# PMATH351 — Real Analysis

Classnotes for Fall 2018

by

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# 1 Lecture 1 Sep 06th

## 1.1 Course Logistics

No content is covered in today's lecture so this chapter will cover some of the important logistical highlights that were mentioned in class.

- Assignments are designed to help students understand the content.
- Due to shortage of manpower, not all assignment questions will be graded; however, students are encouraged to attempt all of the questions.
- To further motivate students to work on ungraded questions, the midterm and final exam will likely recycle some of the assignment questions.
- There are no required text, but the professor has prepared course notes for reading. The course note are self-contained.
- The approach of the class will be more interactive than most math courses.
- Due to the size of the class, students are encouraged to utilize Waterloo Learn for questions, so that similar questions by multiple students can be addressed at the same time.

## 1.2 Preview into the Introduction

How do we compare the size of two sets?

- If the sets are finite, this is a relatively easy task.
- If the sets are infinite, we will have to rely on functions.

- Injective functions tell us that the **domain is of size that is lesser than or equal to the codomain**.
- Surjective functions tell us that the **codomain is of size that is lesser than or equal to the domain**.
- So does a bijective function tell us that the domain and codomain have the same size? Yes, although this is not as intuitive as it looks, as it relies on **Cantor-Schröder-Bernstein Theorem**.

Now, given two arbitrary sets, are we guaranteed to always be able to compare their sizes? It is tempting to immediately say yes, but to do that, one would have to agree on the **Axiom of Choice**. Fortunately, within the realm of this course, the Axiom of Choice is taken for granted.

# 2 Lecture 2 Sep 10th

## 2.1 *Basic Set Theory*

We shall use the following notations for some of the common set of numbers that we are already familiar with:

- $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \ldots\}$;
- $\mathbb{Z}$ denotes the set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$;
- $\mathbb{Q}$ denotes the set of rational numbers $\left\{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N}\right\}$; and
- $\mathbb{R}$ denotes the set of real numbers.

We shall start with having certain basic properties of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$.

We will use the notation $A \subset B$ and $A \subseteq B$ interchangably to mean that $A$ is a subset of $B$ with the possibility that $A = B$. When we wish to explicitly emphasize this possibility, we shall use $A \subseteq B$. When we wish to explicitly state that $A$ is a **proper subset** of $B$, we will either specify that $A \neq B$ or simply $A \subsetneq B$.

**Definition 1 (Universal Set)**

*A universal set, which we shall generally give the label $X$, is a set that contains all the mathematical objects that we are interested in.*

This is a hand-wavy definition, but it is not in the interest of this course to further explore on this topic.

With a universal set in place, we can have the following definitions:

**Definition 2 (Union)**

*Let $X$ be a set. If $\{A_\alpha\}_{\alpha \in I}$ such that $A_\alpha \subset X$, then the* ***union*** *for all $A_\alpha$ is defined as*

$$\bigcup_{\alpha \in I} A_\alpha := \{x \in X \mid \exists \alpha \in I, x \in A_\alpha\}.$$

**Definition 3 (Intersection)**

*Let $X$ be a set. If $\{A_\alpha\}_{\alpha \in I}$ such that $A_\alpha \subset X$, then the* ***intersection*** *for all $A_\alpha$ is defined as*

$$\bigcap_{\alpha \in I} A_\alpha := \{x \in X \mid \forall \alpha \in I, x \in A_\alpha\}.$$

**Definition 4 (Set Difference)**

*Let $X$ be a set and $A, B \subseteq X$. The* ***set difference*** *of $A$ from $B$ is defined as*

$$A \setminus B := \{x \in X \mid x \in A, x \notin B\}.$$

On a similar notion:

**Definition 5 (Symmetric Difference)**

*Let $X$ be a set and $A, B \subseteq X$. The* ***symmetric difference*** *of $A$ and $B$ is defined as*

$$A \Delta B := \{x \in X \mid (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)\}.$$

In words, for an element in the symmetric difference of two sets, the element is either in $A$ or $B$ but not both. We can also think of the symmetric difference as

$$(A \cup B) \setminus (A \cap B)$$

or

$$(A \setminus B) \cup (B \setminus A).$$

We can also talk about the non-members of a set:

**Definition 6 (Set Complement)**

*Let $X$ be a set and $A \subset X$. The set of all non-members of $A$ is called the* ***complement*** *of $A$, which we denote as*

$$A^c := \{x \in X \mid x \notin A\}.$$

**Note**

*Note that*

$$(A^c)^c = \{x \in X \mid x \notin A^c\} = \{x \in X \mid x \in A\} = A.$$

Now taking a step away from that, we define the following:

**Definition 7 (Empty Set)**

*An **empty set**, denoted by $\emptyset$, is a set that contains nothing.*

**Note**

*The empty set is set to be a subset of all sets.*

**Definition 8 (Power Set)**

*Let $X$ be a set. The power set of $X$ is the set that contains all subsets of $X$, i.e.*

$$\mathcal{P}(X) := \{A \mid A \subset X\}.$$

**Note**

*A power set is always non-empty, since $\emptyset \in \mathcal{P}(\emptyset)$, and since $\emptyset \subset X$ for any set $X$, we have $\emptyset \in \mathcal{P}(X)$.*

**Example 2.1.1**

Let $X = \{1, 2, \ldots, n\}$. There are several ways we can show that the size of $\mathcal{P}(X)$ is $2^n$. One of the methods is by using a characteristic function that maps from $A$ to $\{0,1\}$, defined by

$$X_A : A \to \{0,1\}$$
$$X_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

Using this function, each element in $X$ have 2 states: one being in the subset, and the other being not in the subset, which are represented by 1 and 0 respectively. It is then clear that there are $2^n$ of such configurations.

**Theorem 1 (De Morgan's Laws)**

*Let $X$ be a set. Given $\{A_\alpha\}_{\alpha \in I} \subset \mathcal{P}(X)$, we have*

1. $\left(\bigcup_{\alpha \in I} A_\alpha\right)^c = \bigcap_{\alpha \in I} A_\alpha^c$*; and*
2. $\left(\bigcap_{\alpha \in I} A_\alpha\right)^c = \bigcup_{\alpha \in I} A_\alpha^c$.

**Proof**

1. Note that

$$\begin{aligned} x \in \left(\bigcup_{\alpha \in I} A_\alpha\right)^c &\iff \nexists \alpha \in I \;\; x \in A_\alpha \\ &\iff \forall \alpha \in I \;\; x \notin A_\alpha \\ &\iff \forall \alpha \in I \;\; x \in A_\alpha^c \text{ by set complementation} \\ &\iff x \in \bigcap_{\alpha \in I} A_\alpha^c. \end{aligned}$$

2. Observe that, by part 1,

$$\left(\bigcap_{\alpha \in I} A_\alpha\right)^c = \left(\left(\bigcup_{\alpha \in I} A_\alpha^c\right)^c\right)^c = \bigcup_{\alpha \in I} A_\alpha^c.$$

□

**Example 2.1.2**

Suppose $I = \emptyset$. Then what is $\bigcup_{\alpha \in \emptyset} A_\alpha$? It is sensible to think that all we are left with is simply a union of empty sets, and so

$$\bigcup_{\alpha \in \emptyset} A_\alpha = \emptyset. \tag{2.1}$$

And what about $\bigcap_{\alpha \in \emptyset} A_\alpha$? By Theorem 1, it is quite clear from Equation (2.1) that

$$\bigcap_{\alpha \in \emptyset} A_\alpha = X.$$

## 2.2 Products of Sets

**Definition 9 (Product of Sets)**

*Given 2 sets $X$ and $Y$, the* ***product*** *of $X$ and $Y$ is given by*

$$X \times Y := \{(x, y) \mid x \in X, y \in Y\}.$$

*We often refer to elements of $X \times Y$ as* ***tuples****.*

**Note**

*Now if*

$$X = \{x_1, x_2, \ldots, x_n\},$$
$$Y = \{y_1, y_2, \ldots, y_m\},$$

*then*

$$X \times Y = \{(x_i, y_j) \mid i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, m\}$$

*and so the size of $X \times Y$ is $mn$.*

Consequently, we can think of tuples as two elements being in some "relation".

**Definition 10 (Relation)**

*A* ***relation*** *on sets $X$ and $Y$ is a subset $R$ of the product $X \times Y$. We write*

$$xRy \quad \text{if } (x, y) \in R \subset X \times Y.$$

*We call*

- $\{x \in X \mid \exists y \in Y, (x, y) \in R\}$ *as the* ***domain*** *of* $R$*; and*
- $\{y \in Y \mid \exists x \in X, (x, y) \in R\}$ *as the* ***range*** *of* $R$.

In relation to that, functions are, essentially, relations.

**Definition 11 (Function)**

*A* ***function*** *from* $X$ *to* $Y$ *is a relation* $R$ *such that*

$$\forall x \in X \; \exists ! y \in Y \; (x, y) \in R.$$

Suppose $X_1, X_2, \ldots, X_n$ are non-empty[1] sets. We can define

$$X_1 \times X_2 \times \ldots \times X_n = \prod_{i=1}^{n} X_i := \{(x_1, x_2, \ldots, x_n) \mid x_i \in X_i\}.$$

Now if $X_i = X_j = X$ for all $i, j = 1, 2, \ldots, n$, we write

$$\prod_{i=1}^{n} X_i = \prod_{i=1}^{n} X = X^n.$$

And now comes the problem: given a collection $\{X_\alpha\}_{\alpha \in I}$ of non-empty sets[2], what do we mean by

$$\prod_{\alpha \in I} X_\alpha ?$$

To motivate for what comes next, consider

$$\prod_{i=1}^{n} X_i = X_1 \times \ldots \times X_n = \{(x_1, \ldots, x_n) \mid x_i \in X_i\}.$$

Choose $(x_1, \ldots, x_n) \in \prod_{i=1}^{n} X_i$. This induces a function

$$f_{(x_1, \ldots, x_n)} : \{1, \ldots, n\} \to \bigcup_{i=1}^{n} X_i$$

[1] We are typically only interested in non-empty sets, since empty sets usually lead us to vacuous truths, which are not interesting.

[2] i.e. we now talk about arbitrary $\alpha \in I$.

with

$$\begin{aligned} f(1) &= x_1 \in X_1 \\ f(2) &= x_2 \in X_2 \\ &\vdots \\ f(n) &= x_n \in X_n \end{aligned}$$

Now assume for a more general $f$ such that

$$f : \{1, \ldots, n\} \to \bigcup_{i=1}^{n} X_i$$

is defined by

$$f(i) \in X_i.$$

Then, we have

$$(f(1), f(2), \ldots, f(n)) \in \prod_{i=1}^{n} X_i,$$

which leads us to the following notion:

**Definition 12 (Choice Function)**

*Given a collection $\{X_\alpha\}_{\alpha \in I}$ of non-empty sets, let*

$$\prod_{\alpha \in I} X_\alpha = \left\{ f : I \to \bigcup_{\alpha \in I} X_\alpha \right\}$$

*such that $f(\alpha) \in X_\alpha$. Such an $f$ is called a* ***choice function****.*

And so we may ask a similar question as before: if each $X_\alpha$ is non-empty, is $\prod_{\alpha \in I} X_\alpha$ non-empty? Turns out this is not as easy to show. In fact, it is essentially impossible to show, because this is exactly the **Axiom of Choice**.

# 3 Lecture 3 Sep 12th

## 3.1 Axiom of Choice

RECALL our final question of last lecture: If $\{X_\alpha\}_{\alpha \in I}$ is a non-empty collection of non-empty sets, is

$$\prod_{\alpha \in I} X_\alpha \neq \emptyset \text{ ?}$$

Turns out this is widely known (in the world of mathematics) as the **Axiom of Choice**.

**Axiom 2 (Zermelo's Axiom of Choice)**

*If $\{X_\alpha\}_{\alpha \in I}$ is a non-empty collection of non-empty sets, then*

$$\prod_{\alpha \in I} X_\alpha \neq \emptyset.$$

An equivalent statement of the above axiom is:

**Axiom 3 (Zermelo's Axiom of Choice v2)**

$X \neq \emptyset \implies$

$$\exists f : \mathcal{P}(X) \setminus \{\emptyset\} \to X \; \forall A \in \mathcal{P}(X) \setminus \{\emptyset\} \; f(A) \in A$$

*where $f$ is the choice function.*

**Exercise 3.1.1**

*Prove that Axiom 2 and Axiom 3 are equivalent.*

**Proof**

**From Axiom 2 to Axiom 3:**

Since $X \neq \emptyset$, we have that $\mathcal{P}(X) \setminus \{\emptyset\}$ is a non-empty collection of non-empty sets. Therefore,

$$\prod_{A \in \mathcal{P}(X) \setminus \{\emptyset\}} A \neq \emptyset.$$

So we know that

$$\exists (x_A)_{A \in \mathcal{P}(X) \setminus \{\emptyset\}} \in \prod_{A \in \mathcal{P}(X) \setminus \{\emptyset\}} A.$$

We then simply need to choose the choice function $f : \mathcal{P}(X) \setminus \{\emptyset\} \to X$ such that

$$f(A) = x_A \in A.$$

**From Axiom 3 to Axiom 2:**

Let $X_\alpha \in \mathcal{P}(X)$ for $\alpha \in I$, where $I$ is some index set. We know that not all $X_\alpha = \emptyset$ since $X \neq \emptyset$. Choose $J \subseteq I$ such that $\{X_\alpha\}_{\alpha \in J}$ is a non-empty collection of non-empty sets. Let $f : \mathcal{P}(X) \setminus \{\emptyset\}$ be any choice function. By Axiom 3,

$$\forall X_\alpha \in \mathcal{P}(X) \setminus \{\emptyset\} \quad f(X_\alpha) \in X_\alpha.$$

Therefore,

$$(f(X_\alpha))_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha.$$

$\square$

## 3.2 Relations

Now, it is in our interest to start talking about comparisons or relations between the mathematical objects that we have defined.

**Definition 13 (Relations)**

*A relation $R$ on a set $X$ is* [1]

- *(**Reflexive**)* $\forall x \in X \; xRx$;
- *(**Symmetric**)* $\forall x, y \in X \; xRy \iff yRx$;
- *(**Anti-symmetric**)* $\forall x, y \in X \; xRy \land yRx \implies x = y$;
- *(**Transitive**)* $\forall x, y, z \in X \; xRy \land yRz \implies xRz$.

[1] We can look at this definition as $R \subseteq X \times X$. Under such a definition, we would have

- **(Reflexive)** $\forall x \in X \; (x, x) \in R$;
- **(Symmetric)** $\forall x, y \in X \; (x, y) \in R \iff (y, x) \in R$;
- **(Anti-symmetric)** $\forall x, y \in X \; (x, y), (y, x) \in R \implies x = y$;
- **(Transitive)** $\forall x, y, z \in X \; (x, y), (y, z) \in R \implies (x, z) \in R$.

**Example 3.2.1**

Let $X = \mathbb{R}$, and let $xRy \iff x \leq y$, where $\leq$ is the notion of "less than or equal to", which we shall assume that it has the meaning that we know. Observe that $\leq$ is:

- reflexive: $\forall x \in \mathbb{R} \; x \leq x$ is true;
- anti-symmetric: $\forall x, y \in \mathbb{R} \; x \leq y \land y \leq x \implies x = y$; and
- transitive: $\forall x, y, z \in \mathbb{R} \; x \leq y \land y \leq z \implies x \leq z$.

**Example 3.2.2**

Let $Y \neq \emptyset$, $X = \mathcal{P}(Y)$, with $ARB \iff A \subseteq B$. Observe that $\subseteq$ is:

- reflexive: $\forall A \in \mathcal{P}(Y) \; ARA \iff A \subseteq A$ is true;
- anti-symmetric: $\forall A, B \in \mathcal{P}(Y) \; ARB \land BRA \iff A \subseteq B \land B \subseteq A \implies A = B$;
- transitive: $\forall A, B, C \in \mathcal{P}(Y) \; ARB \land BRC \iff A \subseteq B \land B \subseteq C \implies A \subseteq C$.

**Example 3.2.3**

Let $Y \neq \emptyset$, $X = \mathcal{P}(Y)$, with $ARB \iff A \supseteq B$. Observe that $\supseteq$ is:

- reflexive: $\forall A \in \mathcal{P}(Y) \; ARA \iff A \subseteq A$;
- anti-symmetric: $\forall A, B \in \mathcal{P}(Y) \; ARB \land BRA \iff A \supseteq B \land B \supseteq A \implies A = B$;
- transitive: $\forall A, B, C \in \mathcal{P}(Y) \; ARB \land BRC \iff A \supseteq B \land B \supseteq C \implies A \supseteq C$.

All the above examples are also known as *partially ordered sets*.

### Definition 14 (Partially Ordered Sets)

*The set $X$ with the relation $R$ on $X$ is called a* ***partially ordered set*** *(or a* ***poset****) if $R$ is*

- *reflexive;*
- *anti-symmetric; and*
- *transitive.*

*We denote a poset by $(X, R)$.*

The "partial" in 'partially ordered" indicates that not every pair of elements need to be comparable, i.e. there may be pairs for which neither precedes the other (anti-symmetry).

### Note

*If $(X, R)$ is a poset, then if $A \subseteq X$, and $R_1 = R \restriction_{A \times A}$, then $(A, R_1)$ is also a poset.*

**Example 3.2.4**

How many possible relations can we define on these sets to make them into posets?

1. $X = \emptyset$

   **Solution**
   We have that $R = \emptyset \times \emptyset$, and so the only relation we have is an empty relation. Then it is vacuously true that $(X, R)$ a poset.

2. $X = \{x\}$

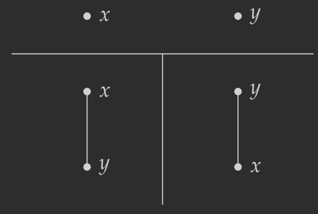
   **Solution**
   We have that $R = X \times X = \{(x, x)\}$. It it clear that $(X, R)$ is a poset.

3. $X = \{x, y\}$

   **Solution**
   There are 3 possible relations:

   - a relation where $xRx$ and $yRy$;
   - a relation where $xRy$; or
   - a relation where $yRx$.

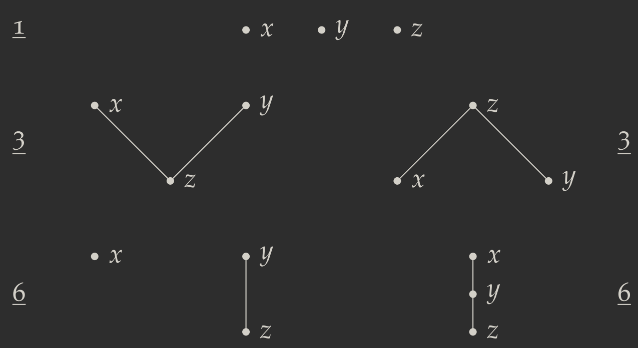
3 possibilities represented as graphs (known as Hasse diagram), separated by lines:



4. $X = \{x, y, z\}$

✎ **Solution**

The following are all the possibilities represented by graphs, where the underlined numbers represent the number of ways we can rearrange the elements for unique relations:




Therefore, we see that there are a total of

$$1 + 3 + 3 + 6 + 6 = 19 \text{ relations.}$$

**Exercise 3.2.1**

*How many possible relations can we define on a set of 6 elements to the set into a poset?*

✎ **Solution**

**to be added**

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**Definition 15 (Totally Ordered Sets / Chains)**

*The set $X$ with the relation $R$ on $X$ is called a* ***totally ordered set*** *(or a* ***chain****) if $(X, R)$ is a poset with the exception that, for any $x, y \in X$, either $xRy$ or $yRx$ but not both.*

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**Definition 16 (Bounds)**

*Let $(X, \leq)$ be a poset. Let $A \subset X$. We say $x_0 \in X$ is an* ***upper bound*** *for $A$ if*

$$\forall a \in A \quad a \leq x_0.$$

*If $A$ has an upper bound, we say that $A$ is* ***bounded above****. If $A$ is bounded above, then $x_0$ is the* ***least upper bound*** *(or* ***supremum****) of*

*$A$ is for any $x_1 \in X$ that is an upper bound of $A$, we have*

$$x_0 \leq x_1.$$

*We write $x_0 = \mathrm{lub}(A) = \sup(A)$. If $\sup(A) \in A$, then $\sup(A) = \max(A)$ is the* ***maximum*** *of $A$.*

*We can analogously define for:*

$$\begin{aligned} \textit{upper bound} &\to \textit{lower bound} \\ \textit{bounded above} &\to \textit{bounded below} \\ \textit{least upper bound}, \ \mathrm{lub} &\to \textit{greatest lower bound}, \ \mathrm{glb} \\ \textit{supremum}, \ \sup &\to \textit{infimum}, \ \inf \\ \textit{maximum}, \ \max &\to \textit{minimum}, \ \min \end{aligned}$$

**Note**
*By* ***anti-symmetry*** *of posets, we have that $\max, \sup, \min, \inf$ are all unique if they exists.*

**Example 3.2.5 (Least Upper Bound Property of $\mathbb{R}$)**

Let $X = \mathbb{R}$, and $\leq$ be the order that we have defined. Every bounded non-empty subset of $X$ has a supremum.

**Example 3.2.6**

Let $Y \neq \emptyset$, and $X = \mathcal{P}(Y)$, and $\subseteq$ the ordering by inclusion. We know that $Y$ is the maximum element of $(X, \subseteq)$. Then the collection $\{A_\alpha\}_{\alpha \in I} \subset \mathcal{P}(Y)$ is bounded above by $Y$, and we have that

$$\sup\left(\{A_\alpha\}_{\alpha \in I}\right) = \bigcup_{\alpha \in I} A_\alpha$$
$$\inf\left(\{A_\alpha\}_{\alpha \in I}\right) = \bigcap_{\alpha \in I} A_\alpha$$

Now if $Y = \emptyset$, we would end up having

$$\sup\left(\{A_\alpha\}_{\alpha \in I}\right) = \emptyset$$
$$\inf\left(\{A_\alpha\}_{\alpha \in I}\right) = X$$

This makes sense, since the empty set would be the least of upper bounds, and since $X = \mathcal{P}(Y)$ would have to be the greatest of lower bounds.

# 4 Lecture 4 Sep 14th

## 4.1 Zorn's Lemma

**Definition 17 (Maximal Element)**

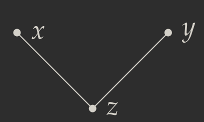
*Let $(X, \leq)$ be a poset. An element $x \in X$ is* ***maximal*** *if whenever $y \in X$ is such that $x \leq y$, we must have $y = x$.*

**Example 4.1.1**

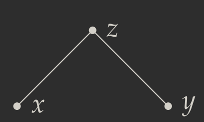
Looking back at Example 3.2.4, on the set $X = \{x, y, z\}$, we have that the maximal element in each possible poset is/are:

This shows to us that the maximal element does not have to be unique.

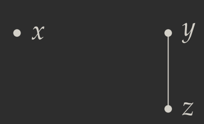

$x, y, z$ are all maximal




$z$ is maximal




$x, y$ are both maximal




$x, z$ are both maximal


$z$ is maximal

**Example 4.1.2**

- Given $X \neq \emptyset$, the maximal element of the poset $(\mathcal{P}(X), \subseteq)$ is $X$.
- Given $X \neq \emptyset$, the maximal element of the poset $(\mathcal{P}(X), \supseteq)$ is $\emptyset$.
- The poset $(\mathbb{R}, \leq)$ has no maximal element.

**Axiom 4 (Zorn's Lemma)**

*If $(X, \leq)$ is a non-empty poset such that every chain $S \subset X$ has an upper bound, then $(X, \leq)$ has a maximal element.*

**Theorem 5 (★ Non-Zero Vector Spaces has a Basis)**

*Every non-zero vector space, $V$, has a basis.*

The flow of this proof is a typical approach when Zorn's Lemma is involved.

**Proof (★)**
Let

$$\mathcal{L} := \{ A \subset V \mid A \text{ is linearly independent } \}.$$

Note that $\mathcal{L} \neq \emptyset$ since $V \neq \{0\}$. Now order elements of $\mathcal{L}$ with $\subseteq$. It suffices to show that $(\mathcal{L}, \subseteq)$ has a maximal element, since this maximal element must be a basis. Otherwise, we would contradict the maximality of such an element.[1]

[1] This is the key to this proof.

Now let $S = \{A_\alpha\}_{\alpha \in I}$ be a chain in $\mathcal{L}$. Let

$$A_0 = \bigcup_{\alpha \in I} A_\alpha.$$

**Require clarification before proceeding...**

**Definition 18 (Well-Ordered)**

*We say that a poset $(X, \leq)$ is **well-ordered** if every non-empty subset $A \subset X$ has a least/minimal element in $A$.*

**Exercise 4.1.1**
*Prove that well-ordered sets are chains.*

**Example 4.1.3**

$(\mathbb{N}, \leq)$ is well-ordered.

**Axiom 6 (Well-Ordering Principle)**

*Every non-empty set can be well-ordered.*

**Theorem 7 (Axioms of Choice and Its Equivalents)**

*TFAE:*

1. *Axiom of Choice, Axiom 2*
2. *Zorn's Lemma, Axiom 4*
3. *Well-Ordering Principle, Axiom 6.*

**Exercise 4.1.2**
*Prove Theorem 7*

**Proof**
(3) $\implies$ (1) is simple; let the choice function be such that we pick the minimal element from each set among a non-empty collection of non-empty sets. It is clear that the product of these sets will always have an element, in particular the tuple where each component is the minimal element of each set.

**The rest will be added once I've worked it out**

**Example 4.1.4**

Let $X = \mathbb{Q}$. Let $\phi : \mathbb{Q} \to \mathbb{N}$ be defined such that

$$\phi\left(\frac{m}{n}\right) = \begin{cases} 2^m 5^n & m > 0 \\ 1 & m = 1 \\ 3^{-m} 7^n & m < 0 \end{cases}$$

By the **unique prime factorization of natural numbers** (or **Fundamental Theorem of Arithmetic**), we have that $\phi$ is injective. In fact,

$$r \leq s \iff \phi(r) \leq \phi(s),$$

showing to us that we have a well-ordering on $\mathbb{Q}$.

## 4.2 Cardinality

### 4.2.1 Equivalence Relation

**Definition 19 (Equivalence Relation)**

*Let $X$ be non-empty set. A relation $\sim$ on $X$ is an* ***equivalence relation*** *if it is*

- *reflexive;*
- *symmetric; and*
- *transitive.*

**Definition 20 (Equivalence Class)**

*Let $X$ be a non-empty set, and $x \in X$. An* ***equivalence class*** *of $x$ under the equivalence relation $\sim$ is defined as*

$$[x] := \{y \in X \mid x \sim y\}.$$

**Note**

*Note that we either have $[x] = [y]$ or $[x] \cap [y] = \emptyset$. This is sensible, since if $w \in [x]$, then $w \sim x$. If $w \in [y]$, then we are done. If $w \notin [y]$, suppose $\exists v \in [y]$ such that $w \sim v$, which then implies $w \in [y]$ which is a contradiction.*

*This results shows to us that*

$$X = \bigcup_{x \in X} [x],$$

*or in words, equivalence classes partition the set.*

**Definition 21 (Partition)**

*Let $X \neq \emptyset$. A **partition** of $X$ is a collection $\{A_\alpha\}_{\alpha \in I} \subset \mathcal{P}(X)$ such that*

1. *$A_\alpha \neq \emptyset$;*
2. *$A_\alpha \cap A_\beta = \emptyset$ if $\alpha \neq \beta$ in $I$; and*
3. *$X = \bigcup_{\alpha \in I} A_\alpha$.*

With this, we have ourselves another method to show that $\sim$ is an equivalence relation.

**Proposition 8 (Characterization of An Equivalence Relation)**

*If $\{A_\alpha\}_{\alpha \in I}$ is a partition of $X$ and $x \sim y \iff x, y \in A_\alpha$, then $\sim$ is an equivalence relation.*

The proof of this statement has been stated above.

Similar to when we defined partial orders, we can ask ourselves the following question:

**Example 4.2.1**

How many equivalence relations are there on the set $X = \{1, 2, 3\}$?[2]

[2] By Proposition 8, this question is equivalent to asking for the number of partitions we can create from the set $X$. The study of counting partitions is what is covered by Bell's Number.

**Solution**

Note that we can partition $X$ as

$$\{\{1\}, \{2\}, \{3\}\}, \{\{1, 2, 3\}\},$$

and

$$\{\{1, 2\}, \{3\}\},$$

which we can rearrange in 3 different ways. Therefore, there are 5 different equivalence relations that we can define on $X$.

**Example 4.2.2**

Let $X$ be any set. Consider $\mathcal{P}(X)$. Define $\sim$ on $\mathcal{P}(X)$ by

$$A \sim B \iff \exists f : A \to B$$

such that $f$ is surjective[3]. It is easy to verify that $\sim$ is an equivalence relation.

[3] $\sim$ partitions $X$ into sets that have the same number of elements.

# 5 Lecture 5 Sep 17th

## 5.1 Cardinality (Continued)

**Definition 22 (Finite Sets)**

*A set $X$ is **finite** if $X = \emptyset$ or if $X \sim \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$, where $\sim$ is the equivalence relation defined in Example 4.2.2.*

**Definition 23 (Cardinality)**

*If $X \sim n$, we say $X$ has **cardinality** $n$ and write $|X| = n$. We also let $|\emptyset| = 0$.*

Now a good question here is: if $n \neq m$, is $\{1, 2, \ldots, n\} \sim \{1, 2, \ldots, m\}$?

