PMATH450 — Lebesgue Integration and Fourier Analysis

Classnotes for Spring 2019

by

Johnson Ng

BMath (Hons), Pure Mathematics major, Actuarial Science Minor

University of Waterloo

Table of Contents

List of Definitions

List of Theorems

List of Procedures

Preface

The pre-requisite to this course is Real Analysis. We will use a lot of the concepts introduced in Real Analysis, at times without explicitly stating it. Refer to notes on PMATH351.

This course is spiritually broken into 2 pieces:

- Lebesgue Integration; and
- Fourier Analysis,

which is as the name of the course.

In this set of notes, we use a special topic environment called **culture** to discuss interesting contents related to the course, but will not be throughly studied and not tested on exams.

1 Lecture 1 May 07th 2019

Since many of our results work for both $\mathbb{C}$ and $\mathbb{R}$, we shall use $\mathbb{K}$ throughout this course to represent either $\mathbb{C}$ or $\mathbb{R}$.

1.1 Riemannian Integration

Definition 1 (Norm and Semi-Norm)

Let V be a vector space over $\mathbb{K}$. We define a ***semi-norm*** *on V as a function*

$$\nu : V \to \mathbb{R}$$

that satisfies

1. *(**Positive Semi-Definite**) $v(x) \geq 0$ for all $x \in V$;*
2. *$\nu(\kappa x) = |\kappa| \, \nu(x)$ for any $\kappa \in \mathbb{K}$ and $x \in V$; and*
3. *(**Triangle Inequality**) $\nu(x+y) \leq \nu(x) + \nu(y)$ for all $x, y \in V$.*

If $\nu(x) = 0 \implies x = 0$, then we say that ν is a ***norm****. In this case, we usually write $\|\cdot\|$ to denote the norm, instead of ν.*

Note 1.1.1

- *We sometimes call a semi-norm a* ***pseudo-length****.*

Remark 1.1.1

Notice that we wrote $\nu(x) = 0 \implies x = 0$ instead of $\nu(x) = 0 \iff x = 0$. This is because if $z = 0 \in V$, then

$$v(z) = v(0z) = 0.$$

Exercise 1.1.1

Show that if ν is a semi-norm on a vector space V, then $\forall x, y \in V$,

$$|\nu(x) - \nu(y)| \leq \nu(x - y).$$

Proof

Notice that by condition (2) and (3), we have

$$\nu(x - y) \leq \nu(x) + \nu(-y) = \nu(x) - \nu(y),$$

and

$$\nu(x - y) = -\nu(y - x) \geq -(\nu(y) - \nu(x)) = \nu(x) - \nu(y).$$

It follows that indeed

$$|\nu(x) - \nu(y)| \leq \nu(x - y). \qquad \square$$

Example 1.1.1

The absolute value $|\cdot|$ is a **norm** on $\mathbb{K}$.

Example 1.1.2 (*p*-norms)

Consider $N \geq 1$ an integer. We define a family of norms on

$$\mathbb{K}^N = \underbrace{K \times K \times \ldots \times K}_{N \text{ times}}.$$

1-norm

$$\left\| (x_n)_{n=1}^N \right\|_1 := \sum_{n=1}^N |x_n|.$$

Infinity-norm, ∞-norm

$$\left\| (x_n)_{n=1}^N \right\|_\infty := \max_{1 \leq n \leq N} |x_n| .$$

Euclidean-norm, 2-norm

$$\left\| (x_n)_{n=1}^N \right\|_2 := \left(\sum_{n=1}^N |x_n|^2 \right)^{\frac{1}{2}}$$

It is relatively easy to check that the above norms are indeed norms, except for the 2-form. In particular, the **triangle inequality** is not as easy to show [1].

[1] See Minkowski's Inequality.

Less obviously so, but true nonetheless, we can define the following p-norms on $\mathbb{K}^N$:

$$\left\| (x_n)_{n=1}^N \right\|_p := \left(\sum_{n=1}^N |x_n|^p \right)^{\frac{1}{p}} ,$$

for $1 \leq p < \infty$.

Culture

*Consider $V = \mathbb{M}_n(\mathbb{C})$, [2] where $n \in \mathbb{N}$ is fixed. For $T \in \mathbb{M}_n(\mathbb{C})$, we define the **singular numbers** of T to be*

$$s_1(T) \geq s_2(T) \geq \ldots \geq s_n(T) \geq 0,$$

where $\sigma(T^*T) = \{s_1(T)^2, s_2(T)^2, \ldots, s_n(T)^2\}$, including multiplicity. Then we can define

$$\|T\|_p := \left(\sum_{i=1}^n s_i(T)^p \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$, which is called the p-norm of T on $\mathbb{M}_n(\mathbb{C})$.

[2] Note that $\mathbb{M}_n(\mathbb{C})$ is the set of $n \times n$ matrices over $\mathbb{C}$.

Example 1.1.3

Let

$$V = \mathcal{C}([0,1], \mathbb{K}) = \{ f : [0,1] \to \mathbb{K} \mid f \text{ is continuous } \}.$$

Then

$$\|f\|_{\sup} := \sup\{|f(x)| \mid x \in [0,1]\}$$

[3] defines a norm on $\mathcal{C}([0,1], \mathbb{K})$.

[3] Some authors use $\|f\|_\infty$, but we will have the notation $\|[f]\|_\infty$ later on, and so we shall use $\|f\|_{\sup}$ for clarity.

A sequence $(f_n)_{n=1)^\infty}$ in V converges in this norm to some $f \in V$, i.e.

$$\lim_{n\to\infty} \|f_n - f\|_{\sup} = 0,$$

which means that $(f_n)_{n=1}^\infty$ converges uniformly to f on $[0,1]$.

