PMATH467 — Algebraic Geometry

Classnotes for Winter 2019

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Preface

The basic goal of the course is to be able to find **algebraic invariants**, which we shall use to classify topological spaces up to homeomorphism.

Other questions that we shall also look into include a uniqueness problem about manifolds; in particular, how many manifolds exist for a given invariant up to homeomorphism? We shall see that for a **2-manifold**, the only such manifold is the 2-dimensional sphere S^2. For a 4-manifold, it is the 4-dimensional sphere S^4. In fact, for any other n-manifold for $n > 4$, the unique manifold is the respective n-sphere. The problem is trickier with the 3-manifold, and it is known as the Poincaré Conjecture, solved in 2003 by Russian Mathematician Grigori Perelman. Indeed, the said manifold is homeomorphic to the 3-sphere.

For this course, you are expected to be familiar with notions from **real analysis**, such as topology, and concepts from **group theory**.

The following topics shall be covered:

1. Point-Set Topology
2. Introduction to Topological Manifolds
3. Simplicial complexes & Introduction to Homology
4. Fundamental Groups & Covering Spaces
5. Classification of Surfaces

Basic Logistics for the Course

I shall leave this here for my own notes, in case something happens to my hard copy.

- OH: (Tue) 1630 - 1800, (Fri) 1245 - 1320
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Part I

Point-Set Topology

1 Lecture 1 Jan 07th

1.1 Euclidean Space

For any $(x_1, \ldots, x_m) \in \mathbb{R}^m$, we can measure its distance from the origin 0 using either

- $\|x\|_\infty = \max\{|x_i|\}$ (the supremum-norm);
- $\|x\|_2 = \sqrt{\sum (x_j)^2}$ (the 2-norm); or
- $\|x\|_p = \left(\sum |x_j|^p \right)^{\frac{1}{p}}$ (the p-norm),

where we may define a "distance" by

$$d_p(x, y) = \|x - y\|_p \,.$$

Definition 1 (Metric)

*Let X be an arbitrary space. A function $d : X \times X \to \mathbb{R}$ is called a **metric** if it satisfies*

1. *(symmetry) $d(x, y) = d(y, x)$ for any $x, y \in X$;*
2. *(positive definiteness) $d(x, y) \geq 0$ for any $x, y \in X$, and $d(x, y) = 0 \iff x = y$; and*
3. *(triangle inequality) $\forall x, y, z \in X$*

$$d(x, y) \leq d(x, z) + d(y, z).$$

Definition 2 (Open and Closed Sets)

Given a space X with a metric d, and $r > 0$, the set

$$B(x,r) := \{w \in X \mid d(x,w) < r\}$$

is called the ***open ball*** *of radius r centered at x. An* ***open set*** *A is such that $\forall a \in A$, $\exists r > 0$ such that*

$$B(a,r) \subseteq A.$$

We say that a set is ***closed*** *if its complement is open.*

Definition 3 (Continuous Map)

A function

$$f : (X,d_1) \to (Y,d_2)$$

is said to be ***continuous*** *if the preimage of an open set in Y is open in X.*

See notes on Real Analysis for why we defined a continuous map in such a way.

Warning

This definition does not imply that a continuous map f maps open sets to open sets.

Exercise 1.1.1

Contruct a function on $[0,1]$ which assumes all values between its maximum and minimum, but is not continuous.

Solution

Consider the piecewise function

$$f(x) = \begin{cases} x & 0 \leq x < \frac{1}{2} \\ x - \frac{1}{2} & x \geq \frac{1}{2}. \end{cases}$$

It is clear that the maximum and minimum are $\frac{1}{2}$ and 0 respectively, and f assumes all values between 0 and $\frac{1}{2}$. However, a piecewise function is not continuous.

Definition 4 (Homeomorphism)

*A function f is a **homeomorphism** if it is a bijection and both f and f^{-1} are continuous.*

Example 1.1.1

The function

$$g : [0, 2\pi) \to \mathbb{R}^2 \text{ given by } \theta \mapsto (\cos\theta, \sin\theta)$$

is not homeomorphic, since if we consider an alternating series that converges to 0 on the unit circle on $\mathbb{R}^2$, we have that the preimage of the series does not converge and f^{-1} is in fact discontinuous.

Now, we want to talk about topologies without referring to a metric.

