PMATH467 — Algebraic Geometry

Classnotes for Winter 2019

by

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Preface

The basic goal of the course is to be able to find **algebraic invariants**, which we shall use to classify topological spaces up to homeomorphism.

Other questions that we shall also look into include a uniqueness problem about manifolds; in particular, how many manifolds exist for a given invariant up to homeomorphism? We shall see that for a **2-manifold**, the only such manifold is the 2-dimensional sphere S^2. For a 4-manifold, it is the 4-dimensional sphere S^4. In fact, for any other n-manifold for $n > 4$, the unique manifold is the respective n-sphere. The problem is trickier with the 3-manifold, and it is known as the Poincaré Conjecture, solved in 2003 by Russian Mathematician Grigori Perelman. Indeed, the said manifold is homeomorphic to the 3-sphere.

For this course, you are expected to be familiar with notions from **real analysis**, such as topology, and concepts from **group theory**. Due to the structure of which this course is designed, each lecture may be much longer than reality, as I am also making heavy references to the recommended text that is Lee's Introduction to Topological Manifolds [1].

[1] Lee, J. M. (2000). *Graduate Texts in Mathematics: Introduction to Topological Manifolds*. Springer

The following topics shall be covered:

1. Point-Set Topology
2. Introduction to Topological Manifolds
3. Simplicial complexes & Introduction to Homology
4. Fundamental Groups & Covering Spaces

Feb 12th I have decided to delve deeper into the recommended text as the organization of the course demands. As so, changes might be made to earlier lectures as I go further down, so as to introduce the definitions and provide propositions at a timing deemed appropriate. To keep track of the changes, please look for **PMATH467** among the commits on `https://gitlab.com/japorized/TeX_notes/commits/master` using the provided filter. If you are unfamiliar with version controlling and writing in LaTeX, once you have found the lecture that you wish to compare, expand the **diff** for `classnotes.tex` if it is collapsed.

Basic Logistics for the Course

I shall leave this here for my own notes, in case something happens to my hard copy.

- OH: (Tue) 1630 - 1800, (Fri) 1245 - 1320
- OR: MC 6457
- EM: aaleyasin

Part I

Point-Set Topology

1 Lecture 1 Jan 07th

We will not be too rigorous in this part.

1.1 Euclidean Space

For any $(x_1, \ldots, x_m) \in \mathbb{R}^m$, we can measure its distance from the origin 0 using either

- $\|x\|_\infty = \max\{|x_i|\}$ (the supremum-norm);
- $\|x\|_2 = \sqrt{\sum (x_j)^2}$ (the 2-norm); or
- $\|x\|_p = \left(\sum |x_j|^p\right)^{\frac{1}{p}}$ (the p-norm),

where we may define a "distance" by

$$d_p(x, y) = \|x - y\|_p .$$

Definition 1 (Metric)

Let X be an arbitrary space. A function $d : X \times X \to \mathbb{R}$ is called a ***metric*** *if it satisfies*

1. *(symmetry) $d(x, y) = d(y, x)$ for any $x, y \in X$;*
2. *(positive definiteness) $d(x, y) \geq 0$ for any $x, y \in X$, and $d(x, y) = 0 \iff x = y$; and*
3. *(triangle inequality) $\forall x, y, z \in X$*

$$d(x, y) \leq d(x, z) + d(y, z).$$

Definition 2 (Open and Closed Sets)

Given a space X with a metric d, and $r > 0$, the set

$$B(x, r) := \{w \in X \mid d(x, w) < r\}$$

is called the ***open ball*** *of radius r centered at x. An* ***open set*** *A is such that $\forall a \in A$, $\exists r > 0$ such that*

$$B(a, r) \subseteq A.$$

We say that a set is ***closed*** *if its complement is open.*

Definition 3 (Continuous Map)

A function

$$f : (X, d_1) \to (Y, d_2)$$

is said to be ***continuous*** *if the preimage of an open set in Y is open in X.*

See notes on Real Analysis for why we defined a continuous map in such a way.

Warning

This definition does not imply that a continuous map f maps open sets to open sets.

Definition 4 (Open and Closed Maps)

A mapping $f : X \to Y$ is said to be ***open*** *if for all open $U \subset X$, $f(U)$ is open. We say that f is a* ***closed map*** *if for all closed $F \subset X$, $f(F)$ is closed.*

Exercise 1.1.2

Contruct a function on $[0, 1]$ which assumes all values between its maximum and minimum, but is not continuous.

Exercise 1.1.1

Suppose $f : X \to Y$ is a bijective continuous map. Then TFAE.

1. *f is a homeomorphism.*
2. *f is open.*
3. *f is closed.*

Solution

Consider the piecewise function

$$f(x) = \begin{cases} 2x & 0 \leq x < \frac{1}{2} \\ x - \frac{1}{2} & x \geq \frac{1}{2}. \end{cases}$$

It is clear that the maximum and minimum are 1 and 0 respectively, and f assumes all values between 0 and 1. However, the function is clearly not continuous, particularly at $\frac{1}{2}$.

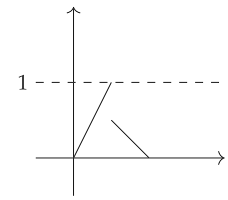


Figure 1.1: Function that assumes all values between its extremas, but is not continuous.

Definition 5 (Homeomorphism)

*A function f is a **homeomorphism** if it is a bijection and both f and f^{-1} are continuous.*

Example 1.1.1

The function

$$f : [0, 2\pi) \to \mathbb{R}^2 \text{ given by } \theta \mapsto (\cos\theta, \sin\theta)$$

is not homeomorphic, since if we consider an alternating series that converges to 0 on the unit circle on $\mathbb{R}^2$, we have that the preimage of the series does not converge and f^{-1} is in fact discontinuous.

Now, we want to talk about topologies without referring to a metric.

Definition 6 (Topology)

*Let X be a space. We say that the set $\mathcal{T} \subseteq \mathcal{P}(X)$ is a **topology** if*

1. $X, \emptyset \in \mathcal{T}$;
2. *if* $\{x_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ *for an arbitrary index set* A, *then*

$$\bigcup_{\alpha \in A} x_\alpha \in \mathcal{T}; \text{ and}$$

3. *If* $\{x_\beta\}_{\beta \in B} \subset \mathcal{T}$ *for some finite index set* B*, then*

$$\bigcap_{\beta \in B} x_\beta \in \mathcal{T}.$$

2 Lecture 2 Jan 09th

2.1 Euclidean Space (Continued)

In the last lecture, from metric topology, we generalized the notion to a more abstract one that is based solely on open sets.

Example 2.1.1

Let X be a set. The following two are uninteresting examples of topologies:

1. The **trivial topology** $\mathcal{T} = \{\emptyset, X\}$.
2. The **discrete topology** $\mathcal{T} = \mathcal{P}(X)$.

WE SHALL NOW continue with looking at more concepts that we shall need down the road.

Definition 7 (Closure of a Set)

Let A be a set. Its ***closure****, denoted as $\overline{A}$, is defined as*

$$\overline{A} = \bigcap_{C \supset A}^{C\text{: closed}} C.$$

It is the smallest closed set that contains A.

Note 2.1.1

In metric topology, one typically defines the closure of a set by taking the union of A and its limit points.

Definition 8 (Interior of a Set)

Let A be a set. Its ***interior****, denoted either as Int (A), A° or* $\mathrm{Int}(A)$*, is defined as*

$$\mathrm{Int}(A) = \bigcup_{G \subseteq A}^{G\text{: open}} G.$$

Definition 9 (Boundary of a Set)

Let A be a set. Its ***boundary****, denoted as ∂A, is defined as*

$$\partial A = \overline{A} \setminus \mathrm{Int}(A).$$

Exercise 2.1.1

Let A be a set. Prove that ∂A is closed.

Proof

Notice that

$$(\partial A)^C = (\overline{A} \setminus \mathrm{Int}(A))^C = X \setminus \overline{A} \cup \mathrm{Int}(A) = X \cap \overline{A}^C \cup \mathrm{Int}(A)$$

which is open. $\square$

Exercise 2.1.2

Let A be a set. Show that

$$\partial(\partial A) = \partial A.$$

Proof

First, notice that $\operatorname{Int}(\partial A) = \emptyset$. Since ∂A is closed, $\overline{\partial A} = \partial A$. Then

$$\partial(\partial A) = \overline{\partial A} \setminus \operatorname{Int}(\partial A) = \partial A \setminus \emptyset = \partial A \qquad \square$$

Example 2.1.2

We know that $\mathbb{Q} \subseteq \mathbb{R}$, and $\overline{\mathbb{Q}} = \mathbb{R}$. We say that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 10 (Dense)

We say that a subset A of a set X is dense if

$$\overline{A} = X.$$

Example 2.1.3

From the last example, we have that $\operatorname{Int}(\mathbb{Q}) = \emptyset$.

Definition 11 (Limit Point)

We say that $p \in X \supseteq A$ is a limit point of A if any neighbourhood of p has a nontrivial intersection with A.

Example 2.1.4 (A Topologist's Circle)

Consider the function

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

on the interval $\left[-\frac{1}{2\pi}, \frac{1}{2\pi}\right]$. Extend the function on both ends such that we obtain Figure 2.1 (See also: Desmos).

The limit points of the graph includes all the points on the straight line from $(0, -1)$ to $(0, 1)$, including the endpoints. This is the case

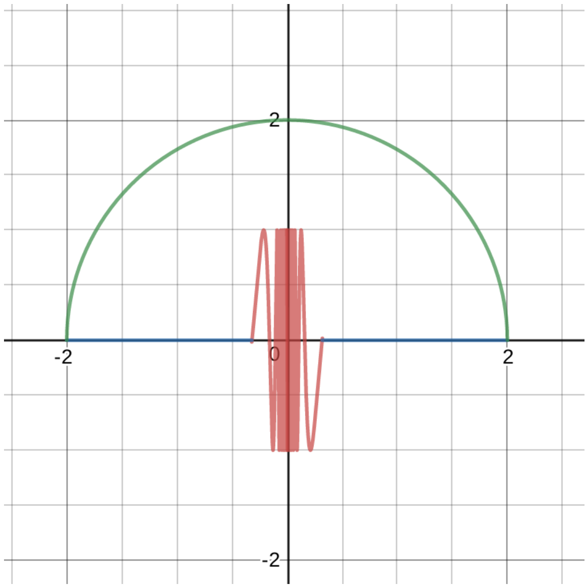


Figure 2.1: A Topologist's Circle

because for any of the points on this line, for any neighbourhood around the point, the neighbourhood intersects the graph f infinitely many times.

GOING BACK TO CONTINUITY, given a function f, how do we know if f^{-1} maps an open set to an open set?

We can actually reduce the problem to only looking at open balls. But why are we allowed to do that?

Definition 12 (Basis of a Topology)

Given a topology $\mathcal{T}$, we say that $\mathcal{B} = \{B_\alpha\}_{\alpha \in I}$ is a ***basis*** *if $\forall T \in \mathcal{T}$, there exists $J \subset I$ such that*

$$T = \bigcup_{\alpha \in J} B_\alpha.$$

Note that while the definition is similar to that of a cover, we are now "covering" over sets and not points.

Example 2.1.5

Let $\mathcal{T}$ be the Euclidean topology on $\mathbb{R}$. Then we can take

$$\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R},\ a \leq b\}.$$

Note that $\mathcal{B}$ is **uncountable**. We can, in fact, have [1]

$$\mathcal{B}_1 = \{(a,b) \mid a,b \in \mathbb{Q}, a \leq b\},$$

which is countable, as a basis for $\mathbb{R}$. Furthermore, we can consider the set

$$\mathcal{B}_2 = \left\{(a,b) \;\middle|\; a \leq b,\, a = \frac{m}{2^p},\, b = \frac{n}{2^q},\, m,n,p,q \in \mathbb{Z}\right\},$$

which is also a countable basis for $\mathbb{R}$. Notice that

$$\mathcal{B}_2 \subseteq \mathcal{B}_1 \subseteq \mathcal{B}.$$

[1] Recall from PMATH 351 that we can write $\mathbb{R}$ as a disjoint union of open intervals with rational endpoints.

Example 2.1.6

In $\mathbb{R}^2$, we can do a similar construction of $\mathcal{B}$, $\mathcal{B}_1$, and $\mathcal{B}_2$ as in the last example and use them as a basis for $\mathbb{R}^2$. In particular, we would have

$$\mathcal{B} = \{(a_1,b_1) \times (a_2,b_2) \mid a_1,a_2,b_1,b_2 \in \mathbb{R}\}.$$

This is called a **dyadic partitioning** of $\mathbb{R}^2$.

Example 2.1.7

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be two topological spaces. Then the Cartesian product $X_1 \times X_2$ has topology induced from $\mathcal{T}_1$ and $\mathcal{T}_2$ by taking the set

$$\mathcal{B} = \{\beta_1 \times \beta_2 \mid \beta_1 \in \mathcal{T}_1,\, \beta_2 \in \mathcal{T}_2\}$$

as the basis.

Exercise 2.1.3

Prove that

1. *β_1 and β_2 can be taken to be elements of bases $\mathcal{B}_1 \subset \mathcal{T}_1$ and $\mathcal{B}_2 \subset \mathcal{T}_2$, respectively.*
2. *the product topology on $\mathbb{R}^2$ is the same as the Euclidean topology.*

3 Lecture 3 Jan 11th

3.1 Euclidean Space (Continued 2)

Let $\tilde{X}$ be a metric space, and $p, q \in \tilde{X}$ with $p \neq q$. Then we have that $d(p,q) = r > 0$.

The student is recommended to do a quick review for the first 3 chapters of the recommended text.

Then we must have that

$$B\left(p, \frac{r}{3}\right) \cap B\left(q, \frac{r}{3}\right) = \emptyset.$$

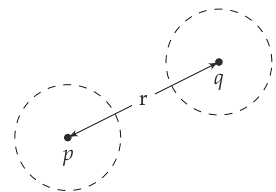


Figure 3.1: Idea of separation

Exercise 3.1.1

Prove that the above claim is true. (Use the triangle inequality)

Proof

Suppose $\exists x \in B\left(p, \frac{r}{3}\right) \cap B\left(q, \frac{r}{3}\right)$. Then

$$d(p,x) + d(q,x) < \frac{2r}{3} < r = d(p,q),$$

which violates the triangle inequality. □

We observe here that the two open sets (or balls) "separate" p and q.

Definition 13 (Hausdorff / T_2)

*Let X be a topological space. X is said to be **Hausdorff** or T_2 iff any 2 distinct points can be separated by disjoint open sets.*

Note 3.1.1

1. *The Hausdorff space (or T_2 space) is an important space; we can only define a metric on spaces that are T_2.*
2. *A space is called T_1 is for any $p, q \in X$ with $p \neq q$, $\exists U \ni p$ open such that $q \notin U$ and $\exists V \ni q$ open such that $p \notin v$. It is worth noting that a T_2 space is also T_1.*

Example 3.1.1 (The Discrete Topology)

Suppose X is a metric space. For any $x \in X$, we have that $\{x\}$ is open. Thus for any $x_1, x_2 \in X$, if $x_1 \neq x_2$, then the open sets $\{x_1\}$ and $\{x_2\}$ separates x_1 and x_2.