Definition 2 (Normed Linear Space)

*A **normed linear space (NLS)** is a pair $(V, \|\cdot\|)$ where V is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is a norm on V.*

Definition 3 (Metric)

*Given an NLS $(V, \|\cdot\|)$, we can define a **metric** d on V (called the **metric induced by the norm**) as follows:*

$$d : V \times V \to \mathbb{R} \quad d(x,y) = \|x - y\|,$$

such that

- $d(x,y) \geq 0$ *for all* $x, y \in V$ *and* $d(x,y) = 0 \iff x = y$;
- $d(x,y) = d(y,x)$; *and*
- $d(x,y) \leq d(x,z) + d(y,z)$.

Note 1.1.2

Norms are all metrics, and so any space that has a norm will induce a metric on the space.

Definition 4 (Banach Space)

We say that an NLS $(V, \|\cdot\|)$ is ***complete*** *or is a* ***Banach Space*** *if the corresponding (V, d), where d is the metric induced by the norm, is complete* [4].

[4] Completeness of a metric space is such that any of its Cauchy sequences converges in the space.

Example 1.1.4

$(\mathcal{C}([0,1], \mathbb{K}), \|\cdot\|_{\sup})$ is a Banach space.

Example 1.1.5

We can define a 1-norm $\|\cdot\|_1$ on $\mathcal{C}([0,1], \mathbb{K})$ via

$$\|f\|_1 := \int_0^1 |f| \,.$$

Then $(\mathcal{C}([0,1], \mathbb{K}), \|\cdot\|_1)$ is an NLS.

Exercise 1.1.2

Show that $(\mathcal{C}([0,1], \mathbb{K}), \|\cdot\|_1)$ is not complete, which will then give us an example of a ***normed linear space that is not Banach****.*

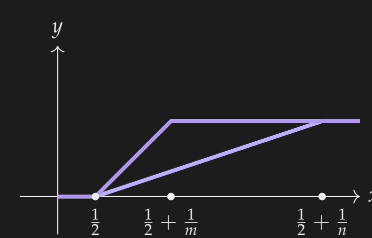

Figure 1.1: Sequence of functions $(f_n)_{n=1}^{\infty}$. We show for two indices $n < m$.

Proof

Consider the sequence $(f_n)_{n=1}^{\infty}$ of continuous functions given by

$$f_n(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ n\left(x + \frac{1}{2}\right) & \frac{1}{2} \leq x \leq \frac{1}{2} + \frac{1}{n} \\ 1 & \text{otherwise} \end{cases}$$

Note that the sequence $(f_n)_{n=1}^{\infty}$ is indeed **Cauchy**: let $\varepsilon > 0$ and $|n - m| < \frac{\varepsilon}{\left|x - \frac{1}{2}\right|}$, and then we have

$$\begin{aligned} |f_n(x) - f_m(x)| &= \left| n\left(x - \frac{1}{2}\right) - m\left(x - \frac{1}{2}\right) \right| \\ &= \left| (n-m)\left(x - \frac{1}{2}\right) \right| = |n - m| \left| x - \frac{1}{2} \right| < \varepsilon. \end{aligned}$$

However, it is clear that the sequence $(f_n)_{n=1}^{\infty}$ converges to the piecewise function (in particular, a non-continuous function)

$$f(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}. \qquad \square$$

Example 1.1.6

If $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ are NLS's, and if $T : \mathfrak{X} \to \mathfrak{Y}$ is a linear map, we define the **operator norm** of T to be

$$\|T\| := \sup\{\|T(x)\|_{\mathfrak{Y}} \mid \|x\|_{\mathfrak{X}} \leq 1\}.$$

We set

$$B(\mathfrak{X}, \mathfrak{Y}) := \{T : \mathfrak{X} \to \mathfrak{Y} \mid T \text{ is linear }, \|T\| < \infty\}.$$

Note that for any such linear map T, $\|T\| < \infty \iff T$ is continuous. Thus $B(\mathfrak{X}, \mathfrak{Y})$ is the set of all continuous functions from $\mathfrak{X}$ into $\mathfrak{Y}$.

Then $(B(\mathfrak{X}, \mathfrak{Y}), \|\cdot\|)$ is an NLS.

It is likely that we have seen this in Real Analysis.

Exercise 1.1.3

Show that $(B(\mathfrak{X}, \mathfrak{Y}), \|\cdot\|)$ is complete iff $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ is complete.

Note 1.1.3

*One example of the last example is when $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}}) = (\mathbb{K}, |\cdot|)$. In this case, $B(\mathfrak{X}, \mathbb{K})$ is known as the **dual space** of $\mathfrak{X}$, or simple the **dual** of $\mathfrak{X}$.*

We are interested in integrating over Banach spaces.

Definition 5 (Partition of a Set)

*Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a, b] \to \mathfrak{X}$ a function, where $a < b \in \mathbb{R}$. A **partition** P of $[a, b]$ is a finite set*

$$P = \{a = p_0 < p_1 < \ldots < p_N = b\}$$

for some $N \geq 1$. The set of all partitions of $[a, b]$ is denoted by $\mathcal{P}[a, b]$.

