PMATH451 — Measure and Integration

Class notes for Fall 2019

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List of Procedures



Assignment problems is introduced in class as we go, so we have the special environment homework for these in this note.

E Lecture 1 Sep 04th, 2019

1.1 Motivation for the Study of Measures

Recall Riemann integration.

■ Definition (Riemann Integration)

Let $f:[a,b] \to \mathbb{R}$ be a bounded function. We call

$$P = \{a = x_0 < x_1 < \ldots < x_n = b\} \subseteq [a, b]$$

a partition of [a,b], and

$$\Delta x_i = x_i - x_{i-1}$$

as the length of the i^{th} interval for i = 1, ..., n.

Let

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

be the supremum of f on the ith interval, and

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$

be the infimum of f on the ith interval. We define the Riemann upper sum as

$$U(f,P) = \sum_{i} M_i \Delta x_i,$$

and the Riemann lower sum as

$$L(f,P) = \sum_{i} m_i \Delta x_i.$$

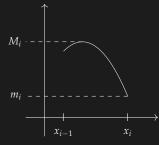


Figure 1.1: Idea of Riemann integration

We define the Riemann upper integral as

$$\int_{a}^{b} f \, dx = \inf_{P} U(f, P)$$

and the Riemann lower integral as

$$\int_{a}^{b} f \, dx = \sup L(f, P).$$

We say that f is Riemann integrable if

$$\int_a^b f \, dx = \int_a^b f \, dx,$$

and we write the integral of f as

$$\int_a^b f \, dx = \overline{\int_a^b} f \, dx = \int_a^b f \, dx.$$

As hyped up as one does earlier in university about Riemann integration, there are functions that are not Riemann integrable!

Example 1.1.1

Consider a function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q} \end{cases}.$$

Then

$$\overline{\int_a^b} f dx = 1$$
 and $\int_a^b f dx = 0$.

Thus f is not Riemann integrable.

66 Note 1.1.1 (Shortcomings of the Riemann integral)

1. We cannot characterize functions that are Riemann integrable, i.e. we do not have a list of characteristics that we can check against to see if a function is Riemann integrable.

This remained an open problem in the earlier 1920s.

- 2. The Riemann integral behaves badly when it comes to pointwise limits of functions. The next example shall illustrate this.
- 3. The Riemann integral is awkward when f is unbounded. In particular, we used to hack our way around by looking at whether the Riemann integral converges to some value the function approaches the unbounded point, and then "conclude" that the integral is the limit of that convergence.
- 4. Recall that the Fundamental Theorem of Calculus states that

$$\frac{d}{dx}\int_{a}^{x}f(t)\,dt=f(x).$$

We know that this works for Riemann integrals. By the first shortcoming, the problem here is that we do not fully know what are the functions that the Fundamental Theorem is true for.

- 5. In PMATH450, we saw that Fourier developed the Fourier series, which is an extremely useful tool in solving Differential Equations using sines and cosines. However, the convergence of the Fourier series remains largely unexplained by Fourier, and we have but developed some roundabout ways of showing some convergence.
- 6. Consider the set R if Riemann integrable functions on the interval [a, b]. The set R has a natural metric:

$$d(f,g) = \int_a^b |f - g| \, dx.$$

However, the metric space (R, d) is not complete. This means many of our favorite results in PMATH351 are not usable!

7. There are many functions that seem like they should have an integral, but turned out that they did not under Riemann integration.

Example 1.1.2 (Pointwise Limits of Riemann Integrable Functions is not necessarily Riemann Integrable)

Let $\mathbb{Q} = \{x_n\}_{n \in \mathbb{N}}$. Then consider a sequence of functions

$$f_n(x) = \begin{cases} 1 & x \in \{x_1, \dots, x_n\} \\ 0 & x \notin \{x_1, \dots, x_n\} \end{cases}.$$

It is rather clear that

$$\int_{a}^{b} f \, dx = \int_{a}^{b} f \, dx = 0.$$

However, the pointwise limit of the f_n 's, and that is

$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is, as mentioned in the last example, not Riemann integrable.

To address the shortcomings of the Riemann integral, Henri Lebesgue developed the **Lebesgue integral**, of which we have seen in PMATH450.

Instead of dividing the *x*-axis, Lebesgue decided to divide the *y*-axis first.

If the range of a function f is [c,d], where c,d can be infinite, then we partition the interval such that

$$P = \{c = y_0 < y_1 < \ldots < y_n = d\},$$

and we define

$$E_i = \{x : f(x) \in [y_{i-1}, y_i]\}.$$

Then if A_i is the area of the "rectangle" for the ith interval of [c,d], we have

$$y_{i-1} \cdot \ell(E_i) \leq A_i \leq y_i \cdot \ell(E_i),$$

where $\ell(E_i)$ is the **Lebesgue measure** of the set E_i . Then if we let $\int_a^b f$ denote the Lebesgue integral of f, we would expect

$$\sum_{i=1}^{n} y_{i-1} \cdot \ell(E_i) \le \int_a^b f \le \sum_{i=1}^{n} y_i \cdot \ell(E_i).$$

However, to truly understand what this means, we need to under-

stand what the Lebesgue measure is.

Furthermore, recall that in PMATH450, we saw that not all sets, in R for example, are measurable, and for 'good' reasons, there always exists non-measurable sets.

1.2 Algebras and σ -Algebra of Sets

■ Definition 1 (Algebra of Sets)

Given X, a non-empty collection of subsets of X, i.e. $\emptyset \neq A \subseteq \mathcal{P}(X)$, is called an algebra of sets of X provided that:

1.
$$A_1, \ldots, A_n \in \mathcal{A} \implies \bigcup_{i=1}^n A_i \in \mathcal{A}$$
; and

2.
$$A \in \mathcal{A} \implies A^C \in \mathcal{A}$$
.

