PMATH467 — Algebraic Geometry

Classnotes for Winter 2019

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Preface

The basic goal of the course is to be able to find **algebraic invariants**, which we shall use to classify topological spaces up to homeomorphism.

Other questions that we shall also look into include a uniqueness problem about manifolds; in particular, how many manifolds exist for a given invariant up to homeomorphism? We shall see that for a **2-manifold**, the only such manifold is the 2-dimensional sphere S^2. For a 4-manifold, it is the 4-dimensional sphere S^4. In fact, for any other n-manifold for $n > 4$, the unique manifold is the respective n-sphere. The problem is trickier with the 3-manifold, and it is known as the Poincaré Conjecture, solved in 2003 by Russian Mathematician Grigori Perelman. Indeed, the said manifold is homeomorphic to the 3-sphere.

For this course, you are expected to be familiar with notions from **real analysis**, such as topology, and concepts from **group theory**.

The following topics shall be covered:

1. Point-Set Topology

2. Introduction to Topological Manifolds

3. Simplicial complexes & Introduction to Homology

4. Fundamental Groups & Covering Spaces

5. Classification of Surfaces

Basic Logistics for the Course

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- OH: (Tue) 1630 - 1800, (Fri) 1245 - 1320

- OR: MC 6457

- EM: aaleyasin

Part I

Point-Set Topology

1 Lecture 1 Jan 07th

We will not be too rigorous in this part.

1.1 Euclidean Space

For any $(x_1, \ldots, x_m) \in \mathbb{R}^m$, we can measure its distance from the origin 0 using either

- $\|x\|_\infty = \max\{|x_i|\}$ (the supremum-norm);

- $\|x\|_2 = \sqrt{\sum (x_j)^2}$ (the 2-norm); or

- $\|x\|_p = \left(\sum |x_j|^p\right)^{\frac{1}{p}}$ (the p-norm),

where we may define a "distance" by

$$d_p(x,y) = \|x - y\|_p.$$

📘 **Definition 1 (Metric)**

Let X be an arbitrary space. A function $d : X \times X \to \mathbb{R}$ is called a **metric** *if it satisfies*

1. *(symmetry) $d(x,y) = d(y,x)$ for any $x,y \in X$;*

2. *(positive definiteness) $d(x,y) \geq 0$ for any $x,y \in X$, and $d(x,y) = 0 \iff x = y$; and*

3. *(triangle inequality) $\forall x,y,z \in X$*

$$d(x,y) \leq d(x,z) + d(y,z).$$

📑 **Definition 2 (Open and Closed Sets)**

Given a space X with a metric d, and r > 0, the set

$$B(x,r) := \{w \in X \mid d(x,w) < r\}$$

*is called the **open ball** of radius r centered at x. An **open set** A is such that $\forall a \in A$, $\exists r > 0$ such that*

$$B(a,r) \subseteq A.$$

*We say that a set is **closed** if its complement is open.*

📑 **Definition 3 (Continuous Map)**

A function

$$f : (X,d_1) \to (Y,d_2)$$

*is said to be **continuous** if the preimage of an open set in Y is open in X.*

See notes on Real Analysis for why we defined a continuous map in such a way.

🐞 **Warning**

This definition does not imply that a continuous map f maps open sets to open sets.

Exercise 1.1.1

Contruct a function on $[0,1]$ which assumes all values between its maximum and minimum, but is not continuous.

✏️ **Solution**

Consider the piecewise function

$$f(x) = \begin{cases} x & 0 \leq x < \frac{1}{2} \\ x - \frac{1}{2} & x \geq \frac{1}{2}. \end{cases}$$

It is clear that the maximum and minimum are $\frac{1}{2}$ and 0 respectively, and f assumes all values between 0 and $\frac{1}{2}$. However, a piecewise function is not continuous.

📔 Definition 4 (Homeomorphism)

*A function f is a **homeomorphism** if it is a bijection and both f and f^{-1} are continuous.*

Example 1.1.1

The function

$$g : [0, 2\pi) \to \mathbb{R}^2 \text{ given by } \theta \mapsto (\cos\theta, \sin\theta)$$

is not homeomorphic, since if we consider an alternating series that converges to 0 on the unit circle on $\mathbb{R}^2$, we have that the preimage of the series does not converge and f^{-1} is in fact discontinuous.

Now, we want to talk about topologies without referring to a metric.

📔 Definition 5 (Topology)

*Let X be a space. We say that the set $\mathcal{T} \subseteq \mathcal{P}(X)$ is a **topology** if*

1. *$X, \varnothing \in \mathcal{T}$;*

2. *if $\{x_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ for an arbitrary index set A, then*

$$\bigcup_{\alpha \in A} x_\alpha \in \mathcal{T}; \text{ and}$$

3. *If $\{x_\beta\}_{\beta \in B} \subset \mathcal{T}$ for some finite index set B, then*

$$\bigcap_{\beta \in B} x_\beta \in \mathcal{T}.$$

2 Lecture 2 Jan 09th

2.1 Euclidean Space (Continued)

In the last lecture, from metric topology, we generalized the notion to a more abstract one that is based solely on open sets.

Example 2.1.1

Let X be a set. The following two are uninteresting examples of topologies:

1. The **trivial topology** $\mathcal{T} = \{\varnothing, X\}$.

2. The **discrete topology** $\mathcal{T} = \mathcal{P}(X)$.

WE SHALL NOW continue with looking at more concepts that we shall need down the road.