Definition 5 (Topology)

*Let X be a space. We say that the set $\mathcal{T} \subseteq \mathcal{P}(X)$ is a **topology** if*

1. $X, \emptyset \in \mathcal{T}$;
2. *if* $\{x_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ *for an arbitrary index set* A*, then*
$$\bigcup_{\alpha \in A} x_\alpha \in \mathcal{T}; \text{ and}$$
3. *If* $\{x_\beta\}_{\beta \in B} \subset \mathcal{T}$ *for some finite index set* B*, then*
$$\bigcap_{\beta \in B} x_\beta \in \mathcal{T}.$$

2 Lecture 2 Jan 09th

2.1 Euclidean Space (Continued)

In the last lecture, from metric topology, we generalized the notion to a more abstract one that is based solely on open sets.

Example 2.1.1

Let X be a set. The following two are uninteresting examples of topologies:

1. The **trivial topology** $\mathcal{T} = \{\emptyset, X\}$.
2. The **discrete topology** $\mathcal{T} = \mathcal{P}(X)$.

We shall now continue with looking at more concepts that we shall need down the road.

Definition 6 (Closure of a Set)

*Let A be a set. Its **closure**, denoted as $\overline{A}$, is defined as*

$$\overline{A} = \bigcap_{C \supset A}^{C:\text{ closed}} C.$$

It is the smallest closed set that contains A.

Note

In metric topology, one typically defines the closure of a set by taking the union of A and its limit points.

Definition 7 (Interior of a Set)

*Let A be a set. Its **interior**, denoted either as $\mathrm{Int}\,(A)$, A° or $\mathring{A}$, is defined as*

$$\mathring{A} = \bigcup_{G \subseteq A}^{G:\text{ open}} G.$$

Definition 8 (Boundary of a Set)

*Let A be a set. Its **boundary**, denoted as ∂A, is defined as*

$$\partial A = \overline{A} \setminus \mathring{A}.$$

Exercise 2.1.1

Let A be a set. Prove that ∂A is closed.

Proof

Notice that

$$(\partial A)^C = (\overline{A} \setminus \mathring{A})^C = X \setminus \overline{A} \cup \mathring{A} = X \cap \overline{A}^C \cup \mathring{A}$$

which is open. $\square$

Exercise 2.1.2

Let A be a set. Show that

$$\partial(\partial A) = \partial A.$$

Proof

First, notice that $\mathring{\partial A} = \emptyset$. Since ∂A is closed, $\overline{\partial A} = \partial A$. Then

$$\partial(\partial A) = \overline{\partial A} \setminus \mathring{\partial A} = \partial A \setminus \emptyset = \partial A$$

Example 2.1.2

We know that $\mathbb{Q} \subseteq \mathbb{R}$, and $\overline{\mathbb{Q}} = \mathbb{R}$. We say that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 9 (Dense)

We say that a subset A of a set X is dense if

$$\overline{A} = X.$$

Example 2.1.3

From the last example, we have that $\mathring{\mathbb{Q}} = \emptyset$.

Definition 10 (Limit Point)

We say that $p \in X \supseteq A$ is a limit point of A if any neighbourhood of p has a nontrivial intersection with A.

Example 2.1.4 (A Topologist's Circle)

Consider the function

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

on the interval $\left[-\frac{1}{2\pi}, \frac{1}{2\pi}\right]$. Extend the function on both ends such that we obtain Figure 2.1 (See also: Desmos).

The limit points of the graph includes all the points on the straight line from $(0, -1)$ to $(0, 1)$, including the endpoints. This is the case because for any of the points on this line, for any neighbourhood around the point, the neighbourhood intersects the graph f infinitely many times.

Going back to continuity, given a function f, how do we know if f^{-1} maps an open set to an open set?

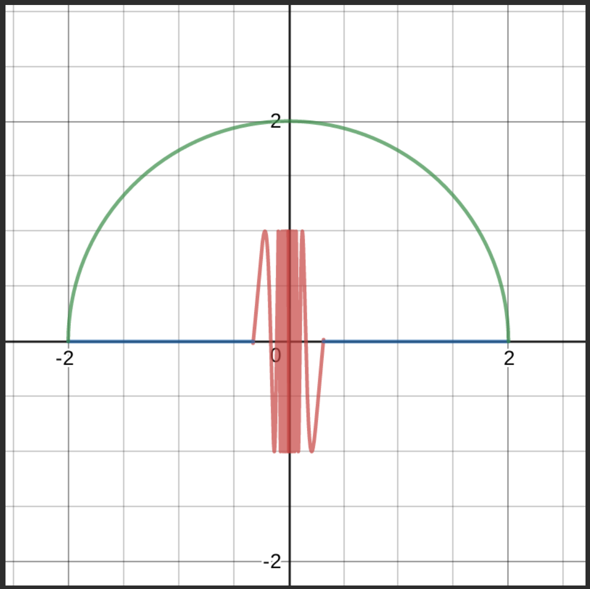

Figure 2.1: A Topologist's Circle

We can actually reduce the problem to only looking at open balls. But why are we allowed to do that?

