PMATH365 — Differential Geometry

Classnotes for Winter 2019

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Preface

This course is a post-requisite of MATH 235/245 (Linear Algebra II) and AMATH 231 (Calculus IV) or MATH 247 (Advanced Calculus III). In other words, familiarity with vector spaces and calculus is expected.

The course is spiritually separated into two parts. The first part shall be called **Exterior Differential Calculus**, which allows for a natural, metric-independent generalization of **Stokes' Theorem**, **Gauss's Theorem**, and **Green's Theorem**. Our end goal of this part is to arrive at Stokes' Theorem, that renders the **Fundamental Theorem of Calculus** as a special case of the theorem.

The second part of the course shall be called in the name of the course: **Differential Geometry**. This part is dedicated to studying geometry using techniques from differential calculus, integral calculus, linear algebra, and multilinear algebra.

Part I

Exterior Differential Calculus

1 Lecture 1 Jan 7th

1.1 Linear Algebra Review

Definition 1 (Linear Map)

*Let V, W be finite dimensional real vector spaces. A map $T : V \to W$ is called **linear** if $\forall a, b \in \mathbb{R}$, $\forall v \in V$ and $\forall w \in W$,*

$$T(av + bw) = aT(v) + bT(w).$$

We define L(U, W) to be the set of all linear maps from V to W.

Note

- *Note that $L(U, W)$ is itself a finite dimensional real vector space.*
- *The structure of the vector space $L(V, W)$ is such that $\forall T, S \in L(V, W)$, and $\forall a, b \in \mathbb{R}$, we have*

  $$aT + bS : V \to W$$

  and

  $$(aT + bS)(v) = aT(v) + bS(v).$$

- *A special case: when $W = V$, we usually write*

  $$L(V, W) = L(V),$$

  *and we call this the **space of linear operators on** V.*

Now suppose $\dim(V) = n$ for some $n \in \mathbb{N}$. This means that there exists a basis $\{e_1, \ldots, e_n\}$ of V with n elements.

Definition 2 (Basis)

A basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ of an n-dimensional vector space V is a subset of V where

1. *$\mathcal{B}$ **spans** V, i.e. $\forall v \in V$*

$$v = \sum_{i=1}^{n} v^i e_i.$$ [1]

2. *$e_1, \ldots, e_n$ are linearly independent, i.e.*

$$v^i e_i = 0 \implies v^i = 0 \text{ for every } i.$$

[1] We shall use a different convention when we write a linear combination. In particular, we use v^i to represent the i^{th} coefficient of the linear combination instead of v_i. Note that this should not be confused with taking powers, and should be clear from the context of the discussion.

Note

We shall abusively write

$$v^i e_i = \sum_i v^i e_i.$$

Again, this should be clear from the context of the discussion.

The two conditions that define a basis implies that any $v \in V$ can be expressed as $v^i e_i$, where $v^i \in \mathbb{R}$.

Definition 3 (Coordinate Vector)

*The n-tuple $(v^1, \ldots, v^n) \in \mathbb{R}^n$ is called the **coordinate vector** $[v]_{\mathcal{B}} \in \mathbb{R}^n$ of v with respect to the basis $\mathcal{B} = \{e_1, \ldots, e_n\}$.*

Note

It is clear that the coordinate vector $[v]_{\mathcal{B}}$ is dependent on the basis $\mathcal{B}$. Note that we shall also assume that the basis is "ordered", which is somewhat important since the same basis (set-wise) with a different "ordering" may give us a completely different coordinate vector.

Example 1.1.1

Let $V = \mathbb{R}^n$, and $\hat{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is the i^{th} compoe-nent of $\hat{e}_1$. Then

$$\mathcal{B}_{\text{std}} = \{\hat{e}_1, \ldots, \hat{e}_n\}$$

is called the **standard basis** of $\mathbb{R}^n$.

Note

Let $v = (v^1, \ldots, v^n) \in \mathbb{R}^n$. Then

$$v = v^1 \hat{e}_1 + \ldots v^n \hat{e}_n.$$

So $\mathbb{R}^n \ni [v]_{\mathcal{B}_{\text{std}}} = v \in V = \mathbb{R}^n$.

This is a privilege enjoyed by the n-dimensional vector space $\mathbb{R}^n$.

Now if we choose a **non-standard basis** of $\mathbb{R}^n$, say $\tilde{\mathcal{B}}$, then $[v]_{\tilde{\mathcal{B}}} \neq v$.

