PMATH347S18 - Groups & Rings

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1 Lecture 1 May 02nd 2018

1.1 Introduction

1.1.1 Numbers

The following are some of the number sets that we are already familiar with:

$$\mathbb{N} = \{1, 2, 3, ...\} \qquad \mathbb{Z} = \{.., -2, -1, 0, 1, 2, ...\}$$
$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N} \right\} \qquad \mathbb{R} = \text{ set of real numbers}$$
$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i = \sqrt{-1}\} = \text{ set of complex numbers}$$

For $n \in \mathbb{Z}$, let $\mathbb{Z}_n$ denote the set of integers modulo n, i.e.

$$\mathbb{Z}_n = \{[0], [1], ..., [n-1]\}$$

where the $[r]$, $0 \leq r \leq n-1$, are the congruence classes, i.e.

$$[r] = \{z \in \mathbb{Z} : z \equiv r \mod n\}$$

These sets share some common properties, e.g. $+$ and $\times$. Let's try to break that down to make further observation.

Note that for $R = \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, or $\mathbb{Z}_n$, R has 2 operations, i.e. addition and multiplication.

Addition If $r_1, r_2, r_3 \in R$, then

- (**closure**) $r_1 + r_2 \in R$
- (**associativity**) $r_1 + (r_2 + r_3) = (r_1 + r_2) + r_3$

Also, if $R \neq \mathbb{N}$, then $\exists 0 \in R$ (the **additive identity**) such that

$$\forall r \in R \quad r + 0 = r = 0 + r.$$

Also, $\forall r \in R$, $\exists (-r) \in R$ such that

$$r + (-r) = 0 = (-r) + r.$$

Multiplication For $r_1, r_2, r_3 \in R$, we have

- (**closure**) $r_1 r_2 \in R$
- (**associativity**) $r_1(r_2 r_3) = (r_1 r_2) r_3$

Also, $\exists 1 \in R$ (a.k.a the **mutiplicative identity**), such that

$$\forall r \in R \quad r \cdot 1 = r = 1 \cdot r.$$

Finally, for $R = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, $\forall r \in R$, $\exists r^{-1} \in R$ such that

$$r \cdot r^{-1} = 1 = r^{-1} \cdot r.$$

Note that for $R = \mathbb{Z}_n$, where $n \in \mathbb{Z}$, not all $[r] \in \mathbb{Z}_n$ have a multiplicative inverse. For example, for $[2] \in \mathbb{Z}_4$, there is no $[x] \in \mathbb{Z}_4$ such that $[2][x] = [1]$.[1]

[1] This is best proven using techniques introduced in MATH135/145.

1.1.2 *Matrices*

For $n \in \mathbb{N} \setminus \{1\}$, an $n \times n$ matrix over $\mathbb{R}$ [2] is an $n \times n$ array that can be expressed as follows:

[2] $\mathbb{R}$ can be replaced by $\mathbb{Q}$ or $\mathbb{C}$.

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

where for $1 \leq i, j \leq n$, $a_{ij} \in \mathbb{R}$. We denote $M_n(\mathbb{R})$ as the set of all $n \times n$ matrices over $\mathbb{R}$.

As in Section 1.1.1, we can perform **addition and multiplication** on $M_n(\mathbb{R})$.

Matrix Addition Given $A = [a_{ij}], B = [b_{ij}], C = [c_{ij}] \in M_n(\mathbb{R})$, we define matrix addition as

$$A + B = [a_{ij} + b_{ij}],$$

which immediately gives the **closure property**, since $a_{ij} + b_{ij} \in \mathbb{R}$ and hence $A + B \in M_n(\mathbb{R})$. Also, by this definition, we also immediately obtain the **associativity property**, i.e.

$$A + (B + C) = (A + B) + C.$$

We define the zero matrix as

$$0 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then we have that 0 is the **additive identity**, i.e.

$$A + 0 = A = 0 + A.$$

Finally, $\forall A \in M_n(\mathbb{R})$, $\exists(-A) \in M_n(\mathbb{R})$ (the **additive inverse**) such that

$$A + (-A) = 0 - (-A) + A.$$

Note that in this case, we also have that that the operation is **commutative**, i.e.

$$A + B = B + A.$$

Matrix Multiplication Given $A = [a_{ij}], B = [b_{ij}], C = [c_{ij}] \in M_n(\mathbb{R})$, we define the matrix multiplication as

$$AB = [d_{ij}] \text{ where } c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \in \mathbb{R}.$$

Clearly, $AB \in M_n(\mathbb{R})$, i.e. it is **closed under matrix multiplication**. Also, we have that, under such a defintion, matrix multiplication is **associative**, i.e.

$$A(BC) = (AB)C.$$

Define the identity matrix, $I \in M_n(\mathbb{R})$, as follows:

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Then we have that I is the **multiplicative identity**, since

$$AI = A = IA.$$

However, contrary to matrix addition, $\forall A \in M_n(\mathbb{R})$, it is not always true that $\exists A^{-1} \in M_n(\mathbb{R})$ such that

This is especially true if the **determinant** of A is 0.

$$AA^{-1} = I = A^{-1}A.$$

Also, we can always find some $A, B \in M_n(\mathbb{R})$ such that

$$AB \neq BA,$$

i.e. matrix multiplication is not always commutative.

