UW W17 PMATH333: Definitions and Theorems

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Appendix A

Zermelo-Fraenkel Set Theory and the Axiom of Choice

A.1 Introduction

Example A.1.1 (Russel's Paradox)
Let X be the set of all sets, and let $S = \{A \in X | A \notin A\}$.
Note for example that $Z \notin Z \implies Z \in S$, and $X \in X \implies X \notin S$.
Thus we have $S \in S \iff S \notin S$.

To ensure that mathematical paradoxes (like the above) can no longer arise, mathematicians considered the following questions, and with these questions, rough answers are provided:

1. What exactly is an allowable mathematical object?
   A: Every mathematical object is a mathematical set, and a mathematical set can be constructed using certain rules, for e.g. the now widely accepted Zermelo-Fraenkel Set Theory and the Axiom of Choice. While the Axiom of Choice is still highly criticized even today (e.g. the highly controversial Banach-Tarski Paradox), the Zermelo-Fraenkel Set Theory is widely welcomed, but not without critics. We shall call the Zermelo-Fraenkel Set Theory and the Axiom of Choice as the ZFC Axioms of Set Theory.

2. What exactly is an allowable mathematical statement?
   A: Every mathematical statement can be expressed in a formal symbolic language, which uses symbols rather than words from any spoken language.

3. What exactly is allowable in a mathematical proof?
   A: Every mathematical proof is a finite list of ordered pairs $(\mathscr{S}_n, \mathscr{F}_n)$ (which we can think of as proven theorems), where each $\mathscr{S}_n$ is a finite set of formulas (called the *premises*) and each $\mathscr{F}_n$ is a single formula (called the *conclusion*), which that each pair $(\mathscr{S}_n, \mathscr{F}_n)$ can be obtained from previous pairs $(\mathscr{S}_i, \mathscr{F}_i)$ with $i < n$, using certain proof rules.

In the remainder of this appendix, we shall look more into the first 2 questions.

Definition A.1.1 (Mathematical Symbols)
We allow ourselves to use only the following symbols from the following symbol set:

$\neg$	*not*
$\wedge$	*and*
$\vee$	*or*
$\implies$	*implies*
$\iff$	*if and only if*
$=$	*equals*
$\in$	*is an element of*
$\forall$	*for all*
$\exists$	*there exists*
$()$ $\{\}$ $[]$	*parenthesis*

along with some variable symbols such as $x, y, z, u, v, w, \ldots$ *or* $x_1, x_2, x_3, \ldots$

Definition A.1.2 (Formula)
A formula (in the formal symbolic language of first order set theory) is a non-empty finite string of symbols, from the above list, which can be obtained using finitely many applications following the three rules below:

1. *If x and y are variable symbols, then each of the following strings are formulas.*

$$x = y, \quad x \in y$$

2. *If F and G are formulas then each of the following strings are formulas.*

$$\neg F, \quad (F \wedge G), \quad (F \vee G), \quad (F \implies G), \quad (F \iff G)$$

3. *If x is a variable symbol and F is a formula then each of the following is a formula.*

$$\forall x \in F, \quad \exists x \in F$$

Definition A.1.3 (Free or Bounded Variable)
Let x be a variable symbol and let F be a formula. For each occurrence of the symbol x, which does not immediately follow a quantifier, in the formula F, we define whether the occurrence of x is free or bound inductively as follows:

1. *If F is a formula of one of the forms $y = z$ or $y \in z$, where y and z are variable symbols (possibly equal to x), then every occurence of x in F is free, and no occurrence is bound.*

2. *If F is a formula of one of the forms $\neg H, (H \wedge G), (H \vee G), (H \implies G), (H \iff G)$, where G and H are formulas, then each occurence of the symbol x is either an occurrence in the formula G or an occurrence in the formula H, and each free (respectively, bound) occurence of x in G remains free (respectively, bound) in F, and similarly for each free (or bound) occurrence of x in G. In other words, wlog, if x is bounded in G, then it is bounded in F, and vice versa.*

3. *If F is a formula of one of the forms $\forall y \in G$ or $\exists y \in G$, where G is a formula and y is a variable symbol. If y is different from x, then each free (or bound) occurrence of x in G remains free (or bound) in the formula G, and if $y = x$ then every free occurrence of x in G becomes bound in F, and every bound occurrence of x in G remains bound in F.*

Definition A.1.4 (Is Bound By and Binds)
When a quantifier symbol occurs in a given formula F, and is followed by the variable symbol x and then by the formula G, any free occurence of x in G will become bound in the given formula F (by the 3rd definition above). We shall say that the occurrence of x is bound by (that occurrence of) the quantifier symbol, or that (the occurrence of) the quantifier symbol binds the occurence of x.