This is true as we can define the following metric on the space: let $d : X \times X \to \mathbb{R}$ such that

$$d(x_1, x_2) = \begin{cases} 0 & x_1 = x_2 \\ 1 & x_1 \neq x_2 \end{cases}$$

This topology is called a **discrete topology**, and it is a metric space.

Let X be a metric space and $A \subseteq X$. Then there is a metric induced by X on A, and this in turn induces a topology on A.

More generally, if $A \subset X$ where X is some arbitrary topological space, then a set $U \subseteq A$ is open iff $U = A \cap V$ for some $V \subseteq X$ that is open. In other words, a subset U of A is said to be open iff we can find an open set V in X such that the intersection of A and V gives us U.

Exercise 3.1.2

Prove that the construction above gives us a topology.

Proof

Let $A \subseteq X$. We shall show that τ_A is a topological space induced by the topological space τ of X. It is clear that $\emptyset \in \tau_A$, since it is open in X, and so $A \cap \emptyset = \emptyset$. Since X is open, we have $A \cap X = A$, and so $A \in \tau_A$.

Now if $\{U_\alpha\}_{\alpha \in I} \subseteq \tau_A$, then $\exists V_\alpha \subseteq X$ such that $U_\alpha = A \cap V_\alpha$. Then

$$\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} A \cap V_\alpha = A \cap \bigcup_{\alpha \in I} V_\alpha, \qquad \square$$

and $\bigcup_{\alpha \in I} V_\alpha$ is open in X by the properties of open sets. Thus $\bigcup_{\alpha \in I} U_\alpha \in \tau_A$.

If $\{U_i\}_{i=1}^n \subset \tau_A$, then again, by the properties of open sets, finite intersection of open sets is open, and so $\bigcap_{i=1}^n U_i \in \tau_A$.

Note 3.1.2

We can say the same can be said about closed sets of A.

Example 3.1.2

Let $A \subseteq X$ and consider the function

$$\iota_A : A \to X \text{ given by } x \mapsto x,$$

which is the inclusion map.

Then ι_A is continuous when the topology on A is chosen to be the induced subspace topology. This is rather clear; notice that the inverse of the inclusion map brings open sets to open sets.

Let Y be an arbitrary topological space. Then let

$$\begin{tikzcd} & X \\ Y \arrow[ur, "\iota_A \circ f"] \arrow[r, "f"'] & A \arrow[u, "\iota_A"'] \end{tikzcd}$$

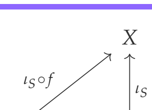
Figure 3.2: Composition of a function and the inclusion map

where f is continuous. Then $\iota_A \circ f$ is continuous.

The converse is also true: if $\iota_A \circ f$ is continuous, then f is continuous. However, we will not prove this. This property is known as the

Theorem 1 (Characteristic Property of the Subspace Topology)

Suppose X is a topological space and $S \subseteq X$ is a subspace. For any topological space T, a map $f : Y \to S$ is continuous iff the composite map $\iota_S \circ f : Y \to X$ is continuous.



characteristic property of the subspace topology.

Lemma 2 (Restriction of a Continuous Map is Continuous)

Let $X \xrightarrow{f} Y$ be continuous, and $A \subseteq X$ [1]. Then

$$f \restriction_A : A \to Y$$

is also continuous.

[1] Here, A is equipped with the subspace topology

Proposition 3 (Other Properties of the Subspace Topology)

Suppose S is a subspace of the topological space X.

1. *If $R \subseteq S$ is a subspace of S, then R is a subspace of X; i.e. the subspace topologies that R inherits from S and from X agree.*
2. *If $\mathcal{B}$ is a basis for the topology of X, then*

$$\mathcal{B}_S = \{B \cap S : B \in \mathcal{B}\}$$

   is a basis for the topology of S.
3. *If $\{p_i\}$ is a sequence of points in S and $p \in S$, then p_i converges to p in S iff p_i converges to p in X.*
4. *Every subspace of a Hausdorff space is Hausdorff.*

3.2 Connected Spaces

Consider the real line $\mathbb{R}$, and consider two disjoint intervals on $\mathbb{R}$.

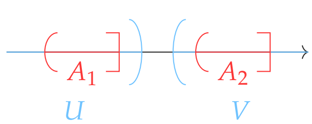


Figure 3.4: Motivation for Connectedness

Observe that we may find two open subsets U and V of $\mathbb{R}$ such that $A_1 \subseteq U$ and $A_2 \subseteq V$, which effectively separates the two intervals on the space $\mathbb{R}$.

Definition 14 (Disconnectedness)

A space X is said to be ***disconnected*** *iff X can be written as a disjoint union*

$$X = A_1 \coprod A_2$$

where $A_1, A_2 \subseteq X$, $A_1 = A_2^C$, that they are both non-empty and open [2].

[2] It goes without saying that the two sets are also simultaneously closed.

Definition 15 (Connectedness)

A space X is said to be ***connected*** *if it is not disconnected.*

Note 3.2.1

By the above definitions, we have that X is connected iff for any partition $X = A \coprod A^C$ with A being open, either A is $\emptyset$ or A is X.

Example 3.2.1

The space $\mathbb{R} \setminus \{0\}$ is disconnected; our disjoint sets are $(-\infty, 0)$ and $(0, \infty)$.

However, $\mathbb{R}^2 \setminus \{0\}$ is connected, but it is not easy to describe why.

Definition 16 (Path)

A ***path*** *in a space X from p to q (both in X) is a continuous map $f : [0,1] \to X$ such that $f(0) = p$ and $f(1) = q$. We say that X is* ***path connected*** *if $\forall p, q \in X$, there is a path in X from p to q.*

Lemma 4 (Path Connectedness implies Connectedness)

If a space X is path connected, then it is connected.

Theorem 5 (From Connected Space to Connected Space)

If $X \xrightarrow{f} Y$ is continuous and X is connected, then $\mathrm{Img}(f)$ is connected.

4 Lecture 4 Jan 14th

4.1 Connected Spaces (Continued)

Definition 17 (Locally Connected)

*We say that X is **locally connected** at x if for every open set V containing x there exists a connected, open set U with $x \in U \subseteq V$. We say that the space X is **locally connected** if it is locally connected $\forall x \in X$.*

Example 4.1.1

The space S generated by the function $\sin \frac{1}{x}$ with 0 at $x = 0$, on the $\mathbb{R}^2$ is not locally connected: consider $(0, y) \in S, y \neq 0$. Then any small open ball at this point will contain infinitely many line segments from S. This cannot be connected, as each one of these constitutes a component, within the neighborhood.

Definition 18 (Connected Component)

*The maximal connected subsets of any topological space X are called **connected components** of the space. The components form a partition of the space.*

4.2 Compactness

Definition 19 (Sequential Compactness)

For $A \subseteq X$, where X is a topological space, if $\{x_i\}_{i \in I} \subseteq A$, arbitrary sequence in A, has a convergent subsequence, we say that A is ***sequentially compact****.*

Definition 20 (Compactness)

We say that a topological space X is ***compact*** *if every open cover of X has a finite subcover.*

Lemma 6 (Compactness implies Sequential Compactness)

Compactness implies sequential compactness.

Example 4.2.1

$[0,1)$ is not compact: consider the open cover $\left\{ \left[0, 1 - \frac{1}{n}\right) \right\}_{n \in \mathbb{N}}$, which contains $[0,1)$ as $n \to \infty$. But whenever n is finite, we have $1 - \frac{1}{n} < 1$, and so any finite collection of the $\left[0, \frac{1}{n}\right)$ is not a cover of $[0,1)$.

Theorem 7 (Continuous Maps map Compact Sets to Compact Images)

Let $f : X \to Y$ be continuous, where X is compact. Then $f(X)$ is compact.

Proof

Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $f(X)$. Since f is continuous, we have that $f^{-1}(U_\alpha)$ is open for each $\alpha \in I$. Since f is bijective between the image set and its domain, we have that $\{f^{-1}(U_\alpha)\}_{\alpha \in I}$ is an open cover of X. Since X is compact, this cover has a finite subcover, say $\{f^{-1}(U_i)\}_{i=1}^n$. Thus

$$X = \bigcup_{i=1}^n f^{-1}(U_i).$$

Thus

$$f(X) = f\left(\bigcup_{i=1}^n f^{-1}(U_i)\right) = \bigcup_{i=1}^n U_i.$$

Hence $\{U_\alpha\}_{\alpha \in I}$ has a finite subcover and so $f(X)$ is compact. □

Corollary 8 (Homeomorphic Maps map Compact Sets to Compact Sets)

Let $X \xrightarrow{f} Y$ be a homeomorphism. Then X is compact iff Y is compact.

Note 4.2.1

Compactness is a topological property.

Lemma 9 (Properties of Compact Sets)

1. *A closed subset of a compact space is compact.*
2. *A compact subset of a topological space is closed provided that the space is Hausdorff.*
3. *In a metric space, a compact set is bounded.*
4. *Finite (Cartesian) product of compact sets is compact.*

The proof for the first item is simple: consider an open cover of the closed subset, and union them with the complement of the closed subset. This covers the entire space, and so it has a finite subcover.

We just need to then remove that complement set, and that would be a finite subcover for the closed subset.

Example 4.2.2

The subset $[-a, b]$, $a, b \in \mathbb{R}$, is compact.

Example 4.2.3

$[0,1]^{\mathbb{N}}$ is not compact: the space is equivalent to ℓ_∞.

Theorem 10 (Heine-Borel)

Let $X \subseteq \mathbb{R}^n$. Then X is compact iff X is closed and bounded.

Proof

$(\implies)$ We say that compactness impliess boundedness. Also, since $\mathbb{R}^n$ is Hausdorff, X is closed.[1].

$(\impliedby)$ X is bounded implies that $X \subseteq [-R, R]^n$ with R sufficiently large. Since X is closed, and $[-R, R]^n$ is compact, X is necessarily compact by Lemma 9. $\square$

[1] Both from Lemma 9

Theorem 11 (Bolzano-Weierstrass)

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Exercise 4.2.1

Prove Theorem 11 as an exercise.

We shall start the next part this lecture.

4.3 Manifolds

Definition 21 (Locally Homeomorphic)

A space is said to be ***locally homeomorphic*** *to* $\mathbb{R}^n$ *provided that* $\forall x \in X, \exists U \ni x$ *open such that* U *is homeomorphic to* $\mathbb{R}^n$*.*

Definition 22 (Manifold)

An n*-dimensional* ***manifold*** *is a* *second countable* [2], *Hausdorff* *topological space that is* *locally homeomorphic* *to* $\mathbb{R}^n$*.*

[2] a topological space is said to be **second countable** if its basis is countable. Note that for a second countable set X, every open cover of X has a countable subcover (see pg 32 of Lee (2000)).

(Show that X is a manifold)
To show that a space is an n*-dimensional manifold, check that the space is*

1. *Hausdorff;*
2. *locally Euclidean; and*
3. *locally homeomorphic to* $\mathbb{R}^n$*.*

This section covers some tools that helps make the process easier.

Note 4.3.1

We may also call the last criterion of being a manifold, that is, to be locally homeomorphic to $\mathbb{R}^n$*, as to be* ***locally Euclidean*** *of dimension* n*.*

Note 4.3.2

One can give an equivalent definition of locally homeomorphic by requiring that U *be homeomorphic to an open ball* $B^n \subseteq \mathbb{R}^n$*. Notice that* B^n *is homeomorphic to* $\mathbb{R}^n$ [3]

[3] By scaling, really.

The following is a quick fact about second countable spaces, which will be helpful when we start creating subspaces from manifolds.

Proposition 12 (Subspaces of Second Countable Spaces are Second Countable)

Every subspace of a second countable space is second countable.

Example 4.3.1

Let $B^n = B^n(0,1) \subseteq \mathbb{R}^n$. Then B^n is homeomorphic to $\mathbb{R}^n$.

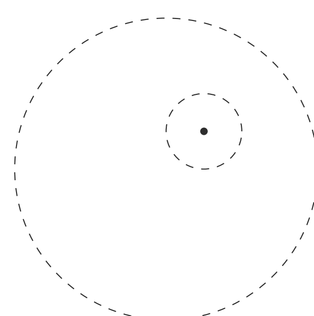


Figure 4.1: Open ball in an open set in $\mathbb{R}^2$

Example 4.3.2

Now consider the closed ball $\overline{B}^n = \overline{B}^n(0,1) \subset \mathbb{R}^n$. This is actually not a manifold, but we are not yet there to prove this. This sort of a structure motivates us to the next definition.

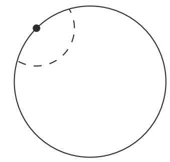


Figure 4.2: Open ball on a point on the boundary of a closed set

Definition 23 (Manifold with Boundary)

An n-dimensional space that is second countable and Hausdorff, such that $\forall x \in X$, there exists a neighbourhood either homeomorphic to $\overline{B}^n \subseteq \mathbb{R}^n$ or $B^n \cap \mathbb{H}^n$.

Note 4.3.3

Note that $\mathbb{H}^n$ is defined as

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) : x_n \geq 0\}.$$

For instance, $\mathbb{H}^2$ has the following graph:

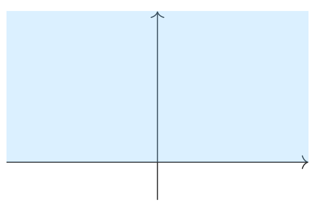


Figure 4.3: Graph of $\mathbb{H}^2$

Part II

Introduction to Topological Manifolds

5 Lecture 5 Jan 16th

5.1 Manifolds (Continued)

Definition 24 (Interior Point)

A point $x \in M$ is called an ***interior point*** *if there is a local homeomorphism*

$$\varphi : \mathcal{U} \to \mathbb{B} \subseteq \mathbb{R}^n,$$

where $\mathcal{U}$ is open.

In the last lecture we asked ourselves the following: how do we know if the idea of 'being on a boundary' is a well-defined notion? In particular, how do well tell the difference between the following two graphs, mathematically?

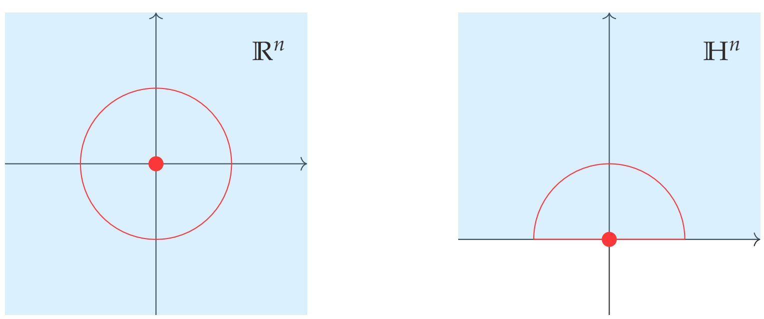


Figure 5.1: How do we tell the difference between the two graphs?

Definition 25 (Boundary Point)

A point x is on the boundary of M, denoted as $x \in \partial M$, if there exists

$\mathcal{U} \ni x$ that is open, and a homeomorphism

$$\psi : \mathcal{U} \to \mathbb{B}_{0,1} \cap \mathbb{H}^n.$$

Note 5.1.1

Note that the definition of a boundary and interior is different from our earlier definitions for the same terminologies. A manifold with boundary will always have an empty boundary, as a subset, despite the fact that its boundary as a manifold may not be empty.