📄 **Definition 6 (Closure of a Set)**

Let A be a set. Its **closure**, denoted as $\overline{A}$, is defined as

$$\overline{A} = \bigcap_{\substack{C \supset A}}^{C:\ closed} C.$$

It is the smallest closed set that contains A.

❝ **Note**
In metric topology, one typically defines the closure of a set by taking the union of A and its limit points.

📖 Definition 7 (Interior of a Set)

Let A be a set. Its *__interior__*, denoted either as $\text{Int}\,(A)$, A° or $\mathring{A}$, is defined as

$$\mathring{A} = \bigcup_{\substack{G \subseteq A}}^{\substack{G:\ open}} G.$$

📖 Definition 8 (Boundary of a Set)

Let A be a set. Its *__boundary__*, denoted as ∂A, is defined as

$$\partial A = \overline{A} \setminus \mathring{A}.$$

Exercise 2.1.1

Let A be a set. Prove that ∂A is closed.

✏️ Proof

Notice that

$$(\partial A)^C = (\overline{A} \setminus \mathring{A})^C = X \setminus \overline{A} \cup \mathring{A} = X \cap \overline{A}^C \cup \mathring{A}$$

which is open. □

Exercise 2.1.2

Let A be a set. Show that
$$\partial(\partial A) = \partial A.$$

✏️ Proof

First, notice that $\mathring{\partial A} = \varnothing$. Since ∂A is closed, $\overline{\partial A} = \partial A$. Then

$$\partial(\partial A) = \overline{\partial A} \setminus \mathring{\partial A} = \partial A \setminus \varnothing = \partial A$$

Example 2.1.2

We know that $\mathbb{Q} \subseteq \mathbb{R}$, and $\overline{\mathbb{Q}} = \mathbb{R}$. We say that $\mathbb{Q}$ is dense in $\mathbb{R}$.

📓 **Definition 9 (Dense)**

We say that a subset A of a set X is dense if

$$\overline{A} = X.$$

Example 2.1.3

From the last example, we have that $\overset{\circ}{\mathbb{Q}} = \varnothing$.

📓 **Definition 10 (Limit Point)**

We say that $p \in X \supseteq A$ is a limit point of A if any neighbourhood of p has a nontrivial intersection with A.

Example 2.1.4 (A Topologist's Circle)

Consider the function

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

on the interval $\left[-\frac{1}{2\pi}, \frac{1}{2\pi}\right]$. Extend the function on both ends such that we obtain Figure 2.1 (See also: Desmos).

The limit points of the graph includes all the points on the straight line from $(0, -1)$ to $(0, 1)$, including the endpoints. This is the case because for any of the points on this line, for any neighbourhood around the point, the neighbourhood intersects the graph f infinitely many times.

GOING BACK TO CONTINUITY, given a function f, how do we know if f^{-1} maps an open set to an open set?
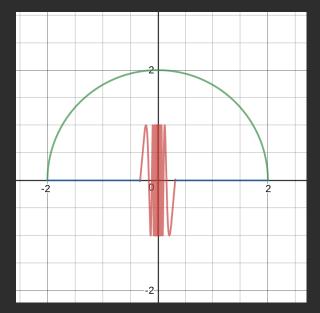
Figure 2.1: A Topologist's Circle

We can actually reduce the problem to only looking at open balls. But why are we allowed to do that?

📘 **Definition 11 (Basis of a Topology)**

*Given a topology $\mathcal{T}$, we say that $\mathcal{B} = \{B_\alpha\}_{\alpha \in I}$ is a **basis** if $\forall T \in \mathcal{T}$, there exists $J \subset I$ such that*

$$T = \bigcup_{\alpha \in J} B_\alpha.$$

Note that while the definition is similar to that of a cover, we are now "covering" over sets and not points.

Example 2.1.5

Let $\mathcal{T}$ be the Euclidean topology on $\mathbb{R}$. Then we can take

$$\mathcal{B} = \{(a,b) \mid a, b \in \mathbb{R},\ a \leq b\}.$$

Note that $\mathcal{B}$ is **uncountable**. We can, in fact, have [1]

$$\mathcal{B}_1 = \{(a,b) \mid a, b \in \mathbb{Q},\ a \leq b\},$$

which is countable, as a basis for $\mathbb{R}$. Furthermore, we can consider the set

$$\mathcal{B}_2 = \left\{ (a,b) \ \middle|\ a \leq b,\ a = \frac{m}{2^p},\ b = \frac{n}{2^q},\ m, n, p, q \in \mathbb{Z} \right\},$$

[1] Recall from PMATH 351 that we can write $\mathbb{R}$ as a disjoint union of open intervals with rational endpoints.

which is also a countable basis for $\mathbb{R}$. Notice that

$$\mathcal{B}_2 \subseteq \mathcal{B}_1 \subseteq \mathcal{B}.$$

Example 2.1.6

In $\mathbb{R}^2$, we can do a similar construction of $\mathcal{B}$, $\mathcal{B}_1$, and $\mathcal{B}_2$ as in the last example and use them as a basis for $\mathbb{R}^2$. In particular, we would have

$$\mathcal{B} = \{(a_1, b_1) \times (a_2, b_2) \mid a_1, a_2, b_1, b_2 \in \mathbb{R}\}.$$

This is called a **dyadic partitioning** of $\mathbb{R}^2$.

