PMATH347S18 - Groups & Rings

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1.1 Introduction

1.1.1 Numbers

The following are some of the number sets that we are already familiar with:

$$\mathbb{N} = \{1, 2, 3, ...\} \qquad \mathbb{Z} = \{.., -2, -1, 0, 1, 2, ...\}$$
$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N} \right\} \qquad \mathbb{R} = \text{ set of real numbers}$$
$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i = \sqrt{-1}\} = \text{ set of complex numbers}$$

For $n \in \mathbb{Z}$, let $\mathbb{Z}_n$ denote the set of integers modulo n, i.e.

$$\mathbb{Z}_n = \{[0], [1], ..., [n-1]\}$$

where the $[r]$, $0 \leq r \leq n-1$, are the congruence classes, i.e.

$$[r] = \{z \in \mathbb{Z} : z \equiv r \mod n\}$$

These sets share some common properties, e.g. $+$ and $\times$. Let's try to break that down to make further observation.

Note that for $R = \mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Z}_n$, R has 2 operations, i.e. addition and multiplication.

Addition If $r_1, r_2, r_3 \in R$, then

- (**closure**) $r_1 + r_2 \in R$
- (**associativity**) $r_1 + (r_2 + r_3) = (r_1 + r_2) + r_3$

Also, if $R \neq \mathbb{N}$, then $\exists 0 \in R$ (the **additive identity**) such that

$$\forall r \in R \quad r + 0 = r = 0 + r.$$

Also, $\forall r \in R$, $\exists (-r) \in R$ such that

$$r + (-r) = 0 = (-r) + r.$$

Multiplication For $r_1, r_2, r_3 \in R$, we have

- (**closure**) $r_1 r_2 \in R$
- (**associativity**) $r_1(r_2 r_3) = (r_1 r_2) r_3$

Also, $\exists 1 \in R$ (a.k.a the **mutiplicative identity**), such that

$$\forall r \in R \quad r \cdot 1 = r = 1 \cdot r.$$

Finally, for $R = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, $\forall r \in R$, $\exists r^{-1} \in R$ such that

$$r \cdot r^{-1} = 1 = r^{-1} \cdot r.$$

Note that for $R = \mathbb{Z}_n$, where $n \in \mathbb{Z}$, not all $[r] \in \mathbb{Z}_n$ have a multiplicative inverse. For example, for $[2] \in \mathbb{Z}_4$, there is no $[x] \in \mathbb{Z}_4$ such that $[2][x] = [1]$.[1]

[1] This is best proven using techniques introduced in MATH135/145.

1.1.2 Matrices

For $n \in \mathbb{N} \setminus \{1\}$, an $n \times n$ matrix over $\mathbb{R}$ [2] is an $n \times n$ array that can be expressed as follows:

[2] $\mathbb{R}$ can be replaced by $\mathbb{Q}$ or $\mathbb{C}$.

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

where for $1 \leq i, j \leq n$, $a_{ij} \in \mathbb{R}$. We denote $M_n(\mathbb{R})$ as the set of all $n \times n$ matrices over $\mathbb{R}$.

As in Section 1.1.1, we can perform **addition and multiplication** on $M_n(\mathbb{R})$.

Matrix Addition Given $A = [a_{ij}], B = [b_{ij}], C = [c_{ij}] \in M_n(\mathbb{R})$, we define matrix addition as

$$A + B = [a_{ij} + b_{ij}],$$

which immediately gives the **closure property**, since $a_{ij}+b_{ij} \in \mathbb{R}$ and hence $A + B \in M_n(\mathbb{R})$. Also, by this definition, we also immediately obtain the **associativity property**, i.e.

$$A + (B + C) = (A + B) + C.$$

We define the zero matrix as

$$0 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then we have that 0 is the **additive identity**, i.e.

$$A + 0 = A = 0 + A.$$

Finally, $\forall A \in M_n(\mathbb{R})$, $\exists(-A) \in M_n(\mathbb{R})$ (the **additive inverse**) such that

$$A + (-A) = 0 - (-A) + A.$$

Note that in this case, we also have that that the operation is **commutative**, i.e.

$$A + B = B + A.$$

Matrix Multiplication Given $A = [a_{ij}], B = [b_{ij}], C = [c_{ij}] \in M_n(\mathbb{R})$, we define the matrix multiplication as

$$AB = [d_{ij}] \text{ where } c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \in \mathbb{R}.$$

Clearly, $AB \in M_n(\mathbb{R})$, i.e. it is **closed under matrix multiplication**. Also, we have that, under such a defintion, matrix multiplication is **associative**, i.e.

$$A(BC) = (AB)C.$$

Define the identity matrix, $I \in M_n(\mathbb{R})$, as follows:

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Then we have that I is the **multiplicative identity**, since

$$AI = A = IA.$$

However, contrary to matrix addition, $\forall A \in M_n(\mathbb{R})$, it is not always true that $\exists A^{-1} \in M_n(\mathbb{R})$ such that

This is especially true if the **determinant** of A is 0.

$$AA^{-1} = I = A^{-1}A.$$

Also, we can always find some $A, B \in M_n(\mathbb{R})$ such that

$$AB \neq BA,$$

i.e. matrix multiplication is not always commutative.

THE COMMON PROPERTIES of the operations from above: **closure, associativity, and existence of an inverse**, are not unique to just addition and multiplication. We shall see in the next lecture that there are other operations where these properties will continue to hold, e.g. **permutations**.

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