Note 5.1.2

*The φ in Definition 24 and ψ in Definition 25 are called **local charts**.*

Also, our definitions do not rule out, e.g.

$$\varphi_2 : \mathcal{U} \to \mathbb{B}^2 \subseteq \mathbb{R}^2$$
$$\varphi_5 : \mathcal{U} \to \mathbb{B}^5 \subseteq \mathbb{R}^5.$$

Definition 26 (Coordinate Chart)

*If M is locally homeomorphic to $\mathbb{R}^n$, a homeomorphism from an open subset $U \subset M$ to an open subset of $\mathbb{R}^n$ is called a **coordinate chart** (or simply a **chart**).*

We shall later on prove that a point cannot simultaneously be a boundary point and an interior point. This property is called **the invariance of the boundary**. [1]

[1] This also means that $\mathrm{Int}(M) \cap \partial M = \emptyset$. Also, by Definition 22, $\mathrm{Int}(M) \cup \partial M = M$.

Note 5.1.3

$\mathrm{Int}(M)$ *is open. Thus, we can use the same definition about open sets as before using* φ.

Note 5.1.4

In contrast, ∂M *is closed; thanks to the invariance of the boundary we have that* $\partial M = M \setminus \mathrm{Int}(M)$ *and* $\mathrm{Int}(M)$ *is open, and so* $(\partial M)^C = \mathrm{Int}(M)$.

We shall also prove the following theorem later on:

Theorem (Invariance of the Dimension)

The n *in* $\mathbb{R}^n$ *is well-defined.*

Example 5.1.1

Consider the equation

$$x^2 - y^2 - z^2 = 0. \tag{5.1}$$

Note that we may write

$$x = \pm\sqrt{y^2 + z^2},$$

and so the graph generated by Equation (5.1) is as shown in Figure 5.2.

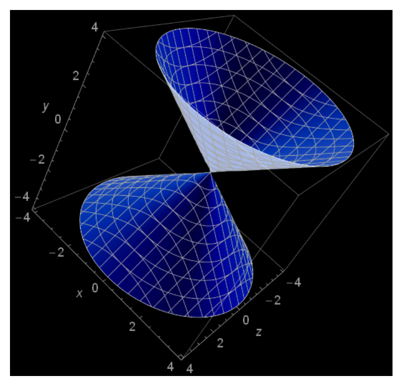


Figure 5.2: A 3D cone in $\mathbb{R}^3$, from WolframAlpha

However, this is not a manifold: if we assume that a ball arounnd the origin is homeomorphic to $\mathbb{R}^2$, then by removing the point at the origin in the cone, the cone becomes two disconnected components, but the ball in $\mathbb{R}^2$ homeomorphic to the aforementioned ball is still connected.

Note 5.1.5

An open subset of a manifold is a manifold, by restriction.

5.1.1 The 1-Sphere S^1

From Example 2.1.5, we have

- $[0, 1) \simeq [0, \infty)$ is a manifold with boundary;
- $(0, 1) \simeq \mathbb{R}$ is a manifold; and
- $[0, 1]$ is a manifold with boundary.

Example 5.1.2 (S^1 is a manifold)

Consider the function $f : [0, 2\pi) \to e^{i\theta}$. The image of f is Consider

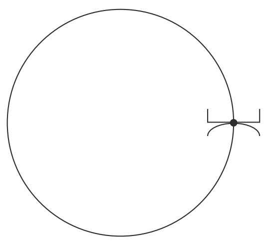


Figure 5.3: S^1 as a manifold

the following two functions $(0, 2\pi) \to \mathbb{C}^2$ by

$$\theta_1 \to e^{i\theta_1} \text{ and } \theta_2 \to e^{i\theta_2 + \pi},$$

which gives us the graphs:

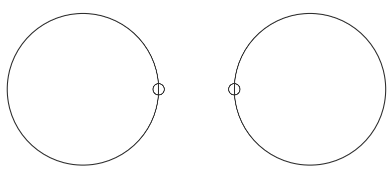


Figure 5.4: Basis for S^1

respectively. Note that the image of both these functions are not compact. Regardless, this gives us a basis for S^1, which we notice is countable, Hausdorff, and locally homeomorphic to $\mathbb{R}^2$.

Theorem 13 (1-Dimensional Manifolds Determined by Its Compactness)

Let M be a connected component of a 1-dimensional manifold. Then either

1. *M is compact, in which case if it is*
   - *without a boundary, then M is homeomorphic to* S^1.
   - *with a boundary, then M is homeomorphic to* $[0, 1]$.
2. *M is not compact, in which case if it is*
   - *without a boundary, then M is homeomorphic to* $[0, 1)$.
   - *with a boundary, then M is homeomorphic to* $(0, 1)$.

6 Lecture 6 Jan 18th

6.1 Manifolds (Continued 2)

6.1.1 The 1-Sphere S^1 (Continued)

Set theoretic view of S^1 We showed that S^1 is a manifold. We can, in fact, set theoretically, look at S^1 as $A = [0, 1]$ glued at the endpoints, i.e. we **identify** the points 0 and 1 as 'the same', and label this notion as $0 \sim 1$.

Topological view of S^1 Topologically, for $0 \sim 1$ in A, we can construct an open set around the point such that the open set is properly contained in A.

Figure 6.1: Topological representation of A

But how can we describe this notion mathematically so?

CONSIDER the real line as follows:

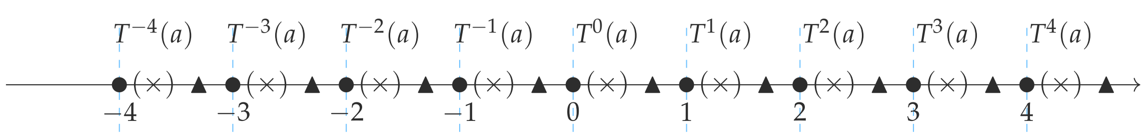


Figure 6.2: Breaking down the real line into parts

Let's define $T : \mathbb{R} \to \mathbb{R}$ such that $x \mapsto x + 1$. Clearly so, T is bijective and, in particular, has an inverse $x \mapsto x - 1$.

Notice that within each interval $[x, x + 1]$, we can find a $\times$ and $\blacktriangle$ at the same distance from x. Also, notice that we can use the same radius for $\times$ such that the open ball around $\times$ sits in $[x, x + 1]$ for each $x \in \mathbb{Z}$.

Thus, instead of studying the entire real line at once, we can reduce our attention only to $[0, 1]$, and simply scale the interval with a 'scalar multiplication' to get to wherever we want on the real line.

Now let

$$G := \left\{ T^k \;\middle|\; k \in \mathbb{Z} \right\},$$

which is evidently a **group**. Furthermore, every element in G is a homeomorphism to $\mathbb{R}$. Let G **act on** $\mathbb{R}$, and for $a \in \mathbb{R}$, consider the **orbit** of a, which is denoted as

$$G \cdot a := \left\{ T^k(a) \;\middle|\; k \in \mathbb{Z} \right\}.$$

Then

$$S^1 \simeq \text{ the space of all orbits of } G \text{ acting on } \mathbb{R} =: \mathbb{R}/G,$$

where $\simeq$ represents homeomorphism. [1] Also, notice that here, G is effectively $\mathbb{Z}$.

[1] $\mathbb{R}/G$ is the **quotient space** of $\mathbb{R}$ over G.

This realization implies the existence of some topology on S^1.

THUS FAR, we have seen that we may look at S^1

- set theoretically: as $A = [0, 1]$ with glued endpoints; and
- topologically: as $\mathbb{R} \setminus \mathbb{Z}$.

S^1 as a topological group Since $\mathbb{C} \simeq \mathbb{R}^2$, we may think of S^1 as a sphere on the complex plane. We see that this 'group' takes on the

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\} = \left\{e^{i\theta} \mid \theta \in \mathbb{R}\right\}$$

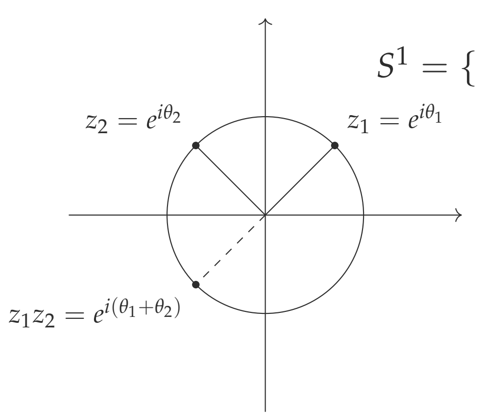


Figure 6.3: S^1 on the complex plane

operation of adding the indices of the exponents. Thus $G = (S^1, \cdot)$. Notice that G is indeed a group equipped with said operation, and for each $z_1 \in G$, there exists $z_1^{-1} = \frac{1}{z_1} \in G$ such that $z_1 \cdot \frac{1}{z_1} = 1$.

Furthermore, the function

$$\iota : S^1 \to S^1 \text{ given by } z \mapsto \frac{1}{z}$$
$$\text{which is } e^{i\theta} \mapsto e^{-i\theta}$$

is continuous.

Also, the function

$$P : S^1 \times S^1 \to S^1 \text{ given by } (z_1, z_2) \mapsto z_1 z_2$$

is continuous.

Definition 27 (Topological Group)

If G is a group, and functions ι and P as defined above, if both ι and P are continuous, then we say that G is a ***topological group****.*

S^1 as a moduli space

Definition 28 (Moduli Space)

A ***moduli space*** *is the space of all lines passing through the origin.*

On $\mathbb{R}^2$, we have

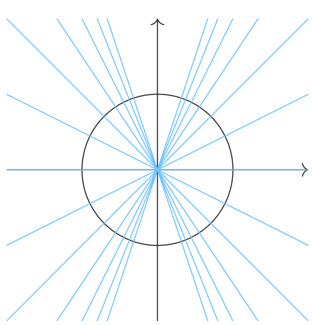


Figure 6.4: The moduli space on $\mathbb{R}^2$

First, how can we understand 'closeness' in a moduli space? We can actually look at the difference in the radians of each line, or really just $x/360$ and compare the x's.

Also, notice that each line passes through S^1 twice. We can indeed avoid this problem by shifting S^1 to one side, as shown in Figure 6.5.

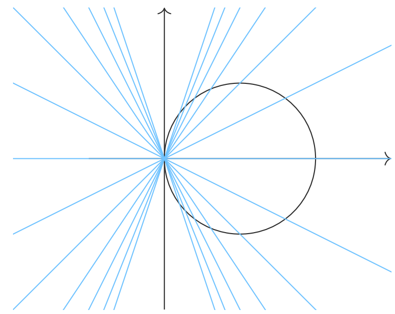


Figure 6.5: Shifted S^1 for the moduli space

Now each line intersects S^1 at the origin and another point on S^1, and this intersection is in fact unique.

6.1.1.1 *The space of $S^1 \times S^1$*

Observe that the product of two manifolds is a manifold [2].

[2] This is called a **product space**.

Consider using the set theoretical viewpoint, with credits to **Felix Klein**, the following figure By joining the sides with $>$, where we

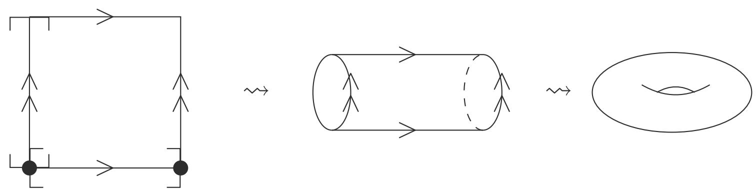


Figure 6.6: $S^1 \times S^1$ becomes a cylinder by identifying the edges

identify the endpoints, we can go from the figure introduced by Klein to a cylinder. Then by identifying the sides with $>>$, we get a **torus**.

Now on a Cartesian plane, observe that

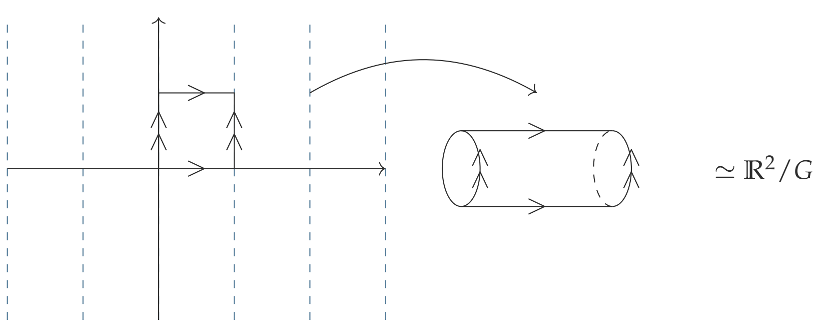


Figure 6.7: Klein's figure on a Cartesian plane to a cylinder

where we define $G = \left\{ T^k \;\middle|\; k \in \mathbb{Z} \right\}$ as before, for $T((x,y)) = (x+1, y)$. Now on a similar note, define $R : \mathbb{R} \to \mathbb{R}$ by $R((x,y)) =$

$(x, y+1)$. Then let $G_2 = \left\{ R^k \;\middle|\; k \in \mathbb{Z} \right\}$. Thus

$$\mathbb{R}^2 / G \oplus G_2 \simeq \mathbb{R}^2 / \mathbb{Z} \oplus \mathbb{Z}.$$

7 Lecture 7 Jan 21st

7.1 Manifolds (Continued 3)

7.1.1 2-dimensional Manifolds

Example 7.1.1

S^2 is a 2-dimensional manifold (w/o boundary).

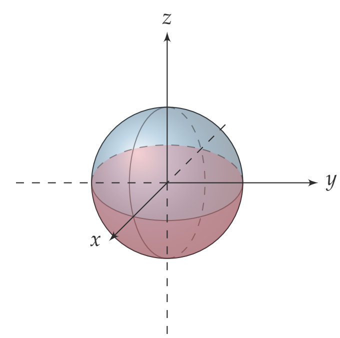


Figure 7.1: The 2-sphere S^2

Note that we may 'cover' this sphere by the function $f_- : \mathbb{D} \subseteq \mathbb{R}^2 \to \mathbb{R}$, where $\mathbb{D}$ is the unit disc in $\mathbb{R}^2$, and f_- is given by

$$(x, y) \mapsto -\sqrt{1 - (x^2 + y^2)}.$$

We see that f_- is a chart for the lower hemisphere for Figure 7.1, shaded red. We can indeed cover the entire 2-sphere with similar hemispheres in different orientations: consider

$$\begin{aligned} f_+(x, y) &= \sqrt{1 - x^2 - y^2} \\ g_+(x, z) &= \sqrt{1 - x^2 - z^2} \\ g_-(x, z) &= -\sqrt{1 - x^2 - z^2} \\ h_+(y, z) &= \sqrt{1 - y^2 - z^2} \\ h_-(y, z) &= -\sqrt{1 - y^2 - z^2}, \end{aligned}$$

which represent the upper hemisphere, eastern hemisphere, western hemisphere, front hemisphere, and back hemisphere, respectively.

We have the following general observation: let $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ be a real-valued continuous function. Then $\Gamma(f)$ is **parameterized** using f, i.e. f constructs the chart $\Gamma(f)$.