THE COMMON PROPERTIES of the operations from above: **closure, associativity, and existence of an inverse**, are not unique to just addition and multiplication. We shall see in the next lecture that there are other operations where these properties will continue to hold, e.g. **permutations**.

2 Lecture 2 May 04th 2018

2.1 Introduction (Continued)

2.1.1 Permutations

Definition 2.1.1 (Injectivity)
*Let $f : X \to Y$ be a function. We say that f is **injective** (or **one-to-one**) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.*

Definition 2.1.2 (Surjectivity)
*Let $f : X \to Y$ be a function. We say that f is **surjective** (or **onto**) if $\forall y \in Y \ \exists x \in X \ \ f(x) = y$.*

Definition 2.1.3 (Bijectivity)
*Let $f : X \to Y$ be a function. We say that f is **bijective** if it is both **injective** and **surjective**.*

Definition 2.1.4 (Permutations)
Given a non-empty set L, a permutation of L is a bijection from L to L. The set of all permutations of L is denoted by S_L.

Example 2.1.1
Consider the set $L = \{1, 2, 3\}$, which has the following 6 different permuta-

tions:

$$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}\quad\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}\quad\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}\quad\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}\quad\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$$

Note

$$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$$

indicates the bijection $\sigma : \{1,2,3\} \to \{1,2,3\}$ *with* $\sigma(1) = 1$, $\sigma(2) = 3$ *and* $\sigma(3) = 2$.

FOR $n \in \mathbb{N}$, we denote $S_n := S_{\{1,2,\ldots,n\}}$, the set of all permutations of $\{1,2,\ldots,n\}$. Example 2.1.1 shows the elements of the set S_3.

Definition 2.1.5 (Order)
The ***order*** *of a set* A, *denoted by* $|A|$, *is the cardinality of the set.*

Example 2.1.2
We have seen that the order of S_3, $|S_3|$ *is* $6 = 3!$.

Proposition 2.1.1
$|S_n| = n!$

Proof
$\forall \sigma \in S_n$, *there are* n *choices for* $\sigma(1)$, $n-1$ *choices for* $\sigma(2)$, ..., 2 *choices for* $\sigma(n-1)$, *and finally* 1 *choice for* $\sigma(n)$. $\square$

Do elements of S_n *share the same properties as what we've seen in the numbers?* Given $\sigma, \tau \in S_n$, we can **compose** the 2 together to get a third element in S_n, namely $\sigma\tau$ (wlog), where $\sigma\tau : \{1,\ldots,n\} \to \{1,\ldots,n\}$ is given by $\forall x \in \{1,\ldots,n\}$, $x \mapsto \sigma(\tau(x))$.

It is important to note that $\because$ σ, τ are **both bijective**, $\sigma\tau$ is also bijective. Thus, together with the fact that $\sigma\tau : \{1,\ldots,n\} \to \{1,\ldots,n\}$, we have that $\sigma\tau \in S_n$ by definition of S_n.

$\therefore \forall \sigma, \tau \in S_n$, $\sigma\tau, \tau\sigma \in S_n$, but $\sigma\tau \neq \tau\sigma$ in general. The following is an example of the stated case:

Example 2.1.3
Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \text{ and } \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}.$$

Compute $\sigma\tau$ and $\tau\sigma$ to show that they are not equal.

Solution

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \text{ but } \tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

Perhaps what is interesting is the question of: **when does commutativity occur?** One such case is when σ and τ have support sets that are disjoint[1].

[1] This is proven in A1

On the other hand, the associative property holds[2], i.e.

$$\forall \sigma, \tau, \mu \in S_n \;\; \sigma(\tau\mu) = (\sigma\tau)\mu$$

[2]

Exercise 2.1.1
Prove this as an exercise.

The set S_n also has an identity element[3], namely

$$\varepsilon = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}$$

[3]

Exercise 2.1.2
Verify that the given identity element is indeed the identity, i.e.

$$\forall \sigma \in S_n \;\; \sigma\varepsilon = \sigma = \varepsilon\sigma.$$

Finally, $\forall \sigma \in S_n$, since σ is a bijection, we have that its inverse function, σ^-1 is also a bijection, and thus satisfies the requirements to be in S_n. We call $\sigma^{-1} \in S_n$ to be the **inverse permutation** of σ, such that

$$\forall x, y \in \{1, ..., n\} \quad \sigma^{-1}(x) = y \iff \sigma(y) = x.$$

It follows, immediately, that

$$\sigma\big(\sigma^{-1}(x)\big) = x \wedge \sigma^{-1}\big(\sigma(y)\big) = y.$$

$\therefore$ We have that

$$\sigma\sigma^{-1} = \varepsilon = \sigma^{-1}\sigma.$$

Example 2.1.4
Find the inverse of

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$$

Solution
By rearranging the image in ascending order, using them now as the object

and their respective objects as their image, construct

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}.$$

It can easily (although perhaps not so prettily) be shown that

$$\sigma\tau = \varepsilon = \tau\sigma.$$

With all the above, we have for ourselves the following proposition:

