

Foreword

Usage

- Notes are presented in two columns: main notes on the left, and sidenotes on the right. Main notes will have a larger margin.
- The following is the color code for the notes:

Blue	Definitions
Red	Important points
Yellow	Points to watch out for / comment for incompleteness
Green	External definitions, theorems, etc.
Light Blue	Regular highlighting
Brown	Secondary highlighting
- The following is the color code for boxes, that begin and end with a line of the same color:

Blue	Definitions
Red	Warning
Yellow	Notes, remarks, etc.
Brown	Proofs
Magenta	Theorems, Propositions, Lemmas, etc.
- Hyperlinks are underlined in magenta. If your PDF reader supports it, you can follow the links to either be redirected to an external website, or a theorem, definition, etc., in the same document. Note that this is only reliable if you have the full set of notes as a single document, which you can find on:
https://japorized.github.io/Tex_notes

14 Lecture 14 Jun 01 2018

14.1 Isomorphism Theorems (Continued 2)

14.1.1 Isomorphism Theorems (Continued)

Note (Recall)

In First Isomorphism Theorem 38, we had that for a group homomorphism $\alpha : G \rightarrow H$ where G and H are groups,

$$G/\ker \alpha \cong \operatorname{im} \alpha$$

Now let $\alpha : G \rightarrow H$ be a group homomorphism, $K = \ker \alpha$, $\phi : G \rightarrow G/K$ be the coset map, and $\bar{\alpha}$ be as defined in the proof of First Isomorphism Theorem 38. We then have the following commutative diagram to illustrate the relationship between the three groups.

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & H \\ \downarrow \phi & \nearrow \bar{\alpha} & \\ G/K & & \end{array}$$

A natural question to ask after seeing the relationship is: Is $\bar{\alpha}\phi = \alpha$? If it is, is the definition of $\bar{\alpha}$ unique? The answer is: **YES!** on both accounts.

Proof

Let $g \in G$. Then

$$\bar{\alpha}\phi(g) = \bar{\alpha}(\phi(g)) = \bar{\alpha}(Kg) = \alpha(g)$$

Suppose $\alpha = \beta\phi$ where $\beta : G/K \rightarrow H$. Then

$$\beta(Kg) \stackrel{(1)}{=} \beta(\phi(g)) = \beta\phi(g) = \alpha(g) = \bar{\alpha}(Kg)$$

where (1) is because ϕ is surjective by Proposition 35. Therefore, we observe that $\beta = \bar{\alpha}$ for any $Kg \in G/K$. This proves that $\bar{\alpha}$ is the unique homomorphism such that $G/K \rightarrow H$ satisfying $\alpha = \bar{\alpha}\phi$. \square

With that, we have the following proposition.

Proposition 39

Let $\alpha : G \rightarrow H$ be a group homomorphism, where G and H are groups. Let $K = \ker \alpha$. Then α factors uniquely as $\alpha = \bar{\alpha}\phi$ where $\phi : G \rightarrow G/K$ is the coset map and $\bar{\alpha} : G/K \rightarrow H$ is defined by

$$\bar{\alpha}(Kg) = \alpha(g).$$

Note that ϕ is surjective and $\bar{\alpha}$ is injective.

In such a scenario, we also say that α **factors through** ϕ .¹

¹ Reference for the terminology: <https://math.stackexchange.com/questions/68941/terminology-a-homomorphism-factors>.

Example 14.1.1

Let $G = \langle g \rangle$ be a cyclic group. Consider $\alpha : \mathbb{Z} \rightarrow G$, defined as

$$\forall k \in \mathbb{Z} \quad \alpha(k) = g^k,$$

which is a group homomorphism. By definition, α is surjective. Note that

$$\ker \alpha = \{k \in \mathbb{Z} : g^k = 1\}.$$

We have, therefore, two cases to consider.

