$  is a norm.

**Proof**

We shall prove these statements together. Suppose that  $x, y \in \ell_p$ . Then

$$\begin{aligned}
\|x + y\|_p &= \left( \sum_{k=1}^{\infty} |x_k + y_k|^p \right)^{\frac{1}{p}} \quad (\text{may be } \infty, \infty^{\frac{1}{p}} = \infty) \\
&= \left( \lim_{n \rightarrow \infty} \sum_{k=1}^n |x_k + y_k|^p \right)^{\frac{1}{p}} \\
&= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n |x_k + y_k|^p \right)^{\frac{1}{p}} \quad \left( \begin{array}{l} x \mapsto x^{\frac{1}{p}} \text{ is cts on } [0, \infty) \\ x \rightarrow \infty \implies x^{\frac{1}{p}} \rightarrow \infty \end{array} \right) \\
&\leq \lim_{n \rightarrow \infty} \left[ \left( \sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} + \left( \sum_{k=1}^n |y_k|^p \right)^{\frac{1}{p}} \right] \quad \text{by Theorem 8.2.1 on each } n \\
&= \left( \lim_{n \rightarrow \infty} \sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} + \left( \lim_{n \rightarrow \infty} \sum_{k=1}^n |y_k|^p \right)^{\frac{1}{p}} \quad \text{cty again} \\
&= \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} + \left( \sum_{k=1}^{\infty} |y_k|^p \right)^{\frac{1}{p}} = \|x\|_p + \|y\|_p < \infty
\end{aligned}$$

Thus  $x + y \in \ell_p$ , and we get subadditivity of  $\|\cdot\|_p$ .

We note that non-negativity and non-degeneracy of  $\|\cdot\|_p$  are obvious properties. Likewise, the  $|\cdot|$ -homogeneity is straightforward.  $\square$

**Theorem 8.3.2** ( $(\ell_{\infty}, \|\cdot\|_{\infty})$  is a normed vector space)

$(\ell_{\infty}, \|\cdot\|_{\infty})$  is a normed vector space.

**Proof**

$x, y \in \ell_{\infty} \implies$

$$\begin{aligned}
\|x + y\|_{\infty} &= \sup_{k \in \mathbb{N}} |x_k + y_k| \leq \sup_{k \in \mathbb{N}} (|x_k| + |y_k|) \\
&\leq \sup_{j, k \in \mathbb{N}} (|x_j| + |y_k|) \\
&= \sup_{j \in \mathbb{N}} |x_j| + \sup_{k \in \mathbb{N}} |y_k| = \|x\|_{\infty} + \|y\|_{\infty}
\end{aligned}$$

Other properties are easy (exercise).  $\square$

## Chapter 9

# Lecture 9: Sep 27, 2017

### 9.1 Last Time

Note

$$1 \leq p < \infty$$

$$\ell_p = \left\{ x = (x_k)_{k=1}^\infty \in \mathbb{R}^\mathbb{N} : \|x\|_p = \left( \sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} \right\}$$
$$\ell_\infty = \left\{ x = (x_k)_{k=1}^\infty \in \mathbb{R}^\mathbb{N} : \|x\|_\infty = \sup_{k \in \mathbb{N}} |x_k| \right\}$$

### 9.2 Continuing with $\ell_p$

$$c_0 = \{x = (x_k)_{k=1}^\infty \in \mathbb{R}^\mathbb{N} : \lim_{k \rightarrow \infty} x_k = 0\}$$

Note that  $c_0$  is a  $\mathbb{R}$ -subspace of  $\mathbb{R}^\mathbb{N} : x, y \in c_0$  and  $\alpha \in \mathbb{R}$ , then

$$x + y = (x_k + y_k)_{k=1}^\infty \in c_0 \left[ x_k + y_k \xrightarrow{k \rightarrow \infty} 0 \right], \alpha x \in c_0$$

. Also  $(0) = (0, 0, \dots) \in c_0$ . Also,  $c_l \subset \ell_\infty$ . Indeed, let  $n_1 \in \mathbb{N}$  such that

$$n \geq n_1 \implies |x_n - 0| = |x_k| < 1 \quad (\text{here, } \epsilon = 1)$$



Then for  $h \in \mathbb{N}$ ,

$$|x_k| \leq \max\{|x_1|, \dots, |x_{n_1-1}|, 1\} = M$$

i.e.  $\|x\|_\infty = \sup_{h \in \mathbb{N}} |x_k| \leq M$ .

