PMATH467 — Algebraic Geometry

Classnotes for Winter 2019

bv

Johnson Ng

BMath (Hons), Pure Mathematics major, Actuarial Science Minor University of Waterloo

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Preface

The basic goal of the course is to be able to find **algebraic invariants**, which we shall use to classify topological spaces up to homeomorphism.

Other questions that we shall also look into include a uniqueness problem about manifolds; in particular, how many manifolds exist for a given invariant up to homeomorphism? We shall see that for a **2-manifold**, the only such manifold is the **2-dimensional sphere** S^2 . For a 4-manifold, it is the 4-dimensional sphere S^4 . In fact, for any other n-manifold for n > 4, the unique manifold is the respective n-sphere. The problem is trickier with the 3-manifold, and it is known as the Poincaré Conjecture, solved in 2003 by Russian Mathematician Grigori Perelman. Indeed, the said manifold is homeomorphic to the 3-sphere.

For this course, you are expected to be familiar with notions from real analysis, such as topology, and concepts from group theory.

The following topics shall be covered:

- 1. Point-Set Topology
- 2. Introduction to Topological Manifolds
- 3. Simplicial complexes & Introduction to Homology
- 4. Fundamental Groups & Covering Spaces
- 5. Classification of Surfaces

Basic Logistics for the Course

I shall leave this here for my own notes, in case something happens to my hard copy.

8 ■ LIST OF THEOREMS - ■ LIST OF THEOREMS

• OH: (Tue) 1630 - 1800, (Fri) 1245 - 1320

• OR: MC 6457

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Part I Point-Set Topology

1 Lecture 1 Jan 07th

We will not be too rigorous in this part.

1.1 Euclidean Space

For any $(x_1, ..., x_m) \in \mathbb{R}^m$, we can measure its distance from the origin 0 using either

- $||x||_{\infty} = \max\{|x_i|\}$ (the supremum-norm);
- $||x||_2 = \sqrt{\sum (x_j)^2}$ (the 2-norm); or
- $||x||_p = \left(\sum |x_j|^p\right)^{\frac{1}{p}}$ (the *p*-norm),

where we may define a "distance" by

$$d_p(x,y) = \|x - y\|_p.$$

Definition 1 (Metric)

Let X be an arbitrary space. A function $d: X \times X \to \mathbb{R}$ is called a **metric** if it satisfies

- 1. (symmetry) d(x,y) = d(y,x) for any $x,y \in X$;
- 2. (positive definiteness) $d(x,y) \ge 0$ for any $x,y \in X$, and $d(x,y) = 0 \iff x = y$; and
- 3. (triangle inequality) $\forall x, y, z \in X$

$$d(x,y) \le d(x,z) + d(y,z).$$

Definition 2 (Open and Closed Sets)

Given a space X with a metric d, and r > 0, the set

$$B(x,r) := \{ w \in X \mid d(x,w) < r \}$$

is called the **open ball** of radius r centered at x. An **open set** A is such that $\forall a \in A, \exists r > 0$ such that

$$B(a,r) \subseteq A$$
.

We say that a set is **closed** if its complement is open.

Definition 3 (Continuous Map)

A function

$$f:(X,d_1)\to (Y,d_2)$$

is said to be continuous if the preimage of an open set in Y is open in X.

See notes on Real Analysis for why we defined a continuous map in such a way.

* Warning

This definition does not imply that a continuous map f maps open sets to open sets.

Exercise 1.1.1

Contruct a function on [0,1] which assumes all values between its maximum and minimum, but is not continuous.

Solution

Consider the piecewise function

$$f(x) = \begin{cases} x & 0 \le x < \frac{1}{2} \\ x - \frac{1}{2} & x \ge \frac{1}{2}. \end{cases}$$

It is clear that the maximum and minimum are $\frac{1}{2}$ and 0 respectively, and f assumes all values between 0 and $\frac{1}{2}$. However, a piecewise function is not continuous.

Definition 4 (Homeomorphism)

A function f is a homeomorphism if it is a bijection and both f and f^{-1} are continuous.