- G is an infinite group

This would imply that $\ker \alpha = \{0\}$ since only $g^0 = 1$. Then by First Isomorphism Theorem 38, we have that

$$\mathbb{Z}/\ker \alpha \cong G$$

Note that²

$$\mathbb{Z}/\ker \alpha = \{(\ker \alpha)k : k \in \mathbb{Z}\} = \{0 + k : k \in \mathbb{Z}\} = \mathbb{Z}.$$

² We are assuming that the group \mathbb{Z} here works under the operation of addition, otherwise, if we employ multiplication, then \mathbb{Z} would not be a group and α would not be a group homomorphism.

Therefore

$$\mathbb{Z} \cong G$$

- G is a finite group

Suppose that $|G| = o(g) = n \in \mathbb{N}$, which is valid by Corollary 24. Then

$$\ker \alpha = n\mathbb{Z}$$

Then by the First Isomorphism Theorem 38, we have

$$\mathbb{Z}/n\mathbb{Z} \cong G.$$

Observe that

$$\mathbb{Z}/n\mathbb{Z} = \{n\mathbb{Z} + k : k \in \mathbb{Z}\} = \mathbb{Z}_n$$

since the set in the middle is the definition of the set of integers modulo n .³ Therefore,

$$\mathbb{Z}_n \cong G$$

Therefore, we have that

$$\mathbb{Z} \cong G \text{ or } \mathbb{Z}_{o(g)} \cong G$$

³ This is why we often see texts from various authors using $\mathbb{Z}/n\mathbb{Z}$ to represent the set of integers modulo n .

Theorem 40 (Second Isomorphism Theorem)

Let H and K be the subgroups of a group G with $K \triangleleft G$. Then

- HK is a subgroup of G ;
- $K \triangleleft HK$;
- $H \cap K \triangleleft H$; and
- $HK/K \cong H/H \cap K$.

Proof

Since $K \triangleleft G$, by Lemma 29 and Proposition 30, we have that $HK = KH$ is a subgroup of G . Consequently, we have $K \triangleleft HK$, since K is clearly a subgroup of HK and $K \triangleleft G$, and so $\forall x \in HK \subseteq G$ we have that $gK = Kg$.

Consider $\alpha : H \rightarrow HK/K$, defined by⁴

⁴ Note that $Kh \in HK/K$ since $h \in H \subseteq HK$.

$$\alpha(h) = Kh$$

Now if $x = kh \in KH = HK$, then

$$Kx = K(kh) = Kh = \alpha(h).$$

Therefore, we have that α is surjective. Now by Proposition 22, observe that

$$\ker \alpha = \{h \in H : Kh = K\} = \{h \in H : h \in K\} = H \cap K.$$

Then by the First Isomorphism Theorem, we have that

$$HK/K \cong H/H \cap K.$$

Since we have that $\ker \alpha = H \cap K$ and $\ker \alpha \triangleleft H$, we have that $H \cap K \triangleleft H$. □

Theorem 41 (Third Isomorphism Theorem)

Let $K \subseteq H \subseteq G$ be groups, with $K \triangleleft G$ and $H \triangleleft G$. Then

$$H/K \triangleleft G/K \text{ and } (G/K) / (H/K) \cong G/H$$

Proof

Define $\alpha : G/K \rightarrow G/H$ by $\alpha(Kg) = Hg$ for all $g \in G$. Clearly, α is surjective. Now if $Kg = Kg_1$, for any $g, g_1 \in G$, then $gg_1^{-1} \in K \subseteq H$. Therefore, $Hg = Hg_1$. Thus α is well-defined. Now

$$\ker \alpha = \{Kg : Hg = H\} = \{Kg : g \in H\} = H/K.$$

Then

$$H/K = \ker \alpha \triangleleft G/K.$$

By the First Isomorphism Theorem, we have

$$(G/K) / (H/K)$$

as required. □

14.2 Group Actions

One reason that we are interested in the symmetric group is that they contain all finite groups.

Theorem (Cayley's Theorem)

If G is a finite group of order n , then G is isomorphic to a subgroup of S_n .