Definition 6 (Test Values)

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a, b] \to \mathfrak{X}$ a function, where $a < b \in \mathbb{R}$. Let $P \in \mathcal{P}[a, b]$. A set

$$P^* := \{p_k^*\}_{k=1}^N$$

satisfying

$$p_{k-1} \leq p_k^* \leq p_k, \text{ for } 1 \leq k \leq n$$

is called a set of ***test values*** *for P.*

Definition 7 (Riemann Sum)

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a, b] \to \mathfrak{X}$ a function, where $a < b \in \mathbb{R}$. Let $P \in \mathcal{P}[a, b]$ and P^* its corresponding set of test values. We define the ***Riemann sum*** *as*

$$S(f, P, P^*) = \sum_{k=1}^N f(p_k^*)(p_k - p_{k-1}).$$

Remark 1.1.2

1. *Note that because Definition 5, $p_k - p_{k-1} > 0$.*
2. *When $(\mathfrak{X}, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, then this is the usual Riemann sum from first-year calculus.*
3. *In general, note that*

$$\frac{1}{b-a} S(f, P, P^*) = \sum_{k=1}^N \lambda_k f(p_k^*),$$

where $0 < \lambda_k = \frac{p_k - p_{k-1}}{b-a} < 1$ and [5]

$$\sum_{k=1}^N \lambda_k = 1.$$

[5] via the fact that the λ_k's form a telescoping sum

So $\frac{1}{b-a}S(f, P, P^*)$ is an ***averaging*** *of f over $[a, b]$. We call $\frac{1}{b-a}S(f, P, P^*)$ the* ***convex combination*** *of the $f(p_k^*)$'s.*

Example 1.1.7 (Silly example)

Let $(\mathfrak{X} = \mathcal{C}([-\pi, \pi], \mathbb{K}), \|\cdot\|_{\sup})$. Let

$$f : [0, 1] \to \mathfrak{X} \text{ such that } x \mapsto e^{2\pi x} \sin 7\theta + \cos x \cos(12\theta),$$

where $\theta \in [-\pi, \pi]$. Now if we consider the partition

$$P = \left\{ -\pi, \frac{1}{10}, \frac{1}{2}, \pi \right\}$$

and its corresponding test value

$$P^* = \left\{ 0, \frac{1}{3}, 2 \right\},$$

then

$$\begin{aligned} S(f, P, P^*) &= f(0) \left(\frac{1}{10} + \pi \right) + f\left(\frac{1}{3} \right) \left(\frac{1}{2} - \frac{1}{10} \right) + f(2) \left(\pi - \frac{1}{2} \right) \\ &= (\sin 7\theta + \cos 12\theta) \left(\pi + \frac{1}{10} \right) \\ &\quad + \left(e^{\frac{2\pi}{3}} \sin 7\theta + \cos \frac{1}{3} \cos 12\theta \right) \left(\frac{2}{5} \right) \\ &\quad + (e^{4\pi} \sin 7\theta + \cos 2 \cos 12\theta) \left(\pi - \frac{1}{2} \right) \end{aligned}$$

Definition 8 (Refinement of a Partition)

Let $a < b \in \mathbb{R}$, and $P \in \mathcal{P}[a, b]$. We say Q is a ***refinement*** *of P is $Q \in \mathcal{P}[a, b]$ and $P \subseteq Q$.*

Note 1.1.4

In simpler words, Q is a "finer" partition that is based on P.

Definition 9 (Riemann Integrable)

Let $a < b \in \mathbb{R}$, $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a,b] \to \mathfrak{X}$ be a function. We say that f is ***Riemann integrable*** *over $[a,b]$ if $\exists x_0 \in \mathfrak{X}$ such that*

$$\forall \varepsilon > 0 \quad \exists P \in \mathcal{P}[a,b],$$

such that if Q is any refinement of P, and Q^* is any set of test values of Q, then

$$\|x_0 - S(f, Q, Q^*)\|_{\mathfrak{X}} < \varepsilon.$$

In this case, we write

$$\int_a^b f = x_0.$$

Proposition 1 (Uniqueness of the Riemann Integral)

If f is Riemann integrable over $[a,b]$, then the value of $\int_a^b f$ is unique.

Proof

Suppose not, i.e.

$$\int_a^b f = x_0 \text{ and } \int_a^b f = y_0$$

for some $x_0 \neq y_0$. Then, let

$$\varepsilon = \frac{\|x_0 - y_0\|}{2},$$

which is > 0 since $\|x_0 - y_0\| > 0$. Let $P_{x_0}, P_{y_0} \in \mathcal{P}[a,b]$ be partitions corresponding to x_0 and y_0 as in the definition of Riemann integrability.

Then, let $R = P_{x_0} \cup P_{y_0}$, so that R is a **common refinement** of P_{x_0} and P_{y_0}. If Q is any refinement of R, then Q is also a common refinement of P_{x_0} and P_{y_0}. Then for any test values Q^* of Q, we have

$$\begin{aligned} 2\varepsilon &= \|x_0 - y_0\| \\ &\leq \|x_0 - S(f, Q, Q^*)\| + \|S(f, Q, Q^*) - y_0\| < \varepsilon + \varepsilon = 2\varepsilon, \end{aligned}$$

which is a contradiction.

