MATH145 Readings

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Introduction

Below is an introduction written by the author of the referred notes, Kenneth R. Davidson.

As a student entering university to study mathematics, you probably have encountered prime bumbers. Chances are great that you believe that every integer factors uniquely into a product of primes but have not seen a proof. This important fact, known as the Fundamental Theorem of Algebra, is of crucial importance in the theory of numbers. It is not easy to prove. More importantly, it is not *intuitively obvious*. Indeed, its significance is only realized with very large numbers beyond our real experience. The crucial fact that enables us to prove this with relative ease is the Euclidean Algorithm for finding greatest common divisors. The first two chapters deal with these basic properties of the integers and modular arithmetic.

It is worth noting that there are number systems not very much different from the intergers in which this unique factorization into primes fails. Far from being a disaster, this is an opportunity to investigate why this phenomena occurs. It shows us which properties of the integers themsevles are crucial to make the theory work. That is why we make a foray into quadratic number domains in chapter 3. Naturally, we merely touch the surface here as this theory lies in a rather deep connection between algebra and number theory.

A nice application of modular arithmetic is the Rivest-Shamir-Adelman (RSA) public key cryptography scheme. This code allows the author to publish the method of *encoding* a message in a public place, while keeping the method of *decoding* the message secret. This is a rather different idea of coding, as for all previously known codes, the method of decoding merely reversed the endoing method. The secret here is that it is very easy (with a computer) to find large primes (say 100-200 digits) but very difficult to factor the product of two large primes. If you believe that checking whether a number is prime involes trial division by all numbers up to the square root, it might be hard to imagine why determining if a number if prime should be any easier than finding factors. So we delve more deeply into methods used to determine whether a number is prime or composite. It quickly becomes clear that simple tests can determine beyond any doubt that a number is

composite without giving any information at all about the factors.

In chapter 5, we introduce the complex numbers. There is a tacit assumption that the student is already reasionably familiar with the real numbers from studying calculus. However, a section is devoted to a brief discussion of how the real numbers are developed. No attempt is made to separate algebra from analysis. Indeed, all of mathematics is integrated and it is foolish to pretend that there are natural walls. In particular, we prove the Fundamental Theorem of Algebra that every complex polynomial factors into a product of linear terms. In spite of its name, this is a theorem of analysis because it relies on the (topological) completeness of the complex numbers. The proof we give is one of the simplest, and relies on the Extreme Value Theorem. We also develop the complex exponential function. This again is really a theorem of analysis. But it is included because it plays such an important role in other applications of the compelx numbers.

In chapter 6, we show that the same theory applies to the algebra of polynomials. In particular, there is a Euclidean Algorithm and unique factorization into irreducible polynomials. We exanime various tests for irreducibility, and study connections with irrationality of the roots. Then we do a few special topics about real and complex polynomials such as Sturm's Theorem for counting real roots, and the formula for solving cubics. Then in chapter 7, we study finite fields in some detail. This is just doing modular arithmetic modulo an irreducible polynomial instead of modulo a prime integer. Many of the results for $\mathbb{Z}_p$ carry over to finite fields with the same ideas. A rather beautiful application of these ideas is an algorithm for factoring polynomials over the rationals. This algorithm is based on a method for factoring polynomial of degree d mod p is much easeier than factoring a d digit base p number.

Contents

List of Definitions

List of Theorems

Chapter 1

The Integers

1.1 Basic Properties

The following list of properties is what a student taught in the 'modern' style would come up with when talking about properties of integers. We give the name Commutative Rings for sets of numbers that satisfy these properties.

Definition 1.1.1 (Commutative Rings)
A commutative ring is a set of numbers that satisfy these properties.