We have not explicitly defined what $\Gamma(f)$ is, but it is recorded in PMATH 733, but we shall provide a short definition here for reference: the **graph** $\Gamma(f)$ of the function f is defined as

$$\Gamma(f) := \{(x_1, \ldots, x_n, y) \mid f(x_1, \ldots, x_n) = y\}.$$

It is interesting to note that $\Gamma(f)$ is homeomorphic to $\mathcal{U}$. Consider the function

$$\Phi : \mathcal{U} \to \Gamma(F) \text{ given by } \Phi(x_1, \ldots, x_n) = (x_1, \ldots, x_n, f(x_1, \ldots, x_n)).$$

It is clear that Φ is continuous since each of its components are continuous. However, it is not as easy to find a continuous map to go from $\Gamma(f)$ to $\mathcal{U}$.

Definition 29 (Projection)

We define function $\pi_{n+1} : \Gamma(f) \to \mathcal{U}$ by

$$\pi_{n+1}(x_1, \ldots, x_n, y) = (x_1, \ldots, x_n),$$

*which is called a **projection**.*

Note 7.1.1

1. *In A1, we showed that π_{n+1} is continuous.*
2. *Furthermore, π_{n+1} is injective.*
3. *Thus, we observe that*

$$\pi_{n+1} \circ \Phi(x_1, \ldots, x_n) = (x_1, \ldots, x_n) = id_{n+1}(x_1, \ldots, x_n).$$

Example 7.1.2

Consider the following figure:

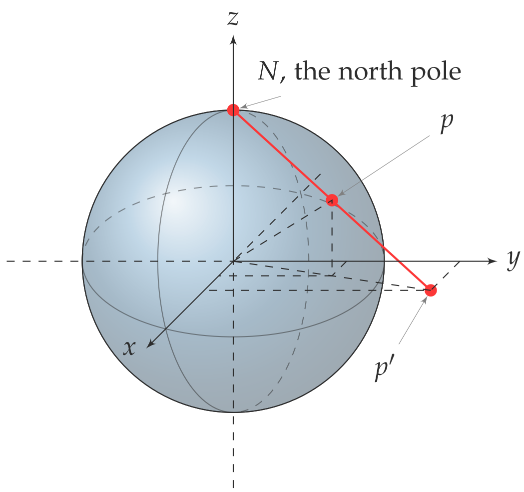


Figure 7.2: Stereographic Projection

We define the **stereographic projection** by

$$\Sigma(p) = p'.$$

7.1.2 *n-spheres*

We can now extend the notion we saw in Example 7.1.1 to higher dimensional manifolds. In particular, we want to be able to 'glue' the boundaries of the hemispheres, as shown in Figure 7.3.

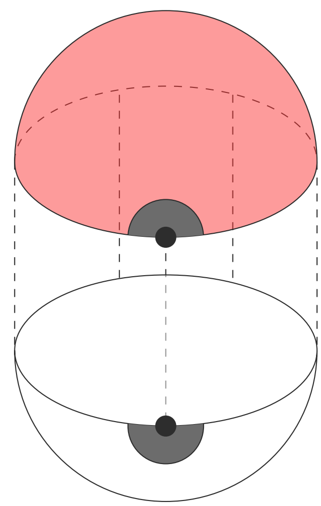


Figure 7.3: Glueing two hemispheres
I was definitely not trying to make a Pokeball.

In a more mathematical sense, we are identifying the lower and upper boundaries of the upper and lower hemisphere, respectively, i.e. we identify

$$S^n = \overline{\mathbb{B}}^n \times \{0\} \cup \overline{\mathbb{B}}^n \times \{1\},$$

where the $\{0\}$ and $\{1\}$ represent the upper and lower hemispheres, respectively. We may also write this as

$$x \in \partial\overline{\mathbb{B}}^n : (x, 0) \sim (x, 1).$$

This calls for the notion of an equivalence class. Recall that an **equivalence relation** on a set A is a relation between elements such that

1. (reflexity) $x \sim x$;
2. (symmetry) $x \sim y \iff y \sim x$; and

3. (transitivity) $x \sim y \wedge y \sim z \implies x \sim z$,

where $\sim$ is our equivalence relation.

Equivalence relations give rise to the notion of **equivalence classes**, where we define an equivalence class as

$$[\beta] := \{\alpha \in A \mid \beta \sim \alpha\}.$$

In this course, we shall sometimes denote equivalence classes in the form of A_β. Notice that

$$A = \coprod_{\beta \in B} A_\beta,$$

where $B \subseteq A$.

THIS BEGS THE QUESTION: do the set of equivalence classes retain the topology of the space?

Here, we look into **quotient topology**. Let A^* be the space of equivalence classes, and let's assume that A is endowed with a topology. Consider the function

$$\pi : A \to A^* \text{ given by } a \mapsto [a]_\sim,$$

where $\sim$ is the equivalence relation, and $[a]_\sim$ is the equivalence class that a belongs to[1].

[1] Note that the function is well-defined as the equivalence classes are disjoint.

$$\begin{tikzcd} A \arrow[d, "\pi"'] \arrow[dr, "g"] & \\ A^* \arrow[r, "\tilde{g}"'] & Z \end{tikzcd}$$

Figure 7.4: Relationship of A, A^* and Z

Consider an arbitrary space Z that is also endowed with a topology, such that $\exists g : A \to Z$ that is continuous, such that g is constant on each of the equivalence classes of A, that is, if $a \sim b$, then $g(a) = g(b)$.

Then $\tilde{g}$ is a well-defined function induced on A^*: we have that

$$\tilde{g}(\pi(a)) = g(a).$$

We want to endow A^* with a topology such that $\tilde{g}$ will be continuous. However,

- if A^* is too 'fine', then π may not be continuous; and
- if A^* is too 'course', then $\tilde{g}$ may not be continuous.

We need to strike a balance in the fineness of the topology of A^* to make sure that both π and $\tilde{g}$ are continuous.

Definition 30 (Strongly Continuous)

Let $\mathcal{V} \subseteq A^*$. $\mathcal{V}$ is open iff $\exists \mathcal{U} \subseteq A$ such that $\pi(\mathcal{U}) = \mathcal{V}$. We say that π is ***strongly continuous****.*

Warning

π may not be an open map!

8 Lecture 8 Jan 23rd

8.1 Quotient Spaces

With the topology that we last introduced for A^*, g is continuous iff $\tilde{f}$ is continuous. However, this is a rather cumbersome way to construct a quotient space. In particular, how do we know what equivalence class should we choose?

This way of constructing a quotient space is intuitive, ring theoretically.

Definition 31 (Saturated)

We say that a subset $S \subseteq A$ is ***saturated*** *(wrt π) provided that it is has a* ***non-empty intersection*** *with a fibre $\pi^{-1}(\{q\})$, where $q \in A^*$, then $S \supseteq \pi^{-1}(\{q\})$.*

Exercise 8.1.1

Let $\pi : X \to Y$ be any map. For a subset $U \subseteq X$, show that TFAE.

1. *U is saturated.*
2. *$U = \pi^{-1}(q(U))$.*
3. *U is a union of fibres.*
4. *If $x \in U$, then every point $x' \in X$ such that $q(x) = q(x')$ is also in U.*

Note 8.1.1

This definition is equivalent to saying that $\pi^{-1} \circ \pi(S) = S$.

With this definition, we may restate the definition of a quotient map in a clearer way.

Definition 32 (Quotient Map)

A map $g : X \to Y$ is called a ***quotient map*** *if it sends saturated open subsets of X to open subsets of Y.*

Example 8.1.1

Let $g : X \to Y$ be surjective. We can define an equivalence relation on X by setting $x_1 \sim x_2$ iff $g(x_1) = g(x_2)$, i.e. x_1 and x_2 belong to the same fibre.

Example 8.1.2

Let $\overline{\mathbb{B}}^2$ be the closed unit disk in $\mathbb{R}^2$, and let $\sim$ be the equivalence relation on $\overline{\mathbb{B}}^2$ defined by $(x, y) \sim (-x, y)$ for all $(x, y) \in \partial\overline{\mathbb{B}}^2$ (See Figure 8.1).

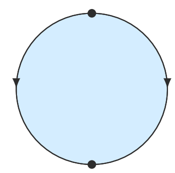


Figure 8.1: A quotient of $\overline{\mathbb{B}}^2$

We can think of this space as one that is obtained from $\mathbb{B}^2$ by "pasting" the left half of the boundary to the right half. It is not difficult to imagine this transformation and notice that we can 'continuously morph' $\mathbb{B}^2$ under this equivalence relation into S^2. We shall prove much later on that this is indeed the case.

The above process is also called 'collapsing $\partial\mathbb{B}^2$ to a point'.

Example 8.1.3

Define an equivalence relation on the square $I \times I$ by setting $(x, 0) \sim (x, 1)$ for all $x \in I$, and $(0, y) \sim (1, y)$ for all $y \in I$ (See Figure 8.2).

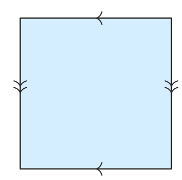


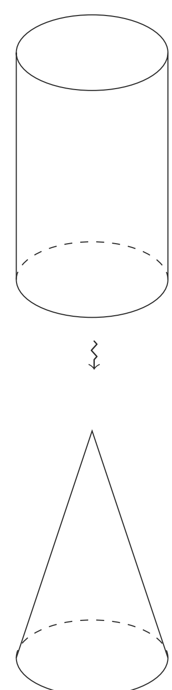
Figure 8.2: A quotient of $I \times I$

Example 8.1.4

Define $\mathbb{P}^n$, the **real projective space of dimension n**, to be the set of 1-dimensional linear subpaces (lines through the origin) in $\mathbb{R}^{n+1}$. There exists a map $q : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, defined by sending a point x to its span. We apply the quotient topology with respect to q on $\mathbb{P}^2$. The 2-dimensional projective space $\mathbb{P}^2$ is usually called the **projective plane**.

Example 8.1.5

Let X be a topological space, the quotient $(X \times I)/(X \times \{0\})$ obtained from the "cylinder" $X \times I$ by collapsing one end to a point is called the **cone** on X. For instance, if $X = S^1$ and $I = (0, 1)$, then taking the quotient $(S^1 \times (0, 1)/(S^1 \times \{0\})$ is exactly the process shown in Figure 8.3.



Example 8.1.6 (Wedge sum of Manifolds)

Let M_1 and M_2 be two manifolds. We define the **wedge sum** of M_1 and M_2 as

$$M_1 \vee M_2 := (M_1 \cup M_2) / p_1 \sim p_2,$$

where $p_1 \in M_1$ and $p_2 \in M_2$. The wedge sum is also sometimes called the **one-point union**.

For instance, the wedge sum of $\mathbb{R} \wedge \mathbb{R}$ is homeomorphic to the union of the x-axis and y-axis on a Cartesian plane (cf Figure 8.4), and the wedge sum $S^1 \wedge S^1$ is homeomorphic to the **figure-eight space**, which is made up by the union of two circles of radius 1 centered at $(0, 1)$ and $(0, -1)$ in the plane (cf Figure 8.5).

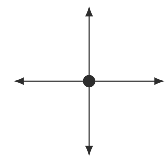


Figure 8.4: Wedge sum of two lines.

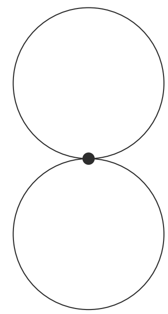


Figure 8.5: Wedge sum of two circles.

Warning

Unlike subspaces and product spaces, quotient spaces do not behave well wrt most topological properties. In particular, none of the definiting properties of manifolds are necessarily inherited by quotient spaces.

For example, a quotient space can be [1]

- *locally Euclidean and second countable, but not Hausdorff;*
- *Hausdorff and second countable, but not locally Euclidean; or*
- *not even first countable.*

[1] **I should provide examples here**.

The following is a proposition that gives us some peace of mind when working within certain spaces.

Proposition 14 (Locally Euclidean Quotient Space of a Second Countable Space is Second Countable)

Suppose M is a second countable space and N is a quotient space of M. If N is locally Euclidean, then it is second countable.

Note 8.1.2

Thus if the original space is, say, a manifold, then for any of its quotient spaces, we only need to check that the quotient space is both Hausdorff and locally Euclidean.

Proof

Let $q : M \to N$ be the quotient map, and suppose N is locally Euclidean. Let $\mathcal{U}$ be a cover of N. Then the set $\{q^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of M, which therefore has a countable subcover. Let $\mathcal{U}' \subseteq \mathcal{U}$ be the countable subset such that $\{q^{-1}(U) : U \in \mathcal{U}'\}$ covers M. Then $\mathcal{U}'$ is a countable subcover of N. $\square$

Exercise 8.1.2

Consider the function $g : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$. Verify that g is a quotient map.

Example 8.1.7

The map $g : S^1 \to S^1 \subseteq \mathbb{R}^2$ as given by the above is indeed a quotient map. Thus we observe that S^1 is a quotient space of itself.

8.1.1 *Characteristic Property and Uniqueness of Quotient Spaces*

Theorem 15 (Characteristic Property of the Quotient Topology)

Suppose X and Y are two topological spaces and $\pi : X \to Y$ is a quotient map. For any topological space Z, a map $f : Y \to Z$ is continuous iff the composite map $f \circ \pi$ is continuous (cf Figure 8.6).

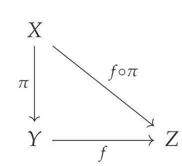


Figure 8.6: Characteristic property of the quotient topology.

Proof

Observe that for any open $U \subseteq Z$, $f^{-1}(U)$ is open in Y iff

$$\pi^{-1}(f^{-1}(U)) = (f \circ \pi)^{-1}(U)$$

is open in X. Our result follows immediately from this observation. $\square$

Theorem 16 (Uniqueness of the Quotient Topology)

Given a topological space X, a set Y and a surjective map $\pi : X \to Y$, the quotient topology is the only topology on Y for which the characterisitc property holds.

Exercise 8.1.3
Prove Theorem 16.

Theorem 17 (Descends to the Quotient)

Suppose $\pi : X \to Y$ is a quotient map, Z a topological space, and $f : X \to Z$ is any continuous map that is constant on the fibres of π [2]. Then there exists a unique continuous map $\tilde{f} : Y \to Z$ such that $f = \tilde{f} \circ \pi$ (cf Figure 8.7).

[2] This means that if $\pi(x) = \pi(x')$, then $f(x) = f(x')$.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ Y \arrow[r, dashed, "\tilde{f}"'] & Z \end{tikzcd}$$

Figure 8.7: Descends to the Quotient

Proof

Since π is surjective, $\forall y \in Y$, $\exists x \in X$ such that $\pi(x) = y$. Then consider $\tilde{f} : Y \to Z$ given by $\tilde{f}(y) = f(x)$ for any x that we discover from the last statement. By the hypothesis on f, $\tilde{f}$ is guaranteed to be unique and well-defined. It follows from Theorem 15 that $\tilde{f}$ is continuous. $\square$

The following theorem, which is a consequence of Theorem 15, gives to us that quotient spaces are unique up to homeomorphism by the identifications made by their quotient maps.

Theorem 18 (Uniqueness of Quotient Spaces)

Suppose $\pi_1 : X \to Y_1$ and $\pi_2 : X \to Y_2$ are quotient maps that make the same identifications, i.e.

$$\pi_1(x) = \pi_1(x') \iff \pi_2(x) = \pi_2(x').$$

Then there exists a unique homeomorphism $\varphi : Y_1 \to Y_2$ such that $\pi \circ \pi_1 = \pi_2$.