Thus $x_0 = y_0$ as required. □

Theorem 2 (Cauchy Criterion of Riemann Integrability)

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space, $a < b \in \mathbb{R}$ and $f : [a,b] \to \mathfrak{X}$ be a function. TFAE:

1. *f is Riemann integrable over $[a,b]$;*
2. *$\forall \varepsilon > 0$, $R \in \mathcal{P}[a,b]$, if P, Q is any refinement of R, and P^* (respectively Q^*) is any test values of P (respectively Q), then*

$$\|S(f,P,P^*) - S(f,Q,Q^*)\|_{\mathfrak{X}} < \varepsilon.$$

Proof

$\Longrightarrow$ This is a rather straightforward proof. Suppose $P, Q \in \mathcal{P}[a,b]$ is some refinement of the given partition $R \in \mathcal{P}[a,b]$, and P^*, Q^* any test values for P, Q, respectively. Then by assumption and Proposition 1, $\exists x_0 \in \mathfrak{X}$ such that

$$\|x_0 - S(f,P,P^*)\|_{\mathfrak{X}} < \frac{\varepsilon}{2} \text{ and } \|x_0 - S(f,Q,Q^*)\|_{\mathfrak{X}} < \frac{\varepsilon}{2}.$$

It follows that

$$\begin{aligned}
&\|S(f,P,P^*) - S(f,Q,Q^*)\|_{\mathfrak{X}} \\
&\leq \|x_0 - S(f,P,P^*)\|_{\mathfrak{X}} + \|x_0 - S(f,Q,Q^*)\|_{\mathfrak{X}} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

$\Longleftarrow$ By hypothesis, wma $\varepsilon = \frac{1}{n}$ for some $n \geq 1$, such that if P, Q are any refinements of the partition $R_n \in \mathcal{P}[a,b]$, and P^*, Q^* are the respective arbitrary test values, then

$$\|S(f,P,P^*) - S(f,Q,Q^*)\|_{\mathfrak{X}} < \frac{1}{n}$$

Now for each $n \geq 1$, define

$$W_n := \bigcup_{k=1}^{n} R_k \in \mathcal{P}[a,b],$$

so that W_n is a common refinement for $R_1, R_2, \ldots, R_n$. For each $n \geq 1$, let W_n^* be an arbitrary set of test values for W_n. For simplicity, let us write

$$x_n = S(f,\, W_n,\, W_n^*), \text{ for each } n \geq 1.$$

[6]

[6] Note that it would be nice if for the finer and finer partitions that we have constructed, i.e. the W_n's, give us a convergent sequence of Riemann sums, since it makes sense that this convergence will give us the final value that we want.

Claim: $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence If $n_1 \geq n_2 > N \in \mathbb{N}$, then

$$\|x_{n_1} - x_{n_2}\|_{\mathfrak{X}} = \left\| S(f, W_{n_1}, W_{n_1}^*) - S(f, W_{n_2}, W_{n_2}^*) \right\| < \frac{1}{N}$$

by our assumption, since W_{n_1}, W_{n_2} are refinements of R_N. Then by picking $N = \frac{1}{\varepsilon}$ for any $\varepsilon > 0$, we have that $(x_n)_{n=1}^{\infty}$ is indeed a Cauchy sequence in $\mathfrak{X}$.

Since $\mathfrak{X}$ is a Banach space, it is complete, and so $\exists x_0 := \lim\limits_{n \to \infty} x_n \in \mathfrak{X}$. It remains to show that, indeed,

$$x_0 = \int_a^b f.$$

Let $\varepsilon > 0$, and choose $N \geq 1$ such that

- $\frac{1}{N} < \frac{\varepsilon}{2}$; and
- $k \geq N$ implies that $\|x_k - x_0\| < \frac{\varepsilon}{2}$.

Then suppose that V is any refinement of W_N, and V^* is an arbitrary set of test values of V. Then we have

$$\begin{aligned}
\|x_0 - S(f,V,V^*)\|_{\mathfrak{X}} &\leq \|x_0 - x_N\|_{\mathfrak{X}} + \|x_N - S(f,V,V^*)\|_{\mathfrak{X}} \\
&< \frac{\varepsilon}{2} + \|S(f, W_N, W_N^*) - S(f,V,V^*)\|_{\mathfrak{X}} \\
&< \frac{\varepsilon}{2} + \frac{1}{N} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

It follows that

$$\int_a^b f = x_0,$$

as desired. $\square$

In first-year calculus, all continuous functions over $\mathbb{R}$ are integrable. A similar result holds in Banach spaces as well. In the next lecture, we shall prove the following theorem.

Theorem (Continuous Functions are Riemann Integrable)
Let $(\mathfrak{X}, \|\cdot\|)$ be a Banach space and $a < b \in \mathbb{R}$. If $f : [a, b] \to \mathfrak{X}$ is continuous, then f is Riemann integrable over $[a, b]$.

2 Lecture 2 May 9th 2019

2.1 *Riemannian Integration (Continued)*

We shall now prove the last theorem stated in class.

Theorem 3 (Continuous Functions are Riemann Integrable)

Let $(\mathfrak{X}, \|\cdot\|)$ be a Banach space and $a < b \in \mathbb{R}$. If $f : [a, b] \to \mathfrak{X}$ is continuous, then f is Riemann integrable over $[a, b]$.

Strategy

This is rather routine should one have gone through a few courses on analysis, and especially on introductory courses that involves Riemannian integration.

We shall show that if $P_N \in \mathcal{P}[a, b]$ is a partition of $[a, b]$ into 2^N subintervals of equal length $\frac{b-a}{2^N}$, and if we use $P_N^* = P_n \setminus \{a\}$ as the set of test values for P_N, which consists of the right-endpoints of each the subintervals in P_N, then the sequence $(S(f, P_N, P_N^*))_{N=1}^{\infty}$ converges in $\mathfrak{X}$ to $\int_a^b f$.

Note that this choice of partition is a valid move, since any of these P_N's, for different N's, is a refinement of some other partition of $[a, b]$, and if we choose a different set of test values, then we may as well consider an even finer partition.