Example 2.1.7

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be two topological spaces. Then the Cartesian product $X_1 \times X_2$ has topology induced from $\mathcal{T}_1$ and $\mathcal{T}_2$ by taking the set

$$\mathcal{B} = \{\beta_1 \times \beta_2 \mid \beta_1 \in \mathcal{T}_1, \beta_2 \in \mathcal{T}_2\}$$

as the basis.

Exercise 2.1.3

Prove that

1. *β_1 and β_2 can be taken to be elements of bases $\mathcal{B}_1 \subset \mathcal{T}_1$ and $\mathcal{B}_2 \subset \mathcal{T}_2$, respectively.*

2. *the product topology on $\mathbb{R}^2$ is the same as the Euclidean topology.*

3 Lecture 3 Jan 11th

3.1 Euclidean Space (Continued 2)

Let $\tilde{X}$ be a metric space, and $p, q \in \tilde{X}$ with $p \neq q$. Then we have that $d(p, q) = r > 0$.

Then we must have that

$$B\left(p, \frac{r}{3}\right) \cap B\left(q, \frac{r}{3}\right) = \varnothing.$$

Exercise 3.1.1

Prove that the above claim is true. (Use the triangle inequality)

Figure 3.1: Idea of separation

✏️ **Proof**

Suppose $\exists x \in B\left(p, \frac{r}{3}\right) \cap B\left(q, \frac{r}{3}\right)$. Then

$$d(p, x) + d(q, x) < \frac{2r}{3} < r = d(p, q),$$

which violates the triangle inequality. □

We observe here that the two open sets (or balls) "separate" p and q.

📘 **Definition 12 (Hausdorff / T_2)**

*Let X be a topological space. X is said to be **Hausdorff** or T_2 iff any 2 distinct points can be separated by disjoint open sets.*

> **❝ Note**
>
> 1. *The Hausdorff space (or T_2 space) is an important space; we can only define a metric on spaces that are T_2.*
>
> 2. *A space is called T_1 is for any $p, q \in X$ with $p \neq q$, $\exists U \ni p$ open such that $q \notin U$ and $\exists V \ni q$ open such that $p \notin v$. It is worth noting that a T_2 space is also T_1.*

Example 3.1.1 (The Discrete Topology)

Suppose X is a metric space. For any $x \in X$, we have that $\{x\}$ is open. Thus for any $x_1, x_2 \in X$, if $x_1 \neq x_2$, then the open sets $\{x_1\}$ and $\{x_2\}$ separates x_1 and x_2.

This is true as we can define the following metric on the space: let $d : X \times X \to \mathbb{R}$ such that

$$d(x_1, x_2) = \begin{cases} 0 & x_1 = x_2 \\ 1 & x_1 \neq x_2 \end{cases}$$

This topology is called a **discrete topology**, and it is a metric space.

Let X be a metric space and $A \subseteq X$. Then there is a metric induced by X on A, and this in turn induces a topology on A.

More generally, if $A \subset X$ where X is some arbitrary topological space, then a set $U \subseteq A$ is open iff $U = A \cap V$ for some $V \subseteq X$ that is open. In other words, a subset U of A is said to be open iff we can find an open set V in X such that the intersection of A and V gives us U.

Exercise 3.1.2

Prove that the construction above gives us a topology.

> **✏ Proof**
>
> Let $A \subseteq X$. We shall show that τ_A is a topological space induced by the topological space τ of X. It is clear that $\varnothing \in \tau_A$, since it is open in X, and so $A \cap \varnothing = \varnothing$. Since X is open, we have $A \cap X = A$, and so $A \in \tau_A$.
>
> Now if $\{U_\alpha\}_{\alpha \in I} \subseteq \tau_A$, then $\exists V_\alpha \subseteq X$ such that $U_\alpha = A \cap V_\alpha$.

Then

$$\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} A \cap V_\alpha = A \cap \bigcup_{\alpha \in I} V_\alpha,$$

and $\bigcup_{\alpha \in I} V_\alpha$ is open in X by the properties of open sets. Thus $\bigcup_{\alpha \in I} U_\alpha \in \tau_A$.

If $\{U_i\}_{i=1}^n \subset \tau_A$, then again, by the properties of open sets, finite intersection of open sets is open, and so $\bigcap_{i=1}^n U_i \in \tau_A$.

$\square$

❝ Note

We can say the same can be said about closed sets of A.

Example 3.1.2
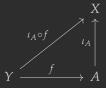
Let $A \subseteq X$ and consider the function

$$\iota_A : A \to X \text{ given by } x \mapsto x,$$

which is the inclusion map.

Then ι_A is continuous when the topology on A is chosen to be the induced subspace topology. This is rather clear; notice that the inverse of the inclusion map brings open sets to open sets.

Let Y be an arbitrary topological space. Then let

Figure 3.2: Composition of a function and the inclusion map

where f is continuous. Then $\iota_A \circ f$ is continuous.

The converse is also true: if $\iota_A \circ f$ is continuous, then f is continuous. However, we will not prove this. This property is known as the **characteristic property of the subspace topology**.

🌲 Lemma 1 (Restriction of a Continuous Map is Continuous)

Let $X \xrightarrow{f} Y$ be continuous, and $A \subseteq X$ [1]. Then

[1] Here, A is equipped with the subspace topology

$$f \upharpoonright_A : A \to Y$$

is also continuous.

Connected Spaces

Consider the real line $\mathbb{R}$, and consider two disjoint intervals on $\mathbb{R}$.
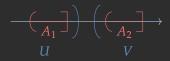
Figure 3.3: Motivation for Connectedness

 Observe that we may find two open subsets U and V of $\mathbb{R}$ such that $A_1 \subseteq U$ and $A_2 \subseteq V$, which effectively separates the two intervals on the space $\mathbb{R}$.

