Foreword

Usage

- Notes are presented in two columns: main notes on the left, and sidenotes on the right. Main notes will have a larger margin.
- The following is the color code for the notes:

Blue	Definitions
Red	Important points
Yellow	Points to watch out for / comment for incompletion
Green	External definitions, theorems, etc.
Light Blue	Regular highlighting
Brown	Secondary highlighting

- The following is the color code for boxes, that begin and end with a line of the same color:

Blue	Definitions
Red	Warning
Yellow	Notes, remarks, etc.
Brown	Proofs
Magenta	Theorems, Propositions, Lemmas, etc.

- Hyperlinks are underlined in magenta. If your PDF reader supports it, you can follow the links to either be redirected to an external website, or a theorem, definition, etc., in the same document. Note that this is only reliable if you have the full set of notes as a single document, which you can find on: `https://japorized.github.io/TeX_notes`

12 Lecture 12 May 28th 2018

12.1 Normal Subgroup (Continued 3)

12.1.1 Normal Subgroup (Continued 2)

Theorem 32
If $H \triangleleft G$ and $K \triangleleft G$ satisfy $H \cap K = \{1\}$, then

$$HK \cong H \times K$$

Proof
Claim 1:

$$H \triangleleft G \wedge K \triangleleft G \wedge H \cap K = \{1\} \implies \forall h \in H \ \forall k \in K \quad hk = kh$$

Consider $x = hkh^{-1}k^{-1}$. Note that since $H \triangleleft G$, by Proposition 27, we have that $\forall g \in G$, $gHg^{-1} = H$. Then $khk^{-1} \in kHk^{-1} = H$. Thus $x = h(kh^{-1}k^{-1}) \in H$. Using a similar argument, we can get that $x \in K$. Since $x \in H \cap K = \{1\}$, we have that $hkh^{-1}k^{-1} = 1$, we have that $hk = kh$ as claimed.

Note that since $H \triangleleft G$, by Proposition 30, we have that HK is a subgroup of G.[1] Define $\sigma : H \times K \to HK$ by

$$\forall h \in H \ \forall k \in K \qquad \sigma\big((h,k)\big) = hk$$

Claim 2: *σ is an isomorphism.*

[1] We do not need the more powerful statement that says that HK is a normal subgroup.

Let $(h,k),(h_1,k_1) \in H \times K$. By Claim 1, note that $h_1 k = k h_1$. Therefore,

$$\begin{aligned}\sigma\big((h,k)\cdot(h_1,k_1)\big) &= \sigma\big((hh_1,kk_1)\big) = hh_1kk_1 \\ &= hkh_1k_1 = \sigma\big((h,k)\big)\sigma\big((h_1,k_1)\big)\end{aligned}$$

Thus we see that σ is a group homomorphism. Note that by the definition of HK, σ is a surjection. Also, if $\sigma\big((h,k)\big) = \sigma\big((h_1,k_1)\big)$, we have that

$$\begin{aligned}hk = h_1k_1 &\implies h_1^{-1}h = k_1k^{-1} \in H \cap K = \{1\} \\ &\implies h_1^{-1}h = 1 = k_1k^{-1} \implies h_1 = h \wedge k_1 = k.\end{aligned}$$

Thus σ is an injection, and hence σ is bijective. Therefore, σ is an isomorphism. This proves that $HK \cong H \times K$. $\square$

An immediate result is the corollary that we were given in the last class but not proven.

Corollary 33

Let G be a finite group, $H, K \triangleleft G$ such that $H \cap K = \{1\}$ and $|H|\,|K| = |G|$. Then $G \cong H \times K$.

Example 12.1.1

Let $m, n \in \mathbb{N}$ with $\gcd(m,n) = 1$. Let G be a cyclic group of order mn. Write $G = \langle\, a \,\rangle$ with $o(a) = mn$. Let $H = \langle\, a^n \,\rangle$ and $K = \langle\, a^m \,\rangle$. Then we have

$$|H| = o(a^n) = m \wedge |K| = o(a^m) = n.$$

It follows that $|H|\,|K| = mn = |G|$. Note that $H \cong C_m$ and $K \cong C_n$. Since $\gcd(m,n) = 1$, by Corollary 26, we have that $H \cap K = \{1\}$.

Also, since G is cyclic and thus abelian, we have that $H, K \triangleleft G$. Then by Corollary 33, we have that $G \cong C_{mn} \cong C_m \times C_n$.

12.2 Isomorphism Theorems

12.2.1 Quotient Groups

Let G be a group and K a subgroup of G. Given a set

$$\{Ka : a \in G\},$$

how can we create a group out of it?

A "natural" way to define an operation on the set of right cosets above is

$$\forall a, b \in G \qquad Ka * Kb = Kab. \tag{$\dagger$}$$

Note that it is entirely possible that for $a_1 \neq a$ and $b_1 \neq b$, we have $Ka = Ka_1$ and $Kb = Kb_1$. In order for Equation (†) to make sense as an operation, it is necessary that

$$Ka = Ka_1 \wedge Kb = Kb_1 \implies Kab = Ka_1b_1.$$

If the condition is satisfied, we say that the "multiplication" $KaKb$ is well-defined.

Lemma 34 (Multiplication of Cosets of Normal Subgroups)
Let K be a subset of G. The following are equivalent:

1. $K \triangleleft G$;
2. $\forall a, b \in G \;\; KaKb = Kab$ *is well-defined.*

Proof
(1) $\implies$ (2) *Suppose $K \triangleleft G$. Suppose $Ka = Ka_1$ and $Kb = Kb_1$. Then $aa_1^{-1} \in K$ and $bb_1^{-1} \in K$. To show that $Kab = Ka_1b_1$, it suffices to show that $(ab)(a_1b_1)^{-1} \in K$. Note that since $K \triangleleft G$, we have that $aKa^{-1} = K$. Therefore,*

$$\begin{aligned} ab(a_1b_1)^{-1} &= ab(b_1^{-1}a_1^{-1}) = a(bb_1^{-1})a_1^{-1} \\ &= \left(a(bb_1^{-1})a^{-1}\right)(aa_1^{-1}) \in K. \end{aligned}$$

Therefore $Kab = Ka_1b_1$ as required.

(2) $\implies$ (1) *If $a \in G$, we need to show that $\forall k \in K$, $aka^{-1} \in K$. Since $Ka = Ka$ and $Kk = K(1)$ [2], by (2), we have that $Kak = Ka(1)$, i.e. $Kak = Ka$. Thus $aka^{-1} = 1 \in K$, implying that $aKa^{-1} \subseteq K$ and hence*

[2] This is cause 1 is in the same coset.

$K \lhd G$. $\square$