Proof

First, note that since $[a, b]$ is closed and bounded in $\mathbb{R}$, it is compact. Also, we have that X is a metric space (via the metric induced by the norm). This means that **any continuous function f on $[a, b]$ is uniformly continuous on $[a, b]$**. In other words,

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall x, y \in [a, b]$$
$$|x - y| < \delta \implies \|f(x) - f(y)\| < \frac{\varepsilon}{2(b-a)}.$$

Claim: $(S(f, P_N, P_N^*))_{N=1}^{\infty}$ is Cauchy Now by picking $P_N \in \mathcal{P}[a, b]$ and set of test values P_N^* as described in the strategy above, we proceed by picking $M > 0$ such that $\frac{b-a}{2^M} < \delta$. Then for any $K \geq L \geq M$, since each of the subintervals have length $\frac{b-a}{2^L}$ and $\frac{b-a}{2^K}$ for P_L and P_K respectively, if we write

$$P_L = \{a = p_0 < p_1 < \ldots < p_{2^L} = b\}$$

and

$$P_K = \{a = q_0 \leq q_1 < \ldots < q_{2^K} = b\},$$

then $p_j = q_j 2^{K-L}$ [1] for all $0 \leq j \leq 2^L$. By uniform continuity, for $1 \leq j \leq 2^L$, wma

$$\left\| f(p_j^*) - f(q_s^*) \right\| < \frac{\varepsilon}{2(b-a)}, \text{ where } (j-1)2^{K-L} < s \leq j2^{K-L}.$$

We can see that

$$\begin{aligned}
&\|S(f, P_L, P_L^*) - S(f, P_K, P_K^*)\| \\
&= \left\| \sum_{j=1}^{2^L} \sum_{s=(j-1)2^{K-L}+1}^{j2^{K-L}} (f(p_j) - f(q_s))(q_s - q_{s-1}) \right\| \\
&\leq \sum_{j=1}^{2^L} \sum_{s=(j-1)2^{K-L}+1}^{j2^{K-L}} \|f(p_j) - f(q_s)\| (q_s - q_{s-1}) \\
&\leq \sum_{j=1}^{2^L} \sum_{s=(j-1)2^{K-L}+1}^{j2^{K-L}} \frac{\varepsilon}{b-a}(q_s - q_{s-1}) \\
&= \frac{\varepsilon}{b-a} \sum_{s=1}^{2^K} (q_s - q_{s-1}) \\
&= \frac{\varepsilon}{2(b-a)}(b-a) = \frac{\varepsilon}{2}.
\end{aligned}$$

This proves our claim.

[1] This is not immediately clear on first read. Think of a as 0.

Since $\mathfrak{X}$ is a Banach space, and hence complete, we have that the sequence $(S(f, P_N, P_N^*))_{N=1}^{\infty}$ has a limit $x_0 \in \mathfrak{X}$.

It remains to show that $\int_a^b f = x_0$. [2]

Let $\varepsilon > 0$, and choose $T \geq 1$ such that $\frac{b-a}{2^T} < \delta$ [3], so that we have

$$\|x_0 - S(f, P_T, P_T^*)\| < \frac{\varepsilon}{2}.$$

Now let $R = \{a = r_0 < r_1 < \ldots < r_J = b\} \in \mathcal{P}[a, b]$ such that $P_T \subseteq R$. Then there exists a sequence

$$0 = j_0 < j_1 < \ldots < j_{2^T} = J$$

such that

$$r_{j_k} = p_k, \text{ where } 0 \leq k \leq 2^T.$$

Let R^* be any set of test values of R. Note that for $j_{k-1} \leq s \leq j_k$, it is clear that

$$|p_k^* - r_s^*| \leq |p_k - p_{k-1}| = \frac{b-a}{2^T} < \delta.$$

Thus

$$\begin{aligned}
&\|S(f, P_T, P_T^*) - S(f, R, R^*)\| \\
&\leq \sum_{k=1}^{2^T} \sum_{s_{j_{k-1}+1}}^{j_k} \|f(p_k^*) - f(r_s^*)\| (r_s - r_{s-1}) \\
&< \frac{\varepsilon}{2(b-a)} \sum_{k=1}^{2^T} \sum_{s_{j_{k-1}+1}}^{j_k} (r_s - r_{s-1}) \\
&= \frac{\varepsilon}{2(b-a)} (b-a) = \frac{\varepsilon}{2}.
\end{aligned}$$

Putting everything together, we have

$$\begin{aligned}
&\|x_0 - S(f, R, R^*)\| \\
&\leq \|x_0 - S(f, P_T, P_T^*)\| + \|S(f, P_T, P_T^*) - S(f, R, R^*)\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

We can also find another refinement of P_T, say Q, that works

[2] The rest of this proof is similar to the above proof.

[3] Note that this is still the same δ as in the first δ in this entire proof.

similarly as in the case of R. It follows from Theorem 2 that

$$x_0 = \int_a^b f,$$

i.e. that f is indeed Riemann integrable over $[a, b]$. $\square$

The following is a corollary whose proof shall be left as an exercise.

Corollary 4 (Piecewise Functions are Riemann Integrable)

A ***piecewise continuous*** *function is also Riemann integrable: if $f : [a, b] \to \mathfrak{X}$ is piecewise continuous, then f is Riemann integrable.*

Exercise 2.1.1

Prove Corollary 4.

LET US exhibit a function that is not Riemann integrable.