📑 **Definition 13 (Disconnectedness)**

*A space X is said to be **disconnected** iff X can be written as a disjoint union*

$$X = A_1 \coprod A_2$$

where $A_1, A_2 \subseteq X$, $A_1 = A_2^C$, that they are both non-empty and open [2].

[2] It goes without saying that the two sets are also simultaneously closed.

📑 **Definition 14 (Connctedness)**

*A space X is said to be **connected** if it is not disconnected.*

❝ **Note**
By the above definitions, we have that X is connected iff for any partition $X = A \coprod A^C$ with A being open, either A is $\varnothing$ or A is X.

Example 3.2.1

The space $\mathbb{R} \setminus \{0\}$ is disconnected; our disjoint sets are $(-\infty, 0)$ and $(0, \infty)$.

However, $\mathbb{R}^2 \setminus \{0\}$ is connected, but it is not easy to describe why.

📓 **Definition 15 (Path)**

*A **path** in a space X from p to q (both in X) is a continuous map f :
$[0,1] \to X$ such that $f(0) = p$ and $f(1) = q$. We say that X is **path
connected** if $\forall p, q \in X$, there is a path in X from p to q.*

🌲 **Lemma 2 (Path Connectedness implies Connectedness)**

If a space X is path connected, then it is connected.

🖥 **Theorem 3 (From Connected Space to Connected Space)**

If $X \xrightarrow{f} Y$ is continuous and X is connected, then $\mathrm{Img}(f)$ is connected.

4 Lecture 4 Jan 14th

4.1 Connected Spaces (Continued)

📖 Definition 16 (Locally Connected)

*We say that X is **locally connected** at x if for every open set V contain-*
ing x there exists a connected, open set U with $x \in U \subseteq V$. We say that
*the space X is **locally connected** if it is locally connected $\forall x \in X$.*

Example 4.1.1

The space S generated by the function $\sin \frac{1}{x}$ with 0 at $x = 0$, on the
$\mathbb{R}^2$ is not locally connected: consider $(0, y) \in S, y \neq 0$. Then any small
open ball at this point will contain infinitely many line segments
from S. This cannot be connected, as each one of these constitutes a
component, within the neighborhood.

📖 Definition 17 (Connected Component)

The maximal connected subsets of any topological space X are called
***connected components** of the space. The components form a partition of*
the space.

4.2 Compactness

📖 Definition 18 (Sequential Compactness)

For $A \subseteq X$, where X is a topological space, if $\{x_i\}_{i \in I} \subseteq A$, arbitrary sequence in A, has a convergent subsequence, we say that A is **sequentially compact**.

📘 **Definition 19 (Compactness)**

We say that a topological space X is **compact** if every **open cover** of X has a finite **subcover**.

🌲 **Lemma 4 (Compactness implies Sequential Compactness)**

Compactness implies sequential compactness.

Example 4.2.1

$[0,1)$ is not compact: consider the open cover $\left\{ \left[0, 1 - \frac{1}{n}\right) \right\}_{n \in \mathbb{N}}$, which contains $[0,1)$ as $n \to \infty$. But whenever n is finite, we have $1 - \frac{1}{n} < 1$, and so any finite collection of the $\left[0, \frac{1}{n}\right)$ is not a cover of $[0,1)$.

🖥 **Theorem 5 (Continuous Maps map Compact Sets to Compact Images)**

Let $f : X \to Y$ be continuous, where X is compact. Then $f(X)$ is compact.

✏️ **Proof**

Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $f(X)$. Since f is continuous, we have that $f^{-1}(U_\alpha)$ is open for each $\alpha \in I$. Since f is bijective between the image set and its domain, we have that $\{f^{-1}(U_\alpha)\}_{\alpha \in I}$ is an open cover of X. Since X is compact, this cover has a finite subcover, say $\{f^{-1}(U_i)\}_{i=1}^{n}$. Thus

$$X = \bigcup_{i=1}^{n} f^{-1}(U_i).$$

Thus

$$f(X) = f\left(\bigcup_{i=1}^{n} f^{-1}(U_i)\right) = \bigcup_{i=1}^{n} U_i.$$

Hence $\{U_\alpha\}_{\alpha \in I}$ has a finite subcover and so $f(X)$ is compact. $\quad\square$

➤ Corollary 6 (Homeomorphic Maps map Compact Sets to Compact Sets)

Let $X \xrightarrow{f} Y$ be a homeomorphism. Then X is compact iff Y is compact.

❝ Note
Compactness is a topological property.

🌲 Lemma 7 (Properties of Compact Sets)

1. A closed subset of a compact space is compact.

2. A compact subset of a topological space is closed provided that the space is Hausdorff.

3. In a metric space, a compact set is bounded.

4. Finite (Cartesian) product of compact sets is compact.

The proof for the first item is simple: consider an open cover of the closed subset, and union them with the complement of the closed subset. This covers the entire space, and so it has a finite subcover. We just need to then remove that complement set, and that would be a finite subcover for the closed subset.

Example 4.2.2

The subset $[-a, b]$, $a, b \in \mathbb{R}$, is compact.

Example 4.2.3

$[0, 1]^{\mathbb{N}}$ is not compact: the space is equivalent to ℓ_∞.

☕ **Theorem 8 (Heine-Borel)**

Let $X \subseteq \mathbb{R}^n$. Then X is compact iff X is closed and bounded.

