PMATH351 — Real Analysis

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Table of Contents

📔　*List of Definitions*

☕ *List of Theorems*

1 Lecture 1 Sep 06th

1.1 Course Logistics

No content is covered in today's lecture so this chapter will cover some of the important logistical highlights that were mentioned in class.

- Assignments are designed to help students understand the content.

- Due to shortage of manpower, not all assignment questions will be graded; however, students are encouraged to attempt all of the questions.

- To further motivate students to work on ungraded questions, the midterm and final exam will likely recycle some of the assignment questions.

- There are no required text, but the professor has prepared course notes for reading. The course note are self-contained.

- The approach of the class will be more interactive than most math courses.

- Due to the size of the class, students are encouraged to utilize Waterloo Learn for questions, so that similar questions by multiple students can be addressed at the same time.

1.2 Preview into the Introduction

How do we compare the size of two sets?

- If the sets are finite, this is a relatively easy task.

- If the sets are infinite, we will have to rely on functions.

- Injective functions tell us that the **domain is of size that is lesser than or equal to the codomain**.

- Surjective functions tell us that the **codomain is of size that is lesser than or equal to the domain**.

- So does a bijective function tell us that the domain and codomain have the same size? Yes, although this is not as intuitive as it looks, as it relies on **Cantor-Schröder-Bernstein Theorem**.

Now, given two arbitrary sets, are we guaranteed to always be able to compare their sizes? It is tempting to immediately say yes, but to do that, one would have to agree on the **Axiom of Choice**. Fortunately, within the realm of this course, the Axiom of Choice is taken for granted.

2 Lecture 2 Sep 10th

2.1 Basic Set Theory

We shall use the following notations for some of the common set of numbers that we are already familiar with:

- $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \ldots\}$;

- $\mathbb{Z}$ denotes the set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$;

- $\mathbb{Q}$ denotes the set of rational numbers $\left\{ \frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N} \right\}$; and

- $\mathbb{R}$ denotes the set of real numbers.

  We shall start with having certain basic properties of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$.

WE WILL USE the notation $A \subset B$ and $A \subseteq B$ interchangably to mean that A is a subset of B with the possibility that $A = B$. When we wish to explicitly emphasize this possibility, we shall use $A \subseteq B$. When we wish to explicitly state that A is a **proper subset** of B, we will either specify that $A \neq B$ or simply $A \subsetneq B$.

📓 **Definition 1 (Universal Set)**

A universal set, which we shall generally give the label X, is a set that contains all the mathematical objects that we are interested in.

This is a hand-wavy definition, but it is not in the interest of this course to further explore on this topic.

With a universal set in place, we can have the following definitions:

📓 **Definition 2 (Union)**

Let X be a set. If $\{A_\alpha\}_{\alpha \in I}$ such that $A_\alpha \subset X$, then the **union** for all A_α is defined as

$$\bigcup_{\alpha \in I} A_\alpha := \{x \in X \mid \exists \alpha \in I, x \in A_\alpha\}.$$

📓 Definition 3 (Intersection)

Let X be a set. If $\{A_\alpha\}_{\alpha \in I}$ such that $A_\alpha \subset X$, then the **intersection** for all A_α is defined as

$$\bigcap_{\alpha \in I} A_\alpha := \{x \in X \mid \forall \alpha \in I, x \in A_\alpha\}.$$

📓 Definition 4 (Set Difference)

Let X be a set and $A, B \subseteq X$. The **set difference** of A from B is defined as

$$A \setminus B := \{x \in X \mid x \in A, x \notin B\}.$$

On a similar notion:

📓 Definition 5 (Symmetric Difference)

Let X be a set and $A, B \subseteq X$. The **symmetric difference** of A and B is defined as

$$A \Delta B := \{x \in X \mid (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)\}.$$

In words, for an element in the symmetric difference of two sets, the element is either in A or B but not both. We can also think of the symmetric difference as
$$(A \cup B) \setminus (A \cap B)$$
or
$$(A \setminus B) \cup (B \setminus A).$$

We can also talk about the non-members of a set:

📓 Definition 6 (Set Complement)

Let X be a set and $A \subset X$. The set of all non-members of A is called the **complement** of A, which we denote as

$$A^c := \{x \in X \mid x \notin A\}.$$

❝ Note

Note that

$$(A^c)^c = \{x \in X \mid x \notin A^c\} = \{x \in X \mid x \in A\} = A.$$

Now taking a step away from that, we define the following:

▤ Definition 7 (Empty Set)

*An **empty set**, denoted by $\emptyset$, is a set that contains nothing.*

❝ Note

The empty set is set to be a subset of all sets.

▤ Definition 8 (Power Set)

Let X be a set. The power set of X is the set that contains all subsets of X, i.e.
$$\mathcal{P}(X) := \{A \mid A \subset X\}.$$

❝ Note

A power set is always non-empty, since $\emptyset \in \mathcal{P}(\emptyset)$, and since $\emptyset \subset X$ for any set X, we have $\emptyset \in \mathcal{P}(X)$.

Example 2.1.1

Let $X = \{1, 2, \dots, n\}$. There are several ways we can show that the size of $\mathcal{P}(X)$ is 2^n. One of the methods is by using a characteristic

function that maps from A to $\{0,1\}$, defined by

$$X_A : A \to \{0,1\}$$

$$X_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

Using this function, each element in X have 2 states: one being in the subset, and the other being not in the subset, which are represented by 1 and 0 respectively. It is then clear that there are 2^n of such configurations.

📺 **Theorem 1 (De Morgan's Laws)**

Let X be a set. Given $\{A_\alpha\}_{\alpha \in I} \subset \mathcal{P}(X)$, we have

1. $\left(\bigcup\limits_{\alpha \in I} A_\alpha \right)^c = \bigcap\limits_{\alpha \in I} A_\alpha^c$; *and*

2. $\left(\bigcap\limits_{\alpha \in I} A_\alpha \right)^c = \bigcup\limits_{\alpha \in I} A_\alpha^c$.

✏️ **Proof**

1. Note that

$$x \in \left(\bigcup_{\alpha \in I} A_\alpha \right)^c \iff \nexists \alpha \in I \ x \in A_\alpha$$

$$\iff \forall \alpha \in I \ x \notin A_\alpha$$

$$\iff \forall \alpha \in I \ x \in A_\alpha^c \text{ by set complementation}$$

$$\iff x \in \bigcap_{\alpha \in I} A_\alpha^c.$$

2. Observe that, by part 1,

$$\left(\bigcap_{\alpha \in I} A_\alpha \right)^c = \left(\left(\bigcup_{\alpha \in I} A_\alpha^c \right)^c \right)^c = \bigcup_{\alpha \in I} A_\alpha^c.$$

$\square$

Example 2.1.2

Suppose $I = \varnothing$. Then what is $\bigcup\limits_{\alpha \in \varnothing} A_\alpha$? It is sensible to think that all we are left with is simply a union of empty sets, and so

$$\bigcup_{\alpha \in \varnothing} A_\alpha = \varnothing. \tag{2.1}$$

And what about $\bigcap\limits_{\alpha \in \varnothing} A_\alpha$? By 🖥 Theorem 1, it is quite clear from Equation (2.1) that

$$\bigcap_{\alpha \in \varnothing} A_\alpha = X.$$

2.2 *Products of Sets*

📄 **Definition 9 (Product of Sets)**

*Given 2 sets X and Y, the **product** of X and Y is given by*

$$X \times Y := \{(x, y) \mid x \in X, y \in Y\}.$$

*We often refer to elements of $X \times Y$ as **tuples**.*

❝ **Note**

Now if

$$X = \{x_1, x_2, \ldots, x_n\},$$
$$Y = \{y_1, y_2, \ldots, y_m\},$$

then

$$X \times Y = \{(x_i, y_j) \mid i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m\}$$

and so the size of $X \times Y$ is mn.

Consequently, we can think of tuples as two elements being in some "relation".

📄 **Definition 10 (Relation)**

*A **relation** on sets X and Y is a subset R of the product $X \times Y$. We write*

$$xRy \quad \text{if } (x, y) \in R \subset X \times Y.$$

We call

- $\{x \in X \mid \exists y \in Y, (x,y) \in R\}$ *as the **domain** of R; and*

- $\{y \in Y \mid \exists x \in X, (x,y) \in R\}$ *as the **range** of R.*

In relation to that, functions are, essentially, relations.

📓 **Definition 11 (Function)**

*A **function** from X to Y is a relation R such that*

$$\forall x \in X \; \exists! y \in Y \; (x,y) \in R.$$

SUPPOSE $X_1, X_2, \ldots, X_n$ are non-empty[1] sets. We can define

$$X_1 \times X_2 \times \ldots \times X_n = \prod_{i=1}^{n} X_i := \{(x_1, x_2, \ldots, x_n) \mid x_i \in X_i\}.$$

Now if $X_i = X_j = X$ for all $i, j = 1, 2, \ldots, n$, we write

$$\prod_{i=1}^{n} X_i = \prod_{i=1}^{n} X = X^n.$$

[1] We are typically only interested in non-empty sets, since empty sets usually lead us to vacuous truths, which are not interesting.

AND NOW COMES THE PROBLEM: given a collection $\{X_\alpha\}_{\alpha \in I}$ of non-empty sets[2], what do we mean by

$$\prod_{\alpha \in I} X_\alpha ?$$

[2] i.e. we now talk about arbitrary $\alpha \in I$.

To motivate for what comes next, consider

$$\prod_{i=1}^{n} X_i = X_1 \times \ldots \times X_n = \{(x_1, \ldots, x_n) \mid x_i \in X_i\}.$$

Choose $(x_1, \ldots, x_n) \in \prod_{i=1}^{n} X_i$. This induces a function

$$f_{(x_1, \ldots, x_n)} : \{1, \ldots, n\} \to \bigcup_{i=1}^{n} X_i$$

with

$$f(1) = x_1 \in X_1$$
$$f(2) = x_2 \in X_2$$
$$\vdots$$
$$f(n) = x_n \in X_n$$

Now assume for a more general f such that

$$f : \{1, \ldots, n\} \to \bigcup_{i=1}^{n} X_i$$

is defined by

$$f(i) \in X_i.$$

Then, we have

$$(f(1), f(2), \ldots, f(n)) \in \prod_{i=1}^{n} X_i,$$

which leads us to the following notion:

📰 **Definition 12 (Choice Function)**

Given a collection $\{X_\alpha\}_{\alpha \in I}$ of non-empty sets, let

$$\prod_{\alpha \in I} X_\alpha = \left\{ f : I \to \bigcup_{\alpha \in I} X_\alpha \right\}$$

such that $f(\alpha) \in X_\alpha$. Such an f is called a choice function.

And so we may ask a similar question as before: if each X_α is non-empty, is $\prod_{\alpha \in I} X_\alpha$ non-empty? Turns out this is not as easy to show. In fact, it is essentially impossible to show, because this is exactly the **Axiom of Choice**.

3 Lecture 3 Sep 12th

3.1 Axiom of Choice

RECALL our final question of last lecture: If $\{X_\alpha\}_{\alpha \in I}$ is a non-empty collection of non-empty sets, is

$$\prod_{\alpha \in I} X_\alpha \neq \emptyset \ ?$$

Turns out this is widely known (in the world of mathematics) as the **Axiom of Choice**.

🛡 **Axiom 2 (Zermelo's Axiom of Choice)**

If $\{X_\alpha\}_{\alpha \in I}$ is a non-empty collection of non-empty sets, then

$$\prod_{\alpha \in I} X_\alpha \neq \emptyset.$$

An equivalent statement of the above axiom is:

🛡 **Axiom 3 (Zermelo's Axiom of Choice v2)**

$X \neq \emptyset \implies$

$$\exists f : \mathcal{P}(X) \setminus \{\emptyset\} \to X \ \forall A \in \mathcal{P}(X) \setminus \{\emptyset\} \ f(A) \in A$$

where f is the *choice function*.

Exercise 3.1.1

Prove that 🛡 *Axiom 2 and* 🛡 *Axiom 3 are equivalent.*

✏ **Proof**

From 🛡 Axiom 2 to 🛡 Axiom 3:

Since $X \neq \varnothing$, we have that $\mathcal{P}(X) \setminus \{\varnothing\}$ is a non-empty collection of non-empty sets. Therefore,

$$\prod_{A \in \mathcal{P}(X) \setminus \{\varnothing\}} A \neq \varnothing.$$

So we know that

$$\exists (x_A)_{A \in \mathcal{P}(X) \setminus \{\varnothing\}} \in \prod_{A \in \mathcal{P}(X) \setminus \{\varnothing\}} A.$$

We then simply need to choose the choice function $f : \mathcal{P}(X) \setminus \{\varnothing\} \to X$ such that

$$f(A) = x_A \in A.$$

From 🛡 Axiom 3 to 🛡 Axiom 2:

Let $X_\alpha \in \mathcal{P}(X)$ for $\alpha \in I$, where I is some index set. We know that not all $X_\alpha = \varnothing$ since $X \neq \varnothing$. Choose $J \subseteq I$ such that $\{X_\alpha\}_{\alpha \in J}$ is a non-empty collection of non-empty sets. Let $f : \mathcal{P}(X) \setminus \{\varnothing\}$ be any choice function. By 🛡 Axiom 3,

$$\forall X_\alpha \in \mathcal{P}(X) \setminus \{\varnothing\} \quad f(X_\alpha) \in X_\alpha.$$

Therefore,

$$(f(X_\alpha))_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha.$$

$\square$

3.2 *Relations*

Now, it is in our interest to start talking about comparisons or relations between the mathematical objects that we have defined.

🗐 **Definition 13 (Relations)**

A relation R on a set X is [1]

- *(**Reflexive**)* $\forall x \in X \;\; xRx$;

- *(**Symmetric**)* $\forall x, y \in X \;\; xRy \iff yRx$;

- *(**Anti-symmetric**)* $\forall x, y \in X \;\; xRy \wedge yRx \implies x = y$;

- *(**Transitive**)* $\forall x, y, z \in X \;\; xRy \wedge yRz \implies xRz$.

[1] We can look at this definition as $R \subseteq X \times X$. Under such a definition, we would have

- (**Reflexive**) $\forall x \in X \;\; (x, x) \in R$;
- (**Symmetric**) $\forall x, y \in X \;\; (x, y) \in R \iff (y, x) \in R$;
- (**Anti-symmetric**) $\forall x, y \in X \;\; (x, y), (y, x) \in R \implies x = y$;
- (**Transitive**) $\forall x, y, z \in X \;\; (x, y), (y, z) \in R \implies (x, z) \in R$.

Example 3.2.1

Let $X = \mathbb{R}$, and let $xRy \iff x \leq y$, where $\leq$ is the notion of "less than or equal to", which we shall assume that it has the meaning that we know. Observe that $\leq$ is:

- reflexive: $\forall x \in \mathbb{R} \;\; x \leq x$ is true;

- anti-symmetric: $\forall x, y \in \mathbb{R} \;\; x \leq y \wedge y \leq x \implies x = y$; and

- transitive: $\forall x, y, z \in \mathbb{R} \;\; x \leq y \wedge y \leq z \implies x \leq z$.

Example 3.2.2

Let $Y \neq \emptyset$, $X = \mathcal{P}(Y)$, with $ARB \iff A \subseteq B$. Observe that $\subseteq$ is:

- reflexive: $\forall A \in \mathcal{P}(Y) \;\; ARA \iff A \subseteq A$ is true;

- anti-symmetric: $\forall A, B \in \mathcal{P}(Y) \;\; ARB \wedge BRA \iff A \subseteq B \wedge B \subseteq A \implies A = B$;

- transitive: $\forall A, B, C \in \mathcal{P}(Y) \;\; ARB \wedge BRC \iff A \subseteq B \wedge B \subseteq C \implies A \subseteq C$.

Example 3.2.3

Let $Y \neq \emptyset$, $X = \mathcal{P}(Y)$, with $ARB \iff A \supseteq B$. Observe that $\supseteq$ is:

- reflexive: $\forall A \in \mathcal{P}(Y) \;\; ARA \iff A \subseteq A$;

- anti-symmetric: $\forall A, B \in \mathcal{P}(Y) \;\; ARB \wedge BRA \iff A \supseteq B \wedge B \supseteq A \implies A = B$;

- transitive: $\forall A, B, C \in \mathcal{P}(Y) \;\; ARB \wedge BRC \iff A \supseteq B \wedge B \supseteq C \implies A \supseteq C$.

All the above examples are also known as *partially ordered sets*.

📘 **Definition 14 (Partially Ordered Sets)**

*The set X with the relation R on X is called a **partially ordered set** (or a **poset**) if R is*

- *reflexive;*

- *anti-symmetric; and*

- *transitive.*

We denote a poset by (X, R).

The "partial" in 'partially ordered' indicates that not every pair of elements need to be comparable, i.e. there may be pairs for which neither precedes the other (anti-symmetry).

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66 **Note**

If (X, R) is a poset, then if $A \subseteq X$, and $R_1 = R \restriction_{A \times A}$, then (A, R_1) is also a poset.

Example 3.2.4

How many possible relations can we define on these sets to make them into posets?

1. $X = \varnothing$

   ✏️ **Solution**
   We have that $R = \varnothing \times \varnothing$, and so the only relation we have is an empty relation. Then it is vacuously true that (X, R) a poset.

2. $X = \{x\}$

   ✏️ **Solution**
   We have that $R = X \times X = \{(x, x)\}$. It it clear that (X, R) is a poset.

