PMATH450 — Lebesgue Integration and Fourier Analysis

Classnotes for Spring 2019

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Preface

The pre-requisite to this course is Real Analysis. We will use a lot of the concepts introduced in Real Analysis, at times without explicitly stating it. Refer to notes on PMATH351.

This course is spiritually broken into 2 pieces:

- Lebesgue Integration; and
- Fourier Analysis,

which is as the name of the course.

In this set of notes, we use a special topic environment called **culture** to discuss interesting contents related to the course, but will not be throughly studied and not tested on exams.

1 Lecture 1 May 07th 2019

Since many of our results work for both $\mathbb{C}$ and $\mathbb{R}$, we shall use $\mathbb{K}$ throughout this course to represent either $\mathbb{C}$ or $\mathbb{R}$.

1.1 Riemannian Integration

Definition 1 (Norm and Semi-Norm)

Let V be a vector space over $\mathbb{K}$. We define a ***semi-norm*** *on V as a function*

$$\nu : V \to \mathbb{R}$$

that satisfies

1. *(**Positive Semi-Definite**) $\nu(x) \geq 0$ for all $x \in V$;*
2. *$\nu(\kappa x) = |\kappa| \, \nu(x)$ for any $\kappa \in \mathbb{K}$ and $x \in V$; and*
3. *(**Triangle Inequality**) $\nu(x+y) \leq \nu(x) + \nu(y)$ for all $x, y \in V$.*

If $\nu(x) = 0 \implies x = 0$, then we say that ν is a ***norm***. *In this case, we usually write $\|\cdot\|$ to denote the norm, instead of ν.*

Note 1.1.1

- *We sometimes call a semi-norm a* ***pseudo-length***.

Remark 1.1.1

Notice that we wrote $\nu(x) = 0 \implies x = 0$ *instead of* $\nu(x) = 0 \iff x = 0$. *This is because if* $z = 0 \in V$*, then*

$$v(z) = v(0z) = 0.$$

Exercise 1.1.1

Show that if ν *is a semi-norm on a vector space* V*, then* $\forall x, y \in V$*,*

$$|\nu(x) - \nu(y)| \leq \nu(x - y).$$

Proof

Notice that by condition (2) and (3), we have

$$\nu(x - y) \leq \nu(x) + \nu(-y) = \nu(x) - \nu(y),$$

and

$$\nu(x - y) = -\nu(y - x) \geq -(\nu(y) - \nu(x)) = \nu(x) - \nu(y).$$

It follows that indeed

$$|\nu(x) - \nu(y)| \leq \nu(x - y). \qquad \square$$

Example 1.1.1

The absolute value $|\cdot|$ is a norm on $\mathbb{K}$.

Example 1.1.2 (p-norms)

Consider $N \geq 1$ an integer. We define a family of norms on

$$\mathbb{K}^N = \underbrace{K \times K \times \ldots \times K}_{N \text{ times}}.$$

1-norm

$$\left\| (x_n)_{n=1}^N \right\|_1 := \sum_{n=1}^N |x_n|.$$

Infinity-norm, ∞-norm

$$\left\| (x_n)_{n=1}^N \right\|_\infty := \max_{1 \leq n \leq N} |x_n| .$$

Euclidean-norm, 2-norm

$$\left\| (x_n)_{n=1}^N \right\|_2 := \left(\sum_{n=1}^N |x_n|^2 \right)^{\frac{1}{2}}$$

It is relatively easy to check that the above norms are indeed norms, except for the 2-form. In particular, the **triangle inequality** is not as easy to show [1].

[1] See Minkowski's Inequality.

Less obviously so, but true nonetheless, we can define the following p-norms on $\mathbb{K}^N$:

$$\left\| (x_n)_{n=1}^N \right\|_p := \left(\sum_{n=1}^N |x_n|^p \right)^{\frac{1}{p}} ,$$

for $1 \leq p < \infty$.

Culture

Consider $V = \mathbb{M}_n(\mathbb{C})$, [2] where $n \in \mathbb{N}$ is fixed. For $T \in \mathbb{M}_n(\mathbb{C})$, we define the ***singular numbers*** *of T to be*

$$s_1(T) \geq s_2(T) \geq \ldots \geq s_n(T) \geq 0,$$

where $\sigma(T^*T) = \{s_1(T)^2, s_2(T)^2, \ldots, s_n(T)^2\}$, including multiplicity. Then we can define

$$\|T\|_p := \left(\sum_{i=1}^n s_i(T)^p \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$, which is called the p-norm of T on $\mathbb{M}_n(\mathbb{C})$.

[2] Note that $\mathbb{M}_n(\mathbb{C})$ is the set of $n \times n$ matrices over $\mathbb{C}$.

Example 1.1.3

Let

$$V = \mathcal{C}([0,1], \mathbb{K}) = \{ f : [0,1] \to \mathbb{K} \mid f \text{ is continuous } \}.$$

Then

$$\|f\|_{\sup} := \sup\{|f(x)| \mid x \in [0,1]\}$$

[3] defines a norm on $\mathcal{C}([0,1], \mathbb{K})$.

[3] Some authors use $\|f\|_\infty$, but we will have the notation $\|[f]\|_\infty$ later on, and so we shall use $\|f\|_{\sup}$ for clarity.

A sequence $(f_n)_{n=1)^\infty}$ in V converges in this norm to some $f \in V$, i.e.

$$\lim_{n\to\infty} \|f_n - f\|_{\sup} = 0,$$

which means that $(f_n)_{n=1}^\infty$ converges uniformly to f on $[0,1]$.

Definition 2 (Normed Linear Space)

A ***normed linear space (NLS)*** *is a pair* $(V, \|\cdot\|)$ *where* V *is a vector space over* $\mathbb{K}$ *and* $\|\cdot\|$ *is a norm on* V.

Definition 3 (Metric)

Given an NLS $(V, \|\cdot\|)$*, we can define a* ***metric*** d *on* V *(called the* ***metric induced by the norm****) as follows:*

$$d : V \times V \to \mathbb{R} \quad d(x,y) = \|x-y\|,$$

such that

- $d(x,y) \geq 0$ *for all* $x, y \in V$ *and* $d(x,y) = 0 \iff x = y$;
- $d(x,y) = d(y,x)$; *and*
- $d(x,y) \leq d(x,z) + d(y,z)$.

Note 1.1.2

Norms are all metrics, and so any space that has a norm will induce a metric on the space.

Definition 4 (Banach Space)

We say that an NLS $(V, \|\cdot\|)$ is ***complete*** *or is a* ***Banach Space*** *if the corresponding (V, d), where d is the metric induced by the norm, is complete* [4].

[4] Completeness of a metric space is such that any of its Cauchy sequences converges in the space.

Example 1.1.4

$(\mathcal{C}([0,1], \mathbb{K}), \|\cdot\|_{\sup})$ is a Banach space.

Example 1.1.5

We can define a 1-norm $\|\cdot\|_1$ on $\mathcal{C}([0,1], \mathbb{K})$ via

$$\|f\|_1 := \int_0^1 |f|.$$

Then $(\mathcal{C}([0,1], \mathbb{K}), \|\cdot\|_1)$ is an NLS.

Exercise 1.1.2

Show that $(\mathcal{C}([0,1], \mathbb{K}), \|\cdot\|_1)$ is not complete, which will then give us an example of a ***normed linear space that is not Banach****.*

Proof

Consider the sequence $(f_n)_{n=1}^{\infty}$ of continuous functions given by

$$f_n(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ n\left(x + \frac{1}{2}\right) & \frac{1}{2} \leq x \leq \frac{1}{2} + \frac{1}{n} \\ 1 & \text{otherwise} \end{cases}$$

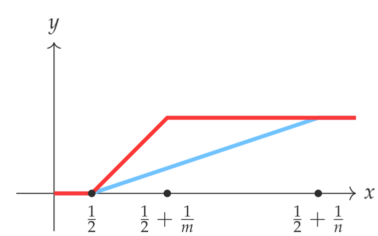

Figure 1.1: Sequence of functions $(f_n)_{n=1}^{\infty}$. We show for two indices $n < m$.

Note that the sequence $(f_n)_{n=1}^{\infty}$ is indeed **Cauchy**: let $\varepsilon > 0$ and $|n - m| < \frac{\varepsilon}{|x - \frac{1}{2}|}$, and then we have

$$\begin{aligned} |f_n(x) - f_m(x)| &= \left| n\left(x - \frac{1}{2}\right) - m\left(x - \frac{1}{2}\right) \right| \\ &= \left| (n - m)\left(x - \frac{1}{2}\right) \right| = |n - m| \left| x - \frac{1}{2} \right| < \varepsilon. \end{aligned}$$

However, it is clear that the sequence $(f_n)_{n=1}^{\infty}$ converges to the piecewise function (in particular, a non-continuous function)

$$f(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}.$$

□

Example 1.1.6

If $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ are NLS's, and if $T : \mathfrak{X} \to \mathfrak{Y}$ is a linear map, we define the **operator norm** of T to be

$$\|T\| := \sup\{\|T(x)\|_{\mathfrak{Y}} \mid \|x\|_{\mathfrak{X}} \leq 1\}.$$

We set

$$B(\mathfrak{X}, \mathfrak{Y}) := \{T : \mathfrak{X} \to \mathfrak{Y} \mid T \text{ is linear }, \|T\| < \infty\}.$$

Note that for any such linear map T, $\|T\| < \infty \iff T$ is continuous. Thus $B(\mathfrak{X}, \mathfrak{Y})$ is the set of all continuous functions from $\mathfrak{X}$ into $\mathfrak{Y}$.

Then $(B(\mathfrak{X}, \mathfrak{Y}), \|\cdot\|)$ is an NLS.

It is likely that we have seen this in Real Analysis.

Exercise 1.1.3

Show that $(B(\mathfrak{X}, \mathfrak{Y}), \|\cdot\|)$ is complete iff $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ is complete.

Note 1.1.3

*One example of the last example is when $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}}) = (\mathbb{K}, |\cdot|)$. In this case, $B(\mathfrak{X}, \mathbb{K})$ is known as the **dual space** of $\mathfrak{X}$, or simple the **dual** of $\mathfrak{X}$.*

We are interested in integrating over Banach spaces.

Definition 5 (Partition of a Set)

*Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a, b] \to \mathfrak{X}$ a function, where $a < b \in \mathbb{R}$. A **partition** P of $[a, b]$ is a finite set*

$$P = \{a = p_0 < p_1 < \ldots < p_N = b\}$$

for some $N \geq 1$. The set of all partitions of $[a, b]$ is denoted by $\mathcal{P}[a, b]$.

Definition 6 (Test Values)

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a, b] \to \mathfrak{X}$ a function, where $a < b \in \mathbb{R}$. Let $P \in \mathcal{P}[a, b]$. A set

$$P^* := \{p_k^*\}_{k=1}^N$$

satisfying

$$p_{k-1} \leq p_k^* \leq p_k, \text{ for } 1 \leq k \leq n$$

is called a set of ***test values*** *for P.*

Definition 7 (Riemann Sum)

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a, b] \to \mathfrak{X}$ a function, where $a < b \in \mathbb{R}$. Let $P \in \mathcal{P}[a, b]$ and P^* its corresponding set of test values. We define the ***Riemann sum*** *as*

$$S(f, P, P^*) = \sum_{k=1}^N f(p_k^*)(p_k - p_{k-1}).$$

Remark 1.1.2

1. *Note that because Definition 5, $p_k - p_{k-1} > 0$.*
2. *When $(\mathfrak{X}, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, then this is the usual Riemann sum from first-year calculus.*
3. *In general, note that*

$$\frac{1}{b-a} S(f, P, P^*) = \sum_{k=1}^N \lambda_k f(p_k^*),$$

where $0 < \lambda_k = \frac{p_k - p_{k-1}}{b-a} < 1$ and [5]

$$\sum_{k=1}^N \lambda_k = 1.$$

[5] via the fact that the λ_k's form a telescoping sum

So $\frac{1}{b-a} S(f, P, P^*)$ is an averaging of f over $[a, b]$. We call $\frac{1}{b-a} S(f, P, P^*)$ the convex combination of the $f(p_k^*)$'s.

Example 1.1.7 (Silly example)

Let $(\mathfrak{X} = \mathcal{C}([-\pi, \pi], \mathbb{K}), \|\cdot\|_{\sup})$. Let

$$f : [0, 1] \to \mathfrak{X} \text{ such that } x \mapsto e^{2\pi x} \sin 7\theta + \cos x \cos(12\theta),$$

where $\theta \in [-\pi, \pi]$. Now if we consider the partition

$$P = \left\{ -\pi, \frac{1}{10}, \frac{1}{2}, \pi \right\}$$

and its corresponding test value

$$P^* = \left\{ 0, \frac{1}{3}, 2 \right\},$$

then

$$\begin{aligned} S(f, P, P^*) &= f(0) \left(\frac{1}{10} + \pi \right) + f\left(\frac{1}{3} \right) \left(\frac{1}{2} - \frac{1}{10} \right) + f(2) \left(\pi - \frac{1}{2} \right) \\ &= (\sin 7\theta + \cos 12\theta) \left(\pi + \frac{1}{10} \right) \\ &\quad + \left(e^{\frac{2\pi}{3}} \sin 7\theta + \cos \frac{1}{3} \cos 12\theta \right) \left(\frac{2}{5} \right) \\ &\quad + (e^{4\pi} \sin 7\theta + \cos 2 \cos 12\theta) \left(\pi - \frac{1}{2} \right) \end{aligned}$$

Definition 8 (Refinement of a Partition)

*Let $a < b \in \mathbb{R}$, and $P \in \mathcal{P}[a, b]$. We say Q is a **refinement** of P is $Q \in \mathcal{P}[a, b]$ and $P \subseteq Q$.*

Note 1.1.4

In simpler words, Q is a "finer" partition that is based on P.

Definition 9 (Riemann Integrable)

Let $a < b \in \mathbb{R}$, $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a, b] \to \mathfrak{X}$ be a function. We say that f is ***Riemann integrable*** *over $[a, b]$ if $\exists x_0 \in \mathfrak{X}$ such that*

$$\forall \varepsilon > 0 \quad \exists P \in \mathcal{P}[a, b],$$

such that if Q is any refinement of P, and Q^* is any set of test values of Q, then

$$\|x_0 - S(f, Q, Q^*)\|_{\mathfrak{X}} < \varepsilon.$$

In this case, we write

$$\int_a^b f = x_0.$$

Proposition 1 (Uniqueness of the Riemann Integral)

If f is Riemann integrable over $[a, b]$, then the value of $\int_a^b f$ is unique.

Proof

Suppose not, i.e.

$$\int_a^b f = x_0 \text{ and } \int_a^b f = y_0$$

for some $x_0 \neq y_0$. Then, let

$$\varepsilon = \frac{\|x_0 - y_0\|}{2},$$

which is > 0 since $\|x_0 - y_0\| > 0$. Let $P_{x_0}, P_{y_0} \in \mathcal{P}[a, b]$ be partitions corresponding to x_0 and y_0 as in the definition of Riemann integrability.

Then, let $R = P_{x_0} \cup P_{y_0}$, so that R is a **common refinement** of P_{x_0} and P_{y_0}. If Q is any refinement of R, then Q is also a common refinement of P_{x_0} and P_{y_0}. Then for any test values Q^* of Q, we have

$$\begin{aligned} 2\varepsilon &= \|x_0 - y_0\| \\ &\leq \|x_0 - S(f, Q, Q^*)\| + \|S(f, Q, Q^*) - y_0\| < \varepsilon + \varepsilon = 2\varepsilon, \end{aligned}$$

which is a contradiction.

Thus $x_0 = y_0$ as required. □

Theorem 2 (Cauchy Criterion of Riemann Integrability)

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space, $a < b \in \mathbb{R}$ and $f : [a, b] \to \mathfrak{X}$ be a function. TFAE:

1. *f is Riemann integrable over $[a, b]$;*
2. *$\forall \varepsilon > 0$, $R \in \mathcal{P}[a, b]$, if P, Q is any refinement of R, and P^* (respectively Q^*) is any test values of P (respectively Q), then*

$$\|S(f, P, P^*) - S(f, Q, Q^*)\|_{\mathfrak{X}} < \varepsilon.$$

Proof

$\implies$ This is a rather straightforward proof. Suppose $P, Q \in \mathcal{P}[a, b]$ is some refinement of the given partition $R \in \mathcal{P}[a, b]$, and P^*, Q^* any test values for P, Q, respectively. Then by assumption and Proposition 1, $\exists x_0 \in \mathfrak{X}$ such that

$$\|x_0 - S(f, P, P^*)\|_{\mathfrak{X}} < \frac{\varepsilon}{2} \text{ and } \|x_0 - S(f, Q, Q^*)\|_{\mathfrak{X}} < \frac{\varepsilon}{2}.$$

It follows that

$$\begin{aligned}
&\|S(f, P, P^*) - S(f, Q, Q^*)\|_{\mathfrak{X}} \\
&\leq \|x_0 - S(f, P, P^*)\|_{\mathfrak{X}} + \|x_0 - S(f, Q, Q^*)\|_{\mathfrak{X}} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

$\impliedby$ By hypothesis, wma $\varepsilon = \frac{1}{n}$ for some $n \geq 1$, such that if P, Q are any refinements of the partition $R_n \in \mathcal{P}[a, b]$, and P^*, Q^* are the respective arbitrary test values, then

$$\|S(f, P, P^*) - S(f, Q, Q^*)\|_{\mathfrak{X}} < \frac{1}{n}$$

Now for each $n \geq 1$, define

$$W_n := \bigcup_{k=1}^{n} R_k \in \mathcal{P}[a, b],$$

so that W_n is a common refinement for $R_1, R_2, \ldots, R_n$. For each $n \geq 1$, let W_n^* be an arbitrary set of test values for W_n. For simplicity, let us write

$$x_n = S(f, W_n, W_n^*), \text{ for each } n \geq 1.$$

[6]

Claim: $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence If $n_1 \geq n_2 > N \in \mathbb{N}$, then

$$\|x_{n_1} - x_{n_2}\|_{\mathfrak{X}} = \left\| S(f, W_{n_1}, W_{n_1}^*) - S(f, W_{n_2}, W_{n_2}^*) \right\| < \frac{1}{N}$$

by our assumption, since W_{n_1}, W_{n_2} are refinements of R_N. Then by picking $N = \frac{1}{\varepsilon}$ for any $\varepsilon > 0$, we have that $(x_n)_{n=1}^{\infty}$ is indeed a Cauchy sequence in $\mathfrak{X}$.