Example 1.1.1

The function

$$g:[0,2\pi)\to\mathbb{R}^2$$
 given by $\theta\mapsto(\cos\theta,\sin\theta)$

is not homeomorphic, since if we consider an alternating series that converges to 0 on the unit circle on \mathbb{R}^2 , we have that the preimage of the series does not converge and f^{-1} is in fact discontinuous.

Now, we want to talk about topologies without referring to a metric.

■ Definition 5 (Topology)

Let X be a space. We say that the set $\mathcal{T} \subseteq \mathcal{P}(X)$ is a **topology** if

- 1. $X,\emptyset \in \mathcal{T}$;
- 2. if $\{x_{\alpha}\}_{\alpha \in A} \subseteq \mathcal{T}$ for an arbitrary index set A, then

$$\bigcup_{\alpha\in A}x_{lpha}\in\mathcal{T}$$
; and

3. If $\{x_{\beta}\}_{\beta \in B} \subset \mathcal{T}$ for some finite index set B, then

$$\bigcap_{\beta\in\mathcal{B}}x_{\beta}\in\mathcal{T}.$$

2 Lecture 2 Jan 09th

2.1 Euclidean Space (Continued)

In the last lecture, from metric topology, we generalized the notion to a more abstract one that is based solely on open sets.

Example 2.1.1

Let *X* be a set. The following two are uninteresting examples of topologies:

- 1. The trivial topology $\mathcal{T} = \{\emptyset, X\}$.
- 2. The discrete topology $\mathcal{T} = \mathcal{P}(X)$.

WE SHALL NOW continue with looking at more concepts that we shall need down the road.

Definition 6 (Closure of a Set)

Let A be a set. Its **closure**, denoted as \overline{A} , is defined as

$$\overline{A} = \bigcap_{C \supset A}^{C: closed} C.$$

It is the smallest closed set that contains A.

66 Note

In metric topology, one typically defines the closure of a set by taking the union of A and its limit points.

Definition 7 (Interior of a Set)

Let A be a set. Its **interior**, denoted either as Int (A), A° or $\overset{\circ}{A}$, is defined as

$$\overset{\circ}{A}=\overset{G:\ open}{\displaystyle\bigcup_{G\subseteq A}}G.$$

Definition 8 (Boundary of a Set)

Let A be a set. Its **boundary**, denoted as ∂A , is defined as

$$\partial A = \overline{A} \setminus \overset{\circ}{A}.$$

Exercise 2.1.1

Let A be a set. Prove that ∂A is closed.

Proof

Notice that

$$(\partial A)^C = (\overline{A} \setminus \overset{\circ}{A})^C = X \setminus \overline{A} \cup \overset{\circ}{A} = X \cap \overline{A}^C \cup \overset{\circ}{A}$$

which is open.

Exercise 2.1.2

Let A be a set. Show that

$$\partial(\partial A) = \partial A$$
.

Proof

First, notice that $\overset{\circ}{\partial A} = \emptyset$. Since ∂A is closed, $\overline{\partial A} = \partial A$. Then

$$\partial(\partial A) = \overline{\partial A} \setminus \overset{\circ}{\partial A} = \partial A \setminus \varnothing = \partial A$$

Example 2.1.2

We know that $\mathbb{Q} \subseteq \mathbb{R}$, and $\overline{\mathbb{Q}} = \mathbb{R}$. We say that \mathbb{Q} is dense in \mathbb{R} .

Definition 9 (Dense)

We say that a subset A of a set X is dense if

$$\overline{A} = X$$
.

Example 2.1.3

From the last example, we have that $\overset{\circ}{\mathbf{Q}} = \varnothing$.

Definition 10 (Limit Point)

We say that $p \in X \supseteq A$ is a limit point of A if any neighbourhood of p has a nontrivial intersection with A.

Example 2.1.4 (A Topologist's Circle)

Consider the function

$$f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0\\ 0 & x = 0 \end{cases}$$

on the interval $\left[-\frac{1}{2\pi}, \frac{1}{2\pi}\right]$. Extend the function on both ends such that we obtain Figure 2.1 (See also: Desmos).

