

PMATH351 - Real Analysis (Class Notes)

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Chapter 1

Lecture 1: Sep 8, 2017

1.1 Logistics

Course Website: <http://www.math.uwaterloo.ca/~nspronk/math351/math351.html>

1.2 Brief Introduction to the Course

1.2.1 Set Theory (Naive, for Real Analysis)

Sets whose existence that we shall take for granted:

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

$$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}, \gcd(m, n) = 1\}$$

Definition 1.2.1 (Inclusion)

Given two sets A , B , write

$$A \subseteq B, \quad A \subset B \text{ or } B \supseteq A, \quad \text{etc.} \tag{1.1}$$

for “ B contains A ”, i.e. $\forall x \in A \implies x \in B$. We shall write

$$A \subsetneq B \text{ if } A \subset B \wedge A \neq B \tag{1.2}$$

Definition 1.2.2 (Power Set)

Let X be a set. Let

$$\mathcal{P}(X) := \{A : A \subseteq X\} \quad (1.3)$$

Note that if $X = \{1, \dots, n\}$, notice that $\mathcal{P}(X)$ has 2^n elements.

Definition 1.2.3 (Unions and Intersections)

Let $A, B \in \mathcal{P}(X)$ where X is the universe, and $\{A_i\}_{i \in I} \subseteq \mathcal{P}(X)$ where $I \neq \emptyset$.

$$\begin{aligned} A \cup B &= \{x \in X : x \in A \vee x \in B\} & \bigcup_{i \in I} A &= \{x \in X : x \in A \text{ for some } i \in I\} \\ A \cap B &= \{x \in X : x \in A \wedge x \in B\} & \bigcap_{i \in I} A &= \{x \in X : x \in A \forall i \in I\} \end{aligned}$$

If we do not have A, B in a common universe, we let the "external union" be

$$A \sqcup B = \{x : x \in A \vee x \in B\} \quad (1.4)$$

Example 1.2.1

Suppose $I \neq \emptyset$. What is the meaning of

$$\bigcup_{i \in I} A_i, \quad \bigcap_{i \in I} A_i? \quad (1.5)$$

Definition 1.2.4 (Difference Set)

If $A, B \in \mathcal{P}(X)$. Let

$$A \setminus B = \{x \in X : x \in A \wedge x \notin B\} \quad (1.6)$$

In particular

$$X \setminus B = \{x \in X : x \notin B\} \text{ (complement)} \quad (1.7)$$

Proposition 1.2.1 (De Morgan's Laws)

If X is a set, with $\{A_i\} \in \mathcal{P}(X)$, then

$$X \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in I} (X \setminus A_i), \quad X \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (X \setminus A_i) \quad (1.8)$$

The proof is straightforward and should be done in two lines.

Definition 1.2.5 (Product Sets)

Let A, B be sets.

$$A \times B = \{(a, b) : a \in A, b \in B\} \quad (\text{ordered pairs}) \quad (1.9)$$

Definition 1.2.6 (Function)

$f \subseteq A \times B$ is called a function if

$$\forall a \in A \quad \exists! b = f(a) \in B \quad (1.10)$$

so that $(a, b) \in f$.

In practice, we write $f : A \rightarrow B$ and the ordered pairs are all denoted $(a, f(a))$.

If X_1, \dots, X_n are sets, where $n \in \mathbb{N}$, then

$$X_1 \times \dots \times X_n = \prod_{j=1}^n X_j = \{(x_1, \dots, x_n) : x_j \in X_j \forall j \in \{1, \dots, n\}\} \quad (1.11)$$

is called the n -tuples of X .

IF $\{X_i\}_{i \in I, I \neq \emptyset}$, is a (or an infinite) family of sets

$$\prod_{i \in I} X_i \{ (x_i)_{i \in I} : x_i \in X_i \forall i \in I \} \quad (1.12)$$

Axiom 1.2.1 (Axiom of Choice)

Given any non empty collection of nonempty sets $\{A_i\}_{i \in I}$, we have $\prod_{i \in I} A_i \neq \emptyset$.

Remark (B. Russell)

1. $\forall n \in \mathbb{N}$, let $S_n = \{l_n, r_n\}$ be a pair of shoes. Surely, $\prod_{i \in I} S_n \neq \emptyset$.

2. $\forall n \in \mathbb{N}$, let $T_n = \{s_n, s'_n\}$ be a pair of socks. Why do we expect $\prod_{i \in I} T_n \neq \emptyset$?

Proposition 1.2.2 (AC')

The AC is equivalent to (AC') given any nonempty set A ,

$$\exists f : \mathcal{P}(A) \setminus \{\emptyset\} \rightarrow A \quad \forall B \in \mathcal{P}(A) \setminus \{\emptyset\} \quad f(B) \in B \quad (1.13)$$

Proof

$(AC) \implies (AC')$

We assume there is

$$(x_B)_{B \in \mathcal{P}(A) \setminus \{\emptyset\}} \in \prod_{B \in \mathcal{P}(A) \setminus \{\emptyset\}} B \quad (1.14)$$

(which is nonempty by assumption).

Then we simply have to let $f(B) = x_B$ for each B .

$(AC') \implies (AC)$

Given a non-empty collection of nonempty sets $\{A_i\}_{i \in I}$, let

$$A = \bigsqcup_{i \in I} A_i \quad (\text{external product}) \quad (1.15)$$

We have a choice function $f : \mathcal{P}(A) \setminus \{\emptyset\} \rightarrow A$, $f(B) \in B$ for each B . Then

$$(f(A_i))_{i \in I} \in \prod_{i \in I} A_i \quad (1.16)$$

1.3 Relations, Ordering and Zorn

Definition 1.3.1 (Relation)

Let X be a nonempty set. A relation on X is any subset

$$R \subseteq X \times X \quad (1.17)$$

We write xRy provided that $(x, y) \in R$.

Example 1.3.1

1. A function $f \subseteq X \times X$ is a relation.
2. In $\mathbb{N} \times \mathbb{N}$, consider

$$mRn \iff \exists p \in \{0\} \cup \mathbb{N} \quad n = m + p \quad (1.18)$$

We write $m \leq n \iff mRn$.

3. On \mathbb{Z} , $m \leq n \iff n - m \in \{0\} \cup \mathbb{N}$.
4. On \mathbb{Q} , $\frac{m}{n} \leq \frac{\mu}{\nu} \iff m\nu \leq \mu n$ in (\mathbb{Z}, \leq) .
5. On $\mathcal{P}(X)$, we have relations

$$A \subseteq B$$

$$A \supseteq B$$

Chapter 2

Lecture 2: Sep 11, 2017

2.1 More on Relations

Definition 2.1.1 (More on Relations)

A relation R on X is

1. **Symmetric** if $xRy \implies yRx$.
2. **Reflexive** if $\forall x \in X \ xRx$
3. **Transitive** if $xRy \wedge yRz \implies xRz$
4. **Anti-Symmetric** if $xRy \wedge yRx \implies x = y \in X$

(i), (ii) and (iii) makes up the **Equivalence Relation**. We usually use notations like \sim, \approx .