**Theorem 9 (Pigeonhole Principle)**

*The set $\{1, 2, \ldots, n\}$ is not equivalent to any of its proper subset.*

**Proof**

We shall prove this by induction on $n$.

**Base case**: $\{1\} \nsim \emptyset$.

This is a **proof by contradiction**, using the fact that we cannot find an injective function from a "larger" set to a "smaller" set.

We can assume that the function $f$ is not surjective, since if the larger set is indeed equivalent to the smaller set, then it should not matter if $f$ is surjective or not. In particular, we only require that there be an injective function.

**Requires clarification and confirmation of proof**

Assume that the statement holds for $\{1, \ldots, k\}$. Suppose we have an injective function

$$f : \{1, 2, \ldots, k, k+1\} \to \{1, 2, \ldots, k, k+1\}$$

that is not surjective.
**Case 1**: $k+1 \notin \operatorname{Range}(f)$, where $\operatorname{Range}(f)$ is the range of $f$. Then we have

Note: $\restriction$ is the restriction sign.

$$f \restriction_{\{1,\ldots,k\}} : \{1, \ldots, k\} \to \{1, \ldots, k\} \setminus \{f(k+1)\}.$$

However, $f$ is an injective function and clearly

$$\{1, \ldots, k\} \setminus \{f(k+1)\} \subseteq \{1, \ldots, k\},$$

a contradiction.

**Case 2**: $k+1 \in \operatorname{Range}(f)$. Then $\exists j_0 \in \{1, \ldots, k, k+1\}$ such that $f(j_0) = k+1$, and since $f$ is not surjective, $\exists m \in \{1, \ldots, k\}$ such that $m \notin \operatorname{Range}(f)$. Then consider a new function $g : \{1, \ldots, k, k+1\} \to \{1, \ldots, k\}$ such that

$$g(a) = \begin{cases} m & a = k+1 \\ f(k+1) & a = j_0 \\ f(a) & a \neq j_0, k+1 \end{cases}$$

---

### Corollary 10 (Pigeonhole Principle (Finite Case))

*If the set $X$ is finite, then $X$ is not equivalent to any proper subset.*

---

Sketch of proof:

$$\begin{tikzcd} \{1, \ldots, n\} \arrow[r] \arrow[d, shift left, "f^{-1}"'] & \{1, \ldots, n\} \\ X \arrow[u, shift left, "f"'] \arrow[r, "1-1"', "\text{onto}"] & A \subsetneq X \arrow[u, "f"'] \end{tikzcd}$$

**Exercise 5.1.1**

*Prove Corollary 10.*

---

### Definition 24 (Infinite Sets)

*$X$ is* ***infinite*** *if it is not finite.*

---

**Example 5.1.1**

Observe that we can construct a function $f : N \to \{2, 3, \ldots\}$ by $f(n) = n + 1$. It is clear that $f$ is a bijective funciton, and so $\mathbb{N} \sim \{2, 3, \ldots\}$.

**Proposition 11 ($\mathbb{N}$ is the Smallest Infinite Set)**

*Every infinite set contains a subset $A \sim \mathbb{N}$.*

**Proof**

Suppose $X$ is infinite. Let

$$f : \mathcal{P}(X) \setminus \{\emptyset\} \to X$$

such that for $S \subset X$ where $S \neq \emptyset$, $f(S) \in S$ [1]. Let $x_1 = f(X)$. Let $x_2 = f(X \setminus \{x_1\})$. Recursively, define

$$x_n = f(X \setminus \{x_1, \ldots, x_{n-1}\}).$$

This gives us a sequence

$$X \supset S = \{x_1, \ldots, x_n, \ldots\}$$

which is equivalent to $\mathbb{N}$ via the map $n \mapsto x_n$. □

[1] Axiom 3 ahoy!

**Corollary 12 (Infinite Sets are Equivalent to Its Proper Subsets)**

*Every infinite set $X$ is equivalent to a proper subset of $X$.*

**Proof**

Given such an $X$, we construct a sequence $\{x_n\}$ as in the previous proof. Define $f : X \to X \setminus \{x_n\}$ by

$$f(x) = \begin{cases} x_{n+1} & x \in \{x_n\} \\ x & x \notin \{x_n\}. \end{cases}$$

Clearly so, $f$ is injective. □

**Definition 25 (Countable)**

*We say that a set is* ***countable*** *(or* ***denumerable****) is either $X$ is finite or if $X \sim \mathbb{N}$. If $X \sim \mathbb{N}$, we can say that $X$ is* ***countably infinite*** *and write $|X| = |\mathbb{N}| = \aleph_0$.*

**Definition 26 (Smaller Cardinality)**

*Given 2 sets $X, Y$, we write*

$$|X| \leq |Y|$$

*if $\exists f : X \to Y$ injective.*

**Proposition 13 (Injectivity is Surjectivity Reversed)**

*TFAE*

1. $\exists f : X \to Y$ *injective*
2. $\exists g : Y \to X$ *surjective*

**Proof**

(1) $\implies$ (2): Define

$$g(y) = \begin{cases} x & \exists x \in X \;\; f(x) = y \\ x_0 & \text{any } x_0 \in X \end{cases}$$

Clearly $g$ is surjective.

(2) $\implies$ (1): Since $g$ is surjective, for each $x \in X$, we have that[2]

$$g^{-1}(|x|) = \{y \in Y : g(y) = x\} \neq \emptyset.$$

By the Axiom of Choice, there exists a choice function $h : \mathcal{P}(Y) \setminus \{\emptyset\} \to Y$ such that for each $A \subset Y$, $h(A) \in A$. Then, let $f : X \to Y$ such that

$$f(x) = h(g^{-1}(\{x\})).$$

[2] The idea here is to collect the preimages into a set, and use the choice function as an injective map.

Clearly so, $f$ is injective. $\square$

**Note**

*Note that we have* $|\mathbb{N}| \leq |\mathbb{Q}|$, *since we can define an injective function* $f : \mathbb{N} \to \mathbb{Q}$ *such that* $f(n) = \frac{n}{1}$.

*We have also shown that* $|\mathbb{Q}| \leq |\mathbb{N}|$ *using our injective function* $g : \mathbb{Q} \to \mathbb{N}$, *given by*

$$g\left(\frac{m}{n}\right) = \begin{cases} 2^m 3^n & m > 0 \\ 1 & m = 0 \\ 5^{-m} 7^n & m < 0 \end{cases}$$

Question: Is $|\mathbb{N} = |\mathbb{Q}||$? In other words, given $|X| \leq |Y| \wedge |Y| \leq |X|$, is $|X| = |Y|$?

# 6 *Lecture 6 Sep 19th*

## 6.1 *Cardinality (Continued 2)*

Before delving into resolving our last question in the previous lecture, note the following:

**Note**

*Suppose $f : X \to Y$ is bijective. Let $A \subseteq B$, then*

$$f(B \setminus A) = f(B) \setminus f(A).$$

Prove this observation as an exercise:

**Exercise 6.1.1**
*Prove the note on the left.*

**Theorem 14 (★★★ Cantor-Schröder-Bernstein Theorem (CSB))**

*Let $A_2 \subset A_1 \subset A_0 = A$. Assume that $A_2 \sim A_0$. Then $A_0 \sim A_1$.*

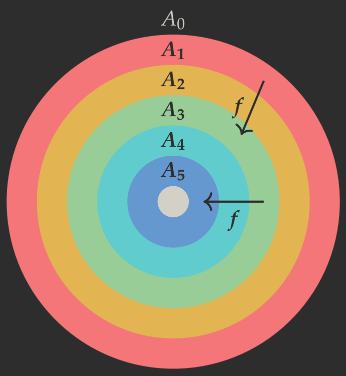



Figure 6.1: The core idea of the proof for Cantor-Schröder-Bernstein Theorem

**Proof**

Let $\phi : A_0 \to A_2$ be bijective, by assumption. Since $A_1 \subset A_1$, let $A_3 = \phi(A_1) \subset A_2$, and since $A_2 \subset A_0$, let $A_4 = \phi(A_2) \subset A_3$. Recursively so, let

$$A_{n+2} = \phi(A_n)$$

Notice that

$$A_0 = (A_0 \setminus A_1) \cup (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup (A_3 \setminus A_4) \cup \ldots \bigcap_{n=0}^{\infty} A_n$$

$$A_1 = (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup (A_3 \setminus A_4) \cup (A_4 \setminus A_5) \cup \ldots \bigcap_{n=1}^{\infty} A_n$$

Observe that

$$\bigcap_{n=0}^{\infty} A_n = \bigcap_{n=1}^{\infty} A_n.$$

[1]Define $f : A \to A_1$ by

$$f(x) = \begin{cases} x & x \in \bigcap_{n=0}^{\infty} A_n \\ x & x \in A_{2k+1} \setminus A_{2k+2},\ k = 0, 1, 2, \ldots \\ \phi(x) & x \in A_{2k} \setminus A_{2k+1},\ k = 0, 1, 2, \ldots \end{cases}$$

[1] Here, we employ the idea from Figure 6.1.

### Corollary 15 (Cantor-Schröder-Bernstein Theorem - Restated)

*If $A_1 \subset A \wedge B_1 \subset B \wedge A \sim B_1 \wedge B \sim A_1$, then $A \sim B$.*[2]

[2] This is equivalent to the statement

$$|A| \leq |B| \wedge |B| \leq |A| \implies |A| = |B|.$$

### Proof

By assumption, let $f : A \to B_1$ be bijective, and let $g : B \to A_1$ be bijective. Let $A_2 = g(B_1) \subseteq A_1 \subset A$ Let $A_2 = g(B_1) \subseteq A_1 \subset A$. Then the composite function $g \circ f : A \to A_2$ is bijective, and so $A \sim A_2$. By Theorem 14, we have

$$A \sim A_2 \sim A_1 \sim B.$$

$\square$

**Example 6.1.1**

Our question from last lecture now has an answer: by Theorem 14, we have that $|\mathbb{Q}| = |\mathbb{N}|$.[3]

[3] Now that we know that they have the same cardinality:

**Exercise 6.1.2**
*Find a bijection between $\mathbb{Q}$ and $\mathbb{N}$.*

### Proposition 16 (Denumerability Check)

*If $X$ is infinite, then*

$$|X| = |\mathbb{N}| = \aleph_0 \iff \exists f : X \to \mathbb{N} \text{ bijective.}$$

**Proof**

$( \implies )$ is immediate. For $( \impliedby )$, suppose $f : X \to \mathbb{N}$, which implies that $|X| \leq |\mathbb{N}|$. By Proposition 11, $|\mathbb{N}| \leq |X|$. Therefore, $|X| = |bbN| = \aleph_0$. □

**Example 6.1.2**

$\mathbb{N} \times \mathbb{N}$ is countable. The function

$$f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \text{ given by } f(m, n) = 2^n 3^m$$

is injective.

**Definition 27 (Uncountable)**

*A set $X$ is* ***uncountable*** *if it is not countable.*

**Theorem 17 (Cantor's Diagonal Argument)**

$(0, 1)$ *is uncountable.*

**Proof**

Suppose, for contradiction, that $(0, 1)$ is countable. Then we can write

$$\begin{aligned} a_1 &= .a_{11}a_{12}a_{13}\ldots \\ a_2 &= .a_{21}a_{22}a_{23}\ldots \\ &\vdots \\ a_n &= .a_{n1}a_{n2}a_{n3}\ldots \\ &\vdots \end{aligned}$$

in $(0, 1)$. This representation is unique if we do not allow the representation to end in a string of 9's. Let $b \in (0, 1)$, expressed as

$b = .b_1b_2b_3\ldots$ such that

$$b_i = \begin{cases} 5 & a_i \neq 5 \\ 2 & a_i = 5 \end{cases}$$

However, $b \notin (0,1)$, otherwise $b$ would be one of the $a_n$'s, a contradiction. □

**Corollary 18 (Uncountability of $\mathbb{R}$)**

$\mathbb{R}$ *is uncountable.*

**Proof**

Let $f : (0,1) \to \mathbb{R}$ be given by

$$f(x) = \tan\left(\pi x - \frac{\pi}{2}\right).$$

Clearly so, $(0,1)$ is bijective. □

**Note**

*We denote* $|\mathbb{R}| = c$.

Question: Given sets $X, Y$, is it always true that either[4]

1. $|X| = |Y|$;
2. $|X| < |Y|$; or
3. $|Y| < |X|$.

[4] As compare to $\leq$, $<$ implies that there is no surjection from the set on the LHS to the RHS.

# 7 Lecture 7 Sep 21st

## 7.1 Cardinality (Continued 3)

**Theorem 19 (★ Comparability of Cardinals)**

*If $X$ and $Y$ are non-empty, then either*

$$|X| \leq |Y| \vee |Y| \leq |X| .$$

**Proof**

Let

$$S = \{(A, B, f) \mid A \subseteq X, B \subseteq Y, f : A \to B \text{ bijective } \}.$$

Note that $S \neq \emptyset$, since $X$ and $Y$ are non-empty, and so we can have $f(a) = b$ for $A = \{a\} \subset X$ and $B = \{b\} \subset Y$.[1] We order $S$ as follows: we say

$$(A_1, B_1, f_1) \leq (A_2, B_2, f_2)$$

if

$$A_1 \subseteq A_2, \;\; B_1 \subseteq B_2, \;\; f_1 = f_2 \restriction_{A_1} .$$

Let $C = \{(A_\alpha, B_\alpha, f_\alpha)\}_{\alpha \in I}$ be a chain in $(S, \leq)$. Let $A_0 = \bigcup_{\alpha \in I} A_\alpha$, $B_0 = \bigcup_{\alpha \in I} B_\alpha$, and define $f_0 : A_0 \to B_0$ by

$$f_0(x) = f_{\alpha_0}(x) \text{ if } x \in A_{\alpha_0}.$$

Now if $x \in A_{\alpha_1}$, $x \in A_{\alpha_2}$ and

$$(A_{\alpha_1}, B_{\alpha_1}, f_{\alpha_1}) \leq (A_{\alpha_2}, B_{\alpha_2}, f_{\alpha_2}),$$

[1] We want to use the maximal element to obtain our result. To that end, we need Zorn's Lemma. So we need $S$ to build this up.

we have that

$$f_{\alpha_1}(x) = f_{\alpha_2} \restriction_{A_{\alpha_1}} (x) = f_{\alpha_2}(x),$$

i.e. $f_0$ is well-defined.

Claim 1: $f_0 : A_0 \to B_0$ is injective.

Let $x_1, x_2 \in A_0$ such that $x_1 \neq x_2$.
$\implies \exists \alpha_1, \alpha_2 \in I \;\; x_1 \in A_{\alpha_1} \wedge x_2 \in A_{\alpha_2} \wedge A_{\alpha_1} \subseteq A_{\alpha_2}$ (wlog)
$\implies x_1 . x_2 \in A_{\alpha_2}$
$\implies (\because f_{\alpha_2}$ injective $\implies f_{\alpha_2}(x_1) \neq f_{\alpha_2}(x))$
$\implies f_0(x_1) \neq f_0(x_2) \implies f_0$ injective.

Claim 2: $f_0 : A_0 \to B_0$ is surjective.

Let $y_0 \in B_0$
$\implies \exists \alpha_0 \in I \;\; y_0 \in B_{\alpha_0}$
$\implies \exists x_0 \in A_{\alpha_0} \;\; f_{\alpha_0}(x_0) = y_0$ ($\because f_{\alpha_0}$ surjective)
$\implies f_0(x_0) = y_0$

$\therefore (A_0, B_0, f_0)$ is an upper bound for $C$. Then by Zorn's Lemma, $(S, \leq)$ has a maximal element $(A, B, f)$.

Case 1: If $A = X$, then injectivity of $f$ implies $|X| \leq |Y|$.

Case 2: If $B = Y$, then surjectivity of $f$ implies $|Y| \leq |A| \leq |X|$.

Case 3: If $A \neq X \wedge B \neq Y$, then $X \setminus A \neq \emptyset \wedge Y \setminus B \neq \emptyset$. Let $x_0 \in X \setminus A$, $y_0 \in Y \setminus B$. Let $A^* = A \cup \{x_0\}$, $B^* = B \cup \{y_0\}$, and $f^* : A^* \to B^*$ such that

$$f^*(x) = \begin{cases} f(x) & x \in A \\ y_0 & x = x_0 \end{cases}$$

Then $(A, B, f) \leq (A^*, B^*, f^*)$, contradicting maximality. $\square$

## 7.1.1 Cardinal Arithmetic

*Sum of Cardinals* Observe that if $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\}$, and $X \cap Y = \emptyset$, then $|X| = n$, $|Y| = m$ and $|X \cup Y| = n + m$. This motivates us to provide the following definition:

**Definition 28 (Sum of Cardinals)**

*Assume that $X$ and $Y$ are such that $X \cap Y = \emptyset$. We define*

$$|X| + |Y| = |X \cup Y|.$$

---

Question: So what about $\aleph_0 + \aleph_0$?

A thought that motivates us to give the following answer lies in the observation that: if $X$ is the set of even natural numbers and $Y$ the odd natural numbers, then $X \cup Y$ is the set of all natural numbers. All three sets are countably infinite, i.e. they have cardinality $\aleph_0$.

Question: What about $c + c$?

A similar motivation comes from the observation that: given $X = (0,1)$ and $Y = (1,2)$, we have

$$c = |X| \leq |X| + |Y| \leq |R| = c,$$

and so $|X| = |Y| = c \implies |X \cup Y| = c$.

---

**Theorem 20 (Sums of Cardinals)**

*Given sets $X$ and $Y$, if $X$ is infinite, then*

1. $|X| + |X| = |X|$
2. $|X| + |Y| = \max(|X|, |Y|)$

---

*Multiplication of Cardinals* Given

$$X = \{x_1, x_2, \ldots, x_n\}$$
$$Y = \{y_1, y_2, \ldots, y_m\}$$

we have that

$$X \times Y = \{(x_i, y_j) \mid i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m\}$$

and so

$$|X \times Y| = nm.$$

**Exercise 7.1.1**
*Prove Theorem 20 as an exercise.*

### Definition 29 (Multiplication of Cardinals)

*Given sets $X$ and $Y$, define*

$$|X|\,|Y| = |X \times Y|\,.$$

**Example 7.1.1**

We have $|\mathbb{N} \times \mathbb{N}| = \aleph_0$ since the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by

$$f(n, m) = 2^n 3^m$$

is injective.

QUESTION: What about $c \cdot c$?

### Theorem 21 (Multiplication of Cardinals)

*If $X$ is infinite and $Y \neq \emptyset$, then*

- $|X \times X| = |X| \implies |X|\,|X| = |X|$;
- $|X \times Y| = \max(|X|\,, |Y|)$.

**Exercise 7.1.2**
*Prove Theorem 21 as an exercise.*

# 8 Lecture 8 Sep 24th

## 8.1 Cardinality (Continued 4)

### 8.1.1 Cardinal Arithmetic (Continued)

*Exponentiation of Cardinals* Recall if $\{Y_x\}_{x \in X}$ is a collection of non-empty sets, we have[1]

$$\prod_{x \in X} Y_x = \{f : X \to \bigcup_{x \in X} Y_x \mid f(x) \in Y_x\}.$$

[1] This should remind you of Axiom 3

Now if $Y = Y_x$ for all $x \in X$, we have

$$Y^X = \prod_{x \in X} = \{f : X \to Y\}.$$

**Example 8.1.1**

Given

$$Y = \{1, \ldots, m\} \quad X = \{1, \ldots, n\}$$

we have

$$Y^X = \{f : \{1, \ldots, n\} \to \{1, \ldots, m\}\}.$$

Observe that $Y^X$ is similar to $Y^n$. So $\left|Y^X\right| = m^n$.[2]

[2] **Need better explanation.**

---

**Definition 30 (Exponentiation of Cardinals)**

*Given sets $X$ and $Y$, define*

$$|Y|^{|X|} := \left|Y^X\right|.$$

---

**Theorem 22 (Exponentiation of Cardinals)**

*If $X, Y, Z$ are non-empty sets, then*

- $|Y|^{|X|} \cdot |Y|^{|Z|} = |Y|^{|X|+|Z|}$;
- $\left(|Y|^{|X|}\right)^{|Z|} = |Y|^{|X| \cdot |Z|}$.

**Exercise 8.1.1**
*Prove Theorem 22.*

### Theorem 23 ($2^{\aleph_0} = c$)

*We have that $2^{\aleph_0} = c$.*

This requires closer studying.

### Proof

Note that $2^{\aleph_0} = \left|\{0,1\}^{\mathbb{N}}\right|$, where[3]

$$\left|\{0,1\}^{\mathbb{N}}\right| = |\{f : \mathbb{N} \to \{0,1\}| = |\{\{a_n\}_{n=1}^{\infty} \mid a_i = 0,1\}|$$

[3] Explain 2nd equality.

Given a sequence $\{a_n\} \in \{0,1\}^{\mathbb{N}}$, define $\phi : \{0,1\}^{\mathbb{N}} \to (0,1)$ such that[4]

$$\phi(\{a_n\}) := \sum_{i=1}^{\infty} \frac{a_n}{3^n}.$$

[4] This is a base 3 representation (**of what?**)

which is injective since there are no trailing 2's. Therefore $2^{\aleph_0} \leq c$.

Given $\alpha \in (0,1)$, let[5]

$$\alpha = \sum_{n=1}^{\infty} \frac{b_n}{2^n},$$

[5] This is a base 2 representation.

where $b_n = 0,1$. Let $\psi : (0,1) \to \{0,1\}^{\mathbb{N}}$ such that

$$\psi(\alpha) = \psi\left(\sum_{i=1}^{\infty} \frac{b_n}{2^n}\right) = \{b_n\}$$

Then $\psi$ is injective, and so $c \leq 2^{\aleph_0}$. Thus $2^{\aleph_0} = c$ as required. $\square$

**Example 8.1.2**

Find $\left|\aleph_0^{\aleph_0}\right|$ and $c^{\aleph_0}$.

### Solution

We have that

$$c = 2^{\aleph_0} \leq \aleph_0^{\aleph_0} \leq c^{\aleph_0} = \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = c$$

**Example 8.1.3**

Show $|\mathcal{P}(X)| = 2^{|X|} = |2^X|$.

**Solution**

Given $f : X \to \{0, 1\}$, let[6]

$$A = \{x \in X \mid f(x) = 1\} \subset X.$$

Define $\Gamma : 2^X \to \mathcal{P}(X)$ by

$$\Gamma(f) = f^{-1}(|1|)$$

$\Gamma$ is injective[7].

Conversely, given $A \subset X$, define the characteristic function

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \in 2^X.$$

Then define $\Phi : P(A) \to 2^X$ such that

$$\Phi(A) = \chi_A.$$

Clearly so, $\Phi$ is injective. □

[6] $A$ is a collection of all $x$'s that gets mapped to $f$.

[7] **Why?**

**Theorem 24 (Russell's Paradox)**

*For any $X$, we have $|X| < |\mathcal{P}(X)| = 2^{|X|}$.*

**Proof**

Let $f : X \to \mathcal{P}(X)$ be $f(X) = \{x\}$. Clearly, $f$ is injective, and so $|X| < |\mathcal{P}(X)|$.

<u>Claim: $\nexists g : X \to \mathcal{P}(X)$ surjective.</u>

Suppose not. Let[8]

$$A = \{x \in X \mid x \notin g(x)\}$$

Pick $x_0 \in X$ with $g(x_0) = A$. Now if $x_0 \in A$, then $x_0 \in g(x_0)$, but this implies that $x \notin A$, a contradiction.

So $x_0 \notin A$, i.e. $x \notin g(x_0)$, which in turn implies that $x \in A$, yet

[8] By the **Bounded Separation Axiom** (see ZF Set Theory), this is a set, and since it is a subset of $X$, it is a valid element of $\mathcal{P}(X)$. Thus, we can consider such a set.

another contradiction. Therefore such a function $g$ cannot exist, as claimed.

Therefore, we have $|X| < |\mathcal{P}(X)|$ as required. □

---

Question: Is there anything between $\aleph_0$ and $c$?

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**Axiom 25 (Continuum Hypothesis)**

*If $\aleph_0 \leq \gamma \leq c$, then either $\gamma = \aleph_0$ or $\gamma = c$.*

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**Axiom 26 (Generalized Continuum Hypothesis)**

*If $|X| \leq \gamma \leq 2^{|X|}$, then either $\gamma = |X|$ or $\gamma = 2^{|X|}$.*

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In this course, we shall assume that the Continuum Hypothesis is true.

# 9 Lecture 9 Sep 26th

## 9.1 Introduction to Metric Spaces

**Definition 31 (Metric & Metric Space)**

*Given a set $X$, a **metric** on $X$ is a function $d : X \times X \to \mathbb{R}$ such that*

1. *(**positive definiteness**) $d(x,y) \geq 0$ and $d(x,y) = 0 \iff x = y$;*
2. *(**symmetry**) $d(x,y) = d(y,x)$; and*
3. *(**triangle inequality**) $d(x,y) \leq d(x,z) + d(z,y)$.*

*The pair $(X, d)$ is called a **metric space**.*

**Remark**
*A metric is an abstract notion of distance.*

**Example 9.1.1 (Standard Metric on $\mathbb{R}$)**

Let $X = \mathbb{R}$, and let $d(x,y) = |x - y|$.

Clearly so, the first 2 criterias are satisfied:

- $|\cdot| \geq 0$ and $|x - y| = 0 \iff x = y$; and
- $|x - y| = |y - x|$.

The triangle inequality property is the usual triangle inequality of the absolute value function, i.e.

$$|x - y| \leq |x| + |y| .$$

Question: For an arbitrary set $X$, can we define a metric on $X$? The following example shows that we can,

**Example 9.1.2 (Discrete Metric)**

Let $X$ be any set. We can simply define

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

This metric clearly satisfies all 3 criterias of being a metric:

- $d : X \times X \to \{0,1\}$ and so $d(x,y) \geq 0$, and by definition, we have $d(x,y) = 0 \iff x = y$;
- By definition, $d(x,y) = d(y,x)$ as it does not matter how the pair is ordered; and
- Since $d(x,y) \geq 0$, we have that $d(x,y) \leq d(x,z) + d(y,z)$.

**Example 9.1.3 (Euclidean Metric / 2-metric on $\mathbb{R}^n$)**

Let $X = \mathbb{R}^n$. Let $\vec{x} = \{x_1, x_2, \ldots, x_n\}$ and $\vec{y} = \{y_1, y_2, \ldots, y_n\}$. Define

$$d_2(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Note that in $\mathbb{R}^2$, this is our regular (Euclidean) distance between two points.

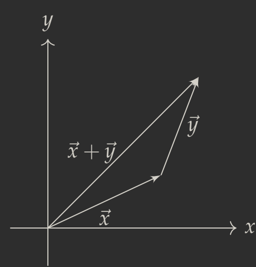


Figure 9.1: A visualization of the triangle inequality in $\mathbb{R}^2$.

It is not difficult to see that $d_2$ satisfies the first 2 criterion to being a metric:

- $d_2$ is the square root of the sum of squares, and so $d_2(\vec{x}, \vec{y}) \geq 0$ for any $\vec{x}, \vec{y} \in \mathbb{R}^n$, and $d_2(\vec{x}, \vec{y}) = 0 \iff \forall i \in \{1, \ldots, n\}\ x_i = y_i \iff \vec{x} = \vec{y}$;
- Since $(x_i - y_i)^2 = (y_i - x_i)^2$ for any $x_i, y_i \in \mathbb{R}$, we have that $d_2(\vec{x}, \vec{y}) = d_2(\vec{y}, \vec{x})$.

However, it is not immediately clear that $d_2$ satisfies the triangle inequality criterion, especially if $n \geq 3$. If $n = 2$, heuristically, the triangle inequality simply tells that the length of any one side of a triangle is less than or equal to the sum of the other two, e.g. Figure 9.1.