The limit points of the graph includes all the points on the straight line from (0, -1) to (0, 1), including the endpoints. This is the case because for any of the points on this line, for any neighbourhood around the point, the neighbourhood intersects the graph f infinitely many times.

Going back to continuity, given a function f, how do we know if f^{-1} maps an open set to an open set?

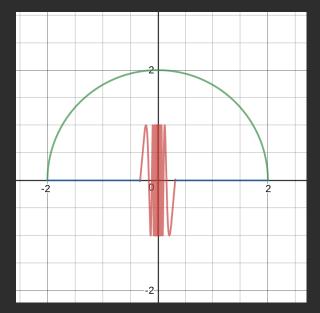


Figure 2.1: A Topologist's Circle

We can actually reduce the problem to only looking at open balls. But why are we allowed to do that?

Definition 11 (Basis of a Topology)

Given a topology \mathcal{T} , we say that $\mathcal{B} = \{B_{\alpha}\}_{\alpha \in I}$ is a **basis** if $\forall T \in \mathcal{T}$, there exists $I \subset I$ such that

$$T=\bigcup_{\alpha\in I}B_{\alpha}.$$

Note that while the definition is similar to that of a cover, we are now "covering" over sets and not points.

Example 2.1.5

Let \mathcal{T} be the Euclidean topology on \mathbb{R} . Then we can take

$$\mathcal{B} = \{(a,b) \mid a,b \in \mathbb{R}, a \leq b\}.$$

Note that \mathcal{B} is **uncountable**. We can, in fact, have ¹

$$\mathcal{B}_1 = \{(a,b) \mid a,b \in \mathbb{Q}, a \leq b\},\,$$

which is countable, as a basis for \mathbb{R} . Furthermore, we can consider the set

$$\mathcal{B}_2 = \left\{ (a,b) \mid a \leq b, a = \frac{m}{2^p}, b = \frac{n}{2^q}, m, n, p, q \in \mathbb{Z} \right\},$$

 1 Recall from PMATH 351 that we can write \mathbb{R} as a disjoint union of open intervals with rational endpoints.

which is also a countable basis for R. Notice that

$$\mathcal{B}_2 \subseteq \mathcal{B}_1 \subseteq \mathcal{B}$$
.

Example 2.1.6

In \mathbb{R}^2 , we can do a similar construction of \mathcal{B} , \mathcal{B}_1 , and \mathcal{B}_2 as in the last example and use them as a basis for \mathbb{R}^2 . In particular, we would have

$$\mathcal{B} = \{(a_1, b_1) \times (a_2, b_2) \mid a_1, a_2, b_1, b_2 \in \mathbb{R}\}.$$

This is called a dyadic partitioning of \mathbb{R}^2 .

Example 2.1.7

Let (X_1, \mathcal{T}_1) and (X_2, \mathcal{T}_2) be two topological spaces. Then the Cartesian product $X_1 \times X_2$ has topology induced from \mathcal{T}_1 and \mathcal{T}_2 by taking the set

$$\mathcal{B} = \{ eta_1 imes eta_2 \mid eta_1 \in \mathcal{T}_1, \, eta_2 \in \mathcal{T}_2 \}$$

as the basis.

Exercise 2.1.3

Prove that

- 1. β_1 and β_2 can be taken to be elements of bases $\mathcal{B}_1 \subset \mathcal{T}_1$ and $\mathcal{B}_2 \subset \mathcal{T}_2$, respectively.
- 2. the product topology on \mathbb{R}^2 is the same as the Euclidean topology.

3 Lecture 3 Jan 11th

3.1 Euclidean Space (Continued 2)

Let \tilde{X} be a metric space, and $p,q\in \tilde{X}$ with $p\neq q$. Then we have that d(p,q)=r>0.

Then we must have that

$$B\left(p,\frac{r}{3}\right)\cap B\left(q,\frac{r}{3}\right)=\emptyset.$$

Exercise 3.1.1

Prove that the above claim is true. (Use the triangle inequality)

The student is recommended to do a quick review for the first 3 chapters of the recommended text.