1. *The **integers** consist of a set $\mathbb{Z}$ together with two binary operations **addition** $(+)$ and **multiplication** $(\cdot)$*
2. *(**commutativity of addition**)* $\forall a, b \in \mathbb{Z} \quad a + b = b + a$
3. *(**associativity of addition**)* $\forall a, b, c \in \mathbb{Z} \quad (a + b) + c = a + (b + c)$
4. *(**additive identity**)* $\exists 0 \in \mathbb{Z} \ \forall a \in \mathbb{Z} \quad a + 0 = a = 0 + a$
5. *(**additive inverse**)* $\forall a \in \mathbb{Z} \ \exists (-a) \in \mathbb{Z} \quad a + (-a) = 0$
6. *(**commutativity of multiplication**)* $\forall a, b \in \mathbb{Z} \quad a \cdot b = b \cdot a$
7. *(**associativity of multiplication**)* $\forall a, b, c \in \mathbb{Z} \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$
8. *(**multiplicative identity**)* $\exists 1 \in \mathbb{Z} \ \forall a \in \mathbb{Z} \quad a \cdot 1 = a = 1 \cdot a$
9. *(**distributive law**)* $\forall a, b, c \in \mathbb{Z} \ (a + b) \cdot c = a \cdot c + b \cdot c$

But this does not fully distinguish integers from many other sets, for example:

1. The real numbers $\mathbb{R}$.
2. The set $\mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\}$
   See Exercise 1 for proof.
3. The set $\mathbb{Z} \oplus \mathbb{Z} := \{(a, b) \mid a, b \in \mathbb{Z}\}$ with coordinate-wise addition and multiplication.

Definition 1.1.2 (Ring)
A ring is a set of numbers that satisfy all properies of a commutative ring except for property 6, ***commutativity of multiplication****, and added with an additional distributive law*

$$\forall a, b, c \in \mathbb{Z} \quad a \cdot (b + c) = a \cdot b + a \cdot c$$

Example 1.1.1
An example of a non-commutative ring is the set of 2×2 *matrices with integer entries. While addition is coordinate-wise, multiplication is defined by the rule:* $\forall a, b, c, d, w, x, y, z \in \mathbb{Z}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

which is clearly non-commutative since

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} aw + cx & bw + dx \\ ay + cz & by + dz \end{bmatrix}$$

Example 1.1.2
Another important example is the set of integers ***modulo*** *n. Consider the ring* $\mathbb{Z}_2$ *consisting of two elements* $\{0, 1\}$ *with operations given by the tables:*

$+$	0	1
0	0	1
1	1	0

$\times$	0	1
0	0	0
1	0	1

The above example gives us a clue about how to add further properties to our list in Definition 1.1.1 to ensure that the integers become our only example.

For instance, we declare the following property so as to distinguish many other commutative rings from the integers.

10. $\mathbb{Z}$ is generated by $\{0, 1\}$

With this, we have laid the foundations of starting our set starting with 0 and 1, then form the rest of the elements of $\mathbb{Z}$ so as to provide the *minimal* collection satisfying all our properties. In particular, since a ring is *closed* under addition and multiplication, we need all the numbers of the form $1, 1 + 1, 1 + 1 + 1, 1 + 1 + 1 + 1, ...$

Note that such a construction ensures that the set is indeed closed under addition and multiplication. The set is closed under addition since

$$\forall a, b \in \mathbb{Z}$$
$$a + b = \underbrace{\underbrace{1 + 1 + \ldots + 1}_{\text{a many 1's}} + \underbrace{1 + 1 + \ldots + 1}_{\text{b many 1's}}}_{\text{(a + b) many 1's}}$$

which thus implies that $a + b \in \mathbb{Z}$ since a ring is closed under addition. The set is closed under multiplication since

$$\begin{aligned} a \cdot b &= (\underbrace{1 + 1 + \ldots + 1}_{\text{a many 1's}}) \cdot (\underbrace{1 + 1 + \ldots + 1}_{\text{b many 1's}}) \\ &= \underbrace{1 \cdot (\underbrace{1 + 1 + \ldots + 1}_{\text{b many 1's}}) + 1 \cdot (\underbrace{1 + 1 + \ldots + 1}_{\text{b many 1's}}) + \ldots + 1 \cdot (\underbrace{1 + 1 + \ldots + 1}_{\text{b many 1's}})}_{\text{a times}} \end{aligned}$$