✏️ **Proof**

$(\implies)$ We say that compactness impliess boundedness. Also, since $\mathbb{R}^n$ is Hausdorff, X is closed.[1].

> [1] Both from Lemma 7

$(\impliedby)$ X is bounded implies that $X \subseteq [-R, R]^n$ with R sufficiently large. Since X is closed, and $[-R, R]^n$ is compact, X is necessarily compact by Lemma 7. □

☕ **Theorem 9 (Bolzano-Weierstrass)**

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Exercise 4.2.1

Prove ☕ *Theorem 9 as an exercise.*

We shall start the next part this lecture.

4.3 Manifolds

📕 **Definition 20 (Locally Homeomorphic)**

*A space is said to be **locally homeomorphic** to $\mathbb{R}^n$ provided that $\forall x \in X, \exists U \ni x$ open such that U is homeomorphic to $\mathbb{R}^n$.*

📕 **Definition 21 (Manifold)**

*An n-dimensional **manifold** is a **second countable**[2], **Hausdorff** topological space that is **locally homeomorphic** to $\mathbb{R}^n$.*

> [2] a topological space is said to be **second countable** if its basis is countable.

❝ Note

One can give an equivalent definition of locally homeomorphic by requiring that U be homeomorphic to an open ball $B^n \subseteq \mathbb{R}^n$. Notice that B^n is homeomorphic to $\mathbb{R}^n$ [3]

[3] By scaling, really.

Example 4.3.1

Let $B^n = B^n(0,1) \subseteq \mathbb{R}^n$. Then B^n is homeomorphic to $\mathbb{R}^n$.

Example 4.3.2

Now consider the closed ball $\bar{B}^n = \bar{B}^n(0,1) \subset \mathbb{R}^n$. This is actually not a manifold, but we are not yet there to prove this. This sort of a structure motivates us to the next definition.

Figure 4.1: Open ball in an open set in $\mathbb{R}^2$

▤ Definition 22 (Manifold on a Boundary)

An n-dimensional space that is second countable and Hausdorff, such that $\forall x \in X$, there exists a neighbourhood either homeomorphic to $B^n \subseteq \mathbb{R}^n$ or $B^n \cap \mathbb{H}^n$.
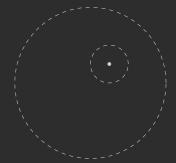

Figure 4.2: Open ball on a point on the boundary of a closed set

❝ Note

Note that $\mathbb{H}^n$ is defined as

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) : x_n \geq 0\}.$$

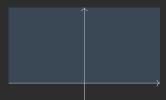*For instance, $\mathbb{H}^2$ has the following graph:*

Figure 4.3: Graph of $\mathbb{H}^2$

Part II

Introduction to Topological Manifolds

5 Lecture 5 Jan 16th

5.1 Manifolds (Continued)

📑 **Definition 23 (Interior Point)**

*A point $x \in M$ is called an **interior point** if there is a local homeomorphism*

$$\phi : \mathcal{U} \to \mathbb{B} \subseteq \mathbb{R}^n,$$

where $\mathcal{U}$ is open.

In the last lecture we asked ourselves the following: how do we know if the idea of 'being on a boundary' is a well-defined notion? In particular, how do well tell the difference between the following two graphs, mathematically?
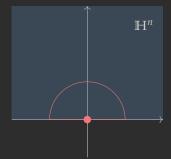

Figure 5.1: How do we tell the difference between the two graphs?

📑 **Definition 24 (Boundary Point)**

A point x is on the boundary of M, denoted as $x \in \partial M$, if there exists

$\mathcal{U} \ni x$ that is open, and a homeomorphism

$$\psi : \mathcal{U} \to \mathbb{B}_{0,1} \cap \mathbb{H}^n.$$

66 Note

The ϕ in ▤ Definition 23 and ψ in ▤ Definition 24 are called local charts.

Also, our definitions do not rule out, e.g.

$$\phi_2 : \mathcal{U} \to \mathbb{B}^2 \subseteq \mathbb{R}^2$$
$$\phi_5 : \mathcal{U} \to \mathbb{B}^5 \subseteq \mathbb{R}^5.$$

We shall later on prove that a point cannot simultaneously be a boundary point and an interior point. This property is called **the invariance of the boundary**. [1]

[1] This also means that $\overset{\circ}{M} \cap \partial M = \varnothing$. Also, by ▤ Definition 21, $\overset{\circ}{M} \cup \partial M = M$.

66 Note

$\overset{\circ}{M}$ is open. Thus, we can use the same definition about open sets as before using ϕ.

66 Note

In contrast, ∂M is closed; thanks to the invariance of the boundary we have that $\partial M = M \setminus \overset{\circ}{M}$ and $\overset{\circ}{M}$ is open, and so $(\partial M)^C = \overset{\circ}{M}$.

We shall also prove the following theorem later on:

🖳 Theorem (Invariance of the Dimension)

The n in $\mathbb{R}^n$ is well-defined.

Example 5.1.1

Consider the equation

$$x^2 - y^2 - z^2 = 0. \tag{5.1}$$

Note that we may write

$$x = \pm\sqrt{y^2 + z^2},$$

and so the graph generated by Equation (5.1) is as shown in Figure 5.2.