The proof is relatively straightforward so I will only jot down the gist of the proof and provide commutative diagrams for visualization of the relationships between these spaces.

Proof (Sketch)

Observe Figure 8.8.

$$\begin{tikzcd} X \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Y_1 \arrow[r, dashed, "\tilde{\pi}_2"'] & Y_2 \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ Y_2 \arrow[r, dashed, "\tilde{\pi}_1"'] & Y_1 \end{tikzcd}$$

Figure 8.8: Relationships of the quotient spaces

Then

$$\tilde{\pi}_1 \circ (\tilde{\pi}_2 \circ \pi_1) = \tilde{\pi}_1 \circ \pi_2 = \pi_1. \qquad \square$$

Thus, we have Figure 8.9.

$$\begin{tikzcd} X \arrow[d, "\pi_1"'] \arrow[dr, "\pi_1"] & \\ Y_1 \arrow[r, dashed, "\tilde{\pi}_1 \circ \tilde{\pi}_2"'] & Y_1 \end{tikzcd}$$

Figure 8.9: Consequence of the relationship between the quotient spaces.

Then by Theorem 16, the map is indeed unique, and it follows from the identity map that the two maps are equal. Similarly, $\tilde{\pi}_2 \circ \tilde{\pi}_1$ is the identity on Y_2.

Then $\varphi = \tilde{\pi}_2$ is the required homeomorphism, and it is unique by Theorem 16.

9 Lecture 9 Jan 25th

9.1 Topological Embeddings

Definition 33 (Topological Embedding)

An injective continuous map $g : S \to Y$ is called a ***topological embedding*** *(or just an* ***embedding****) if it is a homeomorphism onto its image.*

Note 9.1.1

In other words, $g : S \to Y$ is called an embedding if it is a homeomorphism between S and $g(S)$.

Example 9.1.1

Consider the function $f : \mathbb{R} \to \mathbb{R}^3$ given by

$$x \mapsto (\cos x, \sin x, x).$$

We know that f is both bijective and continuous (since each of its components are continuous). Note that

$$f \circ \pi_3 \restriction_{f(\mathbb{R})} = \mathrm{id}_{\mathbb{R}} .$$

It follows that f is an embedding.

Example 9.1.2

The map $f : x \mapsto e^{ix}$ from $[0, 2\pi)$ to $\mathbb{C}$ is continuous and injective, but not a homeomorphism (problem lies on the endpoints). So f is not an embedding, despite being continuous and injective.

However, the restriction of f to any proper subinterval is an embedding, as is the interval $(0, 2\pi)$.

As we've seen above, a continuous injective map is not necessarily an embedding. The following proposition provides us with two sufficient but not necessary conditions to ensure that a continuous injective map is an embedding.

Warning
*Notice that by our definition, and by taking note on Exercise 1.1.1, an embedding is **only a homeomorphism between the domain and its image** under the mapping. It need not be a homeomorphism between the domain and the codomain. Thus, an embedding need not be open or closed.*

Proposition 19 (Sufficient Conditions to be an Embedding)

A continuous injective map that is either open or closed is an embedding.

Proof

Let $f : X \to Y$ be a continuous injective map between 2 topological spaces. Note that f is bijective from X to $f(X)$. It suffices to show that f is open or closed from X to $f(X)$ by Exercise 1.1.1.

By assumption, if $f : X \to Y$ is open, then for any open $A \subset X$, $f(A) \subset f(X)$ is open, simply by definition. It also follows that if f is closed, then for any closed $B \subset X$, $f(B)$ is closed in $f(X)$. $\square$

Example 9.1.3
The map $f : \left(0, \frac{1}{2}\right) \to S^1$ given by $x \to e^{2\pi i x}$ is an embedding but it is neither open nor closed.

Exercise 9.1.1
Give another example of a topological embedding that is neither open nor closed.

Proposition 20 (Surjective Embeddings are Homeomorphisms)

A surjective topological embedding is a homeomorphism.

Exercise 9.1.2
Prove Proposition 20.

Remark 9.1.1

We now have some proper tools to build various examples of manifolds as subspaces of Euclidean spaces. Recall Proposition 3 and Proposition 12, to show that a subspace of $\mathbb{R}^n$ is a manifold, we need only to verify that it is *locally Euclidean.*

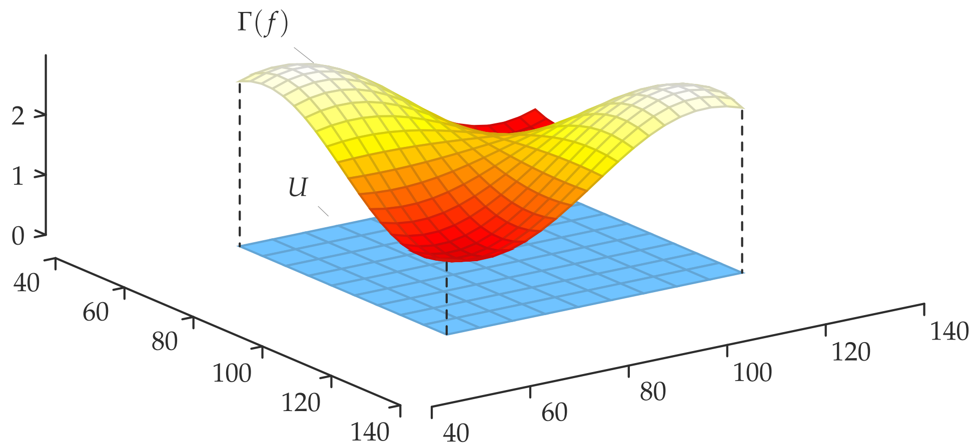


Figure 9.1: Graph of a continuous function

Example 9.1.4

If $U \subseteq \mathbb{R}^n$ is an open subset and $f : U \to \mathbb{R}^k$ is any continuous map, the graph of f (cf Figure 9.1) is the subset $\Gamma(f) \subseteq \mathbb{R}^{n+k}$ defined by

$$\Gamma(f) = \{(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_k) : x \in U,\ y = f(x)\},$$

with the subspace topology inherited from $\mathbb{R}^{n+k}$. To verify that $\Gamma(f)$ is indeed a manifold, we need only to show that $\Gamma(f)$ is homeomorphic to U. Let $\varphi_f : U \to \mathbb{R}^{n+k}$ be the continuous injective map

$$\varphi_f(x) = (x, f(x)).$$

One can verify that φ_f is indeed a continuous bijection from U to $\Gamma(f)$, and for $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$, $\pi_{\Gamma(f)}$ is a continuous inverse for φ_f. It follows that φ_f is an embedding and $\Gamma(f)$ is homeomorphic to U.

Example 9.1.5 (*n*-spheres are Manifolds)

Recall that the n-sphere (or *unit*) n-sphere is the set S^n of unit vectors in $\mathbb{R}^n$. In low dimensions, spheres are easy to visualize:

- S^0 is the two-point discrete space $\{\pm 1\} \subseteq \mathbb{R}$;
- S^1 is the unit circle in $\mathbb{R}^2$; and
- S^2 is the unit spherical surface of radius 1 in $\mathbb{R}^3$.

Since we are working in $\mathbb{R}^n$, by Remark 9.1.1, it suffices for us to show that each of the S^n's are locally Euclidean. We shall show that each point has a neighbourhood in S^n that is the graph of a continuous function.

For each $i \in \{1, \ldots, n+1\}$, let

U_i^+ denote the open subset of $\mathbb{R}^{n+1}$ consisting of points with $x_i > 0$, and

U_i^- denote the open subset of $\mathbb{R}^{n+1}$ consisting of points with $x_i < 0$.

Then for any $x = (x_1, \ldots, x_n) \in S^n$, some coordinate x_i must be nonzero, and so the sets $U_1^\pm, \ldots, U_{n+1}^\pm$ cover S^n. Now for each $U_i^\pm$, we can solve for the equation $|x| = 1$, and find that $x \in S^n \cap U_i^\pm$ iff

$$x_i = \pm \sqrt{1 - \sum_{\substack{j=1 \\ j \neq i}}^{n+1} x_j^2}.$$

Since the square root is a continuous function, it follows that the intersection of S^n with $U_i^\pm$ is the graph of a continuous function. This intersection is therefore locally Euclidean, showing to us that S^n is indeed a manifold.

The following lemma allows us to, essentially, 'glue' surfaces to one another.

Lemma 21 (Glueing Lemma)

Let X and Y be topological spaces, and let $\{A_i\}$ be either an arbitrary open cover of X, or a finite closed cover of X. Suppose that we are given continuous maps $f_i : A_i \to Y$ that agree on overlaps, i.e.

$$f_i \restriction_{A_i \cap A_j} = f_k \restriction_{A_i \cap A_j}.$$

Then there exists a unique continuous map $f : X \to Y$ whose restriction to each A_i is equal to f_i.

With that, we can construct the following space.

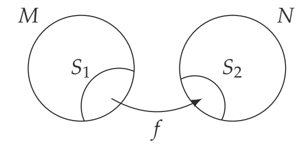


Figure 9.2: Glueing subsets of two disjoint spaces.

Definition 34 (Adjunction Space)

Consider 2 manifolds M and N that are of the same dimension, and let $S_1 \subseteq M$ and $S_2 \subseteq N$. Let $f : S_1 \to S_2$ be a homeomorphism (cf Figure 9.2). Then we define

$$M \cup_f N := M \coprod N \Big/ \left\{ \substack{a \sim f(a) \\ a \in S_1} \right.$$

*as the **adjunction space**, and is said to be formed by **attaching M to N along f**. The map f is called the **attaching map**.*

Remark 9.1.2

By Lemma 21, there exists a continuous map between M and N, and so this allows us to know that this new structure $M \cup_f N$ is indeed a manifold.

Definition 35 (Double)

*If $M = N$, with the identity map $\mathrm{id} \restriction_{\partial M}$ as a homeomorphism between ∂M and ∂N, then we call $M \cup_{\mathrm{id}\restriction_{\partial M}} N$ the **double** of M.*

Lemma 22 (Attaching Manifolds along Their Boundaries)

Suppose M is an n-dimensional manifold with boundary. Then its double $M \cup_{\mathrm{id}\restriction_{\partial M}} M$ is a manifold without boundary. More generally, if M_1 and M_2 are manifolds with non-empty boundaries, then there are topological embeddings $e : M_1 \cup_h M_2$ and $f : M_1 \cup_h M_2$ whose images are closed subsets of $M_1 \cup_h M_2$ satisfying

$$e(M_1) \cup f(M_2) = M_1 \cup_h M_2$$
$$e(M_1) \cap f(M_2) = e(\partial M_1) = f(\partial M_2).$$

The core idea of the proof is illustrated in Figure 9.3.

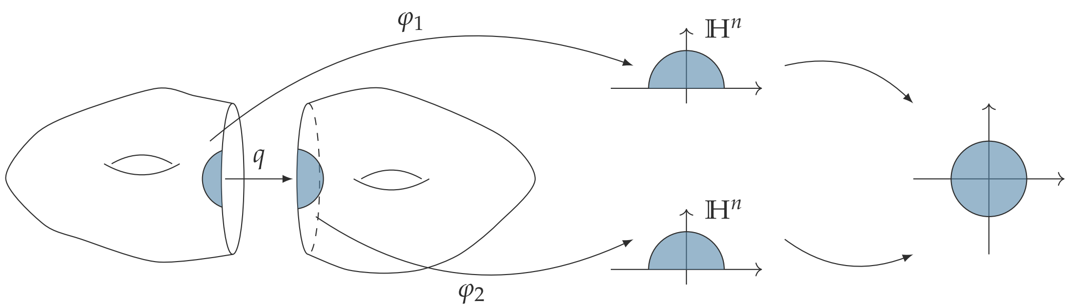


Figure 9.3: Attaching along the boundaries

Proof (Sketch)

Step 1 Find q.

Step 2 $q \restriction_{(} \operatorname{Int}(M_1))$ is an embedding.

Step 3 Define φ_1 and φ_2.

Step 4 Put the two together.

To END this lecture today, we recalled that we can look at the torus as a 2-dimensional rectangle with its sides properly identified. Now if we change the identification of one of the sides by swapping its orientation, we end up with what is known as a **Möbius band**.

Following that, if we also swap the orientation of the other two sides, we get what is known as the **Klein Bottle**.

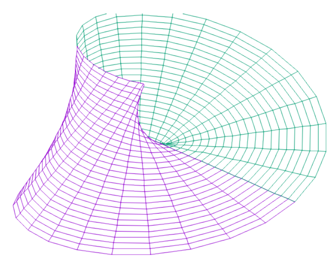


Figure 9.4: Möbius band

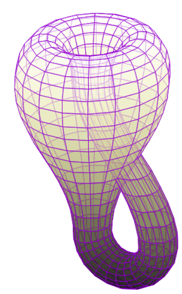


Figure 9.5: Klein Bottle

10 Lecture 10 Jan 28th

10.1 An Enpasse into Orientability

We talked about orientability in the last lecture. We shall now introduce a more concrete way of talking about orientability of a surface.

Definition 36 (Frame)

*A **frame** is a continuous pair of vectors v_1, v_2 defined on the entire surface such that v_1 and v_2 are linearly independent.*

Example 10.1.1

Here are two graphical examples of a frame:

- Cylinder

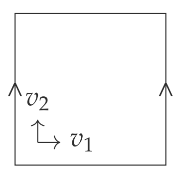


Figure 10.1: Frame on a Cylinder

Note that if we 'move' this frame towards the identified sides, the 'direction' of which the vectors 'point at' remains the same (cf. Figure 10.2).

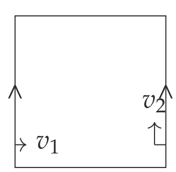


Figure 10.2: Frame on the identified sides of a cylinder

- Möbius strip

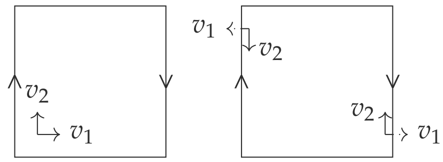


Figure 10.3: Frame on a Möbius strip

The two frames on the second diagram in Figure 10.3 are the same frame. Notice that v_2 is now 'pointing' in the 'opposite direction' simultaneously.

Note 10.1.1

The notion of having a 'side' for a Möbius strip makes sense when we embed it in $\mathbb{R}^3$.

Example 10.1.2

Recall Example 8.1.4, where we introduced the projective space. Since $S^n \simeq \mathbb{R}^{n+1}$, we can define a projective space with respect to n-spheres. Let us consider $n = 2$, and consider the group action $\mathcal{G}_2 = \{-1, 1\}$ given by

$$(-1)(x, y, z) = (-x, -y, -z) \text{ and } (1)(x, y, z) = (x, y, z).$$

Then

$$\mathbb{P}^2 = S^2 / \mathcal{G}_2 \simeq \overline{S^{+}}^2 / {}^{(x,y,z) \sim (-x,-y,-z)}_{(x,y,z) \in \partial S^{+}},$$

where S^+ is the upper hemisphere.