(ii), (iii) and (iv) makes up the **Partial Order** definition. We usually use notations like \leq, \geq

In Example 1.3.1, (ii), (iii), (iv) and (v) are all partial orders. In (i), f is an equivalence relation only if f is an identity function.

Definition 2.1.2 (Total Order)

A total order is a partial order where for x, y we have at least one of

$$x \leq y \quad \text{or} \quad y \leq x \tag{2.1}$$

holds.

Notice that in Example 1.3.1, (ii), (iii) and (iv) are total orders. However, (v) is not if X has at least two elements.

If \sim is an equivalence relation on X , then we denote the equivalence class by $[x] = \{y \in X : y \sim x\}$

Example 2.1.1

On $\mathbb{Z} \times \mathbb{N}$, let $(m, n) \sim (\mu, v)$ if $m\nu = \mu n$ in \mathbb{Z} . Then equivalence classes $[(m, n)]$ are elements of \mathbb{Q} . Generally,

$$\frac{m}{n} = [(m, n)] \quad (2.2)$$

2.2 Construction of the Real Numbers

We provide a sketch of Cantor's construction:

Notation: On \mathbb{Q} , define $|\frac{m}{n}| = \begin{cases} \frac{m}{n} & m > 0 \\ -\frac{m}{n} & m < 0 \end{cases}, n \in \mathbb{Z}$

We have the usual properties (triangle inequalities): for $p, q \in \mathbb{Q}$

$$|p + q| \leq |p| + |q| \quad (2.3)$$

$$||p| - |q|| \leq |p - q| \quad (2.4)$$

Let $\mathbb{Q}_+ = \{q \in \mathbb{Q} : q > 0\}$

$$X = \{(q_n) = (q_n)_{n=1}^\infty \in \mathbb{Q}^\mathbb{N} : \forall \epsilon \in \mathbb{Q}_+ \exists n_\epsilon \in \mathbb{N} \forall n, m \geq n_\epsilon |q_n - q_m| < \epsilon\}$$

(X is set of Cauchy sequences of rationals)

On X we define

$$(q_n) \sim (r_n) \text{ if } \forall \epsilon \in \mathbb{Q} \exists n_\epsilon \in \mathbb{N} |q_n - r_n| < \epsilon \text{ whenever } n \geq n_\epsilon \quad (2.5)$$

(tails become closer together)

Then \sim is an equivalence relation (verify yourselves).

We let

$$\mathbb{R} = \{[(q_n)] : (q_n) \in X\} \quad (2.6)$$

Note

\mathbb{R} is a field.

$$(q_n) \sim (s_n), (r_n) \sim (t_n) \implies (q_n + r_n) \sim (s_n + t_n), (q_n r_n) \sim (s_n t_n) \quad (2.7)$$

(Check! To check for multiplication, observe that elements of X form bounded sets in \mathbb{Q}).

$(r_n) \not\sim (0, 0, \dots) \implies r_n = 0$ for at most finitely many n

\implies define

$$t_n = \begin{cases} 1 & \text{if } r_n = 0 \\ \frac{1}{r_n} & \text{otherwise} \end{cases}$$

$$\implies (r_n)(t_n) \sim (1, 1, 1, \dots)$$

We can define multiplication, addition, etc. on \mathbb{R} and it follows that \mathbb{R} is a field.

Note (Properties)

1. \mathbb{Q} is a subfield:

$$\mathbb{Q} \hookrightarrow \mathbb{R}, \quad q \mapsto [(q, q, \dots)] \quad (2.8)$$

(eq. class of const. seq.)

2. Total order: On X let $(q_n) \leq (r_n)$ if

$$\forall \epsilon \in \mathbb{Q}_+ \exists n_\epsilon \in \mathbb{N} \forall n \geq n_\epsilon \quad q_n \leq r_n + \epsilon \quad (2.9)$$

(Eq. $(1 - \frac{1}{n}) \leq (1, 1, \dots)$)

Then $(q_n) \leq (r_n), (q_n) \sim (s_n), (r_n) \sim (t_n) \implies (s_n) \leq (t_n)$ (check)

Hence, let

$[(q_n)] \leq [(r_n)]$ if $(q_n) \leq (r_n)$.

3. Density of \mathbb{Q} : (HW 1)

If $[(q_n)] < [(r_n)]$ then there is q in \mathbb{Q} s.t.

$$[(q_n)] < [(q, q, \dots)] < [(r_n)] \quad (2.10)$$

4. Absolute value: $|[(q_n)]| = [|q_n|]$

This is the usual absolute value (check)

2.3 Dyadic representation of \mathbb{R}

Like the density of $\mathbb{Q} \in \mathbb{R}$, we can show that for $[(q_n)] \in \mathbb{R}$ there is q in \mathbb{Q} s.t. $[(q_n)] \leq [(q, q, \dots)]$ (HW 1).

Let $X = [(q_n)] \in \mathbb{R}$. Suppose $x \geq 0$. Then there is unique $m \in \mathbb{N}$ s.t.

$$[(m, m, \dots)] \leq x < [(m+1, m+1, \dots)] \quad (2.11)$$

Call $m = \lfloor x \rfloor$.

Define

$$x_1 = \begin{cases} 0 & \text{if } x - \lfloor x \rfloor < \frac{1}{2} = [(\frac{1}{2})] \\ 1 & \text{if } x - \lfloor x \rfloor \geq \frac{1}{2} \end{cases} \quad (2.12)$$

$$\vdots \quad (2.13)$$

$$x_{n+1} = \begin{cases} 0 & \text{if } x - (\lfloor x \rfloor - \sum_{k=1}^n \frac{x_k}{2^k}) < \frac{1}{2^{n+1}} \\ 1 & \text{if } x - (\lfloor x \rfloor - \sum_{k=1}^n \frac{x_k}{2^k}) \geq \frac{1}{2^{n+1}} \end{cases} \quad (2.14)$$

Then, check that

$$x \sim \left(\lfloor x \rfloor + \sum_{k=1}^{\infty} \frac{x_k}{2^k} \right)_{n=1}^{\infty} \quad (2.15)$$

Write $x = \lfloor x \rfloor . x_1 x_2 x_3 \dots$

Similarly, we have decimal (base 10) or ternary representation (base 3).

Chapter 3

Lecture 3: Sep 13, 2017

3.1 Last Time

Definition 3.1.1 (Partial Order)

A partial order is a relation \leq on X which is

- reflexive
- transitive
- anti-symmetric

We write (X, \leq) as a “partially ordered set” or a poset.

3.2 Bounds and Completeness

Definition 3.2.1 (Upper Bound, Supremum)

Let (X, \leq) be a partially ordered set (aka poset). Given $A \subset X$,

- an upper bound is any $u \in X$ s.t. $\forall x \in A, x \leq u$
- a supremum (aka least upper bound) is an upper bound s s.t. $s \leq u$ for any upper bound u .