However, this is not a manifold: if we assume that a ball arounnd the origin is homeomorphic to $\mathbb{R}^2$, then by removing the point at the origin in the cone, the cone becomes two disconnected components, but the ball in $\mathbb{R}^2$ homeomorphic to the aforementioned ball is still connected.
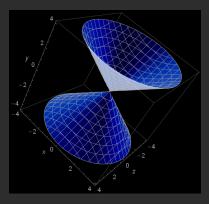

Figure 5.2: A 3D cone in $\mathbb{R}^3$, from WolframAlpha

> **❝ Note**
> *An open subset of a manifold is a manifold, by restriction.*

5.1.1 The 1-Sphere S^1

From Example 2.1.5, we have

- $[0,1) \simeq [0,\infty)$ is a manifold with boundary;

- $(0,1) \simeq \mathbb{R}$ is a manifold; and

- $[0,1]$ is a manifold with boundary.

Example 5.1.2 (S^1 is a manifold)

Consider the function $f : [0,2\pi) \to e^{i\theta}$. The image of f is Consider
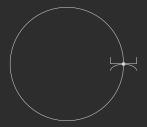

Figure 5.3: S^1 as a manifold

the following two functions $(0, 2\pi) \to \mathbb{C}^2$ by

$$\theta_1 \to e^{i\theta_1} \text{ and } \theta_2 \to e^{i\theta_2 + \pi},$$

which gives us the graphs:

Figure 5.4: Basis for S^1

respectively. Note that the image of both these functions are not compact. Regardless, this gives us a basis for S^1, which we notice is countable, Hausdorff, and locally homeomorphic to $\mathbb{R}^2$.

💻 **Theorem 10 (1-Dimensional Manifolds Determined by Its Compactness)**

Let M be a connected component of a 1-dimensional manifold. Then either

1. *M is compact, in which case if it is*

   - *without a boundary, then M is homeomorphic to S^1.*

   - *with a boundary, then M is homeomorphic to $[0, 1]$.*

2. *M is not compact, in which case if it is*

   - *without a boundary, then M is homeomorphic to $[0, 1)$.*

   - *with a boundary, then M is homeomorphic to $(0, 1)$.*

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6 Lecture 6 Jan 18th

6.1 Manifolds (Continued 2)

6.1.1 The 1-Sphere S^1 (Continued)

Set theoretic view of S^1   We showed that S^1 is a manifold. We can, in fact, set theoretically, look at S^1 as $A = [0, 1]$ glued at the endpoints, i.e. we **identify** the points 0 and 1 as 'the same', and label this notion as $0 \sim 1$.

Topological view of S^1   Topologically, for $0 \sim 1$ in A, we can construct an open set around the point such that the open set is properly contained in A.
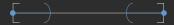

Figure 6.1: Topological representation of A

But how can we describe this notion mathematically so?
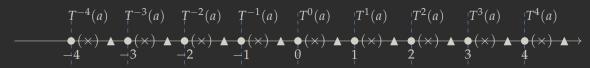
CONSIDER the real line as follows:

Figure 6.2: Breaking down the real line into parts

Let's define $T : \mathbb{R} \to \mathbb{R}$ such that $x \mapsto x + 1$. Clearly so, T is bijective and, in particular, has an inverse $x \mapsto x - 1$.

Notice that within each interval $[x, x + 1]$, we can find a $\times$ and $\blacktriangle$ at the same distance from x. Also, notice that we can use the same radius for $\times$ such that the open ball around $\times$ sits in $[x, x + 1]$ for each $x \in \mathbb{Z}$.

Thus, instead of studying the entire real line at once, we can reduce our attention only to $[0, 1]$, and simply scale the interval with a 'scalar multiplication' to get to wherever we want on the real line.

Now let

$$G := \left\{ T^k \mid k \in \mathbb{Z} \right\},$$

which is evidently a **group**. Furthermore, every element in G is a homeomorphism to $\mathbb{R}$. Let G **act on** $\mathbb{R}$, and for $a \in \mathbb{R}$, consider the **orbit** of a, which is denoted as

$$G \cdot a := \left\{ T^k(a) \mid k \in \mathbb{Z} \right\}.$$

Then

$$S^1 \simeq \text{ the space of all orbits of } G \text{ acting on } \mathbb{R} =: \mathbb{R}/G,$$

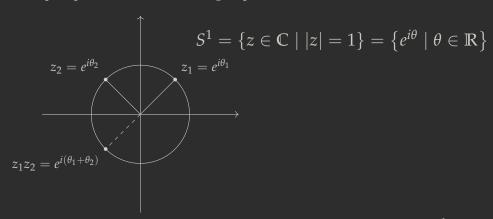where $\simeq$ represents homeomorphism. [1] Also, notice that here, G is effectively $\mathbb{Z}$.

> [1] $\mathbb{R}/G$ is the **quotient space** of $\mathbb{R}$ over G.

This realization implies the existence of some topology on S^1.

THUS FAR, we have seen that we may look at S^1

- set theoretically: as $A = [0, 1]$ with glued endpoints; and

- topologically: as $\mathbb{R} \setminus \mathbb{Z}$.

S^1 *as a topological group*   Since $\mathbb{C} \simeq \mathbb{R}^2$, we may think of S^1 as a sphere on the complex plane. We see that this 'group' takes on the

$$S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} = \left\{ e^{i\theta} \mid \theta \in \mathbb{R} \right\}$$

$z_2 = e^{i\theta_2}$

$z_1 = e^{i\theta_1}$

$z_1 z_2 = e^{i(\theta_1 + \theta_2)}$

Figure 6.3: S^1 on the complex plane

operation of adding the indices of the exponents. Thus $G = (S^1, \cdot)$. Notice that G is indeed a group equipped with said operation, and for each $z_1 \in G$, there exists $z_1^{-1} = \frac{1}{z_1} \in G$ such that $z_1 \cdot \frac{1}{z_1} = 1$.