Definition 10 (Characteristic Function)

Given a subset E of a set $\mathbb{R}$, we define the ***characteristic function*** *of E as a function $\chi_E : \mathbb{R} \to \mathbb{R}$ given by*

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

Example 2.1.1

Consider the set $E = \mathbb{Q} \cap [0, 1] \subseteq \mathbb{R}$. Let $P \in \mathcal{P}[0, 1]$ such that

$$P = \{0 = p_0 < p_1 < \ldots < p_N = 1\},$$

and let

$$P^* = \{p_k^*\}_{k=1}^N \text{ and } P^{**} = \{p_k^{**}\}_{k=1}^N$$

be 2 sets of test values for P, such that we have

$$p_k^* \in \mathbb{Q} \text{ and } p_k^{**} \in \mathbb{R} \setminus \mathbb{Q}.$$

Then we have

$$\begin{aligned} S(\chi_E, P, P^*) &= \sum_{k=1}^{N} \chi_E(p_k^*)(p_k - p_{k-1}) \\ &= \sum_{k=1}^{N} 1 \cdot (p_k - p_{k-1}) \\ &= p_N - p_0 = 1 - 0 = 1, \end{aligned}$$

and

$$\begin{aligned} S(\chi_E, P, P^{**}) &= \sum_{k=1}^{N} \chi_E(p_k^{**})(p_k - p_{k-1}) \\ &= \sum_{k=1}^{N} 0 \cdot (p_k - p_{k-1}) \\ &= 0. \end{aligned}$$

It is clear that the Cauchy criterion fails for χ_E. This shows that χ_E is not Riemann integrable.

Remark 2.1.1

*Let us once again consider $E = \mathbb{Q} \cap [0, 1]$. Note that E is **denumerable** [4]. We may thus write*

$$E = \{q_n\}_{n=1}^{\infty}.$$

Now, for $k \geq 1$, define

$$f_k(x) = \sum_{n=1}^{k} \chi_{\{q_n\}}(x).$$

In other words, $f_k = \chi_{\{q_1, \ldots, q_k\}}$. Furthermore, we have that

$$f_1 \leq f_2 \leq f_3 \ldots \leq \chi_E.$$

Moreover, we have that $\forall x \in [0, 1]$,

$$\chi_E(x) = \lim_{k \to \infty} f_k(x),$$

[4] This means that E is countably infinite.

and

$$\int_0^1 f_k = 0 \text{ for all } k \geq 1.$$

And yet, we have that $\int_0^1 \chi_E$ does not exist!

We want to develop a different integral that will 'cover' for this 'pathological' behavior of where the Riemann integral fails.

The rough idea is as follows.

In Riemann integration, when integrating over an interval $[a, b]$, we partitioned $[a, b]$ into subintervals. This happens on the x-axis.

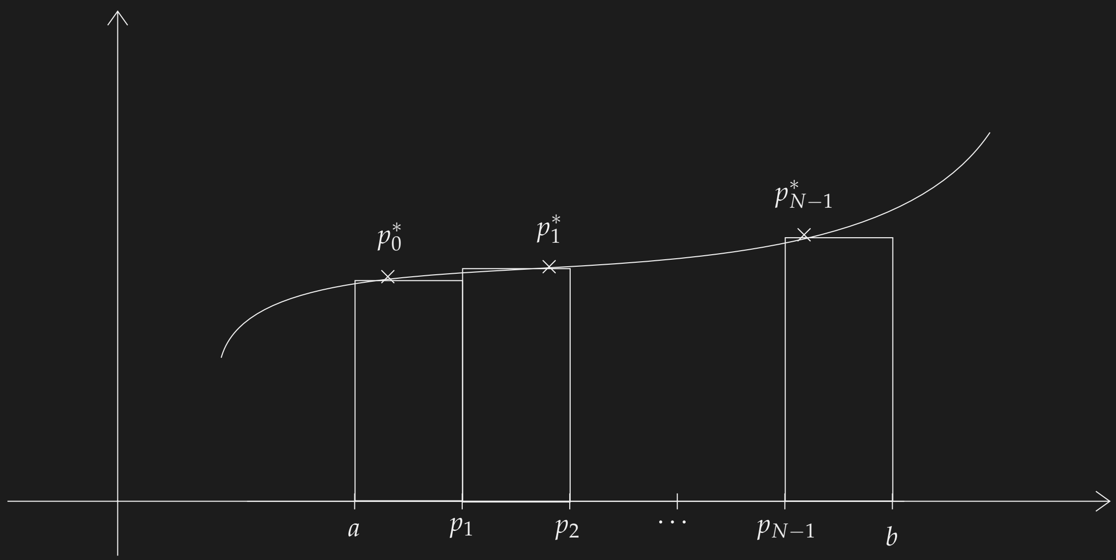

Figure 2.1: Rough illustration of how Riemann's integration works

In each of the subintervals of the partition, we pick out a **test value** p_i^*, and basically draw a rectangle with base at $[p_i, p_{i+1}]$ and height from 0 to p_i^*.

What we shall do now is that we **partition the range of f on the y-axis**, instead of the x-axis as we do in Riemannian integration.

In particular, given a function $f : [a, b] \to \mathbb{R}$, we first partition the range of f into subintervals $[y_{k-1}, y_k]$, where $1 \leq k \leq N$. Then, we set

$$E_k = \{x \in [a, b] : f(x) \in [y_{k-1}, y_k]\} \text{ for } 1 \leq k \leq N.$$