♦ Proposition 1 (Properties of Algebra of Sets)

If A is an algebra of sets of X, then

3.
$$\emptyset$$
, $X \in \mathcal{A}$;

4.
$$A, B \in \mathcal{A} \implies A \setminus B = \{x \in X \mid x \in A \land x \notin B\} \in \mathcal{A} ; and$$

5.
$$A_1, \ldots, A_n \in \mathcal{A} \implies \bigcap_{i=1}^n A_i \in \mathcal{A}$$
.

Proof

3.
$$A \neq \emptyset \implies \exists A \in A \implies A^{C} \in A \implies A \cup A^{C} = X \in A \implies \emptyset = X^{C} \in A$$
.

4.
$$A, B \in \mathcal{A} \implies A^C \in \mathcal{A} \implies A^C \cup B \in \mathcal{A} \implies A \setminus B = (A^C \cup B)^C \in \mathcal{A}$$
.

5. (**De Morgan's Law**) Notice that
$$(A_1 \cap A_2 \cap ... \cap A_n)^C = A_1^C \cup A_2^C \cup ... A_n^C \in \mathcal{A}$$
 since $A_i^C \in \mathcal{A}$. Thus the complement

$$A_1 \cap A_2 \cap \ldots \cap A_n \in \mathcal{A}$$
.

For this course, we shall use the convention that

- the 'ambient' space *X* is always non-empty;
- $\mathcal{P}(X)$, the power set of X, has nontrivial elements; and
- we denote $A^C = \{x \in X : x \notin A\}$ for $A \subseteq X$.

\blacksquare Definition 2 (σ -Algebra of Sets)

Given X and $\emptyset \neq A \subseteq \mathcal{P}(X)$, we say that A is a σ -algebra of sets of X if it is an algebra of sets and

$$\forall A_n \in \mathcal{A}, n \in \mathbb{N}, \quad \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}.$$

Example 1.2.1

- 1. $\mathcal{P}(X)$ is a σ -algebra.
- 2. Consider X as an infinite set. We say that a set A is **cofinite** if A^C is finite. Let

$$A := \{ A \in \mathcal{P}(X) \mid A \text{ is finite or cofinite } \}.$$

Then A is an algebra of sets:

- finite union of finite sets remains finite;
- finite union of finite and cofinite sets remains cofinite; and
- complement of finite sets are the cofinite sets and vice versa.

However, A is **not** a σ -algebra: consider $A_n = \{2^n\} \subseteq X = \mathbb{N}$, which we then realize that

$$\bigcup_{n\in\mathbb{N}}A_n= ext{ set of all even numbers },$$

but the set of all even numbers is clearly not finite, and its complement, which is the set of all odd numbers, is not finite.

3. Consider X as an uncountable set. We say that a set A is co-countable if A^C is countable. ¹ The set

$$A := \{ A \subseteq X \mid A \text{ is countable or co-countable } \}$$

is a σ -algebra:

• countable union of countable sets is countable;

¹ Recall that a set *A* is said to be countable if there is a one-to-one correspondence between elements of *A* and the natural numbers.

- countable union of countable and co-countable sets is cocountable; and
- complement of countable sets are co-countable and vice versa.



E Lecture 2 Sep 06th 2019

2.1 Algebra and σ -algebra of Sets (Continued)

We've seen some examples of σ -algebras. Let's now look at some other important properties of σ -algebras.

\phi Proposition 2 (Closure of σ -algebras under Countable Intersection)

Let X be a set, A a σ -algebra on X. If $A_n \in A$ for each $n \in \mathbb{N}$, then $\bigcap_n A_n \in A$.

This follows rather similarly to • Proposition 1 where we used De Morgan's Law.

Proof

We observe that

$$A_n \in \mathcal{A} \implies A_n^C \in \mathcal{A}$$

$$\implies \bigcup_n A_n^C \in \mathcal{A}$$

$$\implies \bigcap_n A_n = \left(\bigcup_n A_n^C\right)^C \in \mathcal{A}.$$

Let $A_{\alpha} \subseteq \mathcal{P}(X)$, where α is from some index set. We denote

$$\bigcap_{\alpha} \mathcal{A}_{\alpha} = \{ A \subseteq X : A \in \mathcal{A}_{\alpha}, \ \forall \alpha \}.$$

lack Proposition 3 (Existence of the 'Smallest' σ -algebra on a Set)

Let X be a set and $\{A_{\alpha}\}_{\alpha}$ as a collection of σ -algebras on X. Then $\bigcap_{\alpha} A_{\alpha}$ is a σ -algebra.

Proof

$$A \in \bigcap_{\alpha} \mathcal{A}_{\alpha} \implies \forall \alpha, A \in \mathcal{A}_{\alpha}$$

$$\implies \forall \alpha, A^{C} \in \mathcal{A}_{\alpha}$$

$$\implies A^{C} \in \bigcap_{\alpha} \mathcal{A}_{\alpha}$$

and

$$\forall n \in \mathbb{N}, A_n \in \bigcap_{\alpha} \mathcal{A}_{\alpha} \implies \forall n \in \mathbb{N}, \forall \alpha, A_n \in \mathcal{A}_{\alpha}$$

$$\implies \forall \alpha, \bigcup_n A_n \in \mathcal{A}_{\alpha}$$

$$\implies \bigcup_n A_n \in \bigcap_{\alpha} \mathcal{A}_{\alpha}.$$

Due to the above proposition, the following definition is well-defined.