**Definition 9.2.1 ( $C[a, b]$ )**

Let  $a < b \in \mathbb{R}$ , and

$$C[a, b] = \{f \in \mathbb{R}^{[a, b]} : f \text{ is continuous}\} \quad (9.1)$$

Note that  $C[a, b]$  is a  $\mathbb{R}$ -vector space  $f, g \in C[a, b]$ ,  $\alpha \in \mathbb{R}$ , define  $f + g, \alpha f \in \mathbb{R}^{[a, b]}$  by

$$(f + g)(t) = f(t) + g(t), (\alpha f)(t) = \alpha f(t) \quad (9.2)$$

for all  $t \in [a, b]$

**Theorem 9.2.1 (Extreme Value Theorem)**

if  $f \in C[a, b]$  then there exists  $t_{\min}, t_{\max} \in [a, b]$  for which

$$f(t_{\min}) \leq f(t) \leq f(t_{\max}) \quad \text{for all } t \in [a, b] \quad (9.3)$$

Consequently from the [Theorem 9.2.1](#), if  $f \in C[a, b]$ ,  $|f(\cdot)| \in C[a, b]$  and there is  $t_{\max} \in [a, b]$  for which  $|f(t)| \leq |f(t_{\max})|$  for  $t \in [a, b]$ . Define, for  $f \in C[a, b]$ ,  $\|f\|_\infty = \max_{t \in [a, b]} |f(t)|$ .

Just like for  $(\ell_\infty, \|\cdot\|_\infty)$ , we have that  $(C[a, b], \|\cdot\|_\infty)$  is a normed vector space.

We note that  $\|\cdot\|_\infty$  is not the only norm on  $C[a, b]$ . Let  $1 \leq p < \infty$  and let, for  $f \in C[a, b]$

$$\|f\|_p = \left( \int_a^b |f(t)|^p dt \right)^{\frac{1}{p}} \quad (\text{good ol' Riemann integral}) \quad (9.4)$$

**Theorem 9.2.2 ( $(C[a, b], \|\cdot\|_p)$  as a normed vector space)**

$(C[a, b], \|\cdot\|_p)$ ,  $(1 \leq p < \infty)$  is a normed vector space.

**Proof**

First, let us recall right endpoint Riemann sums:  $f, g \in C[a, b]$ , then

$$\int_a^b g(t) dt = \lim_{n \rightarrow \infty} \sum_{k=1}^n g\left(a + \frac{k}{n}(b-a)\right) \frac{b-a}{n} \quad (9.5)$$

Hence if  $f \in C[a, b]$ , then

$$\begin{aligned} \|f\|_p &= \left( \lim_{n \rightarrow \infty} \sum_{k=1}^n |f(b_k)|^p \frac{b-a}{n} \right) \quad \text{where } b_k = a + \frac{k}{n}(b-a) \\ &= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n |f(b_k)|^p \right)^{\frac{1}{p}} \left( \frac{b-a}{n} \right)^{\frac{1}{p}} \end{aligned}$$

Now, suppose,  $f, g \in C[a, b]$

$$\begin{aligned} \|f+g\|_p &= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n |f(b_k) + g(b_k)|^p \right)^{\frac{1}{p}} \left( \frac{b-a}{n} \right)^{\frac{1}{p}} \\ &\leq \lim_{n \rightarrow \infty} \left[ \left( \sum_{k=1}^n |f(b_k)|^p \right)^{\frac{1}{p}} + \left( \sum_{k=1}^n |g(b_k)|^p \right)^{\frac{1}{p}} \right] \left( \frac{b-a}{n} \right)^{\frac{1}{p}} \quad \text{Theorem 8.2.1} \\ &= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n |f(b_k)|^p \right)^{\frac{1}{p}} \left( \frac{b-a}{n} \right)^{\frac{1}{p}} + \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n |g(b_k)|^p \right)^{\frac{1}{p}} \left( \frac{b-a}{n} \right)^{\frac{1}{p}} \\ &= \|f\|_p + \|g\|_p \end{aligned}$$

hence we have subadditivity of  $\|\cdot\|_p$ . It is routine to verify that for  $\alpha \in \mathbb{R}$ ,  $f \in C[a, b]$  we have

$$\|\alpha f\|_p = |\alpha| \|f\|_p \quad (9.6)$$

and  $\|f\|_p \geq 0$  as  $|f(\cdot)|^p \geq 0$  and finally

$$\|f\|_p = 0 \iff \int_a^b |f(t)|^p dx = 0 \stackrel{(1)}{\iff} |f(t)|^p = 0 \text{ for all } t \in [a, b] \iff f = 0 \quad (9.7)$$

((1) as  $|f(t)|^p \geq 0$  for all  $t$ ).

