Foreword

Usage

- Notes are presented in two columns: main notes on the left, and sidenotes on the right. Main notes will have a larger margin.
- The following is the color code for the notes:

Blue	Definitions
Red	Important points
Yellow	Points to watch out for / comment for incompletion
Green	External definitions, theorems, etc.
Light Blue	Regular highlighting
Brown	Secondary highlighting

- The following is the color code for boxes, that begin and end with a line of the same color:

Blue	Definitions
Red	Warning
Yellow	Notes, remarks, etc.
Brown	Proofs
Magenta	Theorems, Propositions, Lemmas, etc.

- Hyperlinks are underlined in magenta. If your PDF reader supports it, you can follow the links to either be redirected to an external website, or a theorem, definition, etc., in the same document. Note that this is only reliable if you have the full set of notes as a single document, which you can find on:
`https://japorized.github.io/TeX_notes`

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14.1 Isomorphism Theorems (Continued 2)

14.1.1 Isomorphism Theorems (Continued)

Note (Recall)

In First Isomorphism Theorem 38, we had that for a group homomorphism $\alpha : G \to H$ where G and H are groups,

$$G/\ker\alpha \cong \operatorname{im}\alpha$$

Now let $\alpha : G \to H$ be a group homomorphism, $K = \ker\alpha$, $\phi : G \to G/K$ be the coset map, and $\bar{\alpha}$ be as defined in the proof of First Isomorphism Theorem 38. We then have the following commutative diagram to illustrate the relationship between the three groups.

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\phi"'] & H \\ G/K \arrow[ur, "\bar{\alpha}"'] & \end{tikzcd}$$

A natural question to ask after seeing the relationship is: Is $\bar{\alpha}\phi = \alpha$? If it is, is the definition of $\bar{\alpha}$ unique? The answer is: **YES!** on both accounts.

Proof

Let $g \in G$. Then

$$\bar{\alpha}\phi(g) = \bar{\alpha}\big(\phi(g)\big) = \bar{\alpha}(Kg) = \alpha(g)$$

Suppose $\alpha = \beta\phi$ where $\beta : {}^{G}\!/\!_{K} \to H$. Then

$$\beta(Kg) \stackrel{(1)}{=} \beta(\phi(g)) = \beta\phi(g) = \alpha(g) = \bar{\alpha}(Kg)$$

where (1) is because ϕ is surjective by Proposition 35. Therefore, we observe that $\beta = \bar{\alpha}$ for any $Kg \in {}^{G}\!/\!_{K}$. This proves that $\bar{\alpha}$ is the unique homomorphism such that ${}^{G}\!/\!_{K} \to H$ satisfying $\alpha = \bar{\alpha}\phi$. $\square$

With that, we have the following proposition.

Proposition 39

Let $\alpha : G \to H$ be a group homomorphism, where G and H are groups. Let $K = \ker\alpha$. Then α factors uniquely as $\alpha = \bar{\alpha}\phi <$ where $\phi : G \to {}^{G}\!/\!_{K}$ is the coset map and $\bar{\alpha} : GK \to H$ is defined by

$$\bar{\alpha}(Kg) = \alpha(g).$$

Note that ϕ is surjective and $\bar{\alpha}$ is injective.

*In such a scenario, we also say that α **factors through** ϕ.*[1]

[1] Reference for the terminology: `https://math.stackexchange.com/questions/68941/terminology-a-homomorphism-factors`.

Example 14.1.1

Let $G = \langle\, g \,\rangle$ be a cyclic group. Consider $\alpha : \mathbb{Z} \to G$, defined as

$$\forall k \in \mathbb{Z} \quad \alpha(k) = g^k,$$

which is a group homomorphism. By definition, α is surjective. Note that

$$\ker\alpha = \{k \in \mathbb{Z} : g^k = 1\}.$$

We have, therefore, two cases to consider.