Furthermore, the function

$$\iota : S^1 \to S^1 \text{ given by } z \mapsto \frac{1}{z}$$
$$\text{which is } e^{i\theta} \mapsto e^{-i\theta}$$

is continuous.

Also, the function

$$P : S^1 \times S^1 \to S^1 \text{ given by } (z_1, z_2) \mapsto z_1 z_2$$

is continuous.

📖 **Definition 25 (Topological Group)**

*If G is a group, and functions ι and P as defined above, if both ι and P are continuous, then we say that G is a **topological group**.*

S^1 as a moduli space

📖 **Definition 26 (Moduli Space)**

*A **moduli space** is the space of all lines passing through the origin.*
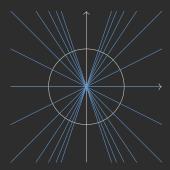

On $\mathbb{R}^2$, we have

Figure 6.4: The moduli space on $\mathbb{R}^2$

First, how can we understand 'closeness' in a moduli space? We can actually look at the difference in the radians of each line, or really just $x/360$ and compare the x's.

Also, notice that each line passes through S^1 twice. We can indeed avoid this problem by shifting S^1 to one side, as shown in Figure 6.5.
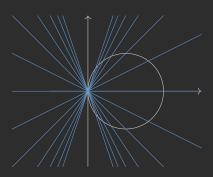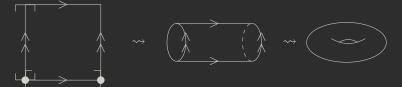

Figure 6.5: Shifted S^1 for the moduli space

Now each line intersects S^1 at the origin and another point on S^1, and this intersection is in fact unique.

6.1.1.1 *The space of $S^1 \times S^1$*
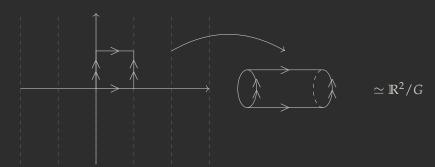
Observe that the product of two manifolds is a manifold [2].

CONSIDER using the set theoretical viewpoint, with credits to **Felix Klein**, the following figure By joining the sides with $>$, where we

Figure 6.6: $S^1 \times S^1$ becomes a cylinder by identifying the edges

identify the endpoints, we can go from the figure introduced by Klein to a cylinder. Then by identifying the sides with $>>$, we get a **torus**.

Now ON a Cartesian plane, observe that

$\simeq \mathbb{R}^2/G$

Figure 6.7: Klein's figure on a Cartesian plane to a cylinder

where we define $G = \left\{ T^k \mid k \in \mathbb{Z} \right\}$ as before, for $T((x,y)) = (x + 1, y)$. Now on a similar note, define $R : \mathbb{R} \to \mathbb{R}$ by $R((x,y)) = (x, y + 1)$. Then let $G_2 = \left\{ R^k \mid k \in \mathbb{Z} \right\}$. Thus

$$\mathbb{R}^2/G \oplus G_2 \simeq \mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z}.$$

7 Lecture 7 Jan 21st

7.1 Manifolds (Continued 3)

7.1.1 2-dimensional Manifolds

Example 7.1.1

S^2 is a 2-dimensional manifold (w/o boundary).

Note that we may 'cover' this sphere by the function $f_- : \mathbb{D} \subseteq \mathbb{R}^2 \to \mathbb{R}$, where $\mathbb{D}$ is the unit disc in $\mathbb{R}^2$, and f_- is given by

$$(x, y) \mapsto -\sqrt{1 - (x^2 + y^2)}.$$

Figure 7.1: The 2-sphere S^2

We see that f_- is a chart for the lower hemisphere for Figure 7.1, shaded red. We can indeed cover the entire 2-sphere with similar hemispheres in different orientations: consider

$$f_+(x, y) = \sqrt{1 - x^2 - y^2}$$
$$g_+(x, z) = \sqrt{1 - x^2 - z^2}$$
$$g_-(x, z) = -\sqrt{1 - x^2 - z^2}$$
$$h_+(y, z) = \sqrt{1 - y^2 - z^2}$$
$$h_-(y, z) = -\sqrt{1 - y^2 - z^2},$$

which represent the upper hemisphere, eastern hemisphere, western hemisphere, front hemisphere, and back hemisphere, respectively.

We have the following general observation: let $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ be a real-valued continuous function. Then $\Gamma(f)$ is **parameterized** using f, i.e. f constructs the chart $\Gamma(f)$.

We have not explicitly defined what $\Gamma(f)$ is, but it is recorded in PMATH 733, but we shall provide a short definition here for refer-
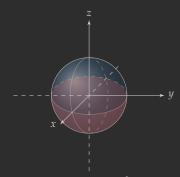
ence: the **graph** $\Gamma(f)$ of the function f is defined as

$$\Gamma(f) := \{(x_1,\ldots,x_n,y) \mid f(x_1,\ldots,x_n) = y\}.$$

It is interesting to note that $\Gamma(f)$ is homeomorphic to $\mathcal{U}$. Consider the function

$$\Phi : \mathcal{U} \to \Gamma(F) \text{ given by } \Phi(x_1,\ldots,x_n) = (x_1,\ldots,x_n,f(x_1,\ldots,x_n)).$$

It is clear that Φ is continuous since each of its components are continuous. However, it is not as easy to find a continuous map to go from $\Gamma(f)$ to $\mathcal{U}$.