3. $X = \{x, y\}$

   ✏️ **Solution**
   There are 3 possible relations:

   - a relation where xRx and yRy;

   - a relation where xRy; or

   - a relation where yRx.
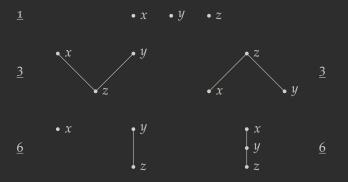
3 possibilities represented as graphs (known as Hasse diagram), separated by lines:

4. $X = \{x, y, z\}$

✏️ **Solution**

The following are all the possibilities represented by graphs, where the underlined numbers represent the number of ways we can rearrange the elements for unique relations:

Therefore, we see that there are a total of

$$1 + 3 + 3 + 6 + 6 = 19 \text{ relations.}$$

Exercise 3.2.1

How many possible relations can we define on a set of 6 elements to the set into a poset?

✏️ **Solution**
to be added

📘 **Definition 15 (Totally Ordered Sets / Chains)**

*The set X with the relation R on X is called a **totally ordered set** (or a **chain**) if (X, R) is a poset with the exception that, for any $x, y \in X$, either xRy or yRx but not both.*

📘 **Definition 16 (Bounds)**

*Let $(X, \leq)$ be a poset. Let $A \subset X$. We say $x_0 \in X$ is an **upper bound** for A if*

$$\forall a \in A \quad a \leq x_0.$$

*If A has an upper bound, we say that A is **bounded above**. If A is bounded above, then x_0 is the **least upper bound** (or **supremum**) of*

A is for any $x_1 \in X$ that is an upper bound of A, we have

$$x_0 \leq x_1.$$

*We write $x_0 = \text{lub}(A) = \sup(A)$. If $\sup(A) \in A$, then $\sup(A) = \max(A)$ is the **maximum** of A.*

We can analogously define for:

$$
\begin{aligned}
upper\ bound\ &\rightarrow\ lower\ bound \\
bounded\ above\ &\rightarrow\ bounded\ below \\
least\ upper\ bound,\ \text{lub}\ &\rightarrow\ greatest\ lower\ bound,\ \text{glb} \\
supremum,\ \text{sup}\ &\rightarrow\ infimum,\ \text{inf} \\
maximum,\ \max\ &\rightarrow\ minimum,\ \min
\end{aligned}
$$

❝ Note

*By **anti-symmetry** of posets, we have that $\max, \sup, \min, \inf$ are all unique if they exists.*

Example 3.2.5 (Least Upper Bound Property of $\mathbb{R}$)

Let $X = \mathbb{R}$, and $\leq$ be the order that we have defined. Every bounded non-empty subset of X has a supremum.

Example 3.2.6

Let $Y \neq \varnothing$, and $X = \mathcal{P}(Y)$, and $\subseteq$ the ordering by inclusion. We know that Y is the maximum element of $(X, \subseteq)$. Then the collection $\{A_\alpha\}_{\alpha \in I} \subset \mathcal{P}(Y)$ is bounded above by Y, and we have that

$$\sup\left(\{A_\alpha\}_{\alpha \in I}\right) = \bigcup_{\alpha \in I} A_\alpha$$

$$\inf\left(\{A_\alpha\}_{\alpha \in I}\right) = \bigcap_{\alpha \in I} A_\alpha$$

Now if $Y = \varnothing$, we would end up having

$$\sup\left(\{A_\alpha\}_{\alpha \in I}\right) = \varnothing$$
$$\inf\left(\{A_\alpha\}_{\alpha \in I}\right) = X$$

This makes sense, since the empty set would be the least of upper bounds, and since $X = \mathcal{P}(Y)$ would have to be the greatest of lower bounds.

4 Lecture 4 Sep 14th

4.1 Zorn's Lemma

📄 **Definition 17 (Maximal Element)**

Let $(X, \leq)$ be a poset. An element $x \in X$ is **maximal** if whenever $y \in X$ is such that $x \leq y$, we must have $y = x$.

Example 4.1.1

Looking back at Example 3.2.4, on the set $X = \{x, y, z\}$, we have that the maximal element in each possible poset is/are:

This shows to us that the maximal element does not have to be unique.

$\bullet\, x \qquad \bullet\, y \qquad \bullet\, z \qquad\qquad x, y, z$ are all maximal

z is maximal

x, y are both maximal

x, z are both maximal

z is maximal

Example 4.1.2

- Given $X \neq \varnothing$, the maximal element of the poset $(\mathcal{P}(X), \subseteq)$ is X.

- Given $X \neq \varnothing$, the maximal element of the poset $(\mathcal{P}(X), \supseteq)$ is $\varnothing$.

- The poset $(\mathbb{R}, \leq)$ has no maximal element.

🛡 **Axiom 4 (Zorn's Lemma)**

If $(X, \leq)$ is a non-empty poset such that every chain $S \subset X$ has an upper bound, then $(X, \leq)$ has a maximal element.

💻 **Theorem 5 (★ Non-Zero Vector Spaces has a Basis)**

Every non-zero vector space, V, has a basis.

The flow of this proof is a typical approach when Zorn's Lemma is involved.

✏ **Proof (★)**

Let
$$\mathcal{L} := \{A \subset V \mid A \text{ is linearly independent }\}.$$

Note that $\mathcal{L} \neq \varnothing$ since $V \neq \{0\}$. Now order elements of $\mathcal{L}$ with $\subseteq$. It suffices to show that $(\mathcal{L}, \subseteq)$ has a maximal element, since this maximal element must be a basis. Otherwise, we would contradict the maximality of such an element.[1]

[1] This is the key to this proof.

Now let $S = \{A_\alpha\}_{\alpha \in I}$ be a chain in $\mathcal{L}$. Let

$$A_0 = \bigcup_{\alpha \in I} A_\alpha.$$

Require clarification before proceeding...

📘 **Definition 18 (Well-Ordered)**

*We say that a poset $(X, \leq)$ is **well-ordered** if every non-empty subset $A \subset X$ has a least/minimal element in A.*

Exercise 4.1.1
Prove that well-ordered sets are chains.

Example 4.1.3

$(\mathbb{N}, \leq)$ is well-ordered.

🛡 **Axiom 6 (Well-Ordering Principle)**

Every non-empty set can be well-ordered.

☕ **Theorem 7 (Axioms of Choice and Its Equivalents)**

TFAE:

1. *Axiom of Choice,* 🛡 *Axiom 2*

2. *Zorn's Lemma,* 🛡 *Axiom 4*

3. *Well-Ordering Principle,* 🛡 *Axiom 6.*

Exercise 4.1.2
Prove ☕ *Theorem 7*

✏ **Proof**

$(3) \implies (1)$ is simple; let the choice function be such that we pick the minimal element from each set among a non-empty collection of non-empty sets. It is clear that the product of these sets will always have an element, in particular the tuple where each component is the minimal element of each set.

The rest will be added once I've worked it out

Example 4.1.4

Let $X = \mathbb{Q}$. Let $\phi : \mathbb{Q} \to \mathbb{N}$ be defined such that

$$\phi\left(\frac{m}{n}\right) = \begin{cases} 2^m 5^n & m > 0 \\ 1 & m = 1 \\ 3^{-m} 7^n & m < 0 \end{cases}$$

By the **unique prime factorization of natural numbers** (or **Fundamental Theorem of Arithmetic**), we have that ϕ is injective. In fact,

$$r \leq s \iff \phi(r) \leq \phi(s),$$

showing to us that we have a well-ordering on Q.

4.2 *Cardinality*

4.2.1 *Equivalence Relation*

📑 **Definition 19 (Equivalence Relation)**

Let X be non-empty set. A relation $\sim$ on X is an **equivalence relation** if it is

- *reflexive;*

- *symmetric; and*

- *transitive.*

📑 **Definition 20 (Equivalence Class)**

Let X be a non-empty set, and $x \in X$. An **equivalence class** of x under the equivalence relation $\sim$ is defined as

$$[x] := \{y \in X \mid x \sim y\}.$$

❝ **Note**

Note that we either have $[x] = [y]$ or $[x] \cap [y] = \emptyset$. This is sensible, since if $w \in [x]$, then $w \sim x$. If $w \in [y]$, then we are done. If $w \notin [y]$, suppose $\exists v \in [y]$ such that $w \sim v$, which then implies $w \in [y]$ which is a contradiction.

This results shows to us that

$$X = \bigcup_{x \in X} [x],$$

or in words, equivalence classes partition the set.

📑 **Definition 21 (Partition)**

Let $X \neq \emptyset$. A **partition** of X is a collection $\{A_\alpha\}_{\alpha \in I} \subset \mathcal{P}(X)$ such that

1. $A_\alpha \neq \emptyset$;

2. $A_\alpha \cap A_\beta = \emptyset$ if $\alpha \neq \beta$ in I; and

3. $X = \bigcup_{\alpha \in I} A_\alpha$.

With this, we have ourselves another method to show that $\sim$ is an equivalence relation.

🌢 **Proposition 8 (Characterization of An Equivalence Relation)**

If $\{A_\alpha\}_{\alpha \in I}$ is a partition of X and $x \sim y \iff x, y \in A_\alpha$, then $\sim$ is an equivalence relation.

The proof of this statement has been stated above.

Similar to when we defined partial orders, we can ask ourselves the following question:

Example 4.2.1

How many equivalence relations are there on the set $X = \{1, 2, 3\}$?[2]

✏️ **Solution**

Note that we can partition X as

$$\{\{1\}, \{2\}, \{3\}\}, \{\{1, 2, 3\}\},$$

and

$$\{\{1, 2\}, \{3\}\},$$

which we can rearrange in 3 different ways. Therefore, there are 5 different equivalence relations that we can define on X.

Example 4.2.2

Let X be any set. Consider $\mathcal{P}(X)$. Define $\sim$ on $\mathcal{P}(X)$ by

$$A \sim B \iff \exists f : A \to B$$

such that f is surjective[3]. It is easy to verify that $\sim$ is an equivalence relation.

[2] By 🌢 Proposition 8, this question is equivalent to asking for the number of partitions we can create from the set X. The study of counting partitions is what is covered by Bell's Number.

[3] $\sim$ partitions X into sets that have the same number of elements.

5 Lecture 5 Sep 17th

5.1 Cardinality (Continued)

📕 **Definition 22 (Finite Sets)**

*A set X is **finite** if $X = \emptyset$ or if $X \sim \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$, where $\sim$ is the equivalence relation defined in Example 4.2.2.*

📕 **Definition 23 (Cardinality)**

*If $X \sim n$, we say X has **cardinality** n and write $|X| = n$. We also let $|\emptyset| = 0$.*

Now a good question here is: if $n \neq m$, is $\{1, 2, \ldots, n\} \sim \{1, 2, \ldots, m\}$?

☕ **Theorem 9 (Pigeonhole Principle)**

The set $\{1, 2, \ldots, n\}$ is not equivalent to any of its proper subset.

✏️ **Proof**

We shall prove this by induction on n.

Base case: $\{1\} \nsim \emptyset$.

This is a **proof by contradiction**, using the fact that we cannot find an injective function from a "larger" set to a "smaller" set.

We can assume that the function f is not surjective, since if the larger set is indeed equivalent to the smaller set, then it should not matter if f is surjective or not. In particular, we only require that there be an injective function.

Requires clarification and confirmation of proof

Assume that the statement holds for $\{1,\ldots,k\}$. Suppose we have an injective function

$$f : \{1,2,\ldots,k,k+1\} \to \{1,2,\ldots,k,k+1\}$$

that is not surjective.

Case 1: $k+1 \notin \text{Range}(f)$, where $\text{Range}(f)$ is the range of f. Then we have

Note: $\restriction$ is the restriction sign.

$$f \restriction_{\{1,\ldots,k\}} : \{1,\ldots,k\} \to \{1,\ldots,k\} \setminus \{f(k+1)\}.$$

However, f is an injective function and clearly

$$\{1,\ldots,k\} \setminus \{f(k+1)\} \subseteq \{1,\ldots,k\},$$

a contradiction.

Case 2: $k+1 \in \text{Range}(f)$. Then $\exists j_0 \in \{1,\ldots,k,k+1\}$ such that $f(j_0) = k+1$, and since f is not surjective, $\exists m \in \{1,\ldots,k\}$ such that $m \notin \text{Range}(f)$. Then consider a new function $g : \{1,\ldots,k,k+1\} \to \{1,\ldots,k\}$ such that

$$g(a) = \begin{cases} m & a = k+1 \\ f(k+1) & a = j_0 \\ f(a) & a \neq j_0, k+1 \end{cases}$$

🏹 **Corollary 10 (Pigeonhole Principle (Finite Case))**

If the set X is finite, then X is not equivalent to any proper subset.

Exercise 5.1.1

Prove 🏹 *Corollary 10.*

📘 **Definition 24 (Infinite Sets)**

*X is **infinite** if it is not finite.*

Example 5.1.1

Sketch of proof:

$$\{1,\ldots,n\} \longrightarrow \{1,\ldots,n\}$$

$$f^{-1} \Big\uparrow f \qquad\qquad \Big\uparrow f$$

$$X \xrightarrow[\text{onto}]{1-1} A \subsetneq X$$

Observe that we can construct a function $f : N \to \{2, 3, \ldots\}$ by $f(n) = n + 1$. It is clear that f is a bijective funciton, and so $\mathbb{N} \sim \{2, 3, \ldots\}$.

💧 Proposition 11 (ℕ is the Smallest Infinite Set)

Every infinite set contains a subset $A \sim \mathbb{N}$.

✏️ Proof

Suppose X is infinite. Let

$$f : \mathcal{P}(X) \setminus \{\emptyset\} \to X$$

such that for $S \subset X$ where $S \neq \emptyset$, $f(S) \in S$ [1]. Let $x_1 = f(X)$. Let $x_2 = f(X \setminus \{x_1\})$. Recursively, define

$$x_n = f(X \setminus \{x_1, \ldots, x_{n-1}\}).$$

This gives us a sequence

$$X \supset S = \{x_1, \ldots, x_n, \ldots\}$$

which is equivalent to $\mathbb{N}$ via the map $n \mapsto x_n$. □

[1] 🛡 Axiom 3 ahoy!

🏹 Corollary 12 (Infinite Sets are Equivalent to Its Proper Subsets)

Every infinite set X is equivalent to a proper subset of X.

✏️ Proof

Given such an X, we construct a sequence $\{x_n\}$ as in the previous proof. Define $f : X \to X \setminus \{x_n\}$ by

$$f(x) = \begin{cases} x_{n+1} & x \in \{x_n\} \\ x & x \notin \{x_n\}. \end{cases}$$

Clearly so, f is injective. □

📃 **Definition 25 (Countable)**

*We say that a set is **countable** (or **denumerable**) is either X is finite or if $X \sim \mathbb{N}$. If $X \sim \mathbb{N}$, we can say that X is **countably infinite** and write $|X| = |\mathbb{N}| = \aleph_0$.*

📃 **Definition 26 (Smaller Cardinality)**

Given 2 sets X, Y, we write

$$|X| \le |Y|$$

if $\exists f : X \to Y$ injective.

💧 **Proposition 13 (Injectivity is Surjectivity Reversed)**

TFAE

1. *$\exists f : X \to Y$ injective*

2. *$\exists g : Y \to X$ surjective*

✏️ **Proof**
$(1) \implies (2)$: Define

$$g(y) = \begin{cases} x & \exists x \in X \ f(x) = y \\ x_0 & \text{any } x_0 \in X \end{cases}$$

Clearly g is surjective.

$(2) \implies (1)$: Since g is surjective, for each $x \in X$, we have that[2]

$$g^{-1}(|x|) = \{y \in Y : g(y) = x\} \ne \emptyset.$$

By the Axiom of Choice, there exists a choice function $h : \mathcal{P}(Y) \setminus \{\emptyset\} \to Y$ such that for each $A \subset Y, h(A) \in A$. Then, let $f : X \to Y$ such that

$$f(x) = h(g^{-1}(\{x\})).$$

[2] The idea here is to collect the preimages into a set, and use the choice function as an injective map.

Clearly so, f is injective. □

> **66 Note**
>
> *Note that we have $|\mathbb{N}| \leq |\mathbb{Q}|$, since we can define an injective function $f : \mathbb{N} \to \mathbb{Q}$ such that $f(n) = \frac{n}{1}$.*
>
> *We have also shown that $|\mathbb{Q}| \leq |\mathbb{N}|$ using our injective function $g : \mathbb{Q} \to \mathbb{N}$, given by*
>
> $$g\left(\frac{m}{n}\right) = \begin{cases} 2^m 3^n & m > 0 \\ 1 & m = 0 \\ 5^{-m} 7^n & m < 0 \end{cases}$$

QUESTION: Is $|\mathbb{N} = |\mathbb{Q}||$? In other words, given $|X| \leq |Y| \wedge |Y| \leq |X|$, is $|X| = |Y|$?

6 Lecture 6 Sep 19th

6.1 Cardinality (Continued 2)

Before delving into resolving our last question in the previous lecture, note the following:

❝ Note

Suppose $f : X \to Y$ is bijective. Let $A \subseteq B$, then

$$f(B \setminus A) = f(B) \setminus f(A).$$

Prove this observation as an exercise:

Exercise 6.1.1
Prove the note on the left.

🖥 Theorem 14 (★★★ Cantor-Schröder-Bernstein Theorem (CSB))

Let $A_2 \subset A_1 \subset A_0 = A$. Assume that $A_2 \sim A_0$. Then $A_0 \sim A_1$.