\blacksquare Definition 3 (Generator of a σ -algebra)

Let X be a set, and $\xi \subseteq \mathcal{P}(X)$ has some non-trivial set(s). Consider all σ -algebras \mathcal{A}_{α} with the property that $\xi \subseteq \mathcal{A}_{\alpha}$. Then we say that $\bigcap_{\alpha} \mathcal{A}_{\alpha}$ is

the σ -algebra generated by ξ , and we denote this generated σ -algebra as

$$\mathfrak{M}(\xi) = \bigcap_{\alpha} \mathcal{A}_{\alpha}.$$

Remark 2.1.1

- 1. It is clear from the definition that if A is a σ -algebra on X and $\xi \subseteq A$, then $\mathfrak{M}(\xi) \subseteq \mathcal{A}$.
- 2. We often say that $\mathfrak{M}(\xi)$ is the "smallest σ -algebra containing ξ ".

The following is an example of such a σ -algebra.

\blacksquare Definition 4 (Borel σ -algebra)

Let X be a metric space (or topological space). The σ -algebra generated by the open subsets of X is called the Borel σ -algebra, of which we denote by $\mathfrak{B}(X)$.

Remark 2.1.2 (Some sets in $\mathfrak{B}(X)$)

Given an arbitrary metric space (or topological space) X. It is often hard to firmly grasp what kind of sets are in the Borel σ -algebra $\mathfrak{B}(X)$. The following are some examples that are in $\mathfrak{B}(X)$.

- 1. Let $\{\mathcal{O}_n\}_{n\in\mathbb{N}}$ denote a countable collection of open sets. By \bullet Proposition 2, $\bigcap_n \mathcal{O}_n \in \mathfrak{B}(X)$. We call these countable union of open sets as G_δ sets.
- 2. Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ denote a countable collection of closed sets. By \bullet Proposition 2, $\bigcup_n \mathcal{F}_n \in \mathfrak{B}(X)$. We call these countable intersection of closed sets as F_{σ} sets.
- 3. Let $\{H_n\}$ be a countable collection of G_δ sets. Then $\bigcup_n H_n \in \mathfrak{B}(X)$. *These are called the* $G_{\delta\sigma}$ *sets.*
- 4. Let $\{K_n\}$ be a countable collection of F_{σ} sets. Then $\bigcap_n K_n \in \mathfrak{B}(X)$. *These are called the* $F_{\sigma\delta}$ *sets.*

We can continue constructing the $G_{\delta\sigma...}$ and $F_{\sigma\delta...}$ similarly, and all these sets belong to the Borel σ -algebra $\mathfrak{B}(X)$.

lack Proposition 4 (Other Formulations of the Borel σ -algebra (aka Proposition 1.2))

The following collection of sets are all equal:

- 1. $\mathfrak{B}_1 = \mathfrak{B}(\mathbb{R})$;
- 2. $\mathfrak{B}_2 = \sigma$ -algebra generated by open intervals (e.g. (a,b));
- 3. $\mathfrak{B}_3 = \sigma$ -algebra generated by closed intervals (e.g. [a,b]);
- 4. $\mathfrak{B}_4 = \sigma$ -algebra generated by half-open intervals (e.g. (a,b]);
- 5. $\mathfrak{B}_5 = \sigma$ -algebra generated by $(-\infty, a)$ and (b, ∞) ; and
- 6. $\mathfrak{B}_6 = \sigma$ -algebra generated by $(-\infty, a]$ and $[b, \infty)$.

As commented before, it is often hard knowing that is in a Borel σ -algebra, and what is not, despite knowing what its generator is. However, when talking about containments, this is a fairly straightforward discussion thanks to its closure under countable unions and \bullet Proposition 2. We simply need to talk about the generators.

Proof

 $\mathfrak{B}_2 \subseteq \mathfrak{B}_1$ Given an arbitrary generator (a,b) in \mathfrak{B}_2 , we know that (a,b) is an open set, and clearly $(a,b) \subseteq \mathbb{R}$. Thus $(a,b) \in \mathfrak{B}_1$, so $\mathfrak{B}_2 \subseteq \mathfrak{B}_1$.

 $\mathfrak{B}_3 \subseteq \mathfrak{B}_2$ Given an arbitrary generator [a,b] of \mathfrak{B}_2 , we have

$$[a,b] = \bigcap_{n} \left(a - \frac{1}{n}, b + \frac{1}{n}\right) \in \mathfrak{B}_2.$$

Thus $\mathfrak{B}_3 \subseteq \mathfrak{B}_2$.

 $\mathfrak{B}_4 \subseteq \mathfrak{B}_3$ Given an arbitrary generator (a,b] of \mathfrak{B}_4 ,

$$(a,b] = \bigcup_{n} \left[a + \frac{1}{n}, b \right] \in \mathfrak{B}_3.$$

Thus $\mathfrak{B}_4 \subseteq \mathfrak{B}_3$.