Note

1. A supremum need not exist.

For example, in (\mathbb{Q}, \leq) ,

- \mathbb{N} is not bounded above

- $A = \{q \in \mathbb{Q} : q^2 \leq 2\}$ is bounded above (e.g. 2 is an upper bound) but admits no supremum.
- 2. If a supremum exists, then it is unique (appeal to the anti-symmetry property of \leq), so we write $s = \sup A$.

Definition 3.2.2 (Complete)

We say that (X, \leq) is complete if any set $A \subset X$ which admits an upper bound has a supremum, $\sup A$.

Example 3.2.1

1. $X \neq \emptyset$, consider $(\mathcal{P}(X), \subseteq)$. Given $A = \{A_i\}_{i \in I} \subseteq \mathcal{P}(X)$, we have $\sup A = \bigcup_{i \in I} A_i$, so $\mathcal{P}(X), \subseteq$ is complete.
2. (\mathbb{R}, \leq) is complete.

(Sketch proof) Suppose $\emptyset \neq A \subset \mathbb{R}$ is bounded above. Based on (HW1), we can find $q_0, r_0 \in \mathbb{Q} [\mathbb{Q} \hookrightarrow \mathbb{R}, q \mapsto [(q, q, \dots)]]$ s.t.

- q_0 is not an upper bound for A
- r_0 is an upper bound for A

Inductively, define for $n \in \{0\} \cup \mathbb{N}$, $(q_{n+1}, r_{n+1}) \in \mathbb{Q}^2$.

$$(q_{n+1}, r_{n+1}) = \begin{cases} (q_n, \frac{1}{2}(q_n + r_n)) & \frac{1}{2}(q_n + r_n) \text{ is an upper bound for } A \\ (\frac{1}{2}(q_n + r_n), r_n) & \text{otherwise} \end{cases} \quad (3.1)$$

Fact (check): $[(q_n)_{n=1}^\infty] = [(r_n)_{n=1}^\infty]$ and is $\sup A$.

Definition 3.2.3 (Maximum)

Further, we call $m \in A (A \subset X, (X, \leq))$ poset a maximum of A if

- $m = \sup A$
- $m \in A$

Definition 3.2.4 (Lower Bound, Infimum, Minimum)

We have symmetric definition for lower bounds, infimums (greatest lower bound) and minimums.

Note: The infimum of A is unique if it exists, denoted as $\inf A$

Proposition 3.2.1 (Infimum of a subset of a space)

If (X, \leq) is a complete partially ordered space, then any $A \subseteq X$ which is bounded below, admits an infimum.

Proof

Let $L = \{x \in X : \forall a \in A \ x \leq a\}$. Notice that $L \neq \emptyset$ (by assumption on A). Also, L is bounded above, since any element of A is an upper bound.

Then $\sup L = \inf A$.

3.3 Chains and Zorn's Lemma**Definition 3.3.1 (Chain)**

Let (X, \leq) be a poset. A chain is any subset $C \subseteq X$ s.t. (C, \leq) is totally ordered.

(Note: Strictly, we should have $(C, \leq \upharpoonright_{C \times C})$).

Definition 3.3.2 (Maximal)

We say an element $m \in X$ is maximal if we have that $\forall x \in X \ m \leq x \implies x = m$.

Axiom 3.3.1 (Zorn's Lemma)

Suppose in a poset (X, \leq) every chain $C \subseteq X$ admits an upper bound, i.e.

$$\exists u \in X \ \forall x \in C \ x \leq u \tag{3.2}$$

Then (X, \leq) admits a maximal element.

Definition 3.3.3 (Linearly Independent, Spanning, Basis)

Let V be a vector space over a field \mathbb{K} ($\mathbb{K} = \mathbb{R}$ or \mathbb{Q}). A subset $L \subseteq V$ is **linearly independent** (aka **lin. ind.**) if for each finite $\{v_1, \dots, v_n\} \subseteq L$,

$$\forall \alpha_n \in \mathbb{K} \ 0 = \sum_{i=1}^n \alpha_i v_i \implies \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

A subset $S \subset V$ is **spanning** if for each $v \in V$ there are finite $\{v_1, \dots, v_n\} \subseteq S, \{\alpha_1, \dots, \alpha_n\} \subseteq \mathbb{K}$ s.t.

$$v = \sum_{i=1}^n \alpha_i v_i$$

A **basis** is a set $B \subset V$ which is both linearly independent and spanning.

Theorem 3.3.1 (Vector space over \mathbb{K} has a basis)

A vector space V over \mathbb{K} always admits a basis.

Proof

Let $\mathcal{L} = \{L \subset V : L \text{ is linearly independent}\}$. We note that (\mathcal{L}, \subseteq) is a poset.

Furthermore, $\{\{v\} : v \in V \setminus \{0\}\} \subseteq \mathcal{L}$. So $\mathcal{L} \neq \emptyset$.

Let $\mathcal{C} = \{L_i\}_{i \in I}$ be a chain in \mathcal{L} , and consider $L = \bigcup_{i \in I} L_i$. If $\{v_1, \dots, v_n\} \subseteq L$, we have $v_k \in L_{i_k}$ for some $k \in [0, n]$, and since \mathcal{C} is a chain, we may relate so $L_{i_1} \subseteq L_{i_2} \subseteq \dots \subseteq L_{i_k}$. Thus $\{v_1, \dots, v_n\} \subseteq L_{i_n}$ and is lin. ind. It follows L is lin. ind. Hence, [Axiom 3.3.1](#) tells us that \mathcal{L} admits a maximal element B .

WTP B is spanning. Suppose B is not spanning. Then there is $v_o \in V$ which cannot be written as a linear combination of finitely many vectors from B . Consider

$$0 = \alpha_0 v_0 + \sum_{i=1}^n \alpha_i v_i \quad (3.3)$$

for $\{v_1, \dots, v_n\} \subseteq B$, and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. If we can have $\alpha_n \neq 0$, then

$$v_0 = \sum_{i=1}^n \left(-\frac{\alpha_i}{\alpha_n} v_i \right) \quad (3.4)$$

which contradicts our assumption on v_o . Hence $\alpha_n = 0$, and thus $0 = \sum_{i=1}^n \alpha_i v_i$, so $\alpha_1 = \dots = \alpha_n = 0$, as well. Hence $B \cup \{v_o\} \in \mathcal{L}$. But $B \subseteq B \cup \{v_o\}$, contradicting maximality.

Remark

An easy modification of the proof shows that any $L = \mathcal{L}$ is a subset of a basis.

Chapter 4

Lecture 4: Sep 15, 2017

4.1 Logistics

Office Hours

- today: 1430 - 1520
- Wed, next week: 1430 - 1630

4.2 Cardinal arithmetic

Definition 4.2.1 (Injection, Surjection, Bijection)

Given nonempty sets X, Y , a function $f : X \rightarrow Y$ is called a(n)

- **injection** $x_1 \neq x_2 \in X \implies f(x_1) \neq f(x_2)$
- **surjection** $\forall y \in Y \exists x \in X f(x) = y$
- **bijection** if it is both an injection and a surjection (aka invertible)

Of course, if $f : X \rightarrow Y$ is a bijection then we can define $f^{-1} : Y \rightarrow X$ by $f^{-1}(f(x)) = x$.

