PMATH451 — Measure and Integration

Class notes for Fall 2019

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1.1 Motivation for the Study of Measures

Recall Riemann integration.

Definition (Riemann Integration)

*Let $f : [a, b] \to \mathbb{R}$ be a **bounded** function. We call*

$$P = \{a = x_0 < x_1 < \ldots < x_n = b\} \subseteq [a, b]$$

*a **partition** of $[a, b]$, and*

$$\Delta x_i = x_i - x_{i-1}$$

*as the **length of the i^{th} interval** for $i = 1, \ldots, n$.*

Let

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

*be the **supremum of f on the i^{th} interval**, and*

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$

*be the **infimum of f on the i^{th} interval**. We define the **Riemann upper sum** as*

$$U(f, P) = \sum_i M_i \Delta x_i,$$

*and the **Riemann lower sum** as*

$$L(f, P) = \sum_i m_i \Delta x_i.$$

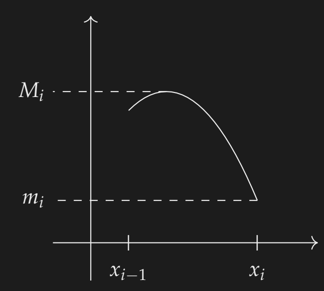

Figure 1.1: Idea of Riemann integration

We define the ***Riemann upper integral*** *as*

$$\overline{\int_a^b} f\,dx = \inf_P U(f, P)$$

and the ***Riemann lower integral*** *as*

$$\underline{\int_a^b} f\,dx = \sup L(f, P).$$

We say that f is ***Riemann integrable*** *if*

$$\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx,$$

and we write the integral of f as

$$\int_a^b f\,dx = \overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx.$$

As hyped up as one does earlier in university about Riemann integration, there are functions that are not Riemann integrable!

Example 1.1.1

Consider a function $f : [0, 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}.$$

Then

$$\overline{\int_a^b} f\,dx = 1 \text{ and } \underline{\int_a^b} f\,dx = 0.$$

Thus f is not Riemann integrable.

Note 1.1.1 (Shortcomings of the Riemann integral)

1. *We cannot characterize functions that are Riemann integrable, i.e. we do not have a list of characteristics that we can check against to see if a function is Riemann integrable.*

   This remained an open problem in the earlier 1920s.

2. *The Riemann integral behaves badly when it comes to pointwise limits of functions. The next example shall illustrate this.*

3. *The Riemann integral is awkward when f is unbounded. In particular, we used to hack our way around by looking at whether the Riemann integral converges to some value the function approaches the unbounded point, and then "conclude" that the integral is the limit of that convergence.*

4. *Recall that the* ***Fundamental Theorem of Calculus*** *states that*

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

   We know that this works for Riemann integrals. By the first shortcoming, the problem here is that we do not fully know what are the functions that the Fundamental Theorem is true for.

5. *In PMATH450, we saw that Fourier developed the Fourier series, which is an extremely useful tool in solving* ***Differential Equations*** *using sines and cosines. However, the convergence of the Fourier series remains largely unexplained by Fourier, and we have but developed some roundabout ways of showing some convergence.*

6. *Consider the set R if Riemann integrable functions on the interval $[a, b]$. The set R has a natural metric:*

$$d(f, g) = \int_a^b |f - g|\,dx.$$

   However, the metric space (R, d) is ***not complete****. This means many of our favorite results in PMATH351 are not usable!*

7. *There are many functions that seem like they should have an integral, but turned out that they did not under Riemann integration.*

Example 1.1.2 (Pointwise Limits of Riemann Integrable Functions is not necessarily Riemann Integrable)

Let $\mathbb{Q} = \{x_n\}_{n \in \mathbb{N}}$. Then consider a sequence of functions

$$f_n(x) = \begin{cases} 1 & x \in \{x_1, \ldots, x_n\} \\ 0 & x \notin \{x_1, \ldots, x_n\} \end{cases}.$$

It is rather clear that

$$\overline{\int_a^b} f \, dx = \underline{\int_a^b} f \, dx = 0.$$

However, the pointwise limit of the f_n's, and that is

$$\lim_{n \to \infty} f_n(x) = f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases},$$

is, as mentioned in the last example, not Riemann integrable.

To address the shortcomings of the Riemann integral, Henri Lebesgue developed the **Lebesgue integral**, of which we have seen in PMATH450.

Instead of dividing the x-axis, Lebesgue decided to divide the y-axis first.