Since $\mathfrak{X}$ is a Banach space, it is complete, and so $\exists x_0 := \lim_{n \to \infty} x_n \in \mathfrak{X}$. It remains to show that, indeed,

$$x_0 = \int_a^b f.$$

Let $\varepsilon > 0$, and choose $N \geq 1$ such that

- $\frac{1}{N} < \frac{\varepsilon}{2}$; and
- $k \geq N$ implies that $\|x_k - x_0\| < \frac{\varepsilon}{2}$.

Then suppose that V is any refinement of W_N, and V^* is an arbitrary set of test values of V. Then we have

$$\begin{aligned} \|x_0 - S(f, V, V^*)\|_{\mathfrak{X}} &\leq \|x_0 - x_N\|_{\mathfrak{X}} + \|x_N - S(f, V, V^*)\|_{\mathfrak{X}} \\ &< \frac{\varepsilon}{2} + \|S(f, W_N, W_N^*) - S(f, V, V^*)\|_{\mathfrak{X}} \\ &< \frac{\varepsilon}{2} + \frac{1}{N} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

It follows that

$$\int_a^b f = x_0,$$

as desired. $\square$

[6] Note that it would be nice if for the finer and finer partitions that we have constructed, i.e. the W_n's, give us a convergent sequence of Riemann sums, since it makes sense that this convergence will give us the final value that we want.

In first-year calculus, all continuous functions over $\mathbb{R}$ are integrable. A similar result holds in Banach spaces as well. In the next lecture, we shall prove the following theorem.

Theorem (Continuous Functions are Riemann Integrable)

Let $(\mathfrak{X}, \|\cdot\|)$ *be a Banach space and* $a < b \in \mathbb{R}$. *If* $f : [a, b] \to \mathfrak{X}$ *is continuous, then* f *is Riemann integrable over* $[a, b]$.

2 Lecture 2 May 9th 2019

2.1 Riemannian Integration (Continued)

We shall now prove the last theorem stated in class.

Theorem 3 (Continuous Functions are Riemann Integrable)

Let $(\mathfrak{X}, \|\cdot\|)$ be a Banach space and $a < b \in \mathbb{R}$. If $f : [a, b] \to \mathfrak{X}$ is continuous, then f is Riemann integrable over $[a, b]$.

Strategy

This is rather routine should one have gone through a few courses on analysis, and especially on introductory courses that involves Riemannian integration.

We shall show that if $P_N \in \mathcal{P}[a, b]$ is a partition of $[a, b]$ into 2^N subintervals of equal length $\frac{b-a}{2^N}$, and if we use $P_N^* = P_n \setminus \{a\}$ as the set of test values for P_N, which consists of the right-endpoints of each the subintervals in P_N, then the sequence $(S(f, P_N, P_N^*))_{N=1}^{\infty}$ converges in $\mathfrak{X}$ to $\int_a^b f$.

Note that this choice of partition is a valid move, since any of these P_N's, for different N's, is a refinement of some other partition of $[a, b]$, and if we choose a different set of test values, then we may as well consider an even finer partition.

Proof

First, note that since $[a, b]$ is closed and bounded in $\mathbb{R}$, it is compact. Also, we have that X is a metric space (via the metric induced by the norm). This means that **any continuous function f on $[a, b]$ is uniformly continuous on $[a, b]$**. In other words,

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall x, y \in [a, b]$$
$$|x - y| < \delta \implies \|f(x) - f(y)\| < \frac{\varepsilon}{2(b-a)}.$$

Claim: $(S(f, P_N, P_N^*))_{N=1}^{\infty}$ is Cauchy Now by picking $P_N \in \mathcal{P}[a, b]$ and set of test values P_N^* as described in the strategy above, we proceed by picking $M > 0$ such that $\frac{b-a}{2^M} < \delta$. Then for any $K \geq L \geq M$, since each of the subintervals have length $\frac{b-a}{2^L}$ and $\frac{b-a}{2^K}$ for P_L and P_K respectively, if we write

$$P_L = \{a = p_0 < p_1 < \ldots < p_{2^L} = b\}$$

and

$$P_K = \{a = q_0 \leq q_1 < \ldots < q_{2^K} = b\},$$

then $p_j = q_j 2^{K-L}$ [1] for all $0 \leq j \leq 2^L$. By uniform continuity, for $1 \leq j \leq 2^L$, wma

[1] This is not immediately clear on first read. Think of a as 0.

$$\left\| f(p_j^*) - f(q_s^*) \right\| < \frac{\varepsilon}{2(b-a)}, \text{ where } (j-1)2^{K-L} < s \leq j2^{K-L}.$$

We can see that

$$\begin{aligned}
&\|S(f, P_L, P_L^*) - S(f, P_K, P_K^*)\| \\
&= \left\| \sum_{j=1}^{2^L} \sum_{s=(j-1)2^{K-L}+1}^{j2^{K-L}} (f(p_j) - f(q_s))(q_s - q_{s-1}) \right\| \\
&\leq \sum_{j=1}^{2^L} \sum_{s=(j-1)2^{K-L}+1}^{j2^{K-L}} \left\| f(p_j) - f(q_s) \right\| (q_s - q_{s-1}) \\
&\leq \sum_{j=1}^{2^L} \sum_{s=(j-1)2^{K-L}+1}^{j2^{K-L}} \frac{\varepsilon}{b-a}(q_s - q_{s-1}) \\
&= \frac{\varepsilon}{b-a} \sum_{s=1}^{2^K} (q_s - q_{s-1}) \\
&= \frac{\varepsilon}{2(b-a)}(b-a) = \frac{\varepsilon}{2}.
\end{aligned}$$

This proves our claim.

Since $\mathfrak{X}$ is a Banach space, and hence complete, we have that the sequence $(S(f, P_N, P_N^*))_{N=1}^{\infty}$ has a limit $x_0 \in \mathfrak{X}$.

It remains to show that $\int_a^b f = x_0$. [2]

Let $\varepsilon > 0$, and choose $T \geq 1$ such that $\frac{b-a}{2^T} < \delta$ [3], so that we have

$$\|x_0 - S(f, P_T, P_T^*)\| < \frac{\varepsilon}{2}.$$

[2] The rest of this proof is similar to the above proof.

[3] Note that this is still the same δ as in the first δ in this entire proof.

Now let $R = \{a = r_0 < r_1 < \ldots < r_J = b\} \in \mathcal{P}[a, b]$ such that $P_T \subseteq R$. Then there exists a sequence

$$0 = j_0 < j_1 < \ldots < j_{2^T} = J$$

such that

$$r_{j_k} = p_k, \text{ where } 0 \leq k \leq 2^T.$$

Let R^* be any set of test values of R. Note that for $j_{k-1} \leq s \leq j_k$, it is clear that

$$|p_k^* - r_s^*| \leq |p_k - p_{k-1}| = \frac{b-a}{2^T} < \delta.$$

Thus

$$\begin{aligned}
&\|S(f, P_T, P_T^*) - S(f, R, R^*)\| \\
&\leq \sum_{k=1}^{2^T} \sum_{s_{j_{k-1}+1}}^{j_k} \|f(p_k^*) - f(r_s^*)\| (r_s - r_{s-1}) \\
&< \frac{\varepsilon}{2(b-a)} \sum_{k=1}^{2^T} \sum_{s_{j_{k-1}+1}}^{j_k} (r_s - r_{s-1}) \\
&= \frac{\varepsilon}{2(b-a)} (b-a) = \frac{\varepsilon}{2}.
\end{aligned}$$

Putting everything together, we have

$$\begin{aligned}
&\|x_0 - S(f, R, R^*)\| \\
&\leq \|x_0 - S(f, P_T, P_T^*)\| + \|S(f, P_T, P_T^*) - S(f, R, R^*)\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

We can also find another refinement of P_T, say Q, that works similarly as in the case of R. It follows from Theorem 2 that

$$x_0 = \int_a^b f,$$

i.e. that f is indeed Riemann integrable over $[a, b]$. $\square$

The following is a corollary whose proof shall be left as an exercise.

Corollary 4 (Piecewise Functions are Riemann Integrable)

A piecewise continuous function is also Riemann integrable: if $f : [a, b] \to \mathfrak{X}$ is piecewise continuous, then f is Riemann integrable.

Exercise 2.1.1

Prove Corollary 4.

Let us exhibit a function that is not Riemann integrable.

Definition 10 (Characteristic Function)

*Given a subset E of a set $\mathbb{R}$, we define the **characteristic function** of E as a function $\chi_E : \mathbb{R} \to \mathbb{R}$ given by*

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

Example 2.1.1

Consider the set $E = \mathbb{Q} \cap [0, 1] \subseteq \mathbb{R}$. Let $P \in \mathcal{P}[0, 1]$ such that

$$P = \{0 = p_0 < p_1 < \ldots < p_N = 1\},$$

and let

$$P^* = \{p_k^*\}_{k=1}^N \text{ and } P^{**} = \{p_k^{**}\}_{k=1}^N$$

be 2 sets of test values for P, such that we have

$$p_k^* \in \mathbb{Q} \text{ and } p_k^{**} \in \mathbb{R} \setminus \mathbb{Q}.$$

Then we have

$$\begin{aligned} S(\chi_E, P, P^*) &= \sum_{k=1}^N \chi_E(p_k^*)(p_k - p_{k-1}) \\ &= \sum_{k=1}^N 1 \cdot (p_k - p_{k-1}) \\ &= p_N - p_0 = 1 - 0 = 1, \end{aligned}$$

and

$$\begin{aligned} S(\chi_E, P, P^{**}) &= \sum_{k=1}^N \chi_E(p_k^{**})(p_k - p_{k-1}) \\ &= \sum_{k=1}^N 0 \cdot (p_k - p_{k-1}) \\ &= 0. \end{aligned}$$

It is clear that the Cauchy criterion fails for χ_E. This shows that χ_E is not Riemann integrable.

Remark 2.1.1

Let us once again consider $E = \mathbb{Q} \cap [0, 1]$. Note that E is ***denumerable*** [4]. *We may thus write*

$$E = \{q_n\}_{n=1}^\infty.$$

Now, for $k \geq 1$, define

$$f_k(x) = \sum_{n=1}^k \chi_{\{q_n\}}(x).$$

In other words, $f_k = \chi_{\{q_1,\ldots,q_k\}}$. Furthermore, we have that

$$f_1 \leq f_2 \leq f_3 \ldots \leq \chi_E.$$

[4] This means that E is countably infinite.

Moreover, we have that $\forall x \in [0,1]$,

$$\chi_E(x) = \lim_{k\to\infty} f_k(x),$$

and

$$\int_0^1 f_k = 0 \text{ for all } k \geq 1.$$

And yet, we have that $\int_0^1 \chi_E$ *does not exist!*

We want to develop a different integral that will 'cover' for this 'pathological' behavior of where the Riemann integral fails.

The rough idea is as follows.

In Riemann integration, when integrating over an interval $[a,b]$, we partitioned $[a,b]$ into subintervals. This happens on the x-axis.

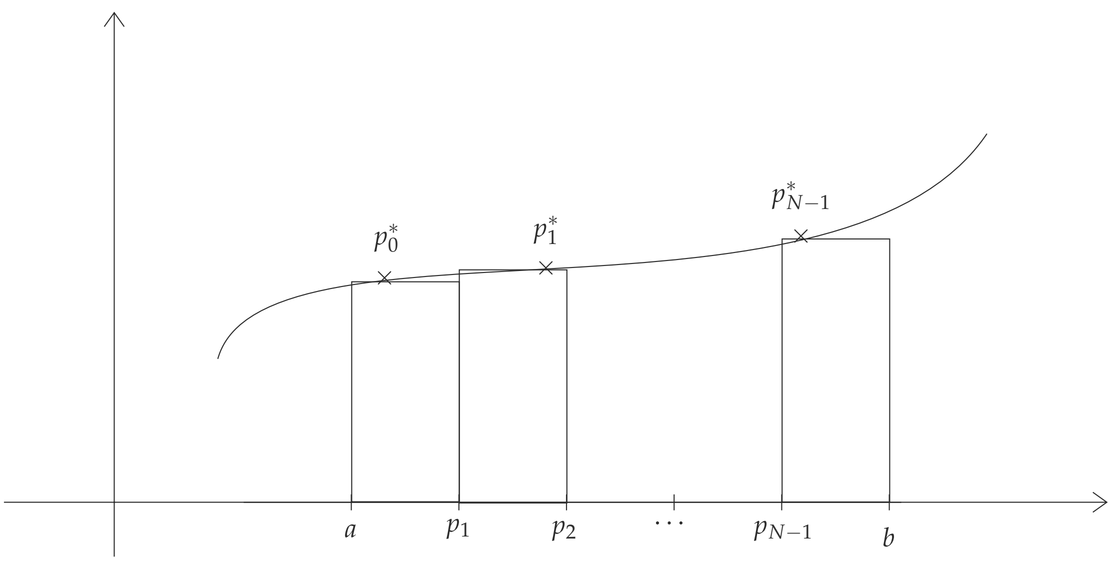

Figure 2.1: Rough illustration of how Riemann's integration works

In each of the subintervals of the partition, we pick out a **test value** p_i^*, and basically draw a rectangle with base at $[p_i, p_{i+1}]$ and height from 0 to p_i^*.

What we shall do now is that we **partition the range of f on the y-axis**, instead of the x-axis as we do in Riemannian integration.

In particular, given a function $f : [a,b] \to \mathbb{R}$, we first partition the range of f into subintervals $[y_{k-1}, y_k]$, where $1 \leq k \leq N$. Then, we set

$$E_k = \{x \in [a,b] : f(x) \in [y_{k-1}, y_k]\} \text{ for } 1 \leq k \leq N.$$

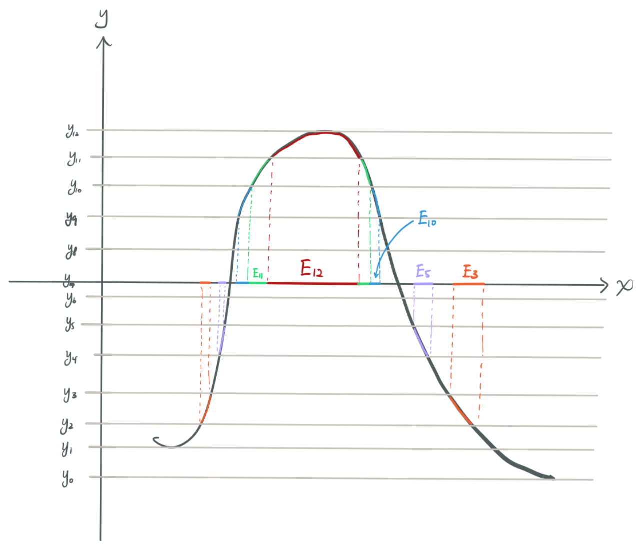

Figure 2.2: A sketch of what's happening with the construction of the E_k's

This will then allow us to estimate the integral of f over $[a,b]$ by the expression

$$\sum_{k=1}^{N} y_k m E_k,$$

where each of the $y_k m E_k$ are called **simple functions**. In the expression, mE_k denotes a "measure" [5] of E_k.

[5] Note that a measure is simply a generalization of the notion of 'length'.

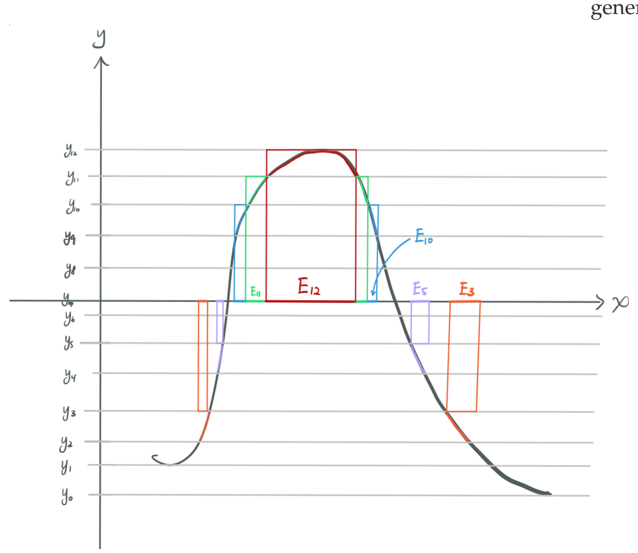

Figure 2.3: Drawing out the rectangles of $y_k m E_k$ from Figure 2.2.

We observe that E_k need not be a particularly well-behaved set.

However, note that we may rearrange the possibly scattered pieces of each E_k together, so as to form a 'continuous' base for the rectangle. We need our definition of a measure to be able to capture this.

The following is an analogy from Lebesgue himself on comparing Lebesgue integration and Riemann integration [6]:

> *I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral.*
> *But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral.*

[6] Siegmund-Schultze, R. (2008). *Henri Lesbesgue, in Timothy Gowers, June Barrow-Green, Imre Leader (eds.), Princeton Companion to Mathematics*. Princeton University Press

The insight here is that one can freely arrange the values of te functions, all the while preserving the value of the integral.

- This requires us to have a better understanding of what a measure is.
- This process of rearrangement converts certain functions which are extremely difficult to deal with, or outright impossible, with the Riemann integral, into easily digestible pieces using Lebesgue integral.

2.2 Lebesgue Outer Measure

Goals of the section

1. Define a "measure of length" on as many subsets of $\mathbb{R}$ as possible.
2. The definition should agree with our intuition of what a 'length' is.

Definition 11 (Length)

*For $a \leq b \in \mathbb{R}$, we define the **length** of the interval (a, b) to be $b - a$, and* *we write*

$$\ell((a,b)) := b - a.$$

We also define

- $\ell(\emptyset) = 0$; *and*
- $\ell((a,\infty)) = \ell((-\infty,b)) = \ell((-\infty,\infty)) = \infty$.

Definition 12 (Cover by Open Intervals)

Let $E \subseteq \mathbb{R}$. A countable collection $\{I_n\}_{n=1}^{\infty}$ of open intervals is said to be a ***cover of E by open intervals*** *if $E \subseteq \bigcup_{n=1}^{\infty} I_n$.*

Note 2.2.1

In this course, the only covers that we shall use are open intervals, and so we shall henceforth refer to the above simply as covers of E.