We write $X \sim Y$ if there exists a bijection $f : X \rightarrow Y$.

Sometimes, we write

$$X \underset{f}{\sim} Y$$

Note (\sim as an equivalence relation)

- (reflexivity) $X \underset{id}{\sim} X$ ($id : X \rightarrow X$ is the identity function)

- (symmetry) $X \underset{f}{\sim} Y \implies Y \underset{f^{-1}}{\sim} X$
- (transitivity) $X \underset{f}{\sim} Y \wedge Y \underset{g}{\sim} Z \implies X \underset{gf}{\sim} Z$

Hence \sim is an equivalence relation on any given family of sets. We let $|X|$ denote the equivalence class. We call this cardinality of X .

Note: $|\emptyset| = 0$, $|\{1, \dots, n\}| = n \in \mathbb{N}$

Example 4.2.1

1.

$$\mathbb{N} \sim \mathbb{Z} \quad \because f(n) = \begin{cases} \frac{n}{2} & n \text{ is even} \\ \frac{1}{n}(1-n) & n \text{ is odd} \end{cases}$$

2.

$$\mathbb{R} \underset{f}{\sim} (-1, 1) \quad \because f(x) = \frac{x}{|x| + 1}$$

Exercise: exhibit f^{-1}

Answer: $f^{-1}(x) = \frac{x}{1-|x|}$

3. $a < b \in \mathbb{R}$ $(0, 1) \underset{g}{\sim} (a, b)$, $g(x) = a + x(b - a)$

Note (Notation)

$$\aleph_0 = |\mathbb{N}| \text{ ("aleph-naught")} \quad c = |\mathbb{R}| \text{ ("continuum")}$$

Note (Arithmetic)

Let A, B be sets.

$$\begin{aligned} |A| + |B| &= |A \sqcup B| \\ |A||B| &= |A \times B| \\ |A|^{|B|} &= |A^B| \quad (B \neq \emptyset, A^B = \{f : B \rightarrow A \mid f \text{ is a function}\}) \end{aligned}$$

Note (Properties)

- (commutativity)

$$\begin{aligned} |A| + |B| &= |B| + |A| \\ |A||B| &= |B||A| \end{aligned}$$

- (distributivity)

$$\begin{aligned} |A|(|B| + |C|) &= |A||B| + |A||C| \\ (A \times (B \sqcup C) &\sim (A \times B) \sqcup (A \times C)) \end{aligned}$$

- (exponential laws)

$$\begin{aligned} (B \neq \emptyset \neq C) \\ (1) \quad |A|^{|B|+|C|} &= |A|^{|B|}|A|^{|C|} \quad (2) \quad |A|^{|B||C|} = \left(|A|^{|B|}\right)^{|C|} \end{aligned}$$

$$\begin{aligned} (1) \quad (A^{B \sqcup C} &\sim A^B \times A^C \text{ via } \phi \mapsto (\phi|_B, \phi|_C)) \\ (2) \quad A^{B \times C} &\sim (A^B)^C \text{ via } \phi \mapsto (\phi(b, \cdot) : C \rightarrow A)_{b \in B} \end{aligned}$$

Definition 4.2.2 (Precedence)

For sets A, B , define

$$A \leq B \text{ if there is an injection } f : A \rightarrow B$$

We sometimes write the above as $A \underset{f}{\leq} B$.

- (reflexivity) $A \leq A$
- (transitivity) $A \leq B, B \leq C \implies A \leq C$

We are one property short of making \leq as an order relation.

Note

It seems reasonable to write $|A| \leq |B|$, in this case, our question is: Is \leq in cardinal numbers anti-symmetric?

Theorem 4.2.1 (Cantor-Bernstein-Schröder)

If, for non-empty set A, B , we have

$$A \leq B \wedge B \leq A \implies A \sim B \quad (4.1)$$

i.e.

$$|A| \leq |B| \wedge |B| \leq |A| \implies |A| = |B| \quad (4.2)$$

Proof

Our assumption is that we have injections

$$A \underset{\phi}{\leq} B, \quad B \underset{\psi}{\leq} A \quad (4.3)$$

To avoid triviality, let us suppose that neither ϕ or ψ is surjective. Thus

$$\phi(A) \subsetneq B \quad \psi \circ \phi(A) \subsetneq \psi(B) \subsetneq A \quad (4.4)$$

Let $A_0 = A$, $A_1 = \psi(B)$, $A_2 = \psi \circ \phi(A)$ and we inductively define

$$A_{n+1} = g(A_n) \text{ where } g = \psi \circ \phi \quad (4.5)$$

Then $A_2 \subsetneq A_1 \subsetneq A_0$, so by applying injection g ,

$$\begin{aligned} A_4 &\subsetneq A_3 \subsetneq A_2 \\ &\vdots \\ A_{n+1} &\subsetneq A_n \subsetneq A_{n-1} \end{aligned}$$

Hence, we may decompose

$$\begin{aligned} A &= A_0 = (A_0 \setminus A_1) \cup A_1 \\ &= (A_0 \setminus A_1) \cup (A_1 \setminus A_2) \cup A_2 \\ &\vdots \\ &= \bigcup_{n=1}^{\infty} (A_{n-1} \setminus A_n) \cup A_{\infty} \end{aligned}$$

where $A_{\infty} = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=2}^{\infty} A_n$, we likewise observe

$$A_i = \bigcup_{n=2}^{\infty} (A_{n-1} \setminus A_n) \cup A_{\infty}$$

Using definitions of the sets A_n ($n \geq 2$) we have

$$g(A_{n-1} \setminus A_n) = A_{n+1} \setminus A_{n+2}$$

Define

$$h : A_0 \rightarrow A_1 \quad h(x) = \begin{cases} g(x) & x \in A_{n-1} \setminus A_n \text{ is odd} \\ x & \text{otherwise} \end{cases} \quad (4.6)$$

Then h is a bijection.

Thus $A = A_0 \underset{h}{\sim} A_1 - \phi(B)$, $B \underset{\phi}{\sim} \psi(B)$ so we conclude that $A \sim B$. □

Example 4.2.2

1. Let $a < b \in \mathbb{R}$. Then

$$[a, b] \leq \mathbb{R} \quad \text{obvious}$$

$$\mathbb{R} \sim (-1, 1) \sim (0, 1) \sim (a, b) \leq [a, b]$$

i.e. $[a, b] \leq \mathbb{R}$ and $\mathbb{R} \leq [a, b]$ so $\mathbb{R} \sim [a, b]$.