✏ Proof

Let $\phi : A_0 \to A_2$ be bijective, by assumption. Since $A_1 \subset A_1$, let $A_3 = \phi(A_1) \subset A_2$, and since $A_2 \subset A_0$, let $A_4 = \phi(A_2) \subset A_3$. Recursively so, let

$$A_{n+2} = \phi(A_n)$$

Notice that

$$A_0 = (A_0 \setminus A_1) \cup (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup (A_3 \setminus A_4) \cup \ldots \bigcap_{n=0}^{\infty} A_n$$

$$A_1 = (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup (A_3 \setminus A_4) \cup (A_4 \setminus A_5) \cup \ldots \bigcap_{n=1}^{\infty} A_n$$
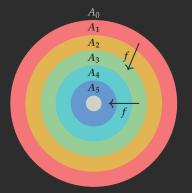
Figure 6.1: The core idea of the proof for Cantor-Schröder-Bernstein Theorem

Observe that

$$\bigcap_{n=0}^{\infty} A_n = \bigcap_{n=1}^{\infty} A_n.$$

[1]Define $f : A \to A_1$ by

$$f(x) = \begin{cases} x & x \in \bigcap_{n=0}^{\infty} A_n \\ x & x \in A_{2k+1} \setminus A_{2k+2}, \ k = 0, 1, 2, \ldots \\ \phi(x) & x \in A_{2k} \setminus A_{2k+1}, \ k = 0, 1, 2, \ldots \end{cases}$$

[1] Here, we employ the idea from Figure 6.1.

➤ Corollary 15 (Cantor-Schröder-Bernstein Theorem - Restated)

If $A_1 \subset A \wedge B_1 \subset B \wedge A \sim B_1 \wedge B \sim A_1$, then $A \sim B$.[2]

[2] This is equivalent to the statement

$$|A| \leq |B| \wedge |B| \leq |A| \implies |A| = |B|.$$

✏ Proof

By assumption, let $f : A \to B_1$ be bijective, and let $g : B \to A_1$ be bijective. Let $A_2 = g(B_1) \subseteq A_1 \subset A$ Let $A_2 = g(B_1) \subseteq A_1 \subset A$. Then the composite function $g \circ f : A \to A_2$ is bijective, and so $A \sim A_2$. By 🖥 Theorem 14, we have

$$A \sim A_2 \sim A_1 \sim B.$$

$\square$

Example 6.1.1

Our question from last lecture now has an answer: by 🖥 Theorem 14, we have that $|\mathbb{Q}| = |\mathbb{N}|$.[3]

[3] Now that we know that they have the same cardinality:

Exercise 6.1.2
Find a bijection between $\mathbb{Q}$ and $\mathbb{N}$.

🌢 Proposition 16 (Denumerability Check)

If X is infinite, then

$$|X| = |\mathbb{N}| = \aleph_0 \iff \exists f : X \to \mathbb{N} \ bijective.$$

✏️ Proof

($\implies$) is immediate. For ($\impliedby$), suppose $f : X \to \mathbb{N}$, which implies that $|X| \leq |\mathbb{N}|$. By 🔸 Proposition 11, $|\mathbb{N}| \leq |X|$. Therefore, $|X| = |bbN| = \aleph_0$. □

Example 6.1.2

$\mathbb{N} \times \mathbb{N}$ is countable. The function

$$f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \text{ given by } f(m, n) = 2^n 3^m$$

is injective.

📘 Definition 27 (Uncountable)

*A set X is **uncountable** if it is not countable.*

🖥 Theorem 17 (Cantor's Diagonal Argument)

$(0, 1)$ *is uncountable.*

✏️ Proof

Suppose, for contradiction, that $(0, 1)$ is countable. Then we can write

$$a_1 = .a_{11}a_{12}a_{13}\ldots$$
$$a_2 = .a_{21}a_{22}a_{23}\ldots$$
$$\vdots$$
$$a_n = .a_{n1}a_{n2}a_{n3}\ldots$$

in $(0, 1)$. This representation is unique if we do not allow the representation to end in a string of 9's. Let $b \in (0, 1)$, expressed as $b = .b_1 b_2 b_3 \ldots$ such that

$$b_i = \begin{cases} 5 & a_i \neq 5 \\ 2 & a_i = 5 \end{cases}$$

Then $b \notin (0,1)$, a contradiction[4]. □

➼ Corollary 18 (Uncountability of $\mathbb{R}$)

$\mathbb{R}$ *is uncountable.*

✎ Proof

Let $f : (0,1) \to \mathbb{R}$ be given by

$$f(x) = \tan\left(\pi x - \frac{\pi}{2}\right).$$

Clearly so, $(0,1)$ is bijective. □

---

66 Note

We denote $|\mathbb{R}| = c.$

QUESTION: Given sets X, Y, is it always true that either[5]

1. $|X| = |Y|$;

2. $|X| < |Y|$; or

3. $|Y| < |X|$.

[5] As compare to $\leq$, $<$ implies that there is no surjection from the set on the LHS to the RHS.

7 Lecture 7 Sep 21st

7.1 Cardinality (Continued 3)

📺 **Theorem 19 (★ Comparability of Cardinals)**

If X and Y are non-empty, then either

$$|X| \leq |Y| \vee |Y| \leq |X|.$$

✏️ **Proof**

Let

$$S = \{(A, B, f) \mid A \subseteq X, B \subseteq Y, f : A \to B \text{ bijective }\}.$$

Note that $S \neq \varnothing$, since X and Y are non-empty, and so we can have $f(a) = b$ for $A = \{a\} \subset X$ and $B = |b| \subset Y$.[1] We order S as follows: we say

$$(A_1, B_1, f_1) \leq (A_2, B_2, f_2)$$

if

$$A_1 \subseteq A_2, \quad B_1 \subseteq B_2, \quad f_1 = f_2 \upharpoonright_{A_1}.$$

Let $C = \{(A_\alpha, B_\alpha, f_\alpha)\}_{\alpha \in I}$ be a chain in $(S, \leq)$. Let $A_0 = \bigcup_{\alpha \in I} A_\alpha$, $B_0 = \bigcup_{\alpha \in I} B_\alpha$, and define $f_0 : A_0 \to B_0$ by

$$f_0(x) = f_{\alpha_0}(x) \text{ if } x \in A_{\alpha_0}.$$

Now if $x \in A_{\alpha_1}$, $x \in A_{\alpha_2}$ and

$$(A_{\alpha_1}, B_{\alpha_1}, f_{\alpha_1}) \leq (A_{\alpha_2}, B_{\alpha_2}, f_{\alpha_2}),$$

[1] We want to use the maximal element to obtain our result. To that end, we need Zorn's Lemma. So we need S to build this up.

we have that

$$f_{\alpha_1}(x) = f_{\alpha_2} \upharpoonright_{A_{\alpha_1}} (x) = f_{\alpha_2}(x),$$

i.e. f_0 is well-defined.

Claim 1: $f_0 : A_0 \to B_0$ is injective.

Let $x_1, x_2 \in A_0$ such that $x_1 \neq x_2$.
$\implies \exists \alpha_1, \alpha_2 \in I \ x_1 \in A_{\alpha_1} \wedge x_2 \in A_{\alpha_2} \wedge A_{\alpha_1} \subseteq A_{\alpha_2}$ (wlog)
$\implies x_1.x_2 \in A_{\alpha_2}$
$\implies (\because f_{\alpha_2}$ injective $\implies f_{\alpha_2}(x_1) \neq f_{\alpha_2}(x))$
$\implies f_0(x_1) \neq f_0(x_2) \implies f_0$ injective.

Claim 2: $f_0 : A_0 \to B_0$ is surjective.

Let $y_0 \in B_0$
$\implies \exists \alpha_0 \in I \ y_0 \in B_{\alpha_0}$
$\implies \exists x_0 \in A_{\alpha_0} \ f_{\alpha_0}(x_0) = y_0 \ (\because f_{\alpha_0}$ surjective)
$\implies f_0(x_0) = y_0$

$\therefore (A_0, B_0, f_0)$ is an upper bound for C. Then by Zorn's Lemma, $(S, \leq)$ has a maximal element (A, B, f).

Case 1: If $A = X$, then injectivity of f implies $|X| \leq |Y|$.

Case 2: If $B = Y$, then surjectivity of f implies $|Y| \leq |A| \leq |X|$.

Case 3: If $A \neq X \wedge B \neq Y$, then $X \setminus A \neq \varnothing \wedge Y \setminus B \neq \varnothing$. Let $x_0 \in X \setminus A$, $y_0 \in Y \setminus B$. Let $A^* = A \cup \{x_0\}$, $B^* = B \cup \{y_0\}$, and $f^* : A^* \to B^*$ such that

$$f^*(x) = \begin{cases} f(x) & x \in A \\ y_0 & x = x_0 \end{cases}$$

Then $(A, B, f) \leq (A^*, B^*, f^*)$, contradicting maximality. $\qquad \square$

7.1.1 *Cardinal Arithmetic*

Sum of Cardinals   Observe that if $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\}$, and $X \cap Y = \varnothing$, then $|X| = n$, $|Y| = m$ and $|X \cup Y| = n + m$. This motivates us to provide the following definition:

📄 **Definition 28 (Sum of Cardinals)**

Assume that X and Y are such that $X \cap Y = \emptyset$. We define

$$|X| + |Y| = |X \cup Y|.$$

QUESTION: So what about $\aleph_0 + \aleph_0$?

A thought that motivates us to give the following answer lies in the observation that: if X is the set of even natural numbers and Y the odd natural numbers, then $X \cup Y$ is the set of all natural numbers. All three sets are countably infinite, i.e. they have cardinality $\aleph_0$.

QUESTION: What about $c + c$?

A similar motivation comes from the observation that: given $X = (0,1)$ and $Y = (1,2)$, we have

$$c = |X| \leq |X| + |Y| \leq |R| = c,$$

and so $|X| = |Y| = c \implies |X \cup Y| = c$.

☕ Theorem 20 (Sums of Cardinals)

Given sets X and Y, if X is infinite, then

1. $|X| + |X| = |X|$

2. $|X| + |Y| = \max(|X|, |Y|)$

Multiplication of Cardinals   Given

$$X = \{x_1, x_2, \ldots, x_n\}$$
$$Y = \{y_1, y_2, \ldots, y_m\}$$

we have that

$$X \times Y = \{(x_i, y_j) \mid i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m\}$$

and so

$$|X \times Y| = nm.$$

Exercise 7.1.1
Prove ☕ Theorem 20 as an exercise.

📔 **Definition 29 (Multiplication of Cardinals)**

Given sets X and Y, define

$$|X|\,|Y| = |X \times Y|\,.$$

Example 7.1.1

We have $|\mathbb{N} \times \mathbb{N}| = \aleph_0$ since the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by

$$f(n,m) = 2^n 3^m$$

is injective.

QUESTION: What about $c \cdot c$?

☕ **Theorem 21 (Multiplication of Cardinals)**

If X is infinite and $Y \neq \varnothing$, then

- $|X \times X| = |X| \implies |X|\,|X| = |X|$;

- $|X \times Y| = \max(|X|, |Y|)$.

Exercise 7.1.2
Prove ☕ *Theorem 21 as an exercise.*

8 Lecture 8 Sep 24th

8.1 Cardinality (Continued 4)

8.1.1 Cardinal Arithmetic (Continued)

Exponentiation of Cardinals Recall if $\{Y_x\}_{x \in X}$ is a collection of non-empty sets, we have[1]

$$\prod_{x \in X} Y_x = \{f : X \to \bigcup_{x \in X} Y_x \mid f(x) \in Y_x\}.$$

Now if $Y = Y_x$ for all $x \in X$, we have

$$Y^X = \prod_{x \in X} = \{f : X \to Y\}.$$

Example 8.1.1

Given
$$Y = \{1, \ldots, m\} \quad X = \{1, \ldots, n\}$$

we have
$$Y^X = \{f : \{1, \ldots, n\} \to \{1, \ldots, m\}\}.$$

Observe that Y^X is similar to Y^n. So $\left|Y^X\right| = m^n$.[2]

📘 **Definition 30 (Exponentiation of Cardinals)**

Given sets X and Y, define

$$|Y|^{|X|} := \left|Y^X\right|.$$

🖥 **Theorem 22 (Exponentiation of Cardinals)**

[1] This should remind you of 🛡 Axiom 3

[2] **Need better explanation.**

If X, Y, Z are non-empty sets, then

- $|Y|^{|X|} \cdot |Y|^{|Z|} = |Y|^{|X|+|Z|}$;

- $\left(|Y|^{|X|}\right)^{|Z|} = |Y|^{|X| \cdot |Z|}$.

☕ **Theorem 23 ($2^{\aleph_0} = c$)**

We have that $2^{\aleph_0} = c$.

This requires closer studying.

✏ **Proof**

Note that $2^{\aleph_0} = \left|\{0, 1\}^{\mathbb{N}}\right|$, where[3]

[3] Explain 2nd equality.

$$\left|\{0, 1\}^{\mathbb{N}}\right| = |\{f : \mathbb{N} \to \{0, 1\}| = |\{\{a_n\}_{n=1}^{\infty} \mid a_i = 0, 1\}|$$

Given a sequence $\{a_n\} \in \{0, 1\}^{\mathbb{N}}$, define $\phi : \{0, 1\}^{\mathbb{N}} \to (0, 1)$ such that[4]

$$\phi(\{a_n\}) := \sum_{i=1}^{\infty} \frac{a_n}{3^n}.$$

[4] This is a base 3 representation (**of what?**)

which is injective since there are no trailing 2's. Therefore $2^{\aleph_0} \leq c$.

Given $\alpha \in (0, 1)$, let[5]

$$\alpha = \sum_{n=1}^{\infty} \frac{b_n}{2^n},$$

[5] This is a base 2 representation.

where $b_n = 0, 1$. Let $\psi : (0, 1) \to \{0, 1\}^{\mathbb{N}}$ such that

$$\psi(\alpha) = \psi\left(\sum_{i=1}^{\infty} \frac{b_n}{2^n}\right) = \{b_n\}$$

Then ψ is injective, and so $c \leq 2^{\aleph_0}$. Thus $2^{\aleph_0} = c$ as required. □

Example 8.1.2

Find $\left|\aleph_0^{\aleph_0}\right|$ and $c^{\aleph_0}$.

✏ **Solution**

We have that

$$c = 2^{\aleph_0} \leq \aleph_0^{\aleph_0} \leq c^{\aleph_0} = \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = c$$

Example 8.1.3

Show $|\mathcal{P}(X)| = 2^{|X|} = |2^X|$.

✏️ **Solution**

Given $f : X \to \{0,1\}$, let[6]

$$A = \{x \in X \mid f(x) = 1\} \subset X.$$

Define $\Gamma : 2^X \to \mathcal{P}(X)$ by

$$\Gamma(f) = f^{-1}(|1|)$$

Γ is injective[7].

Conversely, given $A \subset X$, define the characteristic function

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \in 2^X.$$

Then define $\Phi : P(A) \to 2^X$ such that

$$\Phi(A) = \chi_A.$$

Clearly so, Φ is injective. □

[6] A is a collection of all x's that gets mapped to f.

[7] **Why?**

💻 **Theorem 24 (Russell's Paradox)**

For any X, we have $|X| < |\mathcal{P}(X)| = 2^{|X|}$.

✏️ **Proof**

Let $f : X \to \mathcal{P}(X)$ be $f(X) = \{x\}$. Clearly, f is injective, and so $|X| < |\mathcal{P}(X)|$.

Claim: $\nexists g : X \to \mathcal{P}(X)$ surjective.

Suppose not. Let[8]

$$A = \{x \in X \mid x \notin g(x)\}$$

Pick $x_0 \in X$ with $g(x_0) = A$. Now if $x_0 \in A$, then $x_0 \in g(x_0)$, but this implies that $x \notin A$, a contradiction.

So $x_0 \notin A$, i.e. $x \notin g(x_0)$, which in turn implies that $x \in A$, yet

[8] By the **Bounded Separation Axiom** (see ZF Set Theory), this is a set, and since it is a subset of X, it is a valid element of $\mathcal{P}(X)$. Thus, we can consider such a set.

another contradiction. Therefore such a function g cannot exist, as claimed.

Therefore, we have $|X| < |\mathcal{P}(X)|$ as required. $\qquad\square$

QUESTION: Is there anything between $\aleph_0$ and c?

🛡 **Axiom 25 (Continuum Hypothesis)**

If $\aleph_0 \leq \gamma \leq c$, then either $\gamma = \aleph_0$ or $\gamma = c$.

🛡 **Axiom 26 (Generalized Continuum Hypothesis)**

If $|X| \leq \gamma \leq 2^{|X|}$, then either $\gamma = |X|$ or $\gamma = 2^{|X|}$.

In this course, we shall assume that the Continuum Hypothesis is true.

9 Lecture 9 Sep 26th

9.1 Introduction to Metric Spaces

📖 **Definition 31 (Metric & Metric Space)**

*Given a set X, a **metric** on X is a function $d : X \times X \to \mathbb{R}$ such that*

1. *(**positive definiteness**) $d(x,y) \geq 0$ and $d(x,y) = 0 \iff x = y$;*

2. *(**symmetry**) $d(x,y) = d(y,x)$; and*

3. *(**triangle inequality**) $d(x,y) \leq d(x,z) + d(z,y)$.*

*The pair (X,d) is called a **metric space**.*

Remark
A metric is an abstract notion of distance.