Before giving what immediately follows from the above, I shall present the following notion of an outer measure.

Definition 13 (Outer Measure)

Let $\emptyset \neq X$ be a set. An ***outer measure*** *μ on X is a function*

$$\mu : \mathcal{P}(X) \to [0,\infty] := [0,\infty) \cup \{\infty\}$$

which satisfies

1. $\mu\emptyset = 0$;
2. *(**monotone increment** or **monotonicity**) $E \subseteq F \subseteq X \implies \mu E \leq \mu F$; and*
3. *(**countable subadditivity** or **σ-subadditivity**) $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu E_n.$$

Note 2.2.2

Note that by the monotonicity, the σ-subadditivity condition is equivalent to: given $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$ and $F \subseteq \bigcup_{n=1}^{\infty} E_n$, we have that

$$\mu(F) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

Definition 14 (Lebesgue Outer Measure)

We define the ***Lebesgue outer measure*** *as a function $m^* : \mathcal{P}(X) \to \mathbb{R}$ such that*

$$m^* E := \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : E \subseteq \bigcup_{n=1}^{\infty} I_n \right\}.$$

We cheated a little bit by calling the above an outer measure, so let us now justify our cheating.

Proposition 5 (Validity of the Lebesgue Outer Measure)

m^* is indeed an outer measure.

Proof

$\mu\varnothing = 0$ We consider a sequence of sets $\{I_n\}_{n=1}^{\infty}$ such that $I_n = \varnothing$ for each $n = 1, \ldots, \infty$. It is clear that $\varnothing \subseteq \bigcup_{n=1}^{\infty} I_n$. Also, we have that $\ell(I_n) = 0$ for all $n = 1, \ldots, \infty$. It follows that

$$0 \leq m^*(\varnothing) \leq \sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} 0 = 0,$$

where the inequality is simply by the definition of m^* being an infimum, **not to be confused with σ-subadditivity**. We thus have that

$$m^*(\varnothing) = 0.$$

Monotonicity Suppose $E \subseteq F \subseteq \mathbb{R}$, and $\{I_n\}_{n=1}^{\infty}$ a cover of F. Then

$$E \subseteq F \subseteq \bigcup_{n=1}^{\infty} I_n.$$

In particular, all covers of F are also covers of E, i.e.

$$\left\{ \{J_m\}_{m=1}^{\infty} : E \subseteq \bigcup_{m=1}^{\infty} J_m \right\} \subseteq \left\{ \{I_n\}_{n=1}^{\infty} : F \subseteq \bigcup_{n=1}^{\infty} I_n \right\}.$$

It follows that

$$m^* E \leq m^* F.$$

σ-subaddivitity Consider $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$ such that $E \subseteq \bigcup_{n=1}^{\infty} E_n$. WTS

$$m^* E \leq \sum_{n=1}^{\infty} m^* E_n.$$

Now if the sum of the RHS is infinite, i.e. if any of the $m^* E_n$ is infinite, then the inequality comes for free. Thus WMA $\sum_{n=1}^{\infty} E_n < \infty$, and in particular that $m^* E_n < \infty$ for all $n = 1, \ldots, \infty$.

To do this, let $\varepsilon > 0$. Since $m^* E_n < \infty$ for all n, we can find covers $\left\{ I_k^{(n)} \right\}_{k=1}^{\infty}$ for each of the E_n's such that

$$\sum_{k=1}^{\infty} \ell \left(I_k^{(n)} \right) < m^* E_n + \frac{\varepsilon}{2^n}.$$

Then, we have that

$$E \subseteq \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} I_k^{(n)}.$$

Then by $m^* E$ being the infimum of the sum of lengths of the covering intervals, we have that

$$\begin{aligned} m^* E &\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \ell \left(I_k^{(n)} \right) \\ &\leq \sum_{n=1}^{\infty} \left(m^* E_n + \frac{\varepsilon}{2^n} \right) \\ &= \sum_{n=1}^{\infty} m^* E_n + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} \\ &= \sum_{n=1}^{\infty} m^* E_n + \varepsilon. \end{aligned}$$

Since ε was arbitrary, we have that

$$m^* E_n \leq \sum_{n=1}^{\infty} m^* E_n,$$

as desired. □

Corollary 6 (Lebesgue Outer Measure of Countable Sets is Zero)

If $E \subseteq \mathbb{R}$ is countable, then $m^* E = 0$.

Proof

We shall prove for when E is denumerable, for the finite case follows a similar proof. Let us write $E = \{x_n\}_{n=1}^{\infty}$. Let $\varepsilon > 0$ and

$$I_n = \left(x_n - \frac{\varepsilon}{2^{n+1}}, \, x_n + \frac{\varepsilon}{2^{n+1}} \right).$$

Then it is clear that $\{I_n\}_{n=1}^{\infty}$ is a cover of E.

It follows that

$$0 \leq m^* E \leq \sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Thus as $\varepsilon \to 0$, we have that

$$m^* E = 0,$$

as expected. □

Corollary 7 (Lebesgue Outer Measure of $\mathbb{Q}$ is Zero)

We have that $m^* \mathbb{Q} = 0$.

In the proofs above that we have looked into, and based on the intuitive notion of the length of an open interval, it is compelling to simply conclude that

$$m^*(a,b) = \ell(a,b) = b - a.$$

However, looking back at Definition 14, we know that that is not how $m^*(a,b)$ is defined.

This leaves us with an interesting question:

> *how does our notion of measure $m^*(a,b)$ of an interval compare with the notion of the length of an interval?*

By taking $I_1 = (a,b)$ and $I_n = \emptyset$ for $n \geq 2$, it is rather clear that $\{I_n\}_{n=1}^\infty$ is a cover of (a,b), and so we have

$$m^*(a,b) \leq \ell(a,b) = b - a. \tag{2.1}$$

However, the other side of the game is not as easy to confirm: we would have to consider **all** possible covers of (a,b), which is a lot.

Another question that we can ask ourselves seeing Equation (2.1) is why can't $m^*(a,b)$ be something that is strictly less than the length to give us an even more 'precise' measurement?

To answer these questions, it is useful to first consider the outer measure of a closed and bounded interval, e.g. $[a,b]$, since these intervals are **compact** under the **Heine-Borel Theorem**. This will give us a finite subcover for every infinite cover of the compact interval, which is easy to deal with.

We shall see that with the realization of the outer measure of a compact interval, we will also be able to find the outer measure of intervals that are neither open nor closed.

We shall prove the following proposition in the next lecture. Note that for the sake of presentation, I shall abbreviate the Lebesgue Outer Measure as LOM.

Proposition (LOM of Arbitrary Intervals)

Suppose $a < b \in \mathbb{R}$. *Then*

1. $m^*([a, b]) = b - a$; *and therefore*
2. $m^*((a, b]) = m^*([a, b)) = m^*((a, b)) = b - a$.

3 Lecture 3 May 14th 2019

3.1 Lebesgue Outer Measure Continued

Proposition 8 (LOM of Arbitrary Intervals)

Suppose $a < b \in \mathbb{R}$. Then

1. $m^*([a,b]) = b - a$; *and therefore*
2. $m^*((a,b]) = m^*([a,b)) = m^*((a,b)) = b - a$.

Proof

1. Consider $a < b \in \mathbb{R}$. Let $\varepsilon > 0$, and let

$$I_1 = \left(a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2} \right)$$

and $I_n = \emptyset$ for $n \geq 2$. Then $\{I_n\}_{n=1}^{\infty}$ is a cover of $[a,b]$. This means that

$$m^*([a,b]) \leq \sum_{n=1}^{\infty} \ell(I_n) = b - a + \varepsilon.$$

So for all $\varepsilon \to 0$, we have that

$$m^*([a,b]) \leq b - a.$$

[1] Conversely, if $[a,b]$ is covered by open intervals $\{I_n\}_{n=1}^{\infty}$, then by compactness of $[a,b]$ (via the **Heine-Borel Theorem**), we know that we can cover $[a,b]$ by finitely many of these intervals, and let us denote these as $\{I_n\}_{n=1}^{N}$, for some $1 \leq N < \infty$.

[1] For the converse, we know that $m^*([a,b]) = \inf ★$, where ★ is just a placeholder for you-know-what. So $m^*([a,b])$ is one of the sums. So if we can show that for an arbitrary sum, $\geq$ holds, our work is done.

WTS

$$\sum_{n=1}^{N} \ell(I_n) \geq b - a.$$

If LHS $= \infty$, then our work is done. Thus wlog, WMA each $I_n = (a_n, b_n)$ is a finite interval. Note that we have

$$[a, b] \subseteq \bigcup_{n=1}^{N} (a_n, b_n).$$

In particular, $a \in \bigcup_{n=1}^{N} I_n$. Thus, $\exists 1 \leq n_2 \leq N$ such that $a \in I_{n_1}$. Now if $b_{n_1} > b$, we shall stop this process for our work is done, since then $[a, b] \subseteq I_{n_1}$. Otherwise, if $b_{n_1} \leq b$, then $b_{n_1} \in [a, b] \subseteq \bigcup_{n=1}^{N} I_n$, which means that $\exists 1 \leq n_2 \leq N$ such that $b_{n_1} \in I_{n_2}$. Notice that $n_1 \neq n_2$, since $b_{n_1} \notin I_{n_1}$ but $b_{n_1} \in I_{n_2}$.

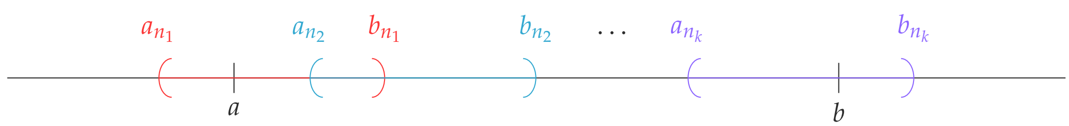

Figure 3.1: Our continual picking of $I_{n_1}, I_{n_2}, \dots, I_{n_k}$

Now once again, if $b_{n_2} > b$, then we shall stop this process since our work is done. Otherwise, we have $a < b_{n_2} \leq b$, and so $\exists 1 \leq n_3 \leq N$, $n_3 \neq n_1, n_2$, such that $b_{n_2} \in I_3$...

We continue with the above process for as long as $b_{n_k} \leq b$. We can thus find, for each k, $I_{n_{k+1}}$, where $n_{k+1} \in \{1, \dots, N\} \setminus \{n_1, n_2, \dots, n_k\}$, such that $b_{n_k} \in I_{n_{k+1}}$.

However, since each of the I_{n_k}'s are different, and since we only have N such intervals, there must exists a $K \leq N$ such that

$$b_{n_{K-1}} \leq b \text{ and } b_{n_K} > b.$$

It now suffices for us to show that

$$\sum_{j=1}^{K} \ell(I_{n_j}) \geq b - a.$$

Observe that

$$\sum_{j=1}^{K} \ell(I_{n_j}) = (b_{n_K} - a_{n_K}) + (b_{n_{K-1}} - a_{n_{K-1}}) + \dots$$

$$
\begin{aligned}
&+ (b_{n_2} - a_{n_2}) + (b_{n_1} - a_{n_1}) \\
&= b_{n_K} + \underbrace{(b_{n_{K-1}} - a_{n_K})}_{\geq 0} + \underbrace{(b_{n_{K-2}} - a_{n_{K-1}})}_{\geq 0} + \dots \\
&+ \underbrace{(b_{n_1} - a_{n_2})}_{\geq 0} - a_{n_1} \\
&\geq b_{n_K} - a_{n_1} \geq b - a.
\end{aligned}
$$

Thus

$$\sum_{n=1}^{\infty} \ell(I_n) \geq \sum_{n=1}^{N} \ell(I_n) \geq \sum_{j=1}^{K} \ell(I_{n_j}) \geq b - a,$$

whence

$$m^*([a,b]) \geq b - a.$$

It follows that, indeed,

$$m^*([a,b]) = b - a.$$

2. First, note that

$$m^*((a,b)) \leq m^*([a,b]) \leq b - a.$$

On the other hand, notice that $\forall 0 < \varepsilon < \frac{b-a}{2}$, we have that

$$[a + \varepsilon, b - \varepsilon] \subseteq (a,b),$$

and so by monotonicity,

$$(b - a) - 2\varepsilon = m^*([a + \varepsilon, b - \varepsilon]) \leq m^*((a,b)).$$

As $\varepsilon \to 0$, we have that

$$b - a \leq m^*((a,b)) \leq b - a.$$

So

$$m^*((a,b)) = b - a$$

as desired.

Finally, we have that

$$b - a = m^*((a,b)) \leq m^*((a,b]) \leq m^*([a,b]) = b - a,$$

and similarly

$$b - a = m^*((a,b)) \leq m^*([a,b)) \leq m^*([a,b]) = b - a.$$

Thus

$$m^*((a,b)) = m^*((a,b]) = m^*([a,b)) = b - a$$

as required. □

Proposition 9 (LOM of Infinite Intervals)

We have that $\forall a, b \in \mathbb{R}$*,*

$$\begin{aligned} m^*((a,\infty)) &= m^*([a,\infty)) \\ &= m^*((-\infty,b)) = m^*((-\infty,b]) \\ &= m^*\mathbb{R} = \infty. \end{aligned}$$

Proof

Observe that

$$(a, a+n) \subseteq (a, \infty)$$

for all $n \geq 1$. Thus

$$n = m^*((a, a+n)) \leq m^*((a,\infty))$$

for all $n \geq 1$. Hence

$$m^*((a,\infty)) = \infty$$

by definition.

All other cases follow similarly. □

Corollary 10 (Uncountability of $\mathbb{R}$)

$\mathbb{R}$ *is uncountable.*

Proof

We have that

$$m^*\mathbb{R} = \infty \neq 0,$$

and so it follows from Corollary 6, we must have that $\mathbb{R}$ is uncountable. $\square$

Definition 15 (Translation Invariant)

*Let μ be an outer measure on $\mathbb{R}$. We say that μ is **translation invariant** if $\forall E \subseteq \mathbb{R}$,*

$$\mu(E) = \mu(E + \kappa)$$

for all $\kappa \in \mathbb{R}$, where

$$E + \kappa := \{x + \kappa : x \in E\}.$$

Proposition 11 (Translation Invariance of the LOM)

The Lebesgue outer measure is translation invariant.

Proof

Let $E \subseteq \mathbb{R}$ and $\kappa \in \mathbb{R}$. Note that E is covered by open intervals $\{I_n\}_{n=1}^{\infty}$ iff $E + \kappa$ is covered by $\{I_n + \kappa\}_{n=1}^{\infty}$.

Claim: $\forall n \geq 1$, $\ell(I_n + \kappa) = \ell(I_n)$ Write

$$I_n = (a_n, b_n).$$

Then

$$I_n + \kappa = (a_n + \kappa, b_n + \kappa).$$

Observe that

$$\ell(I_n + \kappa) = b_n + \kappa - (a_n - \kappa) = b_n - a_n = \ell(I_n),$$

as claimed. $\dashv$

By the claim, it follows that

$$\begin{aligned} m^*(E) &= \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : E \subseteq \bigcup_{n=1}^{\infty} \right\} \\ &= \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n + \kappa) : E + \kappa \subseteq \bigcup_{n=1}^{\infty} (I_n + \kappa) \right\} \\ &= m^*(E + \kappa). \end{aligned}$$

□

Remark 3.1.1

Suppose $E \subseteq \mathbb{R}$ and $E = \bigcup_{n=1}^{\infty} E_n$, where

$$E_i \cap E_j = \emptyset \text{ if } i \neq j.$$

Now by σ-subadditivity of m^*, we have that

$$m^* E \leq \sum_{n=1}^{\infty} m^* E_n.$$

However, equality is not guaranteed. Consider the following case: if $E = [0,1]$, we may have $E_n = [0,1]$ for all $n >= 1$, in which case $E = \bigcup_{n=1}^{\infty} E_n = [0,1]$, but

$$m^* E = m^*[0,1] = 1 < \infty = \sum_{n=1}^{\infty} m^* E_n.$$

It would be desirable to have

$$m^* E = \sum_{n=1}^{\infty} m^* E_n,$$

when the E_i's are pairwise disjoint, i.e. $E = \dot{\bigcup}_{n=1}^{\infty} E_n$. In fact, this would agree with our intuition, that if the outer measure is going to be our 'length'. Consider the example $A = [0,2] \cup [5,7]$. Then we would expect $m^* A = 2 + 2 = 4$.

However, this is actually impossible for an arbitrary number of collections.

Theorem 12 (Non-existence of a sensible Translation Invariant Outer Measure that is also σ-additive)

There does not exist a translation-invariant outer measure μ on $\mathbb{R}$ that satisfies

1. $\mu(\mathbb{R}) > 0$;
2. $\mu[0,1] < \infty$; *and*
3. μ *is* ***σ-additive****; i.e. if* $\{E_n\}_{n=1}^{\infty}$ *is a countable collection of disjoint subsets of* $\mathbb{R}$ *that covers* $E \subseteq \mathbb{R}$*, then*

$$\mu E = \sum_{n=1}^{\infty} \mu E_n.$$

Consequently, the Lebesgue outer measure m^* *is* ***not*** σ*-additive.*

Proof

Suppose to the contrary that such a μ exists.

Step 1 Consider the relation $\sim$ on $\mathbb{R}$ such that $x \sim y$ if $x - y \in \mathbb{Q}$.

Claim: $\sim$ is an equivalence relation

- (**reflexivity**) We know that $0 \in \mathbb{Q}$ and $x - x = 0$. Thus $x \sim x$.
- (**symmetry**) Since $\mathbb{Q}$ is a **field**, it is closed under multiplication, and $-1 \in \mathbb{Q}$. Thus if $x \sim y$, then $x - y \in \mathbb{Q}$, and so $(-1)(x - y) = y - x \in \mathbb{Q}$, which means $y \sim x$.
- (**transitivity**) Again, since $\mathbb{Q}$ is a field, it is closed under (this time) addition. Thus

$$\begin{aligned} x \sim y \land y \sim z &\implies (x - y), (y - z) \in \mathbb{Q} \\ &\implies (x - y) + (y - z) = x - z \in \mathbb{Q}. \end{aligned}$$

Thus $x \sim z$.

This proves the claim. $\dashv$

Let

$$[x] := x + \mathbb{Q} := \{x + q : q \in \mathbb{Q}\}$$

denote the equivalence class of x wrt $\sim$. Note that the set of equivalence classes, which we shall represent as

$$\mathcal{F} := \{[x] : x \in \mathbb{R}\},$$

partitions $\mathbb{R}$, i.e.