**Remark**
*Many of the important examples of metric spaces are vector spaces with an abstract length function, or* ***norm****.*

### Definition 32 (Norm & Normed Linear Space)

*Given a vector space $V$ (usually over $\mathbb{R}$), a* ***norm*** *on $V$ is a function*

$$\|\cdot\| : V \to \mathbb{R}$$

*such that*

1. *(**positive definiteness**)* $\|v\| \geq 0$ *and* $\|v\| = 0 \iff v = 0$*;*
2. *(**scalar multiplication**)* $\|\alpha \cdot v\| = |\alpha| \, \|v\|$*; and*
3. *(**triangle inequality**)* $\|v + w\| \leq \|v\| + \|w\|$*.*

*The pair* $(V, \|\cdot\|)$ *is called a **normed linear space**.*

**Remark**

*Given a normed linear space* $(V, \|\cdot\|)$*, a natural metric,* $d_{\|\cdot\|}$*, on* $V$ *induced by* $\|\cdot\|$ *can be defined as*

$$d_{\|\cdot\|}(x, y) = \|x - y\| .$$

**Exercise 9.1.1**
*Prove that* $d_{\|\cdot\|}$ *is indeed a metric.*

**Proof (Exercise 9.1.1)**

1. (positive definiteness) It is clear from the definition of a norm that $d_{\|\cdot\|}(x, y) = \|x - y\| \geq 0$, and $d_{\|\cdot\|}(x, y) = 0 \iff x - y = 0 \iff x = y$.
2. (symmetry) Symmetry follows simply from definition, as $\|x - y\| = \|y - x\|$.
3. (triangle inequality) For $x, y, z \in V$, we have

$$\begin{aligned} d_{\|\cdot\|}(x, y) &= \|x - y\| = \|x - z + z - y\| \\ &\leq \|x - z\| + \|z - y\| \quad \because \text{ triangle inequality of norms} \\ &= \|x - z\| + \|y - z\| \quad \because \text{ symmetry} \\ &= d_{\|\cdot\|}(x, z) + d_{\|\cdot\|}(y, z) \end{aligned}$$

$\square$

**Example 9.1.4 (Euclidean Norm)**

Let $X = \mathbb{R}^n$, and $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^2$. Define $\|\cdot\|_2$ such that

$$\|(x_1, \ldots, x_n)\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

From Example 9.1.3, we are given the triangle inequality property, in which we have yet to prove. Positive definiteness is clear. For scalar multiplication, let $\vec{x} = x_1, \ldots, x_n$, and notice that

$$\|\alpha \cdot \vec{x}\|_2 = \sqrt{\sum_{i=1}^{n} (\alpha x_i)^2} = \sqrt{\alpha^2 \sum_{i=1}^{n} x_i^2} = |\alpha| \sqrt{\sum_{i=1}^{n} x_i^2} = |\alpha| \, \|\vec{x}\|_2 .$$

Thus $\|\cdot\|_2$ is indeed a norm. We call $\|\cdot\|_2$ the **2-norm** or the **Euclidean norm**.

We observe that, in comparison with Example 9.1.3, we have that

$$d_2(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_2 .$$

**Example 9.1.5 (1-norm)**

Let $X = \mathbb{R}^n$, and $\vec{x} = (x_1, \ldots, x_n)$. Define

$$\|\vec{x}\|_1 := \sum_{i=1}^{n} |x_i| .$$

Clearly so, $\|\cdot\|_1$ is a norm:

- (**positive definiteness**) This is true by the absolute value function, i.e. every $|x_i| \geq 0$, and so the sum over these $x_i$'s is also non-negative, and $\sum_{i=1}^{n} |x_i| = 0 \iff \forall i \in \{1, \ldots, n\}\ x_i = 0 \iff \vec{x} = 0$.
- (**scalar multiplication**) This uses a similar argument as in the previous example.
- (**triangle inequality**) This is true by, again, the triangle inequality on absolute values.

We call $\|\cdot\|_1$ the **1-norm**.

Thus, we can define

$$d_1(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_1$$

and it can easily be verified that $d_1$ is indeed a metric.

**Example 9.1.6**

Let $X = \mathbb{R}^n$ and $\vec{x} = (x_1, \ldots, x_n)$. Define

$$\|\vec{x}\|_\infty = \max\{|x_i|\}$$

Again, it is easy to see that $\|\cdot\|_\infty$ is a norm;

- (**positive definiteness**) $\because \ \forall i \in \{1, \ldots, n\} \quad |x_i| \geq 0 \implies \max\{|x_i\} \geq 0|$ and $\max\{|x_i|\} = 0 \iff x_i = 0 \iff \vec{x} = 0$.
- (**scalar multiplication**) Notice that

$$\|\alpha \cdot \vec{x}\|_\infty = \max\{|\alpha x_i|\} = |\alpha| \max\{|x_i|\} = |\alpha| \, \|\vec{x}\|_\infty .$$

- (**triangle inequality**) This is once again true by the triangle inequality on the absolute value function, i.e.

$$\because \forall i \in \{1, \ldots, n\} \quad |x_i + y_i| \leq |x_i| + |y_i|$$
$$\max\{|x_i + y_i|\} \leq \max\{|x_i|\} + \max\{|y_i|\}.$$

We can then define

$$d_\infty(\vec{x}, \vec{y}) = \max\{|x_i - y_i|\},$$

which we can easily verify that it is indeed a metric[1].

[1] Symmetry holds by the property of the absolute value function.

Here's an interesting notion: let

$$S_i = \{\vec{x} \in \mathbb{R}^2 \mid \|\vec{x}\|_i = 1\}, \quad i = 1, 2, \infty$$

Notice that we would then have the following graph: In fact, it is true

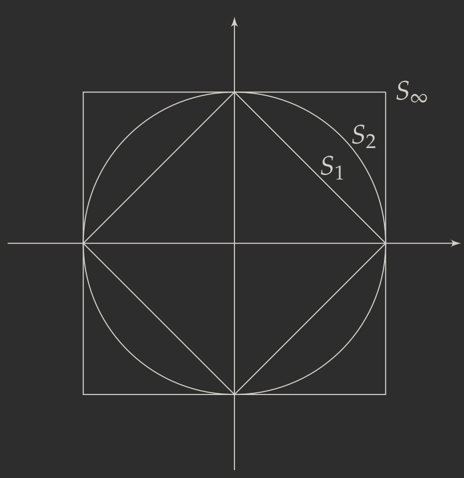


Figure 9.2: Unit ball depending on $\|\vec{x}\|_i$

that if we let $i \in \mathbb{N} \setminus \{0\}$, as suggested by Figure 9.2, we would see that the "diamond" would grow into a "circle" as in $S_2$, and as $i \geq 3$, the unit ball will expand and approach the "square", which is $S_\infty$.

Another observation that we can make is if we can show that a set is open for a "smaller" $S_i$, then the same set is open for any $S_j$ for $j \geq i$.

If we apply these norms into metrics, we have

$$d_\infty \leq d_2 \leq d_1$$

where we say that $d_\infty$ is the **least sensitive**, and $d_1$ being the **most sensitive**[2].

Note that if we allow for $0 < i < 1$, then we would have a graph that looks like the following, which is a convex graph, i.e. we cannot create well-defined norms.

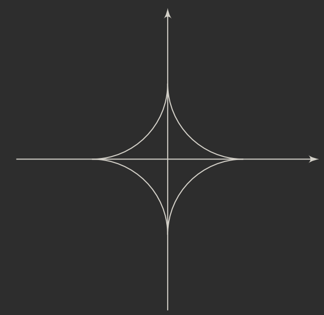

Figure 9.3: $\|\cdot\|_p$ for $0 < p < 1$

[2] For sufficently close points, we see that $d_\infty$ would reflect the least change, while we can see change in $d_1$ for every two points that we take.

**Example 9.1.7**

For $1 < p < \infty$, define on $\mathbb{R}^n$

$$\|\vec{x}\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

Continuing with the same idea as in previous examples, we can let

$$d_p(\vec{x}, \vec{y}) = \left( \sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

In the next lecture, we will go into proving that this is indeed a norm, and so we can define a metric using this norm.

# 10 Lecture 10 Sep 28th

## 10.1 Introduction to Metric Spaces (Continued)

**Definition 33 ($\|\cdot\|_p$-norm)**

*Given $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we define, for $1 < p < \infty$, the $\|\cdot\|_p$-norm to be*

$$\|\vec{x}\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

We asked the question: why is $\|\cdot\|_p$ a norm?

**Lemma 27 (Young's Inequality)**

*If $1 < p < \infty$,*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

*and if $\alpha, beta > 0$, then*

$$\alpha \cdot \beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}.$$

**Proof**

Motivated by Figure 10.1, using notions from calculus, we have from calculus,

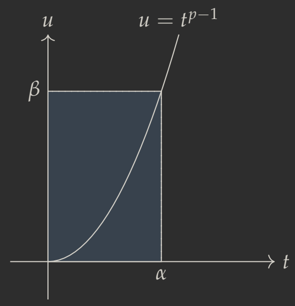


Figure 10.1: Motivation for Lemma 27.

$$\begin{aligned}
\alpha\beta &\leq \int_0^\alpha t^{p-1}\,dt + \int_0^\beta u^{q-1}\,du \\
&= \frac{t^p}{p}\Big|_0^\alpha + \frac{u^q}{q}\Big|_0^\beta \\
&= \frac{\alpha^p}{p} + \frac{\beta^q}{q},
\end{aligned}$$

where we note that

$$\begin{aligned}
\frac{1}{p} + \frac{1}{q} &= 1 \\
\frac{q}{p} &= q - 1 \\
\frac{p}{q} &= p - 1 \\
1 &= (p-1)(q-1)
\end{aligned}$$

□

**Theorem 28 (Hölder's Inequality)**

*For $1 < p < \infty$, let $\frac{1}{p} + \frac{1}{q} = 1$* [1]*. Let*

$$\vec{x} = (x_1, \ldots, x_n), \quad \vec{y} = (y_1, \ldots, y_n).$$

*Then*

$$\sum_{i=1}^n |x_i y_i| \leq \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}.$$

[1] We also call $q$ the conjugate of $p$.

**Note**

*Note that $p = 2$ is just the* ***Cauchy-Schwarz Inequality****:*

$$\begin{gathered}
\sum_{i=1}^n |x_i y_i| \leq \left( \sum_{i=1}^n |x_i|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{i=1}^n |y_i|^2 \right)^{\frac{1}{2}} \implies \\
\left( \sum_{i=1}^n |x_i y_i| \right)^2 \leq \left( \sum_{i=1}^n |x_i|^2 \right) \cdot \left( \sum_{i=1}^n |y_i|^2 \right)
\end{gathered}$$

**Proof**

Since if either $\vec{x}$ or $\vec{y}$ is zero, then we have that the inequality is trivially true, we can suppose that $\vec{x} \neq 0 \neq \vec{y}$. Now, note that for $\alpha, \beta \neq 0$, we have that[2]

$$\sum_{i=1}^{n} |x_i y_i| \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}$$

$$\Updownarrow$$

$$\sum_{i=1}^{n} |\alpha x_i \cdot \beta y_i| \leq \left( \sum_{i=1}^{n} |\alpha x_i|^p \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^{n} |\beta y_i|^q \right)^{\frac{1}{q}}.$$

[2] In the second inequality, notice that we can easily get back to the first equation by dividing both sides by $\alpha\beta$.

So we can assume that

$$\left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} = 1 = \left( \sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}, \tag{10.1}$$

and if not, we can simply choose $\alpha, \beta \neq 0$ to scale these values to become one. By Lemma 27, we have

$$|x_i y_i| \leq \frac{|x_i|^p}{p} + \frac{|y_i|^q}{q}.$$

Hence

$$\begin{aligned} \sum_{i=1}^{n} |x_i y_i| &\leq \sum_{i=1}^{n} \frac{|x_i|^p}{p} + \sum_{i=1}^{n} \frac{|y_i|^q}{q} = \frac{1}{p} + \frac{1}{q} \qquad \because \textit{Equation (10.1)} \\ &= 1 = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}} \end{aligned}$$

as required. $\square$

We are now ready to prove our long-awaited result.

**Theorem 29 (Minkowski's Inequality)**

*Let* $1 < p < \infty$. *If*

$$\vec{x} = (x_1, \ldots, x_n), \quad \vec{y} = (y_1, \ldots, y_n)$$

*in* $\mathbb{R}^n$*, then*

$$\left( \sum_{i=1}^{n} |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}},$$

*i.e.*

$$\|\vec{x} + \vec{y}\|_p \leq \|\vec{x}\|_p + \|\vec{y}\|_p .$$

**Proof**

Let

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Once again, we may assume that $\vec{x} \neq 0 \neq \vec{y}$, as otherwise the inequality is true trivially so. Now, notice that

$$\begin{aligned}
\sum_{i=1}^{n} |x_i + y_i|^p &= \sum_{i=1}^{n} |x_i + y_i| \, |x_i + y_i|^{p-1} \\
&\leq \sum_{i=1}^{n} |x_i| \, |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| \, |x_i + y_i| \quad \because \overset{\text{triangle}}{\text{inequality}} \\
&\leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} |x_i + y_i|^{(p-1)q} \right)^{\frac{1}{q}} \\
&\qquad + \left( \sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} |x_i + y_i|^{(p-1)q} \right)^{\frac{1}{q}}
\end{aligned}$$

where the last step is by Hölder's Inequality. Note that $\frac{1}{p} + \frac{1}{q} = 1 \implies p = q(p-1)$. Thus

$$\begin{aligned}
\sum_{i=1}^{n} |x_i + y_i|^p &\leq \left[ \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}} \right] \cdot \left( \sum_{i=1}^{n} |x_i + y_i|^p \right)^{\frac{1}{q}} \\
\implies \left( \sum_{i=1}^{n} |x_i + y_i|^p \right)^{1 - \frac{1}{q}} &\leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}}
\end{aligned}$$

□

**Note**

*With this we have that* $\|\cdot\|_p$ *satisfies the triangle inequaltiy condition, and so* $\|\cdot\|_p$ *is a norm on* $\mathbb{R}^n$*.*

**Note**

*Given* $1 \leq p \leq q < \infty$, *we have*[3]

$$\|\cdot\|_\infty \leq \|\cdot\|_q \leq \|\cdot\|_p \leq \|\cdot\|_1 .$$

[3] For a visual representation of this result, see Figure 9.2.

**Proof**

It is quite clear that $\forall p \geq 1$,

$$\|\cdot\|_\infty = \max\{|\cdot|\} \leq \left(\sum |\cdot|^p\right)^{\frac{1}{p}} = \|\cdot\|_p .$$

For $1 \leq p \leq q < \infty$, consider Holder's Inequality, where we have

$$\sum_{i=1}^n |a_i| |b_i| \leq \left(\sum_{i=1}^n |a_i|^r\right)^{\frac{1}{r}} \cdot \left(\sum_{i=1}^n |b_i|^{\frac{r}{r-1}}\right)^{1-\frac{1}{r}} .$$

Let $|a_i| = |x_i|^p$, $|b_i| = 1$ and $r = \frac{q}{p} \geq 1$ [4]. Then we have

[4] Note that this is true by $p \leq q$.

$$\begin{aligned} \sum_{i=1}^n |x_i|^p &\leq \left(\sum_{i=1}^n |x_i|^q\right)^{\frac{p}{q}} \cdot \left(\sum_{i=1}^n 1^{\frac{q}{q-p}}\right)^{1-\frac{p}{q}} \\ &= n^{1-\frac{p}{q}} \cdot \left(\sum_{i=1}^n |x_i|^q\right)^{\frac{p}{q}} \end{aligned}$$

Therefore, for $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}$,

$$\begin{aligned} \|\vec{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} &\leq \left(n^{1-\frac{p}{q}} \cdot \left(\sum_{i=1}^n |x_i|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \\ &= n^{\frac{1}{p}-\frac{1}{q}} \cdot \left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}} = n^{\frac{1}{p}-\frac{1}{q}} \cdot \|\vec{x}\|_q . \end{aligned}$$

Thus, we have

$$\|\cdot\|_q \leq \|\cdot\|_p .$$

The chain of inequality follows. $\square$

**Example 10.1.1 (Sequence Spaces)**

1. Let $\ell_1 = \left\{ \{x_i\} \mid \sum_{i=1}^{\infty} |x_i| < \infty \right\}$. Define

$$\|\{x_i\}\|_1 = \sum_{i=1}^{\infty} |x_i|$$

Let $\{x_i\}, \{y_i\} \in \ell_1$. Observe that $\forall n \in \mathbb{N}$, we have

$$\sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| \leq \|\{x_i\}\|_1 + \|\{y_i\}\|_1 .$$

Then by the Monotone Convergence Theorem, we have that

$$\sum_{i=1}^{\infty} |x_i + y_i| \leq \|\{x_i\}\|_1 + \|\{y_i\}\|_1 .$$

Thus $\{x_i + y_i\} \in \ell_1$ and

$$\|\{x_i + y_i\}\|_1 \leq \|\{x_i\}\|_1 + \|\{y_i\}\|_1 .$$

Let $\{x_n\} \in \ell_1$ and $\alpha \in \mathbb{R}$. Then

$$\sum_{i=1}^{\infty} |\alpha x_i| = |\alpha| \sum_{i=1}^{\infty} |x_i| .$$

Therefore $\{\alpha x_n\} \in \ell_1$ and $\|\{\alpha x_n\}\|_1 = |\alpha| \, \|\{x_n\}\|_1$.

Thus, we have that $\ell_1$ is a vector space, and $(\ell_1, \|\cdot\|_1)$ is a normed linear space.

2. Let $\ell_\infty(\mathbb{N}) = \ell_\infty = \{\{x_i\} \mid \{x_i\} \text{ is bounded }\}$. Define

$$\|\{x_i\}\|_\infty = \sup\{|x_1| \mid i \in \mathbb{N}\}.$$

Observe that $\forall \{x_i\}, \{y_i\} \in \ell_\infty$, then $\forall i \in \mathbb{N}$, we have

$$|x_i + y_i| \leq |x_i| + |y_i| \leq \|\{x_i\}\|_\infty + \|\{y_i\}\|_\infty .$$

So $\{x_i + y_i\} \in \ell_\infty$, and

$$\|\{x_i + y_i\}\|_\infty \leq \|\{x_i\}\|_\infty + \|\{y_i\}\|_\infty .$$

Consequently so, $\{\alpha x_i\} \in \ell_\infty$ and

$$\|\{\alpha x_i\}\|_\infty = |\alpha| \, \|\{x_i\}\|_\infty .$$

Question: What about $\ell_p(\mathbb{R})$?

# 11 Lecture 11 Oct 01st

## 11.1 Introduction to Metric Spaces (Continued 2)

We wondered about $\ell_p(\mathbb{R})$ in the last lecture but let us consider a case that is even more general.

Question: Can we define $\ell_p(\Gamma)$ for any set $\Gamma$?

**Example 11.1.1**

Let $\ell_\infty(\Gamma) = \{ f : \Gamma \to \mathbb{R} \mid f(\Gamma) \text{ is bounded } \}$. If $f \in \ell_\infty(\Gamma)$, define

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in \Gamma\}.$$

Notice that for $f, g \in \ell_\infty(\Gamma)$, and $\alpha \in \mathbb{R}$, then we have, by the Triangle Inequality,

$$\begin{aligned}
\|f+g\|_\infty &= \sup\{|(f+g)(x)| \mid x \in \Gamma\} \\
&= \sup\{|f(x)+g(x)| \mid x \in \Gamma\} \\
&\leq \sup\{|f(x)| \mid x \in \Gamma\} + \sup\{|g(x)| \mid x \in \Gamma\} \\
&= \|f\|_\infty \, \|g\|_\infty .
\end{aligned}$$

So $f+g \in \ell_\infty(\Gamma)$, and

$$\|f+g\|_\infty \leq \|f\|_\infty + \|f\|_\infty .$$

Also, we have

$$\begin{aligned}
\|\alpha f\|_\infty &= \sup\{|(\alpha f)(x)| \mid x \in \Gamma\} \\
&= \sup\{|\alpha| \, |f(x)| \mid x \in \Gamma\} \\
&= |\alpha| \sup\{|f(x)| \mid x \in \Gamma\} \\
&= |\alpha| \, \|f\|_\infty .
\end{aligned}$$

So $\alpha f \in \ell_\infty(\Gamma)$, and $\|\alpha f\|_\infty = |\alpha| \, \|f\|$.

Therefore, $(\ell_\infty(\Gamma), \|\cdot\|_\infty)$ is a normed linear space.

**Example 11.1.2**

Let $\ell_1(\Gamma) = \{f : \Gamma \to \mathbb{R} \mid P(f)\}$, where $P(f)$ is the statement

$$\|f\|_1 = \sup\left\{\sum_{i=1}^{n} |f(x_i)| \mid x_1, \ldots, x_n \in \Gamma, n \in \mathbb{N} \setminus \{0\}\right\} < \infty.$$

It is clear that $\ell_1(\Gamma) \subseteq \ell_\infty(\Gamma)$, where $\ell_\infty(\Gamma)$ is from Example 11.1.1. Consequently, $(\ell_1(\Gamma), \|\cdot\|_1)$ is a normed linear space.

**Require clarification** Notice that $\forall f \in \ell_1(\Gamma)$, for each $n \in \mathbb{N}$,

$$A_n = \{x \in \Gamma \mid |f(x)| \geq \frac{1}{n}\} \text{ is finite.}$$

So

$$A_0 = \bigcup_{n=1}^{\infty} A_n \text{ is countable.}$$

and

$$A_0 = \{x \in \Gamma \mid |f(x)| \neq \emptyset\}$$

We can extend the same idea onto $\ell_p$ spaces.

**Example 11.1.3**

Let $X = C[a,b] = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous on } [a,b]\}$, and define[1]

$$\begin{aligned} \|f\|_\infty &= \sup\{|f(x)| \mid x \in [a,b]\} \\ &= \max\{|f(x)| \mid x \in [a,b]\} \end{aligned}$$

[1] Note in this case sup is also max, since we are on a closed interval.

By (regular) Triangle Inequality, for any $f, g \in C[a,b]$, we have

$$\begin{aligned} \|f+g\|_\infty &= \max\{|f(x)+g(x)| \mid x \in [a,b]\} \\ &\leq \max\{|f(x)| \mid x \in [a,b]\} + \max\{|g(x)| \mid x \in [a,b]\} \\ &= \|f\|_\infty + \|g\|_\infty, \end{aligned}$$

and, for $\alpha \in \mathbb{R}$,

$$\begin{aligned} \|\alpha f\|_\infty &= \max\{|\alpha f(x)| \mid x \in [a,b]\} \\ &= |\alpha| \max\{|f(x)| \mid x \in [a,b] \\ &= |\alpha| \, \|f\|_\infty . \end{aligned}$$

Thus $\|\cdot\|_\infty$ is a norm on $C[a,b]$, and $(C[a,b], \|\cdot\|_\infty)$ is a normed linear space[2,3].

[2] This space is complete.

[3] This space is important for us for the purpose of this course.

Also, observe that

$$C[a,b] \subset \ell_\infty([a,b]).$$

**Example 11.1.4**

Let $X = C[a,b]$ [4] have the same definition as the previous example,

[4] Some authors also write this as $L'[a,b]$.

but this time define

$$\|f\|_1 = \int_a^b |f(t)|\,dt.$$

By **linearity of integration**, both the triangle equality and scalar multiplication hold, and so $(C[a,b], \|\cdot\|_1)$ is a normed linear space[5].

[5] Compared to the last example, this is not a complete space.

**Example 11.1.5**

Let $X = C[a,b]$, and $1 < p < \infty$. Define

$$\|f\|_p = \left( \int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

Again, by linearity of integration, scalar multiplication holds. However, it is not as easy to show for the triangle inequality; we are now asking the same question as we did before for $\ell_p$, which we solved using Hölder's Inequality and Minkowski's Inequality. But now, instead of summations, we have integrations.

**Theorem 30 (Hölder's Inequality v2)**

*Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For each $f, g \in C[a,b]$, we have*

$$\int_a^b |f(t)g(t)|\,dt \leq \left( \int_a^b |f(t)|^p \, dt \right)^{\frac{1}{p}} \left( \int_a^b |g(t)|^q \, dt \right)^{\frac{1}{q}}.$$

**Proof**

If either $f(x) = 0$ or $g(x) = 0$ for all $x \in [a,b]$, then the inequality holds trivially so. Thus, we may assume that $\forall x \in [a,b]$, $f(x) \neq 0 \neq g(x)$. By the linearity of integration, we can apply the same reasoning as we did in Theorem 28, and assume that

$$\int_a^b |f(t)|^p \, dt = 1 = \int_a^b |g(t)|^q \, dt$$

By Lemma 27, we have

$$|f(t)g(t)| \leq \frac{|f(t)|^p}{p} + \frac{|g(t)|^q}{q}.$$

Thus

$$\begin{aligned}\int_a^b |f(t)g(t)|\,dt &\leq \int_a^b \left( \frac{|f(t)|^p}{p} + \frac{|g(t)|^q}{q} \right) dt \\ &= \frac{1}{p} + \frac{1}{q} = 1 \\ &= \left( \int_a^b |f(t)|^p \, dt \right)^{\frac{1}{p}} \left( \int_a^b |g(t)|^q \, dt \right)^{\frac{1}{q}}\end{aligned}$$

as required. □

**Theorem 31 (Minkowski's Inequality v2)**

*Let* $1 < p < \infty$. *If* $f, g \in C[a,b]$, *then*

$$\left( \int_a^b |(f+g)(t)|^p \, dt \right)^{\frac{1}{p}} \leq \left( |f(t)|^p \, dt \right)^{\frac{1}{p}} \cdot \left( \int_a^b |g(t)|^p \, dt \right)^{\frac{1}{p}}.$$

**Proof**

The proof is similar to the one we had in Theorem 29; if $\forall x \in [a,b]$, either $f(x) = 0$ or $g(x) = 0$, then the inequality holds trivially so. Thus we may assyme that $\forall x \in [a,b]$, $f(x) \neq 0 \neq g(x)$. Now, notice that by (regular) Triangle Inequality and, later on, Theorem 30,

$$\begin{aligned}&\int_a^b |(f+g)(t)|^p \, dt \\ &= \int_a^b |(f+g)(t)| \, |(f+g)(t)|^{p-1} \, dt \\ &\leq \int_a^b |f(t)| \, |(f+g)(t)|^{p-1} \, dt + \int_a^b |g(t)| \, |(f+g)(t)| \, dt \\ &\leq \left( \int_a^b |f(t)|^p \, dt \right)^{\frac{1}{p}} \left( \int_a^b |(f+g)(t)|^{q(p-1)} \, dt \right)^{\frac{1}{q}} \\ &\qquad + \left( \int_a^b |g(t)|^p \, dt \right)^{\frac{1}{p}} \left( \int_a^b |(f+g)(t)|^{q(p-1)} \, dt \right)^{\frac{1}{q}} \\ &= \left[ \left( \int_a^b |f(t)|^p \, dt \right)^{\frac{1}{p}} + \left( \int_a^b |g(t)|^p \, dt \right)^{\frac{1}{p}} \right] \\ &\qquad \cdot \left( \int_a^b |(f+g)(t)|^p \right)^{\frac{1}{q}}\end{aligned}$$

where we note that $\frac{1}{p} + \frac{1}{q} = 1 \implies p = q(p-1)$. Consequently, since $\frac{1}{p} = 1 - \frac{1}{q}$,

$$\left( \int_a^b |(f+g)(t)|^p \, dt \right)^{\frac{1}{p}} \leq (|f(t)|^p \, dt)^{\frac{1}{p}} \cdot \left( \int_a^b |g(t)|^p \, dt \right)^{\frac{1}{p}},$$

as required. □

---

This shows that our definition of $\|\cdot\|_p$ on $C[a,b]$ is indeed a norm, and so $(C[a,b], \|\cdot\|_p)$ is a normed linear space.

**Exercise 11.1.1**
*Show that there exists an injection from $(C[a,b], \|\cdot\|_2)$ to $\ell_2(\mathbb{N})$. Note that this does not work for $p \geq 3$.*

**Example 11.1.6 (Bounded Operator)**

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces. Let $T : X \to Y$ be linear. Define

$$\|T\| = \sup\{\|T_X\|_Y \mid \|x\|_X < 1\}.$$

We say that $T$ is bounded if $\|T\| < \infty$. Let

$$B(X,Y) = \{T : X \to Y \mid T \text{ is bounded } \}.$$

In the next lecture, we shall show that $(B(X,Y), \|\cdot\|)$ is a normed linear space.

QUESTION: Consider the transformation $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$. What is a norm $\left\| \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \right\|$ that works?

# 12 Lecture 12 Oct 03rd

## 12.1 Introduction to Metric Spaces (Continued 3)

**Example 12.1.1 (Bounded Operator)**

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces. Let $T : X \to Y$ be linear. Define

In this example, we look at how we can apply a translation of norms from $X$ to $Y$ that preserves the norm.

$$\|T\| = \sup\{\|T(x)\|_Y \mid \|x\|_X < 1\}.$$

We say that $T$ is bounded if $\|T\| < \infty$. Let

$$B(X,Y) = \{T : X \to Y \mid T \text{ is bounded }\}.$$

To show that $B(X,Y)$ is a normed linear space, let $S, T \in B(X,Y)$, and let $\|x\|_X \leq 1$. Then

$$\begin{aligned}
\|(S+T)(x)\|_Y &= \|S(x) + T(x)\|_Y \\
&\leq \|S(x)\|_Y + \|T(x)\|_Y \quad \because \|\cdot\|_Y \text{ is a norm} \\
&\leq \|S\| + \|T\|
\end{aligned}$$

and so $S + T \in B(X,Y)$ and $\|S+T\| \leq \|S\| + \|T\|$. For $\alpha \in \mathbb{R}$, we have

$$\begin{aligned}
\|\alpha S\| &= \sup\{\|(\alpha S)(x)\|_Y \mid \|x\|_X \leq 1\} \\
&= |\alpha| \sup\{\|S(x)\|_Y \mid \|x\|_X \leq 1\} \quad \because \|\cdot\|_Y \text{ is a norm} \\
&= |\alpha| \, \|S\| .
\end{aligned}$$

So $(\alpha S) \in B(X,Y)$ and $\|\alpha S\| = |\alpha| \, \|S\|$. It is clear that due to $\|\cdot\|_Y$ being a norm, and so $\|\cdot\|$ is also positive definite. Thus $B(X,Y)$ is a normed linear space as claimed.