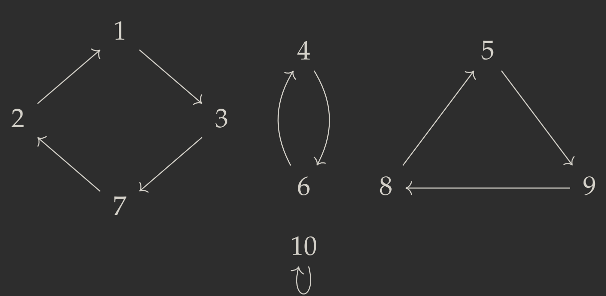
Proposition 2.1.2 (Properties of S_n)
We have

1. $\forall \sigma, \tau \in S_n \;\; \sigma\tau, \tau\sigma \in S_n.$
2. $\forall \sigma, \tau, \mu \in S_n \;\; \sigma(\tau\mu) = (\sigma\tau)\mu.$
3. $\exists \varepsilon \in S_n \;\; \forall \sigma \in S_n \;\; \sigma\varepsilon = \sigma = \varepsilon\sigma.$
4. $\forall \sigma \in S_n \;\; \exists! \sigma^{-1} \in S_n \;\; \sigma\sigma^{-1} = \varepsilon = \sigma^{-1}\sigma.$

CONSIDER

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 7 & 6 & 9 & 4 & 2 & 5 & 8 & 10 \end{pmatrix} \in S_{10}$$

If we represent the action of σ geometrically, we get



We observe that σ can be **decomposed** into one 4-cycle, $\begin{pmatrix} 1 & 3 & 7 & 2 \end{pmatrix}$, one 2-cycle, $\begin{pmatrix} 4 & 6 \end{pmatrix}$, one 3-cycle, $\begin{pmatrix} 5 & 9 & 8 \end{pmatrix}$, and one 1-cycle, $\begin{pmatrix} 10 \end{pmatrix}$.

Note that these cycles are (pairwise) **disjoint**, and we can write[4]

[4] We generally do not include the 1-cycle and assume that by excluding them, it is known that any number that is supposed to appear loops back to themselves.

$$\sigma = \begin{pmatrix} 1 & 3 & 7 & 2 \end{pmatrix} \begin{pmatrix} 4 & 6 \end{pmatrix} \begin{pmatrix} 5 & 9 & 8 \end{pmatrix}$$

Note that we may also write

$$\begin{aligned} \sigma &= \begin{pmatrix} 4 & 6 \end{pmatrix} \begin{pmatrix} 5 & 9 & 8 \end{pmatrix} \begin{pmatrix} 1 & 3 & 7 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 4 \end{pmatrix} \begin{pmatrix} 9 & 8 & 5 \end{pmatrix} \begin{pmatrix} 7 & 2 & 1 & 3 \end{pmatrix} \end{aligned}$$

It is interesting to note that the cycles can rotate their "elements" in a **cyclic** manner, i.e.

$$\begin{pmatrix} 1 & 3 & 7 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 2 & 1 & 3 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 7 & 3 \end{pmatrix}.$$

Although the decomposition of the cycle notation is not unique (i.e. you may rearrange them), each individual cycle is unique, and is proven below[5].

[5] See bonus question of A1. Proof will be included in the notes once the assignment is over.

Theorem 2.1.1 (Cycle Decomposition Theorem)
If $\sigma \in S_n$, $\sigma \neq \varepsilon$, then σ is a product of (one or more) disjoint cycles of length at least 2. This factorization is unique up to the order of the factors.

Note (Convention)
Every permutation in S_n can be regarded as a permutation of S_{n+1} by fixing the permutation of $n+1$. Therefore, we have that

$$S_1 \subseteq S_2 \subseteq \ldots \subseteq S_n \subseteq S_{n+1} \subseteq \ldots$$

3 Lecture 3 May 07th 2018

3.1 Groups

3.1.1 Groups

Definition 3.1.1 (Groups)
*Let G be a set and $*$ an operation on $G \times G$. We say that $G = (G, *)$ is a **group** if it satisfies*[1]

1. ***Closure***: $\forall a, b \in G \quad a * b \in G$
2. ***Associativity***: $\forall a, b, c \in G \quad a * (b * c) = (a * b) * c$
3. ***Identity***: $\exists e \in G \ \forall a \in G \quad a * e = a = e * a$
4. ***Inverse***: $\forall a \in G \ \exists b \in G \quad a * b = e = b * a$

[1] If you wonder why the uniqueness is not specified for **Identity** and **Inverse**, see Proposition 3.1.1.

Definition 3.1.2 (Abelian Group)
A group G is said to be abelian if $\forall a, b \in G$, we have $a * b = b * a$.

Proposition 3.1.1 (Group Identity and Group Element Inverse)
Let G be a group and $a \in G$.

1. *The identity of G is unique.*
2. *The inverse of a is unique.*

Proof

1. *If $e_1, e_2 \in G$ are both identities of G, then we have*

$$e_1 \overset{(1)}{=} e_1 * e_2 \overset{(2)}{=} e_2$$

where (1) is because e_2 is an identity and (2) is because e_1 is an identity.