- $[x] = [y] \iff x - y \in \mathbb{Q}$; and
- $[x] \cap [y] = \emptyset$ otherwise.

Note that since $\mathbb{Q}$ is **dense** in $\mathbb{R}$, we have that $[x] = x + \mathbb{Q}$ is also dense in $\mathbb{R}$, for all $x \in \mathbb{R}$. Then for each [2] $F \in \mathcal{F}$, $\exists x_F \in F$ such that

$$0 \leq x_F \leq 1.$$

[2] Notice that here, we have invoked the Axiom of Choice.

Now consider the set

$$\mathbb{V} := \{x_F : F \in \mathcal{F}\} \subseteq [0, 1],$$

which is called **Vitali's Set**.

Step 2 Since $\mathcal{F}$ partitions $\mathbb{R}$, we have that

$$\begin{aligned}\mathbb{R} &= \dot{\bigcup_{F \in \mathcal{F}}} F = \dot{\bigcup_{F \in \mathcal{F}}} [x_F] \\ &= \dot{\bigcup_{F \in \mathcal{F}}} x_F + \mathbb{Q} \\ &= \mathbb{V} + \mathbb{Q} := \{x + q : q \in \mathbb{Q}, x \in \mathbb{V}\}.\end{aligned}$$

Step 3 Claim: $p \neq q \in \mathbb{Q} \implies (\mathbb{V} + p) \cap (\mathbb{V} + q) = \emptyset$ Suppose not, and suppose $\exists y \in (\mathbb{V} + p) \cap (\mathbb{V} + q)$. Then $\exists F_1, F_2 \in \mathcal{F}$ such that

$$y = x_{F_1} + p = x_{F_2} + q. \tag{3.1}$$

Then we may rearrange the above equation to get

$$x_{F_1} - x_{F_2} = q - p \in \mathbb{Q}.$$

This implies that

$$[x_{F_1}] = [x_{F_2}] \implies F_1 = F_2$$

since $\mathbb{V}$ consists of one unique representative from each of the equivalence classes. However, this would mean that

$$x_{F_1} = x_{F_2}.$$

Since $p \neq q$, we have that

$$x_{F_1} + p \neq x_{F_2} + q,$$

which contradicts Equation (3.1). Thus

$$(\mathbb{V} + p) \cap (\mathbb{V} + q) = \emptyset,$$

as claimed. $\dashv$

This in turn means that the $\mathbb{V} + q$, for each $q \in \mathbb{Q}$, also partitions $\mathbb{R}$. In other words, if we write $\mathbb{Q} = \{p_n\}_{n=1}^{\infty}$, then

$$\mathbb{R} = \mathbb{V} + \mathbb{Q} = \dot{\bigcup_{n=1}^{\infty}} \mathbb{V} + p_n.$$

Now, note that

$$0 \neq \mu\mathbb{R} \overset{(1)}{=} \sum_{n=1}^{\infty} \mu(\mathbb{V} + p_n) \overset{(2)}{=} \sum_{n=1}^{\infty} \mu(\mathbb{V}),$$

where (1) is by μ being σ-additive and (2) is by μ being translation invariant, both directly from our assumptions. This means that

$$\mu\mathbb{V} > 0.$$

Step 4 Now consider $S = \mathbb{Q} \cap [0, 1]$ such that S is denumerable. Write

$$S = \{s_n\}_{n=1}^{\infty}.$$

Note that for all $n \geq 1$,

$$\mathbb{V} \subseteq [0,1] \implies \mathbb{V} + s_n \subseteq [0,2],$$

and as proven above

$$i \neq j \implies (\mathbb{V} + s_i) \cap (\mathbb{V} + s_j) = \emptyset.$$

Thus it follows that

$$\mu\left(\dot{\bigcup_{n=1}^{\infty}} \mathbb{V} + s_n \right) = \sum_{n=1}^{\infty} \mu(\mathbb{V} + s_n) = \sum_{n=1}^{\infty} \mu(\mathbb{V}) = \infty.$$

Also,

$$\begin{aligned} \mu\left(\dot{\bigcup_{n=1}^{\infty}} \mathbb{V} + s_n \right) &= \sum_{n=1}^{\infty} \mu(\mathbb{V} + s_n) \\ &\leq \mu([0,2]) = \mu([0,1] \cup ([0,1] + 1)) \\ &\leq \mu[0,1] + \mu([0,1] + 1) \\ &= 2\mu([0,1]) = 2 < \infty, \end{aligned}$$

contradicting what we have right above.

Therefore, no such μ exists. □

With the realization of Theorem 12, we find ourselves facing a losing dilemma: we may either

1. be happy with the Lebesgue outer measure m^* for all subsets $E \subseteq \mathbb{R}$, which would agree with our intuitive notion of length, at the price of σ-additivity; or
2. restrict the domain of our function m^* to some family of subsets of $\mathbb{R}$, where m^* would have σ-additivity.

We shall adopt the second approach. We shall call the collection of sets where m^* has σ-additivity as the collection of **Lebesgue measurable sets**.

3.2 Lebesgue Measure

We shall first introduce **Carathéodory's** definition of a Lebesgue measurable set.

Definition 16 (Lebesgue Measureable Set)

A set $E \subseteq \mathbb{R}$ is said to be ***Lebesgue measurable*** *if, $\forall X \subseteq \mathbb{R}$,*

$$m^* X = m^*(X \cap E) + m^*(X \setminus E).$$

We denote the collection of all Lebesgue measurable sets as $\mathfrak{M}(\mathbb{R})$.

Remark 3.2.1

Since we shall almost exclusively focus on the Lebesgue measure, we shall hereafter refer to "Lebesgue measurable sets" as simply "measurable sets".

Note 3.2.1

I shall quote and paraphrase this remark from our course notes [3]:

> *Informally, we see that a set $E \subseteq \mathbb{R}$ is measurable provided that it is a "universal slicer", that it "slices" every other set X into two* ***disjoint*** *sets, into where the Lebesgue outer measure is* ***σ-additive***.

Also, note that we get the following inequality for free, simply from σ-subadditivity of m^*:

$$m^* X \leq m^*(X \cap E) + m^*(X \setminus E).$$

Thus, it suffices for us to check if the reverse inequality holds for all sets $X \subseteq \mathbb{R}$.

[3] Marcoux, L. W. (2018). *PMath 450 Introduction to Lebesgue Measure and Fourier Analysis*. (n.p.)

Before ploughing forward to getting out hands dirty with examples, let us first study a result on a structure of $\mathfrak{M}(\mathbb{R})$ that is rather interesting. [4]

[4] For those who has dirtied themselves in the world of probability and statistics, especially probability theory, get ready to get excited!

Definition 17 (Algebra of Sets)

A collection $\Omega \subseteq \mathcal{P}(\mathbb{R})$ is said to be an ***algebra of sets*** *if*

1. $\mathbb{R} \in \Omega$;
2. (**closed under complementation**) $E \in \Omega \implies E^C \in \Omega$; and
3. (**closed under finite union**) given $N \geq 1$ and $\{E_n\}_{n=1}^N \subseteq \Omega$, then
$$\bigcup_{n=1}^N E_n \in \Omega.$$

We say that Ω is a ***σ-algebra of sets*** *if*

1. Ω is an algebra of sets; and
2. (**closed under countable union**) if $\{E_n\}_{n=1}^\infty \subseteq \Omega$, then
$$\bigcup_{n=1}^\infty E_n \in \Omega.$$

Note 3.2.2

We often call a σ-algebra of sets as simply a σ-algebra.

Theorem 13 ($\mathfrak{M}(\mathbb{R})$ is a σ-algebra)

The collection $\mathfrak{M}(\mathbb{R})$ of Lebesgue measurable sets in $\mathbb{R}$ is a σ-algebra.

Due to time constraints, we shall prove the first 2 requirements in this lecture and prove the last requirement next time (which is also really long).

Proof

$\mathbb{R} \in \mathfrak{M}(\mathbb{R})$ Observe that $\forall X \subseteq \mathbb{R}$,

$$m^* X = m^* X + 0 = m^* X + m^* \emptyset = m^*(X \cap \mathbb{R}) + m^*(X \setminus \mathbb{R}) \qquad \square$$

$E \in \mathfrak{M}(\mathbb{R}) \implies E^C \in \mathfrak{M}(\mathbb{R})$ Observe that $\forall X \subseteq \mathbb{R}$, since $E \in \mathfrak{M}(\mathbb{R})$, we have

$$\begin{aligned} m^* X &= m^*(X \cap E) + m^*(X \setminus E) \\ &= m^*(X \cap (E^C)^C) + m^*(X \cap E^C) && A \setminus B = A \cap B^C \\ &= m^*(X \setminus E^C) + m^*(X \cap E^C) \\ &= m^*(X \cap E^C) + m^*(X \setminus E^C) && \textit{rearrangement} \end{aligned}$$

Thus $E^C \in \mathfrak{M}(\mathbb{R})$.

4 Lecture 4 May 16th 2019

4.1 Lebesgue Measure (Continued)

Recalling the last theorem we were in the middle of proving, it remains for us to prove that $\mathfrak{M}(\mathbb{R})$ is closed under arbitrary unions of its elements.

But before we dive in, let's first have a little pep talk.

Strategy

Since m^* is σ-subadditive, given $\{E_n\}_{n=1}^{\infty}$, we need only prove that $\forall X \subseteq \mathbb{R}$,

$$m^* X \geq m^* \left(X \cap \bigcup_{n=1}^{\infty} E_n \right) + m^* \left(X \setminus \bigcup_{n=1}^{\infty} E_n \right).$$

Recall our discussion near the end of Section 3.1. We want σ-additivity, especially when we are given a set of disjoint intervals. However, our E_n's are arbitrary, and so they are not necessarily disjoint.

It helps if one has seen how we can slice $\mathbb{R}$ up into disjoint unions, and consequently we can do so for any of its subsets. We shall not take that for granted and immediately use it, but we shall work through this proof in the spirit of that. We shall see how we can slice $\mathbb{R}$ up in A1.

Once we can, in some way, express $\bigcup_{n=1}^{\infty} E_n$ as a disjoint union of intervals, we will then show that, indeed, we have σ-additivity instead of σ-subadditivity on this disjoint union.

Proof

$\mathfrak{M}(\mathbb{R})$ is closed under arbitrary unions Suppose $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}(\mathbb{R})$. To show that $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{M}(\mathbb{R})$, WTS

$$m^* X = m^* \left(X \cap \bigcup_{n=1}^{\infty} E_n \right) + m^* \left(X \setminus \bigcup_{n=1}^{\infty} E_n \right).$$

Since m^* is σ-subadditive, it suffices for us to show that

$$m^* X \geq m^* \left(X \cap \bigcup_{n=1}^{\infty} E_n \right) + m^* \left(X \setminus \bigcup_{n=1}^{\infty} E_n \right). \tag{4.1}$$

Step 1 Consider

$$H_n = \bigcup_{i=1}^{n} E_i, \quad \forall n \geq 1.$$

Claim: $H_n \in \mathfrak{M}(\mathbb{R}), \forall n \geq 1$ We shall prove this by induction on n.

When $n = 1$, we have $H_1 = E_1 \in \mathfrak{M}(\mathbb{R})$ by assumption, and so we are done. Suppose that $H_k \in \mathfrak{M}(\mathbb{R})$ for some $k \in \mathbb{N}$. Consider $n = k + 1$.

Since we will need the piece $X \cap H_{k+1}$, first, notice that

$$X \cap H_{k+1} = X \cap (H_k \cup E_{k+1}) = (X \cap H_k) \cup ((X \setminus H_k) \cap E_{k+1}),$$

and in particular that

$$X \cap H_{k+1} = X \cap (H_k \cup E_{k+1}) \subseteq (X \cap H_k) \cup ((X \setminus H_k) \cap E_{k+1}). \tag{4.2}$$

This may be (will be) useful later on, and we can guess that we will be using σ-subadditivity on this.

By the IH, since $H_k \in \mathfrak{M}(\mathbb{R})$, we have

$$m^* X = m^*(X \cap H_k) + m^*(X \setminus H_k).$$

Notice the similarity between the above equation and Equation (4.2), where we are just off by that $\cap E_{k+1}$.

Since $E_{k+1} \in \mathfrak{M}(\mathbb{R})$, we have

$$m^*(X \setminus H_k) = m^*(X \setminus H_k \cap E_{k+1}) + m^*(X \setminus H_k \setminus E_{k+1}).$$

To clean the above equation up a little bit, notice that by **De Morgan's Law**,

$$X \setminus H_k \setminus E_{k+1} = X \cap \bigcup_{i=1}^{k} E_i^C \cap E_{k+1}^C = X \setminus H_{k+1}.$$

So

$$m^*(X \setminus H_k) = m^*(X \setminus H_k \cap E_{k+1}) + m^*(X \setminus H_{k+1}).$$

Thus

$$m^* X = m^*(X \cap H_k) + m^*(X \setminus H_k \cap E_{k+1}) + m^*(X \setminus H_{k+1}).$$

Using Equation (4.2) and σ-subadditivity, we have that

$$m^* X \geq m^*(X \cap H_{k+1}) + m^*(X \setminus H_{k+1}),$$

which is what we need. Thus $\forall k \geq 1$, $H_k \in \mathfrak{M}(\mathbb{R})$. $\dashv$

Step 2 Consider $F_1 = H_1 = E_1 \in \mathfrak{M}(\mathbb{R})$, and for $k \geq 2$,

$$F_k = H_k \setminus H_{k-1} = H_k \cap H_{k-1}^C.$$

[1] Claim: $\forall k \geq 2$, $F_k \in \mathfrak{M}(\mathbb{R})$ First, notice that

$$F_k^C = \left(H_k \cap H_{k+1}^C \right)^C = H_k^C \cup H_{k+1}.$$

By **step 1** [2], we have that $F_k^C \in \mathfrak{M}(\mathbb{R})$, and thus by closure under complementation, $F_k \in \mathfrak{M}(\mathbb{R})$.

[1] Note that we cannot assume that $\mathfrak{M}(\mathbb{R})$ is closed under finite intersections because that is part of what we want to prove.

[2] **I need to get this clarified.**

Also, note that the F_i's are pairwise disjoint. Suppose not, i.e. that $\exists x \in F_a \cap F_b$ for some $a, b \geq 1$ and $a \neq b$. Wlog, wma $a < b$. Note that $H_a \subseteq H_b$, since

$$H_a = \bigcup_{i=1}^{a} E_i \subsetneq \bigcup_{i=1}^{b} E_i = H_b.$$

Since $F_b = H_b \setminus H_{b-1}$,

$$x \in F_b \implies x \notin \bigcup_{i=1}^{b-1} E_i \supseteq \bigcup_{i=1}^{a} E_i,$$

and so $x \notin E_i$ for $1 \leq i \leq a \leq b-1$. But we assumed that

$$x \in F_a = H_a \setminus H_{a-1},$$

i.e. it must be that $x \in E_a$, a contradiction.

Step 3 We now have

$$E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} H_i = \dot{\bigcup_{i=1}^{\infty}} F_i.$$

Equation (4.1) becomes [3]

$$m^* X \geq m^* \left(X \cap \left(\dot{\bigcup_{i=1}^{\infty}} F_i \right) \right) + m^* \left(X \setminus E \right).$$

[3] I refrained from changing the second term to the disjoint union. Retrospectively (i.e. once you're done with the proof), it makes sense to not consider this move, since there is no point looking at X take away a bunch of disjoint intervals.

Since the F_i's are disjoint, we expect

$$m^* \left(X \cap \dot{\bigcup_{i=1}^{\infty}} F_i \right) = \sum_{i=1}^{\infty} m^*(X \cap F_i).$$

i.e. for every n,

$$m^* \left(X \cap \dot{\bigcup_{i=1}^{n}} F_i \right) = \sum_{i=1}^{n} m^*(X \cap F_i).$$

Let's prove this inductively. It is clear that case $n = 1$ is trivially true. Suppose that this is true up to some $k \in \mathbb{N}$. Consider case $n = k+1$. Since $F_{k+1} \in \mathfrak{M}(\mathbb{R})$, we have that [4]

[4] This is quite a smart trick!

$$\begin{aligned}
&m^* \left(X \cap \bigcup_{i=1}^{k+1} F_i \right) \\
&= m^* \left(X \cap \bigcup_{i=1}^{k+1} F_i \cap F_{k+1} \right) + m^* \left(\left(X \setminus \bigcup_{i=1}^{k=1} F_i \right) \setminus F_{k+1} \right) \\
&= m^*(X \cap F_{k+1}) + m^* \left(X \cap \bigcup_{i=1}^{k} F_i \right) \qquad \mathit{IH} \\
&= m^*(X \cap F_{k+1}) + \textstyle\sum_{i=1}^{k} m^*(X \cap F_i) \\
&= \sum_{i=1}^{k+1} m^*(X \cap F_i).
\end{aligned}$$

Our claim is complete by induction.

Step 4 With **Step 3**, Equation (4.1) has become

$$m^* X \geq \sum_{i=1}^{\infty} m^*(X \cap F_i) + m^*(X \setminus E).$$

[5] Since $H_k \in \mathfrak{M}(\mathbb{R})$ for each $k \geq 1$, we have

$$m^* X = m^*(X \cap H_k) + m^*(X \setminus H_k). \tag{$*$}$$

[5] This is a reward for the clear-minded, cause I certainly did not find it an obvious step to take.

Since

$$H_k = \bigcup_{i=1}^{k} E_i = \bigcup_{i=1}^{\infty} E_i = E,$$

we have that

$$X \setminus H_k \supseteq X \setminus E,$$

for each $k \geq 1$. Thus by monotonicity, Equation ($*$) becomes

$$\begin{aligned} m^* X &\geq m^*(X \cap H_k) + m^*(X \setminus E) \\ &= m^* \left(X \cap \left(\bigcup_{i=1}^{\infty} F_i \right) \right) + m^*(X \setminus E) \\ &= \sum_{i=1}^{k} m^*(X \cap F_i) + m^*(X \setminus E), \end{aligned}$$

for each $k \geq 1$.