Figure 3.1: Idea of separation

Proof

Suppose $\exists x \in B\left(p, \frac{r}{3}\right) \cap B\left(q, \frac{r}{3}\right)$. Then

$$d(p,x) + d(q,x) < \frac{2r}{3} < r = d(p,q),$$

which violates the triangle inequality.

We observe here that the two open sets (or balls) "separate" \boldsymbol{p} and .

Definition 12 (Hausdorff / T₂)

Let X be a topological space. X is said to be **Hausdorff** or T_2 iff any 2 distinct points can be separated by disjoint open sets.

66 Note

- 1. The Hausdorff space (or T_2 space) is an important space; we can only define a metric on spaces that are T_2 .
- 2. A space is called T_1 is for any $p, q \in X$ with $p \neq q$, $\exists U \ni p$ open such that $q \notin U$ and $\exists V \ni q$ open such that $p \notin v$. It is worth noting that a T_2 space is also T_1 .

Example 3.1.1 (The Discrete Topology)

Suppose X is a metric space. For any $x \in X$, we have that $\{x\}$ is open. Thus for any $x_1, x_2 \in X$, if $x_1 \neq x_2$, then the open sets $\{x_1\}$ and $\{x_2\}$ separates x_1 and x_2 .

This is true as we can define the following metric on the space: let $d: X \times X \to \mathbb{R}$ such that

$$d(x_1, x_2) = \begin{cases} 0 & x_1 = x_2 \\ 1 & x_1 \neq x_2 \end{cases}$$

This topology is called a **discrete topology**, and it is a metric space.

Let *X* be a metric space and $A \subseteq X$. Then there is a metric induced by *X* on *A*, and this in turn induces a topology on *A*.

More generally, if $A \subset X$ where X is some arbitrary topological space, then a set $U \subseteq A$ is open iff $U = A \cap V$ for some $V \subseteq X$ that is open. In other words, a subset U of A is said to be open iff we can find an open set V in X such that the intersection of A and V gives us U.

Exercise 3.1.2

Prove that the construction above gives us a topology.

Proof

Let $A \subseteq X$. We shall show that τ_A is a topological space induced by the topological space τ of X. It is clear that $\emptyset \in \tau_A$, since it is open in X, and so $A \cap \emptyset = \emptyset$. Since X is open, we have $A \cap X = A$, and so $A \in \tau_A$.

Now if $\{U_{\alpha}\}_{\alpha \in I} \subseteq \tau_A$, then $\exists V_{\alpha} \subseteq X$ such that $U_{\alpha} = A \cap V_{\alpha}$.

Then

$$\bigcup_{\alpha\in I}U_{\alpha}=\bigcup_{\alpha\in I}A\cap V_{\alpha}=A\cap\bigcup_{\alpha\in I}V_{\alpha},$$

and $\bigcup_{\alpha \in I} V_{\alpha}$ is open in X by the properties of open sets. Thus $\bigcup_{\alpha\in I}\overline{U_{\alpha}}\in \overline{\tau_{A}}.$

If $\{U_i\}_{i=1}^n \subset \tau_A$, then again, by the properties of open sets, finite intersection of open sets is open, and so $\bigcap_{i=1}^{n} U_i \in \tau_A$.

66 Note

We can say the same can be said about closed sets of A.

Example 3.1.2

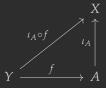
Let $A \subseteq X$ and consider the function

$$\iota_A: A \to X$$
 given by $x \mapsto x$,

which is the inclusion map.

Then ι_A is continuous when the topology on A is chosen to be the induced subspace topology. This is rather clear; notice that the inverse of the inclusion map brings open sets to open sets.

Let *Y* be an arbitrary topological space. Then let



where *f* is continuous. Then $\iota_A \circ f$ is continuous.

The converse is also true: if $\iota_A \circ f$ is continuous, then f is continuous. However, we will not prove this. This property is known as the characteristic property of the subspace topology.

Figure 3.2: Composition of a function and the inclusion map

Lemma 1 (Restriction of a Continuous Map is Continuous)

Let $X \xrightarrow{f} Y$ be continuous, and $A \subseteq X^1$. Then

¹ Here, *A* is equipped with the subspace topology

$$f \upharpoonright_A : A \to Y$$

is also continuous.