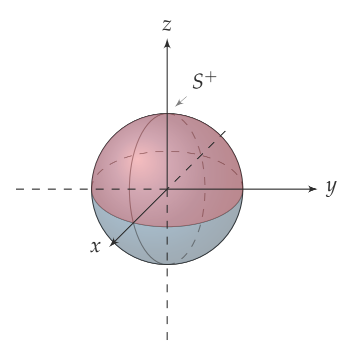


Figure 10.4: Projective Plane from S^2

In a very simple sense, we are 'compressing' the sphere by identifying the upper hemisphere with the lower hemisphere.

Note that because of this construction, $\mathbb{P}^2$ is not orientable: a point where the vector pointing at a direction parallel to y on the upper hemisphere is identified with a point whose corresponding vector points at the opposite direction of y.

Example 10.1.3

Consider the diagram presented in Figure 10.5. Now we have a more

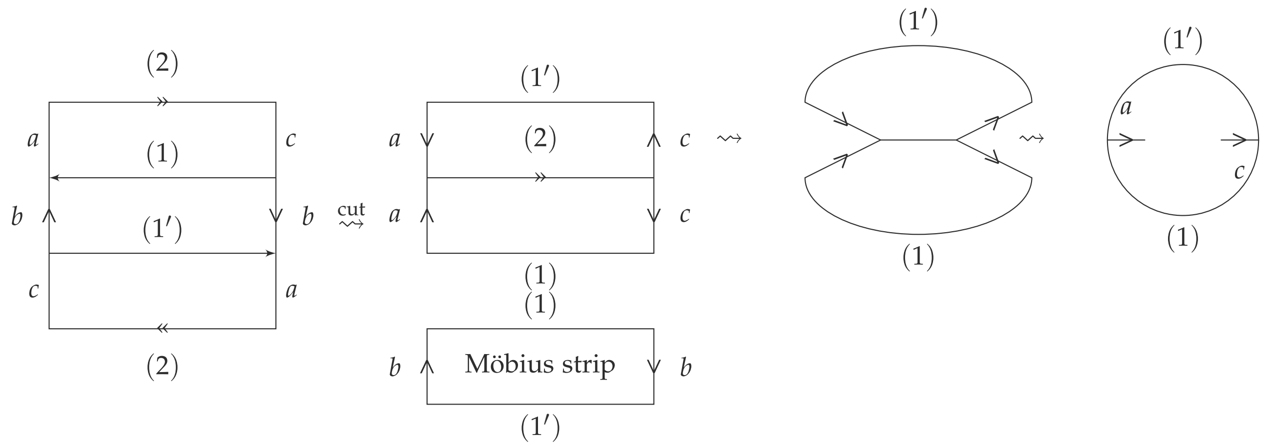


Figure 10.5: Klein Bottle as a Möbius Strip and a Disc

concrete reason to explain why the Klein Bottle is not embeddable into $\mathbb{R}^3$, since the disc itself is not embeddable into $\mathbb{R}^3$.

Recall Figure 6.5 where we saw how S^1 is homeomorphic to the space of all lines in $\mathbb{R}^2$ passing through the origin $(0,0)$.

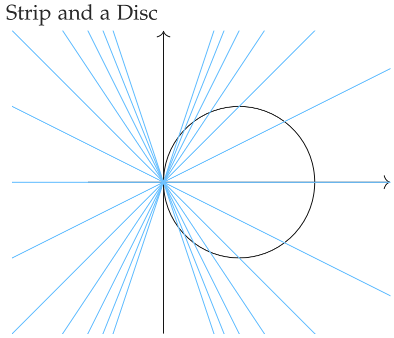


Figure 10.6: Shifted S^1 for the moduli space, as shown in Figure 6.5.

It follows, therefore, that $S^1 \simeq \mathbb{P}^1$.

> *What about the space of all lines in $\mathbb{R}^2$?*

Observing Figure 10.7, we see that since we can rotate γ at the origin, we may, in particular, rotate γ by $180°$ to get the opposite direction. It follows that this space of all lines in $\mathbb{R}^2$ is thus homeomorphic to $\mathfrak{M}$, a Möbius strip. Thus, the space of all lines in $\mathbb{R}^2$ is also not orientable.

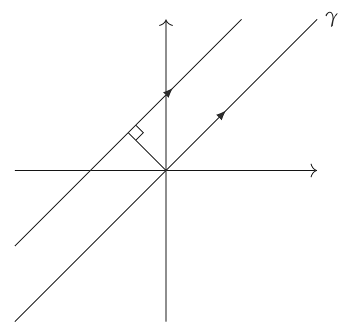


Figure 10.7: Space of all lines in $\mathbb{R}^2$

Following the same question as above, we may ask:

> *What is the space of all lines passing through $(0,0,0)$ in $\mathbb{R}^3$?*

It is not difficult to see that this space is homeomorphic to S^2, via the map $x \mapsto \frac{x}{|x|}$ (which is also a quotient map).

Just right before this, we observed, in Example 10.1.2 that

$$S^2/x \sim (-x) \simeq \mathbb{P}^2,$$

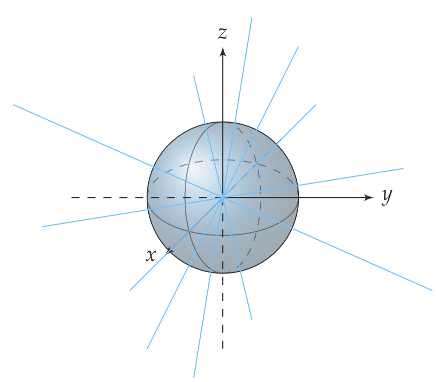


Figure 10.8: Space of all lines passing through $(0,0,0)$ in $\mathbb{R}^3$

where $x \sim (-x)$ via what is called the **antipodal map**.

> *What about the space of all lines in $\mathbb{R}^3$?*

Following a similar observation in Figure 10.7, we have the following:

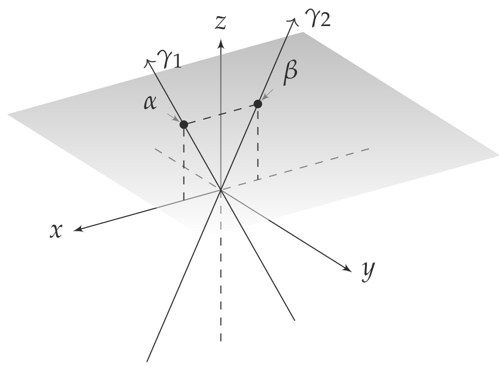


Figure 10.9: Space of all lines in $\mathbb{R}^3$

Observe that if we rotate γ_1 and γ_2 downwards, anchored at α and β respectively, we will eventually get that the two lines are the same line but pointing at different directions. We see that the space of all lines is also not orientable.

Part III

Introduction to Homology

11 Lecture 11 Jan 30th

So far, we have constructed and founded several manifolds and came to know about certain invariances (which we have yet to prove). One may notice that one way to distinguish these manifolds is by counting the number of 'holes' that they have, 'holes' of which there is no continuous deformation that can get rid of it. This brings us into the theory of **homology**.

I shall be using information both from the class lectures, and from a video lecture series on Algebraic Topology prepared by njwildberger, 2012.

11.1 Homology

Consider the following three diagrams:

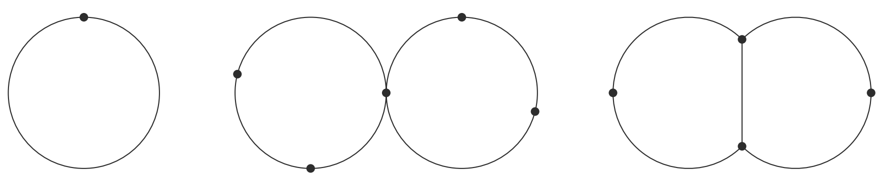


Figure 11.1: Three different kinds of loops

By inspection, we notice that the leftmost diagram has one 'hole', the center diagram has two, while the rightmost diagram also has two. One way we can calculate this number of 'holes' is by counting the number of loops there are. We see that in the first and second diagram, the number of loops and the number of 'holes' coincide. However, in the third diagram, there are a total of 3 loops:

- $z_1 = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = (1\ 2)(2\ 3)(3\ 1)$;
- $z_2 = \begin{pmatrix} 1 & 4 & 3 \end{pmatrix} = (1\ 4)(4\ 3)(3\ 1)$; and
- $z_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} = (1\ 2)(2\ 3)(3\ 4)(4\ 1)$.

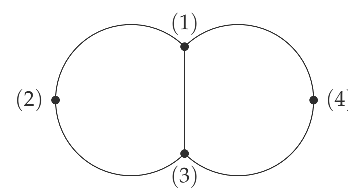


Figure 11.2: Third diagram has three loops

We see that z_3 does not entire capture a 'hole', since it also captures the line, or **edge**, $(1)(3)$.

Also, note that $(1\ 2\ 3)$ is not a unique representation of z_1; we can also describe z_1 as either[1]

$$(3\ 1\ 2),\ (2\ 3\ 1), -(1\ 3\ 2), -(3\ 2\ 1), \text{ or } -(2\ 1\ 3).$$

We shall call these expressions as **cycles**[2].

Notice that geometrically the cycle z_3 looks like $z_1 - z_2$, which one can trace the diagram to see that this 'minusing' between cycles gives us different cycles.

To further refine this approach, consider the following diagram:

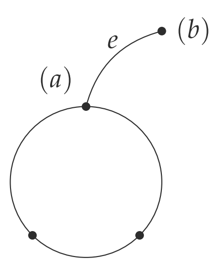


Figure 11.3: Simple diagram with a branching edge

Notice that $(a\ \ b)(b\ \ a)$ also goes back to the same point, forming, in a sense, a closed path. However, this does not enclose a 'hole', despite being algebraically similar to the cycles that we have described above. It is important that we find a way to make this distinction.

To make this distinction, we look at the **boundary** of the edges, which is defined with respect to the **vertices** of an edge.

Definition 37 (Boundary of an Edge)

We define the boundary of an edge e, whose endpoints, or more correctly vertices, are a and b, as follows:

- *if e is an undirected edge, then both a and b are the boundaries of e;*
- *if e is a directed edge, say from a to b, then we denote the boundary of e*

[1] We can look at the following idea from group theory when we talk about **transpositions**, i.e. the swapping of two 'indices', for every one of the descriptions. For instance,

$$\begin{aligned}(1\,2\,3) &= (1\,2)(2\,3)(3\,1) \\ &= -(2\,1)(1\,3)(3\,2) \\ &= -(2\,1\,3).\end{aligned}$$

[2] This is taking a page out of graph theory.

as

$$\partial e = b - a.$$

Remark 11.1.1

There is a good reason behind using notation $\partial e = b - a$, which we shall see later.

Example 11.1.1

Referring back to Figure 11.3, we see that the boundary of e is

$$\partial e = (b) - (a).$$

Example 11.1.2

Looking back at Figure 11.2, we have that

$$\begin{aligned} \partial(1\ 2) &= (2) - (1) & \partial(2\ 3) &= (3) - (2) \\ \partial(3\ 1) &= (1) - (3) & \partial(1\ 4) &= (4) - (1) \\ \partial(4\ 3) &= (3) - (4). \end{aligned}$$

Notice that

$$\partial(1\ 2) + \partial(2\ 3) + \partial(3\ 1) = (2) - (1) + (3) - (2) + (1) - (3) = 0,$$

and $(1\ 2)(2\ 3)(3\ 1)$ is one of the cycles which we have identified that encloses a 'hole'.

For reasons that will become clear later, we shall, from hereon, write

$$(1\ 2)(2\ 3)(3\ 1) = (1\ 2) + (2\ 3) + (3\ 1)$$

to represent the cycle z_1 in Figure 11.2.

Let C_0 denote the **free abelian group**[3] on vertices, and C_1 denote the free abelian group on (directed) edges.

[3] See Section 15.1

Example 11.1.3

Let C_0 be the free abelian group on the vertices x, y, z. Then an element of C_0 would be $2x + 3y - z$, i.e. C_0 is a group that contains linear combinations of x, y, z with integer coefficients.

Following the same 'rule', elements of C_1, the free abelian group on the directed edges e, d, f are of the form $\alpha_1 e + \alpha_2 d + \alpha_3 f$.

We shall now give C_n a proper name.

Definition 38 (n-Chain)

*Elements of C_0 are called **0-dimensional chains**, while elements of C_1 are called **1-dimensional chains**.*

*In general, elements of C_n are called **n-dimensional chains**, or simply n-chains. C_n itself is called a **group of n-chains**.*

We may now make the definition of ∂ more formally so:

Definition 39 (Boundary of Chains)

*The **boundary of an n-chain**, denoted ∂e, where e is an n-chain is a homomorphism*

$$\partial : C_n \to C_{n-1}.$$

Remark 11.1.2

Since ∂ is a homomorphism, it is therefore linear. One can verify that the group of n-chains is indeed an abelian (commutative) group, which explains the notation $e + d$.

Putting the two together, we have

$$\partial(e + d) = \partial e + \partial d.$$

FOLLOWING the observation that we noticed in Example 11.1.2, one can go back on our examples and verify to oneself that the edges that form a cycle will have no boundary, i.e. if $e+d+f$ forms a cycle, then $\partial(e+d+f)=0$.

We shall now try to find all the possible cycles in the following simple example:

Example 11.1.4

Consider Figure 11.4. We want to solve for

$$\partial[\alpha_1(1\ 2)+\alpha_2(2\ 3)+\alpha_3(3\ 1)]=0.$$

Then we have

$$\begin{aligned}\alpha_1[(2)-(1)]+\alpha_2[(3)-(2)]+\alpha_3[(1)-(3)]&=0\\(1)(\alpha_3-\alpha_1)+(2)(\alpha_1-\alpha_2)+(3)(\alpha_2-\alpha_3)&=0.\end{aligned}$$

We thus have the following system of equations

$$\begin{aligned}-\alpha_1+\alpha_3&=0\\\alpha_1-\alpha_2&=0\\\alpha_2-\alpha_3&=0.\end{aligned}$$

We shall solve for this using **row reduction** on matrices: we have

$$\begin{bmatrix}-1&0&1\\1&-1&0\\0&1&-1\end{bmatrix}\sim\begin{bmatrix}1&0&-1\\0&1&-1\\0&0&0\end{bmatrix}.$$

By letting $\alpha_3=r$, we have that the solutions are

$$\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}=r\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

It follows that the null space for this solution is spanned by the vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$, which is therefore isomorphic to $\mathbb{Z}$.

Remark 11.1.3

It is no coincidence that the dimension of $\mathbb{Z}$ agrees with the total number of cycles.

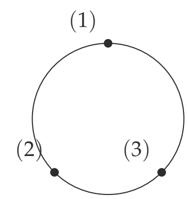


Figure 11.4: Simple example into using boundaries to identify cycles

Example 11.1.4 gives us a procedure for finding the number of cycles:

1. Equate the boundary of an arbitrary chain to 0.
2. Find the solutions to the system of equations, which is just finding the **null space** to the matrix that the system of equations coincide with.

We shall now make the following definition:

Definition 40 (Cycle)

*A chain e is a **cycle** if $\partial e = 0$.*

12 Lecture 12 Feb 01st

12.1 Homology (Continued)

Continuing from before, we may classify chains into equivalence classes if the chains form a boundary.

Definition 41 (Homologous)

Two n-chains a_1, a_2 are said to be ***homologous*** *if*

$$a_1 - a_2 \in \operatorname{Img} \partial_{n+1},$$

where ∂_{n+1} is the boundary operator from C_{n+1} to C_n. We shall write $a_1 \sim a_2$.