By letting $k \to \infty$, we have that

$$m^* X \geq \sum_{i=1}^{\infty} m^*(X \cap F_i) + m^*(X \setminus E).$$

Note that

$$X \cap E = X \cap \bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} (X \cap F_i).$$

By σ-subadditivity, we have that

$$m^*(X \cap E) \leq \sum_{i=1}^{\infty} m^*(X \cap F_i).$$

Therefore

$$m^* X \geq m^*(X \cap E) + m^*(X \setminus E),$$

which is what we want! $\square$

Note 4.1.1 (Post-mortem for proof of Theorem 13)

In steps 1 - 3, we try to slice $\bigcup_{n=1}^{\infty} E_n$ into disjoint ***measurable*** *intervals F_i's. Along the process of constructing them, it is the showing of them being measurable that takes up most of the proof, since we require induction.*

Proposition 14 (Some Lebesgue Measurable Sets)

1. *If $E \subseteq \mathbb{R}$ and $m^* E = 0$, then E is Lebesgue measurable.*
2. $\forall b \in \mathbb{R}$, $(-\infty, b) \in \mathfrak{M}(\mathbb{R})$.
3. *Every open and every closed set is Lebesgue measurable.*

Proof

1. Let $X \subseteq \mathbb{R}$. Note that $X \setminus E \subseteq X$, and so σ-subadditivity gives

$$m^* X \geq m^*(X \setminus E). \tag{4.3}$$

On the other hand, $X \cap E \subseteq E$, and so

$$m^*(X \cap E) \leq m^* E = 0 \implies m^*(X \cap E) = 0.$$

Thus, from Equation (4.3),

$$m^* X \geq ml * (X \setminus E) = m^*(X \cap E) + m^*(X \setminus E).$$

Hence $E \in \mathfrak{M}(\mathbb{R})$ as required.

2. Let $b \in \mathbb{R}$ and $X \subseteq \mathbb{R}$ be arbitrary. WTS

$$m^* X \geq m^*(X \cap (-\infty, b)) + m^*(X \setminus (-\infty, b)).$$

[6] Let $E = (-\infty, b)$. Note that if $m^* X = \infty$, then there is nothing to show. Thus WMA $m^* X < \infty$. In this case, let $\varepsilon > 0$, and

[6] We will look at $X \cap (\infty, b)$ and $X \setminus (-\infty, b)$ more closely, and then realize that since we can cover X, we can "extend" this cover for these disjoint pieces by taking intersections and set removals on each of the covering sets.

$\{I_n\}_{n=1}^{\infty}$ a cover of X by open intervals, where we write

$$I_n = (a_n, b_n)$$

for each $n \geq 1$, so that [7]

$$\sum_{n=1}^{\infty} \ell(I_n) < m^* X + \varepsilon.$$

[7] Note that this is legitimate because $m^* X$ is the infimum of such sums on the LHS, and we can definitely find such a cover as a result. Also, there is no harm in assuming that each of the I_n's are non-empty, since we may simply remove all the empty I_n's from the cover.

For each $n \geq 1$, consider the sets

$$J_n = I_n \cap E + I_n \cap (-\infty, b)$$

and

$$K_n = I_n \setminus E = I_n \setminus (\infty, b) = I_n \cap [b, \infty).$$

The following table captures all possible J_n's and K_n's:

Case	1	2	3
b	$> b_n$	$\in I_n$	$< a_n$
J_n	I_n	(a_n, b)	$\emptyset$
K_n	$\emptyset$	$[b, b_n)$	I_n

Table 4.1: Possible outcomes of J_n and K_n, for each $n \geq 1$

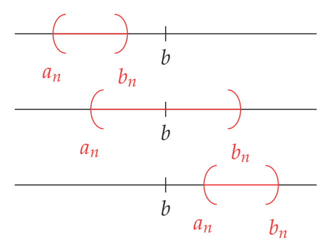

Figure 4.1: Three possible scenarios of where b stands for different I_n's

Notice that $\{J_n\}_{n=1}^{\infty}$ is an open cover for $X \cap E$. $\{K_n\}_{n=1}^{\infty}$ is also a cover of $X \setminus E$ but it is not an open cover (the only covers of which we consider in this course). Thus, we consider a small extension L_n of K_n such that

- if $K_n = \emptyset$, then $L_n = \emptyset$;
- if $K_n = I_n$, then $L_n = I_n$; and
- if $K_n = [b, b_n)$, then $L_n = \left(b - \frac{\varepsilon}{2^n}, b_n\right)$.

Then $\{L_n\}_{n=1}^{\infty}$ is a cover of $X \setminus E$. By σ-subadditivity of m^*, we have that

$$m^*(X \cap E) \leq \sum_{n=1}^{\infty} \ell(J_n)$$

and

$$m^*(X \setminus E) \leq \sum_{n=1}^{\infty} \ell(L_n).$$

Thus

$$m^*(X \cap E) + m^*(X \setminus E) \leq \sum_{n=1}^{\infty} \left(\ell(J_n) + \ell(L_n)\right).$$

Now, notice that in cases 1 and 3,

$$\ell(J_n) + \ell(L_n) = \ell(I_n).$$

In case 2, we have that

$$(\ell(J_n) + \ell(L_n)) - \ell(I_n) < \frac{\varepsilon}{2^n}$$

and so

$$\ell(J_n) + \ell(L_n) < \ell(I_n) + \frac{\varepsilon}{2^n}.$$

Therefore

$$\begin{aligned}
&m^*(X \cap E) + m^*(X \setminus E) \\
&\leq \sum_{n=1}^{\infty} (\ell(J_n) + \ell(L_n)) \\
&\leq \sum_{n=1}^{\infty} \left(\ell(I_n) + \frac{\varepsilon}{2^n} \right) \\
&= \sum_{n=1}^{\infty} \ell(I_n) + \varepsilon \\
&< (m^* X + \varepsilon) + \varepsilon \\
&= m^* X + 2\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have that

$$m^* X \geq m^*(X \cap E) + m^*(X \setminus E),$$

and since X is arbitrary, we have that $E = (-\infty, b) \in \mathfrak{M}(\mathbb{R})$.

3. Wlog, suppose $a < b \in \mathbb{R}$. By part 2, we have that

$$(-\infty, b) \in \mathfrak{M}(\mathbb{R}),$$

and similarly, for $n \geq 1$,

$$\left(\infty, a + \frac{1}{n} \right) \in \mathfrak{M}(\mathbb{R}).$$

Since $\mathfrak{M}(\mathbb{R})$ is a σ-algebra, we have that

$$\left[a + \frac{1}{n}, \infty \right) = \left(-\infty, a + \frac{1}{n} \right)^C \in \mathfrak{M}(\mathbb{R}),$$

for each $n \geq 1$. Consequently,

$$(a, \infty) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, \infty \right) \in \mathfrak{M}(\mathbb{R}).$$

Therefore, we have that

$$(a, b) = (-\infty, b) \cap (a, \infty) \in \mathfrak{M}(\mathbb{R}).$$

[8] Since every open set $G \subseteq \mathbb{R}$ is a countable disjoint union of open intervals in $\mathbb{R}$, it follows that $G \in \mathfrak{M}(\mathbb{R})$ since $\mathfrak{M}(\mathbb{R})$ is a σ-algebra. If $F \subseteq \mathbb{R}$ is closed, notice that

$$F^C = G \in \mathfrak{M}(\mathbb{R})$$

since G is open, and so by closure under complementation of σ-algebras, $F \in \mathfrak{M}(\mathbb{R})$. □

[8] We shall prove this in A1.

Definition 18 (Lebesgue Measure)

Let m^* denote the Lebesgue outer measure on $\mathbb{R}$. We define the ***Lebesgue measure*** *m to be*

$$m = m^* \restriction_{\mathfrak{M}(\mathbb{R})},$$

i.e. $\forall E \in \mathfrak{M}(\mathbb{R})$, we have that

$$mE = m^*E = \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) \mid E \subseteq \bigcup_{n=1}^{\infty} I_n \right\}.$$

In A2, we shall prove that

Theorem 15 (σ-additivity of the Lebesgue Measure on Lebesgue Measurable Sets)

The Lebesgue measure is σ-additive on $\mathfrak{M}(\mathbb{R})$, i.e. if $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}(\mathbb{R})$ with $E_i \cap E_j = \emptyset$ for all $i \neq j$, then

$$m \bigcup_{n=1}^{\infty} E_n = \sum_{n=1}^{\infty} mE_n.$$

Corollary 16 (Existence of Non-Measurable Sets)

There exists non-measurable sets.

Proof

Suppose not, i.e. $\mathfrak{M}(\mathbb{R}) = \mathcal{P}(\mathbb{R})$. Then $m = m^*$ is a translation invariant outer measure on $\mathbb{R}$, with $m^*\mathbb{R} = \infty > 0$, $m^*[0,1] = 1 < \infty$, and m^* is σ-additive, which contradicts Theorem 12. Thus $\mathfrak{M}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$. $\square$

The following proposition is left as an exercise.

Proposition 17 (Non-measurability of the Vitali Set)

The Vitali set $\mathbb{V}$, defined in Theorem 12, is not measurable.

Exercise 4.1.1
Prove Proposition 17.

Definition 19 (σ-algebra of Borel Sets)

The σ-algebra of sets generated by the collection

$$\mathfrak{G} := \{ G \subseteq \mathbb{R} : G \text{ is open } \}$$

*is called the **σ-algebra of Borel sets** of $\mathbb{R}$, and is denoted by*

$$\mathfrak{Bor}(\mathbb{R}).$$

Note 4.1.2

Since $\mathfrak{Bor}(\mathbb{R})$ is generated by open sets in $\mathbb{R}$ and all open subsets of $\mathbb{R}$ are Lebesgue measurable (cf. Proposition 14), we have that

$$\mathfrak{Bor}(\mathbb{R}) \subseteq \mathfrak{M}(\mathbb{R}).$$

Remark 4.1.1

Since $\mathfrak{Bor}(\mathbb{R})$ is a σ-algebra, and it is, in particular, generated by open subsets of $\mathbb{R}$, it also contains all of the closed subsets of $\mathbb{R}$. Thus, we could have instead defined $\mathfrak{Bor}(\mathbb{R})$ to be the σ-algebra of subsets of $\mathbb{R}$ generated by the collection

$$\mathfrak{F} := \{F \subseteq \mathbb{R} : F \text{ is closed }\},$$

and in turn conclude that $\mathfrak{Bor}(\mathbb{R})$ contains $\mathfrak{G}$.

Remark 4.1.2

Let $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$, with $\emptyset, \mathbb{R} \in \mathcal{A}$. Let

$$\mathcal{A}_\sigma := \left\{ \bigcup_{n=1}^{\infty} A_n : A_n \in \mathcal{A}, n \geq 1 \right\}$$
$$\mathcal{A}_\delta := \left\{ \bigcap_{n=1}^{\infty} A_n : A_n \in \mathcal{A}, n \geq 1 \right\}.$$

*We call the elements of $\mathcal{A}_\sigma$ as **$\mathcal{A}$-sigma sets**, and elements of $\mathcal{A}_\delta$ as **$\mathcal{A}$-delta sets**.*

Recalling our definitions

$$\mathfrak{G} = \{G \subseteq \mathbb{R} \mid G \text{ is open }\}$$
$$\mathfrak{F} = \{F \subseteq \mathbb{R} \mid F \text{ is closed }\}$$

from above, notice that

$$\mathfrak{G}_\delta = \left\{ \bigcap_{n=1}^{\infty} G_n \mid G_n \in \mathfrak{G}, n \geq 1 \right\},$$

*which is a **countable intersection of open subsets** of $\mathbb{R}$, and*

$$\mathfrak{F}_\sigma = \left\{ \bigcup_{n=1}^{\infty} F_n \mid F_n \in \mathfrak{F}, n \geq 1 \right\},$$

*which is a **countable union of closed subsets** of $\mathbb{R}$, are both subsets of $\mathfrak{Bor}(\mathbb{R})$.*

As mentioned before, the definition of which we provided for a Lebesgue measurable set is from **Carathéodory**, which is not the most intuitive definition. We shall now show that it is equivalent to the original definition of which Lebesgue himself has provided.

Theorem 18 (Carathéodory's and Lebesgue's Definition of Measurability)

Let $E \subseteq \mathbb{R}$. TFAE:

1. *E is Lebesgue measurable (Carathéodory).*
2. *$\forall \varepsilon > 0$, there exists an open $G \supseteq E$ such that*
$$m^*(G \setminus E) < \varepsilon.$$
3. *There exists a $\mathfrak{G}_\delta$-set H such that $E \subseteq H$ and*
$$m^*(H \setminus E) = 0.$$

Proof

$(1) \implies (2)$ If we can find such a G that is open, then since E is Lebesgue measurable, we have

$$mG = m(G \cap E) + m(G \setminus E) = mE + m(G \setminus E),$$

and so

$$m(G \setminus E) = mG - mE. \tag{4.4}$$

So if we can construct such a G, that is particularly small enough (within ε-bigger) to contain E, our statement is good as done.

Case 1: $mE < \infty$ In this case, we may consider a cover $\{I_n\}_{n=1}^\infty$ of E such that

$$\sum_{n=1}^\infty \ell(I_n) < mE + \varepsilon.$$

Then we may simply let $G = \bigcup_{n=1}^\infty I_n$. Note that since $\mathfrak{M}(\mathbb{R})$ is a

σ-algebra, $G \in \mathfrak{M}(\mathbb{R})$. Thus by monotonicity,

$$mG = m\left(\bigcup_{n=1}^{\infty} I_n\right) \leq \sum_{n=1}^{\infty} mI_n = \sum_{n=1}^{\infty} \ell(I_n) < mE + \varepsilon.$$

With this, Equation (4.4) becomes

$$m(G \setminus E) < mE + \varepsilon - mE = \varepsilon.$$

Case 2: $\forall r \in \mathbb{R},\ mE > r$ Consider

$$E_k = [-k, k] \cap E$$

[9]for each $k \geq 1$. By Proposition 14, closed sets are Lebesgue measurable, and so for each $k \geq 1$, $E_k \in \mathfrak{M}(\mathbb{R})$. Note that

$$E = \bigcup_{k \geq 1} E_k.$$

[10] Note that $E_k \subseteq [-k, k]$, and so

$$mE_k \leq m[-k, k] = 2k < \infty.$$

Using a similar approach as in **Case 1**, we can construct an open set G_k such that $G_k \supseteq E_k$, and

$$m(G_k \setminus E_k) < \frac{\varepsilon}{2^k},$$

for each $k \geq 1$. Now let

$$G := \bigcup_{k \geq 1} G_k \supseteq \bigcup_{k \geq 1} E_k = E.$$

Note that if $x \in G \setminus E$, then $x \notin E_k$ for all $k \geq 1$, and $\exists N \geq 1$ such that $x \in G_N$. In particular, we have that

$$x \in G_N \setminus E_N,$$

and so

$$G \setminus E \subseteq \bigcup_{k \geq 1} G_k \setminus E_k$$

[11]. Therefore

[9] **I should get clarification for my understanding of this approach.** We picked closed intervals instead of open ones so that we deal with the possible quirkiness of E.

[10] It would be a quick job if we take the union of the E_k's but note that the E_k's are not necessarily open!

[11] It is, however, true that equality holds, and it is not difficult to prove so.

$$m(G \setminus E) \leq \sum_{k\geq 1} m(G_k \setminus E_k) \leq \sum_{k\geq 1} \frac{\varepsilon}{2^k} = \varepsilon.$$

(2) $\implies$ (3) By (2), for each $n \geq 1$, let $G_n \supseteq E$ such that

$$m(G_n \setminus E) < \frac{1}{n}.$$

Let $H := \bigcap_{n\geq 1} G_n$, which then $H \in \mathfrak{G}_\delta$. Also, since $E \subseteq G_n$ for all $n \geq 1$, we have $E \subseteq H$. Also, $H \subseteq G_n$ for each n. Thus

$$H \setminus E \subseteq G_n \setminus E,$$

for each $n \geq 1$. By monotonicity,

$$m(H \setminus E) \leq m(G_n \setminus E) < \frac{1}{n}$$

for each $n \geq 1$. Therefore

$$m(H \setminus E) = 0.$$

(3) $\implies$ (1) Notice that $\mathfrak{G}_\delta \subseteq \mathfrak{Bor}(\mathbb{R}) \subseteq \mathfrak{M}(\mathbb{R})$. Suppose $G \in \mathfrak{G}_\delta$, and $E \subseteq H$ such that

$$m(H \setminus E) = 0.$$

By Proposition 14, $H \setminus E \in \mathfrak{M}(\mathbb{R})$. Since $\mathfrak{M}(\mathbb{R})$ is a σ-algebra, notice that

$$E = H \setminus (H \setminus E) = H \cap (H \cap E^C)^C = H \cap H^C \cup E \in \mathfrak{M}(\mathbb{R}). \qquad \square$$

5 Lecture 5 May 21st 2019

5.1 Lebesgue Measure (Continued 2)

Recall from Corollary 6 that any countable subset $E \subseteq \mathbb{R}$ has zero Lebesgue outer measure. From Proposition 14, we have that $E \in \mathfrak{M}(\mathbb{R})$ and so $mE = m^*E = 0$. This shows that every countable set is Lebesgue measurable with Lebesgue measure zero.

> *But is the converse true? I.e., is every Lebesgue measurable set with Lebesgue measure zero countable?*

We shall show that this is not true by giving a counterexample. We shall now construct an **uncountable set** C that has measure zero.

Example 5.1.1 (The Cantor Set)

Let $C_0 = [0, 1]$. Note that C_0 is compact and

$$m^*C_0 = 1 < \infty.$$

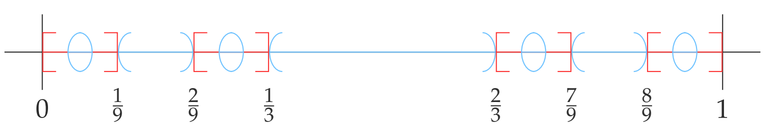

Figure 5.1: Cantor set showing up to $n = 2$, with the excluded interval in $n = 3$ shown.

Let

$$C_1 = C_0 \setminus \left(\frac{1}{3}, \frac{2}{3} \right).$$

Then C_1 is closed [1] and $C_0 \supseteq C_1$.

Let

$$C_2 = C_1 \setminus \left(\left(\frac{1}{9}, \frac{2}{9} \right) \cup \left(\frac{7}{9}, \frac{8}{9} \right) \right).$$

[1] C_1 is an intersection of 2 closed sets.