3.2 Connected Spaces

Consider the real line \mathbb{R} , and consider two disjoint intervals on \mathbb{R} .

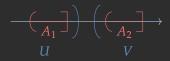


Figure 3.3: Motivation for Connectedness

Observe that we may find two open subsets U and V of \mathbb{R} such that $A_1 \subseteq U$ and $A_2 \subseteq V$, which effectively separates the two intervals on the space \mathbb{R} .

Definition 13 (Disconnectedness)

A space X is said to be **disconnected** iff X can be written as a disjoint union

$$X = A_1 \coprod A_2$$

where $A_1, A_2 \subseteq X$, $A_1 = A_2^C$, that they are both non-empty and open 2 .

² It goes without saying that the two sets are also simultaneously closed.

Definition 14 (Connctedness)

A space X is said to be **connected** if it is not disconnected.

66 Note

By the above definitions, we have that X is connected iff for any partition $X = A \coprod A^C$ with A being open, either A is \emptyset or A is X.

Example 3.2.1

The space $\mathbb{R} \setminus \{0\}$ is disconnected; our disjoint sets are $(-\infty,0)$ and $(0,\infty)$.

However, $\mathbb{R}^2 \setminus \{0\}$ is connected, but it is not easy to describe why.

Definition 15 (Path)

Require clarification

♣ Lemma 2 (Path Connectedness implies Connectedness)

If a space X is path connected, then it is connected.

■ Theorem 3 (From Connected Space to Connected Space)

If $X \stackrel{f}{\to} Y$ is continuous and X is connected, then Img(f) is connected.

4 Lecture 4 Jan 14th

4.1 Connected Spaces (Continued)

Definition 16 (Locally Connected)

We say that X is **locally connected** at x if for every open set V containing x there exists a connected, open set U with xUV. We say that the space X is **locally connected** if it is locally connected $\forall x \in X$.

Example 4.1.1

The space S generated by the function $\sin \frac{1}{x}$ with 0 at x = 0, on the \mathbb{R}^2 is not locally connected: consider $(0,y) \in S, y \neq 0$. Then any small open ball at this point will contain infinitely many line segments from S. This cannot be connected, as each one of these constitutes a component, within the neighborhood.

Definition 17 (Connected Component)

The maximal connected subsets of any topological space X are called **connected components** of the space. The components form a partition of the space.

4.2 Compactness

Definition 18 (Sequential Compactness)

For $A \subseteq X$, where X is a topological space, if $\{x_i\}_{i \in I} \subseteq A$, arbitrary sequence in A, has a convergent subsequence, we say that A is **sequentially compact**.

Definition 19 (Compactness)

We say that a topological space X is **compact** if every **open cover** of X has a finite **subcover**.

Lemma 4 (Compactness implies Sequential Compactness)

Compactness implies sequential compactness.

Example 4.2.1

[0,1) is not compact: consider the open cover $\left\{\left[0,1-\frac{1}{n}\right]\right\}_{n\in\mathbb{N}'}$ which contains [0,1) as $n\to\infty$. But whenever n is finite, we have $1-\frac{1}{n}<1$, and so any finite collection of the $\left[0,\frac{1}{n}\right)$ is not a cover of [0,1).

■ Theorem 5 (Continuous Maps map Compact Sets to Compact Images)

Let $f: X \to Y$ be continuous, where X is compact. Then f(X) is compact.

№ Proof

Let $\{U_{\alpha}\}_{\alpha\in I}$ be an open cover of f(X). Since f is continuous, we have that $f^{-1}(U_{\alpha})$ is open for each $\alpha\in I$. Since f is bijective between the image set and its domain, we have that $\{f^{-1}(U_{\alpha})\}_{\alpha\in I}$ is an open cover of X. Since X is compact, this cover has a finite subcover, say $\{f^{-1}(U_i)\}_{i=1}^n$. Thus

$$X = \bigcup_{i=1}^{n} f^{-1} \left(U_i \right).$$

Thus

$$f(X) = f\left(\bigcup_{i=1}^{n} f^{-1}(U_i)\right) = \bigcup_{i=1}^{n} U_i.$$

Hence $\{U_{\alpha}\}_{{\alpha}\in I}$ has a finite subcover and so f(X) is compact.