2. *Let $a \in G$. If $b_1, b_2 \in G$ are both the inverses of a, then we have*

$$b_1 = b_1 * e = b_1 * (a * b_2) \overset{(1)}{=} e * b_2 = b_2$$

where (1) is by associativity.

Example 3.1.1
The sets $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, and $(\mathbb{C}, +)$ are all abelian, wehre the additive identity is 0, and the additive inverse of an element r is $(-r)$.

Note
$(\mathbb{N}, +)$ is not a group for neither does it have an identity nor an inverse for any of its elements.

Example 3.1.2
*The sets $(\mathbb{Q}, \cdot)$, $(\mathbb{R}, \cdot)$ and $(\mathbb{C}, \cdot)$ are **not** groups, since 0 has no multiplicative inverse in $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.*

We may define that for a set S, let $S^* \subseteq S$ contain all the elements of S that has a multiplicative inverse. For example, $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$. Then, $(\mathbb{Q}, \cdot), (\mathbb{R}, \cdot)$ and $(\mathbb{C}, \cdot)$ are groups and are in fact abelian, where the multiplicative identity is 1 and the multiplicative of an element r is $\frac{1}{r}$.

Example 3.1.3
The set $\big(M_n(\mathbb{R}), +\big)$ is an abelian group, where the additive identity is the zero matrix, $0 \in M_n(\mathbb{R})$, and the additive inverse of an element $M = [a_{ij}] \in M_n(\mathbb{R})$ is $-M = [-a_{ij}] \in M_n(\mathbb{R})$.

Consider the set $M_n(\mathbb{R})$ under the matrix mutiplication operation that we have introduced in Lecture 1 May 02nd 2018. We found that

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the identity matrix is

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \in M_n(\mathbb{R}).$$

But since not all elements of $M_n(\mathbb{R})$ have a multiplicative inverse[2], $(M_n(\mathbb{R}), \cdot)$ is not a group.

[2] The multiplicative inverse of a matrix does not exist if its determinant is 0.

But we can try to do something similar as to what we did before: by excluding the elements that do not have an inverse. In this case, we exclude elements whose determinant is 0. Define the set

$$GL_n(\mathbb{R}) := \{M \in M_n(\mathbb{R}) \,:\, \det M \neq 0\}$$

Note that $\because \det I = 1 \neq 0$, we have that $I \in GL_n(\mathbb{R})$.

Also, $\forall A, B \in GL_n(\mathbb{R})$, we have that $\because \det A \neq 0 \wedge \det B \neq 0$,

$$\det AB = \det A \det B \neq 0,$$

and therefore $AB \in GL_n(\mathbb{R})$. Finally, $\forall M \in GL_n(\mathbb{R})$, $\exists M^{-1} \in GL_n(\mathbb{R})$ such that

$$MM^{-1} = I = M^{-1}M$$

since $\det M \neq 0$. $\therefore (GL_n(\mathbb{R}), \cdot)$ is a group, and is in fact called the **general linear group of degree n over $\mathbb{R}$**.

Since we have introduced permutations in Lecture 2 May 04th 2018, we shall formalize the purpose of its introduction below.

Example 3.1.4

Consider S_n, the set of all permutations on $\{1, 2, ..., n\}$. By Proposition 2.1.2, we know that S_n is a group. We call S_n the ***symmetry group of degree n****. For $n \geq 3$, the group S_n is not abelian*[3].

[3] Let us make this an exercise.

Exercise 3.1.1

For $n \geq 3$, prove that the group S_n is not abelian.

Now that we have a fairly good idea of the basic concept of a group, we will now proceed to look into handling multiple groups. One such operation is known as the **direct product**.

Example 3.1.5

Let G and H be groups. Their direct product is the set $G \times H$ with the *component-wise operation defined by*

$$(g_1, h_1) * (g_2, h_2) = (g_1 *_G g_2, h_1 *_H h_2)$$

where $g_1, g_2 \in G$, $h_1, h_2 \in H$, $*_G$ *is the operation on* G, *and* $*_H$ *is the operation on* H.

The ***closure*** *and* ***associativity*** *property follow immediately from the definition of the operation. The identity is* $(1_G, 1_H)$ *where* 1_G *is the identity of G and* 1_H *is the identity of H. The inverse of an element* $(g_1, h_1) \in G \times H$ *is* (g_1^{-1}, h_1^{-1}).

By induction, we can show that if $G_1, G_2, ..., G_n$ are groups, then so is $G_1 \times G_2 \times \ldots \times G_n$.

To facilitate our writing, use shall use the following notations:

Notation

Given a group G and $g_1, g_2 \in G$, *we often denote its identity by* 1, *and write* $g_1 * g_2 = g_1 g_2$. *Also, we denote the unique inverse of an element* $g \in G$ *as* g^{-1}.