📓 **Definition 27 (Projection)**

We define function $\pi_{n+1} : \Gamma(f) \to \mathcal{U}$ *by*

$$\pi_{n+1}(x_1,\ldots,x_n,y) = (x_1,\ldots,x_n),$$

*which is called a **projection**.*

❝ Note

1. *In A1, we showed that π_{n+1} is continuous.*

2. *Furthermore, π_{n+1} is injective.*

3. *Thus, we observe that*

$$\pi_{n+1} \circ \Phi(x_1,\ldots,x_n) = (x_1,\ldots,x_n) = id_{n+1}(x_1,\ldots,x_n).$$

Example 7.1.2

Consider the following figure:

We define the **stereographic projection** by
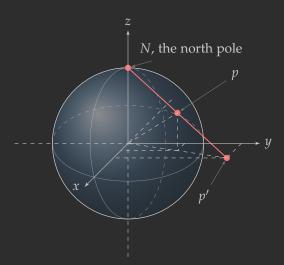
$$\Sigma(p) = p'.$$

N, the north pole

p

z

y

x

p'

7.1.2   *n-spheres*

We can now extend the notion we saw in Example 7.1.1 to higher dimensional manifolds. In particular, we want to be able to 'glue' the boundaries of the hemispheres, as shown in Figure 7.3.

In a more mathematical sense, we are identifying the lower and upper boundaries of the upper and lower hemisphere, respectively, i.e. we identify

$$S^n = \overline{\mathbb{B}}^n \times \{0\} \cup \overline{\mathbb{B}}^n \times \{1\},$$

where the $\{0\}$ and $\{1\}$ represent the upper and lower hemispheres, respectively. We may also write this as

$$x \in \partial \overline{\mathbb{B}}^n : (x, 0) \sim (x, 1).$$

This calls for the notion of an equivalence class. Recall that an **equivalence relation** on a set A is a relation between elements such that

1. (reflexity) $x \sim x$;

2. (symmetry) $x \sim y \iff y \sim x$; and

3. (transitivity) $x \sim y \wedge y \sim z \implies x \sim z$,

where $\sim$ is our equivalence relation.

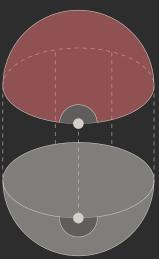Equivalence relations give rise to the notion of **equivalence classes**,

Figure 7.3: Glueing two hemispheres I was definitely not trying to make a Pokeball.

where we define an equivalence class as

$$[\beta] := \{\alpha \in A \mid \beta \sim \alpha\}.$$

In this course, we shall sometimes denote equivalence classes in the form of A_β. Notice that

$$A = \coprod_{\beta \in B} A_\beta,$$

where $B \subseteq A$.

THIS BEGS THE QUESTION: do the set of equivalence classes retain the topology of the space?

Here, we look into **quotient topology**. Let A^* be the space of equivalence classes, and let's assume that A is endowed with a topology. Consider the function

$$\pi : A \to A^* \text{ given by } a \mapsto [a]_\sim,$$

where $\sim$ is the equivalence relation, and $[a]_\sim$ is the equivalence class that a belongs to[1].

[1] Note that the function is well-defined as the equivalence classes are disjoint.

Figure 7.4: Relationship of A, A^* and Z

Consider an arbitrary space Z that is also endowed with a topology, such that $\exists g : A \to Z$ that is continuous, such that g is constant on each of the equivalence classes of A, that is, if $a \sim b$, then $g(a) = g(b)$.

Then $\tilde{g}$ is a well-defined function induced on A^*: we have that

$$\tilde{g}(\pi(a)) = g(a).$$

We want to endow A^* with a topology such that $\tilde{g}$ will be continuous. However,

- if A^* is too 'fine', then π may not be continuous; and

- if A^* is too 'course', then $\tilde{g}$ may not be continuous.

We need to strike a balance in the fineness of the topology of A^* to

make sure that both π and $\tilde{g}$ are continuous.

📘 Definition 28 (Strongly Continuous)

Let $\mathcal{V} \subseteq A^*$. $\mathcal{V}$ is open iff $\exists \mathcal{U} \subseteq A$ such that $\pi(\mathcal{U}) = \mathcal{V}$. We say that π is **strongly continuous**.

🪲 Warning

π may not be an open map!

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8 Lecture 8 Jan 23rd

8.1 Quotient Spaces

With the topology that we last introduced for A^*, g is continuous iff $\tilde{f}$ is continuous. However, this is a rather cumbersome way to construct a quotient space. In particular, how do we know what equivalence class should we choose?

This way of constructing a quotient space is intuitive ring theoretically.

📖 Definition 29 (Saturated)

*We say that a subset $S \subseteq A$ is **saturated** (wrt π) provided that it is has a **non-empty intersection** with a fibre $\pi^{-1}(\{q\})$, where $q \in A^*$, then $S \supseteq \pi^{-1}(\{q\})$.*

---

66 Note
This definition is equivalent to saying that $\pi^{-1} \circ \pi(S) = S$.

With this definition, we may restate the definition of a quotient map in a clearer way.

📖 Definition 30 (Quotient Map)

*A map $g : X \to Y$ is called a **quotient map** if it sends saturated open subsets of X to open subsets of Y.*

Bibliography

Lee, J. M. (2000). *Graduate Texts in Mathematics: Introduction to Topological Manifolds*. Springer.

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