Definition 11 (Basis of a Topology)

Given a topology $\mathcal{T}$, we say that $\mathcal{B} = \{B_\alpha\}_{\alpha \in I}$ is a ***basis*** *if $\forall T \in \mathcal{T}$, there exists $J \subset I$ such that*

$$T = \bigcup_{\alpha \in J} B_\alpha.$$

Note that while the definition is similar to that of a cover, we are now "covering" over sets and not points.

Example 2.1.5

Let $\mathcal{T}$ be the Euclidean topology on $\mathbb{R}$. Then we can take

$$\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}, a \leq b\}.$$

Note that $\mathcal{B}$ is **uncountable**. We can, in fact, have [1]

$$\mathcal{B}_1 = \{(a, b) \mid a, b \in \mathbb{Q}, a \leq b\},$$

which is countable, as a basis for $\mathbb{R}$. Furthermore, we can consider the set

$$\mathcal{B}_2 = \left\{ (a, b) \;\middle|\; a \leq b, a = \frac{m}{2^p}, b = \frac{n}{2^q}, m, n, p, q \in \mathbb{Z} \right\},$$

[1] Recall from PMATH 351 that we can write $\mathbb{R}$ as a disjoint union of open intervals with rational endpoints.

which is also a countable basis for $\mathbb{R}$. Notice that

$$\mathcal{B}_2 \subseteq \mathcal{B}_1 \subseteq \mathcal{B}.$$

Example 2.1.6

In $\mathbb{R}^2$, we can do a similar construction of $\mathcal{B}$, $\mathcal{B}_1$, and $\mathcal{B}_2$ as in the last example and use them as a basis for $\mathbb{R}^2$. In particular, we would have

$$\mathcal{B} = \{(a_1, b_1) \times (a_2, b_2) \mid a_1, a_2, b_1, b_2 \in \mathbb{R}\}.$$

This is called a **dyadic partitioning** of $\mathbb{R}^2$.

Example 2.1.7

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be two topological spaces. Then the Cartesian product $X_1 \times X_2$ has topology induced from $\mathcal{T}_1$ and $\mathcal{T}_2$ by taking the set

$$\mathcal{B} = \{\beta_1 \times \beta_2 \mid \beta_1 \in \mathcal{T}_1,\ \beta_2 \in \mathcal{T}_2\}$$

as the basis.

Exercise 2.1.3

Prove that

1. β_1 *and* β_2 *can be taken to be elements of bases* $\mathcal{B}_1 \subset \mathcal{T}_1$ *and* $\mathcal{B}_2 \subset \mathcal{T}_2$*, respectively.*
2. *the product topology on* $\mathbb{R}^2$ *is the same as the Euclidean topology.*

3 Lecture 3 Jan 11th

3.1 Euclidean Space (Continued 2)

The student is recommended to do a quick review for the first 3 chapters of the recommended text.

Let $\tilde{X}$ be a metric space, and $p, q \in \tilde{X}$ with $p \neq q$. Then we have that $d(p, q) = r > 0$.

Then we must have that

$$B\left(p, \frac{r}{3}\right) \cap B\left(q, \frac{r}{3}\right) = \emptyset.$$

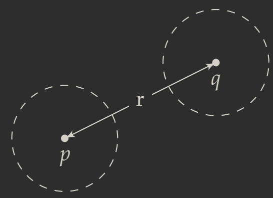

Figure 3.1: Idea of separation

Exercise 3.1.1

Prove that the above claim is true. (Use the triangle inequality)

Proof

Suppose $\exists x \in B\left(p, \frac{r}{3}\right) \cap B\left(q, \frac{r}{3}\right)$. Then

$$d(p, x) + d(q, x) < \frac{2r}{3} < r = d(p, q),$$

which violates the triangle inequality. $\square$

We observe here that the two open sets (or balls) "separate" p and q.

Definition 12 (Hausdorff / T_2)

Let X be a topological space. X is said to be ***Hausdorff*** *or T_2 iff any 2 distinct points can be separated by disjoint open sets.*

Note

1. *The Hausdorff space (or T_2 space) is an important space; we can only define a metric on spaces that are T_2.*
2. *A space is called T_1 is for any $p, q \in X$ with $p \neq q$, $\exists U \ni p$ open such that $q \notin U$ and $\exists V \ni q$ open such that $p \notin v$. It is worth noting that a T_2 space is also T_1.*

Example 3.1.1 (The Discrete Topology)

Suppose X is a metric space. For any $x \in X$, we have that $\{x\}$ is open. Thus for any $x_1, x_2 \in X$, if $x_1 \neq x_2$, then the open sets $\{x_1\}$ and $\{x_2\}$ separates x_1 and x_2.