 $\mathfrak{B}_5 \subseteq \mathfrak{B}_4$ Given an arbitrary generator $(-\infty, a)$ for \mathfrak{B}_5 ,

$$(-\infty,a)=\bigcup_{n}\left(-\infty,a-\frac{1}{n}\right)\in\mathfrak{B}_{4}.$$

On the other hand, for (b, ∞) in \mathfrak{B}_5 ,

$$(b,\infty)=\bigcup_n(b,n)\in\mathfrak{B}_4.$$

 $\mathfrak{B}_6 \subseteq \mathfrak{B}_5$ We have that

$$(-\infty,a]=\bigcap_{n}\left(-\infty,a+\frac{1}{n}\right)\in\mathfrak{B}_{5}$$

and

$$[b,\infty)=\bigcap_n\left(b-\frac{1}{n},\infty\right)\in\mathfrak{B}_5.$$

 $\mathfrak{B}_1 \subseteq \mathfrak{B}_6$ Let $c < d \in \mathbb{R}$. Notice that

$$(-\infty,d]\cap[c,\infty)=[c,d]\in\mathfrak{B}_6.$$

Furthermore,

$$(c,d) = \bigcup_{n} \left[c + \frac{1}{n}, d - \frac{1}{n} \right] \in \mathfrak{B}_6.$$

Recall that given an open set $\mathcal{O} \subseteq \mathbb{R}$, we have

$$\mathcal{O}=\bigcup\{(c,d)\subseteq\mathcal{O}:c,d\in\mathbb{Q}\},$$

which shows that \mathcal{O} is a countable union of open sets (with rational endpoints). It follows that $\mathcal{O} \in \mathfrak{B}_6$ and so $\mathfrak{B}_1 \subseteq \mathfrak{B}_6$.

Exercise 2.1.1

Show that $\mathfrak{B}(\mathbb{R}^2)$ is generated by open rectangles $(a,b) \times (c,d)$.

■ Definition 5 (Infinitely Often)

Given $E_n \subseteq X$ for $n \in \mathbb{N}$, we say that $x \in E_n$ infinitely often (i.o.) if

$${n:x\in E_n}$$

is an infinite set. We typically let

$$A := \{x \in X : x \in E_n \text{ i.o. } \}$$

be the set of x's that are in the E_n 's infinitely often.

■ Definition 6 (Almost always)

Given $E_n \subseteq X$ for $n \in \mathbb{N}$, we say that $x \in E_n$ almost always (a.a.) if

$${n:x \notin E_n}$$

is a finite set. We typically let

$$B := \{x \in X : x \in E_n \ a.a. \}$$

be the set of x's that are in the E_n 's almost always.

♦ Homework (Homework 1)

Let X be a set, A a σ -algebra on X, and $E_n \in A$ for $n \in \mathbb{N}$. Prove that

$$A := \{x \in X : x \in E_n \text{ i.o. } \}$$

and

$$B := \{x \in X : x \in E_n \ a.a. \}$$

are both in A.

■ Definition 7 (Characteristic Function)

Let $E \subseteq X$ *. We call the function*

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

the characteristic function of E.

🗫 Homework (Homework 2 – A review on limsup and liminf)

Let $E_n \subseteq X$ for $n \in \mathbb{N}$, and

$$A := \{x \in X : x \in E_n \text{ i.o. }\}$$
$$B := \{x \in X : x \in E_n \text{ a.a. }\}.$$

Show that

$$\chi_A(x) = \limsup_n \chi_{E_n}(x)$$
 $\chi_B(x) = \liminf_n \chi_{E_n}(x).$

Remark 2.1.3

Due to the above result, some people write

$$A = \limsup E_n$$
$$B = \liminf E_n.$$

2.2 Measures

E Definition 8 (Measure)

Let X be a set and A a σ -algebra of subsets of X. A function $\mu: \mathcal{A} \to$ $[0,\infty]$ is called a measure on A provided that:

- 1. $\mu(\emptyset) = 0$; and
- 2. if $E_n \in A$ for each $n \in \mathbb{N}$, and $\{E_n\}$ is disjoint, we have

$$\mu\left(\bigcup_n E_n\right) = \sum_n \mu(E_n).$$

ELecture 3 Sep 9th 2019

3.1 *Measures* (Continued)

■ Definition 9 (Measure Space)

Let X be a set, \mathfrak{M} a σ -algebra of subsets of X and $\mu: \mathfrak{M} \to [0, \infty]$. We call the 3-tuple (X, \mathfrak{M}, μ) a measure space.

Remark 3.1.1

If $\mu(X) = 1$, we also call (X, \mathfrak{M}, μ) a probability space.

Example 3.1.1

1. (Counting Measure) Let X be a set and $\mathfrak{M} = \mathcal{P}(X)$. For $E \in \mathfrak{M}$, define

$$\mu(E) = \begin{cases} |E| & E \text{ is finite} \\ \infty & \text{otherwise} \end{cases}.$$

We verify that μ is indeed a measure:

- (a) We have that $\mu(\emptyset) = |\emptyset| = 0$.
- (b) Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}$ be a pairwise disjoint set. Notice that if any of the sets are infinite, say E_{N_0} is infinite, then

$$\mu(E_{N_0}) = \infty = |E_{N_0}|.$$

Since $\bigcup_{n=1}^{\infty} E_n$ is infinite in this case, we have

$$\mu\left(\bigcup_{n=1}^{\infty}E_n\right)=\infty=\left|\bigcup_{n=1}^{\infty}E_n\right|.$$

On the other hand, if all the sets are finite, then since the E_n 's are disjoint, we have

$$\mu\left(\bigcup_{n=1}^{\infty}E_n\right)=\left|\bigcup_{n=1}^{\infty}E_n\right|=\sum_{n=1}^{\infty}|E_n|=\sum_{n=1}^{\infty}\mu(E_n).$$

We call μ a counting measure.

2. Let *X* be an uncountable set. Recall that in Example 1.2.1, we showed that

$$\mathfrak{M} := \{ A \subseteq X \mid A \text{ is countable or co-countable } \}$$

is a σ -algebra. There are many measures that we can define on this σ -algebra. For instance,

$$\nu(E) = \begin{cases} 0 & E \text{ is countable} \\ 1 & E \text{ is uncountable} \end{cases},$$

and

$$\delta(E) = \begin{cases} 0 & E \text{ is countable} \\ \infty & E \text{ is uncountable} \end{cases}.$$

Verifying that both ν and δ are indeed measures shall be left to the reader as a straightforward exercise.