► Corollary 6 (Homeomorphic Maps map Compact Sets to **Compact Sets**)

Let $X \xrightarrow{f} Y$ be a homeomorphism. Then X is compact iff Y is compact.

66 Note

Compactness is a topological property.

Lemma 7 (Properties of Compact Sets)

- 1. A closed subset of a compact space is compact.
- 2. A compact subset of a topological space is closed provided that the space is Hausdorff.
- 3. In a metric space, a compact set is bounded.
- 4. Finite (Cartesian) product of compact sets is compact.

The proof for the first item is simple: consider an open cover of the closed subset, and union them with the complement of the closed subset. This covers the entire space, and so it has a finite subcover. We just need to then remove that complement set, and that would be a finite subcover for the closed subset.

Example 4.2.2

The subset [-a, b], $a, b \in \mathbb{R}$, is compact.

Example 4.2.3

 $[0,1]^{\mathbb{N}}$ is not compact: the space is equivalent to ℓ_{∞} .

Theorem 8 (Heine-Borel)

Let $X \subseteq \mathbb{R}^n$. Then X is compact iff X is closed and bounded.

Proof

 (\Longrightarrow) We say that compactness impliess boundedness. Also, since \mathbb{R}^n is Hausdorff, X is closed.¹.

(\iff) X is bounded implies that $X \subseteq [-R, R]^n$ with R sufficiently large. Since X is closed, and $[-R, R]^n$ is compact, X is necessarily compact by Lemma 7.

¹ Both from Lemma 7

■ Theorem 9 (Bolzano-Weierstrass)

Every bounded sequence in \mathbb{R}^n has a convergent subsequence.

Exercise 4.2.1

Prove **Prove** *Theorem 9 as an exercise.*

We shall start the next part this lecture.

4.3 Manifolds

Definition 20 (Locally Homeomorphic)

A space is said to be **locally homeomorphic** to \mathbb{R}^n provided that $\forall x \in X$, $\exists U \ni x$ open such that U is homeomorphic to \mathbb{R}^n .

Definition 21 (Manifold)

An n-dimensional manifold is a second countable², Hausdorff topological space that is locally homeomorphic to \mathbb{R}^n .

² a topological space is said to be **second countable** if its basis is countable.

66 Note

One can give an equivalent definition of locally homeomorphic by requiring that U be homeomorphic to an open ball $B^n \subseteq \mathbb{R}^n$. Notice that B^n is homeomorphic to \mathbb{R}^{n} 3

Example 4.3.1

Let $B^n = B^n(0,1) \subseteq \mathbb{R}^n$. Then B^n is homeomorphic to \mathbb{R}^n .

Example 4.3.2

Now consider the closed ball $\bar{B}^n = \bar{B}^n(0,1) \subset \mathbb{R}^n$. This is actually not a manifold, but we are not yet there to prove this. This sort of a structure motivates us to the next definition.

Definition 22 (Manifold on a Boundary)

An n-dimensional space that is second countable and Hausdorff, such that $\forall x \in X$, there exists a neighbourhood either homeomorphic to $B^n \subseteq \mathbb{R}^n$ or $B^n \cap \mathbb{H}^n$.

³ By scaling, really.

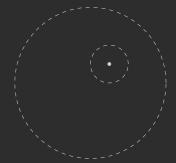


Figure 4.1: Open ball in an open set in



Figure 4.2: Open ball on a point on the boundary of a closed set

66 Note

Note that \mathbb{H}^n *is defined as*

$$\mathbb{H}^n = \{(x_1, \dots, x_n) : x_n \ge 0\}.$$

For instance, \mathbb{H}^2 has the following graph:

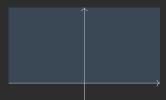


Figure 4.3: Graph of \mathbb{H}^2

Part II

Introduction to Topological Manifolds

5 Lecture 5 Jan 16th

5.1 Manifolds (Continued)

Definition 23 (Interior Point)

A point $x \in M$ is called an **interior point** if there is a local homeomorphism

$$\phi: \mathcal{U} \to \mathbb{B} \subseteq \mathbb{R}^n$$
,

where *U* is open.