Chapter 5

Lecture 5: Sep 18, 2017

5.1 Continuing CBS with examples

Example 5.1.1

2. $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$, i.e. $|\mathcal{P}(\mathbb{N})| = c$

$$\mathcal{P}(\mathbb{N}) \sim \{0, 1\}^{\mathbb{N}} \text{ via } A \mapsto \chi(A) \quad (5.1)$$

where

$$\chi_A(n) = \begin{cases} 1 & n \in A \\ 0 & n \notin A \end{cases} \quad (5.2)$$

is the “characteristic indicator”.

$$\{0, 1\}^{\mathbb{N}} \leq [0, 1) \text{ via } (x_k)_{k=1}^{\infty} \mapsto \sum_{k=1}^{\infty} \frac{x_k}{3^k} = 0.x_1x_2x_3\dots \text{ is the ternary rep'n} \quad (5.3)$$

which is injective.

Claim $[0, 1) \leq \{0, 1\}^{\mathbb{N}}$, $0.x_1x_2x_3\dots = \sum_{k=1}^{\infty} \frac{x_k}{2^k} \mapsto (x_k)_{k=1}^{\infty}$ which is the binary rep'n.
Note that this representation doesn't allow $0.1111\dots = 1$ (see [Lecture 2](#)).

$$\mathcal{P}(\mathbb{N}) \sim \{0, 1\}^{\mathbb{N}} \leq [0, 1) \leq \{0, 1\}^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N})$$

Thus by [Theorem 4.2.1](#),

$$|\mathcal{P}(\mathbb{N})| = |[0, 1)| = c = |\mathbb{R}| \quad (5.4)$$

$$3. \mathbb{N} \sim \mathbb{Q} \sim \mathbb{N}^2$$

- $\mathbb{N} \leq \mathbb{Q}$ (obvious)

- $\mathbb{Q} \leq \mathbb{Z} \times \mathbb{N}$, which we pick $\frac{m}{n} \mapsto (m, n)$ with $\gcd(m, n) = 1$ where $m \in \mathbb{Z}, n \in \mathbb{N}$.
- $\mathbb{Z} \times \mathbb{N} \sim \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ as $\mathbb{Z} \sim \mathbb{N}$
- $\mathbb{N}^2 \sim \mathbb{N}$ via $(m, n) \mapsto 2^m 3^n$

Therefore

$$\mathbb{N} \leq \mathbb{Q} \leq \mathbb{Z} \times \mathbb{N} \sim \mathbb{N}^2 \leq \mathbb{N} \quad (5.5)$$

Thus by [Theorem 4.2.1](#), $\mathbb{N} \sim \mathbb{Q} \sim \mathbb{N}^2$.

Note (Notation)

We say that a set A is

- **countable** if $A \leq \mathbb{N}$, i.e. $|A| \leq \aleph_0$
- **denumerable** if $A \sim \mathbb{N}$, i.e. $|A| = \aleph_0$

5.2 Comparison Theorem

Proposition 5.2.1 (Surjectivity)

Suppose X and Y are non-empty sets and there is a surjection $g : X \rightarrow Y$. Then $Y \leq X$.

Proof

Let $f : \mathcal{P}(X) \setminus \{\emptyset\} \rightarrow X$ be a choice function (by [Axiom 1.2.1 AC](#)). For each $y \in Y$, we have $g^{-1}(\{y\}) = \{x \in X : g(x) = y\} \neq \emptyset$, as g is surjective. Define $h : Y \rightarrow X$ be given by $h(y) = f(g^{-1}(\{y\}))$ and h is injective, as if $y_1 \neq y_2$, $\{y_1\} \cap \{y_2\} = \emptyset$, so we see that

$$g^{-1}(\{y_1\}) \cap g^{-1}(\{y_2\}) = \emptyset \quad (5.6)$$

too. □

Theorem 5.2.1 (Comparison Theorem)

Let X and Y be sets. Then either $X \leq Y$ or $Y \leq X$.

Proof

If $X = \emptyset$ then $X \leq Y$; likewise if $Y = \emptyset$. Hence, assume $X \neq \emptyset \neq Y$. Let

$$\Delta = \{(A, f) : A \in \mathcal{P}(X) \setminus \{\emptyset\}, f \in Y^A \text{ is an injection}\} \quad (5.7)$$

We observe that $\Delta \neq \emptyset$. If $x \in X, y \in Y$, then $(\{x\}, x \mapsto y) \in \Delta$.

On Δ let

$$(A, f) \leq (B, g) \iff \begin{matrix} A \subseteq B \subseteq X \\ g|_A = f \end{matrix} \quad (5.8)$$

Notice that \leq is reflexive, anti-symmetric, and transitive. Thus \leq is a partial order on Δ .

Let $\Gamma = \{(A_i, f_i)\}_{i \in I}$ be a chain in (Δ, \leq) . We let

$$A = \bigcup_{i \in I} A_i \quad (5.9)$$

and $f \in Y^A$ be given by $f(x) = f_i(x)$ provided $x \in A_i$.

Notice that f is well-defined. Say $x \in A_i$ and $x \in A_j$, then since Γ is a chain, without loss of generality, $A_i \subseteq A_j$, and $f_j|_{A_i} = f_i$.

Furthermore, if $x_1 \neq x_2 \in A$, then $x_1 \in A_{i_1}, x_2 \in A_{i_2}$, and we may suppose $A_{i_1} \subseteq A_{i_2}$. Then $f(x_1) = f_{i_1}(x_1) = f_{i_2}(x_1) \neq f_{i_2}(x_2) = f(x_2)$.

So f is an injection. Thus $(A, f) \in \Delta$ and is an upper bound for Γ .

Thus there is a maximal element $(M, g) \in \Delta$, by [Axiom 3.3.1](#) Zorn's Lemma.

1. Case 1: $M = X$. Then $X = M \leq_g Y$.
2. Case 2: $M \subsetneq X$. We wish to see that g is surjective.

Suppose not, i.e. $\exists y_0 \in Y \setminus g(M)$. Since $M \subsetneq X$, $\exists x_0 \in X \setminus M$. Define $h : M \cup \{x_0\} \rightarrow Y$ by

$$h(x) = \begin{cases} g(x) & x \in M \\ y_0 & x = x_0 \end{cases} \quad (5.10)$$

which is injective.

Then $(M \cup \{x_0\}, h) \in \Delta$, and $(M, g) \leq (M \cup \{x_0\}, h)$, contradicting the maximality of $(M, g) \in \Delta$. Thus g is surjective as desired.

Therefore, $Y \leq_{g^{-1}} X$. □

Proposition 5.2.2 (Alternative Definitions of an Infinite Set)

Let A be a set. Then TFAE:

1. $n \leq |A|$ for all $n \in \mathbb{N}$.
2. $\aleph_0 \leq |A|$, i.e. A is infinite
3. $\exists B \subsetneq A$ s.t. $|B| = |A|$.
4. $1 + |A| = |A|$ (Hilbert hotel)
5. $\aleph_0 + |A| = |A|$

Chapter 6

Lecture 6: Sep 20, 2017

6.1 Continuing ordinal arithmetic

Proof

1. $1 \implies 2$

We have that for each $n \in \mathbb{N}$ there is an injection $\phi_n : \{1, \dots, n\} \rightarrow A$. Inductively, define $f : \mathbb{N} \rightarrow A$ by

$$f(1) = \phi_1(1)$$

\vdots

$$f(n+1) = \phi_{n+1}(k) \quad \text{where } k = \min\{j \in \{1, \dots, n+1\} : \phi_{n+1}(j) \notin \{f(1), \dots, f(n)\}\}$$

The f is injective by construction, i.e. $\mathbb{N} \underset{f}{\preceq} A$ or $\aleph_0 \leq |A|$

2. $2 \implies 3$

We have $\mathbb{N} \underset{f}{\preceq} A$. Let $B = A \setminus \{f(1)\}$.

