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Chapter 1

Information from Earlier Chapters

1.1 Real Number System

Definition 1.1.1 (Open and Closed Intervals)
Let a and b be real numbers. A ***closed interval*** *is a set of the form*

$$[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}, \quad [a,\infty) := \{x \in \mathbb{R} : a \leq x\},$$
$$(-\infty,b] := \{x \in \mathbb{R} : x \leq b\}, \quad \text{or } (-\infty,\infty) := \mathbb{R}$$

and an ***open interval*** *is a set of the form*

$$(a,b) := \{x \in \mathbb{R} : a < x < b\}, \quad (a,\infty) := \{x \in \mathbb{R} : a < x\},$$
$$(-\infty,b) := \{x \in \mathbb{R} : x < b\}, \quad \text{or } (-\infty,\infty) := \mathbb{R}$$

Definition 1.1.2 (Degenerate and Non-Degenerate Intervals)
Given the above definition, an interval I with endpoints a, b is called ***degenerate*** *if $a = b$ and* ***non-degenerate*** *if $a < b$.*

A degenerate open interval is the empty set, and a non-degenerate closed interval is a point $a = b$.

1.2 Continuity

Definition 1.2.1 (Continuity)
Let $\emptyset \neq E \subseteq \mathbb{R}$ and $f : E \to R$.

1. *f is said to be* ***continuous*** *at a point $a \in E$ if and only if given $\epsilon > 0$, $\exists \delta > 0$ (which in general depends on ϵ, f and a) such that*

$$|x - a| < \delta \ \wedge \ x \in E \implies |f(x) - f(a)| < \epsilon \tag{1.1}$$

2. *f is said to be* ***continuous*** *on E (notation: $f : E \to \mathbb{R}$ is continuous) if and only if f is continuous at every $x \in E$.*

Chapter 2

Differentiability on $\mathbb{R}$

2.1 The Derivative

Definition 2.1.1 (Differentiable)
A real function f is said to be differentiable at a point $a \in \mathbb{R}$ if and only if f is defined on some open interval I containing a and

$$f'(a) := \lim_{h \to \infty} \frac{f(a+h) - f(a)}{h} \tag{2.1}$$

exists. In this case, $f'(a)$ is called the derivative of f at a.

There are two characterizations of diffrentiability which we shall use to study derivatives. The first one which characterizes the derivatives in terms of the "chord function"

$$F(x) := \frac{f(x) - f(a)}{x - a} \quad x \neq a, \tag{2.2}$$

will be used to establish the Chain Rule.

Theorem 2.1.1 (Differentiability and Continuity)
A real function f is differentiable at some point $a \in \mathbb{R}$ if and only if there exists an open interval I and a function $F : I \to \mathbb{R}$ such that $a \in I, f$ is defined on I, F is continuous at a, and

$$f(x) = F(x)(x - a) + f(a) \tag{2.3}$$

holds for all $x \in I$, in which case $F(a) = f'(a)$.

Proof

*Note that for $x \in I \setminus \{a\}$, Equation 2.2 and Equation 2.3 are equivalent. Suppose f is differentiable at $a \in \mathbb{R}$. By Definition 2.1.1, f is defined on some **open interval** I **containing** a and the limit in Equation 2.1 exists. Define*

$$F(x) = \begin{cases} \frac{f(x)-f(a)}{x-a} & x \neq a \\ f'(a) & x = a \end{cases}$$

Then Equation 2.3 holds for all $x \in I$, F is continuous on a by Equation 2.2 since $f'(a)$ exists.

Conversely, if Equation 2.3 holds, then Equation 2.2 holds for $x \in I \setminus \{a\}$. Taking the limit of Equation 2.2 as $x \to a$, and since F is continuous on a, $F(a) = f'(a)$. Thus by Definition of Differentiability, f is continuous on a.

Theorem 2.1.2

A real function f is differentiable at a if and only if $\exists T(x) := mx$ which is a function, such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - T(h)}{h} = 0 \tag{2.4}$$

Proof

Suppose f is differentiable at a, and let $m := f'(a)$, then

$$\frac{f(a+h) - f(a) - T(h)}{h} = \frac{f(a+h) - f(a)}{h} - f'(a) \to 0$$

as $h \to 0$.

Conversely, suppose Equation 2.4 holds for $T(x) := mx$ and $h \neq 0$. Then

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= m + \frac{f(a+h) - f(a) - mh}{h} \\ &= m + \frac{f(a+h) - f(a) - T(h)}{h} \end{aligned} \tag{2.5}$$

By Equation 2.4, the limit of Equation 2.5 is m. Thus it follows that $(f(a+h) - f(a))/h \to m$ as $h \to 0$; i.e. that $f'(a)$ exists and equals m by Definition of Differentiability, and thus f is differentiable at a.

With Theorem 2.1.1, we will answer a rather interesting question: Are differentiability and continuity related? If so, how?

Theorem 2.1.3 (Differentiability $\implies$ Continuity)

f is differentiable at $a \implies f$ is continuous at a.

Proof

Suppose that f is differentiable at a. By Theorem 2.1.1, there is an open interval I and a function F, that is continuous at a, such that

$$\forall x \in I \;\; f(x) = F(x)(x - a) + f(a). \tag{2.6}$$

So taking the limit of Equation 2.6 as $x \to a$, we observe that

$$\lim_{x \to a} f(x) = F(a) \cdot 0 + f(a) = f(a).$$

In particular, $f(x) \to f(a)$ as $x \to a$, which by Definition of Continuity, f is continuous at a.