**Note (Summary thus far about Normed Vector Spaces)**

$$\begin{aligned} &(\mathbb{R}, |\cdot|) \\ &(\mathbb{R}^{\mathbb{N}}, \|\cdot\|_p), \quad 1 \leq p < \infty \\ &(\ell_p, \|\cdot\|_p), \quad 1 \leq p < \infty \\ &(c_0, \|\cdot\|_{\infty}) \\ &(C[a, b], \|\cdot\|_p), \quad 1 \leq p < \infty \end{aligned}$$

### 9.3 Topology of metric spaces

#### Definition 9.3.1 (Open and Closed Balls (It's Balls AGAIN!!))

Let  $(X, d)$  be a metric space,  $x_0 \in X$ , and  $\epsilon > 0$ . We define

- (open ball)  $B(x_0, \epsilon) = \{x \in X : d(x_0, x) < \epsilon\}$
- (closed ball)  $B[x_0, \epsilon] = \{x \in X : d(x_0, x) \leq \epsilon\}$

#### Example 9.3.1

In  $\mathbb{R}$  we have for  $a < b$

$$(a, b) = B\left(\frac{1}{2}(a+b), \frac{1}{2}(b-a)\right)$$

$$[a, b] = B\left[\frac{1}{2}(a+b), \frac{1}{2}(b-a)\right]$$

#### Definition 9.3.2 (Open and Closed Sets)

Let  $X, d$  be a metric space.

- A set  $U \subseteq X$  is open if

$$\forall x \in U \exists \epsilon_x > 0 B(x, \epsilon_x) \subseteq U \quad (9.8)$$

- A set  $F \subseteq X$  is closed if  $X \setminus F$  is open.

#### Proposition 9.3.1 (Open/Closed Balls are Open/Closed Sets)

Let  $(X, d), x_0, \epsilon$  as above.

1.  $B(x_0, \epsilon)$  is open.
2.  $B[x_0, \epsilon]$  is closed.

#### Proof

1. Let  $x \in B(x_0, \epsilon)$ . Let  $\epsilon_x = \epsilon - d(x_0, x) > 0$ . Then for  $y \in B(x, \epsilon_x)$  and we have

$$\begin{aligned} d(x_0, y) &\leq d(x_0, x) + d(y, x) < d(x_0, x) + \epsilon_x \\ &= d(x_0, x) + \epsilon - d(x_0, x) = \epsilon \end{aligned}$$

So  $y \in B(x_0, \epsilon)$ , i.e.  $B(x, \epsilon_x) \subseteq B(x_0, \epsilon)$ .

2. Let  $x \in X \setminus B[x_0, \epsilon]$ , and let  $\epsilon_x = d(x, x_0) - \epsilon > 0$ . Now if  $y \in B(x, \epsilon_x)$  then

$$\begin{aligned} d(x, x_0) &\leq d(x, y) + d(y, x_0) \\ &< \epsilon_x + d(y, x_0) \\ &= d(x, x_0) - \epsilon + d(y, x_0) \end{aligned}$$

$$\implies \epsilon < d(y, x_0), \text{ i.e. } y \notin B[x_0, \epsilon], \text{ i.e. } y \in X \setminus B[x_0, \epsilon], \text{ so } B(x, \epsilon_x) \subseteq X \setminus B[x_0, \epsilon].$$

**Remark**

We may let

$$B[x_0, 0] = \{x \in X : d(x_0, x) \leq 0\} = \{x_0\} \tag{9.9}$$

As above, singleton sets  $\{x_0\}$  are closed.

# Chapter 10

## Lecture 10: Sep 27, 2017

### 10.1 Continuing with Balls

**Note (Recall)**

$(X, d)$  be a metric space,  $x_0 \in X$ ,  $\epsilon > 0$

$$B(x_0, \epsilon) = \{x \in X : d(x_0, x) < \epsilon\}$$

$$B[x_0, \epsilon] = \{x \in X : d(x_0, x) \leq \epsilon\}$$

**Example 10.1.1**

1.  $X \neq \emptyset$ ,  $|X| \geq 2$ , the discrete metric

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

We have for  $x_0 \in X$ ,

$$B(x_0, \epsilon) = \begin{cases} \{x_0\} & 0 < \epsilon \leq 1 \\ X & \epsilon > 1 \end{cases}$$

$$B[x_0, \epsilon] = \begin{cases} \{x_0\} & 0 < \epsilon < 1 \\ X & \epsilon \geq 1 \end{cases}$$