Figure 5.2: An illustration of the Cantor Set from `https://mathforum.org/mathimages/index.php/Cantor_Set`.

Then C_2 is closed and $C_1 \supseteq C_2$.

We continue this process indefinitely, and construct C_n for each $n \geq 1$, where

$$C_n = \frac{1}{3}C_{n-1} \cup \left(\frac{2}{3} + \frac{1}{3}C_{n-1} \right).$$

Then C_n will consist of 2^n disjoint closed intervals. Thus each C_n is compact and measurable. Moreover,

$$m(C_n) = \left(\frac{2}{3} \right)^n,$$

for each $n \geq 1$.

Also, we have that

$$C_0 \supseteq C_1 \supseteq C_2 \supseteq \ldots$$

is a **descending chain of measurable sets**. Note that the sequence $\{C_n\}_{n=0}^{\infty}$ has the **finite intersection property**, and since $\mathbb{R}$ is compact, the set

$$C := \bigcap_{n=1}^{\infty} C_n,$$

which we shall call it the **Cantor Set**, is non-empty [2].

[2] See FIP and Compactness from PMATH 351

Now from A2, we have that

$$mC = \lim_{n\to\infty} mC_n = \lim_{n\to\infty} \left(\frac{2}{3} \right)^n = 0.$$

We shall now show that C is uncountable. To do this, we shall use the **ternary representation** for each $x \in [0, 1]$. In particular, for each $x \in [0, 1]$, we write

$$x = 0.x_1 x_2 x_3 \ldots,$$

where each $x_i \in \{0, 1, 2\}$ for all $i \geq 1$. Note that in base 10, we can express

$$x = \sum_{k=1}^{\infty} \frac{x_k}{10^k} = 0.x_1 + 0.0x_2 + 0.00x_3 + \dots$$

Thus, we can similarly express

$$x = \sum_{k=1}^{\infty} \frac{x_k}{3^k},$$

in ternary representation. However, just as

$$0.99999\dots \text{ and } 1.00000\dots$$

are indistinguishable, in ternary representation,

$$0.22222\dots \text{ and } 1.00000\dots$$

are indistinguishable. Fortunately, we can find out who exactly are the culprits that cannot be uniquely represented, which shall be left as an exercise.

I shall paraphrase the professor here because I like how the analogy brings good intuition, for me at least.

> *Suppose there's this person that had only 3 fingers and is not aware of the existence of the base-10 system, and in turn invented the ternary system. Then, instead of having 10 regular intervals on $[0,1]$, it had 3 regular intervals.*

Exercise 5.1.1

Show that the ternary expansion of $x \in [0,1)$ is unique except when $\exists N \geq 1$ such that

$$x = \frac{r}{3^N},$$

for some $0 < r < 3^N$, where $3 \nmid r$.

In the cases where we have the above x, we have that [3]

$$x = 0.x_1x_2x_3\dots x_N,$$

where $x_N \in \{1,2\}$.

- If $x_N = 2$, we shall keep this expression; otherwise
- if $x_N = 1$, then we write

$$\begin{aligned} x &= 0.x_1x_2x_3\dots x_{N-2}x_{N-1}1000\dots \\ &= 0.x_1x_2x_3\dots x_{N_2}x_{N-1}0222\dots, \end{aligned}$$

and we shall use the second expression.

[3] Note that the representation terminates somewhere, since it is a fraction, i.e. a rational number.

Also, we shall also use the convention that

$$1 = 0.22222\ldots.$$

With this, we have obtained a **unique** ternary expansion for each $x \in [0,1]$.

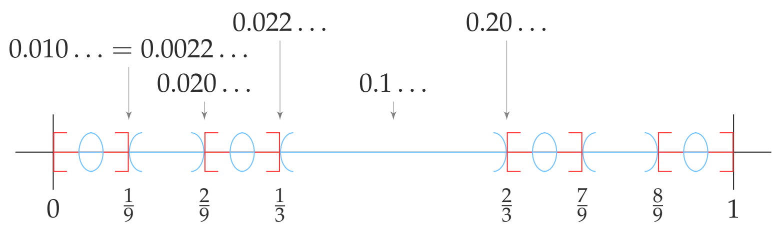

Figure 5.3: Some values on $[0,1]$ in ternary representation

Now, observe that

$$\begin{aligned} C_1 &= [0,1] \setminus \left(\frac{1}{3}, \frac{2}{3} \right) \\ &= \{ x \in [0,1] : x = 0.x_1x_2x_3\ldots, x_1 \neq 1 \}, \end{aligned}$$

i.e. whichever $x \in [0,1]$ with $x_1 = 1$ sits in $\left(\frac{1}{3}, \frac{2}{3}\right)$. Similarly,

$$C_2 = \{ x \in [0,1] : x = 0.x_1x_2x_3\ldots, x_1 \neq 1, x_2 \neq 1 \}.$$

In general, we have that

$$C_N = \{ x \in [0,1] : x = 0.x_1x_2x_3\ldots, x_i \neq 1, \, 1 \leq i \leq N \}.$$

Therefore,

$$\begin{aligned} C &= \bigcap_{n=1}^{\infty} C_n \\ &= \{ x \in [0,1] : x = 0.x_1x_2x_3\ldots, \, x_n \neq 1, n \geq 1 \} \\ &= \{ x \in [0,1] : x = 0.x_1x_2x_3\ldots, \, x_n \in \{0,2\}, n \geq 1 \} \end{aligned}$$

Now, consider the bijection

$$\varphi : C \to [0,1]$$

given by

$$x = 0.x_1x_2x_3\ldots \mapsto y = 0.y_1y_2y_3\ldots,$$

where $x_n \in \{0, 2\}$, for $n \geq 1$, and x is the ternary expansion, while $y_n = \frac{x_n}{2}$ for each $n \geq 1$, and so y is a binary expansion. Then φ is a bijection between C and $[0, 1]$, and therefore

$$|C| = |[0,1]| = |\mathbb{R}| = c = 2^{\aleph_0}.$$

Note 5.1.1

The lesson here is that the Lebesgue measure is not a measure on the cardinality of the set. Rather, it measures the ***distribution of points*** *in the set.*

5.2 Lebesgue Measurable Functions

Note 5.2.1

We used

$$\mathfrak{M}(\mathbb{R}) = \{E \subseteq \mathbb{R} \mid E \text{ is measurable }\}$$

to denote the set of measurable subsets of $\mathbb{R}$.

In general, for $H \subseteq \mathbb{R}$*, set shall denote by* $\mathfrak{M}(H)$ *the collection of all Lebesgue measurable subsets of* H*, i.e.*

$$\mathfrak{M}(H) = \{E \subseteq H \mid E \in \mathfrak{M}(\mathbb{R})\}.$$

In particular, for $E \in \mathfrak{M}(\mathbb{R})$*, we also have*

$$\mathfrak{M}(E) = \{F \subseteq E \mid F \in \mathfrak{M}(\mathbb{R})\}.$$

Exercise 5.2.1

Prove that the above $\mathfrak{M}(E)$ *is a* σ*-algebra of sets.*

Definition 20 (Lebesgue Measurable Function)

Let $E \in \mathfrak{M}(E)$ and (X, d) a metric space. We say that a function

$$f : E \to X$$

is ***Lebesgue measurable*** *(or simply* ***measurable****) if*

$$f^{-1}(G) ::= \{x \in E : f(x) \in G\} \in \mathfrak{M}(E)$$

for every open set $G \subseteq X$.

We write

$$\mathcal{L}(E, X) = \{f : E \to X \mid f \text{ measurable } \}$$

for the set of measurable functions from E to X.

Exercise 5.2.2

Show that we can equivalently define that a function f is Lebesgue measurable if

$$f^{-1}(F) \in \mathfrak{M}(E)$$

for all closed subsets $F \subseteq X$.

Note 5.2.2

Note that we required that the domain of the function is a measurable set in Definition 20. Part of the reason is because we want constant functions to be measurable, and this happens iff the domain of the function is measurable [4].

[4] **Why?**

Proposition 19 (Continuous Functions on a Measurable Set is Measurable)

Let $E \in \mathfrak{M}(\mathbb{R})$ and (X, d) a metric space. If $f : E \to X$ is continuous, then $f \in \mathcal{L}(E, X)$.

Proof

Since f is continuous in a metric space, it implies that for all open $G \subseteq X$, $f^{-1}(G)$ is open in E [5]. This means that $f^{-1}(G) = U_G \cap E$ for some open $U_G \subseteq \mathbb{R}$. Since U_G is open, by Proposition 14, $U_G \in \mathfrak{M}(\mathbb{R})$. Since $E \in \mathfrak{M}(\mathbb{R})$, we have that

$$f^{-1}(G) = U_G \cap E \in \mathfrak{M}(E),$$

and so

$$f \in \mathcal{L}(E, X).$$

□

[5] We say that $f^{-1}(G)$ is **relatively open** in E.

Example 5.2.1

Let $E \in \mathfrak{M}(\mathbb{R})$ and $H \subseteq E$. Consider the characteristic function of H, which is

$$\chi_H : E \to \mathbb{R} \text{ given by } x \mapsto \begin{cases} 1 & x \in H \\ 0 & x \notin H \end{cases}.$$

Let $G \subseteq \mathbb{R}$ be open. Then

$$\chi_H^{-1}(G) = \begin{cases} \emptyset & G \cap \{0,1\} = \emptyset \\ E & G \supseteq \{0,1\} \\ E \setminus H & G \cap \{0,1\} = \{0\} \\ H & G \cap \{0,1\} = \{1\} \end{cases},$$

in which case we observe that all the possible outcomes are measurable subsets of $\mathbb{R}$. Thus χ_H is measurable iff $H \in \mathfrak{M}(\mathbb{R})$.

Proposition 20 (Composition of a Continuous Function and a Measurable Function is Measurable)

Let $E \in \mathfrak{M}(\mathbb{R})$ and (X, d_X), (Y, d_Y) be metric spaces. Suppose that

$$f : E \to X \text{ is measurable and } g : X \to Y \text{ is continuous.}$$

Then

$$g \circ f : E \to Y \text{ is measurable.}$$

The idea is simple: $(gf)^{-1}(G) = f^{-1}g^{-1}(G)$ and continuity of G means that $g^{-1}(G)$ is open in X.

Proof

Let $G \subseteq Y$ be open. Then since g is continuous, we have that

$$g^{-1}(G) \subseteq X \text{ is open.}$$

Then since f measurable, we have that

$$(g \circ f)^{-1}(G) = f^{-1}(g^{-1}(G)) \in \mathfrak{M}(E).$$

Thus $g \circ f \in \mathcal{L}(E, Y)$. □

Example 5.2.2

Let $E \in \mathfrak{M}(E)$ and $f \in \mathcal{L}(E, \mathbb{K})$. Let $g : \mathbb{K} \to \mathbb{R}$ be given by $z \mapsto |z|$. Then g is continuous. By Proposition 20, we have that

$$g \circ f = |f| \text{ is measurable.}$$

Example 5.2.3

Note that the converse to the above is not true, i.e. that if we have that $|f|$ is measurable, it is not necessary that f is not measurable.

Consider $E = \mathbb{R} = \mathbb{K}$. If we take $H \subseteq \mathbb{R}$ that is not measurable, which we know exists, and then consider the function

$$f : E \to \mathbb{R} \text{ given by } f(x) = \begin{cases} 1 & x \in H \\ -1 & x \notin H \end{cases},$$

which is constructed by summing up two characteristic functions over H and then minus 1. Then $|f| = 1$, but

$$f^{-1}(\{1\}) = H \notin \mathfrak{M}(\mathbb{R}).$$

Proposition 21 (Component-wise Measurability)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $f, g : E \to \mathbb{K}$. Then TFAE:

1. *$f, g \in \mathcal{L}(E, \mathbb{K})$;*
2. *$h : E \to \mathbb{K}^2$ given by $x \mapsto (f(x), g(x))$ is measurable.*

Proof

$(2) \implies (1)$ [6] Let

$$\pi_1 : \mathbb{K}^2 \to \mathbb{K} \text{ given by } (w, z) \mapsto w$$
$$\pi_2 : \mathbb{K}^2 \to \mathbb{K} \text{ given by } (w, z) \mapsto z$$

so that π_1, π_2 are continuous. Then by Proposition 20, we have that

$$\pi_1 \circ h = f \text{ and } \pi_2 \circ h = g$$

are both measurable.

$(1) \implies (2)$ Let $G \subseteq \mathbb{K}^2$ be open. We can write G as a countable union of open sets [7], i.e.

$$G = \bigcup_{n=1}^{\infty} A_n \times B_n,$$

where $A_n, B_n \subseteq \mathbb{K}$ are open. Then

$$\begin{aligned} h^{-1}(G) &= h^{-1}\left(\bigcup_{n=1}^{\infty} A_n \times B_n \right) \\ &= \bigcup_{n=1}^{\infty} \underbrace{f^{-1}(A_n)}_{\in \mathfrak{M}(\mathbb{K})} \cap \underbrace{g^{-1}(B_n)}_{\in \mathfrak{M}(\mathbb{K})} \in \mathfrak{M}(\mathbb{K}) \end{aligned}$$

Thus $h \in \mathcal{L}(E, \mathbb{K}^2)$. □

[6] Awareness about projective maps is a plus here.

[7] If you are unsure about this, think $\varepsilon - \delta$.

Proposition 22 ($\mathcal{L}(E, \mathbb{K})$ is a Unital Algebra)

Let $E \in \mathfrak{M}(\mathbb{R})$. Then $\mathcal{L}(E, \mathbb{K})$ is a unital algebra, i.e. if $f, g \in \mathcal{L}(E, \mathbb{K})$, then

1. *$f + g \in \mathcal{L}(E, \mathbb{K})$;*

2. $fg \in \mathcal{L}(E, \mathbb{K})$ [8];
3. $g(x) \neq 0, \forall x \in E \implies \frac{f}{g} \in \mathcal{L}(E, \mathbb{K})$; *and*
4. *if* $h : E \to \mathbb{K}$ *is constant, then* $h \in \mathcal{L}(E, \mathbb{K})$.

[8] Here, it's multiplication of two functions, not compositions

Proof

We shall make use of this clever trick [9]. Let $\mu : E \to \mathbb{K}^2$ given by $x \mapsto (f(x), g(x))$. Note that since $f, g \in \mathcal{L}(E, \mathbb{K})$, by Proposition 21, $\mu \in \mathcal{L}(E, \mathbb{K}^2)$.

[9] "Clever trick" = "Trick you should learn".

1. Consider the function

$$\sigma : \mathbb{K}^2 \to \mathbb{K} \text{ given by } (w, z) \mapsto w + z.$$

   It is clear that σ is continuous. Then

$$\sigma \circ \mu : x \mapsto f(x) + g(x)$$

   is measurable by Proposition 20.

2. Consider the function

$$\sigma : \mathbb{K}^2 \to \mathbb{K} \text{ given by } (w, z) \mapsto wz.$$

   Again, we see that σ is continuous. Then

$$\sigma \circ \mu : x \mapsto f(x)g(x)$$

   is measurable by Proposition 20.

3. Consider the function

$$\sigma : \mathbb{K} \times (\mathbb{K} \setminus \{0\}) \to \mathbb{K} \text{ given by } (w, z) \mapsto \frac{w}{z}.$$

   Again, σ is continuous. Thus

$$\sigma \circ \mu : x \mapsto \frac{f(x)}{g(x)}$$

   is measurable by Proposition 20.

4. Suppose $h : E \to \mathbb{K}$ is a constant, and we have $h(x) = \alpha_0$ for all $x \in E$. Then for any $G \subseteq \mathbb{K}$ that is open, we have that

$$h^{-1}(G) = \begin{cases} \emptyset & a_0 \notin G \\ E & a_0 \in G \end{cases},$$

both of which are measurable sets. Thus h is indeed measurable.

□

Remark 5.2.1

Note that $(\mathbb{C}, d)$, where $d(w, z) = |w - z|$, is a metric space. Moreover, the map

$$\gamma : \mathbb{C} \to \mathbb{R}^2 \text{ given by } x + iy \mapsto (x, y),$$

where $x, y \in \mathbb{R}$ is a homeomorphism, which, in particular, is continuous. Then given a $E \in \mathfrak{M}(\mathbb{R})$ with a measurable $f \in E \to \mathbb{C}$, then

$$\gamma \circ f : E \to \mathbb{R}^2 \in \mathcal{L}(E, \mathbb{R}^2).$$

Also, notice that

$$\gamma \circ f = (\Re f, \Im f).$$

By Proposition 21, $\Re f, \Im f \in \mathcal{L}(E, \mathbb{R})$. This also means that

$$h : x \mapsto (\Re f(x), \Im f(x)) \in \mathcal{L}(E, \mathbb{R}^2).$$

Conversely, if $\Re f, \Im f \in \mathcal{L}(E, \mathbb{R})$, then

$$f = \gamma^{-1} \circ h \in \mathcal{L}(E, \mathbb{C})$$

by Proposition 21.

This means that a ***complex-valued function is measurable iff its real and imaginary parts are both measurable****. Consequently, to study about complex-valued functions, it is sufficient for us to study about real-valued functions.*

Proposition 23 (Measurable Function Broken Down into an

Absolute Part and a Scaling Part)

Let $E \in \mathfrak{M}(\mathbb{R})$ and suppose that $f : E \to \mathbb{C}$ is measurable. Then there exists a measurable function $\Theta : E \to \mathbb{T}$, where

$$\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\},$$

such that

$$f = \Theta \cdot |f|.$$

Proof

Since $\{0\} \subseteq \mathbb{C}$ is closed and f is measurable, we have that

$$K := f^{-1}(\{0\}) \in \mathfrak{M}(E).$$

Since χ_K is a measurable function, we have that $f + \chi_K$ is also measurable (cf. Proposition 22).

Claim: $f + \chi_K \neq 0$ over E.

- If $x \in E$ such that $f(x) = 0$, then $x \in K$, and so $\chi_K(x) = 1$.
- If $x \in E$ such that $\chi_K(x) = 0$, then $x \notin K$, which means $f(x) \neq 0$.