3. Let's make a non-example. Let X be an infinite set, and $\mathfrak{M} = \mathcal{P}(X)$. Define

$$\mu(E) = \begin{cases} 0 & E \text{ is finite} \\ \infty & E \text{ is infinite} \end{cases}.$$

Consider $X = \mathbb{N}$ and a sequence of sets with singletons,

$$E_n = \{2n+1\}, \quad \text{for } n \in \mathbb{N}.$$

Clearly,

$$\bigcup_{n=1}^{\infty} E_n = \text{ set of all odd numbers },$$

and clearly

$$\mu\left(\bigcup_{n=1}^{\infty}E_n\right)=\infty.$$

However, notice that

$$\mu(E_n) = 0$$
 for each $n \in \mathbb{N}$.

Since each of the E_n 's are pairwise disjoint, we should have

$$\infty = \mu \left(\bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \mu(E_n) = 0,$$

which is impossible. Thus μ is **not** a measure.

Remark 3.1.2 (Finite additivity)

Given a finite set of pairwise disjoint sets $\{E_n\}_{n=1}^N\subseteq\mathfrak{M}$ for some σ -algebra \mathfrak{M} of some set X. By the definition of a σ -algebra, we may set $E_n=\emptyset$ for n > N. Then

$$\mu\left(\bigcup_{n=1}^{N} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{N} \mu(E_n).$$

We call this the finite additivity of a measure.

\blacksquare Definition 10 (Finitivity, σ -finitivity, and Semi-finitivity of a Measure)

Let (X, \mathfrak{M}, μ) be a measure space.

- 1. We say that μ is finite if $\mu(E) < \infty$ for every $E \in \mathfrak{M}$.
- 2. If $X = \bigcup_{n=1}^{\infty} X_n$ with $X_n \in \mathfrak{M}$, we say that μ is σ -finite if

$$\mu(X_n) < \infty$$
 for every $n \in \mathbb{N}$.

3. We say that μ is semi-finite if for every $E \in \mathfrak{M}$ with $\mu(E) = \infty$,

 $\exists F \subseteq E \in \mathfrak{M} \text{ such that }$

$$0 < \mu(F) < \infty$$
.

Exercise 3.1.1

- 1. Show that the counting measure is finite iff the ambient space X is a finite set.
- 2. Show that δ in Example 3.1.1 is neither finite, σ -finite, nor semi-finite.

☐ Theorem 5 (Properties of a Measure)

Let (X, \mathfrak{M}, μ) be a measure space. Then

- 1. (Monotonicity) If $E \subseteq F$ and $E, F \in \mathfrak{M}$, then $\mu(E) \leq \mu(F)$.
- 2. (Subadditivity) If $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}$, then

$$\mu\left(\bigcup_n E_n\right) \leq \sum_n \mu(E_n).$$

3. (Continuity from below) If $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}$ is an increasing sequence of sets, i.e.

$$E_1 \subseteq E_2 \subseteq \ldots \subseteq E_n \subseteq \ldots$$

then

$$\mu\left(\bigcup_{n=1}^{\infty}E_n\right)=\lim_{n\to\infty}\mu(E_n).$$

4. (Continuity from above) If $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}$ is a decreasing sequence of sets, i.e.

$$E_1 \supseteq E_2 \supseteq \ldots \supseteq E_n \supseteq \ldots$$

and $\exists n_0 \in \mathbb{N}$ such that $\mu(E_{n_0}) < \infty$, then

$$\mu\left(\bigcap_{n=1}^{\infty}E_n\right)=\lim_{n\to\infty}\mu(E_n).$$

Remark 3.1.3 (A comment on the condition for the 4th statement)

It may seem that the extra condition of a finite measure seem extravagant. However, it is necessary, as demonstrated below.

Consider $X = \mathbb{N}$, with μ as the counting measure. Then, consider the sequence of sets

$$E_1 = \{1, 2, 3, \ldots\},\$$
 $E_2 = \{2, 3, 4, \ldots\},\$
 $E_3 = \{3, 4, 5, \ldots\},\$
 \vdots
 $E_n = \{n, n+1, n+2, \ldots\},\$
 \vdots

Then $\bigcap_{n=1}^{\infty} E_n = \emptyset$, which then $\mu(\bigcap_{n=1}^{\infty} E_n) = 0$. However,

$$\mu(E_n) = \infty$$
 for each $n \in \mathbb{N}$.

♦ Homework (Homework 3)

Let (X, \mathfrak{M}, μ) be a measure space. Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}$, and

$$A := \{x \in X \mid x \in E_n \text{ i.o. } \}.$$

Prove that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$ implies that $\mu(A) = 0$.

4 \(\mathcal{E} \) Lecture 4 Sep 11th 2019

4.1 Measures (Continued 2)

We shall now prove __Theorem 5.



1. Notice that

$$F = (F \cap E) \cup (F \setminus E),$$

and $F \cap E$ and $F \setminus E$ are disjoint. Thus

$$\mu(F) = \mu(F \cap E) + \mu(F \setminus E) = \mu(E) + \mu(F \setminus E).$$

Since $\mu(F \setminus E) \ge 0$, we have

$$\mu(F) \ge \mu(E)$$
.