This is true as we can define the following metric on the space: let $d : X \times X \to \mathbb{R}$ such that

$$d(x_1, x_2) = \begin{cases} 0 & x_1 = x_2 \\ 1 & x_1 \neq x_2 \end{cases}$$

This topology is called a **discrete topology**, and it is a metric space.

Let X be a metric space and $A \subseteq X$. Then there is a metric induced by X on A, and this in turn induces a topology on A.

More generally, if $A \subset X$ where X is some arbitrary topological space, then a set $U \subseteq A$ is open iff $U = A \cap V$ for some $V \subseteq X$ that is open. In other words, a subset U of A is said to be open iff we can find an open set V in X such that the intersection of A and V gives us U.

Exercise 3.1.2

Prove that the construction above gives us a topology.

Proof

Let $A \subseteq X$. We shall show that τ_A is a topological space induced by the topological space τ of X. It is clear that $\emptyset \in \tau_A$, since it is open in X, and so $A \cap \emptyset = \emptyset$. Since X is open, we have $A \cap X = A$, and so $A \in \tau_A$.

Now if $\{U_\alpha\}_{\alpha \in I} \subseteq \tau_A$, then $\exists V_\alpha \subseteq X$ such that $U_\alpha = A \cap V_\alpha$.

Then

$$\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} A \cap V_\alpha = A \cap \bigcup_{\alpha \in I} V_\alpha,$$

and $\bigcup_{\alpha \in I} V_\alpha$ is open in X by the properties of open sets. Thus $\bigcup_{\alpha \in I} U_\alpha \in \tau_A$.

If $\{U_i\}_{i=1}^n \subset \tau_A$, then again, by the properties of open sets, finite intersection of open sets is open, and so $\bigcap_{i=1}^n U_i \in \tau_A$.

□

Note

We can say the same can be said about closed sets of A.

Example 3.1.2

Let $A \subseteq X$ and consider the function

$$\iota_A : A \to X \text{ given by } x \mapsto x,$$

which is the inclusion map.

Then ι_A is continuous when the topology on A is chosen to be the induced subspace topology. This is rather clear; notice that the inverse of the inclusion map brings open sets to open sets.

Let Y be an arbitrary topological space. Then let

$$\begin{tikzcd} & X \\ Y \arrow[ur, "\iota_A \circ f"] \arrow[r, "f"'] & A \arrow[u, "\iota_A"'] \end{tikzcd}$$

Figure 3.2: Composition of a function and the inclusion map

where f is continuous. Then $\iota_A \circ f$ is continuous.

The converse is also true: if $\iota_A \circ f$ is continuous, then f is continuous. However, we will not prove this. This property is known as the **characteristic property of the subspace topology**.

Lemma 1 (Restriction of a Continuous Map is Continuous)

Let $X \xrightarrow{f} Y$ be continuous, and $A \subseteq X$ [1]*. Then*

[1] Here, A is equipped with the subspace topology

$$f\restriction_A : A \to Y$$

is also continuous.

3.2 Connected Spaces

Consider the real line $\mathbb{R}$, and consider two disjoint intervals on $\mathbb{R}$.

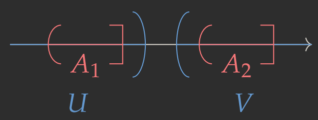

Figure 3.3: Motivation for Connectedness

Observe that we may find two open subsets U and V of $\mathbb{R}$ such that $A_1 \subseteq U$ and $A_2 \subseteq V$, which effectively separates the two intervals on the space $\mathbb{R}$.

Definition 13 (Disconnectedness)

A space X is said to be ***disconnected*** *iff X can be written as a disjoint union*

$$X = A_1 \coprod A_2$$

where $A_1, A_2 \subseteq X$, $A_1 = A_2^C$, that they are both non-empty and open [2].

[2] It goes without saying that the two sets are also simultaneously closed.

Definition 14 (Connctedness)

A space X is said to be ***connected*** *if it is not disconnected.*

Note

By the above definitions, we have that X is connected iff for any partition $X = A \coprod A^C$ with A being open, either A is $\emptyset$ or A is X.

Example 3.2.1

The space $\mathbb{R} \setminus \{0\}$ is disconnected; our disjoint sets are $(-\infty, 0)$ and $(0, \infty)$.

However, $\mathbb{R}^2 \setminus \{0\}$ is connected, but it is not easy to describe why.

Definition 15 (Path)

Require clarification

Lemma 2 (Path Connectedness implies Connectedness)

If a space X is path connected, then it is connected.

Theorem 3 (From Connected Space to Connected Space)

If $X \xrightarrow{f} Y$ is continuous and X is connected, then (f) is connected.

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