This will then allow us to estimate the integral of f over $[a, b]$ by

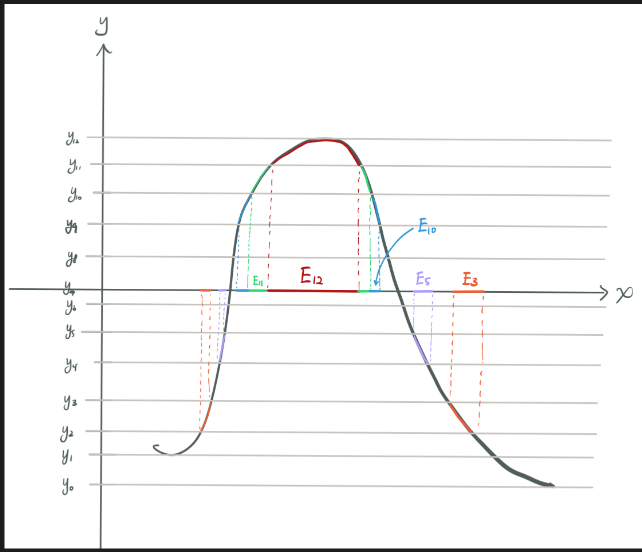

Figure 2.2: A sketch of what's happening with the construction of the E_k's

the expression

$$\sum_{k=1}^{N} y_k m E_k,$$

where each of the $y_k m E_k$ are called **simple functions**. In the expression, mE_k denotes a "measure" [5] of E_k.

[5] Note that a measure is simply a generalization of the notion of 'length'.

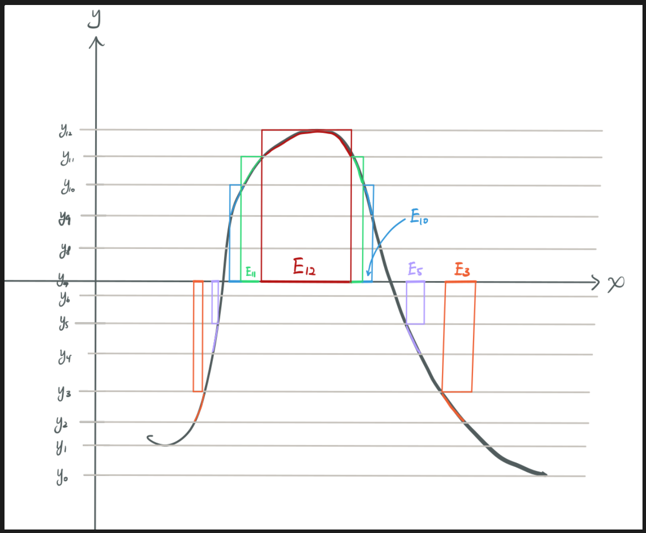

Figure 2.3: Drawing out the rectangles of $y_k m E_k$ from Figure 2.2.

We observe that E_k need not be a particularly well-behaved set. However, note that we may rearrange the possibly scattered pieces of each E_k together, so as to form a 'continuous' base for the rectangle. We need our definition of a measure to be able to capture this.

The following is an analogy from Lebesgue himself on comparing

Lebesgue integration and Riemann integration [6]:

> *I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral.*
> *But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral.*

[6] Siegmund-Schultze, R. (2008). *Henri Lesbesgue, in Timothy Gowers, June Barrow-Green, Imre Leader (eds.), Princeton Companion to Mathematics*. Princeton University Press

The insight here is that one can freely arrange the values of te functions, all the while preserving the value of the integral.

- This requires us to have a better understanding of what a measure is.
- This process of rearrangement converts certain functions which are extremely difficult to deal with, or outright impossible, with the Riemann integral, into easily digestible pieces using Lebesgue integral.

2.2 *Lebesgue Outer Measure*

Goals of the section

1. Define a "measure of length" on as many subsets of $\mathbb{R}$ as possible.
2. The definition should agree with our intuition of what a 'length' is.

Definition 11 (Length)

For $a \leq b \in \mathbb{R}$, we define the ***length*** *of the interval (a, b) to be $b - a$, and we write*

$$\ell((a, b)) := b - a.$$

We also define

- $\ell(\emptyset) = 0$; *and*
- $\ell((a, \infty)) = \ell((-\infty, b)) = \ell((-\infty, \infty)) = \infty$.

Definition 12 (Cover by Open Intervals)

Let $E \subseteq \mathbb{R}$. A countable collection $\{I_n\}_{n=1}^{\infty}$ of open intervals is said to be a ***cover of E by open intervals*** *if $E \subseteq \bigcup_{n=1}^{\infty} I_n$.*

Note 2.2.1

In this course, the only covers that we shall use are open intervals, and so we shall henceforth refer to the above simply as covers of E.

Before giving what immediately follows from the above, I shall present the following notion of an outer measure.

Definition 13 (Outer Measure)

Let $\emptyset \neq X$ be a set. An ***outer measure*** *μ on X is a function*

$$\mu : \mathcal{P}(X) \to [0, \infty] := [0, \infty) \cup \{\infty\}$$

which satisfies

1. $\mu\emptyset = 0$;
2. (***monotone increment*** *or* ***monotonicity***) $E \subseteq F \subseteq X \implies \mu E \leq \mu F$; *and*
3. (***countable subadditivity*** *or* ***σ-subadditivity***) $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu E_n.$$

Note 2.2.2

Note that by the monotonicity, the σ-subadditivity condition is equivalent *to: given* $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$ *and* $F \subseteq \bigcup_{n=1}^{\infty} E_n$*, we have that*

$$\mu(F) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

Definition 14 (Lebesgue Outer Measure)

We define the ***Lebesgue outer measure*** *as a function* $m^* : \mathcal{P}(X) \to \mathbb{R}$ *such that*

$$m^* E := \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : E \subseteq \bigcup_{n=1}^{\infty} I_n \right\}.$$

We cheated a little bit by calling the above an outer measure, so let us now justify our cheating.