Definition A.1.5 (Free Variable, Statement, Statement About)
*A **free variable** in a formula F is any variable symbol that has at least one free occurence in F. A formula F with no free variables is called a **statement**. When the free variables in F all lie in the set $\{x_1, x_2, ..., x_n\}$, we shall write F as $F(x_1, x_2, ..., x_n)$ and we shall say that F is a **statement about** the variables $x_1, x_2, ..., x_n$.*

Definition A.1.6 (Unique Existence)
When F(x) is a statement about x, we sometimes write F(y) as a short form for the formula $\forall x(x = y \implies F(x))$, and we sometimes write

$$\exists! y \quad F(y)$$

which we read as "there exists a unique y such that F(y)", as a short form for the formula

$$(\exists y \quad F(y) \wedge \forall z \quad F(z)) \implies z = y))$$

which is, in turn, for the formula

$$\exists y\Big(\forall x\Big(x = y \implies F(x)\Big) \wedge \forall z\Big(\forall x(x = z \implies F(x)) \implies z = y\Big)\Big)$$

Remark (The ZFC Axioms of Set Theory (informal))
Every mathematical set can be constructed using specific rules, which we shall use the ZFC Axioms of Set Theory. Below is a list of the ZFC Axioms, stated informally.

- *Empty Set Axiom: There exists an empty set $\emptyset$ with no elements.*
- *Extension Axiom: 2 sets are equal if and only if they have the same elements.*
- *Separation Axiom: If u is a set and F(x) is a statement about x, $\{x \in u : F(x)\}$ is a set.*
- *Pair Axiom: If u and v are sets then u, v is a set.*
- *Union Axiom: If u is a set then $\cup u = \bigcup_{v \in u} v$ is a set.*
- *Power Set Axiom: If u is a set then $\mathcal{P}(u) = \{v : v \in u\}$ is a set.*
- *Axiom of Infinity: If we define the natural numbers to be the sets $0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\}, 3 = \{0, 1, 2\}$ and so on, then $\mathbb{N} = \{0, 1, 2, 3, ...\}$ is a set.*
- *Replacement Axiom: If u is a ste and F(x, y) is a statement about x and y with the property that $\forall x\, \exists! y\, F(x, y)$ then $\{y : \exists x \in u\, F(x, y)\}$ is a set.*
- *Axiom of Choice: Given a set u of non-empty pairwise disjoint sets, there exists a set which contains exactly one element from each of the sets in u, i.e.*

We may write the Axiom of Choice symbolically as:

$$\forall i \in \mathbb{N} \quad u_i \neq \emptyset \quad \forall j \neq i \in \mathbb{N} \quad u_i \cap u_j = \emptyset$$
$$\exists v = \{x_1, x_2, x_3, ... : \forall k \in \mathbb{N}, x_k \in u_k\}$$

Definition A.1.7 (Empty Set Axiom)
The Empty Set Axiom is the formula

$$\exists u\, \forall x \quad \neg x \in u$$

Definition A.1.8 (Extension Axiom)
The Extension Axiom is the formula

$$\forall u\, \forall v\, \Big(u = v \iff \forall x\, (x \in u \iff x \in v)\Big)$$

Theorem A.1.1
The empty set is unique.

Definition A.1.9 ($\emptyset$)
We denote the unique empty set by $\emptyset$.

Definition A.1.10 (Subset)
Given sets u and v, we say that u is a ***subset*** *of v, and write $u \subseteq v$, when $\forall x(x \in u \implies x \in v)$*

Definition A.1.11 (Separation Axiom)
For any statement $F(x)$ about x, the following formula is an axiom.

$$\forall u \, \exists v \, \forall x \Big(x \in v \iff (x \in u \wedge F(x)) \Big)$$

More generally, for any statement $F(x, u_1, u_2, ..., u_n)$ about $x, u_1, u_2, ..., u_n$ where $n \geq 0$, the following formula is an axiom.

$$\forall u \, \forall u_1 \ldots \forall u_n \, \exists v \, \forall x \Big(x \in v \iff (x \in i \wedge F(x, u_1, ..., u_n)) \Big)$$

Any axiom of this form is called the Separation Axiom.

Note
It is important to realize that a Separation Axiom only allows us to construct a subset of a given set u. So, e.g., we cannot use the Separation Axiom to show that the collection $S = \{x : \neg x \in x\}$, which is used to formulate Russel's Paradox, is a set.

Definition A.1.12 (Pair Axiom)
The Pair Axiom is the formula

$$\forall u \, \forall v \, \exists w \, \forall x \Big(x \in w \iff (x = u \vee x = v) \Big)$$

Definition A.1.13 (Union Axiom)
The Union Axiom is the formula

$$\forall u \, \exists w \, \forall x \Big(x \in w \iff \exists v (v \in u \wedge x \in v) \Big)$$

Definition A.1.14 (Union)
Given a set u, by the Union Axiom there exists a set w with the property that $\forall x \Big(x \in w \iff \exists v (v \in u \wedge x \in v) \Big)$, and by the Extension Axiom, this set w is unique. We call

*the set w the **union** of the elements in u, and denote it by*

$$\cup u = \bigcup_{v \in u} v.$$

Given two sets u and v, we define the union of u and v to be the set

$$u \cup v := \bigcup \{u, v\}.$$

Given three sets u, v, and w, note that $\{z\} = \{z,\ z\}$ is a set and so $\{x, y, z\} = \{x, y\} \cup \{z\}$ is also a set. More generally, if $u_1, u_2, ..., u_n$ are sets then $\{u_1, u_2, ..., u_n\}$ is a set and we define the union of the sets $u_1, u_2, ..., u_n$ to be

$$u_1 \cup u_2 \cup \ldots \cup u_n = \bigcup_{k=1}^{n} u_k = \bigcup \{u_1, u_2, ..., u_n\}$$