Two n-cycles z_1, z_2 are said to be ***homologous*** *if*

$$z_1 - z_2 = \partial_{n+1}\sigma$$

for some $(n+1)$-chain, and we also label this as $z_1 \sim z_2$.

Example 12.1.1

Consider a disc in $\mathbb{R}^2$. We know that the disc is homeomorphic to a triangle whose interior is filled. This process of identifying a 'shape' with triangles is called **triangulation**. We see that

$$b_1 \sim (b_2 + b_3) \quad b_2 \sim (b_3 + b_1) \quad b_3 \sim (b_1 + b_2),$$

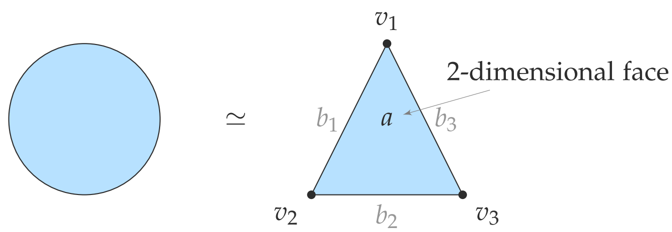


Figure 12.1: Triangulation of a disc

which forms the boundaries of a, and

$$v_1 \sim v_2 \quad v_2 \sim v_3 \quad v_3 \sim v_1,$$

which forms the boundary of b_1, b_2 and b_3 respectively. However, a is not homeomorphic to any other faces.

Example 12.1.2

Similar to the above, we can triangulate a sphere using a crystal-like shape (cf. Figure 12.2).

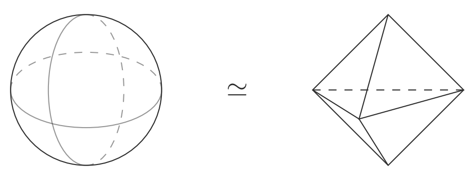


Figure 12.2: Triangulation of a sphere

Under the maps ∂_n, for $n \geq 0$, we can relate the chains as follows:

Definition 42 (Chain Complex)

A sequence of groups of chains

$$\ldots \overset{\partial_{n+1}}{\to} C_n \overset{\partial_n}{\to} C_{n-1} \overset{\partial_{n-1}}{\to} \ldots \overset{\partial_0}{\to} 0$$

is called a ***chain complex*** *if* $\partial_i \circ \partial_{i+1} = 0$*, for* $i \geq 0$.

Example 12.1.3

In Example 12.1.1, we have the following chain complex

$$\ldots \to C_3 = 0 \overset{\partial_3}{\to} C_2 \overset{\partial_2}{\to} C_1 \overset{\partial_1}{\to} C_0 \overset{\partial_0}{\to} 0,$$

Note that the n-chains are constructed using a specific triangulation. However, it does not matter, which is something that shall be proven much later.

where we note that $C_n = 0$ for $n \geq 3$ since there are no 3-chains and above, we have

$$\begin{aligned} C_2 &= \{na \mid n \in \mathbb{Z}\} \simeq \mathbb{Z} \\ C_1 &= \left\{ \sum_{i=1}^{3} n_i b_i \mid n_i \in \mathbb{Z} \right\} \simeq \mathbb{Z}^3 \\ C_0 &= \left\{ \sum_{i=1}^{3} n_i a_i \mid n_i \in \mathbb{Z} \right\} \simeq \mathbb{Z}^3 \end{aligned}$$

We are now ready to provide a more specific definition of Definition 40 that is more useful to us.

Definition 43 (k-cycles)

*A **k-cycle** is a k-chain z such that $\partial_k z = 0$, i.e. $z \in \ker \partial_k$. We denote the set of k-cycles as Z_k.*

Note that chains are free abelian groups, and so $Z_k \triangleleft C_k$. By the property $\partial_k \circ \partial_{k+1}$, we have that $\operatorname{Img} \partial_{k+1} \triangleleft Z_k$.

Definition 44 (Homology Groups)

The k-th Homology is defined as

$$H_k := \ker \partial_k \big/ \operatorname{Img} \partial_{k+1}.$$

Now WHY is it true that $\partial_k \circ \partial_{k+1} = 0$?

Proposition 23 (Boundary of a Boundary is Zero)

For any $k \geq 0$, we have

$$\partial_k \circ \partial_{k+1} = 0.$$

We shall prove this for the $k = 1$ case. One can prove for the more general case later on using **simplicial homology**.

Proof

Consider the 2-face $[(1)\ (2)\ (3)]$ as shown in Figure 12.3.

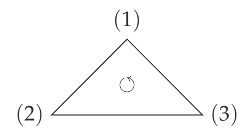


Figure 12.3: Your everyday 2-face

We have

$$\partial_2[(1)\ (2)\ (3)] = [(3)\ (1)] + [(1)\ (2)] + [(2)\ (3)].$$

Then

$$\begin{aligned}&\partial_1([(3)\ (1)] + [(1)\ (2)] + [(2)\ (3)])\\ &= (1) - (3) + (2) - (1) + (3) - (2) = 0.\end{aligned}$$

$\square$

13 Lecture 13 Feb 04th

I missed this lecture and I have not been able to obtain any notes for it. This lecture seems to have introduced simplicial complexes, simplices, and did some homology on some of the basic structures that we have came across. I will try to fill these in.

14 Lecture 14 Feb 08th

14.1 Simplicial Homology (Continued)

In the last lecture, we showed the the following:

- The homology group of the filled triangle $\blacktriangle$ is

$$\begin{aligned} H_0(\blacktriangle) &= \mathbb{Z} \\ H_1(\blacktriangle) &= 0 \\ H_2(\blacktriangle) &= 0 \end{aligned}$$

- The homology group of the **tetrahedron** $\triangle$ is

$$\begin{aligned} H_0(\triangle) &= \mathbb{Z} \\ H_1(\triangle) &= 0 \\ H_3(\triangle) &= \mathbb{Z} \end{aligned}$$

We shall now turn to studying the homology groups of the torus.

Example 14.1.1 (Homology of the Torus)

Our goal is to show that the homology group of the torus $\mathbb{T}^2$, as shown in Figure 14.1, is given by

$$\begin{aligned} H_0(\mathbb{T}^2) &= \mathbb{Z} \\ H_1(\mathbb{T}^2) &= \operatorname{span}_{\mathbb{Z}}\{\bigcirc, \bigcirc\} = \mathbb{Z} \oplus \mathbb{Z} \\ H_2(\mathbb{T}^2) &= \mathbb{Z} \end{aligned}$$

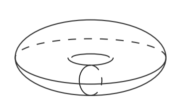


Figure 14.1: Homology of a torus

We may triangulate $\mathbb{T}^2$ as in Figure 14.2.

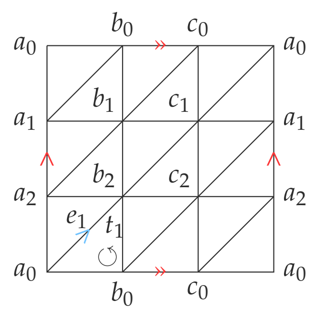


Figure 14.2: Triangulating $\mathbb{T}^2$

Note that the red arrows and blue arrows indicate different things:

- > indicates the sides which are identified with one another;
- > indicates an orientation on the 1-chain.

$H_0(\mathbb{T}^2)$ Notice that there are a total of 10 vertices in Figure 14.2. Thus $\ker \partial_0 = \mathbb{Z}^{10}$.

Notice that one may fill in the orientations on the 1-chains in a way illustrated in Figure 14.3. It can be shown that given any lin-

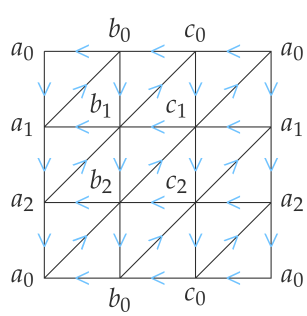


Figure 14.3: An set of orientations for the 1-chains on $\mathbb{T}^2$

ear combination of 1-chains, one of the coefficients will end up being dependent on all others. In other words, $\operatorname{Img} \partial_1 = \mathbb{Z}^9$. Thus $H_0(\mathbb{T}^2) = \mathbb{Z}$, as required.

$H_2(\mathbb{T}^2)$ Note that $C_3 = 0$, and so $\operatorname{Img} \partial_3 = 0$. By the triangulation in Figure 14.2, we know that $C_2 \simeq \mathbb{Z}^{18}$. Now by applying the same orientation on each of the 2-faces, we notice that in order for the 1-cycles to cancel out each other (i.e. in order for $\partial_2 \Delta$, where Δ is an arbitrary 2-chain, to be 0), we need the coefficients of all of the 2-chains to be the same. Thus $\ker \partial_2 = \mathbb{Z}$. It follows that $H_2(\mathbb{T}^2) = \mathbb{Z}$.

$H_1(\mathbb{T}^2)$ [1]

[1] This is extremely tedious to deal with if we go down the normal route of calculating the nullity and rank. We shall use the following procedure to arrive at $H_1(\mathbb{T}^2)$.

(Figuring out $H_1(\mathbb{T}^2)$)

1. *Show that every 1-cycle is homologous to the 1-cycle that lies entirely on the boundary of the rectangle in Figure 14.2.*
2. *Show that every 1-cycle supported on the boundary of the rectangle is of the form $m_1 z_1 + m_2 z_2$, where $m_1, m_2 \in \mathbb{Z}$ and z_1, z_2 are as labelled in Figure 14.4.*
3. *Show that z_1 and z_2 are not homologous.*

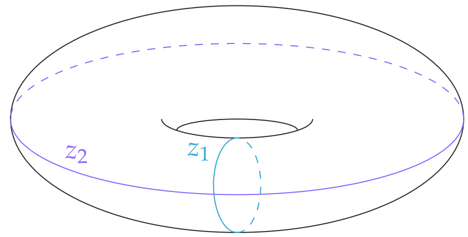


Figure 14.4: z_1 and z_2 on $\mathbb{T}^2$, as a basis for the null space of 1-chains.

I will fill in the details later when I've worked it out. Some steps are probably wrong or not well-justified.

Example 14.1.2 (Homology of the Klein Bottle)

I will have to revisit the compution of homology.

15 Lecture 15 Feb 11th

15.1 Free Abelian Groups

We shall make a little detour and look at free abelian groups, and look at some of the results that are useful to us.

Definition 45 (Free Abelian Groups)

*A group G is called a **free abelian group** if every $g \in G$ can be uniquely written as a finite sum*

$$g = \sum n_\alpha g_\alpha,$$

*where n_α are (integer) coefficients, and $\{g_\alpha\}$ is a **basis** for the group.*

Example 15.1.1

Consider

$$\tilde{G} = \bigoplus_{j \in \mathbb{N}} \mathbb{Z},$$

a sequence of finitely many non-zero components. As a set, $\tilde{G}$ is the **infinite Cartesian product** of infinitely many copies of $\mathbb{Z}$. We may thus consider the basis elements of $\tilde{G}$ as

$$g_j = (0, \ldots, 0, \underset{\substack{\uparrow \\ \text{j-th component}}}{1}, 0, \ldots).$$

Definition 46 (Generators of a Group)

If we remove the condition of unique representation, i.e. for $g \in G$,

$$g = \sum n_\alpha g_\alpha$$

*not necessarily uniquely, we then call $\{g_\alpha\}$ a set of **generators** for G.*

Definition 47 (Finitely Generated)

*We say that a group G is **finitely generated** if it has a finite set of generators.*

Example 15.1.2

Consider $\tilde{G}$ as in the last example. We have that $\tilde{G}$ is not finitely generated.

Example 15.1.3

$\mathbb{Z}^2$ is finitely generated: its generators are

$$\{g_1 = (1,0),\, g_2 = (0,1)\}.$$

Note 15.1.1

*Note that the above two examples are free abelian groups. This means that free abelian groups **do not have to be finitely generated**.*

Exercise 15.1.1

Show that any subgroup of a finitely generated abelian group is finitely generated.

Proof

Let G be a finitely generated abelian group, with generators

$\{g_i\}_{i=1}^n$. Let $H \leq G$. Then $\forall h \in H$, since $h \in G$, we have that

$$h = \sum n_i g_i$$

for some coefficients n_i. Thus H is finitely generated. $\square$

Note 15.1.2

Observe that if $\{g_\alpha\}$ and $\{h_\alpha\}$ are two finite basis for the finitely generated group G, then

$$|\{g_\alpha\}| = |\{h_\alpha\}| .$$

Definition 48 (Order)

An element $g \in G$ is said to have ***order*** *p if*

$$p \cdot g = 0$$

and $\forall q$, if $q \cdot g = 0$, then $p \leq q$.

Definition 49 (Torsion Group)

A group G is said to be ***torsion*** *if all of its elements have finite order.*

Example 15.1.4

Finite groups are necessarily torsion.

Exercise 15.1.2

Let G be an abelian group and

$$H := \{g \in G : g \text{ has finite order } \} .$$

Show that $H < G$. We say that H is a *torsion subgroup* *of G.*

Theorem 24 (Fundamental Theorem of Finitely Generated Abelian Groups)

*Let G be a finitely generated abelian group, and T a torsion subgroup. Then G **splits**, i.e.*

$$G \simeq K \oplus T,$$

where K is a free abelian group. Furthermore,

$$T \simeq T_1 \oplus \ldots \oplus T_k,$$

where T_j is of order m_j and $m_1 \mid m_2 \mid \ldots \mid m_k$.

16 Lecture 16 Feb 13th

16.1 Simplices and Simplicial Complexes

We defined what the homology of a triangulated surface was, and we used these triangles as the 'building blocks' of any shape.

Example 16.1.1

Recall that a 2-face is as shown in Figure 16.1.

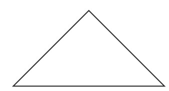


Figure 16.1: Typical 2-face

This is actually called a 2-simplex.

We did not give an explicit definition as to what a **face** is, but we shall do that in this section.

Definition 50 (Simplices)

*The **k-simplex** is defined as a set*

$$\sigma^{(k)} = \left\{ v_0 + \sum_{j=1}^{k} t_j(v_j - v_0) : 0 \leq t_j \leq 1, \sum_{j=1}^{k} t_j = 1 \right\}.$$

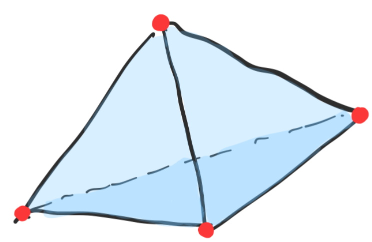


Figure 16.2: A typical simplicial complex

Definition 51 (Face)

Let σ be a k-simplex spanned by the vertices

$$\mathcal{A} = \{a_0, a_1, \ldots, a_k\}.$$

*Any simplex spanned by a subset of $\mathcal{A}$ of size $l \leq k$ is called an **l-face**.*

Definition 52 (Simplicial Complex)

A collection, K, of k-simplices is called a ***simplicial complex*** *if*

1. *every face of a simplex in K is also in K; and*
2. *the intersection of any two simplices in K is a face of both of them.*

Example 16.1.2

The diagram in Figure 16.4 is not a simplicial complex. For example, the edge $[a_3\, a_4]$ is not an intersection of two 2-faces.