If the range of a function f is $[c, d]$, where c, d can be infinite, then we partition the interval such that

$$P = \{c = y_0 < y_1 < \ldots < y_n = d\},$$

and we define

$$E_i = \{x : f(x) \in [y_{i-1}, y_i]\}.$$

Then if A_i is the area of the "rectangle" for the i^{th} interval of $[c, d]$, we have

$$y_{i-1} \cdot \ell(E_i) \leq A_i \leq y_i \cdot \ell(E_i),$$

where $\ell(E_i)$ is the **Lebesgue measure** of the set E_i. Then if we let $\int_a^b f$ denote the Lebesgue integral of f, we would expect

$$\sum_{i=1}^{n} y_{i-1} \cdot \ell(E_i) \leq \int_a^b f \leq \sum_{i=1}^{n} y_i \cdot \ell(E_i).$$

However, to truly understand what this means, we need to understand what the Lebesgue measure is.

Furthermore, recall that in PMATH450, we saw that not all sets, in $\mathbb{R}$ for example, are measurable, and for 'good' reasons, there always exists non-measurable sets.

1.2 *Algebras and σ-Algebra of Sets*

For this course, we shall use the convention that

- the 'ambient' space X is always non-empty;
- $\mathcal{P}(X)$, the power set of X, has non-trivial elements; and
- we denote $A^C = \{x \in X : x \notin A\}$ for $A \subseteq X$.

Definition 1 (Algebra of Sets)

*Given X, a non-empty collection of subsets of X, i.e. $\emptyset \neq \mathcal{A} \subseteq \mathcal{P}(X)$, is called an **algebra of sets** of X provided that:*

1. $A_1, \ldots, A_n \in \mathcal{A} \implies \bigcup_{i=1}^n A_i \in \mathcal{A}$; *and*
2. $A \in \mathcal{A} => A^C \in \mathcal{A}$.

Proposition 1 (Properties of Algebra of Sets)

If $\mathcal{A}$ is an algebra of sets of X, then

3. $\emptyset, X \in \mathcal{A}$;
4. $A, B \in \mathcal{A} \implies A \setminus B = \{x \in X \mid x \in A \wedge x \notin B\} \in \mathcal{A}$; *and*
5. $A_1, \ldots, A_n \in \mathcal{A} \implies \bigcap_{i=1}^n A_i \in \mathcal{A}$.

Proof

3. $\mathcal{A} \neq \emptyset \implies \exists A \in \mathcal{A} \implies A^C \in \mathcal{A} \implies A \cup A^C = X \in \mathcal{A} \implies \emptyset = X^C \in \mathcal{A}$.
4. $A, B \in \mathcal{A} \implies A^C \in \mathcal{A} \implies A^C \cup B \in \mathcal{A} \implies A \setminus B = (A^C \cup B)^C \in \mathcal{A}$.
5. (**De Morgan's Law**) Notice that $(A_1 \cap A_2 \cap \ldots \cap A_n)^C = A_1^C \cup A_2^C \cup \ldots A_n^C \in \mathcal{A}$ since $A_i^C \in \mathcal{A}$. Thus the complement

$$A_1 \cap A_2 \cap \ldots \cap A_n \in \mathcal{A}. \qquad \square$$

Definition 2 (σ-Algebra of Sets)

*Given X and $\emptyset \neq \mathcal{A} \subseteq \mathcal{P}(X)$, we say that $\mathcal{A}$ is a **σ-algebra of sets** of X if it is an algebra of sets and*

$$\forall A_n \in \mathcal{A}, n \in \mathbb{N}, \quad \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}.$$

Example 1.2.1

1. $\mathcal{P}(X)$ is a σ-algebra.
2. Consider X as an infinite set. We say that a set A is **cofinite** if A^C is finite. Let

   $$\mathcal{A} := \{ A \in \mathcal{P}(X) \mid A \text{ is finite or cofinite } \}.$$

   Then $\mathcal{A}$ is an algebra of sets:

   - finite union of finite sets remains finite;
   - finite union of finite and cofinite sets remains cofinite; and
   - complement of finite sets are the cofinite sets and vice versa.

   However, $\mathcal{A}$ is **not** a σ-algebra: consider $A_n = \{2^n\} \subseteq X = \mathbb{N}$, which we then realize that

   $$\bigcup_{n \in \mathbb{N}} A_n = \text{ set of all even numbers },$$

   but the set of all even numbers is clearly not finite, and its complement, which is the set of all odd numbers, is not finite.
3. Consider X as an uncountable set. We say that a set A is **co-countable** if A^C is countable. [1] The set

   $$\mathcal{A} := \{ A \subseteq X \mid A \text{ is countable or co-countable } \}$$

   is a σ-algebra:

   - countable union of countable sets is countable;

[1] Recall that a set A is said to be countable if there is a one-to-one correspondence between elements of A and the natural numbers.

- countable union of countable and co-countable sets is co-countable; and
- complement of countable sets are co-countable and vice versa.

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