by the distributive law, and since each $1 \cdot (\underbrace{1 + 1 + \ldots + 1}_{\text{b many 1's}}) \in \mathbb{Z}$, we have, once again, that

$$a \cdot b = \underbrace{1 \cdot (\underbrace{1 + 1 + \ldots + 1}_{\text{b many 1's}}) + 1 \cdot (\underbrace{1 + 1 + \ldots + 1}_{\text{b many 1's}}) + \ldots + 1 \cdot (\underbrace{1 + 1 + \ldots + 1}_{\text{b many 1's}})}_{\text{a times}} \quad \in \mathbb{Z}$$

and thus our construction is closed under multiplication.

Next, in order to satisfy the **additive inverse property**, we *may* have to add in $-1, -(1+1), -(1+1+1), ...$ The proof that such a collection is enough to satisfy all the properties of a commutative ring is similar to our exercise above.

Our legwork is not done yet, since we still have, for example, $\mathbb{Z}_2$ also being generated by its 0 and 1. We can, fortunately, eliminate this group of sets by stating that $1, 1+1, 1+1+1, ...$ are all unique. If such a property holds in any ring, then the collection $S = \{1, 1+1, 1+1+1, ...\}$ will be indistinguishable from the **natural numbers** $\mathbb{N}$ (without the 0) by any mathematical property. In fact, to ensure that all the elements are distinct, it is important that none of them are 0, since for example:

$$0 = 1 + 1 + 1 + 1 \implies (1 + 1 + 1 + 1) + 1 = 1 \text{ but } (1 + 1) + (1 + 1 + 1) = 5$$

by the distributive law.

With that said, then $-S \cap S = \emptyset$, and thus $R = S \cup \{0\} \cup -S$ is a ring that has indistinguishable properties from $\mathbb{Z}$. To clarify the explanation above, we state the following as a property:

11. No nontrivial sum of 1's is equal to 0.

Exercises

1.1 Show that $\mathbb{Z}[\sqrt{3}]$ is a commutative ring.

Let $a, b, c, d, f, g \in \mathbb{Z}$. Note that $a + b\sqrt{3}, c + d\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$

Commutativity of Addition

$$\begin{aligned} a + b\sqrt{3} + c + d\sqrt{3} &= \underbrace{1 + 1 + \ldots + 1}_{\text{a many 1's}} + (\underbrace{1 + 1 + \ldots + 1}_{\text{b many 1's}})\sqrt{3} \\ &\quad + \underbrace{1 + 1 + \ldots + 1}_{\text{c many 1's}} + (\underbrace{1 + 1 + \ldots + 1}_{\text{d many 1's}})\sqrt{3} \\ &= \underbrace{1 + 1 + \ldots + 1}_{\text{c many 1's}} + (\underbrace{1 + 1 + \ldots + 1}_{\text{d many 1's}})\sqrt{3} \\ &\quad + \underbrace{1 + 1 + \ldots + 1}_{\text{a many 1's}} + (\underbrace{1 + 1 + \ldots + 1}_{\text{b many 1's}})\sqrt{3} \quad (*) \\ &= c + d\sqrt{3} + a + b\sqrt{3} \end{aligned}$$

where $(*)$ above is by commutativity of addition for $\mathbb{Z}$.

Associativity of Addition

$$a + b\sqrt{3} + c + d\sqrt{3} + f + g\sqrt{3}$$

$$\begin{aligned} (a + b\sqrt{3}) \cdot (c + d\sqrt{3}) &= ac + ad\sqrt{3} + bc\sqrt{3} + 3bd \\ &= (ac + 3bd) + (ad + bc)\sqrt{3} \quad \in \mathbb{Z}[\sqrt{3}] \end{aligned}$$