Define $g : A \rightarrow B$ by

$$g(x) = \begin{cases} f(n+1) & x = f(n), n \in \mathbb{N} \\ x & \text{otherwise} \end{cases} \tag{6.1}$$

Then $A \underset{g}{\sim} B$, i.e. $|A| = |B|$.

3. $3 \implies 4$

We suppose that there is $x_0 \in A \setminus B$ and $B \sim A$. Thus,

$$A \sim B \leq B \cup \{x_0\} \leq A \quad (6.2)$$

Then by **Theorem 4.2.1**, $A \sim B$ and furthermore $A \sim B \cup \{x_0\} \sim A \sqcup \{1\}$, i.e. $|A| = |A| + 1$.

4. $4 \implies 5$

We have $\{1\} \sqcup A \sim A$. Then $\phi(A) \subsetneq A$. Thus $\phi \circ \phi(A) \subsetneq \phi(A) \subsetneq A$, and by induction

$$\underbrace{\phi^{\circ n}}_{\phi \text{ composed with itself } n \text{ times}}(A) \subsetneq \phi^{\circ(n-1)}(A) \subsetneq \dots \subsetneq A \quad (6.3)$$

Hence $|A| \geq |A \setminus \phi^{\circ n}(A)| \geq n$ (at each stage above, we gain at least one point).

5. $2 \implies 5$

We have $\mathbb{N} \leq_f A$. Let

$$g : \mathbb{N} \sqcup A \rightarrow A, \quad g(x) = \begin{cases} f(2n) & x = n, n \in \mathbb{N} \\ f(2n+1) & x = f(n) \in A, n \in \mathbb{N} \\ x & \text{otherwise} \end{cases} \quad (6.4)$$

6. $5 \implies 2$

$$\aleph_0 \leq \aleph_0 + |A| \underset{\text{by assumption}}{=} |A|.$$

Note

Any set satisfying 1 to 5 of the above is called infinite.

Corollary 6.1.1 (A set is either finite or denumerable)

If $A \in \mathcal{P}(\mathbb{N})$, then either A is finite or denumerable.

Proof

Either $n \leq |A|$ for all $n \in \mathbb{N}$, or $|A| < n$ for some $n \in \mathbb{N}$.

Theorem 6.1.1 (Cantor)

For any set X

$$|X| \leq |\mathcal{P}(X)|, \text{ i.e. } X \leq \mathcal{P}(X) \wedge X \not\sim \mathcal{P}(X) \quad (6.5)$$

Proof

If $X = \emptyset$, $0 = |\emptyset| \leq 1 = |\{\emptyset\}|$.

If $X \neq \emptyset$, then $x \mapsto \{x\} : X \rightarrow \mathcal{P}(X)$ shows $X \leq \mathcal{P}(X)$.

Now suppose $X \neq \emptyset$, $f : X \rightarrow \mathcal{P}(X)$. We will show that f cannot be surjective. Let

$$E = \{x \in X : x \notin f(x)\} \quad (6.6)$$

i.e. E is a set that is not in the range of f .

If we had $E \subseteq f(X)$, i.e. $E = f(x)$ for some $x \in X$, then either

- $x \in E$, i.e. $x \notin f(x)$, which means that $E \neq f(x)$, or
- $x \notin E = f(x)$, so $x \in E$.

These contradictions show that $E \not\subseteq f(X)$.

Hence there is no surjection $f : X \rightarrow \mathcal{P}(X)$.

Example 6.1.1

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = c$$

Theorem 6.1.2 (Cantor's Continuum Hypothesis)

This is no set A such that

$$\aleph_0 < |A| < c \quad (6.7)$$

Remark

This theorem has recently been proven (about a month ago from Sep 20, 2017). This theorem is independent of ordinary set theory.

Theorem 6.1.3 (Generalized Continuum Hypothesis)

Given an infinite set C , there is no set A such that

$$|C| < |A| < |\mathcal{P}(C)| \quad (6.8)$$

Theorem 6.1.4 (Cantor's Paradox)

There is no "set" of all sets.

Suppose there was a universal set \mathcal{U} , i.e. any set $A \subseteq \mathcal{U}$. But then,

$$|\mathcal{U}| < |\mathcal{P}(\mathcal{U})|, \text{ so } \mathcal{P}(\mathcal{U}) \not\subseteq \mathcal{U} \quad (6.9)$$

so \mathcal{U} cannot exist.

Axiom 6.1.1 (Well-Ordering)

Given a non-empty set X , a **well-order** is a partial order on X such that any $\emptyset \neq A \subseteq X$ admits a minimal element, i.e.

$$\exists m_A \in A \forall a \in A \ m_A \leq a \quad (6.10)$$

Remark

Well-order VS total order: $x, y \in X$ consider $A = \{x, y\}$.

Example 6.1.2

1. (\mathbb{N}, \leq) is well-ordered (principle of mathematical induction).
2. \mathbb{N}^2 . Let us consider two well-orders.

(pyramid) $(m, n) \leq (\mu, \nu) \iff$

$$\begin{cases} \text{either } m + n < \mu + \nu \\ m + n = \mu + \nu \text{ and } m \leq \mu \end{cases} \quad (6.11)$$

(lexicographic) $(m, n) \leq_l (\mu, \nu) \iff$

$$\begin{cases} \text{either } m < \mu \text{ or} \\ m = \mu \text{ and } n \leq \nu \end{cases} \quad (6.12)$$

Notice that $(m, n) \leq_l (\mu, \nu) \iff 2m - \frac{1}{n} \leq 2\mu - \frac{2}{\nu} \in (\mathbb{Q}, \leq)$

Chapter 7

Lecture 7: Sep 22, 2017

7.1 Metric Spaces

Note

We can use \mathbb{R} in any reasonable manner.

Definition 7.1.1 (Metric and Metric Space)

Let X be a nonempty set. A metric $d : X \times X \rightarrow \mathbb{R}$ is a function which satisfies, for $x, y, z \in X$

- **(non-negativity)** $d(x, y) \geq 0$
- **(non-degeneracy)** $d(x, y) = 0 \iff x = y$
- **(symmetry)** $d(x, y) = d(y, x)$
- **(triangle inequality)** $d(x, z) \leq d(x, y) + d(y, z)$

We often call the pair (X, d) a metric space.