2. (Geometry of balls in  $\mathbb{R}^2$ )

$$1 \leq p < \infty, B_p(0, 1) = \{x \in \mathbb{R}^2, d_p(0, x) = \|x\|_p < 1\}$$

*Pictures*

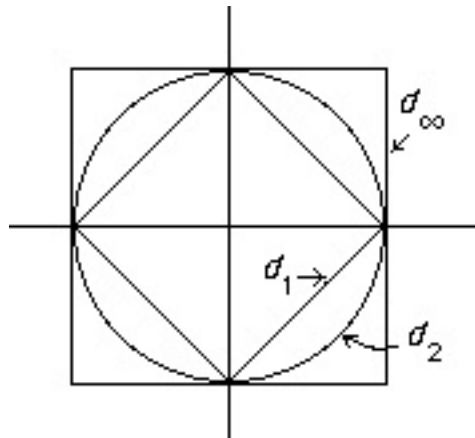
$B_1(0, 1) : x_1 + x_2 = 1$  is a diamond-shaped “ball”

$B_2(0, 1)$  is a round “ball”

$B_\infty(0, 1)$  is a squared “ball”

$B_p(0, 1) \ 1 < p < 2$  the “ball” is inscribed inside the circle

$B_p(0, 1) \ 2 < p < \infty$  : circle is inscribed within (a square with rounded corners)



**Proposition 10.1.1**

Let  $(X, d)$  be a metric space.

1.  $X, \emptyset$  are both open and closed.
2. If  $\{U_i\}_{i \in I}$  is a family of open sets, then

$$\bigcup_{i \in I} U_i \text{ is open} \quad (10.1)$$

3. If  $\{U_1, \dots, U_n\}$  is a finite family of open sets, then

$$\bigcap_{i=1}^n U_i \text{ is open} \quad (10.2)$$

4. If  $\{F_i\}_{i \in I}$  is a family of closed sets, then

$$\bigcap_{i \in I} F_i \text{ is closed} \quad (10.3)$$

5. Of  $\{F_1, \dots, F_n\}$  is a finite family of closed sets, then

$$\bigcup_{i=1}^n F_i \text{ is closed} \quad (10.4)$$

[Recall that singleton sets are closed, hence (5) implies that finite sets are closed]

**Proof**

1. Let  $x \in X$ . Then  $x \in B(x, 1) \subseteq X$ , so  $X$  is open. The test for openness of  $\emptyset$  is vacuously true (i.e. there are no points to speak of: there are no  $x \in \emptyset$  at all, hence for any such  $x$ , we have  $x$  is “contained” in a ball in  $\emptyset$ ).

We have  $\emptyset = X \setminus X$ ,  $X = X \setminus \emptyset$  are closed.

2. Let  $x \in U = \bigcup_{i \in I} U_i$ . Then there is some  $i_0 \in I$  so  $x \in U_{i_0}$ , which is open, so there is an  $\epsilon_x > 0$  such that

$$x \in B(x, \epsilon_x) \subseteq U_{i_0} \subseteq U \quad (10.5)$$

3. Let  $x \in V = \bigcap_{i=1}^n U_i$ . Then for each  $i = 1, \dots, n$ , there is  $\epsilon_i > 0$  so  $B(x, \epsilon_i) \subseteq U_i$ . Let  $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\} > 0$  and  $B(x, \epsilon) \subseteq \bigcap_{i=1}^n B(x, \epsilon_i) \subseteq V$

For (4) and (5), use De Morgan’s Laws and (2) & (3) from above.

**Definition 10.1.1 (Boundary)**

Given a metric space  $(X, d)$ ,  $A \subseteq X$ , we define the boundary of  $A$  as

$$\partial A = \{x \in X : \forall \epsilon > 0 \ B(x, \epsilon) \cap A \neq \emptyset, \underbrace{B(x, \epsilon) \setminus A}_{B(x, \epsilon) \cap (X \setminus A)} \neq \emptyset\} \quad (10.6)$$

**Remark**

$$\partial A = \partial(X \setminus A)$$

**Definition 10.1.2 (Interior)**

We let the interior of  $A$

$$A^\circ = \bigcup \{U \subseteq X : U \subseteq A \wedge U \text{ is open}\} \quad (10.7)$$