In the last lecture we asked ourselves the following: how do we know if the idea of 'being on a boundary' is a well-defined notion? In particular, how do well tell the difference between the following two graphs, mathematically?



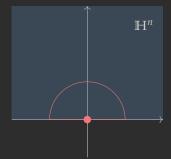


Figure 5.1: How do we tell the difference between the two graphs?

Definition 24 (Boundary Point)

A point x is on the boundary of M, denoted as $x \in \partial M$, if there exists

 $U \ni x$ that is open, and a homeomorphism

$$\psi: \mathcal{U} \to \mathbb{B}_{0,1} \cap \mathbb{H}^n$$
.

66 Note

The ϕ in \blacksquare Definition 23 and ψ in \blacksquare Definition 24 are called **local** charts.

Also, our definitions do not rule out, e.g.

$$\phi_2:\mathcal{U}\to\mathbb{B}^2\subseteq\mathbb{R}^2$$

$$\phi_5:\mathcal{U}\to\mathbb{B}^5\subseteq\mathbb{R}^5.$$

We shall later on prove that a point cannot simultaneously be a boundary point and an interior point. This property is called **the invariance of the boundary**. ¹

66 Note

M is open. Thus, we can use the same definition about open sets as before using ϕ .

66 Note

In contrast, ∂M is closed; thanks to the invariance of the boundary we have that $\partial M = M \setminus \overset{\circ}{M}$ and $\overset{\circ}{M}$ is open, and so $(\partial M)^C = \overset{\circ}{M}$.

We shall also prove the following theorem later on:

■ Theorem (Invariance of the Dimension)

The n in \mathbb{R}^n is well-defined.

Example 5.1.1

Consider the equation

$$x^2 - y^2 - z^2 = 0. (5.1)$$

Note that we may write

$$x=\pm\sqrt{y^2+z^2},$$

and so the graph generated by Equation (5.1) is as shown in Fig-

However, this is not a manifold: if we assume that a ball arounnd the origin is homeomorphic to \mathbb{R}^2 , then by removing the point at the origin in the cone, the cone becomes two disconnected components, but the ball in \mathbb{R}^2 homeomorphic to the aforementioned ball is still connected.

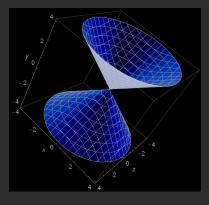


Figure 5.2: A 3D cone in \mathbb{R}^3 , from WolframAlpha

66 Note

An open subset of a manifold is a manifold, by restriction.

The 1-Sphere S^1 5.1.1

From Example 2.1.5, we have

- $[0,1) \simeq [0,\infty)$ is a manifold with boundary;
- $(0,1) \simeq \mathbb{R}$ is a manifold; and
- [0,1] is a manifold with boundary.

Example 5.1.2 (S^1 is a manifold)

Consider the function $f:[0,2\pi)\to e^{i\theta}$. The image of f is Consider

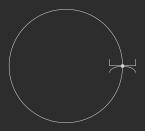


Figure 5.3: S^1 as a manifold

the following two functions $(0,2\pi) \to \mathbb{C}^2$ by

$$\theta_1 \to e^{i\theta_1}$$
 and $\theta_2 \to e^{i\theta_2 + \pi}$,

which gives us the graphs:



Figure 5.4: Basis for S^1

respectively. Note that the image of both these functions are not compact. Regardless, this gives us a basis for S^1 , which we notice is countable, Hausdorff, and locally homeomorphic to \mathbb{R}^2 .

■ Theorem 10 (1-Dimensional Manifolds Determined by Its Compactness)

Let M be a connected component of a 1-dimensional manifold. Then either

- 1. M is compact, in which case if it is
 - without a boundary, then M is homeomorphic to S^1 .
 - with a boundary, then M is homeomorphic to [0,1].
- 2. M is not compact, in which case if it is
 - without a boundary, then M is homeomorphic to [0,1).
 - with a boundary, then M is homeomorphic to (0,1).

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