Example 7.1.1

1. On \mathbb{R} , $d(x, y) = |x - y|$
2. Let $X \neq \emptyset$ any set. Define the “discrete” metric

$$d : X \times X \rightarrow \{0, 1\} \subseteq \mathbb{R}, \quad d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases} \quad (7.1)$$

Note that non-degeneracy and symmetry are obvious. The triangle inequality is sat-

isfied since

$$\begin{aligned} \text{Case: } x \neq y \neq z \neq x \\ 1 = d(x, z) \leq 2 = d(x, y) + d(y, z) \end{aligned}$$

3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be strictly increasing. Let

$$d_f : \mathbb{R}^2 \rightarrow [0, \infty) \quad d_f(x, y) = |f(x) - f(y)| \quad (7.2)$$

Exercise: check for its properties.

$$\text{E.g. } f(x) = \frac{x}{|x|+1}.$$

4. (French railroad metric) Suppose we have a set $X \neq \emptyset$, and a function $f : X \rightarrow [0, \infty)$ which satisfies $f^{-1}(\{0\}) = \{p_0\}$. Notice that $f(x) > 0$ if $x \in X \setminus \{p_0\}$.

$$d_f : X \times X \rightarrow [0, \infty) \quad d_f(x, y) = \begin{cases} 0 & x = y \\ f(x) + f(y) & x \neq y \end{cases} \quad (7.3)$$

Easy exercise: This is a metric.

Definition 7.1.2 (Norm, Normed Vector Space)

Let V be a vector space over \mathbb{R} . A **norm** is a function $\|\cdot\| : V \rightarrow \mathbb{R}$ which satisfies, for $x, y \in V$, $\alpha \in \mathbb{R}$

1. (**non-negativity**) $\|x\| \geq 0$
2. (**non-degeneracy**) $\|x\| = 0 \iff x = 0$
3. (**$\|\cdot\|$ -homogeneity**) $\|\alpha x\| = |\alpha| \|x\|$
4. (**subadditivity**) $\|x + y\| \leq \|x\| + \|y\|$

We call the pair $(V, \|\cdot\|)$ a **normed vector space**.

Note

If $(V, \|\cdot\|)$ is a normed vector space, then

$$d : V \times V \rightarrow [0, \infty) \quad d(x, y) = \|x - y\| \quad (7.4)$$

is always a metric on V . Everything is easy to check; subadditivity of $\|\cdot\| \implies$ triangle inequality of d .

Example 7.1.2

1. $(\mathbb{R}, |\cdot|)$ is a normed vector space.

2. On \mathbb{R}^n , for $x = (x_1, \dots, x_n)$

$$\|x\|_2 = \sqrt{x_1^2 + \dots + x_n^2} \quad (7.5)$$

This is the Euclidean norm.

Consider, also

$$\begin{aligned} \|x\|_1 &= |x_1| + \dots + |x_n| \\ \|x\|_\infty &= \max\{|x_1|, \dots, |x_n|\} \end{aligned}$$

Note

non-degeneracy and $|\cdot|$ -homogeneity are obvious for $\|\cdot\|_1$, $\|\cdot\|_\infty$

Let us consider subadditivity

$$\begin{aligned} \|x + y\|_1 &= |x_1 + y_1| + \dots + |x_n + y_n| \\ &\leq |x_1| + |y_1| + \dots + |x_n| + |y_n| \\ &= |x_1| + \dots + |x_n| + |y_1| + \dots + |y_n| \\ &= \|x\|_1 + \|y\|_1 \end{aligned}$$

$$\begin{aligned} \|x + y\|_\infty &= \max\{|x_i + y_i| : i = 1, \dots, n\} \\ &= \max\{|x_i| + |y_i| : i = 1, \dots, n\} \\ &= \max\{|x_i| + |y_j| : i, j = 1, \dots, n\} \\ &= \max\{|x_i| : i = 1, \dots, n\} + \max\{|y_j| : j = 1, \dots, n\} \\ &= \|x\|_\infty + \|y\|_\infty \end{aligned}$$

Now for $1 < p < \infty$ consider

$$x^p = \begin{cases} e^{p \log x} & x > 0 \\ 0 & x = 0 \end{cases} \quad (7.6)$$

$$\|x\|_p = (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}}$$

Remark (Cauchy-Bunyakovsky-Schwartz)

$$|x \cdot y| \leq \|x\|_2 \|y\|_2$$

Lemma 7.1.1 ($\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$)

Let $\alpha, \beta \leq 0 \in \mathbb{R}$, $1 < p < \infty$ and q is chosen such that $\frac{1}{p} + \frac{1}{q} = 1$ (i.e. $q = \frac{p}{p-1}$) then

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q} \quad (7.7)$$

with the equality when $\alpha^p = \beta^q$.

Proof

Consider the graph of $y = x^{p-1}$ (assume $p \geq 2$). Then

$$\begin{aligned}\alpha\beta &\leq \int_0^\alpha x^{p-1} dx + \int_0^b y^{q-1} dy \\ &= \frac{\alpha^p}{p} + \frac{\beta^q}{q}\end{aligned}$$

Equality holds only if $\beta = \alpha^{p-1} \implies \beta^{\frac{1}{p-1}} = \alpha \implies \beta^q = \alpha^p$

Theorem 7.1.1 (Holder's Inequality)

Let $x, y \in \mathbb{R}^n$, $1 < p < \infty$ and q be so $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left| \sum_{j=1}^n x_j y_j \right| \leq \sum_{j=1}^n |x_j| |y_j| \leq \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |y_j|^q \right)^{\frac{1}{q}} = \|x\|_p \|y\|_q \quad (7.8)$$

Chapter 8

Lecture 8: Sep 25, 2017

8.1 Logistics

Expect assignment 2 to be up tonight!

8.2 Continuing Normed Vector Space

Proof (Holder's Inequality)

$\|x\|_p \|y\|_q = 0 \implies (x = 0 \vee y = 0) \wedge$ the inequality is trivial. Let us assume $\|x\|_p \|y\|_q \neq 0$.
For $j = 1, \dots, n$

$$\alpha_j = \frac{|x_j|}{\|x\|_p}, \quad \beta_j = \frac{|y_j|}{\|y\|_q}$$

Then

$$\begin{aligned} \frac{1}{\|x\|_p \|y\|_q} \sum_{j=1}^n |x_j| |y_j| &= \sum_{j=1}^n \alpha_j \beta_j \stackrel{(1)}{\leq} \sum_{j=1}^n \left(\frac{\alpha_j^p}{p} + \frac{\beta_j^q}{q} \right) \\ &= \frac{1}{p} \sum_{j=1}^n \alpha_j^p + \frac{1}{q} \sum_{j=1}^n \beta_j^q \\ &= \frac{1}{p \|x\|_p^p} \sum_{j=1}^n |x_j|^p + \frac{1}{q \|y\|_q^q} \sum_{j=1}^n |y_j|^q \\ &= \frac{1}{p \|x\|_p^p} \|x\|_p^p + \frac{1}{q \|y\|_q^q} \|y\|_q^q = \frac{1}{p} + \frac{1}{q} \stackrel{(2)}{=} 1 \end{aligned}$$

where (1) is by [Lemma 7.1.1](#) and (2) is by choice of q .