Example 9.1.1 (Standard Metric on $\mathbb{R}$)

Let $X = \mathbb{R}$, and let $d(x,y) = |x - y|$.

Clearly so, the first 2 criterias are satisfied:

- $|\cdot| \geq 0$ and $|x - y| = 0 \iff x = y$; and

- $|x - y| = |y - x|$.

The triangle inequality property is the usual triangle inequality of the absolute value function, i.e.

$$|x - y| \leq |x| + |y|.$$

QUESTION: For an arbitrary set X, can we define a metric on X? The following example shows that we can,

Example 9.1.2 (Discrete Metric)

Let X be any set. We can simply define

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

This metric clearly satisfies all 3 criterias of being a metric:

- $d : X \times X \to \{0,1\}$ and so $d(x,y) \geq 0$, and by definition, we have $d(x,y) = 0 \iff x = y$;

- By definition, $d(x,y) = d(y,x)$ as it does not matter how the pair is ordered; and

- Since $d(x,y) \geq 0$, we have that $d(x,y) \leq d(x,z) + d(y,z)$.

Example 9.1.3 (Euclidean Metric / 2-metric on $\mathbb{R}^n$)

Let $X = \mathbb{R}^n$. Let $\vec{x} = \{x_1, x_2, \ldots, x_n\}$ and $\vec{y} = \{y_1, y_2, \ldots, y_n\}$. Define

$$d_2(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Note that in $\mathbb{R}^2$, this is our regular (Euclidean) distance between two points.

It is not difficult to see that d_2 satisfies the first 2 criterion to being a metric:

- d_2 is the square root of the sum of squares, and so $d_2(\vec{x}, \vec{y}) \geq 0$ for any $\vec{x}, \vec{y} \in \mathbb{R}^n$, and $d_2(\vec{x}, \vec{y}) = 0 \iff \forall i \in \{1, \ldots, n\}\ x_i = y_i \iff \vec{x} = \vec{y}$;

- Since $(x_i - y_i)^2 = (y_i - x_i)^2$ for any $x_i, y_i \in \mathbb{R}$, we have that $d_2(\vec{x}, \vec{y}) = d_2(\vec{y}, \vec{x})$.

However, it is not immediately clear that d_2 satisfies the triangle inequality criterion, especially if $n \geq 3$. If $n = 2$, heuristically, the triangle inequality simply tells that the length of any one side of a triangle is less than or equal to the sum of the other two, e.g. Figure 9.1.

Figure 9.1: A visualization of the triangle inequality in $\mathbb{R}^2$.

Remark
*Many of the important examples of metric spaces are vector spaces with an abstract length function, or **norm**.*

📃 **Definition 32 (Norm & Normed Linear Space)**

*Given a vector space V (usually over $\mathbb{R}$), a **norm** on V is a function*

$$\|\cdot\| : V \to \mathbb{R}$$

such that

1. *(**positive definiteness**)* $\|v\| \geq 0$ *and* $\|v\| = 0 \iff v = 0$;

2. *(**scalar multiplication**)* $\|\alpha \cdot v\| = |\alpha| \, \|v\|$; *and*

3. *(**triangle inequality**)* $\|v + w\| \leq \|v\| + \|w\|$.

The pair $(V, \|\cdot\|)$ *is called a **normed linear space**.*

Remark

Given a normed linear space $(V, \|\cdot\|)$, *a natural metric,* $d_{\|\cdot\|}$, *on V induced by* $\|\cdot\|$ *can be defined as*

$$d_{\|\cdot\|}(x, y) = \|x - y\|.$$

Exercise 9.1.1
Prove that $d_{\|\cdot\|}$ is indeed a metric.

✏️ **Proof (Exercise 9.1.1)**

1. (positive definiteness) It is clear from the definition of a norm that $d_{\|\cdot\|}(x, y) = \|x - y\| \geq 0$, and $d_{\|\cdot\|}(x, y) = 0 \iff x - y = 0 \iff x = y$.

2. (symmetry) Symmetry follows simply from definition, as $\|x - y\| = \|y - x\|$.

3. (triangle inequality) For $x, y, z \in V$, we have

$$
\begin{aligned}
d_{\|\cdot\|}(x, y) = \|x - y\| &= \|x - z + z - y\| \\
&\leq \|x - z\| + \|z - y\| \quad \because \text{ triangle inequality of norms} \\
&= \|x - z\| + \|y - z\| \quad \because \text{ symmetry} \\
&= d_{\|\cdot\|}(x, z) + d_{\|\cdot\|}(y, z)
\end{aligned}
$$

$\square$

Example 9.1.4 (Euclidean Norm)

Let $X = \mathbb{R}^n$, and $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^2$. Define $\|\cdot\|_2$ such that

$$\|(x_1, \ldots, x_n)\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

From Example 9.1.3, we are given the triangle inequality property, in

which we have yet to prove. Positive definiteness is clear. For scalar multiplication, let $\vec{x} = x_1, \ldots, x_n$, and notice that

$$\|\alpha \cdot \vec{x}\|_2 = \sqrt{\sum_{i=1}^{n} (\alpha x_i)^2} = \sqrt{\alpha^2 \sum_{i=1}^{n} x_i^2} = |\alpha| \sqrt{\sum_{i=1}^{n} x_i^2} = |\alpha| \, \|\vec{x}\|_2 \, .$$

Thus $\|\cdot\|_2$ is indeed a norm. We call $\|\cdot\|_2$ the **2-norm** or the **Euclidean norm**.

We observe that, in comparison with Example 9.1.3, we have that

$$d_2(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_2 \, .$$

Example 9.1.5 (1-norm)

Let $X = \mathbb{R}^n$, and $\vec{x} = (x_1, \ldots, x_n)$. Define

$$\|\vec{x}\|_1 := \sum_{i=1}^{n} |x_i| \, .$$

Clearly so, $\|\cdot\|_1$ is a norm:

- **(positive definiteness)** This is true by the absolute value function, i.e. every $|x_i| \geq 0$, and so the sum over these x_i's is also non-negative, and $\sum_{i=1}^{n} |x_i| = 0 \iff \forall i \in \{1, \ldots, n\} \ x_i = 0 \iff \vec{x} = 0$.

- **(scalar multiplication)** This uses a similar argument as in the previous example.

- **(triangle inequality)** This is true by, again, the triangle inequality on absolute values.

We call $\|\cdot\|_1$ the **1-norm**.

Thus, we can define

$$d_1(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_1$$

and it can easily be verified that d_1 is indeed a metric.

Example 9.1.6

Let $X = \mathbb{R}^n$ and $\vec{x} = (x_1, \ldots, x_n)$. Define

$$\|\vec{x}\|_\infty = \max\{|x_i|\}$$

Again, it is easy to see that $\|\cdot\|_\infty$ is a norm;

- **(positive definiteness)** $\because$ $\forall i \in \{1, \ldots, n\}$ $|x_i| \geq 0$ $\implies$ $\max\{|x_i\} \geq 0|$ and $\max\{|x_i|\} = 0 \iff x_i = 0 \iff \vec{x} = 0$.

- **(scalar multiplication)** Notice that

$$\|\alpha \cdot \vec{x}\|_\infty = \max\{|\alpha x_i|\} = |\alpha| \max\{|x_i|\} = |\alpha| \|\vec{x}\|_\infty.$$

- **(triangle inequality)** This is once again true by the triangle inequality on the absolute value function, i.e.

$$\because \forall i \in \{1, \ldots, n\} \quad |x_i + y_i| \leq |x_i| + |y_i|$$
$$\max\{|x_i + y_i|\} \leq \max\{|x_i|\} + \max\{|y_i|\}.$$

We can then define

$$d_\infty(\vec{x}, \vec{y}) = \max\{|x_i - y_i|\},$$

which we can easily verify that it is indeed a metric[1].

[1] Symmetry holds by the property of the absolute value function.

HERE's an interesting notion: let

$$S_i = \{\vec{x} \in \mathbb{R}^2 \mid \|\vec{x}\|_i = 1\}, \quad i = 1, 2, \infty$$

Notice that we would then have the following graph: In fact, it is true
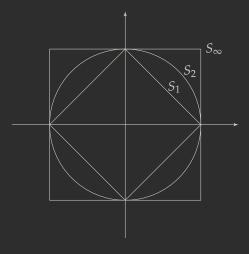
Figure 9.2: Unit ball depending on $\|\vec{x}\|_i$

that if we let $i \in \mathbb{N} \setminus \{0\}$, as suggested by Figure 9.2, we would see that the "diamond" would grow into a "circle" as in S_2, and as $i \geq 3$, the unit ball will expand and approach the "square", which is S_∞.

Another observation that we can make is if we can show that a set

is open for a "smaller" S_i, then the same set is open for any S_j for $j \geq i$.

If we apply these norms into metrics, we have

$$d_\infty \leq d_2 \leq d_1$$

where we say that d_∞ is the **least sensitive**, and d_1 being the **most sensitive**[2].

Example 9.1.7

For $1 < p < \infty$, define on $\mathbb{R}^n$

$$\|\vec{x}\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

Continuing with the same idea as in previous examples, we can let

$$d_p(\vec{x}, \vec{y}) = \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}$$

In the next lecture, we will go into proving that this is indeed a norm, and so we can define a metric using this norm.

Note that if we allow for $0 < i < 1$, then we would have a graph that looks like the following, which is a convex graph, i.e. we cannot create well-defined norms.

Figure 9.3: $\|\cdot\|_p$ for $0 < p < 1$
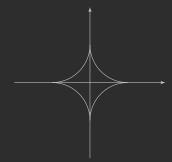
[2] For sufficently close points, we see that d_∞ would reflect the least change, while we can see change in d_1 for every two points that we take.

10 *Lecture 10 Sep 28th*

10.1 *Introduction to Metric Spaces (Continued)*

📓 **Definition 33 ($\|\cdot\|_p$-norm)**

Given $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we define, for $1 < p < \infty$, the $\|\cdot\|_p$-norm to be

$$\|\vec{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

We asked the question: why is $\|\cdot\|_p$ a norm?

🌲 **Lemma 27**

If $1 < p < \infty$,

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and if $\alpha, beta > 0$, then

$$\alpha \cdot \beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}.$$

✏️ **Proof**

Motivated by Figure 10.1, using notions from calculus, we have from calculus,

Figure 10.1: Motivation for Lemma 27.

$$\alpha\beta \le \int_0^\alpha t^{p-1}\, dt + \int_0^\beta u^{q-1}\, du$$
$$= \frac{t^p}{p}\Big|_0^\alpha + \frac{u^q}{q}\Big|_0^\beta$$
$$= \frac{\alpha^p}{p} + \frac{\beta^q}{q},$$

where we note that

$$\frac{1}{p} + \frac{1}{q} = 1$$
$$\frac{q}{p} = q - 1$$
$$\frac{p}{q} = p - 1$$
$$1 = (p-1)(q-1)$$

$\square$

🖥 Theorem 28 (Hölder's Inequality)

For $1 < p < \infty$, let $\frac{1}{p} + \frac{1}{q} = 1$ [1]. Let

$$\vec{x} = (x_1, \ldots, x_n), \quad \vec{y} = (y_1, \ldots, y_n).$$

Then

$$\sum_{i=1}^n |x_i y_i| \le \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}.$$

[1] We also call q the conjugate of p.

---

66 Note

Note that $p = 2$ is just the *Cauchy-Schwarz Inequality*:

$$\sum_{i=1}^n |x_i y_i| \le \left(\sum_{i=1}^n |x_i|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^n |y_i|^2 \right)^{\frac{1}{2}} \implies$$
$$\left(\sum_{i=1}^n |x_i y_i| \right)^2 \le \left(\sum_{i=1}^n |x_i|^2 \right) \cdot \left(\sum_{i=1}^n |y_i|^2 \right)$$

✏ Proof

Since if either $\vec{x}$ or $\vec{y}$ is zero, then we have that the inequality is trivially true, we can suppose that $\vec{x} \neq 0 \neq \vec{y}$. Now, note that for $\alpha, \beta \neq 0$, we have that[2]

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}$$

$$\Updownarrow$$

$$\sum_{i=1}^{n} |\alpha x_i \cdot \beta y_i| \leq \left(\sum_{i=1}^{n} |\alpha x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |\beta y_i|^q \right)^{\frac{1}{q}}.$$

So we can assume that

$$\left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} = 1 = \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}, \tag{10.1}$$

and if not, we can simply choose $\alpha, \beta \neq 0$ to scale these values to become one. By Lemma 27, we have

$$|x_i y_i| \leq \frac{|x_i|^p}{p} + \frac{|y_i|^q}{q}.$$

Hence

$$\sum_{i=1}^{n} |x_i y_i| \leq \sum_{i=1}^{n} \frac{|x_i|^p}{p} + \sum_{i=1}^{n} \frac{|y_i|^q}{q} = \frac{1}{p} + \frac{1}{q} \quad \because \textit{Equation (10.1)}$$

$$= 1 = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}$$

as required. □

> [2] In the second inequality, notice that we can easily get back to the first equation by dividing both sides by $\alpha\beta$.

We are now ready to prove our long-awaited result.

🖥 Theorem 29 (Minkowski's Inequality)

Let $1 < p < \infty$. If

$$\vec{x} = (x_1, \ldots, x_n), \quad \vec{y} = (y_1, \ldots, y_n)$$

in $\mathbb{R}^n$, then

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}},$$

i.e.

$$\|\vec{x} + \vec{y}\|_p \le \|\vec{x}\|_p + \|\vec{y}\|_p.$$

✏️ **Proof**

Let

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Once again, we may assume that $\vec{x} \ne 0 \ne \vec{y}$, as otherwise the inequality is true trivially so. Now, notice that

$$\sum_{i=1}^n |x_i + y_i|^p = \sum_{i=1}^n |x_i + y_i| \, |x_i + y_i|^{p-1}$$

$$\le \sum_{i=1}^n |x_i| \, |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| \, |x_i + y_i| \quad \because \begin{array}{c}\text{triangle}\\\text{inequality}\end{array}$$

$$\le \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^{(p-1)q}\right)^{\frac{1}{q}}$$

$$+ \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^{(p-1)q}\right)^{\frac{1}{q}}$$

where the last step is by Hölder's Inequality. Note that $\frac{1}{p} + \frac{1}{q} = 1 \implies p = q(p-1)$. Thus

$$\sum_{i=1}^n |x_i + y_i|^p \le \left[\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}\right] \cdot \left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{q}}$$

$$\implies \left(\sum_{i=1}^n |x_i + y_i|^p\right)^{1-\frac{1}{q}} \le \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}$$

$\square$

❝ **Note**

With this we have that $\|\cdot\|_p$ satisfies the triangle inequaltiy condition, and so $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$.

> **❝ Note**
>
> *Given $1 \le p \le q < \infty$, we have*[3]
>
> $$\|\cdot\|_\infty \le \|\cdot\|_q \le \|\cdot\|_p \le \|\cdot\|_1 .$$

[3] For a visual representation of this result, see Figure 9.2.

✎ Proof

It is quite clear that $\forall p \ge 1$,

$$\|\cdot\|_\infty = \max\{|\cdot|\} \le \left(\sum |\cdot|^p\right)^{\frac{1}{p}} = \|\cdot\|_p .$$

For $1 \le p \le q < \infty$, consider Holder's Inequality, where we have

$$\sum_{i=1}^n |a_i|\,|b_i| \le \left(\sum_{i=1}^n |a_i|^r\right)^{\frac{1}{r}} \cdot \left(\sum_{i=1}^n |b_i|^{\frac{r}{r-1}}\right)^{1-\frac{1}{r}} .$$

Let $|a_i| = |x_i|^p$, $|b_i| = 1$ and $r = \frac{q}{p} \ge 1$ [4]. Then we have

[4] Note that this is true by $p \le q$.

$$\sum_{i=1}^n |x_i|^p \le \left(\sum_{i=1}^n |x_i|^q\right)^{\frac{p}{q}} \cdot \left(\sum_{i=1}^n 1^{\frac{q}{q-p}}\right)^{1-\frac{p}{q}}$$

$$= n^{1-\frac{p}{q}} \cdot \left(\sum_{i=1}^n |x_i|^q\right)^{\frac{p}{q}}$$

Therefore, for $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}$,

$$\|\vec{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \le \left(n^{1-\frac{p}{q}} \cdot \left(\sum_{i=1}^n |x_i|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}$$

$$= n^{\frac{1}{p}-\frac{1}{q}} \cdot \left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}} = n^{\frac{1}{p}-\frac{1}{q}} \cdot \|\vec{x}\|_q .$$

Thus, we have

$$\|\cdot\|_q \le \|\cdot\|_p .$$

The chain of inequality follows. □

Example 10.1.1 (Sequence Spaces)

1. Let $\ell_1 = \left\{ \{x_i\} \mid \sum\limits_{i=1}^{\infty} |x_i| < \infty \right\}$. Define

$$\|\{x_i\}\|_1 = \sum_{i=1}^{\infty} |x_i|$$

Let $\{x_i\}, \{y_i\} \in \ell_1$. Observe that $\forall n \in \mathbb{N}$, we have

$$\sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| \leq \|\{x_i\}\|_1 + \|\{y_i\}\|_1 .$$

Then by the Monotone Convergence Theorem, we have that

$$\sum_{i=1}^{\infty} |x_i + y_i| \leq \|\{x_i\}\|_1 + \|\{y_i\}\|_1 .$$

Thus $\{x_i + y_i\} \in \ell_1$ and

$$\|\{x_i + y_i\}\|_1 \leq \|\{x_i\}\|_1 + \|\{y_i\}\|_1 .$$

Let $\{x_n\} \in \ell_1$ and $\alpha \in \mathbb{R}$. Then

$$\sum_{i=1}^{\infty} |\alpha x_i| = |\alpha| \sum_{i=1}^{\infty} |x_i| .$$

Therefore $\{\alpha x_n\} \in \ell_1$ and $\|\{\alpha x_n\}\|_1 = |\alpha| \, \|\{x_n\}\|_1$.