2. Consider a sequence of sets defined as such: 1

$$F_1 = E_1$$

$$F_2 = E_2 \setminus E_1$$

$$\vdots$$

$$F_n = E_n \setminus \bigcup_{j=1}^{n-1} E_j.$$

First, note that $F_n \subseteq E_n$ for each $n \in \mathbb{N}$. So by the last part, we have

$$\mu(F_n) \leq \mu(E_n)$$
 for each $n \in \mathbb{N}$.

¹ ★ This is a common technique in measure theory. We will see this repeatedly so in this course.

Secondly,

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n.$$

Also, $\{F_n\}_{n=1}^{\infty}$ is a pairwise disjoint collection of sets. It follows that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n) \le \sum_{n=1}^{\infty} \mu(E_n).$$

3. Consider a sequence of sets defined as such:

$$F_1 = E_1$$

$$F_2 = E_2 \setminus E_1$$

$$F_3 = E_3 \setminus E_2$$

$$\vdots$$

$$F_n = E_n \setminus E_{n-1}.$$

We see that

•
$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$$
;

•
$$\bigcup_{n=1}^{N} F_n = \bigcup_{n=1}^{N} E_n = E_N$$
; and

• $\{F_n\}_n$ is a collection pairwise disjoint sets.

Thus we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n)$$
$$= \lim_{N \to \infty} \sum_{n=1}^{N} \mu(F_n) = \lim_{N \to \infty} \mu\left(\bigcup_{n=1}^{N} F_n\right)$$
$$= \lim_{N \to \infty} \mu(E_N).$$

4. First, it is important that we notice that

$$\bigcap_{n=1}^{\infty} E_n = \bigcap_{n=m}^{\infty} E_n$$

for any $m \in \mathbb{N}$, since $\{E_n\}_n$ is a decreasing sequence of sets.

Suppose $n_0 \in \mathbb{N}$ is such that $\mu(E_{n_0}) < \infty$. Consider a sequence

of sets defined as follows: for $n_0 \le j \in \mathbb{N}$, we let $F_j = E_{n_0} \setminus E_j$. Then we have

$$\emptyset = F_{n_0} \subseteq F_{n_0+1} \subseteq \ldots \subseteq F_{n_0+k} \subseteq \ldots$$

i.e. $\{F_n\}_{n=n_0}^{\infty}$ is an increasing sequence of sets. By the last part, we have

$$\mu\left(\bigcup_{n=n_0}^{\infty} F_n\right) = \lim_{n \to \infty} \mu(F_{n_0+n}) = \lim_{n \to \infty} \mu(E_{n_0} \setminus E_{n_0+n})$$
$$= \mu(E_{n_0}) - \lim_{n \to \infty} \mu(E_{n_0+n})$$
$$= \mu(E_{n_0}) - \lim_{n \to \infty} \mu(E_n).$$

Furthermore, we observe that

$$\bigcup_{n=1}^{\infty} F_n = E_{n_0} \setminus \bigcap_{n=n_0}^{\infty} E_n.$$

Thus

$$\mu\left(\bigcup_{n=n_0}^{\infty} F_n\right) = \mu\left(E_{n_0} \setminus \bigcap_{n=n_0}^{\infty} E_n\right) = \mu(E_{n_0}) - \mu\left(\bigcap_{n=n_0}^{\infty} E_n\right)$$
$$= \mu(E_{n_0}) - \mu\left(\bigcap_{n=1}^{\infty} E_n\right).$$

It follows that indeed

$$\mu\left(\bigcap_{n=1}^{\infty}E_{n}\right)=\lim_{n\to\infty}\mu(E_{n}).$$

Exercise 4.1.1

Let (X, \mathfrak{M}, μ) be a measure space. Show that

- 1. μ is finite iff $\mu(X) < \infty$.
- 2. μ is σ -finite implies that μ is semi-finite.

Solution

- 1. This is rather simple.
 - (\Longrightarrow) μ is finite implies that each $E \in \mathfrak{M}$ has a finite measure. In

particular, $X \in \mathfrak{M}$, and so $\mu(X) < \infty$.

 (\longleftarrow) $\forall E \in \mathfrak{M}, E \subseteq X$, thus by the first item in \blacksquare Theorem 5, we have $\mu(E) \leq \mu(X) < \infty$. Thus μ is finite.

2. μ being σ -finite means that if $X = \bigcup_{n=1}^{\infty} X_n$ where $X_n \in \mathfrak{M}$, then $\mu(X_n) < \infty$ for each n. Let $E \in \mathfrak{M}$ such that $\mu(E) = \infty$. If we take

$$E_n = X_n \cap E$$
,

then $\mu(E_n) < \infty$ for each $n \in \mathbb{N}$. Then, taking a union of any finite number of these E_n 's will give us a subset of E with a finite measure. Hence, μ is indeed semi-finite.

■ Definition 11 (Null Set of a Measure)

Let (X, \mathfrak{M}, μ) be a measure space. The set

$$\mathcal{N} := \{ N \in \mathfrak{M} : \mu(N) = 0 \}$$

is called the μ -null set, or the null set of the measure μ .

Remark 4.1.1

1. If $N_j \in \mathcal{N}$, then $\bigcup_{n=1}^{\infty} N_j \in \mathcal{N}$. ²

² Requires elab

2. If $N \in \mathcal{N}$, and $E \in \mathfrak{M}$ and $E \subseteq N$, then $E \in \mathcal{N}$.

It is important to note there that the highlighted condition is required, since not all subsets of N are measurable.

3. \mathcal{N} is not a σ -algebra. If we picked an X such that $\mu(X) \neq 0$, then $\emptyset \in \mathcal{N}$ but $X \notin \mathcal{N}$.