## 12.2 Topology on Metric Spaces

### Definition 34 (Open & Closed)

*Let $X(, d)$ be a metric space. If $x_0 \in X$, then*

$$B(x_o, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$$

*is called the* ***open ball*** *centered at $x_0$ with radius $\varepsilon > 0$.*

$$B[x_0, \varepsilon] = \{y \in X \mid d(x, y) \leq \varepsilon\}$$

*is called the* ***closed ball*** *centered at $x_0$ with radius $\varepsilon > 0$.*

*We say that $U \subset X$ is* ***open*** *if*

$$\forall x \in U \;\; \exists \varepsilon_0 > 0 \quad B(x_0, \varepsilon_0) \subset U.$$

*We say that $F \subset X$ is* ***closed*** *if $F^C$ is open.*

### Proposition 32 (Properties of Open Sets)

*Let $(X, d)$ be a metric space.*

1. *$X, \emptyset$ are open,*
2. *If $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets, then $U = \bigcup_{\alpha \in I}$ is open.*
3. *If $\{U_1, \ldots, U_n\}$ is a collection of open sets, then $U = \bigcap_{i=1}^n U_i$ is open.*

### Proof

1. If $x_0 \in X$, then $B(x_0, 1) \subseteq X$, and so $X$ is open. The empty set is open vacuously so.
2. Let $U = \bigcup_{\alpha \in I} U_\alpha$ and $x_0 \in U$. Then $\exists \alpha_0 \in I$ such that $x_0 \in U_{\alpha_0}$. Then $\exists \varepsilon_0 > 0$ such that
$$B(x_0, \varepsilon_0) \subset U_{\alpha_0} \subset U.$$
3. Let $x_0 \in U = \bigcap_{i=1}^n$. Then for each $i \in \{1, \ldots, n\}$, $\exists \varepsilon_i > 0$ such

that $B(x_0, \varepsilon_i) \subset U_i$. Let

$$\varepsilon_0 = \min\{\varepsilon_1, \ldots, \varepsilon_n\}.$$

Then we have that $\forall i \in \{1, \ldots, n\}$, $\varepsilon_0 \leq \varepsilon_i$. Thus

$$B(x_0, \varepsilon_0) \subset B(x_0, \varepsilon_i) \subset U_i$$

for each $i$. Therefore $B(x_0, \varepsilon_0) \subset U$.

□

**Corollary 33 (Properties of Closed Sets)**

*Let $(X, d)$ be a metric space.*

1. *$X, \emptyset$ are closed.*
2. *If $\{F_\alpha\}_{\alpha \in I}$ is a collection of closed sets, then $F = \bigcap_{\alpha \in I} F_\alpha$ is closed.*
3. *If $\{F_1, \ldots, F_n\}$ is a collection of closed sets, then $F = \bigcup_{i=1}^{n} F_i$ is closed.*

**Proof**

The proof follows from De Morgan's Laws, Proposition 32, and by taking set complements. □

**Exercise 12.2.1**
*Write out the full proof for Corollary 33 as an exercise.*

**Example 12.2.1**

Let $X$ be any set and $d$ the discrete metric

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

We want to know what sets are open on $X$ under $d$. Notice that any set of just a singleton is open, since

$$B\left(x_0, \frac{1}{2}\right) \subset X.$$

Consequently, any $A \in \mathcal{P}(X)$ is an arbitrary union of open sets, i.e.

$$A = \bigcup_{x \in A} \{x\}.$$

Thus by Proposition 32, $A$ is open.

**Note**

*On $\mathbb{R}$, only $\emptyset$ and $\mathbb{R}$ itself are both open and closed. This can be proven using the* ***Intermediate Value Theorem****.*

**Definition 35 (Topology)**

*Given any $X$, a set $\tau \subset \mathcal{P}(X)$ is called a* ***topology*** *on $X$ is*

1. $X, \emptyset \in \tau$
2. *If $\{U_\alpha\}_{\alpha \in I}$ such that for each $\alpha \in I$, $U_\alpha \in \tau$, then*

$$U = \bigcup_{\alpha \in I} U_\alpha \in \tau.$$

3. *If $\{U_1, \ldots, U_n\}$ such that $U_i \in \tau$ for each $i \in \{1, \ldots, n\}$, then*

$$U = \bigcap_{i=1}^{n} U_i \in \tau.$$

*If $(X, d)$ is a metric space, then*

$$\tau_d = \{U \subset X \mid U \text{ open in } (X, d)\}$$

*is called a* ***metric topology****, or* ***$d$-topology****, associated with the metric $d$. We call $(X, \tau)$ a* ***topological space****.*

**Example 12.2.2**

Given $X$,

1. $\mathcal{P}(X)$ is a topology on $X$, and it is called the **discrete topology**;
2. $\{\emptyset, X\}$ is a topology on $X$, and it is called the **indiscrete topology**.

# 13 Lecture 13 Oct 05th

## 13.1 Topology on Metric Spaces (Continued)

**Theorem 34 (Open Balls are Open)**

1. *$B(x_0, \varepsilon)$ is open.*
2. *$B[x_0, \varepsilon]$ is closed.*
3. *Every open set is the union of open balls.*
4. *$\forall x \in X$, $\{x\}$ is closed.*

**Proof**

1. Consider $x \in B(x_0, \varepsilon)$ and let $r = d(x, x_0)$.

   Let $\alpha = \varepsilon - r$. Assume that $y \in B(x, \alpha)$. By the Triangle Inequality,
   $$d(x_0, y) \leq d(x_0, x) + d(x, y) < r + \alpha = \varepsilon.$$

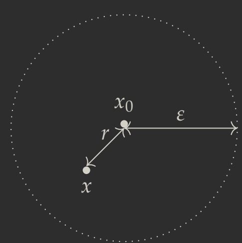


Figure 13.1: Idea of proof for 1. in $\mathbb{R}^2$.

2. Let $y \in B[x_0, \varepsilon]^C$, and let $r = d(x_0, y)$.

   Let $\alpha = r - \varepsilon$. Assume $z \in B(y, \alpha)$, and suppose, for contradiction, that $z \in B[x_0, \varepsilon]$. Then
   $$r = d(x_0, y) \leq d(x_0, z) + d(z, y) < \varepsilon + \alpha = r,$$
   but $r < r$ contradicts the fact that $r = r$.

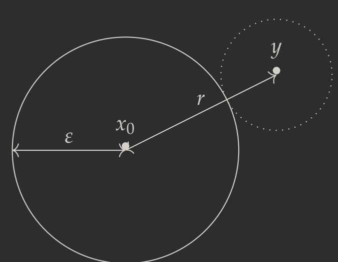


Figure 13.2: Idea of proof for 2. in $\mathbb{R}^2$.

3. Let $U \subset X$ be open. $\forall x \in U$, let $\varepsilon_x > 0$ be such that $B(x, \varepsilon_x) \subset U$. Then
   $$U = \bigcup_{x \in U} B(x, \varepsilon_x).$$

4. Let $y \in X$ such that $y \neq x$. Let $r = d(y, x)$. Then $x \notin B\left(y, \frac{r}{2}\right)$, and so

$$B\left(y, \frac{r}{2}\right) \subset \{x\}^C.$$

$\square$

---

**Example 13.1.1 (Open Intervals are Open)**

Let $X = \mathbb{R}$, and $d(x, y) = |x - y|$, the standard metric. Let $I = (a, b)$, for some $a, b \in \mathbb{R} \cup \{\pm\infty\}$. Let $x \in I$. Now let

$$\varepsilon = \min\{1, |x - a|, |x - b|\}.$$

Then, clearly so, $B(x, \varepsilon) \subset I$.

If $U \subset \mathbb{R}$ is open, and if we define $\sim$ on $U$ by $x \sim y$ [1]. if $(x, y), (y, x) \subset U$. We proved that $\sim$ is an equivalence relation. Let $I_x = [x]$ be the interval defined by $\sim$. We proved that $I_x$ is an open interval.

[1] This is what we did in Q1.

Consequently, if we have $U$ being open in $\mathbb{R}$, then $U$ can be expressed as the union of a countable collection $\{I_\alpha, \alpha \in I\}$ of open intervals, which are pairwise disjoint.

Question: Given $U = \{(x, y) \mid |x|, |y| < 1\}$, can we do the same as above, i.e. can we use a countable collection of disjoint open sets to express $U$, or, in other words, cover $U$?

**Example 13.1.2 (Cantor Set)**

Let's consider the closed interval $[0, 1]$, of which we shall label as $P_0$.

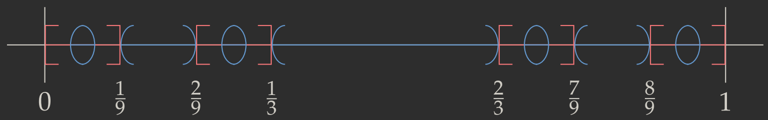


Figure 13.3: Cantor set showing up to $n = 2$, with the excluded interval in $n = 3$ shown.

Define $P_1$ by removing an open interval of length $\frac{1}{3}$ sitting in the middle of $P_0$, i.e.

$$P_1 = [0, 1] \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

Define $P_2$ by removing an open interval of length $\frac{1}{3^2}$ sitting in the middle of each of the 2 closed intervals in $P_1$, ie.

$$P_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

Recursively so, define $P_{n+1}$ by removing an open interval of length $\frac{1}{3^{n+1}}$ sitting in the middle of each of the $2^n$ closed intervals in $P_n$.

Let $P$, the **Cantor Set** (or **Cantor Ternary Set**), be defined as

$$P = \bigcap_{n=0}^{\infty} P_n.$$

The following are some properties of $P$:

1. $P$ is closed, since it is closed under an arbitrary number of closed sets (see Corollary 33).
2. We have
$$x \in P \iff x = \sum_{i=1}^{\infty} \frac{a_n}{3^n}$$
where $a_n = 0, 2$. In other words, every element of $P$ is a ternary number.
3. $|P| = 2^{\aleph_0} = c$.
4. $P_n$ does not contain any interval of length greater than or equal to $\frac{1}{3^n}$.
5. Consequently, the length of $P$ is 0.

# 14 Lecture 14 Oct 12th

## 14.1 Topology on Metric Spaces (Continued 2)

**Definition 36 (Closure)**

*Let $A \subseteq (X, d)$. We define the* ***closure*** $\overline{A}$ *of $A$ to be*

$$\overline{A} = \cap \{ F \subset X \mid F \text{ is closed }, A \subset F \}.$$

*$\overline{A}$ is the smallest closed set that contains $A$.*

**Definition 37 (Interior)**

*Let $A \subseteq (X, d)$. We define the* ***interior*** $A^\circ$ *of $A$ to be*

$$A^\circ = \cup \{ U \subset X \mid U \text{ is open }, U \subset A \}.$$

*$A^\circ$ is the largest open set contained in $A$.*

**Remark**

*We have that*

$$A^\circ \subset A \subset \overline{A}$$

**Definition 38 (Neighbourhood)**

*We say that a set $A$ is a* ***neighbourhood*** *of a point $x \in X$ if $x \in A^\circ$.*[1]

[1] A neighbourhood is **not necessarily open**; the definition applies to elements in the interior after all.

**Note**

*$A$ is a neighourhood of $x \in X$ if and only if $\exists \varepsilon > 0$ such that $B(x, \varepsilon) \subset A$.*

**Definition 39 (Boundary Point)**

*Given $A \subset (X, d)$, a point $x$ is called a **boundary point** for $A$ if*

$$\forall \varepsilon > 0 \quad B(x, \varepsilon) \cap A \neq \emptyset \wedge B(x, \varepsilon) \cap A^C \neq \emptyset.$$

*We denote the collection of all boundary points of $A$ by $\operatorname{bdy}(A)$.*

**Proposition 35 (Closed Sets Include Its Boundary Points)**

*Let $(X, d)$ be a metric space and $A \subset X$. Then $A$ is closed $\iff$ $\operatorname{bdy}(A) \subset A$.*

**Proof**

(1) $\implies$ (2): Suppose $x \in A^C$, which is open. Then $\exists \varepsilon > 0$ such that $B(x, \varepsilon) \subset A^C$. Then $x \notin \operatorname{bdy}(A)$, i.e. $\operatorname{bdy}(A) \subset A$.[2]

(2) $\implies$ (3): [3]Let $x \in A^C$. Then, by assumption, $x \notin \operatorname{bdy}(A)$. Then $\exists \varepsilon > 0$ such that either $B(x, \varepsilon) \subset A$ or $B(x, \varepsilon) \subset A^C$. But since $x \notin A$, we must have $B(x, \varepsilon) \subset A^C$, i.e. $A^C$ is open. □

[2] The idea of this proof is to show that it is impossible for the boundary to be in $A^C$.

[3] To show that $A$ is closed, we should show that $A^C$ is open.

**Proposition 36 (Closures include the Boundary Points of a Set)**

*Given $A \subset (X, d)$, we have $\overline{A} = A \cup \operatorname{bdy}(A)$.*

**Proof**

By definition, $A \subseteq \overline{A}$, so it suffices to show that $\operatorname{bdy}(A) \subset \overline{A}$ to show that $A \cup \operatorname{bdy}(A) \subseteq \overline{A}$.

[4]Assume that $x \notin \overline{A}$, i.e. $x \in \overline{A}^C$, which is open since $\overline{A}$ is closed by definition. Then $\exists \varepsilon > 0$ such that $B(x, \varepsilon) \subset \overline{A}^C$. Since

[4] Here, we employ the same proof as the previous proposition.

$x \notin A \subset \overline{A}$, we have that $B(x,\varepsilon) \cap A = \emptyset$, i.e. $x \notin \mathrm{bdy}(A)$. Therefore $\mathrm{bdy}(A) \subset \overline{A}$, and so $A \cup \mathrm{bdy}(A) \subseteq \overline{A}$ as claimed.

[5]Let $x \in \mathrm{bdy}(A \cup \mathrm{bdy}(A))$. Then $\forall \varepsilon > 0$, we have

$$B(x,\varepsilon) \cap (A \cup \mathrm{bdy}(A)) \neq \emptyset \tag{14.1}$$

$$\wedge$$

$$B(x,\varepsilon) \cap (A \cup \mathrm{bdy}(A))^C \neq \emptyset. \tag{14.2}$$

[5] For this part, if we can show that $A \cup \mathrm{bdy}(A)$ is closed, then by definition, $\overline{A} \subseteq A \cup \mathrm{bdy}(A)$ since $\overline{A}$ is the smallest such set that contains $A$. To show that $A \cup \mathrm{bdy}(A)$ is closed, we can either show that $(A \cup \mathrm{bdy}(A))^C$ is open, or use Proposition 35 to show that $\mathrm{bdy}(A \cup \mathrm{bdy}(A)) \subset (A \cup \mathrm{bdy}(A))$. We shall show for the more complicated expression.

Note that by De Morgan's Laws, we have that

$$(A \cup \mathrm{bdy}(A))^C = A^C \cap \mathrm{bdy}(A)^C.$$

Then (14.2) would be

$$B(x,\varepsilon) \cap A^C \cap \mathrm{bdy}(A)^C \neq \emptyset,$$

and so

$$B(x,\varepsilon) \cap A^C \neq \emptyset \tag{14.3}$$

$$\wedge$$

$$B(x,\varepsilon) \cap \mathrm{bdy}(A)^C \neq \emptyset. \tag{14.4}$$

From (14.1), we have

$$B(x,\varepsilon) \cap A \neq \emptyset \ \vee \ B(x,\varepsilon) \cap \mathrm{bdy}(A) \neq \emptyset.$$

If $B(x,\varepsilon) \cap A \neq \emptyset$, then $\because$ (14.3), $x \in \mathrm{bdy}(A)$, and so

$$\mathrm{bdy}(A \cup \mathrm{bdy}(A)) \subseteq (A \cup \mathrm{bdy}(A)). \tag{†}$$

If $B(x,\varepsilon) \cap \mathrm{bdy}(A) \neq \emptyset$, let $z \in B(x,\varepsilon) \cap \mathrm{bdy}(A)$. $\because z \in B(x,\varepsilon)$, let $r = d(x,z)$, and $\alpha = \varepsilon - r > 0$. Let $z_0 \in B(z,\alpha)$. Then by the Triangle Inequality

$$d(x,z_0) \leq d(x,z) + d(z,z_0) < r + \alpha = \varepsilon.$$

Thus $z_0 \in B(x,\varepsilon) \implies (B(z,\alpha) \subseteq B(x,\varepsilon))$. Then $\because z \in \mathrm{bdy}(A)$, we have $B(z,\alpha) \cap A \neq \emptyset$, and so $B(x,\varepsilon) \cap A \neq \emptyset$. Then we can just follow the argument we did in (†) and arrive as the same conclusion. Consequently, by Proposition 35, $A \cup \mathrm{bdy}(A)$ is closed as claimed. □

**Example 14.1.1**

Let $X = \mathbb{R}$ and $A = [0, 1)$. We have that

- $\operatorname{bdy}(A) = \{0, 1\}$;
- $A^\circ = (0, 1)$; and
- $\overline{A} = [0, 1]$.

**Example 14.1.2**

Let $X = \mathbb{R}$ and $A = \mathbb{Q}$. We have that

- $\operatorname{bdy}(A) = \mathbb{R}$ since every open ball around $a \in A$ will always contain elements in $\mathbb{Q}$ and $\mathbb{Q}^C$;
- $A^\circ = \emptyset$ since $A^\circ = A \setminus \operatorname{bdy}(A)$; and
- $\overline{A} = \mathbb{R}$ since $\overline{A} = A \cup \operatorname{bdy}(A)$.

---

**Definition 40 (Separable)**

*A metric space $(X, d)$ is* ***separable*** *if there exists a countable set $A \subset X$ such that $\overline{A} = X$, and call the metric space* ***non-separable*** *otherwise.*

---

**Example 14.1.3**

Every finite metric space $(X, d)$ is separable.

This is true since every subset $A$ of $X$ is countable since $X$ itself is countable. Consequently, if we pick $A$ to be a subset that takes every other element in $X$, then it is clear that $\overline{A} = X$, and so $(X, d)$ is separable.

**Example 14.1.4**

$\mathbb{R}$ is separable as shown in Example 14.1.2.[6]

[6]

**Exercise 14.1.1**
*Prove that $\overline{\mathbb{Q}} = \mathbb{R}$ using the* ***Archimedean Property*** *of $\mathbb{R}$.*

**Example 14.1.5**

$\mathbb{R}^n$ is separable if $d_p$ for all $1 \leq p \leq \infty$. We can apply the same argument that we had for Example 14.1.2 and apply it component-wise. Consequently, $\overline{\mathbb{Q}^n} = \mathbb{R}^n$. In other words, for any $(x_1, \ldots, x_n) \in (\mathbb{R}^n, d_p)$, we can pick a $(r_1, \ldots, r_n) \in \mathbb{Q}^n$ that is as close to $(x_1, \ldots, x_n)$ as possible.

**Remark**

*Notice that*

$$\overline{A} = X \iff \forall x \in X \forall \, \varepsilon > 0 \; B(x, \varepsilon) \cap A \neq \emptyset.$$

---

**Definition 41 (Dense)**

*A is* ***dense*** *in* $(X, d)$ *if* $\overline{A} = X$*. Equivalently, A is dense if for every open set* $W \subset X$*,* $W \cap A \neq \emptyset$*.*

---

Question: Is $(\ell_1, \|\cdot\|_1)$ separable? Is $(\ell_\infty, \|\cdot\|_\infty)$ separable?

Recall Example 10.1.1.

# 15 Lecture 15 Oct 15th

## 15.1 Topology on Metric Spaces (Continued 3)

**Definition 42 (Limit Points)**

*Let $(X, d)$ be a metric space, and $A \subset X$. We say that $x_0$ is a* ***limit point*** *for $A$ if for any neighbourhood of $x_0$, we have that*

$$N \cap (A \setminus \{x_0\}) \neq \emptyset.$$

*Equivalently, $\forall \varepsilon > 0$, $\exists x \in A$, where $x \neq x_0$, such that $x \in B(x_0, \varepsilon)$.*[1] *We sometimes call limit points as* ***cluster points****. We denote the set of limit points of $A$ as $\mathrm{Lim}(A) \subset X$*[2]

[1] This also means that $B(x_0, \varepsilon)$ must have infinitely many points close to $x_0$, for otherwise, we would be able to find some $\varepsilon > \varepsilon_0 > 0$ such that $B(x_0, \varepsilon_0) \cap A = \emptyset$.

[2] Note that the set of limit points is not necessarily a subset of $A$.

**Example 15.1.1**

Let $X = \mathbb{R}$, and $A = [0, 1) \subset \mathbb{R}$. We have that

$$\mathrm{Lim}[0, 1) = [0, 1].$$

**Example 15.1.2**

Let $X = \mathbb{R}$ and $A = \mathbb{N} \subset \mathbb{R}$. Since $\forall n \in \mathbb{N}$, $\exists \varepsilon = \frac{1}{2}$ such that $\forall m \in \mathbb{N} \setminus \{n\}$, we have that $m \notin B\left(n, \frac{1}{2}\right)$, we have

$$\mathrm{Lim}\, \mathbb{N} = \emptyset.$$

**Proposition 37 (Closed Sets Include Its Limit Points)**

*Let $A \subset (X, d)$. Then*

1. *$A$ is closed* $\iff \mathrm{Lim}(A) \subset A$;
2. $\overline{A} = A \cup \mathrm{Lim}(A)$.

**Proof**

1. For the ($\implies$) direction, suppose $A$ is closed. [3]Let $x_0 \in A^C$. Then $\exists \varepsilon > 0$ such that $B(x_0, \varepsilon) \cap A = \emptyset$. Thus, by definition, we have that $x_0 \notin \mathrm{Lim}(A)$ [4]. Therefore, $\mathrm{Lim}(A) \subset A$.

   For the ($\impliedby$) direction, suppose $\mathrm{Lim}(A) \subset A$. Let $x_0 \in A^C$. Then $x_0 \notin \mathrm{Lim}(A)$, which means that $\exists \varepsilon > 0$ such that $B(x_0, \varepsilon) \cap A = \emptyset$, i.e. $B(x_0, \varepsilon) \subset A^C$. Thus $A$ is closed.

2. [5]It is clear that $A \subset \overline{A}$. Let $x_0 \in \overline{A}^C$. Then $\exists \varepsilon > 0$ such that $B(x_0, \varepsilon) \subset \overline{A}^C$. In particular, we have that $B(x_0, \varepsilon) \cap A = \emptyset$, i.e. $x_0 \notin \mathrm{Lim}(A)$. Thus $\mathrm{Lim}(A) \subset \overline{A}$.

   Again, it suffices to show that $A \cup \mathrm{Lim}(A)$ is closed to CTP. Let $x_0 \in (A \cup \mathrm{Lim}(A))^C$ [6]. Then $\exists \varepsilon > 0$ such that $B(x_0, \varepsilon) \cap A = \emptyset$. If $z \in \mathrm{Lim}(A)$ and $z \in B(x_0, \varepsilon)$, then we have $B(x_0, \varepsilon)$ is a neighbourhood of $z$, and so we must have that $B(x_0, \varepsilon) \cap A \neq \emptyset$, which is a contradiction. Thus $(A \cup \mathrm{Lim}(A))^C$ is open, and so $A \cup \mathrm{Lim}(A)$ is closed, as required. □

[3] This uses a reversed way of thinking: if we want to show that $\mathrm{Lim}(A) \subset A$, then instead of trying to directly show the containment, we show that all elements in $A^C$ are in fact not limit points due to $A$ being closed.

[4] Notice there that there are no elements in $A$ that are **close to** $x_0$, and so it's not a limit point.

[5] This proof is similar to that of Proposition 36.

[6] It is clear by De Morgan's Law that $x_0 \in A^C$ and $x_0 \notin \mathrm{Lim}(A)$, which implies that $\mathrm{Lim}(A) \subset A$. But this does not give us a clear geometrical picture of the notion.

**Proposition 38 (Mixing the notions)**

*Let $A \subseteq B \subseteq (X, d)$.*

1. $\overline{A} \subseteq \overline{B}$;
2. $A^\circ \subset B^\circ$;
3. $A^\circ = A \setminus \mathrm{bdy}(A)$;
4. $\mathrm{bdy}(A) = \mathrm{bdy}(A^C)$;
5. $A^\circ = \left(\overline{A^C}\right)^C$.

**Exercise 15.1.1**
*Prove Proposition 38.*

**Proof**

1. It is clear that $A \subset B \subset \overline{B}$. Suppose $\mathrm{Lim}(A)$ is not a subset of $\overline{B}$.

Then $\exists x \in \mathrm{Lim}(A) \setminus \overline{B}$, i.e. $x \in \overline{B}^C$. Since $\overline{B}$ is closed, $B^C$ is open and so $\exists \varepsilon > 0$ such that $B(x, \varepsilon) \subset B^C$. Since $x \in \mathrm{Lim}(A)$, $\exists a \in A$ such that $a \in B(x, \varepsilon) \subset B^C$, but $A \subset B$, a contracdiction. Thus $\mathrm{Lim}(A) \subset \overline{B}$.

2. $a \in A^\circ \implies \exists \varepsilon > 0 \; B(a, \varepsilon) \subset A \subset B \implies a \in B^\circ \dashv$
3. $x \in A \setminus \mathrm{bdy}(A) \implies \exists \varepsilon > 0 \; B(x, \varepsilon) \cap A^C = \emptyset \implies x \in A^\circ \dashv$
   $x \in A^\circ \implies \exists \varepsilon_0 > 0 \; B(x, \varepsilon_0) \subset A$
   Sps $x \in \mathrm{bdy}(A)$. Then $\forall \varepsilon > 0 \; B(x, \varepsilon) \cap A^C \neq \emptyset \implies B(x, \varepsilon_0) \cap A^C = \emptyset \lightning B(x, \varepsilon_0) \subset A \dashv$
4. $x \in \mathrm{bdy}(A) \implies \forall \varepsilon > 0 \; B(x, \varepsilon) \cap A \neq \emptyset \wedge B(x, \varepsilon) \cap A^C \neq \emptyset$
   $x \notin \mathrm{bdy}(A^C) \implies \exists \varepsilon_0 > 0 \; B(x, \varepsilon_0) \cap A = \emptyset \vee B(x, \varepsilon_0) \cap A^C = \emptyset$
   But $B(x, \varepsilon_0) \cap A = \emptyset \lightning \forall \varepsilon > 0 \; B(x, \varepsilon) \cap A^C \neq \emptyset$
   and $B(x, \varepsilon_0) \cap A^C = \emptyset \lightning \forall \varepsilon > 0 \; B(x, \varepsilon) \cap A^C \neq \emptyset$
   $\implies x \in \mathrm{bdy}(A^C) \dashv$. The converse is a similar argument.
5. $\left(\overline{A^C}\right)^C = \left(A^C \cup \mathrm{bdy}(A^C)\right)^C = A \cap \mathrm{bdy}\,(A)^C = A \setminus \mathrm{bdy}(A) = A^\circ \dashv$

$\square$

### Proposition 39 (More on Closures and Interiors)

*Let $A, B \subseteq (X, d)$.*

1. $\overline{A \cup B} = \overline{A} \cup \overline{B}$;
2. $(A \cap B)^\circ = A^\circ \cap B^\circ$.

**Exercise 15.1.2**
*Prove Item 2 for Proposition 39.*

### Proof

1. We have that $A \subset \overline{A}$ and $B \subset \overline{B}$, so $A \cup B \subset \overline{A} \cup \overline{B}$. Since $\overline{A} \cup \overline{B}$ is closed, we must have that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Similarly so, we have

$$A \subseteq A \cup B \implies \overline{A} \subseteq \overline{A \cup B}$$
$$B \subseteq A \cup B \implies \overline{B} \subseteq \overline{A \cup B}$$

   and so $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

2. Since $A^\circ \subseteq A$ and $B^\circ \subseteq B$, and $A^\circ \cap B^\circ$ is open, we must have that $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$. On the other hand, since $(A \cap B)^\circ \subset$

$A^\circ$ and $(A \cap B)^\circ \subset B^\circ$, we have that $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$.

$\square$

---

QUESTION: Is $\overline{A \cap B} = \overline{A} \cap \overline{B}$? No.

**Example 15.1.3**

Let $X = \mathbb{R}$, $A = \mathbb{Q}$ and $B = \mathbb{Q}^C$. We know that $\overline{A} = \mathbb{R} = \overline{B}$. But, observe that

$$\overline{A \cap B} = \emptyset \text{ while } \overline{A} \cap \overline{B} = \mathbb{R}.$$

However, we do have that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

QUESTION: Given $(X, d)$ a metric space, is

$$B(x_0, \varepsilon) = B[x_0, \varepsilon]$$

true? Again, no.

**Example 15.1.4**

Let $X$ be a set with $|X| \geq 2$, and $d$ the discrete metric. We have that

$$B(x_0, 1) = \{x_0\} \text{ but } B[x_0, 1] = X.$$

## 15.2 *Convergences of Sequences*

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**Definition 43 (Convergence)**

*Given a sequence $\{x_n\} \subset (X, d)$ and $x_0 \in X$, we say that the sequence **converges** to $x_0$ if*

$$\forall \varepsilon > 0 \; \exists N_0 \in \mathbb{N} \; \forall n \geq N_0 \; d(x_n, x_0) < \varepsilon.$$

*This is equivalent to saying that the sequence $\{d(x_n, x_0)\}$ converges to $0$ in $X$. We denote this by*

$$x_0 = \lim_{n \to \infty} x_n \text{ or } x_n \to x_0.$$

*If no such $x_0$ exists, we say that the sequence **diverges**.*

**Theorem 40 (Uniqueness of Limits of Sequences)**

*If $\{x_n\}$ is a sequence in $(X, d)$ with $x_n \to x_0$ and $x_n \to y_0$, then $x_0 = y_0$.*

**Proof**

$x_0 \neq y_0 \implies \exists \varepsilon = d(x_0, y_0) \implies B\left(x_0, \frac{\varepsilon}{2}\right) \cap B\left(y_0, \frac{\varepsilon}{2}\right) = \emptyset$

However, $\exists N_0 \in \mathbb{N}\ \forall n \geq N_0$

$$x_n \in B\left(x_0, \frac{\varepsilon}{2}\right) \wedge x_n \in B\left(y_0, \frac{\varepsilon}{2}\right)$$

which is impossible. Thus $x_0 = y_0$. □

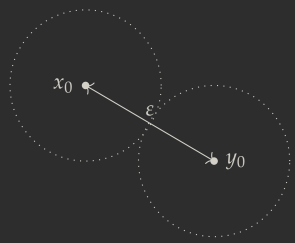


Figure 15.1: A geometric representation of the proof for Theorem 40.