**Proposition 10.1.2 (Characterizations of the Interior)**

If  $(X, d)$ ,  $A$  are as above, then

$$A^\circ = \{x \in X : \exists \epsilon_x > 0 \ B(x, \epsilon_x) \subseteq A\} \quad (10.8)$$

$$= A \setminus \partial A \quad (10.9)$$

**Proof**

Let  $x \in A$ . Then we have either

- for some  $\epsilon_x > 0$ ,  $x \in \underbrace{B(x, \epsilon_x)}_{\text{open}} \subseteq A \implies x \in A^\circ$ ; or
- $\forall \epsilon > 0$ ,  $B(x, \epsilon) \setminus A \neq \emptyset \implies$  since  $x \in A \cap B(x, \epsilon)$ , we have  $x \in \partial A$ . Since  $A^\circ \subseteq A$ , we see that the two equalities in [Equation 10.9](#) coincide.

**Definition 10.1.3 ()**

Let  $(X, d)$  be a metric space,  $(x_n)_{n=1}^\infty \subseteq X$  and  $x_0 \in X$ . Then we say that  $(x_n)_{n=1}^\infty$  converges to the limit  $x_0$ , written

$$x_0 = \lim_{n \rightarrow \infty} x_n \quad (10.10)$$

or

$$x_n \xrightarrow[n \rightarrow \infty]{} x_0 \quad (10.11)$$

if

$$\begin{aligned} \forall \epsilon > 0 \exists n_\epsilon \in \mathbb{N} \\ n \geq n_\epsilon \implies d(x_0, x_n) < \epsilon \end{aligned}$$

**Remark**

The limit, if it exists, is unique. Indeed, since

$$x_0 = \lim_{n \rightarrow \infty} x_n \wedge y_0 = \lim_{n \rightarrow \infty} x_n$$

then

$$\begin{aligned} \forall \epsilon > 0 \exists n_\epsilon, n'_\epsilon \in \mathbb{N} \\ n \geq n_\epsilon \implies d(x_0, x_n) < \frac{\epsilon}{2} \\ n \geq n'_\epsilon \implies d(y_0, x_n) < \frac{\epsilon}{2} \end{aligned}$$

But then if  $n \geq \max\{n_\epsilon, n'_\epsilon\}$  we have

$$d(x_0, y_0) \leq d(x_0, x_n) + d(x_n, y_0) < \epsilon$$

If this holds for all  $\epsilon > 0$ ,  $d(x_0, y_0) = 0$  so  $x_0 = y_0$ .

**Example 10.1.2**

Let  $(V, \|\cdot\|)$  be a normed vector space. A subset  $\{e_n\}_{n=1}^\infty \subseteq V$  is a **Schauder basis** provided that

$$\begin{aligned} \forall x \in V \exists! \{x_n\}_{n=1}^\infty \\ x = \lim_{n \rightarrow \infty} \sum_{k=1}^n x_k e_k \in V \end{aligned}$$



*Example:* In  $\ell_p$  ( $1 \leq p < \infty$ ), let  $e_n = (0, \dots, 0, \underset{n\text{-th place}}{1}, 0, \dots)$

**Definition 10.1.4 (Accumulation points)**

We let  $(X, d)$  is a metric space,  $A \subseteq X$  as above, the set of accumulation points (or cluster points) be given

$$A' = \{x \in X : \forall \epsilon > 0 \ (B(x, \epsilon) \setminus \{x\}) \cap A \neq \emptyset\} \quad (10.12)$$

(aka a punctured ball).

Furthermore, we call elements of  $A \setminus A'$  as isolated points.

**Proposition 10.1.3**

Given  $(X, d)$  as a metric space,  $A \subseteq X$  as above, the set of all accumulation points

$$A' = \{x \in X : x = \lim_{n \rightarrow \infty} x_n, \text{ where } (x_n)_{n=1}^{\infty} \subseteq A \setminus \{x\}\}$$

**Proof**

If  $x \in A'$ , let  $x_1 \in (B(x, 1) \setminus \{x\}) \cap A$ , and inductively let

$$x_{n+1} \in (B(x, \epsilon_n) \setminus \{x\}) \cap A$$

where  $\epsilon + m = \min\{\frac{1}{n}, d(x, x_n)\}$ .

Then we have (exercise) that  $x = \lim_{n \rightarrow \infty} x_n$ , while  $(x_n)_{n=1}^{\infty} \subseteq A \setminus \{x\}$ . [Notice the points  $x_1, x_2, \dots, x_n$  are distinct]

The converse inclusion just uses the definition of limits. □