  Therefore, consider the function

$$\Theta = \frac{f + \chi_K}{|f + \chi_K|} : E \to \mathbb{T}.$$

By Proposition 22, Θ is measurable, and clearly

$$f = \Theta \cdot |f|. \qquad \square$$

Remark 5.2.2

As of now, given a set $E \in \mathfrak{M}(\mathbb{R})$, to verify that a function $f \in \mathcal{L}(E, \mathbb{R})$, we need to check that

$$\forall G \subseteq \mathbb{R} \text{ open }, f^{-1}(G) \in \mathfrak{M}(E).$$

Since there is an obscene amount of open (respectively closed) subsets of $\mathbb{R}$, we want to be able to reduce our workload. This shall be the first thing we do in the next lecture.

6 Lecture 6 May 23rd 2019

6.1 Lebesgue Measurable Functions (Continued)

Proposition 24 (Function Measurability Check)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $f : E \to \mathbb{R}$ be a function. TFAE:

1. *f is measurable, i.e. $\forall G \subseteq \mathbb{R}$ that is open, $f^{-1}(G) \in \mathfrak{M}(E)$.*
2. $\forall a \in \mathbb{R}$, $f^{-1}((a, \infty)) \in \mathfrak{M}(E)$.
3. $\forall b \in \mathbb{R}$, $f^{-1}((-\infty, b]) \in \mathfrak{M}(E)$.
4. $\forall b \in \mathbb{R}$, $f^{-1}((-\infty, b)) \in \mathfrak{M}(E)$.
5. $\forall a \in \mathbb{R}$, $f^{-1}([a, \infty)) \in \mathfrak{M}(E)$.

Proof

$(1) \implies (2)$ This is trivially true since $\forall a \in \mathbb{R}$, (a, ∞) is open in $\mathbb{R}$, and so since f is measurable, we must have that $f^{-1}((a, \infty)) \in \mathfrak{M}(E)$.

$(2) \implies (3)$ Notice that $\forall b \in \mathbb{R}$,

$$f^{-1}((-\infty, b]) = f^{-1}(\mathbb{R} \setminus (b, \infty)) = E \setminus f^{-1}((b, \infty))$$

and $f^{-1}((b, \infty)) \in \mathfrak{M}(E)$ by assumption. Since $\mathfrak{M}(E)$ is a σ-algebra, $f^{-1}((-\infty, b]) \in \mathfrak{M}(E)$.

(3) $\implies$ (4) Notice that $\forall b \in \mathbb{R}$,

$$f^{-1}((-\infty, b)) = \bigcup_{n=1}^{\infty} f^{-1}\left(\left(-\infty, b - \frac{1}{n}\right]\right),$$

and by assumption, for each $n \geq 1$, $f^{-1}\left(\left(-\infty, b - \frac{1}{n}\right]\right) \in \mathfrak{M}(E)$. It follows that $f^{-1}((-\infty, b)) \in \mathfrak{M}(E)$.

(4) $\implies$ (5) Observe that $\forall a \in \mathbb{R}$, we have

$$f^{-1}([a, \infty)) = f^{-1}(\mathbb{R} \setminus (-\infty, a)) \in \mathfrak{M}(E)$$

by assumption.

(5) $\implies$ (1) [1] Notice that $\forall a \in \mathbb{R}$,

$$f^{-1}((a, \infty)) = \bigcup_{n=1}^{\infty} f^{-1}\left(\left[a + \frac{1}{n}, \infty\right)\right) \in \mathfrak{M}(E)$$

by assumption. Furthermore, we have that $\forall b \in \mathbb{R}$,

$$f^{-1}((-\infty, b)) = E \setminus f^{-1}([b, \infty)) \in \mathfrak{M}(E),$$

also by assumption. Thus

$$f^{-1}((a, b)) = f^{-1}((a, \infty)) \cap f^{-1}((-\infty, b)) \in \mathfrak{M}(E),$$

for any $a, b \in \mathbb{R}$.

Since for any open $G \subseteq \mathbb{R}$ can be written as a countable union of open intervals, i.e.

$$G = \bigcup_{n=1}^{\infty} I_n,$$

where each I_n is an open interval, we have that

$$f^{-1}(G) = \bigcup_{n=1}^{\infty} f^{-1}(I_n) \in \mathfrak{M}(E).$$

Thus f is measurable. □

[1] This uses the same idea as in Proposition 14.

The proof of the following result is left to A2.

Corollary 25 (Measurability Check on the Borel Set)

If $E \in \mathfrak{M}(\mathbb{R})$ and $f : E \to \mathbb{R}$ is a function, then TFAE:

1. *f is measurable.*
2. *$\forall B \in \mathfrak{Bor}(\mathbb{R})$, $f^{-1}(B) \in \mathfrak{M}(E)$.*

Remark 6.1.1

Let $E \in \mathfrak{M}(\mathbb{R})$ and $f : E \to \mathbb{R}$. Define

$$f^+(x) = \max\{f(x), 0\},\ x \in E$$
$$f^-(x) = \max\{-f(x), 0\},\ x \in E$$

Then $f^+, f^- \geq 0$, and

$$f = f^+ - f^- \text{ and } |f| = f^+ + f^-.$$

Moreover,

$$f^+ = \frac{|f| + f}{2} \text{ and } f^- = \frac{|f| - f}{2},$$

and so both f^+ and f^- are measurable.

By Remark 5.2.1, every complex-valued measurable function is a linear combination of 4 non-negative, real-valued measurable functions.

We shall now examine a number of results dealing with pointwise limits of sequences of measurable, real-valued functions. We shall include the case where the limit of a given point is allowed to be an **extended real number**; i.e. the sequence diverges either to ∞ or $-\infty$.

Definition 21 (Extended Real Numbers)

*We define the **extended real numbers** to be the set*

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}.$$

We also write $\overline{\mathbb{R}} = [-\infty, \infty]$.

By convention, we shall define

- $\infty + \infty = \infty$, $-\infty - \infty = -\infty$;
- $\forall \alpha \in \mathbb{R} \cup \{\infty\}$, $\alpha + \infty = \infty = \infty + \alpha$;
- $\forall \alpha \in \mathbb{R}$, $\alpha + (-\infty) = -\infty = -\infty + \alpha$;
- $\forall 0 < \alpha \in \overline{\mathbb{R}}$, $a \cdot \infty = \infty \cdot \alpha = (-\infty) \cdot (-\alpha) = (-\alpha) \cdot (-\infty) = \infty$;
- $\forall \alpha < 0 \in \overline{\mathbb{R}}$, $a \cdot \infty = \infty \cdot \alpha = (-\infty) \cdot (-\alpha) = (-\alpha) \cdot (-\infty) = -\infty$; *and*
- $0 = 0 \cdot \infty = \infty \cdot 0 = 0 \cdot (-\infty) = (-\infty) \cdot 0$.

Warning

Notice that we do not define $\infty - \infty$ and $-\infty + \infty$.

Note 6.1.1

While the space of extended real numbers is useful for treating measure-theoretic and analytic properties of sequences of functions, it has poor algebraic properties. In particular, it is no longer a vector space, since ∞ and $-\infty$ do not have their additive inverses.

Definition 22 (Extended Real-Valued Function)

Given $H \subseteq \mathbb{R}$, the function $f : H \to \overline{\mathbb{R}}$ is called an ***extended real-valued function****.*

Definition 23 (Measurable Extended Real-Valued Function)

If $E \in \mathfrak{M}(\mathbb{R})$ and $f : E \to \overline{\mathbb{R}}$ is an extended real-valued function, we say that f is ***Lebesgue measurable*** *(or simply* ***measurable****) if*

1. $\forall G \subseteq \mathbb{R}$ *open*, $f^{-1}(G) \in \mathfrak{M}(E)$ *; annd*
2. $f^{-1}(\{-\infty\}), f^{-1}(\{\infty\}) \in \mathfrak{M}(E)$.

We denote the set of Lebesgue measurable extended real-valued functions on E by

$$\mathcal{L}(E, \overline{\mathbb{R}}) = \{ f : E \to \overline{\mathbb{R}} : f \text{ is measurable } \}.$$

Since we shall often refer to only the non-negative elements of $\mathcal{L}(E, \overline{\mathbb{R}})$, *we also define the notation*

$$\mathcal{L}(E, [0, \infty]) = \{ f \in \mathcal{L}(E, \overline{\mathbb{R}}) : \forall x \in E, \, 0 \leq f(x) \}.$$

Note 6.1.2

Note that we can also replace the first condition of Lebesgue measurability of extended real-valued functions by

$$\forall F \subseteq \mathbb{R} \text{ closed }, \, f^{-1}(F) \in \mathfrak{M}(E).$$

Just as in the case with regular real-valued measurable functions, we have the following shortcuts in testing whether an extended real-valued function is measurable.

Notation

We write

- $(a, \infty] = (a, \infty) \cup \{\infty\}$; *and*
- $[-\infty, b) = (-\infty, b) \cup \{-\infty\}$,

for all $a, b \in \mathbb{R}$.

Proposition 26 (Measurability Check for Extended Real-Valued Functions)

Let $E \in \mathfrak{M}(\mathbb{R})$ *and suppose* $f : E \to \overline{\mathbb{R}}$ *is a function. Then TFAE:*

1. f *is Lebesgue measurable.*

2. $\forall a \in \mathbb{R}$, $f^{-1}((a, \infty]) \in \mathfrak{M}(E)$.

3. $\forall b \in \mathbb{R}$, $f^{-1}([-\infty, b)) \in \mathfrak{M}(E)$.

Exercise 6.1.1

Prove Proposition 26.

Proposition 27 (Measurability of Limits and Extremas)

Let $E \in \mathfrak{M}(\mathbb{R})$ and suppose that $(f_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{L}(E, \overline{\mathbb{R}})$. Then the following extended real-valued functions are also measurable:

1. $g_1 := \sup_{n \geq 1} f_n$;
2. $g_2 := \inf_{n \geq 1} f_n$;
3. $g_3 := \limsup_{n \geq 1} f_n$; *and*
4. $g_4 := \liminf_{n \geq 1} f_n$.

Proof

1. Let $a \in \mathbb{R}$. Then

$$g_1^{-1}((a, \infty]) = \bigcup_{n \geq 1} \underbrace{f_n^{-1}((a, \infty])}_{\in \mathfrak{M}(E)} \in \mathfrak{M}(E).$$

It follows from Proposition 26 that $g_1 \in \mathcal{L}(E, \overline{\mathbb{R}})$.

2. [2] For any $b \in \mathbb{R}$, we have

$$g_2^{-1}([-\infty, b)) = \bigcap_{n \geq 1} f_n^{-1}([-\infty, b)) \in \mathfrak{M}(E).$$

Thus by Proposition 26, $g_2 \in \mathcal{L}(E, \overline{\mathbb{R}})$.

3. Let $h_n = \sup_{k \geq n} f_n$ for each $n \geq 1$. Then by part (1), $h_n \in \mathcal{L}(E, \overline{\mathbb{R}})$ for each $n \geq 1$. Also, notice that $h_1 \geq h_2 \geq h_3 \geq \ldots$, i.e. $\{h_n\}_{n=1}^{\infty}$ is an increasing sequence of functions. Then by part

[2] Both notes and lecture notes used union, but should it not be intersection?

(2),

$$g_3 = \lim_{n\to\infty} h_n = \inf_{n\geq 1} h_n \in \mathcal{L}(E, \overline{\mathbb{R}}).$$

4. Let $h_n = \inf_{k\geq n} f_n$ for each $n \geq 1$. Then by part (2), each $h_n \in \mathcal{L}(E, \overline{\mathbb{R}})$. Also, $\{h_n\}_{n=1}^{\infty}$ is a decreasing sequence of functions. Then by part (1), we have that

$$g_4 = \lim_{n\to\infty} h_n = \sup_{n\geq 1} h_n \in \mathcal{L}(E, \overline{\mathbb{R}}).$$ □

Corollary 28 (Extended Limit of Real-Valued Functions)

Let $E \in \mathfrak{M}(\mathbb{R})$ and suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of real-valued functions such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists as an extended real-valued number for all $x \in E$. Then

$$f \in \mathcal{L}(E, \overline{\mathbb{R}}).$$

Proof

By A2, when the said limit exists, we have that

$$f = \limsup_{n\geq 1} f_n = \liminf_{n\geq 1} f_n,$$

and so $f \in \mathcal{L}(E, \overline{\mathbb{R}})$ by Proposition 27. □

Definition 24 (Simple Functions)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $\varphi : E \to \overline{\mathbb{R}}$. We say that φ is ***simple*** *if* range φ *is finite. Furthermore, we denote the set of all simple, real-valued,* ***measurable*** *functions on E as*

$$\mathrm{SIMP}(E, \mathbb{R}).$$

Definition 25 (Standard Form)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $\varphi : E \to \overline{\mathbb{R}}$. Suppose that

$$\operatorname{range}\varphi = \{\alpha_1 < \alpha_2 < \ldots < a_N\},$$

and set

$$E_n := \varphi^{-1}(\{\alpha_n\}), \text{ for } 1 \leq n \leq N.$$

We say that

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{E_n}$$

is the ***standard form*** *of φ.*

Proposition 29 (Measurability of Simple Functions with Measurable Support)

Let $E \in \mathfrak{M}(\mathbb{R})$. Suppose $\varphi : E \to \overline{\mathbb{R}}$ is simple with

$$\operatorname{range}\varphi = \{\alpha_1 < \alpha_2 < \ldots < \alpha_N\}.$$

TFAE:

1. *φ is measurable.*
2. *If $\varphi = \sum_{n=1}^{N} \alpha_n \chi_{E_n}$ is the standard form of φ, then $E_n \in \mathfrak{M}(E)$, for all $n \in \{1, \ldots, N\}$.*

Proof

$(\implies)$ Since φ is measurable, notice that for each $n \in \{1, \ldots, N\}$,

- if $\alpha_n \in \mathbb{R}$, then $\{\alpha_n\}$ is closed, and so
$$E = \varphi^{-1}(\{\alpha_n\}) \in \mathfrak{M}(E); \text{ and}$$
- if $\alpha_1 = -\infty$, and similarly if $\alpha_N = \infty$, then by Definition 23, $\varphi^{-1}(\{\alpha_1\}), \varphi^{-1}(\{\alpha_N\}) \in \mathfrak{M}(E)$.

($\impliedby$) By Example 5.2.1, $\forall n \geq 1$, $E_n \in \mathfrak{M}(E) \implies \forall n \geq 0 \chi_{E_n} \in \mathfrak{M}(E)$. Notice that $\forall a \in \mathbb{R}$,

$$\varphi^{-1}((a, \infty]) = \bigcup \{E_n : a < \alpha_n\},$$

and so $\varphi^{-1}((a, \infty])$ is a finite (or empty) union of measurable sets, and is hence measurable. $\square$

THE STANDARD FORM is not a unique way of expressing a simple function as a finite linear combination of characteristic functions.

Example 6.1.1

Consider the function $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi = \chi_{\mathbb{Q}} + 9\chi_{[2,6]}.$$

Then range $\varphi = \{0, 1, 9, 10\}$; we see that

$$x \mapsto \begin{cases} 0 & x \in \mathbb{Q}^C \cap [2,6]^C \\ 1 & x \in \mathbb{Q} \cap [2,6]^C \\ 9 & x \in \mathbb{Q}^C \cap [2,6] \\ 10 & x \in \mathbb{Q} \cap [2,6] \end{cases}.$$

Then we may write φ as

$$\varphi = 0\chi_{\mathbb{Q}^C \cap [2,6]^C} + 1\chi_{\mathbb{Q} \cap [2,6]^C} + 9\chi_{\mathbb{Q}^C \cap [2,6]} + 10\chi_{\mathbb{Q} \cap [2,6]}.$$

Definition 26 (Real Cone)

*Let $\mathcal{V}$ be a vector space over $\mathbb{K}$. A subset $C \subseteq \mathcal{V}$ is said to be a **(real) cone** is*

1. *$C \cap -C = \{0\}$, where $-C = \{-w : w \in C\}$; and*
2. *$y, z \in C$ and $0 \leq \kappa \in \mathbb{R}$ imply that*

$$\kappa y + z \in C.$$

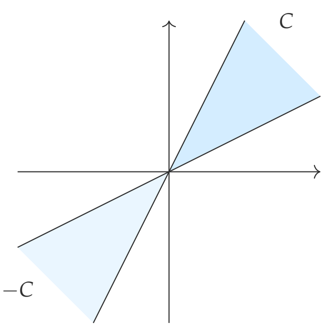

Figure 6.1: Typical figure of a cone

Example 6.1.2

1. Let $\mathcal{V} = \mathbb{R}^3$ and
$$C = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x, y, z\}.$$
Then C is a (real) cone.

2. Let $\mathcal{V} = \mathbb{C}$ and
$$C = \left\{ w \in \mathbb{C} : w = re^{i\theta}, \, \frac{\pi}{6} \leq \theta \leq \frac{2\pi}{6}, \, 0 \leq r < \infty \right\}.$$
The C is a cone in $\mathbb{C}$. Note that in both the above examples, C is not closed.

3. Let $\mathcal{V} = \mathcal{C}([0,1], \mathbb{C})$, and
$$C = \{ f \in \mathcal{V} : 0 \leq f(x), \, \forall x \in [0,1] \},$$
where we note that the condition means that we only look at those functions that return real positive values. Then C is a (real) cone in $\mathcal{V}$.

Exercise 6.1.2

Show that $\mathrm{SIMP}(E, \mathbb{R})$ *is an algebra, and hence a vector space over* $\mathbb{R}$.

Remark 6.1.2

1. *Note that*
$$\mathrm{SIMP}(E, \overline{\mathbb{R}}) = \{ f : E \to \overline{\mathbb{R}} : f \text{ is simple and measurable } \}.$$
is not a vector space. In fact, it is neither a field nor a ring.

2. *We shall adopt the following notation:*
$$\mathrm{SIMP}(E, [0, \infty)) := \{ \varphi \in \mathrm{SIMP}(E, \mathbb{R}) : 0 \leq \varphi(x) \text{ for all } x \in E \}.$$
Observe that this is a real cone in $\mathrm{SIMP}(E, \mathbb{R})$.

In A3, we will show the following proposition.

Proposition 30 (Increasing Sequence of Simple Functions that Converges to a Measurable Function)

Let $E \in \mathfrak{M}(E)$ *and* $f \in \mathcal{L}(E, [0, \infty])$. *Then there exists an increasing sequence*

$$\varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \ldots \leq f$$

of simple, real-valued functions φ_n *such that*

$$f(x) = \lim_{n \to \infty} \varphi_n(x)$$

for all $x \in E$.