# 16 Lecture 16 Oct 17th

## 16.1 Convergences of Sequences (Continued)

**Example 16.1.1**

Let $X = \mathbb{R}^n$, $d = d_p$, for $1 \leq p \leq \infty$, and $\vec{x}_k = \{(x_{k,1}, x_{k,2}, \ldots, x_{k,n})\}$.
**Claim**:

$$\vec{X}_k \xrightarrow{\ell_p} \vec{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,n}) \iff \forall j \in \{1, \ldots, n\}\ x_{k,j} \to x_{0,j}.$$

**Note**: In general, we have

$$\left| x_{k,j} - x_{0,j} \right| \leq \| \vec{x}_k - \vec{x}_0 \|_p$$

So it is clear that the ( $\implies$ ) direction is true, i.e.

$$\vec{X}_k \to \vec{x}_0 \implies \forall j \in \{1, \ldots, n\}\ x_{k,j} \to x_{0,j}.$$

For the other direction, we look at the different $p$'s to see how it works differently: in all cases, assume that $x_{k,j} \to x_{0,j}$ for all $j$, and that $\varepsilon > 0$

$p = \infty$ : we have that $\exists k_0 \in \mathbb{N}$ such that $\forall k \geq k_0$,

$$\left| x_{k,j} - x_{0,j} \right| < \varepsilon \text{ for } j \in \{1, \ldots, n\},$$

and so

$$\| \vec{x}_k - \vec{x}_0 \|_\infty = \max \left\{ \left| x_{k,j} - x_{0,j} \right| : 1 \leq j \leq n \right\} < \varepsilon.$$

$p = 1$ : if we assume that for each $j$,

$$\left| x_{k,j} - x_{0,j} \right| < \frac{\varepsilon}{n},$$

then

$$\|\vec{x}_k - \vec{x}_0\|_1 = \sum_{j=1}^{n} \left| x_{k,j} - x_{0,j} \right| < \sum_{j=1}^{n} \frac{\varepsilon}{n} = \varepsilon.$$

$1 < p < \infty$ : this time, we assume that for each $j$,

$$\left| x_{k,j} - x_{0,j} \right| < \frac{\varepsilon}{\sqrt[p]{n}}.$$

Then

$$\|\vec{x}_k - \vec{x}_0\|_p = \left( \sum_{j=1}^{n} \left| x_{k,j} - x_{0,j} \right|^p \right)^{\frac{1}{p}} < \left( \sum_{j=1}^{n} \left( \frac{\varepsilon}{\sqrt[p]{n}} \right)^p \right)^{\frac{1}{p}} = \varepsilon.$$

This completes the proof of our claim.

**Example 16.1.2**

Let $X = (C[a,b], \|\cdot\|_\infty)$. Then

$$f_n \to f \iff \|f_n - f\|_\infty \to 0.$$

Notice that for the ( $\implies$ ) direction,[1]

$$\begin{aligned} &(\forall \varepsilon > 0 \; \exists N_0 \in \mathbb{N} \; \forall n \geq N_0 \;\; |f_n - f| < \varepsilon) \\ &\implies \|f_n - f\|_\infty = \max\{|f_n(x) - f(x)| : x \in [a,b]\} < \varepsilon. \end{aligned}$$

[1] Note that this is **uniform convergence**, which implies **pointwise convergence**.

The ( $\impliedby$ ) direction is easy, since

$$|f_n(x) - f(x)| \leq \max\{|f_n(x) - f(x)| : x \in [a,b]\} < \varepsilon.$$

**Theorem 41 (Sequential Characterizations of Limit Points, Boundaries, and Closedness)**

*Given $A \subset (X, d)$,*

1. $x_0 \in \mathrm{Lim}(A) \iff \exists \{x_n\} \subset A \; (x_n \neq x_0) \wedge (x_n \to x_0)$;
2. $x_0 \in \mathrm{bdy}(A) \iff \exists \{x_n\} \subset A, \{y_n\} \subset A^C \; (x_n \to x_0) \wedge (y_n \to x_0)$;
3. *$A$ is closed* $\iff (\forall \{x_n\} \subset A \; x_n \to x_0 \in X \implies x_0 \in A)$

**Proof**

1. $x_0 \in \text{Lim}(A) \implies \forall n \in \mathbb{N}\ x_n \in B\left(x_0, \frac{1}{n}\right) \setminus \{x_0\} \implies d(x_n, x_0) < \frac{1}{n} \implies x_n \to x_0 \dashv$

   $\{x_n\} \subset A\ (x_n \to x_0) \wedge (x_n \neq x_0) \implies \forall \varepsilon > 0\ \exists N_0 \in \mathbb{N}\ \forall n \geq N_0\ x_n \in B(x_0, \varepsilon) \dashv$

2. $x \in \text{bdy}(A) \implies$
   $(\because \forall \varepsilon > 0\ A \cap B(x, \varepsilon) \neq \emptyset)\ \exists x_n \in A \cap B\left(x, \frac{1}{n}\right) \wedge$
   $(\because \forall \varepsilon > 0\ A^C \cap B(x, \varepsilon) \neq \emptyset)\ \exists y_n \in A^C \cap B\left(x, \frac{1}{n}\right)$
   $\implies (\{x_n\} \subset A \wedge x_n \to x_0) \wedge (\{y_n\} \subset A^C \wedge y_n \to x_0) \dashv$

   $(\{x_n\} \subset A \wedge x_n \to x_0) \wedge (\{y_n\} \subset A^C \wedge y_n \to x_0)$
   $\implies \forall \varepsilon > 0\ \exists N_0 \in \mathbb{N}\ \forall n \geq N_0\ \ x_n, y_n \in B(x, \varepsilon)$
   $\implies x_0 \in \text{bdy}(A) \dashv$

3. Sps $A$ is closed and $(\{x_n\} \subset A) \wedge (x_n \to x_0 \in X)$.
   $x_0 \in A^C \implies \exists \varepsilon > 0\ B(x_0, \varepsilon) \subset A^C \implies x_n \notin B(x_0, \varepsilon) \lightning\ x_n \to x_0$
   $\implies x_0 \in A$

   Sps $A$ is $\neg$ closed $\implies$ ($\because$ Proposition 37) $\exists x_0 \in \text{Lim}(A) \setminus A$
   $\implies$ ($\because$ Item 1 )$\exists \{x_n\} \subset A\ (x_n \neq x_0) \wedge (x_n \to x_0 \notin A)$, showing that RHS is false $\dashv$

$\square$

---

**Example 16.1.3**

Let $X$ be a set and $d$ a discrete metric. Then

$$x_n \to x_0 \iff \exists N \in \mathbb{N}\ \forall l n \geq N\ x_n = x_0.$$

**Example 16.1.4**

Let $c_0 = \{\{x_n\} \mid \lim_{n\to\infty} x_n = 0\} \subset \ell_\infty$.

**Claim** : $c_0$ is closed in $\ell_\infty$.

Assume $\vec{x}_k = \{x_{k,j}\}_{j=1}^{\infty} \subset c_0$, and let

$$\vec{x}_k \overset{\|\cdot\|_\infty}{\to} \vec{x}_0 = \{x_{0,j}\}_{j=1}^{\infty} \subset \ell_\infty,$$

i.e.

$$\forall \varepsilon > 0 \, \exists N_0 \in \mathbb{N} \, \forall k \geq N_0 \quad \|\vec{x}_k - \vec{x}_0\|_\infty < \frac{\varepsilon}{2}$$

Let $k_0 \geq N_0$. $\because \vec{x}_{k_0} \in c_0$, $\exists J_0 \in \mathbb{N}$ such that $\forall j \geq J_0$, we have $\left| x_{k_0,j} \right| < \frac{\varepsilon}{2}$, and so

$$\left| x_{0,j} \right| \leq \left| x_{k_0,j} - x_{0,j} \right| + \left| x_{k_0,j} \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus we have that

$$\lim_{j \to \infty} x_{0,j} = 0$$

and so $\vec{x}_0 \in c_0$. Therefore, by ☕ Theorem 41 Item 3, $c_0$ is closed in $\ell_\infty$.

Note, however, that $c_{00} \subset \ell_1 \subset c_0$ is not closed. Also $\ell_p$ is not closed in $c_0$.

# 17 Lecture 17 Oct 19th

## 17.1 Induced Metric and Topologies

**Definition 44 (Induced Metric & Induced Topology)**

*Given $(X, d)$ and $A \subset X$, we define the* ***induced metric*** *$d_A$ on $A$ by*

$$d_A : A \times A \to \mathbb{R}$$

*where $d_A(x, y) = d(x, y)$, for all $x, y \in A$, i.e. $d_A = d \restriction_{A \times A}$.*

*We define $\tau_A$, the* ***induced topology*** *on $A$ by*

$$\tau_A = \{W \subset A \mid W = U \cap A, U \subset X \text{ is open } \}$$

**Note**

*Note that $\tau_A$ is indeed a topology: it is clear that $\emptyset \in \tau_A$. Also, $A \in \tau_A$, since $X$ is open and $A = X \cap A$.*

*For an arbitrary collection $\{U_\alpha\}_{\alpha \in I} \subset \tau_A$, we know that each $U_\alpha \subset A$, and so $\bigcup_{\alpha \in I} U_\alpha \subset A$. Since each $U_\alpha \in \tau_A$, $\exists F_\alpha \subset X$ that is an open set such that $U_\alpha = F_\alpha \cap A$. Then*

$$\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} F_\alpha \cap A.$$

*Thus $\bigcup_{\alpha \in I} U_\alpha \in \tau_A$.*

*For a finite collection $\{U_1, U_2, ..., U_n\} \subset \tau_A$, we have that for each $U_i$, $\exists F_i \subset X$ that is open such that $U_i = F_i \cap A$. By Proposition 32, we* *have that*

$$\bigcap_{i=1}^{n} F_i \subset X$$

*is open, and so*

$$\bigcap_{i=1}^{n} U_i = \bigcap_{i=1}^{n} F_i \cap A \subset A.$$

*Therefore,* $\bigcap_{i=1}^{n} U_i \in \tau_A$.

**Theorem 42 (The Metric Topology of a Subset is Its Induced Topology)**

*We have*

$$\tau_A = \tau_{d_A}.$$

**Proof**

$\subseteq$ : $W \in \tau_A \implies \exists U \subset X$ open such that $W = U \cap A$

$\implies \forall x_0 \in W \; \exists > 0 \; B_X(x_0, \varepsilon) \subset U$

$\implies B_A(x_0, \varepsilon) = B_X(x_0, \varepsilon) \cap A \subset W \implies W \in \tau_{d_A} \dashv$

$\supseteq$ : $W \in \tau_{d_A} \implies \forall x_0 \in W \; \exists \varepsilon_x > 0 \; B_A(x_0, \varepsilon_x) \subset W$

$\implies W = \bigcup_{x_0 \in W} B_A(x_0, \varepsilon_x)$

Let $U = \bigcup_{x_0 \in W} B_X(x_0, \varepsilon_x)$, which is open

$\implies zW = \bigcup_{x_0 \in W} B_X(x_0, \varepsilon_0) \cap A = U \cap A$

$\implies W \in \tau_A \dashv$ $\square$

## 17.2 *Continuity on Metric Spaces*

**Definition 45 (Continuity)**

*Given metric spaces* $(X, d_X), (Y, d_Y)$, *and* $f : X \to Y$, *we say that* $f$ *is* ***continuous*** *at* $x_0$ *if*

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in X \; d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon.$$

**Theorem 43 (Continuity and Neighbourhoods)**

*Given metric spaces $(X, d_X)$ and $(Y, d_Y)$, and $f : X \to Y$, then TFAE:*

1. *$f$ is continuous at $x_0 \in X$;*
2. *if $W$ is a neighbourhood of $f(x_0) \in Y$, then $f^{-1}(W)$ is a neighbourhood of $x_0 \in X$, where*

$$f^{-1}(W) = \{x \in X : f(x) \in W\}.$$

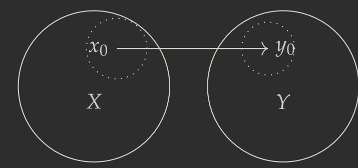


Figure 17.1: Visual representation of Theorem 43

**Proof**

$(1) \implies (2)$ : Sps $f$ is continuous at $x_0 \in X$ and $W$ a neighbourhood of $y_0 = f(x_0)$

$$\implies f(x_0) = y_0 \in W^\circ$$
$$\implies \exists \varepsilon > 0 \; B(f(x_0), \varepsilon) \subset W$$

$\because$ $f$ is continuous,

$$\exists \delta > 0 \; \forall x \in X \;\; x \in B_X(x_0, \delta) \implies d_Y(f(x), f(x_0)) < \varepsilon$$
$$\implies f(x) \in B_Y(y_0, \varepsilon) \subset W$$
$$\implies x \in f^{-1}(W) \implies x_0 \in f^{-1}(W)^\circ \dashv$$

$(2) \implies (1)$ : Sps $f^{-1}(W)$ is a neighbourhood of $x \in X$ for each neighbourhood $W$ of $y_0 = f(x_0)$

$\implies \forall \varepsilon > 0 \; W = B_Y(f(x_0), \varepsilon)$ is a neighbourhood of $f(x_0)$

$\implies U = f^{-1}(W)$ is a neighbourhood of $x_0 \in X$

$$\implies x_0 \in U$$
$$\implies \exists \delta > 0 \; B(x_0, \delta) \subset U = f^{-1}(W)$$
$$\implies (d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon) \dashv$$

$\square$

**Theorem 44 (★ Sequential Characterization of Continuity)**

*For metric spaces $(X, d_X)$ and $(Y, d_Y)$, and $f : X \to Y$, TFAE*

1. *$f$ is continuous at $x_0 \in X$;*
2. $\{x_n\} \subset X \; x_n \xrightarrow{X} x_0 \implies f(x_n) \xrightarrow{Y} f(x_0)$

**Proof**

$(1) \implies (2)$ : Sps $f$ is continuous at $x_0 \in X$.

$x_n \to x_0 \iff$

$\forall \varepsilon > 0 \, \exists \delta > 0 \; x \in B_X(x_0, \delta) \implies f(x) \in B_Y(f(x_0), \varepsilon)$

$x_n \to x_0 \implies \exists N_0 \in \; \forall n \geq N_0$

$d_X(x_0, x) < \delta \implies x_n \in B_X(x_0, \delta) \implies f(x) \in B_Y(f(x_0), \varepsilon) \dashv$

$(2) \implies (1)$ (Prove by Contrapositive) : Sps $f$ is $\neg$ continuous at $x_0 \in X$

$$\implies \exists \varepsilon_0 > 0 \, \forall \delta > 0 \; (x_\delta \in B_X(x_0, \delta)) \wedge (f(x_\delta) \notin B_Y(f(x_0), \varepsilon_0))$$

$$\implies \forall n \in \mathbb{N} \, \exists x_n \in B_X\left(x_0, \frac{1}{n}\right) \wedge f(x_n) \notin B_Y(f(x_0), \varepsilon_0)$$

$$\implies x_n \to x_0 \wedge f(x_n) \not\to f(x_0) \dashv$$

$\square$

# 18 Lecture 18 Oct 22nd

## 18.1 Continuity on Metric Spaces (Continued)

**Definition 46 (Continuity on a Space)**

*We say that*

$$f : (X, d_X) \to (Y, d_Y)$$

*is **continuous** on $X$ if $f$ is continous at each $x_0 \in X$.*

*We let*

$$C(X, Y) := \{f : X \to Y \mid f \text{ is continous on } X\},$$

*be the set of all continuous functions on $X$.*

**Note**

*In the case where $Y = \mathbb{R}$, we will simply write $C(X, X)$ as $C(X)$.*

**Remark**

*We can also define the following set*

$$C_b(X) = \{f \in C(X) \mid f \text{ is bounded } \}.$$

*We can define $\|\cdot\|_\infty$ on $C_b(X)$ by*

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in X\}.$$

*Then we have that $C_b(X) \subseteq \ell_\infty(X)$.*

**Theorem 45 (Analogue of Sequential Characterization of Continuity on a Space, and Continuity and Neighbourhoods)**

*Let $f : (X, d_X) \to (Y, d_Y)$. TFAE*

1. *$f$ is continuous;*
2. *$f^{-1}(W)$ is open for every open set $W \subset Y$;*
3. *$x_n \to x_0 \in X \implies f(x_n) \to f(x_0) \in Y$.*

**Proof**

$(1) \implies (2)$ : Let $W \subset Y$ be open, and $V = f^{-1}(W)$.

$x_0 \in V \implies f(x_0) = y_0 \in W \implies W$ is a neighbourhood of $y_0$

$\implies (\because$ Theorem 43 $) V$ is a neighbourhood of $x_0$

$\implies x_0 \in V^\circ \implies V$ is open $\dashv$

$(2) \implies (3)$ : $x_n \to x_0 \in X$

$\implies \forall \varepsilon > 0$ $(\because B_Y(f(x_0), \varepsilon)$ open $)$

$\implies x_0 \in V = f^{-1}(B_Y(f(x_0), \varepsilon))$, which is open

$\implies \exists \delta > 0\ B_X(x_0, \delta) \subset V$

$x_n \to x_0 \implies \exists N_0 \in \mathbb{N}\ \forall n \geq N_0\ x_n \in B_X(x_0, \delta)$

$\implies f(x_n) \in B_Y(f(x_0), \varepsilon) \implies f(x_n) \to f(x_0) \dashv$

$(3) \implies (1)$ : Sps $f \neg$ continuous, i.e.

$\exists \varepsilon_0 > 0\ \forall \delta \geq 0\ \exists x_\delta \in X \quad d_X(x_\delta, x_0) < \delta \wedge d_Y(f(x_\delta), f(x_0)) > \varepsilon_0$

$\implies \forall n \in \mathbb{N}\ \exists x_n \in d_X(x_0, x_n) < \frac{1}{n} \wedge d_Y(f(x_0), f(x_n)) > \varepsilon_0 \dashv$ $\square$

**Remark**

*Note that if $f : X \to Y$ and $B \subset Y$, then*

$$\left(f^{-1}(B)\right)^C = f^{-1}\left(B^C\right).$$

*Thus we have that $f : (X, d_X) \to (Y, d_Y)$ is continuous iff $f^{-1}(F)$ is closed for each closed $F \subset Y$.*

Question: For the forward direction[1], if $f : (X, d_X) \to (Y, d_Y)$ is continuous, and if $U \subset X$ is open, is $f(U)$ open? **No**.

[1] instead of talking about the pullback

**Example 18.1.1**

Consider $f : \mathbb{R} \to \mathbb{R}$ such that $\forall x \in X$, $f(x) = 1$. Then $f(\mathbb{R})$ is not open.

This motivates us to consider such "nice" functions that allow us to bring open sets to open sets, and closed to their closed counterpart.

**Definition 47 (Homeomorphism)**

*A function $\phi : (X, d_X) \to (Y, d_Y)$ is a* ***homeomorphism*** *if $\phi$ is bijective and if both $\phi$ and $\phi^{-1}$ are continuous.*

**Note**

*If $\phi$ is a homeomorphism, then we have*

- *$\phi(W) \subset Y$ is open $\iff$ $W \subset X$ is open;*
- *$\phi(F) \subset Y$ is closed $\iff$ $F \subset X$ is closed.*

**Definition 48 (Equivalent Metric Spaces)**

*We say that $(X, d_X)$ and $(Y, d_Y)$ are* ***equivalent metric spaces*** *if there exists a bijective $\phi : X \to Y$, and $c_1, c_2 \geq 0$ such that*

$$c + 1 d_X(x_1, x_2) \leq d_Y(\phi(x_1), \phi(x_2)) \leq c_2 d_X(x_1, x_2).$$

**Exercise 18.1.1**

*Show that the $\phi$ in Definition 48 is a* ***homeomorphism****.*

**Example 18.1.2**

Let $(X, d)$ be a metric space, where $X$ is any set and $d$ is the discrete metric. Let $f : (X, d) \to (Y, d_Y)$, where $(Y, d_Y)$ is another metric space that is arbitrary. Since $(X, d)$ is discrete, it is clear that if $W \subset Y$ is open, then $f^{-1}(W)$ is open.

Question: Suppose that $f : (\mathbb{R}, |\cdot|) \to (Y, d)$. When is $f$ continuous?

**Exercise 18.1.2**
*Use the* ***Intermediate Value Theorem*** *to prove that the only open and closed sets in $\mathbb{R}$ are $\emptyset$ and $\mathbb{R}$.*

Let $y_0 \in Y$. We know that $\{y_0\}$ is both open and closed. Then if $f$ is continuous, we must have that $f^{-1}(\{y_0\})$ is both open and closed. Therefore, $f$ must be a constant function.

---

**Definition 49 (Continuity on a set)**

*Let $A \subset (X, d)$ and $f : X \to (Y, d_Y)$. We say that $f$ is* ***continuous*** *on $A$ iff $f \restriction_A$ is continuous on $(A, d_A)$, where $d_A$ is the induced metric, and $f \restriction_A$ is the restriction of $f$ on $A$.*

---

**Remark**

*From the sequential characterization of continuity, we have that $(A, d_A)$ is the induced metric iff whenever $\{x_n\} \subset A$ is a sequence with $x_n \to x_0$, then $f(x_n) \to f(x_0)$.*

# 19 Lecture 19 Oct 24th

## 19.1 Completeness of Metric Spaces

Question: Is there an intrinsic way for us to tell if a sequence $\{x_n\} \subset (X, d)$ converges?

Observation Assume that $x_n \to x_0$. Then

$$\forall \varepsilon > 0 \, \exists N_0 \in \mathbb{N} \, \forall n \geq N_0 \;\; d(x_0, x_n) < \frac{\varepsilon}{2}.$$

Thus if $m, n \geq N_0$, we have

$$d(x_m, x_n) < d(x_m, x_0) + d(x_0, x_n) < \varepsilon.$$

**Definition 50 (Cauchy)**

*We say that a sequence $\{x_n\} \subset (X, d)$ is* ***Cauchy*** *if*

$$\forall \varepsilon > 0 \, \exists N_0 \in \mathbb{N} \, \forall m, n \geq N_0 \; d(x_m, x_n) < \varepsilon.$$

**Theorem 46 (Convergent Sequences are Cauchy)**

*Every convergent sequence is Cauchy.*

We proved this in our observation.

Question: Is the converse true? **No**.

**Example 19.1.1**

Let $X = (0, 1)$ with the usual metric. Let $x_n = \frac{1}{n}$. It is clear that $\{x_n\}$ is Cauchy in $(X, d)$, but the sequence does not converge.[1]

[1] The flaw here lies in the fact that $X$ is open. Should we have chosen $X = [0, 1]$, then the limit point 0 would have been included, allowing the sequence to actually converge.

**Definition 51 (Complete Metric Spaces)**

*A metric space $(X, d)$ is **complete** if each Cauchy sequence $\{x_n\} \subset X$ converges in $(X, d)$.*

## 19.1.1 *Basic Properties of Cauchy Sequences*

Observation Given a sequence $\{x_n\} \subset (X, d)$, it is possible that $\{x_n\}$ diverges but $\{x_n\}$ has a subsequence $\{x_{n,k}\}$ that converges.

**Example 19.1.2**

The sequence $\{x_n\}$ defined by $x_n = (-1)^{n-1}$, i.e.

$$\{x_n\} = \{1, -1, 1, -1, \ldots\},$$

is divergent. However, $x_{2k} \to -1$ and $x_{2k+1} \to 1$.

**Theorem 47 (★★★ Convergent Cauchy Subsequences)**

*Let $\{x_n\} \subset (X, d)$ be Cauchy and assume $x_{n,k} \to x_0$ for some subsequence $\{x_{n,k}\}_{k=1}^{\infty}$. Then $x_n \to x_0$.*

**Proof (★★★ )**

$\forall \varepsilon > 0 \; \exists N_0 \in \mathbb{N} \; \forall m, n \in N_0 \;\; d(x_n, x_m) < \frac{\varepsilon}{2}$

$x_n \to x_0 \implies \exists k_0 \in \mathbb{N} \; n_{k_0} \geq N_0 \;\; d(x_0, x_{k_0}) < \frac{\varepsilon}{2}$

$\therefore n \geq N_0 \implies$

$$d(x_n, x_0) \leq d(x_n, x_{k_0}) + d(x_{k_0}, x_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore x_n \to x_0$ $\square$

**Definition 52 (Boundedness)**

*Let $A \subset (X, d)$. $A$ is **bounded** if*

$$\exists M > 0 \; \exists x_0 \in X \; A \subset B[x_0, M].$$

**Proposition 48 (Cauchy Sequences are Bounded)**

*If $\{x_n\} \subset (X, d)$ is Cauchy, then $\{x_n\}$ is bounded.*

**Proof**

Let $\varepsilon = 1$. $\exists N_0 \in \mathbb{N} \; \forall m, n \geq N_0 \;\; d(x_n, x_m) < \varepsilon$. In particular, if $n \geq N_0$, then $d(x_n, x_{N_0}) < 1$. Then, let

$$M = \max\{d\left(x_1, x_{N_0}\right), d\left(x_2, x_{N_0}\right), \ldots, d\left(x_{N_0 - 1}, d_{N_0}\right), 1\}$$

Then it is clear that $\{x_n\} \subset B[x_{N_0}, M]$. □

### 19.1.2 Examples of Complete Spaces

#### 19.1.2.1 Completeness of $\mathbb{R}$

**Theorem 49 (Bolzano-Weierstrass)**

*Every bounded sequence $\{x_n\} \subset \mathbb{R}$ has a convergent subsequence.*

**Be sure to review a proof of this and add it here.**

**Theorem 50 ($\mathbb{R}$ is complete)**

*$\mathbb{R}$ is complete.*

**Proof**

If $\{x_n\} \subset \mathbb{R}$ is Cauchy, then it is bounded by Proposition 48, and so by Bolzano-Weierstrass, $\{x_n\}$ has a convergent subsequence $\{x_{n,k}\}$ such that $x_{n,k} \to x_0$. Since $\{x_n\}$ is Cauchy, by ☕ Theorem 47, $x_n \to x_0$. □

---

**Example 19.1.3**

Consider $(\mathbb{R}^n, \|\cdot\|_p)$, with $1 \leq p \leq \infty$. Let $\{\vec{x}_k\} = \{(x_{k,1}, x_{k,2}, \ldots, x_{k,n})\}$ be Cauchy in $(\mathbb{R}^n, \|\cdot\|_p)$.

$$\because \left| x_{k,j} - x_{m,j} \right| \leq \|\vec{x}_k - \vec{x}_m\|_p$$
$$\implies \{x_{k,j}\} \text{ is Cauchy for each } j = 1, \ldots, n$$
$$\implies x_{k,j} \to x_{0,j} \text{ for each } j = 1, \ldots, n \ \because \text{☕ Theorem 47}$$
$$\implies \vec{x}_k \to \vec{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,n})$$
$$\implies (\mathbb{R}, \|\cdot\|_p) \text{ is complete.}$$

**Example 19.1.4**

Let $(X, d)$ be discrete[2]. If $\{x_n\}$ is Cauchy, then $\exists N_0 \in \mathbb{N}$ such that $\forall m, n \geq N_0$, we have $x_n = x_m$, i.e. $\{x_n\}$ converges. Therefore, $(X, d)$ is complete.

[2] By discrete, we mean a discrete metric space, i.e. $d$ is a discrete metric.

**Example 19.1.5 (★ )**

Let $X = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\} \subset \mathbb{R}$ with the induced standard metric. Recall that each of the singleton $\{\frac{1}{n}\}$ is open.

Note that given $Y = \{1, 2, \ldots, n, \ldots\} = \mathbb{N}$ with the discrete metric, if we define $\phi : \mathbb{N} \to \left\{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right\}$ by $\phi(n) = \frac{1}{n}$, then $\phi$ is a homeomorphism, and so $(Y, d)$, where $d$ is the discrete metric, is complete.

However, as shown before, since $\{\frac{1}{n}\}$ is Cauchy but not convergent, $X = \left\{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right\}$ is not complete.