We will write $g^0 = 1$. *Also, for* $n \in \mathbb{N}$, *we define*

$$g^n = \underbrace{g * g * \ldots * g}_{n \text{ times}}$$

and

$$g^{-n} = (g^{-1})^n$$

With the above notations,

Proposition 3.1.2

Let G be a group and $g, h \in G$. *We have*

1. $(g^{-1})^{-1} = g$
2. $(gh)^{-1} = h^{-1} g^{-1}$
3. $g^n g^m = g^{n+m}$ *for all* $n, m \in \mathbb{Z}$
4. $(g^n)^m = g^{nm}$ *for all* $n, m \in \mathbb{Z}$

Exercise 3.1.2
Prove Proposition 3.1.2 as an exercise.

Warning

In general, it is not true that if $g, h \in G$, then $(gh)^n = g^n h^n$. For example,

$$(gh)^2 = ghgh \quad \text{but} \quad g^2h^2 = gghh.$$

The two are only equal if and only if G is abelian.

4 Lecture 4 May 09 2018

4.1 Groups (Continued)

4.1.1 Groups (Continued)

Proposition 4.1.1 (Cancellation Laws)
Let G be a group and $g, h, f \in G$. Then

1. (a) (***Right Cancellation***) $gh = gf \implies h = f$

   (b) (***Left Cancellation***) $hg = fg \implies h = f$

2. *The equation $ax = b$ and $ya = b$ have unique solution for $x, y \in G$.*

Proof

1. (a) *By left multiplication and associativity,*

$$gh = gf \iff g^{-1}gh = g^{-1}gf \iff h = f$$

   (b) *By right multiplication and associativity,*

$$hg = fg \iff hgg^{-1} = fgg^{-1} \iff h = f$$

2. *Let $x = a^{-1}b$. Then*

$$ax = a(a^{-1}b) = (aa^{-1})b = b.$$

   If $\exists u \in G$ that is another solution, then

$$au = b = ax \implies u = x$$

by Left Cancellation. The proof for $ya = b$ is similar by letting $y = ba^{-1}$.

□

4.1.2 *Cayley Tables*

For a finite group, defining its operation by means of a table is sometimes convenient.

Definition 4.1.1 (Cayley Table)
Let G be a group. Given $x, y \in G$, let the product xy be an entry of a table in the row corresponding to x and column corresponding to y. Such a table is called a ***Cayley Table****.*

Note
By Cancellation Laws 4.1.1, the entries in each row (and respectively, column) of a Cayley Table are all distinct.

Example 4.1.1
Consider the group $(\mathbb{Z}_2, +)$. Its Cayley Table is

$\mathbb{Z}_2$	$[0]$	$[1]$
$[0]$	$[0]$	$[1]$
$[1]$	$[1]$	$[0]$

where note that we must have $[1] + [1] = [0]$; otherwise if $[1] + [1] = [1]$ then $[1]$ does not have its additive inverse, which contradicts the fact that it is in the group.

If we replace 1 by $[0]$ and -1 by $[1]$, the Cayley Tables of $\mathbb{Z}_2$ and $\mathbb{Z}^*$ are the same. In thie case, we say that $\mathbb{Z}_2$ and $\mathbb{Z}^*$ are **isomorphic**, which we denote by $\mathbb{Z}_2 \cong \mathbb{Z}^*$.

Example 4.1.2
Consider the group $\mathbb{Z}^* = \{1. -1\}$. Its Cayley Table (under multiplication) is

$\mathbb{Z}^*$	1	-1
1	1	-1
-1	-1	1

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Example 4.1.3

Given $n \in \mathbb{N}$, the ***cyclic group*** *of order n is defined by*

$$C_n = \{1, a, a^2, ..., a^{n-1}\} \quad \text{with } a^n = 1.$$

We write $C_n = \langle a : a^n = 1 \rangle$ and a is called a generator of C_n. The Cayley Table of C_n is

C_n	1	a	a^2	$\ldots$	a^{n-2}	a^{n-1}
1	1	a	a^2	$\ldots$	a^{n-2}	a^{n-1}
a	a	a^2	a^3	$\ldots$	a^{n-1}	1
a^2	a^2	a^3	a^4	$\ldots$	1	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^{n-2}	a^{n-2}	a^{n-1}	1	$\ldots$	a^{n-4}	a^{n-3}
a^{n-1}	a^{n-1}	1	a	$\ldots$	a^{n-3}	a^{n-2}

Proposition 4.1.2

Let G be a group. Up to isomorphism, we have

1. *if $|G| = 1$, then $G \cong \{1\}$.*
2. *if $|G| = 2$, then $G \cong C_2$.*
3. *if $|G| = 3$, then $G \cong C_3$.*
4. *if $|G| = 4$, then either $G \cong C_4$ or $G \cong K_4 \cong C_2 \times C_2$.*

K_n is known as the **Klein n-group**

Proof

1. *If $|G| = 1$, then it can only be $G = \{1\}$ where 1 is the identity element.*
2. *$|G| = 2 \implies G = \{1, g\}$ with $g \neq 1$. The Cayley Table of G is thus*

G	1	g
1	1	g
g	g	1

where we note that $g^2 = 1$; otherwise if $g^2 = g$, then we would have $g = 1$ by Cancellation Laws 4.1.1, which contradicts the fact that $g \neq 1$. Comparing the above Cayley Table with that of C_2, we see that $G = \langle g : g^2 = 1 \rangle \cong C_2$.