Thus, we have that ℓ_1 is a vector space, and $(\ell_1, \|\cdot\|_1)$ is a normed linear space.

2. Let $\ell_\infty(\mathbb{N}) = \ell_\infty = \{\{x_i\} \mid \{x_i\} \text{ is bounded }\}$. Define

$$\|\{x_i\}\|_\infty = \sup\{|x_1| \mid i \in \mathbb{N}\}.$$

Observe that $\forall \{x_i\}, \{y_i\} \in \ell_\infty$, then $\forall i \in \mathbb{N}$, we have

$$|x_i + y_i| \leq |x_i| + |y_i| \leq \|\{x_i\}\|_\infty + \|\{y_i\}\|_\infty .$$

So $\{x_i + y_i\} \in \ell_\infty$, and

$$\|\{x_i + y_i\}\|_\infty \leq \|\{x_i\}\|_\infty + \|\{y_i\}\|_\infty .$$

Consequently so, $\{\alpha x_i\} \in \ell_\infty$ and

$$\|\{\alpha x_i\}\|_\infty = |\alpha| \, \|\{x_i\}\|_\infty .$$

QUESTION: What about $\ell_p(\mathbb{R})$?

11 Lecture 11 Oct 01st

11.1 Introduction to Metric Spaces (Continued 2)

We wondered about $\ell_p(\mathbb{R})$ in the last lecture but let us consider a case that is even more general.

QUESTION: Can we define $\ell_p(\Gamma)$ for any set Γ?

Example 11.1.1

Let $\ell_\infty(\Gamma) = \{f : \Gamma \to \mathbb{R} \mid f(\Gamma) \text{ is bounded }\}$. If $f \in \ell_\infty(\Gamma)$, define

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in \Gamma\}.$$

Notice that for $f, g \in \ell_\infty(\Gamma)$, and $\alpha \in \mathbb{R}$, then we have, by the Triangle Inequality,

$$
\begin{aligned}
\|f + g\|_\infty &= \sup\{|(f + g)(x)| \mid x \in \Gamma\} \\
&= \sup\{|f(x) + g(x)| \mid x \in \Gamma\} \\
&\leq \sup\{|f(x)| \mid x \in \Gamma\} + \sup\{|g(x)| \mid x \in \Gamma\} \\
&= \|f\|_\infty \|g\|_\infty.
\end{aligned}
$$

So $f + g \in \ell_\infty(\Gamma)$, and

$$\|f + g\|_\infty \leq \|f\|_\infty + \|f\|_\infty.$$

Also, we have

$$
\begin{aligned}
\|\alpha f\|_\infty &= \sup\{|(\alpha f)(x)| \mid x \in \Gamma\} \\
&= \sup\{|\alpha| \, |f(x)| \mid x \in \Gamma\} \\
&= |\alpha| \sup\{|f(x)| \mid x \in \Gamma\} \\
&= |\alpha| \, \|f\|_\infty.
\end{aligned}
$$

So $\alpha f \in \ell_\infty(\Gamma)$, and $\|\alpha f\|_\infty = |\alpha| \, \|f\|$.

Therefore, $(\ell_\infty(\Gamma), \|\cdot\|_\infty)$ is a normed linear space.

Example 11.1.2

Let $\ell_1(\Gamma) = \{f : \Gamma \to \mathbb{R} \mid P(f)\}$, where $P(f)$ is the statement

$$\|f\|_1 = \sup \left\{ \sum_{i=1}^{n} |f(x_i)| \mid x_1, \ldots, x_n \in \Gamma, n \in \mathbb{N} \setminus \{0\} \right\} < \infty.$$

It is clear that $\ell_1(\Gamma) \subseteq \ell_\infty(\Gamma)$, where $\ell_\infty(\Gamma)$ is from Example 11.1.1.
Consequently, $(\ell_1(\Gamma), \|\cdot\|_1)$ is a normed linear space.

We can extend the same idea onto ℓ_p spaces.

Example 11.1.3

Let $X = C[a,b] = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous on } [a,b]\}$, and
define[1]

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in [a,b]\}$$
$$= \max\{|f(x)| \mid x \in [a,b]\}$$

By (regular) Triangle Inequality, for any $f, g \in C[a,b]$, we have

$$\|f + g\|_\infty = \max\{|f(x) + g(x)| \mid x \in [a,b]\}$$
$$\leq \max\{|f(x)| \mid x \in [a,b]\} + \max\{|g(x)| \mid x \in [a,b]\}$$
$$= \|f\|_\infty + \|g\|_\infty,$$

and, for $\alpha \in \mathbb{R}$,

$$\|\alpha f\|_\infty = \max\{|\alpha f(x)| \mid x \in [a,b]\}$$
$$= |\alpha| \max\{|f(x)| \mid x \in [a,b]$$
$$= |\alpha| \, \|f\|_\infty.$$

Thus $\|\cdot\|_\infty$ is a norm on $C[a,b]$, and $(C[a,b], \|\cdot\|_\infty)$ is a normed linear
space[2,3].

Also, observe that
$$C[a,b] \subset \ell_\infty([a,b]).$$

Example 11.1.4

Let $X = C[a,b]$ [4] have the same definition as the previous example,

---

[1] Note in this case sup is also max, since we are on a closed interval.

[2] This space is complete.
[3] This space is important for us for the purpose of this course.

[4] Some authors also write this as $L'[a,b]$.

but this time define

$$\|f\|_1 = \int_a^b |f(t)|\,dt.$$

By **linearity of integration**, both the triangle equality and scalar multiplication hold, and so $(C[a,b], \|\cdot\|_1)$ is a normed linear space[5].

Example 11.1.5

Let $X = C[a,b]$, and $1 < p < \infty$. Define

$$\|f\|_p = \left(\int_a^b |f(x)|^p\,dx \right)^{\frac{1}{p}}.$$

Again, by linearity of integration, scalar multiplication holds. However, it is not as easy to show for the triangle inequality; we are now asking the same question as we did before for ℓ_p, which we solved using Hölder's Inequality and Minkowski's Inequality. But now, instead of summations, we have integrations.

☕ **Theorem 30 (Hölder's Inequality v2)**

Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For each $f, g \in C[a,b]$, we have

$$\int_a^b |f(t)g(t)|\,dt \leq \left(\int_a^b |f(t)|^p\,dt \right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q\,dt \right)^{\frac{1}{q}}.$$

✏️ **Proof**

If either $f(x) = 0$ or $g(x) = 0$ for all $x \in [a,b]$, then the inequality holds trivially so. Thus, we may assume that $\forall x \in [a,b]$, $f(x) \neq 0 \neq g(x)$. By the linearity of integration, we can apply the same reasoning as we did in ☕ Theorem 28, and assume that

$$\int_a^b |f(t)|^p\,dt = 1 = \int_a^b |g(t)|^q\,dt$$

By Lemma 27, we have

$$|f(t)g(t)| \leq \frac{|f(t)|^p}{p} + \frac{|g(t)|^q}{q}.$$

Thus

$$\int_a^b |f(t)g(t)|\,dt \le \int_a^b \left(\frac{|f(t)|^p}{p} + \frac{|g(t)|^q}{q} \right) dt$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

$$= \left(\int_a^b |f(t)|^p\,dt \right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q\,dt \right)^{\frac{1}{q}}$$

as required. ☐

☕ **Theorem 31 (Minkowski's Inequality v2)**

Let $1 < p < \infty$. If $f, g \in C[a,b]$, then

$$\left(\int_a^b |(f+g)(t)|^p\,dt \right)^{\frac{1}{p}} \le (|f(t)|^p\,dt)^{\frac{1}{p}} \cdot \left(\int_a^b |g(t)|^p\,dt \right)^{\frac{1}{p}}.$$

✏️ **Proof**

The proof is similar to the one we had in ☕ Theorem 29; if $\forall x \in [a,b]$, either $f(x) = 0$ or $g(x) = 0$, then the inequality holds trivially so. Thus we may assyme that $\forall x \in [a,b]$, $f(x) \ne 0 \ne g(x)$.

Now, notice that by (regular) Triangle Inequality and, later on, ☕ Theorem 30,

$$\int_a^b |(f+g)(t)|^p\,dt$$

$$= \int_a^b |(f+g)(t)|\,|(f+g)(t)|^{p-1}\,dt$$

$$\le \int_a^b |f(t)|\,|(f+g)(t)|^{p-1}\,dt + \int_a^b |g(t)|\,|(f+g)(t)|\,dt$$

$$\le \left(\int_a^b |f(t)|^p\,dt \right)^{\frac{1}{p}} \left(\int_a^b |(f+g)(t)|^{q(p-1)}\,dt \right)^{\frac{1}{q}}$$

$$\quad + \left(\int_a^b |g(t)|^p\,dt \right)^{\frac{1}{p}} \left(\int_a^b |(f+g)(t)|^{q(p-1)}\,dt \right)^{\frac{1}{q}}$$

$$= \left[\left(\int_a^b |f(t)|^p\,dt \right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p\,dt \right)^{\frac{1}{p}} \right]$$

$$\quad \cdot \left(\int_a^b |(f+g)(t)|^p \right)^{\frac{1}{q}}$$

where we note that $\frac{1}{p} + \frac{1}{q} = 1 \implies p = q(p-1)$. Consequently, since $\frac{1}{p} = 1 - \frac{1}{q}$,

$$\left(\int_a^b |(f+g)(t)|^p \, dt \right)^{\frac{1}{p}} \leq (|f(t)|^p \, dt)^{\frac{1}{p}} \cdot \left(\int_a^b |g(t)|^p \, dt \right)^{\frac{1}{p}},$$

as required. $\qquad\square$

This shows that our definition of $\|\cdot\|_p$ on $C[a,b]$ is indeed a norm, and so $(C[a,b], \|\cdot\|_p)$ is a normed linear space.

Example 11.1.6 (Bounded Operator)

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces. Let $T : X \to Y$ be linear. Define

$$\|T\| = \sup\{\|T_X\|_Y \mid \|x\|_X < 1\}.$$

We say that T is bounded if $\|T\| < \infty$. Let

$$B(X, Y) = \{T : X \to Y \mid T \text{ is bounded }\}.$$

In the next lecture, we shall show that $(B(X, Y), \|\cdot\|)$ is a normed linear space.

QUESTION: Consider the transformation $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$. What is a norm $\left\| \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \right\|$ that works?

Exercise 11.1.1
Show that there exists an injection from $(C[a,b], \|\cdot\|_2)$ to $\ell_2(\mathbb{N})$. Note that this does not work for $p \geq 3$.

12 *Lecture 12 Oct 03rd*

12.1 *Introduction to Metric Spaces (Continued 3)*

Example 12.1.1 (Bounded Operator)

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces. Let $T : X \to Y$ be linear. Define

$$\|T\| = \sup\{\|T(x)\|_Y \mid \|x\|_X < 1\}.$$

We say that T is bounded if $\|T\| < \infty$. Let

$$B(X, Y) = \{T : X \to Y \mid T \text{ is bounded }\}.$$

To show that $B(X, Y)$ is a normed linear space, let $S, T \in B(X, Y)$, and let $\|x\|_X \leq 1$. Then

$$
\begin{aligned}
\|(S + T)(x)\|_Y &= \|S(x) + T(x)\|_Y \\
&\leq \|S(x)\|_Y + \|T(x)\|_Y \quad \because \|\cdot\|_Y \text{ is a norm} \\
&\leq \|S\| + \|T\|
\end{aligned}
$$

and so $S + T \in B(X, Y)$ and $\|S + T\| \leq \|S\| + \|T\|$. For $\alpha \in \mathbb{R}$, we have

$$
\begin{aligned}
\|\alpha S\| &= \sup\{\|(\alpha S)(x)\|_Y \mid \|x\|_X \leq 1\} \\
&= |\alpha| \sup\{\|S(x)\|_Y \mid \|x\|_X \leq 1\} \quad \because \|\cdot\|_Y \text{ is a norm} \\
&= |\alpha| \|S\|.
\end{aligned}
$$

So $(\alpha S) \in B(X, Y)$ and $\|\alpha S\| = |\alpha| \|S\|$. It is clear that due to $\|\cdot\|_Y$ being a norm, and so $\|\cdot\|$ is also positive definite. Thus $B(X, Y)$ is a normed linear space as claimed.

In this example, we look at how we can apply a translation of norms from X to Y that preserves the norm.

📑 **Definition 34 (Open & Closed)**

Let $X(,d)$ be a metric space. If $x_0 \in X$, then

$$B(x_o, \varepsilon) = \{y \in X \mid d(x,y) < \varepsilon\}$$

*is called the **open ball** centered at x_0 with radius $\varepsilon > 0$.*

$$B[x_0, \varepsilon] = \{y \in X \mid d(x,y) \leq \varepsilon\}$$

*is called the **closed ball** centered at x_0 with radius $\varepsilon > 0$.*

*We say that $U \subset X$ is **open** if*

$$\forall x \in U \ \exists \varepsilon_0 > 0 \quad B(x_0, \varepsilon_0) \subset U.$$

*We say that $F \subset X$ is **closed** if F^C is open.*

🌢 **Proposition 32 (Properties of Open Sets)**

Let (X,d) be a metric space.

1. *$X, \emptyset$ are open,*

2. *If $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets, then $U = \bigcup_{\alpha \in I}$ is open.*

3. *If $\{U_1, \ldots, U_n\}$ is a collection of open sets, then $U = \bigcap_{i=1}^{n} U_i$ is open.*

✏️ **Proof**

1. If $x_0 \in X$, then $B(x_0, 1) \subseteq X$, and so X is open. The empty set is open vacuously so.

2. Let $U = \bigcup_{\alpha \in I} U_\alpha$ and $x_0 \in U$. Then $\exists \alpha_0 \in I$ such that $x_0 \in U_{\alpha_0}$. Then $\exists \varepsilon_0 > 0$ such that

$$B(x_0, \varepsilon_0) \subset U_{\alpha_0} \subset U.$$

3. Let $x_0 \in U = \bigcap_{i=1}^{n}$. Then for each $i \in \{1, \ldots, n\}$, $\exists \varepsilon_i > 0$ such

that $B(x_0, \varepsilon_i) \subset U_i$. Let

$$\varepsilon_0 = \min\{\varepsilon_1, \ldots, \varepsilon_n\}.$$

Then we have that $\forall i \in \{1, \ldots, n\}$, $\varepsilon_0 \leq \varepsilon_i$. Thus

$$B(x_0, \varepsilon_0) \subset B(x_0, \varepsilon_i) \subset U_i$$

for each i. Therefore $B(x_0, \varepsilon_0) \subset U$.

$\square$

➤ Corollary 33 (Properties of Closed Sets)

Let (X, d) be a metric space.

1. *$X, \emptyset$ are closed.*

2. *If $\{F_\alpha\}_{\alpha \in I}$ is a collection of closed sets, then $F = \bigcap_{\alpha \in I} F_\alpha$ is closed.*

3. *If $\{F_1, \ldots, F_n\}$ is a collection of closed sets, then $F = \bigcup_{i=1}^{n} F_i$ is closed.*

✎ Proof

The proof follows from De Morgan's Laws, ♦ Proposition 32, and by taking set complements. $\square$

Example 12.2.1

Let X be any set and d the discrete metric

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

We want to know what sets are open on X under d. Notice that any set of just a singleton is open, since

$$B\left(x_0, \frac{1}{2}\right) \subset X.$$

Consequently, any $A \in \mathcal{P}(X)$ is an arbitrary union of open sets, i.e.

$$A = \bigcup_{x \in A} \{x\}.$$

Thus by 🔹 Proposition 32, A is open.

> 📖 **Note**
>
> *On $\mathbb{R}$, only $\emptyset$ and $\mathbb{R}$ itself are both open and closed. This can be proven using the **Intermediate Value Theorem**.*

📘 **Definition 35 (Topology)**

*Given any X, a set $\tau \subset \mathcal{P}(X)$ is called a **topology** on X is*

1. *$X, \emptyset \in \tau$*

2. *If $\{U_\alpha\}_{\alpha \in I}$ such that for each $\alpha \in I$, $U_\alpha \in \tau$, then*

$$U = \bigcup_{\alpha \in I} U_\alpha \in \tau.$$

3. *If $\{U_1, \ldots, U_n\}$ such that $U_i \in \tau$ for each $i \in \{1, \ldots, n\}$, then*

$$U = \bigcap_{i=1}^{n} U_i \in \tau.$$

If (X, d) is a metric space, then

$$\tau_d = \{U \subset X \mid U \text{ open in } (X, d)\}$$

*is called a **metric topology**, or **d-topology**, associated with the metric d. We call (X, τ) a **topological space**.*

Example 12.2.2

Given X,

1. $\mathcal{P}(X)$ is a topology on X, and it is called the **discrete topology**;

2. $\{\emptyset, X\}$ is a topology on X, and it is called the **indiscrete topology**.