Hence, we multiply by $\|x\|_p \|y\|_q$ and see that

$$\sum_{j=1}^n |x_j| |y_j| \leq \|x\|_p \|y\|_q \quad (8.1)$$

□

Theorem 8.2.1 (Minkowski's Inequality)

Let $x, y \in \mathbb{R}^n$ and $1 < p < \infty$. Then

$$\|x + y\| \leq \|x\|_p + \|y\|_p \quad (8.2)$$

Proof

If $x + y = 0$, this is trivial, hence suppose $x + y \neq 0$. Compute

$$\begin{aligned} \|x + y\|_p^p &= \sum_{j=1}^n |x_j + y_j|^p = \sum_{j=1}^n |x_j + y_j| |x_j + y_j|^{p-1} \\ &= \sum_{j=1}^n (|x_j| + |y_j|) |x_j + y_j|^{p-1} \\ &= \sum_{j=1}^n |x_j| |x_j + y_j|^{p-1} + \sum_{j=1}^n |y_j| |x_j + y_j|^{p-1} \\ &\leq \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}} \\ &\quad + \left(\sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}} \\ &= (\|x\|_p + \|y\|_p) \left(\sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}} \end{aligned}$$

We have $\frac{1}{p} + \frac{1}{q} = 1 \implies \frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p} \implies p = q(p-1)$, and thus

$$\begin{aligned} \|x + y\|_p^p &\leq (\|x\|_p + \|y\|_p) \left(\sum_{j=1}^n |x_j + y_j|^p \right)^{\frac{1}{q}} \\ &= (\|x\|_p + \|y\|_p) \|x + y\|_p^{\frac{p}{q}} \end{aligned}$$

Now divide $\|x + y\|_p^{\frac{p}{q}} \neq 0$, we get

$$\|x + y\|_p = \|x + y\|_p^{p - \frac{p}{q}} \leq \|x\|_p + \|y\|_p \quad (\text{since } p - \frac{p}{q} = p(1 - \frac{1}{q}) = \frac{p}{p} = 1) \quad (8.3)$$

Corollary 8.2.1 ($\|\cdot\|_p$ is a norm)

Given $1 < p < \infty$, $\|\cdot\|_p$ is a norm on \mathbb{R}^n .

Proof

Clearly, $\|\cdot\|_p$ is non-negative and non-degenerate. If $\alpha \in \mathbb{R}, x \in \mathbb{R}^n$ then

$$\begin{aligned} \|\alpha x\|_p &= \left(\sum_{j=1}^n |\alpha x_j|_p^p \right)^{\frac{1}{p}} = \left(\sum_{j=1}^n |\alpha|^p |x_j|^p \right)^{\frac{1}{p}} \\ &= |\alpha| \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} = |\alpha| \|x\|_p \end{aligned}$$

Finally, subadditivity is provided by [Theorem 8.2.1](#).

8.3 ℓ_p -spaces

Consider $\mathbb{R}^n = \{x = (x_k)_{k=1}^\infty : x_k \in \mathbb{R}\}$ which is a \mathbb{R} -vector space:

$$(x_k)_{k=1}^\infty + (y_k)_{k=1}^\infty = (x_k + y_k)_{k=1}^\infty, \quad \alpha(x_k)_{k=1}^\infty = 1^\infty = (\alpha x_k)_{k=1}^\infty = 1^\infty \quad (8.4)$$

We let, for $1 \leq p < \infty$,

$$\bullet \ell_p = \{x = (x_k)_{k=1}^\infty \in \mathbb{R}^\mathbb{N} : \sum_{k=1}^\infty |x_k|^p = \lim_{n \rightarrow \infty} \sum_{k=1}^n |x_k|^p < \infty\}$$

and

$$\ell_\infty = \{x = (x_k)_{k=1}^\infty : \sup_{k \in \mathbb{N}} |x_k| < \infty\}$$

On ℓ_p we define

$$\|x\|_p = \begin{cases} \left(\sum_{k=1}^\infty |x_k|^p \right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \sup_{k \in \mathbb{N}} |x_k| & p = \infty \end{cases} \quad (8.5)$$

Theorem 8.3.1 (ℓ_p is a \mathbb{R} -subspace)

Let $1 \leq p < \infty$. Then ℓ_p is a \mathbb{R} -subspace of $\mathbb{R}^\mathbb{N}$ and $\|\cdot\|_p$ is a norm.

Proof

We shall prove these statements together. Suppose that $x, y \in \ell_p$. Then

$$\begin{aligned}
\|x + y\|_p &= \left(\sum_{k=1}^{\infty} |x_k + y_k|^p \right)^{\frac{1}{p}} \quad (\text{may be } \infty, \infty^{\frac{1}{p}} = \infty) \\
&= \left(\lim_{n \rightarrow \infty} \sum_{k=1}^n |x_k + y_k|^p \right)^{\frac{1}{p}} \\
&= \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n |x_k + y_k|^p \right)^{\frac{1}{p}} \quad \left(\begin{array}{l} x \mapsto x^{\frac{1}{p}} \text{ is cts on } [0, \infty) \\ x \rightarrow \infty \implies x^{\frac{1}{p}} \rightarrow \infty \end{array} \right) \\
&\leq \lim_{n \rightarrow \infty} \left[\left(\sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^n |y_k|^p \right)^{\frac{1}{p}} \right] \quad \text{by Theorem 8.2.1 on each } n \\
&= \left(\lim_{n \rightarrow \infty} \sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} + \left(\lim_{n \rightarrow \infty} \sum_{k=1}^n |y_k|^p \right)^{\frac{1}{p}} \quad \text{cty again} \\
&= \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |y_k|^p \right)^{\frac{1}{p}} = \|x\|_p + \|y\|_p < \infty
\end{aligned}$$

Thus $x + y \in \ell_p$, and we get subadditivity of $\|\cdot\|_p$.

We note that non-negativity and non-degeneracy of $\|\cdot\|_p$ are obvious properties. Likewise, the $|\cdot|$ -homogeneity is straightforward. \square

Theorem 8.3.2 ($(\ell_{\infty}, \|\cdot\|_{\infty})$ is a normed vector space)

$(\ell_{\infty}, \|\cdot\|_{\infty})$ is a normed vector space.

Proof

$x, y \in \ell_{\infty} \implies$

$$\begin{aligned}
\|x + y\|_{\infty} &= \sup_{k \in \mathbb{N}} |x_k + y_k| \leq \sup_{k \in \mathbb{N}} (|x_k| + |y_k|) \\
&\leq \sup_{j, k \in \mathbb{N}} (|x_j| + |y_k|) \\
&= \sup_{j \in \mathbb{N}} |x_j| + \sup_{k \in \mathbb{N}} |y_k| = \|x\|_{\infty} + \|y\|_{\infty}
\end{aligned}$$

Other properties are easy (exercise). \square