Proposition 5 (Validity of the Lebesgue Outer Measure)

m^* *is indeed an outer measure.*

Proof

$\mu\emptyset = 0$ We consider a sequence of sets $\{I_n\}_{n=1}^{\infty}$ such that $I_n = \emptyset$ for each $n = 1, \ldots, \infty$. It is clear that $\emptyset \subseteq \bigcup_{n=1}^{\infty} I_n$. Also, we have that $\ell(I_n) = 0$ for all $n = 1, \ldots, \infty$. It follows that

$$0 \leq m^*(\emptyset) \leq \sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} 0 = 0,$$

where the inequality is simply by the definition of m^* being an infimum, **not to be confused with σ-subadditivity**. We thus have that

$$m^*(\emptyset) = 0.$$

Monotonicity Suppose $E \subseteq F \subseteq \mathbb{R}$, and $\{I_n\}_{n=1}^{\infty}$ a cover of F. Then

$$E \subseteq F \subseteq \bigcup_{n=1}^{\infty} I_n.$$

In particular, all covers of F are also covers of E, i.e.

$$\left\{ \{J_m\}_{m=1}^{\infty} : E \subseteq \bigcup_{m=1}^{\infty} J_m \right\} \subseteq \left\{ \{I_n\}_{n=1}^{\infty} : F \subseteq \bigcup_{n=1}^{\infty} I_n \right\}.$$

It follows that

$$m^* E \leq m^* F.$$

σ-subaddivitity Consider $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$ such that $E \subseteq \bigcup_{n=1}^{\infty} E_n$. WTS

$$m^* E \leq \sum_{n=1}^{\infty} m^* E_n.$$

Now if the sum of the RHS is infinite, i.e. if any of the $m^* E_n$ is infinite, then the inequality comes for free. Thus WMA $\sum_{n=1}^{\infty} E_n < \infty$, and in particular that $m^* E_n < \infty$ for all $n = 1, \ldots, \infty$.

To do this, let $\varepsilon > 0$. Since $m^* E_n < \infty$ for all n, we can find covers $\left\{ I_k^{(n)} \right\}_{k=1}^{\infty}$ for each of the E_n's such that

$$\sum_{k=1}^{\infty} \ell \left(I_k^{(n)} \right) < m^* E_n + \frac{\varepsilon}{2^n}.$$

Then, we have that

$$E \subseteq \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} I_k^{(n)}.$$

Then by $m^* E$ being the infimum of the sum of lengths of the covering intervals, we have that

$$\begin{aligned} m^* E &\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \ell \left(I_k^{(n)} \right) \\ &\leq \sum_{n=1}^{\infty} \left(m^* E_n + \frac{\varepsilon}{2^n} \right) \\ &= \sum_{n=1}^{\infty} m^* E_n + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} \\ &= \sum_{n=1}^{\infty} m^* E_n + \varepsilon. \end{aligned}$$

Since ε was arbitrary, we have that

$$m^* E_n \leq \sum_{n=1}^{\infty} m^* E_n,$$

as desired. □

Corollary 6 (Lebesgue Outer Measure of Countable Sets is Zero)

If $E \subseteq \mathbb{R}$ is countable, then $m^* E = 0$.

Proof

We shall prove for when E is denumerable, for the finite case follows a similar proof. Let us write $E = \{x_n\}_{n=1}^{\infty}$. Let $\varepsilon > 0$ and

$$I_n = \left(x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}} \right).$$

Then it is clear that $\{I_n\}_{n=1}^{\infty}$ is a cover of E.

It follows that

$$0 \leq m^* E \leq \sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Thus as $\varepsilon \to 0$, we have that

$$m^* E = 0,$$

as expected. □

Corollary 7 (Lebesgue Outer Measure of $\mathbb{Q}$ is Zero)

We have that $m^* \mathbb{Q} = 0$.

In the proofs above that we have looked into, and based on the intuitive notion of the length of an open interval, it is compelling to simply conclude that

$$m^*(a, b) = \ell(a, b) = b - a.$$

However, looking back at Definition 14, we know that that is not how $m^*(a,b)$ is defined.

This leaves us with an interesting question:

> *how does our notion of measure $m^*(a,b)$ of an interval compare with the notion of the length of an interval?*

By taking $I_1 = (a,b)$ and $I_n = \emptyset$ for $n \geq 2$, it is rather clear that $\{I_n\}_{n=1}^{\infty}$ is a cover of (a,b), and so we have

$$m^*(a,b) \leq \ell(a,b) = b - a. \tag{2.1}$$

However, the other side of the game is not as easy to confirm: we would have to consider **all** possible covers of (a,b), which is a lot.

Another question that we can ask ourselves seeing Equation (2.1) is why can't $m^*(a,b)$ be something that is strictly less than the length to give us an even more 'precise' measurement?

To answer these questions, it is useful to first consider the outer measure of a closed and bounded interval, e.g. $[a,b]$, since these intervals are **compact** under the **Heine-Borel Theorem**. This will give us a finite subcover for every infinite cover of the compact interval, which is easy to deal with.

We shall see that with the realization of the outer measure of a compact interval, we will also be able to find the outer measure of intervals that are neither open nor closed.

We shall prove the following proposition in the next lecture. Note that for the sake of presentation, I shall abbreviate the Lebesgue Outer Measure as LOM.

Proposition (LOM of Arbitrary Intervals)
Suppose $a < b \in \mathbb{R}$. Then

1. $m^*([a,b]) = b - a$*; and therefore*
2. $m^*((a,b]) = m^*([a,b)) = m^*((a,b)) = b - a$.

Bibliography

Siegmund-Schultze, R. (2008). *Henri Lesbesgue, in Timothy Gowers, June Barrow-Green, Imre Leader (eds.), Princeton Companion to Mathematics*. Princeton University Press.

Index