7 Lecture 7 May 28th 2019

7.1 Lebesgue Integration

We shall first begin by defining integration over simple, non-negative, extended real-valued functions. We shall then use this definition to define the integral of $f \in \mathcal{L}(E, [0, \infty])$, and derive several consequences of our definition. Furthermore, we shall also build the Lebesgue integral such that it is linear, which will require us to impose certain conditions to the range of functions which will retain this desirable property.

Definition 27 (Integration of Simple Functions)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $\varphi \in \mathrm{SIMP}(E, [0, \infty])$, such that its standard form is denoted as

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{E_n}.$$

We define

$$\int_E \varphi := \sum_{n=1}^{N} \alpha_n m E_n \in [0, \infty].$$

If $F \subseteq E$ is measurable, we define

$$\int_F \varphi = \int_E \varphi \cdot \chi_F = \sum_{n=1}^{N} \alpha_n m(F \cap E_n).$$

Note 7.1.1

Note that since φ is measurable, so is each E_n for $1 \leq n \leq N$.

Example 7.1.1

1. Let $\varphi = 0\chi_{[4,\infty)} + 17\xi_{\mathbb{Q}\cap[0,4)} + 29\chi_{[2,4)\setminus\mathbb{Q}}$. Then

$$\begin{aligned}\int_{[0,\infty)} \varphi &= 0m[4,\infty) + 17m(\mathbb{Q}\cap[0,4)) + 29m([2,4)\setminus\mathbb{Q}) \\ &= 0 + 17\cdot 0 + 29(2) = 58.\end{aligned}$$

2. Let $C \subseteq [0,1]$ be the Cantor set from Example 5.1.1 and $\varphi = 1\chi_C + 2\chi_{[5,9]}$. Then

$$\begin{aligned}\int_{[0,6]} \varphi &= 1m(C\cap[0,6]) + 2m([5,9]\cap[0,6]) \\ &= 1\cdot 0 + 2m([5,6]) \\ &= 2.\end{aligned}$$

Since our definition is fairly limited since it requires that our simple function be in standard form, let us try to relax that condition.

Definition 28 (Disjoint Representation)

Let $E \in \mathfrak{M}(E)$ and $\varphi \in \mathrm{SIMP}(E,[0,\infty])$. Suppose

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{H_n},$$

*where $H_n \subseteq E$ is **measurable** and $\alpha_n \in \overline{\mathbb{R}}$ for each $1 \le n \le N$.* [1] *We shall say that the above decomposition of φ is a **disjoint representation** of φ if*

$$H_i \cap H_j = \emptyset, \text{ for } 1 \le i \ne j \le N.$$

[1] Note that we did not require that the α_n's be distinct, nor do we require that they be written in any particular order, nor do we require that $E = \bigcup_{n=1}^N H_n$, unlike in the definition of simple functions.

Lemma 31 (Common Disjoint Representation of Simple Functions over a Common Domain)

Let $E \in \mathfrak{M}(\mathbb{R})$ and suppose that $\varphi, \psi \in \mathcal{L}(E,\mathbb{R})$. Then there exists

1. $N \in \mathbb{N}$;

2. *$H_1, H_2, \ldots, H_n \in \mathfrak{M}(E)$ with $H_i \cap H_j = \emptyset$ for all $i \neq j$; and*

3. *$\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N$ such that*

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{H_n} \quad \text{and} \quad \psi = \sum_{n=1}^{N} \beta_N \chi_{H_n}$$

are disjoint representations of φ and ψ.

Proof

Since φ and ψ are simple, from Definition 25, if we write

$$\varphi = \sum_{m=1}^{M_1} a_m \chi_{E_m} \quad \text{and} \quad \psi = \sum_{m=1}^{M_2} b_m \chi_{F_m}$$

in their standard forms, we have that the E_m's and F_m's are respectively pairwise disjoint [2]. Then

$$\{E_i \cap F_j : 1 \leq i \leq M_1,\, 1 \leq j \leq M_2\}$$

is also a pairwise disjoint family of measurable sets, which we shall relabel them as

$$\{H_n\}_{n=1}^{N}, \text{ where } N = M_1 M_2.$$

Then

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{H_n},$$

where $\alpha_n = a_i$ if $H_n = E_i \cap F_j$ for some $1 \leq j \leq M_2$, and

$$\psi = \sum_{n=1}^{N} \beta_N \chi_{H_n},$$

where $\beta_n = b_j$ if $H_n = E_i \cap F_j$ for some $1 \leq i \leq M_1$. □

[2] It is important to note here that the E_m's and F_m's are pairwise disjoint on E, which is why the next step is a sensible and correct one.

Lemma 32 (Integral of a Simple Funciton Using Its Disjoint Representation)

Let $E \in \mathfrak{M}(\mathbb{R})$ and suppose $\varphi \in \mathrm{SIMP}(E, [0, \infty])$. If

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{H_n}$$

is any disjoint representation, then

$$\int_E \varphi = \sum_{n=1}^{n} \alpha_n m H_n.$$

Proof

[3] If $\bigcup_{n=1}^{N} H_n \neq E$, then we set

$$H_{N+1} = E \setminus \bigcup_{n=1}^{N} H_n \text{ and } \alpha_{N+1} = 0.$$

[3] One of the problems here is that the disjoint H_n's may not cover the entire domain φ, but we can fill it up with zeros.

Then

$$\sum_{n=1}^{N} \alpha_n m H_n = \sum_{n=1}^{N+1} \alpha_n m H_n.$$

Thus, wlog, wma

$$\bigcup_{n=1}^{N} H_n = E.$$

Now since the H_n's are mutually disjoint, wma

$$\operatorname{range} \varphi = \{\alpha_1, \ldots, \alpha_N\},$$

where we note that the above set may contain repeated elements, i.e. some $\alpha_i = \alpha_j$. We may thus rewrite this set such that

$$\{\alpha_1, \ldots, \alpha_N\} = \{\beta_1 < \beta_2 < \ldots < \beta_M\}$$

and set

$$E_i = \bigcup \{H_j : \alpha_j = \beta_i\}.$$

Note that since $H_i \cap H_j = \emptyset$ for $1 \leq i \neq j \leq N$, for $1 \leq k \leq M$, we have

$$mE_k = \sum_{\alpha_j = \beta_k} m(H_j).$$

Then by definition,

$$\begin{aligned}\int_E \varphi &= \sum_{k=1}^{M} \beta_k \xi_{E_k} \\ &= \sum_{i=1}^{M} \beta_i \sum_{\alpha_j = \beta_i} mH_j \\ &= \sum_{n=1}^{N} \alpha_j mH_j,\end{aligned}$$

as desired. □

Proposition 33 (Linearity and Monotonicity of the Integral of Simple Functions)

Let $E \in \mathfrak{M}(\mathbb{R})$. If $\varphi, \psi \in \mathrm{SIMP}(E, [0, \infty])$ and $\kappa \in [0, \infty)$, then

1. $\int_E \kappa\varphi + \psi = \kappa \int_E \varphi + \int_E \psi$*; and*
2. $\varphi \leq \psi$ *on E implies*
$$\int_E \varphi \leq \int_E \psi.$$

Proof

1. By Lemma 31, we can find a common disjoint representation of φ and ψ, say

$$\varphi = \sum_{n=1}^{N} a_n \chi_{H_n} \quad \text{and} \quad \psi = \sum_{n=1}^{N} b_n \chi_{H_n},$$

where the H_n's are pairwise disjoint. Then

$$\kappa\varphi + \psi = \sum_{n=1}^{N} (\kappa a_n + b_n) \chi_{H_n}.$$

Thus by Lemma 32,

$$\begin{aligned}\int_E (\kappa\varphi + \psi) &= \sum_{n=1}^{N} (\kappa a_n + b_n) mH_n \\ &= \kappa \sum_{n=1}^{N} a_n mH_n * \sum_{n=1}^{N} b_n mH_n\end{aligned}$$

$$= \kappa \int_E \varphi + \int_E \psi.$$

2. Using the disjoint representation, if $\varphi \leq \psi$, then $a_n \leq b_n$ for all $1 \leq n \leq N$, and so by Lemma 32,

$$\int_E \varphi = \sum_{n=1}^{N} a_n m H_n \leq \sum_{n=1}^{N} b_n m H_n = \psi. \qquad \square$$

We are now ready to define the Lebesgue integral for arbitrary measurable functions.

Definition 29 (Lebesgue Integral)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $f \in \mathcal{L}(E, [0, \infty])$. We define the ***Lebesgue integral*** *of f as*

$$\int_E^{NEW} f = \sup \left\{ \int_E \varphi : \varphi \in \mathrm{SIMP}(e, [0, \infty)),\, 0 \leq \varphi \leq f \right\}.$$

Note 7.1.2

- *We can actually allow $\varphi \in \mathrm{SIMP}(E, [0, \infty])$.*
- *We put "NEW" in the above integral because we now have "two" definitions for the integral of $\varphi \in \mathrm{SIMP}(E, [0, \infty])$. Writing $\varphi = \sum_{n=1}^{N} \alpha_n \chi_{H_n}$ in its standard form, by Definition 27,*

$$\int_E \varphi = \sum_{n=1}^{N} \alpha_n m H_n,$$

while Definition 29 gives us

$$\int_E^{NEW} \varphi = \sup \left\{ \int_E \psi : \psi \in \mathrm{SIMP}(E, [0, \infty)),\, 0 \leq \psi \leq \varphi \right\}.$$

Remark 7.1.1

Let us try reconciling these two definitions, which will allow us to drop the *dumb-looking "NEW" notation. First, note that*

$$\varphi \in \{\psi \in \mathrm{SIMP}(E, [0, \infty]) : 0 \leq \psi \leq \varphi\},$$

and so by Definition 29*, then*

$$\int_E \varphi \leq \int_E^{NEW} \varphi.$$

On the other hand, by Proposition 33*, if* $\varphi \in \mathrm{SIMP}(E, [0, \infty])$ *and* $0 \leq \psi \leq \varphi$*, we have that*

$$\int_E \psi \leq \int_E \varphi,$$

and so

$$\int_E^{NEW} \varphi = \sup \left\{ \int_E \psi : \psi \in \mathrm{SIMP}(E, 0, \infty]),\ \psi \leq \varphi \right\} \leq \int_E \varphi.$$

Thus

$$\int_E^{NEW} \varphi = \int_E \varphi.$$

With that we shall drop the "NEW" notation from here on.

Definition 30 (Almost Everywhere (a.e.))

Let $E \in \mathfrak{M}(\mathbb{R})$*. We say that a property* (P) *holds* ***almost everywhere (a.e.)*** *on* E *if the set*

$$B := \{x \in E : (P) \text{ does not hold }\}$$

has Lebesgue measure zero.

Example 7.1.2

Let $E \in \mathfrak{M}(\mathbb{R})$. Given $f, g \in \mathcal{L}(E, \overline{\mathbb{R}})$, we say that $f = g$ a.e. on E if

$$B := \{x \in E : f(x) \neq g(x)\}$$

has measure zero, i.e. $mB = 0$.

An example of this is

$$\chi_{\mathbb{Q}} = 0 = \chi_C$$

a.e. on $\mathbb{R}$, where C is the Cantor set.

Lemma 34 (Linearity, Monotonicity of the Lebesgue Integral)

Let $E \in \mathfrak{M}(\mathbb{R})$ and let $f, g, h : E \to [0, \infty]$ be functions. Suppose that g and h are measurable.

1. *Suppose further that $E = X \mathbin{\dot\cup} Y$, where $X, Y \in \mathfrak{M}(E)$. Set $f_1 := f \restriction_X$ and $f_2 := f \restriction_Y$. Then $f \in \mathcal{L}(E, [0, \infty])$ iff f_1 and f_2 are measurable. When this is the case, then*
$$\int_E f = \int_X f_1 + \int_Y f_2.$$
2. *If $g \leq h$, then*
$$\int_E g \leq \int_E h.$$
3. *If $H \in \mathfrak{M}(E)$, then*
$$\int_H g = \int_E g \cdot \chi_H \leq \int_E g.$$

Exercise 7.1.1

Prove Lemma 34.

Proof

1. f is measurable $\iff$ f_1 and f_2 are measurable ($\implies$) Note that
$$f_1 = f \cdot \chi_X \text{ and } f_2 = f \cdot \chi_Y,$$
and since X, Y are measurable, by Proposition 20, we have that f_1 and f_2 are measurable.

   ($\impliedby$) Suppose f_1 and f_2 are measurable and $X \mathbin{\dot\cup} Y$. We have that
$$f = f_1 + f_2.$$
I will spare the details, but it is not difficult to see that $\forall a \in \mathbb{R}$,

breaking $(a, \infty]$ into disjoint pieces if necessary, $f^{-1}((a, \infty])$ is measurable, and hence f is indeed measurable.

The integral [4] By Definition 27 and Proposition 33, we have

$$\begin{aligned}
\int_E f &= \sup \left\{ \int_E \varphi : \varphi \in \mathrm{SIMP}(E, [0, \infty]),\ \varphi \leq f \right\} \\
&= \sup \left\{ \int_E \varphi \cdot \chi_X + \varphi \cdot \chi_Y : \varphi \in \mathrm{SIMP}(E, [0, \infty]),\ \varphi \leq f \right\} \\
&= \sup \left\{ \int_X \varphi + \int_Y \varphi : \varphi \in \mathrm{SIMP}(E, [0, \infty]),\ \varphi \leq f \right\} \\
&\leq \sup \left\{ \int_X \varphi : \varphi \in \mathrm{SIMP}(X, [0, \infty]),\ \varphi \leq f_1 \right\} \\
&\qquad + \sup \left\{ \int_Y \psi : \psi \in \mathrm{SIMP}(Y, [0, \infty]),\ \psi \leq f_2 \right\} \\
&= \int_X f_1 + \int_Y f_2.
\end{aligned}$$

[4] This proof is iffy.

On the other hand, since $f_1 = f$ on X and $f_2 = f$ on Y, and X and Y are disjoint,

$$\begin{aligned}
&\int_X f_1 + \int_Y f_2 \\
&= \sup \left\{ \int_X \varphi : \varphi \in \mathrm{SIMP}(X, [0, \infty]),\ \varphi \leq f_1 = f \restriction_X \right\} \\
&\qquad + \sup \left\{ \int_Y \psi : \psi \in \mathrm{SIMP}(Y, [0, \infty]),\ \psi \leq f_2 = f \restriction_Y \right\} \\
&= \sup \left\{ \int_X \varphi + \int_Y \psi : \varphi \in \mathrm{SIMP}(X, [0, \infty]), \right. \\
&\qquad \left. \psi \in \mathrm{SIMP}(Y, [0, \infty]),\ \varphi \leq f \restriction_X,\ \psi \leq f \restriction_Y \right\} \\
&= \sup \left\{ \int_E \varphi \cdot \chi_X + \psi \cdot \chi_Y : \varphi, \psi \in \mathrm{SIMP}(E, [0, \infty]), \right. \\
&\qquad \left. \varphi + \psi \leq f \restriction_X + f \restriction_Y = f \right\} \\
&= \int_E f.
\end{aligned}$$

2. By Proposition 30, there exists sequences $\{\varphi_n\}_n$ and $\{\psi_n\}_n$ such that

$$\lim_{n \to \infty} \varphi_n = g \leq h = \lim_{n \to \infty} \psi_n.$$

In particular,

$$\sup_{n\geq 1} \varphi_n = g \leq h = \sup_{n\geq 1} \psi_n.$$

Since the leftmost and rightmost terms are simple functions, by Proposition 33,

$$\begin{aligned}
\int_E g &= \sup \left\{ \int_E \varphi : \varphi \in \mathrm{SIMP}(E, [0, \infty]),\ \varphi \leq g \right\} \\
&\leq \sup \left\{ \int_E \psi : \psi \in \mathrm{SIMP}(E, [0, \infty]),\ \psi \leq h \right\} \\
&= \int_E h.
\end{aligned}$$

3. [5] For the first equality, by Definition 27, we have that

$$\begin{aligned}
\int_H g &= \sup \left\{ \int_H \varphi : \varphi \in \mathrm{SIMP}(H, [0, \infty]),\ \varphi \leq g \right\} \\
&= \sup \left\{ \int_E \varphi \cdot \chi_H : \varphi \in \mathrm{SIMP}(E, [0, \infty]),\ \varphi \leq g \right\} \\
&= \int_E g \cdot \chi_H.
\end{aligned}$$

Note that we have $g \cdot \chi_H \leq g$, and so by part (2),

$$\int_E g \cdot \chi_H \leq \int_E g. \qquad \square$$

[5] This is also iffy.

Proposition 35 (Integration over Domains of Measure Zero and Integration of Functions Agreeing Almost Everywhere)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $f, g \in \mathcal{L}(E, [0, \infty])$.

1. *If $mE = 0$, then $\int_E f = 0$.*
2. *If $f = g$ a.e. on E, then $\int_E f = \int_E g$.*

Proof

1. $\forall \varphi \in \mathrm{SIMP}(E, [0, \infty])$ written in its standard form

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{E_n},$$

by monotonicity,

$$0 \leq \int_E \varphi = \sum_{n=1}^{N} \alpha_n m E_n \leq \sum_{n=1}^{N} \alpha_n m E = 0,$$

and so

$$\int_E \varphi = 0.$$

Thus

$$\int_E f = \sup \left\{ \int_E \varphi : \varphi \in \mathrm{SIMP}(E, [0, \infty]),\ \varphi \leq f \right\} = \sup\{0\} = 0.$$

2. Let $B := \{x \in E : f(x) \neq g(x)\}$ so that $mB = 0$. Then by Lemma 34 and part (1), we have

$$\begin{aligned} \int_E f &= \int_{E \setminus B} f + \int_B \\ &= \int_{E \setminus B} f + 0 \\ &= \int_{E \setminus B} g + \int_B g \\ &= \int_E g. \end{aligned}$$ □

We are now in a position to prove the following important theorem, which we shall do so next lecture.

Theorem (The Monotone Convergence Theorem)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $(f_n)_n$ be a sequence in $\mathcal{L}(E, [0, \infty])$ such that $f_n \leq f_{n+1}$ a.e. on E. Suppose further that

$$f : E \to [0, \infty]$$

is a function such that $f(x) = \lim_{n\to\infty} f_n(x)$ a.e. on E. Then $f \in \mathcal{L}(E, [0, \infty])$ and

$$\int_E f = \lim_{n \to \infty} \int_E f_n.$$

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