13 *Lecture 13 Oct 05th*

13.1 *Topology on Metric Spaces (Continued)*

📖 Theorem 34 (Open Balls are Open)

1. $B(x_0, \varepsilon)$ *is open.*

2. $B[x_0, \varepsilon]$ *is closed.*

3. *Every open set is the union of open balls.*

4. $\forall x \in X$, $\{x\}$ *is closed.*

✏️ Proof

1. Consider $x \in B(x_0, \varepsilon)$ and let $r = d(x, x_0)$.

   Let $\alpha = \varepsilon - r$. Assume that $y \in B(x, \alpha)$. By the Triangle Inequality,
   $$d(x_0, y) \leq d(x_0, x) + d(x, y) < r + \alpha = \varepsilon.$$

Figure 13.1: Idea of proof for 1. in $\mathbb{R}^2$.

2. Let $y \in B[x_0, \varepsilon]^C$, and let $r = d(x_0, y)$.

   Let $\alpha = r - \varepsilon$. Assume $z \in B(y, \alpha)$, and suppose, for contradiction, that $z \in B[x_0, \varepsilon]$. Then

   $$r = d(x_0, y) \leq d(x_0, z) + d(z, y) < \varepsilon + \alpha = r,$$

Figure 13.2: Idea of proof for 2. in $\mathbb{R}^2$.

   but $r < r$ contradicts the fact that $r = r$.

3. Let $U \subset X$ be open. $\forall x \in U$, let $\varepsilon_x > 0$ be such that $B(x, \varepsilon_x) \subset U$. Then
   $$U = \bigcup_{x \in U} B(x, \varepsilon_x).$$

4. Let $y \in X$ such that $y \neq x$. Let $r = d(y, x)$. Then $x \notin B\left(y, \frac{r}{2}\right)$, and so

$$B\left(y, \frac{r}{2}\right) \subset \{x\}^C.$$

□

Example 13.1.1 (Open Intervals are Open)

Let $X = \mathbb{R}$, and $d(x, y) = |x - y|$, the standard metric. Let $I = (a, b)$, for some $a, b \in \mathbb{R} \cup \{\pm\infty\}$. Let $x \in I$. Now let

$$\varepsilon = \min\{1, |x - a|, |x - b|\}.$$

Then, clearly so, $B(x, \varepsilon) \subset I$.

If $U \subset \mathbb{R}$ is open, and if we define $\sim$ on U by $x \sim y$ [1]. if $(x, y), (y, x) \subset$ [1] This is what we did in Q1.
U. We proved that $\sim$ is an equivalence relation. Let $I_x = [x]$ be the interval defined by $\sim$. We proved that I_x is an open interval.

Consequently, if we have U being open in $\mathbb{R}$, then U can be expressed as the union of a countable collection $\{I_\alpha, \alpha \in I\}$ of open intervals, which are pairwise disjoint.

QUESTION: Given $U = \{(x, y) \mid |x|, |y| < 1\}$, can we do the same as above, i.e. can we use a countable collection of disjoint open sets to express U, or, in other words, cover U?

Example 13.1.2 (Cantor Set)

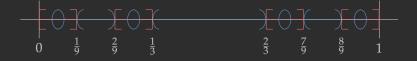Let's consider the closed interval $[0, 1]$, of which we shall label as P_0.

Figure 13.3: Cantor set showing up to $n = 2$, with the excluded interval in $n = 3$ shown.

Define P_1 by removing an open interval of length $\frac{1}{3}$ sitting in the middle of P_0, i.e.

$$P_1 = [0, 1] \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

Define P_2 by removing an open interval of length $\frac{1}{3^2}$ sitting in the

middle of each of the 2 closed intervals in P_1, ie.

$$P_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

Recursively so, define P_{n+1} by removing an open interval of length $\frac{1}{3^{n+1}}$ sitting in the middle of each of the 2^n closed intervals in P_n.

Let P, the **Cantor Set** (or **Cantor Ternary Set**), be defined as

$$P = \bigcap_{n=0}^{\infty} P_n.$$

The following are some properties of P:

1. P is closed, since it is closed under an arbitrary number of closed sets (see ➤ Corollary 33).

2. We have

$$x \in P \iff x = \sum_{i=1}^{\infty} \frac{a_n}{3^n}$$

   where $a_n = 0, 2$. In other words, every element of P is a ternary number.

3. $|P| = 2^{\aleph_0} = c$.

4. P_n does not contain any interval of length greater than or equal to $\frac{1}{3^n}$.

5. Consequently, the length of P is 0.

14 Lecture 14 Oct 12th

14.1 Topology on Metric Spaces (Continued 2)

◫ Definition 36 (Closure)

Let $A \subseteq (X, d)$. We define the **closure** $\overline{A}$ of A to be

$$\overline{A} = \cap \{F \subset X \mid F \text{ is closed }, A \subset F\}.$$

$\overline{A}$ is the smallest closed set that contains A.

◫ Definition 37 (Interior)

Let $A \subseteq (X, d)$. We define the **interior** A° of A to be

$$A^\circ = \cup \{U \subset X \mid U \text{ is open }, U \subset A\}.$$

A° is the largest open set contained in A.

Remark
We have that
$$A^\circ \subset A \subset \overline{A}$$

◫ Definition 38 (Neighbourhood)

We say that a set A is a **neighbourhood** of a point $x \in X$ if $x \in A^\circ$.[1]

[1] A neighbourhood is **not necessarily open**; the definition applies to elements in the interior after all.

❝ Note

A is a neighourhood of $x \in X$ if and only if $\exists \varepsilon > 0$ such that $B(x, \varepsilon) \subset A$.

▣ Definition 39 (Boundary Point)

*Given $A \subset (X, d)$, a point x is called a **boundary point** for A if*

$$\forall \varepsilon > 0 \quad B(x, \varepsilon) \cap A \neq \emptyset \wedge B(x, \varepsilon) \cap A^C \neq \emptyset.$$

We denote the collection of all boundary points of A by $\mathrm{bdy}(A)$.

♦ Proposition 35 (Closed Sets Include Its Boundary Points)

Let (X, d) be a metric space and $A \subset X$. Then A is closed $\iff$ $\mathrm{bdy}(A) \subset A$.

✎ Proof

(1) $\implies$ (2): Suppose $x \in A^C$, which is open. Then $\exists \varepsilon > 0$ such that $B(x, \varepsilon) \subset A^C$. Then $x \notin \mathrm{bdy}(A)$, i.e. $\mathrm{bdy}(A) \subset A$.[2]

(2) $\implies$ (3): [3]Let $x \in A^C$. Then, by assumption, $x \notin \mathrm{bdy}(A)$. Then $\exists \varepsilon > 0$ such that either $B(x, \varepsilon) \subset A$ or $B(x, \varepsilon) \subset A^C$. But since $x \notin A$, we must have $B(x, \varepsilon) \subset A^C$, i.e. A^C is open. □

[2] The idea of this proof is to show that it is impossible for the boundary to be in A^C.

[3] To show that A is closed, we should show that A^C is open.

♦ Proposition 36 (Closures include the Boundary Points of a Set)

Given $A \subset (X, d)$, we have $\overline{A} = A \cup \mathrm{bdy}(A)$.

✎ Proof

By definition, $A \subseteq \overline{A}$, so it suffices to show that $\mathrm{bdy}(A) \subset \overline{A}$ to show that $A \cup \mathrm{bdy}(A) \subseteq \overline{A}$.

[4]Assume that $x \notin \overline{A}$, i.e. $x \in \overline{A}^C$, which is open since $\overline{A}$ is closed by definition. Then $\exists \varepsilon > 0$ such that $B(x, \varepsilon) \subset \overline{A}^C$. Since

[4] Here, we employ the same proof as the previous proposition.

$x \notin A \subset \overline{A}$, we have that $B(x, \varepsilon) \cap A = \varnothing$, i.e. $x \notin \mathrm{bdy}(A)$. Therefore $\mathrm{bdy}(A) \subset \overline{A}$, and so $A \cup \mathrm{bdy}(A) \subseteq \overline{A}$ as claimed.

[5]Let $x \in \mathrm{bdy}(A \cup \mathrm{bdy}(A))$. Then $\forall \varepsilon > 0$, we have

$$B(x, \varepsilon) \cap (A \cup \mathrm{bdy}(A)) \neq \varnothing \tag{14.1}$$

$$\wedge$$

$$B(x, \varepsilon) \cap (A \cup \mathrm{bdy}(A))^C \neq \varnothing. \tag{14.2}$$

Note that by De Morgan's Laws, we have that

$$(A \cup \mathrm{bdy}(A))^C = A^C \cap \mathrm{bdy}(A)^C.$$

Then (14.2) would be

$$B(x, \varepsilon) \cap A^C \cap \mathrm{bdy}(A)^C \neq \varnothing,$$

and so

$$B(x, \varepsilon) \cap A^C \neq \varnothing \tag{14.3}$$

$$\wedge$$

$$B(x, \varepsilon) \cap \mathrm{bdy}(A)^C \neq \varnothing. \tag{14.4}$$

From (14.1), we have

$$B(x, \varepsilon) \cap A \neq \varnothing \ \vee \ B(x, \varepsilon) \cap \mathrm{bdy}(A) \neq \varnothing.$$

If $B(x, \varepsilon) \cap A \neq \varnothing$, then $\because$ (14.3), $x \in \mathrm{bdy}(A)$, and so

$$\mathrm{bdy}(A \cup \mathrm{bdy}(A)) \subseteq (A \cup \mathrm{bdy}(A)). \tag{†}$$

If $B(x, \varepsilon) \cap \mathrm{bdy}(A) \neq \varnothing$, let $z \in B(x, \varepsilon) \cap \mathrm{bdy}(A)$. $\because z \in B(x, \varepsilon)$, let $r = d(x, z)$, and $\alpha = \varepsilon - r > 0$. Let $z_0 \in B(z, \alpha)$. Then by the Triangle Inequality

$$d(x, z_0) \leq d(x, z) + d(z, z_0) < r + \alpha = \varepsilon.$$

Thus $z_0 \in B(x, \varepsilon) \implies (B(z, \alpha) \subseteq B(x, \varepsilon))$. Then $\because z \in \mathrm{bdy}(A)$, we have $B(z, \alpha) \cap A \neq \varnothing$, and so $B(x, \varepsilon) \cap A \neq \varnothing$. Then we can just follow the argument we did in (†) and arrive as the same conclusion. Consequently, by ◊ Proposition 35, $A \cup \mathrm{bdy}(A)$ is closed as claimed. □

[5] For this part, if we can show that $A \cup \mathrm{bdy}(A)$ is closed, then by definition, $\overline{A} \subseteq A \cup \mathrm{bdy}(A)$ since $\overline{A}$ is the smallest such set that contains A. To show that $A \cup \mathrm{bdy}(A)$ is closed, we can either show that $(A \cup \mathrm{bdy}(A))^C$ is open, or use ◊ Proposition 35 to show that $\mathrm{bdy}(A \cup \mathrm{bdy}(A)) \subset (A \cup \mathrm{bdy}(A))$. We shall show for the more complicated expression.

Example 14.1.1

Let $X = \mathbb{R}$ and $A = [0,1)$. We have that

- $\mathrm{bdy}(A) = \{0,1\}$;

- $A^\circ = (0,1)$; and

- $\overline{A} = [0,1]$.

Example 14.1.2

Let $X = \mathbb{R}$ and $A = \mathbb{Q}$. We have that

- $\mathrm{bdy}(A) = \mathbb{R}$ since every open ball around $a \in A$ will always contain elements in $\mathbb{Q}$ and $\mathbb{Q}^C$;

- $A^\circ = \varnothing$ since $A^\circ = A \setminus \mathrm{bdy}(A)$; and

- $\overline{A} = \mathbb{R}$ since $\overline{A} = A \cup \mathrm{bdy}(A)$.

📓 **Definition 40 (Separable)**

*A metric space (X,d) is **separable** if there exists a countable set $A \subset X$ such that $\overline{A} = X$, and call the metric space **non-separable** otherwise.*

Example 14.1.3

Every finite metric space (X,d) is separable.

This is true since every subset A of X is countable since X itself is countable. Consequently, if we pick A to be a subset that takes every other element in X, then it is clear that $\overline{A} = X$, and so (X,d) is separable.

Example 14.1.4

$\mathbb{R}$ is separable as shown in Example 14.1.2.[6]

Example 14.1.5

$\mathbb{R}^n$ is separable if d_p for all $1 \leq p \leq \infty$. We can apply the same argument that we had for Example 14.1.2 and apply it component-wise. Consequently, $\overline{\mathbb{Q}^n} = \mathbb{R}^n$. In other words, for any $(x_1, \ldots, x_n) \in (\mathbb{R}^n, d_p)$, we can pick a $(r_1, \ldots, r_n) \in \mathbb{Q}^n$ that is as close to $(x_1, \ldots, x_n)$ as possible.

[6]

Remark

Notice that

$$\overline{A} = X \iff \forall x \in X \forall \, \varepsilon > 0 \; B(x, \varepsilon) \cap A \neq \emptyset.$$

📓 **Definition 41 (Dense)**

A is dense in (X, d) if $\overline{A} = X$.

QUESTION: Is $(\ell_1, \|\cdot\|_1)$ separable? Is $(\ell_\infty, \|\cdot\|_\infty)$ separable?

Recall Example 10.1.1.

15 Lecture 15 Oct 15th

15.1 Topology on Metric Spaces (Continued 3)

📘 **Definition 42 (Limit Points)**

*Let (X,d) be a metric space, and $A \subset X$. We say that x_0 is a **limit point** for A if for any neighbourhood of x_0, we have that*

$$N \cap (A \setminus \{x_0\}) \neq \emptyset.$$

*Equivalently, $\forall \varepsilon > 0$, $\exists x \in A$, where $x \neq x_0$, such that $x \in B(x_0, \varepsilon)$.[1] We sometimes call limit points as **cluster points**. We denote the set of limit points of A as $\mathrm{Lim}(A) \subset X$ [2]*

[1] This also means that $B(x_0, \varepsilon)$ must have infinitely many points close to x_0, for otherwise, we would be able to find some $\varepsilon > \varepsilon_0 > 0$ such that $B(x_0, \varepsilon_0) \cap A = \emptyset$.

[2] Note that the set of limit points is not necessarily a subset of A.

Example 15.1.1

Let $X = \mathbb{R}$, and $A = [0,1) \subset \mathbb{R}$. We have that

$$\mathrm{Lim}[0,1) = [0,1].$$

Example 15.1.2

Let $X = \mathbb{R}$ and $A = \mathbb{N} \subset \mathbb{R}$. Since $\forall n \in \mathbb{N}$, $\exists \varepsilon = \frac{1}{2}$ such that $\forall m \in \mathbb{N} \setminus \{n\}$, we have that $m \notin B\left(n, \frac{1}{2}\right)$, we have

$$\mathrm{Lim}\,\mathbb{N} = \emptyset.$$

🔶 **Proposition 37 (Closed Sets Include Its Limit Points)**

Let $A \subset (X,d)$. Then

1. *A is closed $\iff$ $\mathrm{Lim}(A) \subset A$;*

2. $\overline{A} = A \cup \mathrm{Lim}(A)$.

✏ Proof

1. For the ($\implies$) direction, suppose A is closed. [3]Let $x_0 \in A^C$. Then $\exists \varepsilon > 0$ such that $B(x_0, \varepsilon) \cap A = \emptyset$. Thus, by definition, we have that $x_0 \notin \mathrm{Lim}(A)$ [4]. Therefore, $\mathrm{Lim}(A) \subset A$.

    For the ($\impliedby$) direction, suppose $\mathrm{Lim}(A) \subset A$. Let $x_0 \in A^C$. Then $x_0 \notin \mathrm{Lim}(A)$, which means that $\exists \varepsilon > 0$ such that $B(x_0, \varepsilon) \cap A = \emptyset$, i.e. $B(x_0, \varepsilon) \subset A^C$. Thus A is closed.

2. [5]It is clear that $A \subset \overline{A}$. Let $x_0 \in \overline{A}^C$. Then $\exists \varepsilon > 0$ such that $B(x_0, \varepsilon) \subset \overline{A}^C$. In particular, we have that $B(x_0, \varepsilon) \cap A = \emptyset$, i.e. $x_0 \notin \mathrm{Lim}(A)$. Thus $\mathrm{Lim}(A) \subset \overline{A}$.

    Again, it suffices to show that $A \cup \mathrm{Lim}(A)$ is closed to CTP. Let $x_0 \in (A \cup \mathrm{Lim}(A))^C$ [6]. Then $\exists \varepsilon > 0$ such that $B(x_0, \varepsilon) \cap A = \emptyset$. If $z \in \mathrm{Lim}(A)$ and $z \in B(x_0, \varepsilon)$, then we have $B(x_0, \varepsilon)$ is a neighbourhood of z, and so we must have that $B(x_0, \varepsilon) \cap A \neq \emptyset$, which is a contradiction. Thus $(A \cup \mathrm{Lim}(A))^C$ is open, and so $A \cup \mathrm{Lim}(A)$ is closed, as required. □

[3] This uses a reversed way of thinking: if we want to show that $\mathrm{Lim}(A) \subset A$, then instead of trying to directly show the containment, we show that all elements in A^C are in fact not limit points due to A being closed.

[4] Notice there that there are no elements in A that are **close to** x_0, and so it's not a limit point.

[5] This proof is similar to that of ♦ Proposition 36.

[6] It is clear by De Morgan's Law that $x_0 \in A^C$ and $x_0 \notin \mathrm{Lim}(A)$, which implies that $\mathrm{Lim}(A) \subset A$. But this does not give us a clear geometrical picture of the notion.