# 20 Lecture 20 Oct 26th

## 20.1 Completeness of Metric Spaces (Continued)

### 20.1.1 Examples of Complete Spaces (Continued)

#### 20.1.1.1 Completeness of $\ell_p$

**Theorem 51 (★ Completeness of $\ell_p$)**

*$\ell_p$ is complete for every $1 \leq p \leq \infty$.*

**Proof**

$p = \infty$: Let $\{\vec{x}_k\} \subset \ell_\infty$ be Cauchy in $\|\cdot\|_\infty$. We have

$$\vec{x}_k = \{x_{k,1}, x_{k,2}, \ldots, x_{k,j}, \ldots\}$$

$\implies \forall \varepsilon > 0 \; \exists N_0 \in \mathbb{N} \; \forall m, n \geq N_0 \quad \|\vec{x}_n - \vec{x}_m\|_\infty < \frac{\varepsilon}{2}$

$\because \left| x_{n,j} - x_{m,j} \right| \leq \|\vec{x}_n - \vec{x}_m\|_\infty < \frac{\varepsilon}{2}$,

each of the $\vec{x}_k$, for $k \geq N_0$, is Cauchy in $\mathbb{R}$.

$\implies \exists x_{0,j} \in \mathbb{R} \; x_{k,j} \to x_{0,j} \quad \because \mathbb{R}$ is complete

Let $\vec{x}_0 = \{x_{0,1}, x_{0,2}, \ldots, x_{0,j}, \ldots\}$ and $x_{0,j} = \lim\limits_{k\to\infty} x_{k,j}$.

By our argument on Line 4, we have that

$$\left| x_{n,j} - x_{0,j} \right| = \lim_{m\to\infty} \left| x_{n,j} - x_{m,j} \right| \leq \frac{\varepsilon}{2} < \varepsilon \tag{20.1}$$

$\implies \{x_{n,j} - x_{0,j}\}_{j=1}^\infty \in \ell_\infty$

$\implies \{x_{0,j}\}_{j=1}^\infty \in \ell_\infty$

Also, by Equation (20.1), we have

$$\|\vec{x}_n - \vec{x}_0\|_\infty \leq \frac{\varepsilon}{2} < \varepsilon,$$

so $\vec{x}_k \to \vec{x}_0$. $\dashv$.

$1 \leq p < \infty$: Let $\{\vec{x}_k\} \subset \ell_p$ be Cauchy. By the same argument as above, $|x_{n,j} - x_{m,j}| \leq \|\vec{x}_n - \vec{x}_m\|_p \implies \left\{x_{k,j}\right\}_{j=1}^{\infty}$ is Cauchy for each $j$. Since $\mathbb{R}$ is complete, let $x_{0,j} = \lim_{k\to\infty} x_{k,j}$, and

$$\vec{x}_0 = \{x_{0,1}, x_{0,2}, \ldots, x_{0,j}, \ldots\}.$$

Now $\forall \varepsilon > 0\ \exists N_0 \in \mathbb{N}\ \forall n, m \geq N_0\ \ \|\vec{x}_n - \vec{x}_m\| < \frac{\varepsilon}{2}$. Thus for $j \in \mathbb{N}$,

$$\left(\sum_{i=1}^{j} |x_{n,i} - x_{m,i}|^p\right)^{\frac{1}{p}} \leq \|\vec{x}_n - \vec{x}_m\|_p < \frac{\varepsilon}{2}.$$

Then for $n \geq N_0$,

$$\left(\sum_{i=1}^{j} |x_{k,i} - x_{0,i}|^p\right)^{\frac{1}{p}} = \lim_{m\to\infty} \left(\sum_{i=1}^{j} |x_{n,i} - x_{m,i}|^p\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{2}$$

for each $j$, and so

$$\lim_{j\to\infty} \left(\sum_{i=1}^{j} |x_{n,i} - x_{0,i}|^p\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{2}$$

$\implies \vec{x}_0 \in \ell_p$ and $\|\vec{x}_n - \vec{x}_0\|_p \leq \frac{\varepsilon}{2} < \varepsilon$. $\square$

### 20.1.1.2 *Completeness of $(C_b(X), \|\cdot\|_\infty)$*

**Definition 53 (Convergence of Functions)**

*A sequence of functions $f_n : (X, d_X) \to (Y, d_Y)$ is said to* ***converge pointwise*** *to some function $f_0 : (X, d_X) \to (Y, d_Y)$ if for each $x_0 \in X$,*

$$\forall \varepsilon > 0\ \exists N_0 \in \mathbb{N}\ \forall n \geq N_0\ \ d_Y(f_n(x_0) - f_0(x_0)) < \varepsilon.$$

*The sequence $f_n$ is said to* ***converge uniformly*** *if*

$$\forall \varepsilon > 0\ \exists N_0 \in \mathbb{N}\ \forall n \geq N_0\ \forall x \in X\ \ d_Y(f_n(x) - f_)(x)) < \varepsilon.$$

**Remark**
*It is clear that uniform convergence implies pointwise convergence.*

**Example 20.1.1 (Pointwise Convergent but not Uniformly Convergent)**

Let $X = [0,1]$, $Y = \mathbb{R}$, $f_n(x) = x^n$ for each $n \in \mathbb{N}$. It is quite clear that

$$f_n(x) \to f_0(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}.$$

$f_n$ is pointwise convergent but not uniformly convergent; just take $\varepsilon = \frac{1}{2}$.

**Theorem 52 (★★★ Uniformly Convergent Pointwise Continuous Functions have a Pointwise Continuous Limit)**

*Assume that $f_n : (X, d_X) \to (Y, d_Y)$ converges uniformly to $f_0 : (X, d_X) \to (Y, d_Y)$. If each $f_n$ is continuous at $x_0 \in X$, then $f_0$ is continuous at $x_0$.*

This is a classic $\frac{\varepsilon}{3}$ argument.

**Proof**

$\forall \varepsilon > 0\ \exists N_0 \in \mathbb{N}\ \forall n \geq N_0\ \forall x \in X\ \ d_Y(f_n(x) - f_0(x)) < \frac{\varepsilon}{3}$

$f_n$ is continuous at $x_0 \implies \exists \delta > 0\ \forall x \in X\ \ x \in B(x_0, \delta)$

$\implies \forall n_0 \geq N_0\ \ d_Y(f_{n_0}(x) - f_{n_0}(x_0)) < \frac{\varepsilon}{3}$

$$\begin{aligned} \implies d_Y\left(f_0(x), f_0(x_0)\right) \\ &\leq d_Y\left(f_0(x), f_{n_0}(x)\right) + d_Y\left(f_{n_0}(x), f_{n_0}(x_0)\right) + d_Y\left(f_{n_0}(x_0), f_0(x_0)\right) \\ &< \varepsilon \end{aligned}$$

$\implies f_0$ is continuous at $x_0$. □

# 21 Lecture 21 Oct 31st

## 21.1 Completeness of Metric Spaces (Continued 2)

### 21.1.1 Examples of Complete Spaces (Continued 2)

#### 21.1.1.1 Completeness of $(C_b(X), \|\cdot\|_\infty)$ (Continued)

**Note**

*A normed linear space $V$ is called a* ***Banach space*** *if $(V, \|\cdot\|)$ is complete with respect to $d_V$.*

**Theorem 53 (★★★ Completeness for $C_b(X)$)**

*The space $(C_b(X), \|\cdot\|_\infty)$ is Banach (i.e. complete).*

This will come out in the final.

**Proof**

Let $\{f_n\} \subset C_b(X)$ be Cauchy.

$$\implies \forall \varepsilon > 0 \; \exists N_0 \in \mathbb{N} \; \forall n, m \geq N_0 \quad \|f_n - f_m\|_\infty < \frac{\varepsilon}{2}, \qquad (*)$$

and

$$\forall x \in X \quad |f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \frac{\varepsilon}{2}.$$

$\therefore \{f_n(x)\}$, for every $x \in X$ is Cauchy, and so $\{f_n(x)\}$ is complete.

Let $f_0(x) = \lim_{n\to\infty} f_n(x)$, and in particular, $\forall n \geq N_0$, $\forall x \in X$, we have

$$|f_n(x) - f_0(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \frac{\varepsilon}{2} < \varepsilon.$$

So $f_n \to f_0$ uniformly. By Theorem 52, $f_0$ is continuous.

It remains to show that $f_0$ is bounded: we have that $\{f_n\}$ is bounded.

Let $M > 0$ such that $\|f_n\|_\infty \leq M$ for all $n \in \mathbb{N}$. Let $x \in X$.

From ($*$), we can find $n_0 \in \mathbb{N}$ such that $|f_{n_0}(x) - f_0(x)| \leq 1$.

$\implies |f_0(x)| \leq |f_0(x) - f_{n_0}(x)| + |f_{n_0}| \leq 1 + M$

$\therefore f_0(x) \in C_b(X)$. □

**Note**

*Given any set $X$, if $(X, d)$ is a metric space with the discrete metric, then*

$$(C_b(X), \|\cdot\|_\infty) = (\ell_\infty, \|\cdot\|_\infty).$$

### 21.1.2 Characteriztions of Completeness

We shall state the following without proving it, although the proof is straightforward: view $\{a_n\}$ and $\{b_n\}$ as increasing and decreasing sequences respectively and use the monotone convergence theorem.

**Theorem 54 (Nested Interval Theorem)**

*If $\{[a_n, b_n]\}$ with $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$, then*

$$\bigcap_{n=1}^\infty [a_n, b_n] \neq \emptyset.$$

We know that this works for $\mathbb{R}$, but does this work for $(X, d)$? In particular, we conjecture that:

> *If $\{F_n\}$ is a sequence of non-empty closed sets in $(X, d)$, with $F_{n+1} \subseteq F_n$, then*
> $$\bigcap_{n=1}^\infty F_n \neq \emptyset.$$

However, this is not true, as shown in the following example.

**Example 21.1.1**

Let $X = \mathbb{R}$, and $F_n[n, \infty)$, and $F_{n+1} \subsetneq F_n$. Note that $F_n$ is indeed closed since its complement, $(-\infty, n)$, is open. We notice that

$$\bigcap_{n=1}^{\infty} F_n = \emptyset.$$

**Example 21.1.2**

Let $X = (0, 1)$, and $F_n \left(0, \frac{1}{n}\right]$, which is closed in $X$, and that $F_{n+1} \subsetneq F_n$. However, once again, we notice that

$$\bigcap_{n=1}^{\infty} F_n = \emptyset.$$

Of course, one would ask the question as to why does such a property not hold. The following notion will explain why.

**Definition 54 (Diameter of a Set)**

*Given a subset $A \subset (X, d)$, we define the **diameter** of $A$ as*

$$\mathrm{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}.$$

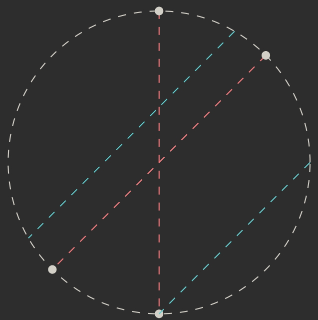

Figure 21.1: Intuitive illustration of Definition 54. Red lines are the diameters, as captured by the sup function. Blue lines are other possible candidates, but none of them can be a supremum.

**Proposition 55 (Diameters of Subsets)**

*Let $A \subseteq B \subset (X, d)$. Then*

1. $\mathrm{diam}(A) \leq \mathrm{diam}(B)$;
2. $\mathrm{diam}(A) = \mathrm{diam}(\overline{A})$.

**Proof**

1. If $A = B$, then there is nothing to proof. Suppose $A \subsetneq B$. Suppose to the contrary that $\mathrm{diam}(A) > \mathrm{diam}(B)$. Let $x_A, y_A \in A$ such that $d(x_A, y_A) = \mathrm{diam}(A)$ and $x_B, y_B \in B$ such that $d(x_B, y_B)$. By our assumption, we have

$$d(x_A, y_A) > d(x_B, y_B).$$

However, $x_A, y_A \in A \subseteq B$, and by definition of a diameter, we have

$$d(x_B, y_B) \geq d(x_A, y_A),$$

which is a contradiction. This proves the statement.

2. If $\operatorname{diam}(A) = \infty$, then we must have $\operatorname{diam}(\overline{A}) = \infty$ since $A \subseteq \overline{A}$. Thus WMA $\operatorname{diam}(A) = d < \infty$. Let $x_0, y_0 \in \overline{A}$. Then given any $\varepsilon > 0$, by definition of limits, we can find $x_1, y_1 \in A$ such that

$$d(x_0, x_1) < \frac{\varepsilon}{2} \text{ and } d(y_0, y_1) < \frac{\varepsilon}{2}.$$

Hence

$$\begin{aligned} d(x_0, y_0) &\leq d(x_0, x_1) + d(x_1, y_1) + d(y_1, y_0) \\ &< \frac{\varepsilon}{2} + d + \frac{\varepsilon}{2} = d + \varepsilon. \end{aligned}$$

Thus $\operatorname{diam}(\overline{A}) \leq d + \varepsilon$, for any $\varepsilon > 0$. Therefore by the earlier part,

$$\operatorname{diam}(\overline{A}) \leq d = \operatorname{diam}(A) \leq \operatorname{diam}(\overline{A}).$$

$\square$

With this notion, we have a partial equivalence to the nested interval theorem, of which we shall prove in the next lecture.

# 22 Lecture 22 Nov 02nd

## 22.1 Completeness of Metric Spaces (Continued 3)

### 22.1.1 Characterizations of Completeness (Continued)

We are now ready to prove the following statement.

**Theorem 56 (Cantor's Intersection Principle)**

*Let $(X, d)$ be a metric space. TFAE:*

1. *$(X, d)$ is complete.*
2. *If $\{F_n\}$ is a sequence of non-empty closed subsets such that $F_{n+1} \subset F_n$ for all $n \in \mathbb{N}$, and if $\lim_{n\to\infty} \operatorname{diam}(F_n) = 0$, then*

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

**Proof**

$(1) \implies (2)$: [1]For each $n \in \mathbb{N}$, pick $x_n \in F_n$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence formed from these $x_n$'s.

By the assumption that $\lim_{n\to\infty} \operatorname{diam}(F_n) = 0$, we have that

$$\forall \varepsilon > 0 \; \exists N_0 \in \mathbb{N} \;\; \operatorname{diam}\left(F_{N_0}\right) < \varepsilon.$$

In particular, for $n, m \geq N_0$, we have that $x_n, x_m \in F_{N_0}$, as $F_n, F_m \subset F_{N_0}$, and so

$$d(x_n, x_m) \leq \operatorname{diam}\left(F_{N_0}\right) < \varepsilon.$$

Thus $\{x_n\}$ is Cauchy. By assumption that $(X, d)$ is complete, $x_n \to x_0 \in X$. Thus $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $d(x_n, x_0) < \varepsilon$. Thus,

[1] Since we have a sequence of non-empty closed subsets, we can, by using Axiom 2, form a sequence of elements in $X$ from each of the $F_n$'s. By proving that this sequence of elements is Cauchy, we obtain a limit point from the assumption that $X$ is complete. From there, it remains to show that the limit point lives in all of the $F_n$'s.

for any such $n$, since $F_{n+1} \subset F_n$, $\{x_n, x_{n+1}, x_{n+2}, \ldots\} \subset F_n$, and the sequence converges to $x_0$. Since $F_n$ is closed, we must have $x_0 \in F_n$. This forces $x_0 \in F_n$ for every $n \in \mathbb{N}$. This completes ( $\implies$ ).

(2) $\implies$ (1) : Let $\{x_n\} \subset X$ be Cauchy. Let $F_n = \{x_n, x_{n+1}, x_{n+2}, \ldots\}$. We have that $F_n$ is closed: given any $y \notin F_n$, we can pick $\delta = \frac{1}{2}\min\{d(x_i, x_j) : n \leq i < j\}$ and we would have that $B(y, \delta) \cap F_n = \emptyset$.

Note that $F_{n+1} \subset F_n$.

$\because \{x_n\}$ is Cauchy, $\forall \varepsilon > 0\ \exists N_0 \in \mathbb{N}\ \forall n, m \geq N_0\ d(x_n, x_m) < \frac{\varepsilon}{2}$. Consequently,

$$\operatorname{diam}\left(\{x_{N_0}, x_{N_0+1}, \ldots\}\right) = \operatorname{diam}\left(F_{N_0}\right) \leq \frac{\varepsilon}{2} < \varepsilon.$$

$\therefore \operatorname{diam}(F_n) \to 0$, which, along with assumption, implies that[2]

$$\bigcap_{n=1}^{\infty} F_n = \{x_0\}.$$

[2] Note that the intersection can only contain one element, since $\operatorname{diam}(F_n) \to 0$.

Also, since $\operatorname{diam}(F_n) \to 0$, we have that for any $k > 0$, $F_{i_k} \subseteq B\left(x_0, \frac{1}{k}\right)$ [3]. This implies that for each $k$, $B\left(x_0, \frac{1}{k}\right)$ contains the tail of the sequence $\{x_n\}$. Then, inductively so, we have

[3] Otherwise, $x_0$ cannot be a limit point.

$$\begin{aligned}
k = 1 &\implies \exists n_1 > 0\ x_{n_1} \in B\left(x_0, 1\right) \\
k = 2 &\implies \exists n_2 > 0\ x_{n_2} \in B\left(x_0, \frac{1}{2}\right) \\
&\ \ \vdots \\
k = m &\implies \exists n_m > 0\ x_{n_m} \in B\left(x_0, \frac{1}{m}\right) \\
&\ \ \vdots
\end{aligned}$$

$\therefore x_{n_m} \to x_0$.

Then since $\{x_n\}$ is Cauchy, and $\{x_{n_m}\}$ is a subsequence of $\{x_n\}$, we have $x_n \to x_0$. $\square$

### Definition 55 (Formal Sum)

*Let $(X, \|\cdot\|)$ be a normed linear space. A series in $X$ is called a* ***formal***

***sum***, *expressed as*

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \ldots + x_n + \ldots, \tag{22.1}$$

*where* $\{x_n\} \subseteq X$. *For each* $k \in \mathbb{N}$, *the* $k^{th}$ ***partial sum*** *of Equation (22.1) is*

$$S_k = \sum_{n=1}^{k} x_n = x_1 + x_2 + \ldots + x_k.$$

*We say that* $\sum_{n=1}^{\infty} x_n$ *converges in* $(X, \|\cdot\|)$ *if* $\{S_k\}_{k=1}^{\infty}$ *converges. In this case, we write*

$$\sum_{n=1}^{\infty} = \lim_{k \to \infty} S_k.$$

*Otherwise,* $\sum_{n=1}^{\infty} x_n$ *is said to diverge.*

**Theorem 57 (★★ Weierstrass M-test)**

*Let* $(X, \|\cdot\|)$ *be a normed linear space. TFAE:*

1. $(X, \|\cdot\|)$ *is complete, i.e.* $(X, \|\cdot\|)$ *is a Banach space.*
2. *If* $\sum_{n=1}^{\infty} x_n$ *is such that* $\sum_{n=1}^{\infty} \|x_n\|$ *converges, then* $\sum_{n=1}^{\infty} x_n$ *converges.*

**Proof**

$(1) \implies (2)$: Given $\sum_{n=1}^{\infty} x_n$, let

$$S_k = \sum_{n=1}^{k} x_n \text{ and } T_k = \sum_{n=1}^{k} \|x_n\|.$$

Suppose $T_k$ converges. Then in particular, $\{T_k\}$ is Cauchy. Thus

$$\forall \varepsilon > 0 \; \exists N_0 \in \mathbb{N} \; \forall n > m \geq N_0$$
$$T_n - T_m = \sum_{k=1}^{n} \|x_k\| - \sum_{k=1}^{m} \|x_k\| = \sum_{k=m+1}^{n} \|x_k\| < \varepsilon.$$

$\therefore N_0 \leq m < n \implies$

$$\begin{aligned} \|S_n - S_m\| &= \left\| \sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k \right\| = \left\| \sum_{k=m+1}^{n} x_k \right\| \\ &\leq \sum_{k=m+1}^{n} \|x_k\| \quad \because \text{ Triangle Ineq.} \\ &< \varepsilon. \end{aligned}$$

$\therefore$ $\{S_k\}$ is Cauchy, and since $(X, \|\cdot\|)$ is complete, $\{S_k\}$ is convergent.

$(2) \implies (1)$: Suppose $\{x_n\}$ is Cauchy in $(X, \|\cdot\|)$. We can find an increasing sequence

$$N_0 < n_1 < n_2 < \ldots < n_j < \ldots \in \mathbb{N},$$

for some $N_0 \in \mathbb{N}$ such that

$$\left\| x_{n_j} - x - n_{j+1} \right\| < \frac{1}{2^j}.$$

Then by the **infinite geometric series**,

$$\sum_{j=1}^{\infty} \left\| x_{n_j} - x_{n_{j+1}} \right\| \leq \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty.$$

$\therefore \sum\limits_{j=1}^{\infty} (x_{n_j} - x_{n_{j+1}})$ converges to some $x_0 \in X$. In particular, notice that the partial sums are **telescoping series**:

$$S_k = \sum_{j=1}^{k} \left( x_{n_j} - x_{n_{j+1}} \right) = x_{n_1} - x_{n_{k+1}} \to x_0.$$

It follows that as $k \to \infty$,

$$x_{n_{k+1}} \to x_{n_1} - x_0.$$

We have that the subsequence $\{x_{n_k}\}$ of our Cauchy sequence $\{x_n\}$ has a limit point. $\square$

# 23 *Lecture 23 Nov 05th*

## 23.1 *Completeness of Metric Spaces (Continued 4)*

### 23.1.1 *Characterizations of Completeness (Continued 2)*

**Example 23.1.1**

Let

$$\phi(x) = \begin{cases} x & x \in [0,1] \\ 2-x & x \in [1,2] \end{cases}.$$

Extend $\phi$ to $\mathbb{R}$ by

$$\phi(x+2) = \phi(x) \qquad \text{for all } x \in \mathbb{R}.$$

Define

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \phi\left(4^n x\right).$$

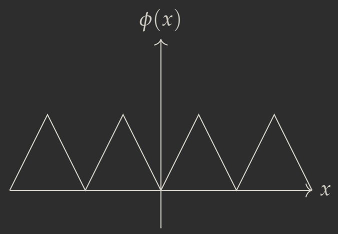


Figure 23.1: Sawtooth-like graph from $\phi$

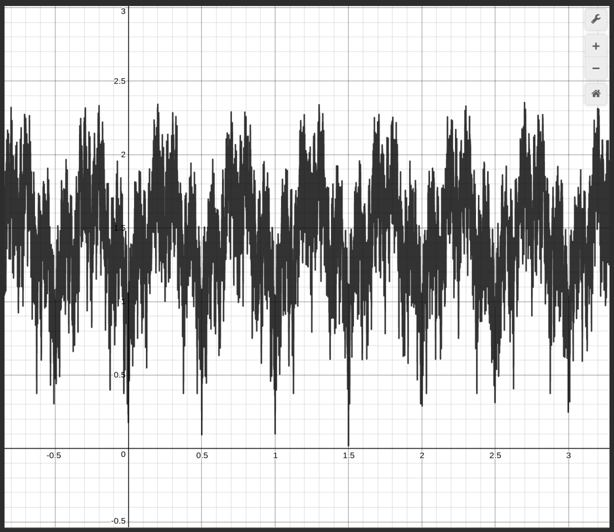


Figure 23.2: Function of $f$ as generated on Desmos. See it live.

Figure 23.2 is a simplified graph of $f$, drawn using the online tool Desmos.

It is clear that $\phi \in C_b(\mathbb{R})$, and $\|\phi\|_\infty = 1$. Thus

$$\sum_{n=1}^{\infty} \left\| \left(\frac{3}{4}\right)^n \phi\left(4^n x\right) \right\|_\infty = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n < \infty,$$

and so

$$f(x) = \lim_{L \to \infty} \sum_{n=1}^{L} \left(\frac{3}{4}\right)^n \phi\left(4^n x\right) = \lim_{L \to \infty} S_L(x),$$

uniformly so. Since the partial sums are continuous, $f \in C_b(\mathbb{R})$.

However, $f$ is not **differentiable**. Let $x \in \mathbb{R}$. For each $m \in \mathbb{N}$, we can find $k \in \mathbb{Z}$ such that

$$k \leq 4^m x \leq k + 1.$$

Let

$$p_m = \frac{k}{4^m} \text{ and } q_m = \frac{k+1}{4^m},$$

and for any $n \in \mathbb{N}$,

$$\alpha = 4^n p_m = 4^{n-m} k \text{ and } \beta = 4^n q_m = 4^{n-m}(k+1).$$

Now

- if $n > m$, then since $\alpha$ and $\beta$ differ by an even integer, $|\phi(\alpha) - \phi(\beta)| = 0$;
- if $n = m$, then $\alpha$ and $\beta$ differs by 1, and so $|\phi(\alpha) - \phi(\beta)| = 1$;
- if $n < m$, then there are no integers between $\alpha$ and $\beta$, and so

$$|\phi(\alpha) - \phi(\beta)| = |4^n p_m - 4^n q_m| \,{}^{1} = \left|4^{n-m} k - 4^{n-m}(k+1)\right| = 4^{n-m}.$$

[1] Note that if we have $1 \leq \alpha, \beta \leq 2$, we still get the same formula.

For large enough $m$, consider

$$
\begin{aligned}
|f(p_m) - f(q_m)| &= \left| \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \left(\phi\left(4^n p_m\right) - \phi\left(4^n q_m\right)\right) \right| \\
&= \left| \sum_{n=1}^{m} \left(\frac{3}{4}\right)^n \left(\phi\left(4^n p_m\right) - \phi\left(4^n q_m\right)\right) \right| \qquad (23.1) \\
&\geq \left| \left(\frac{3}{4}\right)^n - \sum_{n=1}^{m-1} \left(\frac{3}{4}\right)^n \left|\phi\left(4^n p_m\right) - \phi\left(4^n q_m\right)\right| \right| \qquad (23.2) \\
&= \left| \left(\frac{3}{4}\right)^n - \sum_{n=1}^{m-1} \left(\frac{3}{4}\right)^n 4^{n-m} \right| \qquad (23.3) \\
&= \left| \left(\frac{3}{4}\right)^n - \frac{1}{4^m} \sum_{n=1}^{m-1} 3^n \right| \\
&= \left| \left(\frac{3}{4}\right)^n - \frac{1}{4^m} \left[\frac{3^m - 1}{2}\right] \right| \qquad (23.4) \\
&= \frac{1}{4^m} \left[\frac{3^m + 1}{2}\right] > \frac{1}{2} \cdot \left(\frac{3}{4}\right)^m
\end{aligned}
$$

where we note that

(23.1) terms after $m$ are eliminated as they are 0 as argued previously;

(23.2) by the reverse Triangle ineq. and the case where $n = m$;

(23.3) using the argument for when $n < m$;

(23.4) using the formula for a finite geometric sum.

Hence we observe that

$$\frac{|f(p_m) - f(q_m)|}{|p_m - q_m|} > 4^m \cdot \frac{3^m}{2 \cdot 4^m} = \frac{3^m}{2}.$$

Now if $p_m = x$, then

$$\frac{|f(x) - f(q_m)|}{|x - q_m|} > \frac{3^m}{2}.$$

If $p_m \neq x$, then

$$
\begin{aligned}
\frac{3^m}{2} < \frac{|f(p_m) - f(q_m)|}{|p_m - q_m|} &\leq \frac{|f(p_m) - f(x)| + |f(x) - f(q_m)|}{|p_m - q_m|} \\
&\leq \frac{|f(p_m) - f(x)|}{|p_m - x|} + \frac{|f(x) - f(q_m)|}{|x - q_m|},
\end{aligned}
$$

which implies that either

$$\frac{|f(x) - f(q_m)|}{|x - q_m|} > \frac{3^m}{2},$$

or

$$\frac{|f(p_m) - f(x)|}{|p_m - x|} > \frac{3^m}{2}.$$

Then for any sequence $\{t_m\}$ such that $t_m \to x$, and $t_m \neq x$, we have that

$$\frac{|f(x) - f(t_m)|}{|x - t_m|} \geq \frac{3^m}{4} \to \infty$$

as $m \to \infty$. Thus the function $f$ is not differentiable at any $x$.

# 24 Lecture 24 Nov 07th

## 24.1 Completions of Metric Spaces

**Definition 56 (Isometry)**

*A map $\phi : (X, d_X) \to (Y, d_Y)$ is called an* ***isometry*** *if*

$$d_Y(\phi(x_1), \phi(x_2)) = d_X(x_1, x_2).$$

**Definition 57 (Completion)**

*A* ***completion*** *of a metric space $(X, d)$ is a pair $((Y, d_Y), \phi)$ where $(Y, d_Y)$ is a complete metric space, $\phi : X \to Y$ is an isometry, and $\overline{\phi(X)} = Y$.*

**Proposition 58 (Subsets of Complete Spaces are Complete if they are Closed)**

*Let $(X, d)$ be a complete metric space. Let $A \subset X$. Then $(A, d_A)$ is complete iff $A$ is closed.*

**Proof**

$(\implies)$: $(A, d_A)$ is complete

$\implies \{x_n\} \subset A$ Cauchy $\implies x_n \to x_0 \implies x_0 \in A \implies \mathrm{Lim}(A) \subseteq A$

$\implies A$ is closed.

$(\impliedby)$ Let $\{x_n\} \subset A$ be Cauchy in $(A, d_A)$

$\implies \{x_n\}$ is Cauchy in $(X, d)$

$\implies x_n \to x_0 \in X$

$\implies (\because A \text{ is closed }) x_0 \in A$

$\implies (A, d_A)$ is complete. $\square$

A natural question arises: does every space have a completion?