3. *$|G| = 3 \implies G = \{1, g, h\}$ with $g \neq 1 \neq h$ and $g \neq h$. We can then*

start with the following Cayley Table:

G	1	g	h
1	1	g	h
g	g		
h	h		

We know that by Cancellation Laws 4.1.1, $gh \neq g$ and $gh \neq h$. Thus $gh = 1$. Similarly, we get that $hg = 1$.

Claim: *Entries in a row (or column) must be distinct. Suppose not. Then say $g^2 = 1$. But since $gh = 1$, by Cancellation Laws 4.1.1, we have that $h = g$, which is a contradiction.*

With that, we can proceed to fill in the rest of the entries: with $g^2 = h$ and $h^2 = g$. Therefore,

G	1	g	h
1	1	g	h
g	g	h	1
h	h	1	g

Recall that the Cayley Table for C_3 is:

C_3	1	a	a^2
1	1	a	a^2
a	a	a^2	1
a^2	a^2	1	a

$\therefore G \cong C_3$ (by identifying $g = a$ and $h = a^2$).

4. ***Proof will be added once assignment 1 is over***

4.2 Subgroups

4.2.1 Subgroups

Definition 4.2.1 (Subgroup)
Let G be a group and $H \subseteq G$. If H itself is a group, then we say that H is a subgroup of G

5 Lecture 5 May 11th 2018

5.1 Subgroups (Continued)

5.1.1 Subgroups (Continued)

Note (Recall: definition of a subgroup)

Let G be a group and $H \subseteq G$. If H itself is a group, then we say that H is a subgroup of G.

Note

Since G is a group, $\forall h_1, h_2, h_3 \in H \subseteq G$, we have $h_1(h_2h_3) = (h_1h_2)h_3$. So H is a subgroup of G if it satisfies the following conditions, which we shall hereafter refer to as the Subgroup Test.

Subgroup Test

1. $h_1h_2 \in H$
2. $1_G \in H$
3. $\exists h_1^{-1} \in H$ *such that* $h_1h_1^{-1} = 1_G$

Note that the identity in H must also be the identity in G. This is because if $h_1, h_1^{-1} \in H$, then $h_1h_1^{-1} = 1_H$, but $h_1, h_1^{-1} \in G$ as well, and so $h_1h_1^{-1} = 1_G$. Thus $1_H = 1_G$.

Example 5.1.1

Given a group G, it is clear that $\{1\}$ and G are both subgroups of G.

Example 5.1.2

We have the following chain of groups:

$$(\mathbb{Z}, +) \subseteq (\mathbb{Q}, +) \subseteq (\mathbb{R}, +) \subseteq (\mathbb{C}, +)$$

Recall that the general linear group is defined as:

$$GL_n(\mathbb{R}) = (GL_n(\mathbb{R}), \cdot) = \{A \in M_n(\mathbb{R}) : \det A \neq 0\}$$

Definition 5.1.1 (Special Linear Group)
The ***special linear group*** *of order n of* $\mathbb{R}$ *is defined as*

$$SL_n(\mathbb{R}) = (SL_n(\mathbb{R}), \cdot) = \{A \in M_n(\mathbb{R}) : \det A = 1\}$$

Example 5.1.3
Clearly, $SL_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$. *Note that the identity matrix* I *must be in* $SL_n(\mathbb{R})$ *since* $\det I = 1$. *Also,* $\forall A, B \in SL_n(\mathbb{R})$, *we have that*

$$\det AB = \det A \det B = 1$$

$\therefore AB \in SL_n(\mathbb{R})$. *Also, since* $\det A^{-1} = \frac{1}{\det A} = 1$, *we also have that* $^{-1} \in SL_n(\mathbb{R})$. *We see that* $SL_n(\mathbb{R})$ *satisfies the* ***Subgroup Test****, and hence it is a subgroup of* $GL_n(\mathbb{R})$.

Definition 5.1.2 (Center of a Group)
Given a group G, *the* ***the center of a group*** G *is defined as*

$$Z(G) = \{z \in G : \forall g \in G \;\; zg = gz\}$$

Example 5.1.4
For a group G, $Z(G)$ *is an abelian subgroup of* G.

Proof
Clearly, $1_G \in Z(G)$. *Let* $y, z \in G$. $\forall g \in G$, *we have that*

$$(yz)g = y(zg) = y(gz) = (yg)z = (gy)z = g(yz)$$

Therefore $yz \in Z(G)$ *and so* $Z(G)$ *is closed under its operation. Also,* $\forall hinG$, *we can write* $h = (h^{-1})^{-1} = g^{-1}$. *Since* $z \in Z(G)$, *we have that*

$\forall g \in G$,

$$zg = gz \iff (zg)^{-1} = (gz)^{-1} \iff g^{-1}z^{-1} = z^{-1}g^{-1}$$
$$\iff hz^{-1} = z^{-1}h$$

Therefore $z^{-1} \in Z(G)$. By the ***Subgroup Test****, it follows that $Z(G)$ is a subgroup of G.*

Finally, since $Z(G) \subseteq G$, by its definition, we have that $\forall x, y \in Z(G)$, $x, y \in G$ as well, and we have that $xy = yx$. Therefore, $Z(G)$ is abelian. □

Proposition 5.1.1 (Intersection of Subgroups is a Subgroup)
Let H and K be subgroups of a group G. Then their intersection

$$H \cap K = \{g \in G : g \in H \land g \in K\}$$

is also a subgroup of G.