♦ Proposition 38 (Mixing the notions)

Let $A \subseteq B \subseteq (X, d)$.

1. $\overline{A} \subseteq \overline{B}$;

2. $A^\circ \subset B^\circ$;

3. $A^\circ = A \setminus \mathrm{bdy}(A)$;

4. $\mathrm{bdy}(A) = \mathrm{bdy}(A^C)$;

5. $A^\circ = \left(\overline{A^C} \right)^C$.

Exercise 15.1.1
Prove ♦ Proposition 38.

✏ Proof

1. It is clear that $A \subset B \subset \overline{B}$. Suppose $\mathrm{Lim}(A)$ is not a subset of $\overline{B}$.

Then $\exists x \in \mathrm{Lim}(A) \setminus \bar{B}$, i.e. $x \in \bar{B}^C$. Since $\bar{B}$ is closed, B^C is open and so $\exists \varepsilon > 0$ such that $B(x, \varepsilon) \subset B^C$. Since $x \in \mathrm{Lim}(A)$, $\exists a \in A$ such that $a \in B(x, \varepsilon) \subset B^C$, but $A \subset B$, a contracdiction. Thus $\mathrm{Lim}(A) \subset \bar{B}$.

2. $a \in A^\circ \implies \exists \varepsilon > 0\ B(a, \varepsilon) \subset A \subset B \implies a \in B^\circ\ \dashv$

3. $x \in A \setminus \mathrm{bdy}(A) \implies \exists \varepsilon > 0\ B(x, \varepsilon) \cap A^C = \emptyset \implies x \in A^\circ\ \dashv$
   $x \in A^\circ \implies \exists \varepsilon_0 > 0\ B(x, \varepsilon_0) \subset A$
   Sps $x \in \mathrm{bdy}(A)$. Then $\forall \varepsilon > 0\ B(x, \varepsilon) \cap A^C \neq \emptyset \implies B(x, \varepsilon_0) \cap A^C = \emptyset \notni B(x, \varepsilon_0) \subset A\ \dashv$

4. $x \in \mathrm{bdy}(A) \implies \forall \varepsilon > 0\ B(x, \varepsilon) \cap A \neq \emptyset \wedge B(x, \varepsilon) \cap A^C \neq \emptyset$
   $x \notin \mathrm{bdy}(A^C) \implies \exists \varepsilon_0 > 0\ B(x, \varepsilon_0) \cap A = \emptyset \vee B(x, \varepsilon_0) \cap A^C = \emptyset$
   But $B(x, \varepsilon_0) \cap A = \emptyset \notni \forall \varepsilon > 0\ B(x, \varepsilon) \cap A^C \neq \emptyset$
   and $B(x, \varepsilon_0) \cap A^C = \emptyset \notni \forall \varepsilon > 0\ B(x, \varepsilon) \cap A^C \neq \emptyset$
   $\implies x \in \mathrm{bdy}(A^C)\ \dashv$. The converse is a similar argument.

5. $\left(\overline{A^C} \right)^C = \left(A^C \cup \mathrm{bdy}(A^C) \right)^C = A \cap \mathrm{bdy}(A)^C = A \setminus \mathrm{bdy}(A) = A^\circ\ \dashv$

$\square$

💧 **Proposition 39 (More on Closures and Interiors)**

Let $A, B \subseteq (X, d)$.

1. $\overline{A \cup B} = \bar{A} \cup \bar{B}$;

2. $(A \cap B)^\circ = A^\circ \cap B^\circ$.

✏️ **Proof**

1. We have that $A \subset \bar{A}$ and $B \subset \bar{B}$, so $A \cup B \subset \bar{A} \cup \bar{B}$. Since $\bar{A} \cup \bar{B}$ is closed, we must have that $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$. Similarly so, we have

$$A \subseteq A \cup B \implies \bar{A} \subseteq \overline{A \cup B}$$
$$B \subseteq A \cup B \implies \bar{B} \subseteq \overline{A \cup B}$$

and so $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$.

2. Since $A^\circ \subseteq A$ and $B^\circ \subseteq B$, and $A^\circ \cap B^\circ$ is open, we must have that $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$. On the other hand, since $(A \cap B)^\circ \subset$

A° and $(A \cap B)^\circ \subset B^\circ$, we have that $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$.

$\square$

QUESTION: Is $\overline{A \cap B} = \overline{A} \cap \overline{B}$? No.

Example 15.1.3

Let $X = \mathbb{R}$, $A = \mathbb{Q}$ and $B = \mathbb{Q}^C$. We know that $\overline{A} = \mathbb{R} = \overline{B}$. But, observe that
$$\overline{A \cap B} = \varnothing \text{ while } \overline{A} \cap \overline{B} = \mathbb{R}.$$

However, we do have that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

QUESTION: Given (X, d) a metric space, is
$$B(x_0, \varepsilon) = B[x_0, \varepsilon]$$
true? Again, no.

Example 15.1.4

Let X be a set with $|X| \geq 2$, and d the discrete metric. We have that
$$B(x_0, 1) = \{x_0\} \text{ but } B[x_0, 1] = X.$$

15.2 *Convergences of Sequences*

📖 **Definition 43 (Convergence)**

Given a sequence $\{x_n\} \subset (X, d)$ and $x_0 \in X$, we say that the sequence converges *to x_0 if*
$$\forall \varepsilon > 0 \; \exists N_0 \in \mathbb{N} \; \forall n \geq N_0 \; d(x_n, x_0) < \varepsilon.$$

This is equivalent to saying that the sequence $\{d(x_n, x_0)\}$ converges to 0 in X. We denote this by
$$x_0 = \lim_{n \to \infty} x_n \text{ or } x_n \to x_0.$$

If no such x_0 exists, we say that the sequence diverges.

⬛ Theorem 40 (Uniqueness of Limits of Sequences)

If $\{x_n\}$ is a sequence in (X, d) with $x_n \to x_0$ and $x_n \to y_0$, then $x_0 = y_0$.

✏️ Proof

$$x_0 \neq y_0 \implies \exists \varepsilon = d(x_0, y_0) \implies B\left(x_0, \tfrac{\varepsilon}{2}\right) \cap B\left(y_0, \tfrac{\varepsilon}{2}\right) = \emptyset$$

However, $\exists N_0 \in \mathbb{N} \; \forall n \geq N_0$

$$x_n \in B\left(x_0, \frac{\varepsilon}{2}\right) \wedge x_n \in B\left(y_0, \frac{\varepsilon}{2}\right)$$
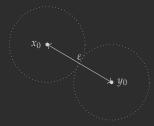
which is impossible. Thus $x_0 = y_0$. □

Figure 15.1: A geometric representation of the proof for ⬛ Theorem 40.

16 Lecture 16 Oct 17th

16.1 Convergences of Sequences (Continued)

Example 16.1.1

Let $X = \mathbb{R}^n$, $d = d_p$, for $1 \le p \le \infty$, and $\vec{x}_k = \{(x_{k,1}, x_{k,2}, \ldots, x_{k,n})\}$.

Claim :

$$\vec{X}_k \xrightarrow{\ell_p} \vec{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,n}) \iff \forall j \in \{1, \ldots, n\} \; x_{k,j} \to x_{0,j}.$$

Note : In general, we have

$$\left| x_{k,j} - x_{0,j} \right| \le \| \vec{x}_k - \vec{x}_0 \|_p$$

So it is clear that the ($\implies$) direction is true, i.e.

$$\vec{X}_k \to \vec{x}_0 \implies \forall j \in \{1, \ldots, n\} \; x_{k,j} \to x_{0,j}.$$

For the other direction, we look at the different p's to see how it works differently: in all cases, assume that $x_{k,j} \to x_{0,j}$ for all j, and that $\varepsilon > 0$

$\boxed{p = \infty}$: we have that $\exists k_0 \in \mathbb{N}$ such that $\forall k \ge k_0$,

$$\left| x_{k,j} - x_{0,j} \right| < \varepsilon \text{ for } j \in \{1, \ldots, n\},$$

and so

$$\| \vec{x}_k - \vec{x}_0 \|_\infty = \max \left\{ \left| x_{k,j} - x_{0,j} \right| : 1 \le j \le n \right\} < \varepsilon.$$

$\boxed{p = 1}$: if we assume that for each j,

$$\left| x_{k,j} - x_{0,j} \right| < \frac{\varepsilon}{n},$$

then

$$\|\vec{x}_k - \vec{x}_0\|_1 = \sum_{j=1}^{n} \left| x_{k,j} - x_{0,j} \right| < \sum_{j=1}^{n} \frac{\varepsilon}{n} = \varepsilon.$$

$1 < p < \infty$: this time, we assume that for each j,

$$\left| x_{k,j} - x_{0,j} \right| < \frac{\varepsilon}{\sqrt[p]{n}}.$$

Then

$$\|\vec{x}_k - \vec{x}_0\|_p = \left(\sum_{j=1}^{n} \left| x_{k,j} - x_{0,j} \right|^p \right)^{\frac{1}{p}} < \left(\sum_{j=1}^{n} \left(\frac{\varepsilon}{\sqrt[p]{n}} \right)^p \right)^{\frac{1}{p}} = \varepsilon.$$

This completes the proof of our claim.

Example 16.1.2

Let $X = (C[a,b], \|\cdot\|_\infty)$. Then

$$f_n \to f \iff \|f_n - f\|_\infty \to 0.$$

Notice that for the ($\implies$) direction,[1]

$$(\forall \varepsilon > 0 \ \exists N_0 \in \mathbb{N} \ \forall n \geq N_0 \ |f_n - f| < \varepsilon)$$
$$\implies \|f_n - f\|_\infty = \max\{|f_n(x) - f(x)| : x \in [a,b]\} < \varepsilon.$$

The ($\impliedby$) direction is easy, since

$$|f_n(x) - f(x)| \leq \max\{|f_n(x) - f(x)| : x \in [a,b]\} < \varepsilon.$$

[1] Note that this is **uniform convergence**, which implies **pointwise convergence**.

💻 **Theorem 41 (Sequential Characterizations of Limit Points, Boundaries, and Closedness)**

Given $A \subset (X,d)$,

1. *$x_0 \in \text{Lim}(A) \iff \exists \{x_n\} \subset A \ (x_n \neq x_0) \wedge (x_n \to x_0)$;*

2. *$x_0 \in \text{bdy}(A) \iff \exists \{x_n\} \subset A, \{y_n\} \subset A^C \ (x_n \to x_0) \wedge (y_n \to x_0)$;*

3. *A is closed $\iff (\forall \{x_n\} \subset A \ x_n \to x_0 \in X \implies x_0 \in A)$*

✎ Proof

1. $x_0 \in \mathrm{Lim}(A) \implies \forall n \in \mathbb{N} \; x_n \in B\left(x_0, \frac{1}{n}\right) \setminus \{x_0\} \implies$
   $d(x_n, x_0) < \frac{1}{n} \implies x_n \to x_0 \dashv$

   $\{x_n\} \subset A \; (x_n \to x_0) \wedge (x_n \neq x_0) \implies$
   $\forall \varepsilon > 0 \; \exists N_0 \in \mathbb{N} \; \forall n \geq N_0 \; x_n \in B(x_0, \varepsilon) \dashv$

2. $x \in \mathrm{bdy}(A) \implies$
   $(\because \forall \varepsilon > 0 \; A \cap B(x, \varepsilon) \neq \emptyset) \; \exists x_n \in A \cap B\left(x, \frac{1}{n}\right) \wedge$
   $(\because \forall \varepsilon > 0 \; A^C \cap B(x, \varepsilon) \neq \emptyset) \; \exists y_n \in A^C \cap B\left(x, \frac{1}{n}\right)$
   $\implies (\{x_n\} \subset A \wedge x_n \to x_0) \wedge (\{y_n\} \subset A^C \wedge y_n \to x_0) \dashv$

   $(\{x_n\} \subset A \wedge x_n \to x_0) \wedge (\{y_n\} \subset A^C \wedge y_n \to x_0)$
   $\implies \forall \varepsilon > 0 \; \exists N_0 \in \mathbb{N} \; \forall n \geq N_0 \; x_n, y_n \in B(x, \varepsilon)$
   $\implies x_0 \in \mathrm{bdy}(A) \dashv$

3. Sps A is closed and $(\{x_n\} \subset A) \wedge (x_n \to x_0 \in X)$.
   $x_0 \in A^C \implies \exists \varepsilon > 0 \; B(x_0, \varepsilon) \subset A^C \implies x_n \notin B(x_0, \varepsilon) \;\not\!\!\not\to\; x_n \to x_0$
   $\implies x_0 \in A$

   Sps A is $\neg$ closed $\implies (\because \; ⬩ \text{ Proposition 37}) \; \exists x_0 \in \mathrm{Lim}(A) \setminus A$
   $\implies (\because \text{ Item 1 }) \exists \{x_n\} \subset A \; (x_n \neq x_0) \wedge (x_n \to x_0 \notin A), \text{ showing}$
   that RHS is false $\dashv$

   $\square$

Example 16.1.3

Let X be a set and d a discrete metric. Then

$$x_n \to x_0 \iff \exists N \in \mathbb{N} \; \forall l n \geq N \; x_n = x_0.$$

Example 16.1.4

Let $c_0 = \{\{x_n\} \mid \lim_{n \to \infty} x_n = 0\} \subset \ell_\infty$.

Claim : c_0 is closed in ℓ_∞.

Assume $\vec{x}_k = \{x_{k,j}\}_{j=1}^\infty \subset c_0$, and let

$$\vec{x}_k \overset{\|\cdot\|_\infty}{\to} \vec{x}_0 = \{x_{0,j}\}_{j=1}^\infty \subset \ell_\infty,$$

i.e.

$$\forall \varepsilon > 0 \ \exists N_0 \in \mathbb{N} \ \forall k \geq N_0 \ \ \|\vec{x}_k - \vec{x}_0\|_\infty < \frac{\varepsilon}{2}$$

Let $k_0 \geq N_0$. $\because \vec{x}_{k_0} \in c_0$, $\exists J_0 \in \mathbb{N}$ such that $\forall j \geq J_0$, we have $\left| x_{k_0,j} \right| < \frac{\varepsilon}{2}$, and so

$$\left| x_{0,j} \right| \leq \left| x_{k_0,j} - x_{0,j} \right| + \left| x_{k_0,j} \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus we have that

$$\lim_{j \to \infty} x_{0,j} = 0$$

and so $\vec{x}_0 \in c_0$. Therefore, by ☕ Theorem 41 Item 3, c_0 is closed in ℓ_∞.

Note, however, that $c_{00} \subset \ell_1 \subset c_0$ is not closed. Also ℓ_p is not closed in c_0.

17 Lecture 17 Oct 19th

17.1 Induced Metric and Topologies

📖 **Definition 44 (Induced Metric & Induced Topology)**

Given (X, d) and $A \subset X$, we define the **induced metric** d_A on A by

$$d_A : A \times A \to \mathbb{R}$$

where $d_A(x, y) = d(x, y)$, for all $x, y \in A$, i.e. $d_A = d \restriction_{A \times A}$.

We define τ_A, the **induced topology** on A by

$$\tau_A = \{W \subset A \mid W = U \cap A, U \subset X \text{ is open }\}$$

❝ **Note**

Note that τ_A is indeed a topology: it is clear that $\varnothing \in \tau_A$. Also, $A \in \tau_A$, since X is open and $A = X \cap A$.

For an arbitrary collection $\{U_\alpha\}_{\alpha \in I} \subset \tau_A$, we know that each $U_\alpha \subset A$, and so $\bigcup_{\alpha \in I} U_\alpha \subset A$. Since each $U_\alpha \in \tau_A$, $\exists F_\alpha \subset X$ that is an open set such that $U_\alpha = F_\alpha \cap A$. Then

$$\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} F_\alpha \cap A.$$

Thus $\bigcup_{\alpha \in I} U_\alpha \in \tau_A$.

For a finite collection $\{U_1, U_2, ..., U_n\} \subset \tau_A$, we have that for each U_i, $\exists F_i \subset X$ that is open such that $U_i = F_i \cap A$. By ♦ *Proposition 32*, we

have that

$$\bigcap_{i=1}^{n} F_i \subset X$$

is open, and so

$$\bigcap_{i=1}^{n} U_i = \bigcap_{i=1}^{n} F_i \cap A \subset A.$$

Therefore, $\bigcap_{i=1}^{n} U_i \in \tau_A$.