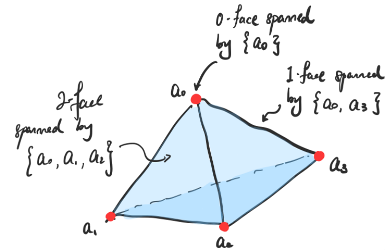


Figure 16.3: Faces on a simplicial complex

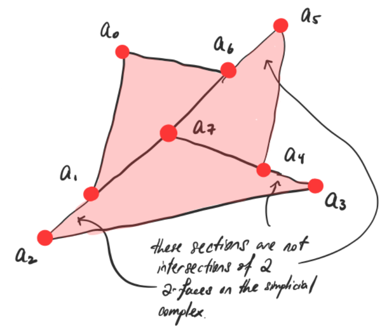


Figure 16.4: Non-example for a simplicial complex

Recall how it is common for us to represent a vertex as simply a_0 and an edge as $[a_0\, a_1]$. We may, for example, represent the edges in Figure 16.3 as

$$[a_0\, a_1],\ [a_1\, a_2],\ [a_2\, a_3],\ [a_3\, a_0],\ [a_0\, a_2],\ [a_3\, a_1],$$

and the 2-faces as

$$[a_0\, a_1\, a_2],\ [a_0\, a_2\, a_3],\ [a_0,\, a_3\, a_1],\ [a_1\, a_2\, a_3].$$

Note that the order of which we write the vertices in $[\ldots]$ indicates an orientation. For instance,

- $[a_0\, a_1]$ is the edge that has a_0 as its starting point, and a_1 its ending point;
- $[a_0\, a_1\, a_2]$ is the 2-face that is oriented anticlockwise.

Note 16.1.1

In relation to how we write the above, for edges, we represent the opposite orientation by simply writing the vertices in reverse. E.g. the opposite of $[a_0\, a_1]$ is $[a_1\, a_0]$. We relate the two together by denoting one as the negative of the other:

$$[a_0\, a_1] = -[a_1\, a_0].$$

This notation is consistent with the use of the boundary operator introduced earlier; in particular

$$\partial([a_0\, a_1]) = a_1 - a_0 = -(a_0 - a_1) = -\partial([a_1\, a_0]).$$

We may also do so for other simplices.

Definition 53 (Boundary of Simplicies)

Let $[a_0 \ldots a_k]$ be a k-simplex. We define the boundary of $[a_0 \ldots a_k]$ as

$$\partial[a_0 \ldots a_k] = \sum_{j=0}^{k} (-1)^j [a_0 \ldots \hat{a}_j \ldots a_k],$$

where $\hat{a}_j$ indicates the removal of a_j from the notation.

Example 16.1.3

Let $\sigma = [a_0\, a_1\, a_2]$ be a 2-face. The boundary of σ is

$$\partial\sigma = [a_1\, a_2] - [a_0\, a_2] + [a_0\, a_1] = [a_1\, a_2] + [a_2\, a_0] + [a_0\, a_1].$$

Notice that the last expression expresses the geometric boundary of σ, with an induced orientation.

Also, notice that we may also write σ as

$$[a_1\, a_2\, a_0],\ [a_2\, a_0\, a_1].$$

In Note 16.1.1, we learned that we can represent

$$[a_0\, a_1] = -[a_1\, a_0].$$

We can do the same for k-simplices for any k; in particular,

$$[a_0\, \ldots\, a_k] = (\operatorname{sgn} \tau)[a_{\tau(0)} \ldots a_{\tau(k)}]$$

for $\tau \in S_k$, where $\operatorname{sgn} \tau$ is the sign of τ, and S_k is the **permutation group on $\{1, \ldots, k\}$**.

16.2 *Relative Homology*

Definition 54 (Simplicial Subcomplex)

Given a simplicial complex K, a ***simplicial subcomplex*** *of K is a subset of the simplicies in K that still forms a simplicial complex.*

Since each of the sets of p-chains form abelian groups, any of its subgroups are normal, and in particular, if K_0 is a subcomplex of K, then we can take

$$C_p(K) \big/ C_p(K_0).$$

Definition 55 (Relative Chain)

Given a simplicial complex K and a subcomplex K_0,

17 Lecture 17 Feb 15th

A Assignment Problems

A.1 Assignment #1

1. (a) Find an explicit homeomorphism between the interval $(0, 1)$ and the real line $\mathbb{R}$.

   (a') Extend the last proof to show that the unit ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$.

   (b) Show that the circle, S^1, is not homeomorphic to any S^m for $m \geq 2$.

   (b') Show that S^1 is not homeomorphic to $[0, 1)$.

   (c) Prove that $[0, 1]$ is not homeomorphic to $\mathbb{R}$.

   (d) Show that the interval $[0, 1)$ is not homeomorphic to the real line.

2. Recall that the topologist's sine curse (similar to the topologist's circle) was defined as

$$f(x) := \begin{cases} \sin\left(\frac{1}{x}\right) & x \in (0, 1] \\ 0 & x = 0 \end{cases}.$$

   Show that the space given as the graph of this function, $\Gamma(f)$, is connected, but not path-connected or even locally connected.

3. (a) Show that in a Hausdorff space, the limit of a convergent sequence is unique.

   (b) Show that the subspace topology induced on a subset of a Hausdorff space is also Hausdorff.

4.(a) Recall that the projection operator $\pi_1 : Y_1 \times Y_2 \to Y_1$ is defined by $\pi_1 : (y_1, y_2) \mapsto y_1$. Show that π_1, and similarly π_2, are continuous when $Y_1 \times Y_2$ is equipped with the product topology.

(b) Let $f : X \to Y_1 \times Y_2$ be a continuous map. Define $f_j : \pi_j \circ f$. Show that f is continuous iff f_1 and f_2 are continuous.

(c) Let $g : X_1 \times X_2 \to Y$ be a function. We say that g is continuous in each variable iff for any given x_0 and y_0, the functions $g(x_0, \cdot)$ and $g(\cdot, y_0)$ are continuous. Show that if g is continuous, then it is continuous in each variable.

(d) Show that the converse of the last statement does not hold.

(e) Is the projection map π_1 an open map, a closed map, or neither?

5.(a) Show that the image of a connected set is connected.

(b) Prove the intermediate value theorem for functions defined on the interval.

(c) Show that any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point, that is a point x such that $f(x) = x$.

6. Consider the metric given by $\tilde{d}(x, y) := \arctan(|x - y|)$. Determine if the topology induced is equivalent to the standard topology of $\mathbb{R}^n$. Further, determine if this metric satisfies the Heine-Borel theorem, i.e. is compactness of a subset equivalent to it being closed and bounded with respect to $\tilde{d}$.

A.2 Assignment #2

1.(a) Let M be a manifold with or without boundary. Show that $\operatorname{Int} M$ is closed iff $\partial M = \emptyset$.

(b) Show that the boundary of an n-dimensional manifold is an $(n-1)$-dimensional manifold *without* boundary.

(c) Show that $[0, 1)$ is not homeomorphic to $(0, 1)$ and use that idea to prove that the invariance of boundary for 1-dimensional manifolds.

(d) Prove the invariance of dimension for 1-dimensional manifolds.

2.(a) Show that the closed quadrant $M := \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ is a manifold with boundary. Describe the coordinate charts explicitly.

(a') Generalise what you proved in part (a) to $N := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1, \ldots, x_n \geq 0\}$,

(b) Let M_1^m and M_2^n be two topological manifolds of dimensions m and n respectively. Take $N := M_1 \times M_2$ with its product topology. Show that N is an $(n + m)$-dimensional manifold whose boundary is $\partial N = M_1 \times \partial M_2 \cup \partial M_1 \times M_2$. Conclude, in particuar, that N has no boundary iff M_1 and M_2 have empty boundary.

3. Let P be an arbitrary n-gon, that is to say, a union of finite number of line segments such that each line end point is shared by exatly two segments. Show that P is homeomorphic to S^1. (You might find the glueing lemma stated as Lemma 3.23 of J. Lee's book useful.)

4. Let $f : M_1 \to M_2$ be a homeomorphism between two manifolds with a boundary. Show that under f the boundary of M_1 is sent to the boundary of M_2. Conclude that in such a case we must have $\partial M_1 \simeq \partial M_2$.

5. Let $\mathbb{T}^2$ be the standard torus obtained by taking the quotient $\mathbb{R}^2/(\mathbb{Z} \oplus \mathbb{Z})$ with the projection map $\Pi : \mathbb{R}^2 \to \mathbb{T}^2$. Define L to be the line $y = \sqrt{2}x$ in $\mathbb{R}^2$.

(a) Determine if the image of L under this projection, $\Pi(L)$, is a manifold or not.

(b) Show that $\Pi(L)$ is the image of $\mathbb{R}$ under a bijective continuous map, but it is not homeomorphic to $\mathbb{R}$. Conclude that $\Pi \restriction_L$ is not an embedding of L.

6. Recall that we showed in class how to obtain the cylinder as the quotient of the plane by the action of $\mathbb{Z}$. Define a group action of G on $\mathbb{R}^2$ such that $\mathbb{R}^2/G$ will be homeomorphic to the infinite Möbius band.

A.3 *Assignment #3*

1. Find an injective open map between topological spaces which is not continuous.

2. Consider the Möbius band with boundary, $\mathfrak{M}$.

   (a) Find an explicit embedding $f : \mathfrak{M} \to \mathbb{R}^3$ using elementary functions.

   (b) Generalise that embedding to the case of the space obtained by k times twisting and then glueing the unit cell $[0,1] \times [0,1]$ along $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$.

   (c) Conclude that all bands with an even number of twists are homeomorphic to the cylinder $S^1 \times [0,1]$ and the ones with an odd number of twists are homeomorphic to $\mathfrak{M}$.

3. Consider the action of $\mathbb{Z}$ on $M := \mathbb{R}^n \setminus \{0\}$ defined by $k \cdot (x_1, \ldots, x_n) := (2^k x_1, \ldots, 2^k x_n)$.

   - Show that for any element $k \in \mathbb{Z}$, the action on M by k has no fixed points. (Recall that for an element $g \in G$ acting on a space X, we say that $x \in X$ is a fixed point provided that $g \cdot x = x$.)

   - Can you identify the quotient space $M/\mathbb{Z}$? (It might help to start from lower dimensions.)

4. (Embedding the projective plane) Recall how we defined the projective plane $\mathbb{P}^2$ as the quotient $S^2/((x,y,z) \sim -(x,y,z))$. This can be generalized to the case of $\mathbb{P}^n$, which is defined as the quotient of $S^n \subset \mathbb{R}^{n+1}$ by the antipodal map $w \in \mathbb{R}^{n+1} \mapsto -w$.

   (a) Show that any $\mathbb{P}^n$ embeds in $\mathbb{P}^{n+1}$. (Hint: It can be done in two lines.)

   (b) Consider the map $F : \mathbb{R}^3 \to \mathbb{R}^4$ defined as $(x,y,z) \mapsto (xy, yz, xz, x^2 - y^2)$. Show that the map $f := F \restriction_{S^2}$ has the property that $f(x,y,z) = f(-x,-y,-z)$. Use this observation to show that this map descends to a continuous map $\tilde{f} : \mathbb{P}^2 \to \mathbb{R}^4$.

   (c) Show that $\tilde{f}$ is injective. Notice that $\mathbb{P}^2$ is compact, hence, any continuous map is a closed map. (Indeed, we stated this as a general lemma: any continuous injection from a compact space is an embedding.) Combine these two observations to prove that $\tilde{f}$ gives an embedding of $\mathbb{P}^2$ into $\mathbb{R}^4$.

5. Recall that we found the homology groups of sphere S^2 by finding the homology groups of a (boundary of) a tetrahedron (cf. Figure A.1). We can see this tetrahedron as the manifold boundary of a 3-simplex. Let K_n denote the simplicial complex formed by the

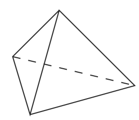


Figure A.1: 3-simplex, tetradedron

boundary of an $(n+1)$-simplex. Generalise what we did in class in case of the tetrahedron to show that we have

$$H_0(K_n) = \mathbb{Z}, H_1(K_n) = 0, \ldots, H_{n-1}(K_n) = 0, H_n(K_n) = \mathbb{Z}.$$

Let us assume the result that says homology is invariant under homeomorphism. Conclude that we have for the n-dimensional sphere, S^n:

$$H_0(S^n) = \mathbb{Z}, H_1(S^n) = 0, \ldots, H_{n-1}(S^n) = 0, H_n(S^n) = \mathbb{Z}. \quad \text{(A.1)}$$

6. Verify that the triangulated surface in the figure below is homeomorphic to the Möbius strip and find its homology groups.

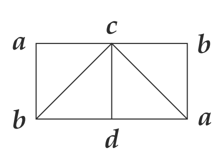


Figure A.2: Möbius band

7. Recall that a holomorphic map that is not constant is an open map. More generally, a meromorphic function can be seen as an open map when the target space is taken to be the extended plane $\mathbb{C} \cup \{\infty\}$. This is the content of the open mapping theorem in complex analysis. Conoclude that a non-constant holomorphic (meromorphic) map is a quotient map. In particular, study the

case of the following function wherein k is an integer.

$$\begin{cases} f_k : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\} \\ z \mapsto z^k \end{cases}$$

Show that f_k is holomorphic to the extended complex plane and may, hence, be seen as a quotient map $S^2 \to S^2$. Identify the various types of fibres over various points of the target.

(a) Consider the action on $\mathbb{C}$ of $\mathbb{Z}_m$ given by multiplication by ζ_m, the m-th root of unity. Describe the corresponding action of $\mathbb{Z}_m$ on the sphere S^2. Show that under this action $\mathbb{Z}_m$ on S^2, we have $S^2/\mathbb{Z}_m \simeq S^2$.
(**Remark**: This equivalence is only at the level of homeomorphism and does not hold at the level of diffeomorphism.)

(b) Let Λ be a lattice in $\mathbb{R}^2 = \mathbb{C}$. Show that a function can be seen as a continuous function on the torus $\mathbb{C}/\Lambda$ iff it is Λ-periodic.

(b′) (Optional) Consider the following function defined on $\mathbb{C}$. Show that it is doubly periodic where the lattice Λ is generated by 1 and i.

$$\wp(\xi) := \frac{1}{\xi^2} + \sum_{\mu^2+\nu^2 \neq 0} \left(\frac{1}{(\xi + m + ni)^2} - \frac{1}{(m+ni)^2} \right)$$

(You can do so by showing that the derivative $\frac{d\wp}{dz}$ is a doubly periodic function. Indeed, it is easy to show that the derivative converges away from the poles. Further, $\wp$ is an even function. This, along with the double periodicity of the derivative, gives periodicity of $\wp$.) This is an example of a Weierstrass $\wp$-function.

(c) Use the previous observations to conclude that this function descends to a continuous function on the torus T obtained from $[0,1] \times [0,1] \subset \mathbb{R}^2$. Indeed, more is true. We shall not go into the details in full generality, but since $\wp$ is meromorphic on the complex plane, it descends to a meromorphic function on T. You can verify meromorphicity by studying the function along the boundaries of the torus.

(d) Show that $\wp$ provides a meromorphic function with two poles on $\mathbb{T}^2$. In particular, it is a quotient map from the torus $\mathbb{T}^2$ to the sphere S^2.

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