Proof
Since H and K are subgroups, we have that $1 \in H$ and $1 \in K$ and hence $1 \in H \cap K$. Let $a, b \in H \cap K$. Since H and K are subgroups, we have that $ab \in H$ and $ab \in K$. Therefore, $ab \in H \cap K$. Similarly, since $a^{-1} \in H$ and $a^{-1} \in K$, $a^{-1} \in H \cap K$. By the ***Subgroup Test****, $H \cap K$ is a subgroup of G.* □

Proposition 5.1.2 (Finite Subgroup Test)
If H is a finite nonempty subset of a group G, then H is a subgroup if and only if H is closed under its operation.

This result says that if H is a finite nonempty subset, then we only need to prove that it is closed under its operation to prove that it is a subgroup. The other two conditions in the **Subgroup Test** are automatically implied.

Proof
The forward direction of the proof is trivially true, since H must satisfy the closure property for it to be a subgroup.

For the converse, since $H \neq \emptyset$, let $h \in H$. Since H is closed under its *operation, we have that*

$$h, h^2, h^3, \ldots$$

are all in H*. Since* H *is finite, not all of the* h^n*'s are distinct. Then,* $\forall n \in \mathbb{N}$*, there must* $\exists m \in \mathbb{N}$ *such that* $h^n = h^{n+m}$*. Then by Finite Subgroup Test 4.1.1,* $h^m = 1$ *and so* $1 \in H$*. Also, because* $1 = h^{m-1}h$*, we have that* $h^{-1} = h^{m-1}$*, and thus the inverse of* h *is also in* H*. Therefore,* H *is a subgroup of* G *as requried.* $\square$

6 Lecture 6 May 14th 2018

6.1 Subgroups (Continued 2)

6.1.1 Alternating Groups

Recall that $\forall \sigma \in S_n$, with $\sigma \neq \varepsilon$, σ can be uniquely decomposed (up to the order) as disjoint cycles of length at least 2. We will now present a related concept.

Definition 6.1.1 (Transposition)
*A **transposition** $\sigma \in S_n$ is a cycle of length 2, i.e. $\sigma = \begin{pmatrix} a & b \end{pmatrix}$, where $a, b \in \{1, ..., n\}$ and a neqb.*

Example 6.1.1
We have that[1]

$$\begin{pmatrix} 1 & 2 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 4 \end{pmatrix} \begin{pmatrix} 4 & 5 \end{pmatrix}$$

Also, we can show that[2]

$$\begin{pmatrix} 1 & 2 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 4 \end{pmatrix} \tag{6.1}$$

Observe that the factorization into transpositions are **not unique or disjoint**. However, the following property is true.

Theorem 6.1.1 (Parity Theorem)
If a permutations σ has 2 factorizations

$$\sigma = \gamma_1 \gamma_2 \ldots \gamma_r = \mu_1 \mu_2 \ldots \mu_s,$$

[1] If we apply the permutations on the right hand side, we have that

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ & & \downarrow & & \\ 1 & 2 & 3 & 5 & 4 \\ & & \downarrow & & \\ 1 & 4 & 3 & 5 & 2 \\ & & \downarrow & & \\ 2 & 4 & 3 & 5 & 1 \end{array}$$

[2]

Exercise 6.1.1
Show that Equation 6.1 is true.

Exercise 6.1.2
Play around with the same idea and create a few of your own transpositions. Note that you will only be able to get an odd number of tranpositions (why?).

where each γ_i and μ_j are transpositions, then $r \equiv s \mod 2$.

Proof

This is the bonus question in A2. Proof shall be included after the end of the assignment.

Definition 6.1.2 (Odd and Even Permutations)

A permutation σ is even (or odd) if it can be written as a product of an even (or odd) number of transpositions. By Parity Theorem 6.1.1, a permutation must either be even or odd, but not both.

Theorem 6.1.2 (Alternating Group)

For $n \geq 2$, let A_n denote the set of all even permutations in S_n. Then

1. $\varepsilon \in A_n$
2. $\forall \sigma, \tau \in A_n \;\; \sigma\tau \in A_n$ *and* $\exists \sigma^{-1} \in A_n$ *such that* $\sigma\sigma^{-1} = \varepsilon = \sigma^{-1}\sigma$
3. $|A_n| = \frac{1}{2}n!$

Note

*From items 1 and 2, we know that A_n si a subgroup of S_n. A_n is called the **alternating subgroup of degree n**.*

Proof

1. *We have that* $\varepsilon = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}$. *Thus ε is even and so $\varepsilon \in A_n$.*
2. $\forall \sigma, \tau \in A_n$, *we may write*

$$\begin{aligned} \sigma &= \sigma_1\sigma_2 \ldots \sigma_r \quad \text{and} \\ \tau &= \tau_1\tau_2 \ldots \tau_s, \end{aligned}$$

where σ_i, τ_j are transpositions, and r, s are even integers. Then

$$\sigma\tau = \sigma_1\sigma_2 \ldots \sigma_r\tau_1\tau_2 \ldots \tau_s$$

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is a product of $(r+s)$ *transpositions, and thus* $\sigma\tau$ *is even. THus* $\sigma\tau \in A_n$.