■ Definition 12 (Complete Measure Space)

Let (X, \mathfrak{M}, μ) be a measure space. We say that the space is complete if $N \in \mathcal{N}$ and $E \subseteq N$, then $E \in \mathfrak{M}$. In this case, we also say that μ is a complete measure on \mathfrak{M} .

Remark 4.1.2

By the first item in \blacksquare Theorem 5, we have that if $\mu(E) = 0$, and so $E \in \mathcal{N}$ as well.

■ Theorem 6 (Extending the Measurable Sets)

Let (X, \mathfrak{M}, μ) be a measure space and

$$\mathcal{N} := \{ N \in \mathfrak{M} \mid \mu(N) = 0 \}.$$

Consider

$$\overline{\mathfrak{M}} := \{ E \cup F \mid E \in \mathfrak{M}, F \subseteq N \in \mathcal{N} \}.$$

Then $\overline{\mathfrak{M}}$ is a σ -algebra which contains \mathfrak{M} . Furthermore, if we define $\overline{\mu}:\overline{\mathfrak{M}}\to [0,\infty]$ as

$$\overline{\mu}(E \cup F) = \mu(E)$$
,

then $\overline{\mu}$ is a well-defined measure on $\overline{\mathfrak{M}}$.

Moreover, if $\nu : \overline{\mathfrak{M}} \to [0, \infty]$ *is any measure such that* $\nu(E) = \mu(E)$ *for all* $E \in \mathfrak{M}$ *, then* $\nu = \overline{\mu}$ *.*

ELecture 5 Sep 13th 2019

5.1 Measures (Continued 3)

Proof (Theorem 6)

 $\overline{\mathfrak{M}}$ is a σ -algebra Since $\emptyset \in \mathfrak{M}$ and $\emptyset \subseteq N$ for any $N \in \mathcal{N}$, it is clear that $\emptyset \in \overline{\mathfrak{M}}$.

Now, for $E \cup F \in \overline{\mathfrak{M}}$, if we suppose $F \subseteq N \in \mathcal{N}$, then

$$(E \cup F)^{C} = (E \cup N)^{C} \cup (N \setminus E \cup F) \in \overline{\mathfrak{M}}$$

since $E \cup N \in \mathfrak{M}$ and $N \setminus (E \cup F) \in \mathcal{N}$.

Let $\{E_n \cup F_n\}_{n=1}^{\infty} \subseteq \overline{\mathfrak{M}}$. Then we observe that

$$\bigcup_{n=1}^{\infty} (E_n \cup F_n) = \bigcup_{\substack{n=1 \ \in \mathfrak{M}}}^{\infty} E_n \cup \bigcup_{\substack{n=1 \ \in \mathcal{N}}}^{\infty} F_n \in \overline{\mathfrak{M}}.$$

Well-definedness of $\overline{\mu}$ Let $E_1 \cup F_1 = E_2 \cup F_2 \in \overline{\mathfrak{M}}$. Suppose $F_1 \subseteq N_1, F_2 \subseteq N_2 \in \mathcal{N}$. WTS

$$\mu(E_1) = \overline{\mu}(E_1 \cup F_1) = \overline{\mu}(E_2 \cup F_2) = \mu(E_2)$$

Notice that

$$E_1 \subseteq E_1 \cup F_1 = E_2 \cup F_2 \subseteq E_2 \cup N_2$$
,

and

$$E_2 \subseteq E_2 \cup F_2 = E_1 \cup F_1 \subseteq E_1 \cup N_1$$
.

By Prheorem 5, in particular, by subadditivity, we have that

$$\mu(E_1) \le \mu(E_2 \cup N_2) \le \mu(E_2) + 0 = \mu(E_2)$$

and

$$\mu(E_2) \le \mu(E_1 \cup N_1) \le \mu(E_1) + 0 = \mu(E_1).$$

It follows that $\mu(E_1) = \mu(E_2)$, as required.

 $\overline{\mu}$ is a measure

1. Since $\emptyset \in \mathfrak{M}$ and $\emptyset \in \mathcal{N}$, $\overline{\mu}$ is defined for \emptyset , and

$$\overline{\mu}(\emptyset) = \mu(\emptyset) = 0.$$

2. Let $\{E_n \cup F_n\}_{n=1}^{\infty} \subseteq \overline{\mathfrak{M}}$ be a pairwise disjoint collection. We observe that

$$\overline{\mu}\left(\bigcup_{n=1}^{\infty}(E_n \cup F_n)\right) = \overline{\mu}\left(\bigcup_{n=1}^{\infty}E_n \cup \bigcup_{n=1}^{\infty}F_n\right)$$

$$= \mu\left(\bigcup_{n=1}^{\infty}E_n\right)$$

$$= \sum_{n=1}^{\infty}\mu(E_n),$$

and

$$\sum_{n=1}^{\infty} \overline{\mu}(E_n \cup F_n) = \sum_{n=1}^{\infty} \mu(E_n).$$

Hence

$$\overline{\mu}\left(\bigcup_{n=1}^{\infty}(E_n\cup F_n)\right)=\sum_{n=1}^{\infty}\overline{\mu}(E_n\cup F_n).$$

 $\nu = \overline{\mu}$ Let $E \cup F \in \overline{\mathfrak{M}}$. Suppose $F \subseteq N \in \mathfrak{M}$ By monotonicity,

$$\overline{\mu}(E \cup F) = \mu(E) = \nu(E) \le \nu(E \cup F).$$

By subadditivity,

$$\nu(E \cup F) \le \nu(E) + \nu(F) \le \mu(E) + \nu(N) \le \overline{\mu}(E \cup F) + \mu(N) = \overline{\mu}(E \cup F) + 0.$$

Thus, indeed,

$$\nu(E \cup F) = \overline{\mu}(E \cup F).$$

5.2 *The Outer Measure*

In this section, we will show that one way we can construct a measure is by using an outer measure.

