Foreword

Usage

- Notes are presented in two columns: main notes on the left, and sidenotes on the right. Main notes will have a larger margin.
- The following is the color code for the notes:

Blue	Definitions
Red	Important points
Yellow	Points to watch out for / comment for incompletion
Green	External definitions, theorems, etc.
Light Blue	Regular highlighting
Brown	Secondary highlighting

- The following is the color code for boxes, that begin and end with a line of the same color:

Blue	Definitions
Red	Warning
Yellow	Notes, remarks, etc.
Brown	Proofs
Magenta	Theorems, Propositions, Lemmas, etc.

- Hyperlinks are underlined in magenta. If your PDF reader supports it, you can follow the links to either be redirected to an external website, or a theorem, definition, etc., in the same document. Note that this is only reliable if you have the full set of notes as a single document, which you can find on: `https://japorized.github.io/TeX_notes`

5 Lecture 5 May 11th 2018

5.1 Subgroups (Continued)

5.1.1 Subgroups (Continued)

Note (Recall: definition of a subgroup)
Let G be a group and $H \subseteq G$. If H itself is a group, then we say that H is a subgroup of G.

Note
Since G is a group, $\forall h_1, h_2, h_3 \in H \subseteq G$, we have $h_1(h_2h_3) = (h_1h_2)h_3$. So H is a subgroup of G if it satisfies the following conditions, which we shall hereafter refer to as the Subgroup Test.

Subgroup Test

1. $h_1h_2 \in H$
2. $1_G \in H$
3. $\exists h_1^{-1} \in H$ *such that* $h_1h_1^{-1} = 1_G$

Note that the identity in H must also be the identity in G. This is because if $h_1, h_1^{-1} \in H$, then $h_1h_1^{-1} = 1_H$, but $h_1, h_1^{-1} \in G$ as well, and so $h_1h_1^{-1} = 1_G$. Thus $1_H = 1_G$.

Example 5.1.1
Given a group G, it is clear that $\{1\}$ and G are both subgroups of G.

Example 5.1.2
We have the following chain of groups:

$$(\mathbb{Z}, +) \subseteq (\mathbb{Q}, +) \subseteq (\mathbb{R}, +) \subseteq (\mathbb{C}, +)$$

Recall that the general linear group is defined as:

$$GL_n(\mathbb{R}) = (GL_n(\mathbb{R}), \cdot) = \{A \in M_n(\mathbb{R}) : \det A \neq 0\}$$

Definition 11 (Special Linear Group)
The special linear group of order n of $\mathbb{R}$ is defined as

$$SL_n(\mathbb{R}) = (SL_n(\mathbb{R}), \cdot) = \{A \in M_n(\mathbb{R}) : \det A = 1\}$$

Example 5.1.3
Clearly, $SL_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$. Note that the identity matrix I must be in $SL_n(\mathbb{R})$ since $\det I = 1$. Also, $\forall A, B \in SL_n(\mathbb{R})$, we have that

$$\det AB = \det A \det B = 1$$

*$\therefore AB \in SL_n(\mathbb{R})$. Also, since $\det A^{-1} = \frac{1}{\det A} = 1$, we also have that $^{-1} \in SL_n(\mathbb{R})$. We see that $SL_n(\mathbb{R})$ satisfies the **Subgroup Test**, and hence it is a subgroup of $GL_n(\mathbb{R})$.*

Definition 12 (Center of a Group)
Given a group G, the the center of a group G is defined as

$$Z(G) = \{z \in G : \forall g \in G \;\; zg = gz\}$$

Example 5.1.4
For a group G, $Z(G)$ is an abelian subgroup of G.

Proof
Clearly, $1_G \in Z(G)$. Let $y, z \in G$. $\forall g \in G$, we have that

$$(yz)g = y(zg) = y(gz) = (yg)z = (gy)z = g(yz)$$

Therefore $yz \in Z(G)$ and so $Z(G)$ is closed under its operation. Also, $\forall hinG$, we can write $h = (h^{-1})^{-1} = g^{-1}$. Since $z \in Z(G)$, we have that

$\forall g \in G$,

$$\begin{aligned} zg = gz \iff (zg)^{-1} = (gz)^{-1} &\iff g^{-1}z^{-1} = z^{-1}g^{-1} \\ &\iff hz^{-1} = z^{-1}h \end{aligned}$$

Therefore $z^{-1} \in Z(G)$. By the Subgroup Test, it follows that $Z(G)$ is a subgroup of G.

Finally, since $Z(G) \subseteq G$, by its definition, we have that $\forall x, y \in Z(G)$, $x, y \in G$ as well, and we have that $xy = yx$. Therefore, $Z(G)$ is abelian. □

Proposition 8 (Intersection of Subgroups is a Subgroup)
Let H and K be subgroups of a group G. Then their intersection

$$H \cap K = \{g \in G : g \in H \wedge g \in K\}$$

is also a subgroup of G.

Proof
Since H and K are subgroups, we have that $1 \in H$ and $1 \in K$ and hence $1 \in H \cap K$. Let $a, b \in H \cap K$. Since H and K are subgroups, we have that $ab \in H$ and $ab \in K$. Therefore, $ab \in H \cap K$. Similarly, since $a^{-1} \in H$ and $a^{-1} \in K$, $a^{-1} \in H \cap K$. By the Subgroup Test, $H \cap K$ is a subgroup of G. □

Proposition 9 (Finite Subgroup Test)
If H is a finite nonempty subset of a group G, then H is a subgroup if and only if H is closed under its operation.

This result says that if H is a finite nonempty subset, then we only need to prove that it is closed under its operation to prove that it is a subgroup. The other two conditions in the Subgroup Test are automatically implied.

Proof
The forward direction of the proof is trivially true, since H must satisfy the closure property for it to be a subgroup.

For the converse, since $H \neq \emptyset$, let $h \in H$. Since H is closed under its *operation, we have that*

$$h, h^2, h^3, \ldots$$

are all in H. Since H is finite, not all of the h^n's are distinct. Then, $\forall n \in \mathbb{N}$, there must $\exists m \in \mathbb{N}$ such that $h^n = h^{n+m}$. Then by Cycle Decomposition Theorem 6, $h^m = 1$ and so $1 \in H$. Also, because $1 = h^{m-1}h$, we have that $h^{-1} = h^{m-1}$, and thus the inverse of h is also in H. Therefore, H is a subgroup of G as requried. $\square$