To answer this, we need the following concept:

**Definition 58 (Uniformly Continuous Functions)**

*We say that a function $f : (X, d_X) \to (Y, d_Y)$ is **uniformly continuous** if*

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x_1, x_2 \in X$$
$$d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

**Example 24.1.1**

Given $(X, d)$, and $x_0 \in X$, define

$$g_{x_0}(x) = d(x, x_0).$$

Note that $|d(x_0, x) - d(x_0, y)| \leq d(x, y)$.[1] Thus

[1] Proved in A3

$$|g_{x_0}(x_1) - g_{x_0}(x_2)| \leq d(x_1, x_2).$$

Then $\forall \varepsilon > 0 \; \exists \delta = \varepsilon > 0$, we have

$$d(x_1, x_2) < \delta \implies |g_{x_0}(x_1) - g_{x_0}(x_2)| < \varepsilon.$$

Thus $g_{x_0}$ is uniformly continuous.

**Theorem 59 (Completion Theorem)**

*Every metric space $(X, d)$ has a completion.*

**Proof**
Let $a \in X$. Define $\phi : X \to C_b(X)$ by

$$(\phi(u))\,(x) = f_u(x) = d(u,x) - d(x,a).$$

By our earlier example, $\phi(u)$ is continuous. Notice that we have

$$|f_u(x)| = |d(u,x) - d(x,a)| \leq d(u,a).$$

Thus $\phi(u) in C_b(X)$, proving that $\phi$ is well-defined.

WTS $\phi$ is an isometry. Let $u, v \in X$. Then

$$\begin{aligned} |f_u(x) - f_v(x)| &= |d(u,x) - d(x,a) - d(v,x) - +d(x,a)| \\ &= |d(u,x) - d(v,x)| \\ &\leq d(u,v). \end{aligned}$$

Thus $\|f_u - f_v\|_\infty \leq d(u,v)$ by definition of $\|\cdot\|_\infty$. Notice that

$$|f_u(v) - f_v(v)| = d(u,v),$$

which gives us the greatest possible value. Thus

$$\|\phi(u) - \phi(v)\|_\infty = \|f_u - f_v\|_\infty = d(u,v).$$

Thus $\phi$ is an isometry.

Since $(C_b(X), \|\cdot\|_\infty)$ is a complete metric space, let $Y = \overline{\phi(X)}$. The proof is complete by Proposition 58. □

---

Question: If $(X,d)$ has 2 completions, how are they related?

Suppose $(X,d)$ is a metric space that has 2 completions through the functions $\phi$ and $\psi$.

Since we have that $\phi$ is bijective from $X$ to $\phi(X)$, we can take its inverse. Consequently, we have that the function $\Gamma = \psi \circ \phi^{-1}$ is an isometry.

Now for some $\{x_n\} \subset X$ that is Cauchy, we know that in $\phi(X)$, $\phi(x_n) \to y_0 \in \phi(X)$. Note that $y_0$ is a limit point of $\phi(X)$. Through $\Gamma$, we have that

$$\Gamma(\phi(x_n)) = \psi(x_n).$$

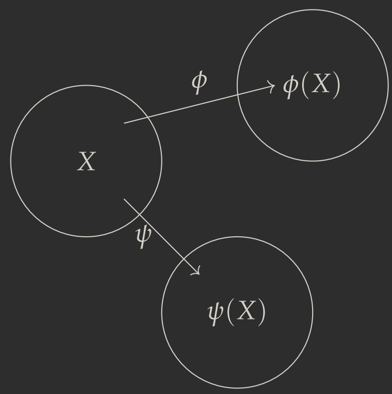


Figure 24.1: Relation of the 2 completions of a metric space.

If $\psi(x_n) \to z_0 \in \psi(X)$, then we must have

$$\Gamma(y_0) = z_0,$$

and in particular $z_0$ is a limit point of $\psi(X)$. This forces limits point of $\phi(X)$ to also be limit points of $\psi(X)$, and interior to interior. Thus the two completions are isomorphic.

## 24.2 *Banach Contractive Mapping Theorem*

QUESTION: Does there exist a function $f \in C[0,1]$ such that

$$f(x) = e^x + \int_0^x \frac{\sin t}{2} f(t)\, dt \quad ? \tag{24.1}$$

Let $\Gamma : C[0,1] \to C[0,1]$ such that

$$\Gamma(f)(x) = e^x + \int_0^x \frac{\sin t}{2} f(t)\, dt.$$

Then $f_0$ is a solution to Equation (24.1) iff $\Gamma(f_0) = f_0$.

This is known as an **integral transform**.

**Definition 59 (Fixed Point)**

*Given $(X, d)$, $\Gamma : X \to X$, we say that $x_0$ is a fixed point of $\Gamma$ if $\Gamma(x_0) = x_0$.*

# 25 Lecture 25 Nov 09th

## 25.1 Banach Contractive Mapping Theorem (Continued)

**Definition 60 (Lipschitz)**

*A function $f : (X, d_X) \to (Y, d_Y)$ is said to be Lipschitz if there exists $\alpha \geq 0$ such that $\forall x_1, x_2 \in X$,*

$$d_Y(f(x_1), f(x_2)) \leq \alpha d_X(x_1, x_2)$$

**Definition 61 (Contraction)**

*A function $f : X \to Y$ is called a* ***contraction*** *if there exists $0 \leq k < 1$ with*

$$d_Y(f(x_1), f(x_2)) \leq k d_X(x_1, x_2)$$

*for all $x_1, x_2 \in X$.*

**Note**

*Notice that a Lipschitz function is uniformly continuous: choose $\delta = \frac{\varepsilon}{\alpha}$.*

**Exercise 25.1.1**

*Prove that if $f : [a, b] \to \mathbb{R}$ and $f'$ is continuous, then by the* ***Extreme Value Theorem*** *and the* ***Mean Value Theorem****, $f$ is Lipschitz.*

**Theorem 60 (Banach Contractive Mapping Theorem)**

*Assume that $(X, d)$ is complete. If $\Gamma : X \to X$ is contractive, then there exists a unique $x_0 \in X$ such that $\Gamma(x_0) = x_0$.*

**Proof**

Pick $x_1 \in X$. Then, let

$$x_2 = \Gamma(x_1),\ x_3 = \Gamma(x_2),\ \ldots,\ x_{n+1} = \Gamma(x_n),\ \ldots.$$

**Claim**: $\{x_n\}$ is Cauchy[1]

Let $k \in \mathbb{R}$ such that $0 < k < 1$, so that we have

$$d\left(\Gamma(x), \Gamma(y)\right) \leq kd(x, y)$$

for any $x, y \in X$. Then

$$\begin{aligned}
d(x_3, x_2) &= d(\Gamma(x_2), \Gamma(x_1)) \leq kd(x_2, x_1) \\
d(x_4, x_3) &= d(\Gamma(x_3), \Gamma(x_1)) \leq kd(x_3, x_2) \leq k^2 d(x_2, x_1) \\
&\vdots \\
d(x_{n+1}, x_n) &= d(\Gamma(x_{n+1}), \Gamma(x_n)) \leq k^{n-1} d(x_2, x_1) \\
&\vdots
\end{aligned}$$

Also, notice that if $m > n$, then

$$\begin{aligned}
d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \ldots + d(x_{n+1}, x_n) \\
&\leq k^{m-2} d(x_2, x_1) + k^{m-3} d(x_2, x_1) + \ldots + k^{n-1} d(x_2, x_1) \\
&= \sum_{j=n-1}^{m-2} k^j d(x_2, x_1) = \frac{k^{n-1}}{1-k} d(x_2, x_1).
\end{aligned}$$

Since $k^{n-1} \to 0$, we have that $\{x_n\}$ is Cauchy. Since $(X, d)$ is complete, $\exists x_0 \in X$ such that $x_n \to x_0$.

In particular, we have that $x_{n+1} \to x_0$, i.e. $\Gamma(x_n) \to x_0$. Since $\Gamma$ is continuous, we mst have that $\Gamma(x_n) \to \Gamma(x_0)$. Therefore $\Gamma(x_0) = x_0$ as required.

**(Uniqueness)** Suppose there exists another point $y_0 \in X$ such

[1] This will CTP since $(X, d)$ is complete, i.e. it will give us a limit point at which $\Gamma$ must converge to, and thus forcing its iteration to be terminated at the limit point due to $\Gamma$ being contractive.

that $\Gamma(y_0) = y_0$. Then

$$d(x_0, y_0) = d(\Gamma(x_0), \Gamma(y_0)) \leq kd(x_0, y_0),$$

which implies that $d(x_0, y_0) = 0$. □

**Example 25.1.1**

Show that the equation

$$f_0(x) = e^x + \int_0^x \frac{\sin t}{2} f_0(t)\, dt$$

has a unique solution in $C[0, 1]$.

**Solution**

Define $\Gamma : C[0, 1] \to C[0, 1]$ by

$$\Gamma(f)(x) = e^x + \int_0^x \frac{\sin t}{2} f(t)\, dt.$$

Let $f, g \in C[0, 1]$. We have that

$$\begin{aligned}
|\Gamma(f)(x) - \Gamma(g)(x)| &= \left| \int_0^x \frac{\sin t}{2} f(t)\, dt - \int_0^x \frac{\sin t}{2} g(t)\, dt \right| \\
&= \left| \int_0^x \frac{\sin t}{2} \left( f(t) - g(t) \right) dt \right| \\
&\leq \int_0^x \left| \frac{\sin t}{2} \right| |f(t) - g(t)|\, dt \\
&\leq \|f - g\|_\infty \int_0^1 \frac{1}{2}\, dt \\
&= \frac{1}{2} \|f - g\|_\infty
\end{aligned}$$

Thus $\|\Gamma(f) - \Gamma(g)\|_\infty \leq \frac{1}{2} |f - g|_\infty$. Thus $\Gamma$ is contractive. By Theorem 60, the unique fixed point is the solution.

**Example 25.1.2**

Show that the equation

$$f(x) = x + \int_0^x t^2 f(t)\, dt \tag{25.1}$$

has a unique solution.

**Solution**

Let $\Gamma(f)(x) = x + \int_0^x t^2 f(t)\,dt$. Then

$$\begin{aligned} |\Gamma(f)(x) - \Gamma(g)(x)| &= \leq \int_0^1 t^2 \left\| f - g \right\|_\infty dt \\ &= \frac{1}{3} \left\| f - g \right\|_\infty . \end{aligned}$$

By the Banach Contractive Mapping Theorem, Equation (25.1) has a unique solution. In particular,

$$\begin{aligned} f_1(x) &= x \\ f_2(x) &= \Gamma(f_1)(x) = x + \int_0^x t^2 t_1(t)\,dt \\ &= x + \int_0^x t^3\,dt = x + \frac{1}{4}x^4 \\ f_3(x) &= \Gamma(f_2)(x) = x + \int_0^x t^2 \left( t + \frac{1}{4}t^4 \right) dt \\ &= x + \int_0^x t^3 + \frac{1}{4}t^6\,dt = x + \frac{1}{4}x^4 + \frac{1}{4 \cdot 7}x^7 \\ &\vdots \\ f_n(x) &= \frac{x}{1} + \frac{x^4}{4} + \frac{x^7}{4 \cdot 7} + \ldots + \frac{x^{3n-2}}{4 \cdot 7 \cdot \ldots \cdot (3n-2)} \end{aligned}$$

and so the limit is

$$f_0(x) = \sum_{k=1}^{\infty} \frac{x^{3k-2}}{4 \cdot 7 \cdot \ldots \cdot (3k-2)}.$$

**Example 25.1.3 (Other Applications)**

1. Newton's Method.
2. (**Picard's Theorem**) Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be Lipschitz in $\mathbb{R}$, i.e. $\exists \alpha \geq 0$ such that

$$|f(t, y_1) - f(t, y_2)| \leq \alpha \left|_1 - y_2\right|$$

for any $y_1, y_2 \in \mathbb{R}$. If $y_0 \in \mathbb{R}$, then there exists a unique $\phi \in C[a, b]$ such that

$$\phi'(t) = f(t, \phi(t))$$

for all $t \in (a, b)$ with $\phi(a) = y_0$.

## 25.2 *Baire Category Theorem*

**Example 25.2.1 (Dirchlet Function)**

Consider the function

$$f(x) = \begin{cases} 0 & x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & x = 0 \\ \frac{1}{m} & x \in \mathbb{Q} \end{cases}$$

The function $g$ is continuous at each $x \in \mathbb{R} \setminus \mathbb{Q}$, and discontinuous otherwise.

Question: Does there exist a function function $f$ such that $f$ is continuous on $\mathbb{Q}$ but not on $\mathbb{R} \setminus \mathbb{Q}$? **No!**

However, to prove that there is need no such function, we need more machinery. In particular, the set of discontinuities of a function $f : (X, d) \to \mathbb{R}$ has a particular topological nature.

**Definition 62 (Points of Discontinuity)**

*Let $f : X \to \mathbb{R}$. For each $n \in \mathbb{N}$, the **points of discontinuity** is a set defined as*

$$D_N(f) = \left\{ x_0 \in X : \forall \delta > 0 \; \exists x_1, y_1 \in B(x_0, \delta) \;\; |f(x_1) - f(y_1)| \geq \frac{1}{n} \right\}.$$

**Note**

1. *For each $n \in \mathbb{N}$, $D_n$ is closed.*
2. *$f$ is continuous at $x_0 \iff x_0 \notin \bigcap_{n=1}^{\infty} D_n$.*

**Remark**

*Recall the definition of an $F_\sigma$-set from the midterm (definition also provided in next lecture).*

*The set*

$$D(f) = \{x_0 \in X \mid f \text{ is discontinuous at } x_0\} = \bigcap_{n=1}^{\infty} D_n(f)$$

*is an $F_\sigma$-set.*

A natural question to ask is:

QUESTION: Is $\mathbb{R} \setminus \mathbb{Q}$ an $F_\sigma$-set?

# 26 Lecture 26 Nov 12th

## 26.1 Baire Category Theorem (Continued)

**Definition 63 ($F_\sigma$ Sets)**

*Let $(X, d)$ be a metric space. We say that $A \subseteq X$ is $F_\sigma$ if there exists a sequence $\{F_n\}_{n=1}^\infty$ of closed sets with*

$$A = \bigcup_{n=1}^\infty F_n.$$

**Definition 64 ($G_\delta$ Sets)**

*Let $(X, d)$ be a metric space. We say that $A \subseteq X$ is $G_\delta$ if there exists a sequence $\{U_n\}_{n=1}^\infty$ of open sets such that*

$$A = \bigcap_{n=1}^\infty U_n.$$

**Example 26.1.1**

The interval $[0, 1) \subset \mathbb{R}$ is $G_\delta$, since

$$[0, 1) = \bigcap_{n=1}^\infty \left( \frac{1}{n}, 1 \right)$$

**Remark**

*$A$ is $F_\sigma$ iff $A^C$ is $G_\delta$.*

Recall the definition of a dense set. We have the following complementary definition.

**Definition 65 (Nowhere Dense)**

*Given a metric space $(X, d)$, we say that $A \subseteq X$ is **nowhere dense** if $\overline{A}^\circ = \emptyset$.*

**Remark**

*The above definition is equivalent to saying that $\overline{A}^C$ is dense.*

**Definition 66 (First Category)**

*We say that a set $A$ is of **first category** if*

$$A = \bigcup_{n=1}^{\infty} A_n$$

*where each $A_n$ is nowhere dense.*

**Definition 67 (Second Category)**

*We say that $A$ is of **second category** is $A$ is not of first category.*

**Remark**

*We colloquially refer to a set of first category as being **topologically thin**, and a set of second category as being **topologically thick**.*

**Definition 68 (Residual)**

*We say that $A \subseteq (X, d)$ is a **residual** in $X$ if $A^C$ is of first category.*

**Theorem 61 (Set of Points of Discontinuity is $F_\sigma$)**

*Let $f : (X, d_X) \to (Y, d_Y)$. Then for each $n \in \mathbb{N}$, $D_N(f)$ is closed in $X$.*

*Moreover,*

$$D(f) = \bigcup_{n=1}^{\infty} D_N(f).$$

*In particular, $D(f)$ is $F_\sigma$.*

**Exercise 26.1.1**

*Prove Theorem 61.*

**Example 26.1.2**

If $F \subset (X,d)$ is closed, then $f$ is $G_\delta$. In particular, notice that

$$F = \bigcap_{n=1}^{\infty} \left( \bigcup_{x \in F} B\left(x, \frac{1}{n}\right) \right),$$

where we note that each of the $B\left(x, \frac{1}{n}\right)$ is $F_\delta$.

**Theorem 62 (Baire Category Theorem I)**

*Let $(X,d)$ be complete. Let $\{U_n\}_{n=1}^{\infty}$ be a countable collection of dense open sets. Then*[1]

$$\bigcap_{n=1}^{\infty} U_n \text{ is dense in } X.$$

*In particular, it is not empty.*

[1] Note that we have ourselves a dense $G_\delta$ set.

**Proof**

Assume that $\{U_n\}_{n=1}^{\infty}$ is a sequence of open and dense sets. Let $W \subset X$ be open and non-empty. Since $U_1$ is dense, we have that $W \cap U_1 \neq \emptyset$. Then $\exists x_1 \in W \cap U_1$ such that $\exists 0 < r_1 \leq 1$ so that

$$B(x_1, r_1) \subset B(x_1, r_1) \subset W \cap U_1.$$

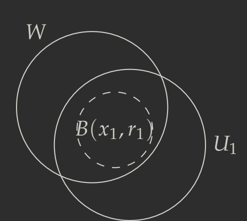


Figure 26.1: Visualization of proof for Baire Category Theorem I

Similarly,

we can find $x_2 \in X$ such that for some $0 < r_2 \leq \frac{1}{2}$,

$$B(x_2, r_2) \subset B[x_2, r] \subset B(x_1, r_1) \cap U_2.$$

We can proceed recursively and find, for $n \in \mathbb{N}$, an $x_n \in X$ with

$0 < r_n \leq \frac{1}{n}$ such that

$$B(x_n, r_n) \subset B[x_n, r_n] \subset B(x_{n-1}, r_{n-1}) \cap U_n.$$

Now since $(X, d)$ is complete, $\{\mathrm{diam}(B[x_n, r_n])\} = \{r_n\}$ is a decreasing sequence such that $r_n \to 0$, by Cantor's Intersection Principle,

$$\exists x_0 \in \bigcap_{n=1}^{\infty} B[x_n, r_n].$$

Then by this construction, we must have $x_0 \in B[x_1, r_1] \subset W \cap U_1$, and $x_0 \in B[x_n, r_n] \subset U_n$ for each $n \in \mathbb{N}$. Thus

$$x_0 \in W \cap \left( \bigcap_{n=1}^{\infty} U_n \right).$$

□

Note that the statement does not hold if we have an uncountable collection of dense open sets.

**Example 26.1.3**

Consider $U_x = \mathbb{R} \setminus \{x\}$, where $x \in \mathbb{R}$. This is clearly a dense and open set. Notice, however, that

$$\bigcap_{x \in \mathbb{R}} U_x = \emptyset.$$

**Remark**

*Theorem 62 shows that given a countable sequence $\{U_n\}_{n=1}^{\infty}$ of open dense sets of $X$, the countable intersection of these sets, $\bigcap_{n=1}^{\infty} U_n$, is a dense $G_\delta$.*

**Theorem 63 (★ Baire Category Theorem II)**

*If $(X, d)$ is complete, then $X$ is of second category.*

**Proof**

Suppose to the contrary that $X = \bigcup_{n=1}^{\infty} A_n$ where each $A_n$ is nowhere dense. Note that by definition of being nowhere dense, we have that $A_n = \overline{A_n}$. Let $U_n = \overline{A_n}^C$, which would then be open and dense. However, by De Morgan's Laws, we have that

$$\left( \bigcap_{n=1}^{\infty} U_n \right)^C = \bigcup_{n=1}^{\infty} U_n^C = \bigcup_{n=1}^{\infty} \overline{A_n} = X$$

and so

$$\bigcup_{n=1}^{\infty} U_n = \emptyset,$$

which is impossible by Theorem 62. □

**Example 26.1.4**

$\mathbb{R}$ and $\mathbb{R} \setminus \mathbb{Q}$ are of second category. In fact, $\mathbb{R} \setminus \mathbb{Q}$ is a residual, since $\mathbb{Q}$ is of first category.

Question: Is

$$Q = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \left( r_n - \frac{1}{2^{k+n}}, r_n + \frac{1}{2^{k+n}} \right),$$

where $Q = \{r_1, r_2, \ldots\}$, $\mathbb{Q}$? No. Notice that this is fairly close, but it is not.[2]

[2] It should be $\mathbb{R}$?

**Corollary 64 ($\mathbb{Q}$ is not $G_\delta$)**

*$\mathbb{Q}$ is not a $G_\delta$ set.*

**Proof**

Suppose to the contrary that $\mathbb{Q}$ is $G_\delta$, i.e. there exists a countable sequence of open sets $\{U_n\}$ such that

$$\mathbb{Q} = \bigcap_{n=1}^{\infty} U_n.$$

Let $F_n = U_n^C$. Since $\mathbb{Q}$ is dense, it follows that each of the $U_n$'s is also dense. Thus $F_n$ is nowhere dense and closed.

Let $\mathbb{Q} = \{r_1, r_2, \ldots\}$, an enumeration on $\mathbb{Q}$, and $S_n = F_n \cup \{r_n\}$. Then $S_n$ is closed and nowhere dense. However, we would then have

$$\mathbb{R} = \bigcup_{n=1}^{\infty} S_n,$$

which contradicts the fact that $\mathbb{R}$ is of second category. □

Consequently:

**Corollary 65 (There are no Functions Discontinuous on all Irrational Numbers)**

*There is no function $f : \mathbb{R} \to \mathbb{R}$ for which $D(f) = \mathbb{R} \setminus \mathbb{Q}$.*

We are now able to show that for a sequence $\{f_n\} \subset C[a, b]$ that converges pointwise, the limit function must be continuous at each point on a residual set. We require the following notion:

**Definition 69 (Uniformly Convergent Sequence of Functions on a Point)**

*We say that a sequence of functions $\{f_n\}$ where,*

$$f_n : (X, d_X) \to (Y, d_Y),$$

***converges uniformly*** *at $x_0 \in X$ if*

$$\begin{gathered} \forall \varepsilon > 0 \; \exists \delta > 0 \; \exists N \in \mathbb{N} \; \forall n, m \geq N \\ x \in B(x_0, \delta) \implies d_Y(f_n(x), f_m(x)) < \varepsilon. \end{gathered}$$

The proof of the following theorem is left as an exercise.

**Theorem 66 (Limit of Sequence of Continuous Functions that Converges Pointwise is Continuous)**

*Let $(X, d_X)$ and $)Y, d_Y)$ be metric spaces. Let $\{f_n : X \to Y\}$ be a sequence of functions that converges pointwise on $X$ to $f_0$. Assume that $\{f_n\}$ converges uniformly at $x_0 \in X$. If each $f_n$ is continuous at $x_0$, then so is $f_0$.*

# 27 Lecture 27 Nov 14th

## 27.1 Baire Category Theorem (Continued 2)

**Theorem 67 (Uniform Convergence of A Sequence of Continuous Functions that Converges Pointwise)**

*Let $f_n : (a,b) \to \mathbb{R}$ be a sequence of continuous functions that converges pointwise to $f(x)$. Then there exists an $x_0 \in (a,b)$ such that $f_n \to f$ uniformly at $x_0$.*

**Proof**

Assume that $f_n \to f_0$ on $(a,b)$, pointwise.

**Claim** There exists $[\alpha_1, \beta_1] \subset (a,b)$ and $N_1 \in \mathbb{N}$ such that if $x \in [\alpha_1, \beta_1]$ and $n, m \geq N_1$, then $|f_n(x) - f_m(x)| \leq 1$.

Suppose not. Then $\exists t_1 \in (a,b)$ and $n_1, m_1 \in \mathbb{N}$ such that $|f_{n_1}(t_1) - f_{m_1}(t_1)| > 1$. Since $f_{n_1} - f_{m_1}$ is continuous, there exists an open interval $I_1 \subsetneq \bar{I}_1 \subsetneq (a,b)$ such that $|f_{n_1}(x) - f_{m_1}(x)| > 1$ for all $x \in I_1$.

Similarly, $\exists t_2 \in I_1$ and $n_2, m_2 \geq \max\{n_1, m_1\}$ such that $|f_{n_2}(t_2) - f_{m_2}(t_2)| > 1$. Again, since $f_{n_2} - f_{m_2}$ is continuous, there exists an open interval $I_2 \subsetneq \bar{I}_2 \subsetneq I_1$ such that $|f_{n_2}(x) - f_{m_2}(x)| > 1$ for all $x \in I_2$.

Recursively so, we get a sequence $\{I_n\}$ of open interval with $I_{n+1} \subset \bar{I}_{n+1} \subset \bar{I}_k$, and two sequence of integers $\{n_k\}$ and $\{m_k\}$, with $n_{k+1}, m_{k+1} \geq \max\{n_k, m_k\}$ and if $x \in I_k$, we have $|f_{n_k}(x) - f_{m_k}(x)| > 1$.

Then, by the Nested Interval Theorem, we have

$$\bigcap_{k=1}^{\infty} \bar{I}_k \neq \emptyset.$$

Let $x^* \in \bigcap_{k=1}^{\infty} \bar{I}_k$. Then by construction, we have that for any $k$, $\left|f_{n_k}(x^*) - f_{m_k}(x^*)\right| > 1$. However, since $\{f_n\}$ converges pointwise, $\{f_n(x^*)\}$ is Cauchy and hence we have a contradiction. This proves the claim $\dashv$.

In a similar manner, we can find a sequence $\{[\alpha_k, \beta_k]\}$ of closed sets, where $\alpha_k < \beta_k$, such that

$$(\alpha_{k+1}, \beta_{k+1}) \subseteq [\alpha_{k+1}, \beta_{k+1}] \subseteq (\alpha_k, \beta_k) \subseteq \ldots \subseteq (a, b),$$

and a sequence

$$N_1 < N_2 < \ldots < N_k < \ldots,$$

such that if $x \in [\alpha_k, \beta_k]$ and $n, m \geq N_k$, then $|f_n(x) - f_m(x)| \leq \frac{1}{k}$. Then, once again, by the Nested Interval Theorem, let $x_0 \in \bigcap_{k=1}^{\infty} [\alpha_k, \beta_k]$. Let $\varepsilon > 0$. Now if $\frac{1}{k} < \varepsilon$, then if $n, m \geq N_k$, then we have

$$|f_n(x) - f_m(x)| \leq \frac{1}{k} < \varepsilon.$$

Since $x_0 \in \bigcap_{k=1}^{\infty} [\alpha_k, \beta_k]$ and $\alpha_k < \beta_k$, we can choose $\delta = \min\{\beta_k - \alpha_k : k \in \mathbb{N} \setminus \{0\}\} > 0$, so that $(x_0 - \delta, x_0 + \delta) \subset (\alpha_k, \beta_k)$, then for any $x \in (x_0 - \delta, x_0 + \delta)$, we have

$$|f_n(x) - f_m(x)| < \varepsilon.$$

□

**Corollary 68 (Continuity of the Limit of a Sequence of Pointwise Convergent Functions on a Residual Set)**

*Let $\{f_n\} \subset C[a, b]$ be such that $f_n \to f_0$ pointwise on $[a, b]$. Then there exists a residual set $A \subset [a, b]$ such that $f_0(x)$ is continuous at each $x \in A$.*

**Proof**

Theorem 67 shows that the set $A$ of which $f_0$ is continuous on is dense in $[a, b]$. However, from XXX that $D(f_0)$ is $F_\sigma$, and so $A$ is a dense $G_\delta$.

**Remark**

*Thus we have that $D(f_0)$ is a nowhere dense $F_\sigma$, i.e. it is of first category.*

**Corollary 69 (Derivative of a Function is Continuous on a dense $G_\delta$ set in $\mathbb{R}$)**

*Assume that $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Then $f'(x)$ is continuous for every point on a dense $G_\delta$-subset of $\mathbb{R}$.*

**Proof**

Using notions from the first principles of calculus, notice that $f'(x)$ is a pointwise limit of the sequence of continuous functions

$$\left\{ \frac{f\left(x + \frac{1}{n}\right) - f(x)}{\frac{1}{n}} \right\}.$$

$\square$

## 27.2 Compactness

In this section, we study 3 important properties of a topological space, namely:

- compactness;
- sequential compactness; and
- the Bolzano-Weierstrass Property.

We shall see that, in fact, the three properties are equivalent.

**Definition 70 (Cover)**

*Given $(X, d)$ a metric space, an (open)* ***cover*** *of $X$ is a collection $\{U_\alpha\}_{\alpha \in I}$*

*of open sets with*

$$X = \bigcup_{\alpha \in I} U_\alpha.$$

*A* ***subcover*** *is a subset (or subcollection)* $\{U_\alpha\}_{\alpha \in J \subset I}$ *such that*

$$X = \bigcup_{\alpha \in J} U_\alpha.$$

*If* $A \subset X$*, then we say that* $\{U_\alpha\}_{\alpha \in I}$ ***covers*** $A$ *if* $A \subset \bigcup_{\alpha \in I} U_\alpha$*, or, equivalently, if* $\{U_\alpha \cap A\}_{\alpha \in I}$ *is a cover of* $(A, d_A)$*.*

---

**Definition 71 (Compact)**

*We say that* $(X, d)$ *is* ***compact*** *iff each cover of* $X$*,* $\{U_\alpha\}_{\alpha \in I}$*, has a finite subcover.*

*We say that* $A \subset (X, d)$ *is* ***compact*** *if every cover* $\{U_\alpha\}_{\alpha \in I}$ *of* $A$ *has a finite subcover (or, equivalently, if* $(A, d_A)$ *is compact).*

---

From earlier courses in Calculus, recall:

---

**Theorem 70 (Heine-Borel Theorem)**

*$A \subset \mathbb{R}^n$ is compact iff $A$ is closed and bounded.*

---

**Example 27.2.1**

$[0, 1] \subset \mathbb{R}$ is compact, but $(0, 1) \subset \mathbb{R}$ is not compact.