- *G is an infinite group*
  This would imply that $\ker\alpha = \{0\}$ since only $g^0 = 1$. Then by First Isomorphism Theorem 38, we have that

  $$ {}^{\mathbb{Z}}\!/\!_{\ker\alpha} \cong G$$

  Note that[2]

[2] We are assuming that the group $\mathbb{Z}$ here works under the operation of addition, otherwise, if we employ multiplication, then $\mathbb{Z}$ would not be a group and α would not be a group homomorphism.

$$\mathbb{Z} / \ker \alpha = \{(\ker \alpha)k : k \in \mathbb{Z}\} = \{0 + k : k \in \mathbb{Z}\} = \mathbb{Z}.$$

Therefore

$$\mathbb{Z} \cong G$$

- *G is a finite group*
  Suppose that $|G| = o(g) = n \in \mathbb{N}$, *which is valid by Corollary 24. Then*

  $$\ker \alpha = n\mathbb{Z}$$

  Then by the First Isomorphism Theorem 38, we have

  $$\mathbb{Z} / n\mathbb{Z} \cong G.$$

  Observe that

  $$\mathbb{Z} / n\mathbb{Z} = \{n\mathbb{Z} + k : k \in \mathbb{Z}\} = \mathbb{Z}_n$$

  since the set in the middle is the definition of the set of integers modulo n.[3] *Therefore,*

  $$\mathbb{Z}_n \cong G$$

  Therefore, we have that

  $$\mathbb{Z} \cong G \text{ or } \mathbb{Z}_{o(g)} \cong G$$

[3] This is why we often see texts from various authors using $\mathbb{Z} / n\mathbb{Z}$ to represent the set of integers modulo n.

Theorem 40 (Second Isomorphism Theorem)

Let H and K be the subgroups of a group G with $K \triangleleft G$. *Then*

- HK *is a subgroup of* G;
- $K \triangleleft HK$;
- $H \cap K \triangleleft H$; *and*
- $HK / K \cong H / H \cap K$.

Proof

Since $K \triangleleft G$, *by Lemma 29 and Proposition 30, we have that* $HK = KH$ *is a subgroup of* G. *Consequently, we have* $K \triangleleft HK$, *since* K *is clearly a subgroup of* HK *and* $K \triangleleft G$, *and so* $\forall x \in HK \subseteq G$ *we have that* $gK = Kg$.

Consider $\alpha : H \to {}^{HK}\!/_{K}$*, defined by*[4]

$$\alpha(h) = Kh$$

Now if $x = kh \in KH = HK$*, then*

$$Kx = K(kh) = Kh = \alpha(h).$$

Therefore, we have that α *is surjective. Now by Proposition 22, observe that*

$$\ker \alpha = \{h \in H : Kh = K\} = \{h \in Hh \in K\} = H \cap K.$$

Then by the First Isomorphism Theorem, we have that

$${}^{HK}\!/_{K} \cong {}^{H}\!/_{H \cap K}.$$

Since we have that $\ker \alpha = H \cap K$ *and* $\ker \alpha \lhd H$*, we have that* $H \cap K \lhd H$. □

[4] Note that $Kh \in {}^{HK}\!/_{K}$ since $h \in H \subseteq HK$.

Theorem 41 (Third Isomorphism Theorem)

Let $K \subseteq H \subseteq G$ *be groups, with* $K \lhd G$ *and* $H \lhd G$*. Then*

$${}^{H}\!/_{K} \lhd {}^{G}\!/_{K} \text{ and } \left({}^{G}\!/_{K}\right) \Big/ \left({}^{H}\!/_{K}\right) \cong {}^{G}\!/_{H}$$

Proof

Define $\alpha : {}^{G}\!/_{K} \to {}^{G}\!/_{H}$ *by* $\alpha(Kg) = Hg$ *for all* $g \in G$*. Clearly,* α *is surjective. Now if* $Kg = Kg_1$*, for any* $g, g_1 \in G$*, then* $gg_1 \in K \subseteq H$*. Therefore,* $Hg = Hg_1$*. Thus* α *is well-defined. Now*

$$\ker \alpha = \{Kg : Hg = H\} = \{Kg : g \in H\} = {}^{H}\!/_{K}.$$

Then

$${}^{H}\!/_{K} = \ker \alpha \lhd {}^{G}\!/_{K}.$$

By the First Isomorphism Theorem, we have

$$\left({}^{G}\!/_{K}\right) \Big/ \left({}^{H}\!/_{K}\right)$$

as required. □

One reason that we are interested in the symmetric group is that they contain all finite groups.

Theorem (Cayley's Theorem)

If G is a finite group of order n, then G is isomorphic to a subgroup of S_n.