■ Definition 13 (Outer Measure)

Given a set X, a function

$$\mu^*: \mathcal{P}(X) \implies [0, \infty]$$

is called an outer measure if

- 1. $\mu^*(\emptyset) = 0$;
- 2. (monotonicity) if $E \subseteq F$, then $\mu^*(E) \le \mu^*(F)$; and
- 3. (countable subadditivity) if $\{A_n\}_n \subseteq \mathcal{P}(X)$, then

$$\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \le \sum_{n=1}^{\infty} \mu^*(A_n).$$

Coming from PMATH450, we have seen an example of an outer measure.

♦ Proposition 7 (Lebesgue's Outer Measure)

Given $E \subseteq \mathbb{R}$, consider

$$\mu^*(E) := \inf \left\{ \sum_{n=1}^{\infty} (b_n - a_n) : E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n) \right\}.$$

 μ^* is Lebesgue's outer measure.

Proof

- 1. It is clear that $\mu^*(\emptyset) = \emptyset$, since we can pick all $(a_n, b_n) = \emptyset$.
- 2. Suppose $A \subseteq B \subseteq \mathbb{R}$. It is clear that any collection of intervals whose union contain B will contain A, but there are such collections for A that do not contain B. This means that

$$\mu^*(A) \leq \mu^*(B)$$

by the property of the infimum.

3. Let $E = \bigcup_{i=1}^{\infty} E_i$. WTS $\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$.

Now if $\mu^*(E_i) = \infty$ for any i, then the inequality is trivially true. Thus, wma $\mu^*(E_i) < \infty$ for all i.

¹ Let $\varepsilon > 0$. By the definition of the infimum, for each i, we can pick a countable sequence $\{(a_n^i,b_n^i)\}_{n=1}^\infty \subseteq \mathcal{P}(X)$ such that $E_1 \subseteq \bigcup_{n=1}^\infty (a_n^i,b_n^i)$ and

¹ This is also a common trick in measure

theory.

$$\sum_{n=1}^{\infty} (b_n^i - a_n^i) \le \mu^*(E_i) + \frac{\varepsilon}{2^i}.$$

Then

$$E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} (a_n^i, b_n^i).$$

And so it follows that

$$\mu^*(E) \le \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} (b_n^i - a_n^i)$$
$$\le \sum_{i=1}^{\infty} \mu^*(E_i) + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i}$$
$$= \sum_{i=1}^{\infty} \mu^*(E_i) + \varepsilon.$$

Since ε was arbitrary, it follows that

$$\mu^*(E) \le \sum_{i=1}^{\infty} \mu^*(E_i).$$

Show that had we defined Lebesgue's outer measure with closed intervals, i.e.

$$\widetilde{\mu}^*(E) := \inf \left\{ \sum_{n=1}^{\infty} (b_n - a_n) : E \subseteq \bigcup_{n=1}^{\infty} [a_n, b_n] \right\},$$

 $\tilde{\mu}^*$ is still an outer measure.

In fact, we can do so for half-open intervals.

Example 5.2.1 (Lebesgue-Stieltjes Outer Measure)

Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function that is continuous from the right. Let

$$\mu^*(E) := \inf \left\{ \sum_{n=1}^{\infty} (F(b_n) - F(a_n)) : E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n] \right\}.$$

Then μ^* is an outer measure.

Remark 5.2.1

Again, we could have defined the above outer measure using open or closed intervals.

Example 5.2.2 (Lebesgue's Outer Measure on \mathbb{R}^2)

Let $E \subseteq \mathbb{R}^2$, and

$$\mu^*(E) := \inf \left\{ \sum_{n=1}^{\infty} A(R_n) : E \subseteq \bigcup_{n=1}^{\infty} R_n \right\},$$

where *A* is the 'area' function, and $R_n = (a_n, b_n) \times (c_n, d_n)$ are open rectangles. Then μ^* is an outer measure.

Remark 5.2.2

- 1. Again, we can define the above outer measure using closed rectangles, or half-open rectangles.
- 2. We can continue defining an outer measure for \mathbb{R}^3 using cubes, for \mathbb{R}^4 using hypercubes, and so on.

We want to now show that given an outer measure, we can always construct a measure. This is known as Carathéodory's Theorem.

This requires the following definition:

\blacksquare Definition 14 (μ^* -measurability)

A set $A \subseteq X$ is said to be μ^* -measurable if $\forall E \subseteq X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C).$$

Remark 5.2.3

1. By subadditivity, we always have

$$\mu^*(E) \le \mu^*(E \cap A) + \mu^*(E \cap A^C),$$

since
$$E = (E \cap A) \cup (E \cap A^C)$$
.

2. Note that $E \cap A^C = E \setminus A$. In a sense, A is said to be μ^* -measurable if it can slice any subset of X such that we have additivity of the sliced parts. We may also say that A is a 'universal slicer'.

□Theorem 8 (Carathéodory's Theorem)

If μ^* is an outer measure on a set X, let

$$\mathfrak{M} := \{ A \subseteq X : A \text{ is } \mu^*\text{-measureable} \}.$$

Then \mathfrak{M} is a σ -algebra, and we set

$$\mu:\mathfrak{M}\to[0,\infty]$$

such that

$$\mu(A) = \mu^*(A).$$

Then μ is a complete measure on \mathfrak{M} .





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