However, the Heine-Borel Theorem is not true for arbitrary metric spaces.

**Example 27.2.2 (★ )**

Let

$$A = \{\{x_n\} \in \ell_\infty \mid \|x_n\|_\infty \leq 1\}.$$

It is clear that $A$ is closed and bounded. However, consider $U_{\{x_n\}} =$

$B\left(\{x_n\}, \frac{1}{2}\right)$. It is then clear that

$$A \subset \bigcup_{\{x_n\} \in A} U_{\{x_n\}}.$$

Let $S = \{\{x_n\} \mid x_n = 1 \vee x_n = 0\}$, which is infinite. Then we notice that $\left| S \cap B\left(\{x_n\}, \frac{1}{2}\right)\right| \leq 1$, showing to us that we cannot find a finite subcover for $S$ itself is infinite.

However, we do have the following implication.

**Proposition 71 (Compact Spaces are Closed and Bounded)**

*If $A \subset (X, d)$ is compact, then $A$ is closed and bounded.*

**Proof**

Suppose $A$ is not closed. Then $\exists x_0 \in \mathrm{bdy}(A) \setminus A$. Let

$$U_n = \left( B\left[x_0, \frac{1}{n}\right]\right)^C.$$

Since $x_0 \notin A$, we have that $A \subset \bigcup_{n=1}^{\infty} U_n$. However, $\{U_n\}_{n=1}^{\infty}$ has no finite subcover. Otherwise, if it does have some finite subcover, say $\{U_n\}_{n=1}^{N}$, then for any $n_0 > N$, we would have that

$$\left( B\left[x_0, \frac{1}{n_0}\right]\right) \supsetneq \bigcup_{n=1}^{N} U_n,$$

and so $\exists x_1 \in B\left[x_0, \frac{1}{n_0}\right]$ such that $x_1 \in A$ but $x_0 \notin \bigcup_{n=1}^{N} U_n$. This contradicts the assumption that a subcover exists. But $A$ must have some subcover for we assumed that $A$ is compact. Therefore $A$ must be closed.

For boundedness, let $x_0 \in X$. Then $\{B(x_0, n)\}_{n=1}^{\infty}$ is an open cover of $A$. Since $A$ is compact, $\{B(x_0, n)\}_{n=1}^{\infty}$ must have some finite subcover $\{B(x_0, n_1), B(x_0, n_2), \ldots, B(x_0, n_k)\}$. WMA $n_1 < n_2 < \ldots < n_k$, for we may rearrange the radii. It follows that $A \subset B(x_0, n_k)$, and so $A$ is bounded as required. $\square$

# 28 Lecture 28 Nov 16th

## 28.1 Compactness (Continued)

We also have the following relation between compact sets and their closed subsets.

**Proposition 72 (Closed Subsets of Compact Sets are Compact)**

*If $(X, d)$ is compact and $A$ is closed, then $A$ is compact.*

**Proof**

Let $\{U_\alpha\}_{\alpha \in I}$ be a cover of $A$. Then

$$\{U_\alpha\}_{\alpha \in I} \cup A^C \tag{$*$}$$

is a cover of $X$. Since $X$ is compact, Equation $(*)$ has a finite subcover $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_k}, A^C\}$ such that

$$\left( \bigcup_{i=1}^{k} U_{\alpha_i} \right) \cup A^C = X.$$

Since $A \subset X$ and $A \cap A^C = \emptyset$, we must have

$$A \subset \bigcup_{i=1}^{k} U_{\alpha_i}.$$

$\square$

We have the following 2 variants of compactness:

### Definition 72 (Sequential Compactness)

*A set $A \subset (X, d)$ is said to be **sequentially compact** if every sequence*[1] *$\{x_n\} \subset A$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x_0 \in A$.*

[1] Beware that this is not the same as completeness.

### Definition 73 (Bolzano-Weierstrass Property (BWP))

*Let $(X, d)$ be a metric space. We say that X has the **Bolzano-Weierstrass Property (BWP)** if every infinite subset of X has a limit point in the subset.*

**Exercise 28.1.1**

*Show that for $A \subset \mathbb{R}^n$, $A$ is compact iff $A$ is sequentially compact.*

**Proof**

$(\implies)$ Suppose $A$ is not sequentially compact. Then

$$\exists \{x_n\} \subset A \ \forall \{x_{n_k}\} \subset \{x_n\} \ \forall x_0 \in A \ \ x_{n_k} \not\to x_0.$$

Let this $\{x_n\} = \{x_1, x_2, \ldots, x_n, \ldots\}$. Let

$$U_n = A \setminus \{x_j \mid j \geq n\}.$$

Then it is clear that

$$\bigcup_{n=1}^{\infty} U_n = A,$$

i.e. $\{U_n\}$ is a cover of $A$. Since $A$ is compact, $\{U_n\}$ has a finite subcover, say $\{U_{n_1}, U_{n_2}, \ldots, U_{n_k}\}$. WMA $n_1 < n_2 < \ldots < n_k$. Then

$$A = \bigcup_{m=1}^{k} U_{n_m} = A \setminus \{x_j \mid j \geq n_k\}.$$

But that is impossible since $x_{n_k+1} \notin \bigcup_{m=1}^{k} U_{n_k}$. Thus $A$ must be sequentially compact.

$(\impliedby)$ Suppose $A$ is sequentially compact. Then

$$\forall \{x_n\} \subset A \ \exists \{x_{n_k}\} \subset \{x_n\} \ \exists x_0 \in A \ \ x_{n_k} \to x_0.$$

Let $\{U_\alpha\}_{\alpha \in I}$ be a cover of $A$. **Yet to figure out where to go from**

**here**. Tried looking into trying to construct a finite subcover using the convergent subsequence, but that actually leads to nowhere.

**Theorem 73 (Sequential Compactness is Equivalent to BWP)**

*Let $(X, d)$ be a metric space. TFAE:*

1. *$(X, d)$ is sequentially compact.*
2. *$(X, d)$ has the BWP.*

**Proof**

$(\implies)$ Let $(X, d)$ be sequentially compact. Let $A \subset (X, d)$ be infinite. By sequential compactness, every sequence $\{x_n\} \subset A$ has a convergent subsequence $\{x_{n_k}\}$, such that $x_{n_k} \to x_0 \in A$. $\dashv$

$(\impliedby)$ Suppose $(X, d)$ has the BWP. Let $\{x_n\}$ be a sequence in $X$. If $\{x_n\}$ is not infinite (as a set), then it has a subsequence $\{x_{n_k}\}$ such that $x_{n_{k_1}} = x_{n_{k_2}}$ for all $k_1, k_2$, which is convergent. WMA $\{x_n\}$ is infinite (as a set). By the BWP, $\{x_n\}$ (as a set) has a limit point $x_0 \in \{x_n\}$. Then for $k \in \mathbb{N} \setminus \{0\}$, let

$$x_{n_k} \in B\left(x_0, \frac{1}{k}\right).$$

Clearly then $x_{n_k} \to x_0$, and $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. $\square$

**Definition 74 (Finite Intersection Property (FIP))**

*A collection $\{A_\alpha\}_{\alpha \in I}$ of subsets of $X$ is said to have the* ***finite intersection property (FIP)*** *if*

$$\bigcap_{i=1}^{n} A_n \neq \emptyset$$

*for all finite subcollections $\{A_1, \ldots, A_n\}$.*

**Example 28.1.1**

Let $F_n = [n, \infty)$. Then $\{F_n\}_{n=1}^{\infty}$ has the FIP, but $\bigcap_{n=1}^{\infty} F_n = \emptyset$.

The following theorem can be seen as an upgrade to Cantor's Intersection Principle for compact metric spaces: instead of allowing only a countably infinite intersection, we can now take an arbitrary number of intersections.

**Theorem 74 (FIP and Compactness)**

*Let $(X, d)$ be a metric space. TFAE:*

1. *$(X, d)$ is compact.*
2. *If $\{F_\alpha\}_{\alpha \in I}$ is a non-empty collection of closed sets with the FIP, then*

$$\bigcap_{\alpha \in I} F_\alpha \neq \emptyset.$$

**Remark**

*As compared to Cantor's Intersection Principle, we do not need the notion of a diameter of a set to achieve this result in a compact set.*

**Proof**

(1) $\implies$ (2) Suppose to the contrary that for a non-empty collection $\{F_\alpha\}_{\alpha \in I}$ of closed sets with the FIP, we have

$$\bigcap_{\alpha \in I} F_\alpha = \emptyset.$$

Let $U_\alpha = F_\alpha^C$. Then by De Morgan's Laws, we have $X = \bigcup_{\alpha \in I} U_\alpha$. Since $(X, d)$ is compact, $\exists \{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ such that

$$\bigcup_{i=1}^{n} U_{\alpha_i} = X.$$

But that implies that

$$\emptyset = X^C = \left( \bigcup_{i=1}^{n} U_{\alpha_i} \right)^C = \bigcap_{i=1}^{n} F_{\alpha_i},$$

contradicting FIP.

(2) $\implies$ (1) Suppose to the contrary that $\{U_\alpha\}_{\alpha \in I}$, a cover of $X$,

has no finite subcover. Then $\forall \{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$, we must have

$$X \setminus \bigcup_{i=1}^{n} U_{\alpha_i} \neq \emptyset,$$

i.e., by De Morgan's Laws, $\bigcap_{i=1}^{n} U_{\alpha_i}^C \neq \emptyset$. Then $\{F_\alpha\}_{\alpha \in I}$, where $F_\alpha = U_\alpha^C$, is a non-empty collection of closed sets with the FIP (by our argument), but via De Morgan's Laws, we have

$$\bigcap_{\alpha \in I} F_\alpha = \emptyset,$$

contradicting our assumption. □

**Corollary 75 (Generalized Nested Interval Theorem for Compact Metric Spaces)**

*Let $(X, d)$ be compact and $\{F_N\}_{n=1}^{\infty}$ be a sequence of non-empty closed sets such that $F_{n+1} \subset F_n$. Then*

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

**Corollary 76 (Compact Metric Spaces are Complete)**

*If $(X, d)$ is compact, then $(X, d)$ is complete.*

**Note**

Recall *the definition for compactness, in which we may then have the following notion: for a compact set $(X, d)$, for $\varepsilon > 0$, since $\{B(x, \varepsilon)\}_{x \in X}$ is an open cover of $X$, we know that there exists $x_1, \ldots, x_n \in X$ such that they form a finite subcover on $X$.*

$$X = \bigcup_{i=1}^{n} B(x_i, \varepsilon).$$

We use the same idea and make the following definition:

**Definition 75 ($\varepsilon$-net)**

*Given $A \subset (X, d)$ and $\varepsilon > 0$. An **$\varepsilon$-net** for $A$ is a set $\{x_\alpha\}_{\alpha \in I} \subset X$ such that*

$$A \subset \bigcup_{\alpha \in I} B(x_i, \varepsilon).$$

**Definition 76 (Totally Bounded)**

*We say that a subset $A \subset (X, d)$ is **totally bounded** if $A$ has a **finite** $\varepsilon$-net for every $\varepsilon > 0$.*

**Theorem 77 (Compact Sets are Totally Bounded)**

*If $(X, d)$ is compact, then $(X, d)$ is totally compact.*

**Proof**

The proof immediately follows from the definition of compactness, as discussed in Note 23.

Note that bounded and totally bounded are **not equivalent**.

**Example 28.1.2**

Let

$$S = \{\{x_n\} \in \ell_\infty \mid \|\{x_n\}\|_\infty \leq 1\}.$$

We have that $S$ is bounded, but it does not have a $\frac{1}{2}$-net.

**Proposition 78 (A Set is Totally Bounded iff Its Closure is Totally Bounded)**

*$A \subset (X, d)$ is totally bounded iff $\overline{A}$ is totally bounded.*

**Exercise 28.1.2**
*Prove Proposition 78.*

**Proof**

The ( $\impliedby$ ) direction is immediate, since $A \subset \overline{A}$. It suffices to show for ( $\implies$ ). Suppose $A$ is totally bounded. If $A$ is closed, then we are done, so WMA $A$ is open. Then $\mathrm{Lim}(A) \not\subseteq A$. Let $x_0 \in \mathrm{Lim}(A) \setminus A$. Since $x_0$ is a limit point, for any $\varepsilon > 0$, $B(x_0, \varepsilon) \cap A \neq \emptyset$. **Need to verify definition of an $\varepsilon$-net.**

# 29 Lecture 29 Nov 19th

## 29.1 Compactness (Continued 2)

**Theorem 79 (Compact Sets have BWP)**

*If $(X, d)$ is compact, then $(X, d)$ has the BWP.*

**Proof**

Suppose $S \subset X$ is infinite. Then we can obtain a sequence $\{x_n\} \subset S$ such that for $n \neq m$, $x_n \neq x_m$. Then, consider

$$F_n = \{x_n, x_{n+1}, \ldots\}.$$

We have that $F_{n+1} \subseteq F_n$ and we observe that $\{F_n\}$ has the FIP, i.e.

$$\exists x_0 \in \bigcap_{n=1}^{\infty} F_n.$$

Then for any $\varepsilon > 0$, for any $n \in \mathbb{N}$, we have that

$$B(x_0, \varepsilon) \subset F_n.$$

In fact, $B(x_0, \varepsilon) \cap \{x_n\} \neq \emptyset$ is also infinite. Thus $x_0 \in \mathrm{Lim}(S)$. $\square$

**Proposition 80 (Sequential Compactness $\implies$ Completeness and Total Boundedness)**

*If $(X, d)$ is sequentially compact, then $(X, d)$ is both complete and totally bounded.*

**Proof**

**Completeness** Let $\{x_n\} \subset X$ be Cauchy. Then by the assumption that $X$ is sequentially compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x_0 \in X$. Then by Theorem 47, $x_n \to x_0$. $\dashv$

**Totally Bounded** Suppose to the contrary that $X$ is not totally bounded, i.e. $\exists \varepsilon_0 > 0$ such that $X$ has no finite $\varepsilon_0$-net. Then we can find $x_1 \in X$ such that $B(x_1, \varepsilon_0) \neq X$, an $x_2 \in X \setminus B(x_1, \varepsilon_0)$, $x_3 \in X \setminus (B(x_1, \varepsilon_0) \cup B(x_2, \varepsilon_2))$, and so on. In other words, we can construct a sequence $\{x_n\} \subset X$ such that $d(x_n, x_m) > \varepsilon$ for all $n \neq m$. Then by construction, $\{x_n\}$ has no convergent subsequences, i.e. $X$ is not sequentially compact. $\square$

**Theorem 81 (Continuity Preserves Sequential Compactness)**

*If $(X, d)$ is sequentially compact and if $f : (X, d_X) \to (Y, d_Y)$ is continuous, then $f(X)$ is sequentially compact.*

**Proof**

Let $\{y_n\} \subset f(X)$. Consider $\{x_n\}$ such that $f(x_n) = y_n$. Since $X$ is sequentially compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with $x_{n_k} \to x_0$. Then by continuity,

$$y_{n_k} = f(x_{n_k}) \to f(x_0) = y_0.$$

$\square$

**Corollary 82 (Extreme Value Theorem)**

*If $(X, d)$ is sequentially compact and $f : X \to \mathbb{R}$ is continuous, then $\exists c, d \in X$ such that*

$$f(c) \leq f(x) \leq f(d)$$

*for all $x \in X$.*

**Proof**

By Theorem 81, $f(X)$ is sequentially compact in $\mathbb{R}$, and by Proposition 80, $f(X)$ is complete, and so by Heine-Borel, $f(X)$ is closed and bounded. Thus

$$\sup(f(X)), \inf(f(X)) \in f(X).$$

□

**Theorem 83 (Lesbesgue)**

*Let $(X, d)$ be sequentially compact. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $X$. Then $\exists \varepsilon > 0$ such that for every $0 < \delta < \varepsilon$, and every $x \in X$ such that for some $\alpha_0 \in I$*

$$B(x_0, \delta) \subset U_{\alpha_0}.$$

**Proof**

If $U_{\alpha_0} = X$, then any $\varepsilon > 0$ will work. WMA $U_\alpha \neq X$ for any $\alpha \in I$. Let $\phi : X \to \mathbb{R}$ be defined by

$$\phi(x) = \sup\left\{\delta > 0 : B(x, \delta) \subseteq U_{\alpha_0}, \alpha_0 \in I\right\}.$$

Since $\{U_\alpha\}_{\alpha \in I}$ is an open cover of $X$, every $x$ must be in one of the $U_\alpha$'s, and so the set

$$\{\delta > 0 : B(x, \delta) \subseteq U_{\alpha_0}, \alpha_0 \in I\}$$

is non-empty and $\phi(x) > 0$. Also, $\phi(x) < \infty$, since $X$ is bounded (as $X$ is sequentially compact) and $U_\alpha \neq X$ for any $\alpha \in I$.

Now for any $x, y \in X$, [1] we have that

$$\phi(x) \leq \phi(y) + d(x, y)$$

by the Triangle Inequality. Thus

$$\phi(x) - \phi(y) \leq d(x, y)$$

[1] **I should check in with the professor on how to show this**

and by symmetry we have

$$|\phi(x) - \phi(y)| \leq d(x, y).$$

Thus $\phi$ is Lipschitz, and so $\phi$ is uniformly continuous[2]. Then by the Extreme Value Theorem, $\exists \varepsilon > 0$ such that $\exists \varepsilon > 0$ such that $\phi(x) \geq \varepsilon$ for all $x \in X$. □

[2] see note on definition of Lipschitz.

**Note**

*The $\varepsilon$ in Lesbesgue's Theorem is also called a* ***Lesbesgue Number****.*

**Theorem 84 (Lesbesgue-Borel)**

*Let $(X, d)$ be a metric space. TFAE:*

1. *$(X, d)$ is compact.*
2. *$(X, d)$ has BWP.*
3. *$(X, d)$ is seqentially compact.*

**Proof**

We already have $(1) \implies (2)$ and $(2) \iff (3)$. It suffices to prove $(3) \implies (1)$. Let $\{U_\alpha\}_{\alpha \in I}$ be a cover of $X$. By Lesbesgue's Theorem, let $\varepsilon_0 > 0$, and fix $0 < \delta < \varepsilon_0$. Since $(X, d)$ is totally bounded (as it sequentially compact), there exists $\{x_1, \ldots, x_n\}$ with

$$X = \bigcup_{i=1}^{n} B(x_i, \delta).$$

Then for each $i$, we have that $B(x_i, \delta) \subset U_{\alpha_i}$ for some $\alpha_i \in I$. Then

$$X = \bigcup_{i=1}^{n} U_{\alpha_i}$$

is a finite subcover of the cover $\{U_\alpha\}_{\alpha \in I}$. □

### Theorem 85 (Compactness $\iff$ Completeness + Totally Bounded)

*Let $(X, d)$ be a metric space. TFAE:*

1. *$(X, d)$ is compact.*
2. *$(X, d)$ is complete and totally bounded.*

### Proof

By Theorem 84 and Proposition 80, we have $\boxed{(1) \implies (2)}$. Thus it suffices to show for $\boxed{(2) \implies (1)}$. Notice that we only need to show that $(X, d)$ is sequentially compact. Let $\{x_n\} \subset (X, d)$.

Since $(X, d)$ is totally bounded, $X$ can be covered by finitely many open balls of radius 1. Thus one such ball $S_1 = B(y_1, 1)$, for some $y_1 \in X$, contains infinitely many terms in $\{x_n\}$ [3].

[3] Note that sequences are infinitary by nature in our context.

Similarly, $X$ can be covered by finitelym any open balls of radius $\frac{1}{2}$, and we can pick one of these open balls $S_2 = B\left(y_2, \frac{1}{2}\right)$ which contains infinitely many terms in $\{x_n\} \cap S_1$.

Recursively, we may construct a sequence of open balls $\left\{S_k = B\left(y_k, \frac{1}{k}\right)\right\}$ with the property that each $S_{k+1}$ contains infinitely many terms is

$$\{x_n\} \cap \left( \bigcap_{i=1}^{k} S_i \right).$$

Note that

$$\mathrm{diam}(S_k) = \frac{2}{k} \to 0$$

as $k \to \infty$, and since can pick

$$n_1 < n_2 < \ldots < n_k < \ldots$$

such that

$$x_{n_{k+1}} \in \bigcap_{i=1}^{k} S_i.$$

WMA for some $N \in \mathbb{N}$, for any $k, m \geq N$, we ahve that $x_{n_k}, x_{n_m} \in S_N$, i.e.

$$\mathrm{diam}(x_{n_k}, x_{n_m}) \leq \mathrm{diam}(S_N).$$

Thus $\{x_{n_k}\} \subset \{x_n\}$ is Cauchy. Since $(X, d)$ is complete, $x_{n_k} \to x_0$, and therefore $X$ is sequentially compact by definition. $\square$

# 30 Lecture 30 Nov 21st

## 30.1 Compactness (Continued 3)

The proof of the following theorem was left as an exercise:

**Theorem 86 (Continuity Preserves Compactness)**

*If $(X, d_X)$ is compact and $f : (X, d_X) \to (Y, d_Y)$ is continuous, then $f(X)$ is compact in $Y$.*

**Proof**

The proof easily follows from Theorem 84 and Theorem 81.

□

## 30.2 Finite Dimensional Normed Linear Spaces

**Definition 77 (Bounded Linear Map)**

*A linear map $T : (V, \|\cdot\|_V) \to (W, \|\cdot\|_W)$ is said to be **bounded** if*

$$\|T\|_T = \sup\{\|T(v)\|_W \mid \|v\|_V \leq 1\} < \infty.$$

In assignment 3, we proved the following important result about linear maps in finite dimensional normed lienar spaces.

**Theorem 87 (Boundedness is Equivalent to Continuity in Finite Dimensional Normed Linear Spaces)**

*Let $T : (V, \|\cdot\|_V) \to (W, \|\cdot\|_W)$ be a linear map. TFAE:*

1. *$T$ is bounded.*
2. *$T$ is continuous.*
3. *$T$ is continuous at $0$.*

**Lemma 88 (Continuity of the Norm)**

*The function $f : (V, \|\cdot\|) \to \mathbb{R}$ given by $f(x) = \|x\|$ is continuous.*

**Proposition 89 (Linear Map Between Spaces of Different Dimensions is Bounded)**

*Let $T : (\mathbb{R}^n, \|\cdot\|_2) \to (\mathbb{R}^m, \|\cdot\|_2)$ be linear. Then $T$ is bounded.*

**Proof**

Since $T$ is a linear map, we may represent $T$ using a matrix $A$ such that

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix}.$$

If $\|x\| \leq 1$, then

$$\begin{aligned} \|T(x)\|_2 &= \left\| \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right\| = \left\| \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vec{a}_2 \cdot \vec{x} \\ \vdots \\ \vec{a}_m \cdot \vec{x} \end{bmatrix} \right\| \\ &= \left( \sum_{i=1}^m (\vec{a}_i \cdot \vec{x})^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i=1}^m \|\vec{a}_i\|^2 \|\vec{x}\|^2 \right)^{\frac{1}{2}} \\ &\leq \left( \sum_{i=1}^m \|\vec{a}_i\|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

This completes the proof. □

**Theorem 90 (Boundedness of Functions between $n$-dimensional Vector Spaces and $n$-dimensional Normed Linear Spaces)**

*Let $(V, \|\cdot\|_V)$ be an $n$-dimensional normed linear space with basis $\{v_1, \ldots, v_n\}$. Let $\Gamma_n : \mathbb{R}^n \to V$ be given by*

$$\Gamma_n(\alpha_1, \ldots, \alpha_n) = \alpha_1 v_1 + \ldots + \alpha_n v_n.$$

*Then $\Gamma_n$ and $\Gamma_n^{-1}$ are both bounded. Furthermore, they are both continuous by Theorem 87.*

**Proof**

**$\Gamma_n$ is bounded** Suppose $\|(\alpha_1, \ldots, \alpha_n)\|_2 \leq 1$. Then

$$\begin{aligned} \|\Gamma_n(\alpha_1, \ldots, \alpha_n)\|_V &= \|\alpha_1 v_1 + \ldots \alpha_n v_n\|_V \\ &\leq |\alpha_1| \|v_1\|_V + \ldots |\alpha_n| \|v_n\|_V \\ &\leq \sum_{i=1}^n \|v_i\|_V . \end{aligned}$$

**$\Gamma_n^{-1}$ is bounded** Note that since $\Gamma_n$ is bounded, it is continuous. Consider

$$S = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \mid \|(\alpha_1, \ldots, \alpha_n) = 1\|_2 = 1\}.$$

Since $S$ is closed and bounded, and is a subset of $\mathbb{R}^n$, $S$ is compact by the Heine-Borel Theorem, and so $\Gamma(S)$ is compact in $V$ by Theorem 86. Since the mapping $v \to \|v\|_V$ is continuous, by the Extreme Value Theorem,

$$\min\{\|\Gamma_n(\alpha_1, \ldots, \alpha_n)\|_V \mid (\alpha_1, \ldots, \alpha_n) \in S\} = \alpha > 0.$$

It follows y continuity that if $\|v\|_V \leq \alpha$, then $\left\|\Gamma_n^{-1}(v)\right\|_2 \leq 1$. Therefore, we have that $\left\|\Gamma_n^{-1}\right\| \leq \frac{1}{\alpha}$. □

**Note**

1. $\Gamma_n$ *is a homeomorphism.*
2. *As a consequence of* $\Gamma$ *being continuous, we have that* $\{x_n\}$ *is Cauchy in* $\mathbb{R}^n$ *iff* $\{\Gamma(x_n)\}$ *is Cauchy in* $(V, \|\cdot\|_V)$.
3. *As a result,* $(V, \|\cdot\|_V)$ *is complete by the Heine-Borel Theorem. Since* $V$ *is arbitrary, we have that* ***all finite dimensional normed linear spaces are complete****.*

**Theorem 91 (The Basis of a Infinite Dimensional Banach Spaces is Uncountable)**

*Suppose* $(W, \|\cdot\|)$ *is a infinite dimensional Banach Space. If* $\{w_\alpha\}_{\alpha \in I}$ *is a basis of* $W$*, then* $I$ *is uncountable.*

**Exercise 30.2.1**

*Prove Theorem 91 (see also in A3).*

**Theorem 92 (All Linear Maps Between Finite Dimensional Normed Linear Spaces are Bounded)**

*If* $(V, \|\cdot\|_V)$ *and* $(W, \|\cdot\|_W)$ *are finite dimensional normed linear spaces, and* $T : V \to W$ *is linear, then* $T$ *is bounded.*

**Proof**

Consider the following diagram that illustrates the relationship between each of the spaces: Then, we define $S : (\mathbb{R}^n, \|\cdot\|_2) \to$

$$\begin{tikzcd}
(V, \|\cdot\|_V) \arrow[r, "T"] \arrow[d, shift right, "\Gamma_n"'] & (W, \|\cdot\|_W) \arrow[d, shift right, "\Gamma_m"'] \\
(\mathbb{R}^n, \|\cdot\|_2) \arrow[u, shift right, "\Gamma_n^{-1}"'] & (\mathbb{R}^m, \|\cdot\|_2) \arrow[u, shift right, "\Gamma_m^{-1}"']
\end{tikzcd}$$

Figure 30.1: Relationship between the finite dimensional normed linear spaces.

$(\mathbb{R}^m, \|\cdot\|_2)$ such that $S = \Gamma_m \circ T \circ \Gamma_n^{-1}$. By Proposition 89, $S$ is continuous. Consequently, we have that $T = \Gamma_m{-1} \circ S \circ \Gamma_n$, which is a composition of continuous functions. Thus $T$ is continuous, and hence bounded. $\square$

**Corollary 93 (All Linear Maps from A Finite Dimensional Normed Linear Space to Any Normed Linear Space is Bounded)**

*If $(V, \|\cdot\|_V)$ is a finite dimensional normed lienar space, and $T : (V, \|\cdot\|_V) \to (W, \|\cdot\|_W)$ is linear, then $T$ is bounded.*

# A Useful Theorems from Earlier Calculus

**Theorem 94 (Monotone Convergence Theorem)**

*Let $\{x_k\}$ be a sequence in $\mathbb{R}$.*

1. *Suppose $\{x_k\}$ is increasing.*
   - *If $\{x_k\}$ is bounded above, then $x_k \to \sup\{x_k\}$ as $k \to \infty$.*
   - *If $\{x_k\}$ is not bounded above, then $x_k \to \infty$ as $k \to \infty$.*
2. *Suppose $\{x_k\}$ is decreasing.*
   - *If $\{x_k\}$ is bounded below, then $x_k \to \inf\{x_k\}$ as $k \to \infty$.*
   - *If $\{x_k\}$ is not bounded below, then $x_k \to -\infty$.*

# *Bibliography*

Forrest, B. E. (2018). Pmath351, real analysis.

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