For the inverse, note that since σ_i *is a transposition, we have that* $\sigma_i^2 = \varepsilon$ *and thus* $\sigma_i^{-1} = \sigma_i$. *It follows that*

$$\begin{aligned} \sigma^{-1} &= (\sigma_1\sigma_2 \dots \sigma_r)^{-1} \\ &= \sigma_r^{-1}\sigma_{r-1}^{-1} \dots \sigma_2^{-1}\sigma_1^{-1} \\ &= \sigma_r\sigma_{r-1} \dots \sigma_2\sigma_1 \end{aligned}$$

which is an even permutation and

$$\sigma\sigma^{-1} = \sigma_1\sigma_2 \dots \sigma_r\sigma_r \dots \sigma_2\sigma_1 = \varepsilon.$$

Thus $\exists \sigma^{-1} \in A_n$ *such that it is the inverse of* σ.

3. *Let* O_n *denote the set of odd permutations in* S_n. *Then we have* $S_n = A_n \cup O_n$, *and by the Parity Theorem, we have that* $A_n \cap O_n = \emptyset$. *Since* $|S_n| = n!$, *to prove that* $|A_n| = \frac{1}{2}n!$, *it suffices to show that* $|A_n| = |O_n|$.

   Let $\gamma = \begin{pmatrix} 1 & 2 \end{pmatrix}$ *and* $f : A_n \to O_n$ *such that* $f(\sigma) = \gamma\sigma$. *Since* σ *is even,* $\gamma\sigma$ *is odd, and so* f *is well-defined.*

   Also, if $\gamma\sigma_1 = \gamma\sigma_2$, *then by Cancellation Laws,* $\sigma_1 = \sigma_2$, *and hence* f *is injective.*

   Finally, $\forall \tau \in O_n$, *we have that* $\gamma\tau = \sigma \in A_n$. *Note that*

$$f(\sigma) = \gamma\sigma = \gamma\gamma\tau = \tau.$$

   Therefore, f *is surjective.*

   It follows that $|A_n| = |O_n|$. □

For the proof of 3, we know that $|S_n| = n!$, which is twice of the suggested order of A_n. Since we took out the even permutations of S_n, we just need to make the rest of the permutations, the odd permutations, into a set and prove that A_n and this new set has the same size. One way to show this is by creating a bijection between the two.

Also, note that the set of all odd permutations of S_n is not a group, since

- there is no identity element in this set; and
- this set is not closed under map composition.

We have shown that ε is an even permutation, and so by the Parity Theorem, it cannot be an odd permutation, and there is only one identity in S_n. The set is not closed under map composition since if we compose two odd permutations, we would get an even permutation, which does not belong to this set.

6.1.2 *Order of Elements*

Notation

If G is a group and $g \in G$, *we denote*

$$\langle\, g \,\rangle = \{g^k : k \in \mathbb{Z}\}.$$

Note that $1 = g^0 \in \langle g \rangle$.

If $x = g^m, y = g^n \in \langle g \rangle$ *where* $m, n \in \mathbb{Z}$, *then*

$$xy = g^m g^n = g^{m+n} \in \langle g \rangle$$

and we have $\exists x^{-1} = g^{-m} \in \langle g \rangle$ *such that*

$$xx^{-1} = g^m g^{-m} = g^0 = 1.$$

Along with the **Subgroup Test**, we have the following proposition:

Proposition 6.1.1
If G *is a group and* $g \in G$, *then* $\langle g \rangle$ *is a subgroup of* G.

Definition 6.1.3 (Cyclic Groups)
Let G *be a group and* $g \in G$. *Then we call* $\langle g \rangle$ *the* ***cyclic subgroup*** *of* G *generated by* g. *If* $G = \langle g \rangle$ *for some* $g \in G$, *then we say that* G *is a* ***cyclic group***, *and* g *is a* ***generator*** *of* G.

7 Index

List of Symbols

$M_n(\mathbb{R})$	set of $n \times n$ matrices over $\mathbb{R}$
$\mathbb{Z}_n^*$	set of integers modulo n; each element has its multiplicative inverse
S_n	symmetry group of degree n
D_{2n}	dihedral group of degree n; a subset of S_n
K_n	Klein n-group
A_n	alternating group of degree n; a subset of S_n
$\|D_{2n}\|$	order of the dihedral group; the size of the dihedral group
$\begin{pmatrix} 1 & 2 & \dots & n \end{pmatrix}$	An n-cycle
$\det A$	determinant of matrix A
$GL_n(\mathbb{R})$	general linear group of degree n; the set that contains elements of $M_n(\mathbb{R})$ with non-zero determinant
$SL_n(\mathbb{R})$	special linear group of order n; the set that contains elements of $GL_n(\mathbb{R})$ with determinant of 1
$Z(G)$	center of group G
$\langle g \rangle$	cyclic group with generator g