💻 **Theorem 42 (The Metric Topology of a Subset is Its Induced Topology)**

We have

$$\tau_A = \tau_{d_A}.$$

✏️ **Proof**

$\subseteq$: $W \in \tau_A \implies \exists U \subset X$ open such that $W = U \cap A$

$\implies \forall x_0 \in W \; \exists > 0 \; B_X(x_0, \varepsilon) \subset U$

$\implies B_A(x_0, \varepsilon) = B_X(x_0, \varepsilon) \cap A \subset W \implies W \in \tau_{d_A}$ ⊣

$\supseteq$: $W \in \tau_{d_A} \implies \forall x_0 \in W \; \exists \varepsilon_x > 0 \; B_A(x_0, \varepsilon_x) \subset W$

$\implies W = \bigcup_{x_0 \in W} B_A(x_0, \varepsilon_x)$

Let $U = \bigcup_{x_0 \in W} B_X(x_0, \varepsilon_x)$, which is open

$\implies zW = \bigcup_{x_0 \in W} B_X(x_0, \varepsilon_0) \cap A = U \cap A$

$\implies W \in \tau_A$ ⊣　　　　　　　　　　　□

17.2　*Continuity on Metric Spaces*

📑 **Definition 45 (Continuity)**

*Given metric spaces $(X, d_X), (Y, d_Y)$, and $f : X \to Y$, we say that f is **continuous** at x_0 if*

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in X \; d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon.$$

☕ Theorem 43 (Continuity and Neighbourhoods)

Given metric spaces (X, d_X) and (Y, d_Y), and $f : X \to Y$, then TFAE:

1. *f is continuous at $x_0 \in X$;*

2. *if W is a neighbourhood of $f(x_0) \in Y$, then $f^{-1}(W)$ is a neighbourhood of $x_0 \in X$, where*

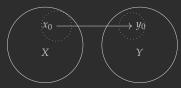$$f^{-1}(W) = \{x \in X : f(x) \in W\}.$$

Figure 17.1: Visual representation of ☕ Theorem 43

✏️ Proof

$(1) \implies (2)$: Sps f is continuous at $x_0 \in X$ and W a neighbourhood of $y_0 = f(x_0)$

$\implies f(x_0) = y_0 \in W^\circ$

$\implies \exists \varepsilon > 0 \ B(f(x_0), \varepsilon) \subset W$

$\because f$ is continuous,

$\exists \delta > 0 \ \forall x \in X \ \ x \in B_X(x_0, \delta) \implies d_Y(f(x), f(x_0)) < \varepsilon$

$\implies f(x) \in B_Y(y_0, \varepsilon) \subset W$

$\implies x \in f^{-1}(W) \implies x_0 \in f^{-1}(W)^\circ \dashv$

$(2) \implies (1)$: Sps $f^{-1}(W)$ is a neighbourhood of $x \in X$ for each neighbourhood W of $y_0 = f(x_0)$

$\implies \forall \varepsilon > 0 \ W = B_Y(f(x_0), \varepsilon)$ is a neighbourhood of $f(x_0)$

$\implies U = f^{-1}(W)$ is a neighbourhood of $x_0 \in X$

$\implies x_0 \in U$

$\implies \exists \delta > 0 \ B(x_0, \delta) \subset U = f^{-1}(W)$

$\implies (d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon) \dashv$ $\qquad\qquad \square$

☕ Theorem 44 (★ Sequential Characterization of Continuity)

For metric spaces (X, d_X) and (Y, d_Y), and $f : X \to Y$, TFAE

1. *f is continuous as $x_0 \in X$;*

2. *$\{x_n\} \subset X \ x_n \xrightarrow{X} x_0 \implies f(x_n) \xrightarrow{Y} f(x_0)$*

✏ Proof

$(1) \implies (2)$: Sps f is continuous at $x_0 \in X$.

$x_n \to x_0 \iff$

$\forall \varepsilon > 0 \; \exists \delta > 0 \; x \in B_X(x_0, \delta) \implies f(x) \in B_Y(f(x_0), \varepsilon)$

$x_n \to x_0 \implies \exists N_0 \in \; \forall n \geq N_0$

$d_X(x_0, x) < \delta \implies x_n \in B_X(x_0, \delta) \implies f(x) \in B_Y(f(x_0), \varepsilon) \; \dashv$

$(2) \implies (1)$ (Prove by Contrapositive) : Sps f is $\neg$ continuous at $x_0 \in X$

$\implies \exists \varepsilon_0 > 0 \; \forall \delta > 0 \; (x_\delta \in B_X(x_0, \delta)) \wedge (f(x_\delta) \notin B_Y(f(x_0), \varepsilon_0))$

$\implies \forall n \in \mathbb{N} \; \exists x_n \in B_X \left(x_0, \frac{1}{n} \right) \wedge f(x_n) \notin B_Y(f(x_0), \varepsilon_0)$

$\implies x_n \to x_0 \wedge f(x_n) \nrightarrow f(x_0) \; \dashv$ $\qquad\qquad\qquad\qquad \square$

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18 *Lecture 18 Oct 22nd*

18.1 *Continuity on Metric Spaces (Continued)*

📓 **Definition 46 (Continuity on a Space)**

We say that

$$f : (X, d_X) \to (Y_{d_Y})$$

is **continuous** *on X if f is continuous at each $x_0 \in X$.*

We let

$$C(X, Y) := \{ f : X \to Y \mid f \text{ is continuous on } X \},$$

be the set of all continuous functions on X.

❝ **Note**

In the case where $Y = \mathbb{R}$, we will simply write $C(X, X)$ as $C(X)$.

Remark

We can also define the following set

$$C_b(X) = \{ f \in C(X) \mid f \text{ is bounded } \}.$$

We can define $\|\cdot\|_\infty$ on $C_b(X)$ by

$$\|f\|_\infty = \sup\{ |f(x)| \mid x \in X \}.$$

Then we have that $C_b(X) \subseteq \ell_\infty(X)$.

🖥 **Theorem 45 (Analogue of Sequential Characterization of Continuity on a Space, and Continuity and Neighbourhoods)**

Let $f : (X, d_X) \to (Y, d_Y)$. TFAE

1. *f is continuous;*

2. *$f^{-1}(W)$ is open for every open set $W \subset Y$;*

3. *$x_n \to x_0 \in X \implies f(x_n) \to f(x_0) \in Y$.*

✏ **Proof**

$(1) \implies (2)$: Let $W \subset Y$ be open, and $V = f^{-1}(W)$.

$x_0 \in V \implies f(x_0) = y_0 \in W \implies W$ is a neighbourhood of y_0

$\implies (\because$ 🖥 Theorem 43 $)V$ is a neighbourhood of x_0

$\implies x_0 \in V^\circ \implies V$ is open $\dashv$

$(2) \implies (3)$: $x_n \to x_0 \in X$

$\implies \forall \varepsilon > 0 \ (\because B_Y(f(x_0), \varepsilon)$ open $)$

$\implies x_0 \in V = f^{-1}(B_Y(f(x_0), \varepsilon))$, which is open

$\implies \exists \delta > 0 \ B_X(x_0, \delta) \subset V$

$x_n \to x_0 \implies \exists N_0 \in \mathbb{N} \ \forall n \geq N_0 \ x_n \in B_X(x_0, \delta)$

$\implies f(x_n) \in B_Y(f(x_0), \varepsilon) \implies f(x_n) \to f(x_0) \dashv$

$(3) \implies (1)$: Sps $f \ \neg$ continuous, i.e.

$\exists \varepsilon_0 > 0 \ \forall \delta \geq 0 \ \exists x_\delta \in X \quad d_X(x_\delta, x_0) < \delta \wedge d_Y(f(x_\delta), f(x_0)) > \varepsilon_0$

$\implies \forall n \in \mathbb{N} \ \exists x_n \in d_X(x_0, x_n) < \frac{1}{n} \wedge d_Y(f(x_0), f(x_n)) > \varepsilon_0 \dashv \ \square$

Remark

Note that if $f : X \to Y$ and $B \subset Y$, then

$$(f^{-1}(B))^C = f^{-1}\left(B^C\right).$$

Thus we have that $f : (X, d_X) \to (Y, d_Y)$ is continuous iff $f^{-1}(F)$ is closed for each closed $F \subset Y$.

QUESTION: For the forward direction[1], if $f : (X, d_X) \to (Y, d_Y)$ is continuous, and if $U \subset X$ is open, is $f(U)$ open? **No**.

[1] instead of talking about the pullback

Example 18.1.1

Consider $f : \mathbb{R} \to \mathbb{R}$ such that $\forall x \in X, f(x) = 1$. Then $f(\mathbb{R})$ is not open.

This motivates us to consider such "nice" functions that allow us to bring open sets to open sets, and closed to their closed counterpart.

📓 **Definition 47 (Homeomorphism)**

*A function $\phi : (X, d_X) \to (Y, d_Y)$ is a **homeomorphism** if ϕ is bijective and if both ϕ and ϕ^{-1} are continuous.*

❝ Note

If ϕ is a homeomorphism, then we have

- *$\phi(W) \subset Y$ is open $\iff W \subset X$ is open;*

- *$\phi(F) \subset Y$ is closed $\iff F \subset X$ is closed.*

📓 **Definition 48 (Equivalent Metric Spaces)**

*We say that (X, d_X) and (Y, d_Y) are **equivalent metric spaces** if there exists a bijective $\phi : X \to Y$, and $c_1, c_2 \geq 0$ such that*

$$c + 1 d_X(x_1, x_2) \leq d_Y(\phi(x_1), \phi(x_2)) \leq c_2 d_X(x_1, x_2).$$

Exercise 18.1.1

*Show that the ϕ in 📓 Definition 48 is a **homeomorphism**.*

Example 18.1.2

Let (X, d) be a metric space, where X is any set and d is the discrete metric. Let $f : (X, d) \to (Y, d_Y)$, where (Y, d_Y) is another metric space that is arbitrary. Since (X, d) is discrete, it is clear that if $W \subset Y$ is open, then $f^{-1}(W)$ is open.

QUESTION: Suppose that $f : (\mathbb{R}, |\cdot|) \to (Y, d)$. When is f continuous?

Exercise 18.1.2

*Use the **Intermediate Value Theorem** to prove that the only open and closed sets in $\mathbb{R}$ are $\emptyset$ and $\mathbb{R}$.*

Let $y_0 \in Y$. We know that $\{y_0\}$ is both open and closed. Then if f is continuous, we must have that $f^{-1}(\{y_0\})$ is both open and closed. Therefore, f must be a constant function.

📰 **Definition 49 (Continuity on a set)**

*Let $A \subset (X,d)$ and $f : X \to (Y,d_Y)$. We say that f is **continuous** on A iff $f \restriction_A$ is continuous on (A,d_A), where d_A is the induced metric, and $f \restriction_A$ is the restriction of f on A.*

Remark

From the sequential characterization of continuity, we have that (A,d_A) is the induced metric iff whenever $\{x_n\} \subset A$ is a sequence with $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

19 Lecture 19 Oct 24th

19.1 Completeness of Metric Spaces

QUESTION: Is there an intrinsic way for us to tell if a sequence $\{x_n\} \subset (X, d)$ converges?

OBSERVATION Assume that $x_n \to x_0$. Then

$$\forall \varepsilon > 0 \; \exists N_0 \in \mathbb{N} \; \forall n \geq N_0 \;\; d(x_0, x_n) < \frac{\varepsilon}{2}.$$

Thus if $m, n \geq N_0$, we have

$$d(x_m, x_n) < d(x_m, x_0) + d(x_0, x_n) < \varepsilon.$$

📕 **Definition 50 (Cauchy)**

*We say that a sequence $\{x_n\} \subset (X, d)$ is **Cauchy** if*

$$\forall \varepsilon > 0 \; \exists N_0 \in \mathbb{N} \; \forall m, n \geq N_0 \; d(x_m, x_n) < \varepsilon.$$

🖥 **Theorem 46 (Convergent Sequences are Cauchy)**

Every convergent sequence is Cauchy.

We proved this in our observation.

QUESTION: Is the converse true? **No**.

Example 19.1.1

Let $X = (0,1)$ with the usual metric. Let $x_n = \frac{1}{n}$. It is clear that $\{x_n\}$ is Cauchy in (X,d), but the sequence does not converge.[1]

[1] The flaw here lies in the fact that X is open. Should we have chosen $X = [0,1]$, then the limit point 0 would have been included, allowing the sequence to actually converge.

📓 **Definition 51 (Complete Metric Spaces)**

*A metric space (X,d) is **complete** if each Cauchy sequence $\{x_n\} \subset X$ converges in (X,d).*

19.1.1 *Basic Properties of Cauchy Sequences*

OBSERVATION Given a sequence $\{x_n\} \subset (X,d)$, it is possible that $\{x_n\}$ diverges but $\{x_n\}$ has a subsequence $\{x_{n,k}\}$ that converges.

Example 19.1.2

The sequence $\{x_n\}$ defined by $x_n = (-1)^{n-1}$, i.e.

$$\{x_n\} = \{1, -1, 1, -1, \ldots\},$$

is divergent. However, $x_{2k} \to -1$ and $x_{2k+1} \to 1$.

🖥 **Theorem 47 (★★★ Convergent Cauchy Subsequences)**

Let $\{x_n\} \subset (X,d)$ be Cauchy and assume $x_{n,k} \to x_0$ for some subsequence $\{x_{n,k}\}_{k=1}^{\infty}$. Then $x_n \to x_0$.

✏️ **Proof (★★★)**

$\forall \varepsilon > 0 \ \exists N_0 \in \mathbb{N} \ \forall m, n \in N_0 \ \ d(x_n, x_m) < \frac{\varepsilon}{2}$

$x_n \to x_0 \implies \exists k_0 \in \mathbb{N} \ n_{k_0} \geq N_0 \ \ d(x_0, x_{k_0}) < \frac{\varepsilon}{2}$

$\therefore n \geq N_0 \implies$

$$d(x_n, x_0) \leq d(x_n, x_{k_0}) + d(x_{k_0}, x_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore x_n \to x_0$ □

📖 **Definition 52 (Boundedness)**

Let $A \subset (X, d)$. A is **bounded** *if*

$$\exists M > 0 \; \exists x_0 \in X \; A \subset B[x_0, M].$$

💧 **Proposition 48 (Cauchy Sequences are Bounded)**

If $\{x_n\} \subset (X, d)$ is Cauchy, then $\{x_n\}$ is bounded.

✏️ **Proof**

Let $\varepsilon = 1$. $\exists N_0 \in \mathbb{N} \; \forall m, n \geq N_0 \; d(x_n, x_m) < \varepsilon$. In particular, if $n \geq N_0$, then $d(x_n, x_{N_0}) < 1$. Then, let

$$M = \max\{d\left(x_1, x_{N_0}\right), d\left(x_2, x_{N_0}\right), \ldots, d\left(x_{N_0 - 1}, d_{N_0}\right), 1\}$$

Then it is clear that $\{x_n\} \subset B[x_{N_0}, M]$. $\qquad\square$

19.1.2　*Examples of Complete Spaces*

🖥 **Theorem 49 (Bolzano-WeierStrass)**

Every bounded sequence $\{x_n\} \subset \mathbb{R}$ has a convergent subsequence.

Be sure to review a proof of this and add it here.

🖥 **Theorem 50 ($\mathbb{R}$ is complete)**

$\mathbb{R}$ *is complete.*

✏️ **Proof**

If $\{x_n\} \subset \mathbb{R}$ is Cauchy, then it is bounded by 💧 Proposition 48, and so by Bolzano-Weierstrass, $\{x_n\}$ has a convergent subse-

quence $\{x_{n,k}\}$ such that $x_{n,k} \to x_0$. Since $\{x_n\}$ is Cauchy, by
💻 Theorem 47, $x_n \to x_0$. □

Example 19.1.3

Consider $(\mathbb{R}^n, \|\cdot\|_p)$, with $1 \leq p \leq \infty$. Let $\{\vec{x}_k\} = \{(x_{k,1}, x_{k,2}, \ldots, x_{k,n})\}$
be Cauchy in $(\mathbb{R}^n, \|\cdot\|_p)$.

$\because \left| x_{k,j} - x_{m,j} \right| \leq \|\vec{x}_k - \vec{x}_m\|_p$

$\implies \{x_{k,j}\}$ is Cauchy for each $j = 1, \ldots, n$

$\implies x_{k,j} \to x_{0,j}$ for each $j = 1, \ldots, n$ $\because$ 💻 Theorem 47

$\implies \vec{x}_k \to \vec{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,n})$

$\implies (\mathbb{R}, \|\cdot\|_p)$ is complete.

Example 19.1.4

Let (X, d) be discrete[2]. If $\{x_n\}$ is Cauchy, then $\exists N_0 \in \mathbb{N}$ such that
$\forall m, n \geq N_0$, we have $x_n = x_m$, i.e. $\{x_n\}$ converges. Therefore, (X, d) is
complete.

[2] By discrete, we mean a discrete metric space, i.e. d is a discrete metric.

Example 19.1.5 (★)

Let $X = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots \right\} \subset \mathbb{R}$ with the induced standard metric.
Recall that each of the singleton $\{\frac{1}{n}\}$ is open.

Note that given $Y = \{1, 2, \ldots, n, \ldots\} = \mathbb{N}$ with the discrete metric,
if we define $\phi : \mathbb{N} \to \left\{ 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots \right\}$ by $\phi(n) = \frac{1}{n}$, then ϕ is a
homeomorphism, and so (Y, d), where d is the discrete metric, is
complete.

However, as shown before, since $\{\frac{1}{n}\}$ is Cauchy but not conver-
gent, $X = \left\{ 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots \right\}$ is not complete.

A Useful Theorems from Earlier Calculus

☕ **Theorem 51 (Monotone Convergence Theorem)**

Let $\{x_k\}$ be a sequence in $\mathbb{R}$.

1. *Suppose $\{x_k\}$ is increasing.*

   - *If $\{x_k\}$ is bounded above, then $x_k \to \sup\{x_k\}$ as $k \to \infty$.*

   - *If $\{x_k\}$ is not bounded above, then $x_k \to \infty$ as $k \to \infty$.*

2. *Suppose $\{x_k\}$ is decreasing.*

   - *If $\{x_k\}$ is bounded below, then $x_k \to \inf\{x_k\}$ as $k \to \infty$.*

   - *If $\{x_k\}$ is not bounded below, then $x_k \to -\infty$.*

Bibliography

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Index