Note

It does not make sense to ask if a standard basis exists for an arbitrary space, as we have seen above. A geometrical way of wrestling with this notion is as follows:

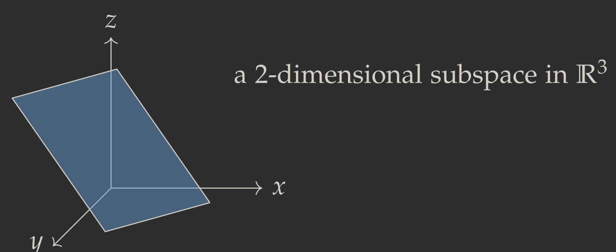

Figure 1.1: An arbitrary 2-dimensional subspace in a 3-dimensional space

While the subspace is embedding in a vector space of which has a standard basis, we cannot establish a "standard" basis for this 2-dimensional subspace. In laymen terms, we cannot tell which direction is up or down, positive or negative for the subspace, without making assumptions.

However, since we are still in a finite-dimensional vector space, we can still make a connection to a Euclidean space of the same dimension.

Definition 4 (Linear Isomorphism)

Let V be n-dimensional, and $\mathcal{B} = \{e_1, \ldots, e_n\}$ be some basis of V. The map

$$v = v^i e_i \mapsto [v]_{\mathcal{B}}$$

from V to $\mathbb{R}^n$ is a ***linear isomorphism*** *of vector spaces.*

Exercise 1.1.1

Prove that the said linear isomorphism is indeed linear and bijective[2].

[2] i.e. we are right in calling it linear and being an isomorphism

Note

Any n-dimensional real vecotr space is isomorphic to $\mathbb{R}^n$, but not ***canonically*** *so, as it requires the knowledge of the basis that is arbitrarily chosen. In other words, a different set of basis would give us a different isomorphism.*

1.2 *Orientation*

Consider an n-dimensional vector space V. Recall that for any linear operator $T \in L(V)$, we may associate a real number $\det(T)$, called the **determinant** of T, such that T is said to be **invertible** iff $\det(T) \neq 0$.

Definition 5 (Same and Opposite Orientations)

Let

$$\mathcal{B} = \{e_1, \ldots, e_n\} \quad \text{and} \quad \tilde{\mathcal{B}} = \{\tilde{e}_1, \ldots, \tilde{e}_n\}$$

be two ordered bases of V. Let $T \in L(V)$ be the linear operator defined by

$$T(e_i) = \tilde{e}_i$$

for each $i = 1, 2, \ldots, n$. This mapping is clearly invertible, and so

$\det(T) \neq 0$, *and* T^{-1} *is also linear, such that* $T^{-1}(\tilde{e}_i) = e_i$, *for each* i.

We say that $\mathcal{B}$ *and* $\tilde{\mathcal{B}}$ *determine the* ***same orientation*** *if* $\det(T) > 0$, *and we say that they determine the* ***opposite orientations*** *if* $\det(T) < 0$.

Note

- *This notion of orientation only works in real vector spaces, as, for instance, in a complex vector space, there is no sense of "positivity" or "negativity".*
- *Whenever we talk about same and opposite orientation(s), we are usually talking about 2 sets of bases. It makes sense to make a comparison to the standard basis in a Euclidean space, and determine that the compared (non-)standard basis is "positive" (same direction) or "negative" (opposite), but, again, in an arbitrary space, we do not have this convenience.*

Exercise 1.2.1 (A1Q1)

Show that any n*-dimensional real vector space* V *admits exactly 2 orientations.*

Example 1.2.1

On $\mathbb{R}^n$, consider the standard basis

$$\mathcal{B}_{\text{std}} = \{\hat{e}_1, \ldots, \hat{e}_n\}.$$

The orientation determined by $\mathcal{B}_{\text{std}}$ is called the **standard orientation** of $\mathbb{R}^n$.

Definition 6 (Dual Space)

Let V *be an* n*-dimensional vector space. Then* $\mathbb{R}$ *is a* 1*-dimensional real vector space. Thus we have that* $L(V, \mathbb{R})$ *is also a real vector space*[3]*. The* ***dual space*** V^* *of* V *is defined to be*

$$V^* := L(V, \mathbb{R}).$$

[3] Note that $L(V, \mathbb{R})$ is also finite dimensional since both the domain and codomain are finite dimensional.

Let $\mathcal{B}$ be a basis of V. For all $i = 1, 2, \dots, n$, let $e^i \in V^*$ such that

$$e^i(e_j) = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

This δ^i_j is known as the **Kronecker Delta**.

In general, we have that for every $v = v^j e_j \in V$, where $v^i \in \mathbb{R}$, by the linearity of e^i, we have

$$e^i(v) = e^i(v^j e_j) = v^j e^i(e_j) = v_j \delta^i_j = v^i.$$

So each of the e^i, when applied on v, gives us the i^{th} component of $[v]_{\mathcal{B}}$, where $\mathcal{B}$ is a basis of V.

Bibliography

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