PMATH467 — Algebraic Geometry

Classnotes for Winter 2019

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List of Theorems

Preface

The basic goal of the course is to be able to find **algebraic invariants**, which we shall use to classify topological spaces up to homeomorphism.

Other questions that we shall also look into include a uniqueness problem about manifolds; in particular, how many manifolds exist for a given invariant up to homeomorphism? We shall see that for a **2-manifold**, the only such manifold is the 2-dimensional sphere S^2. For a 4-manifold, it is the 4-dimensional sphere S^4. In fact, for any other n-manifold for $n > 4$, the unique manifold is the respective n-sphere. The problem is trickier with the 3-manifold, and it is known as the Poincaré Conjecture, solved in 2003 by Russian Mathematician Grigori Perelman. Indeed, the said manifold is homeomorphic to the 3-sphere.

For this course, you are expected to be familiar with notions from **real analysis**, such as topology, and concepts from **group theory**.

The following topics shall be covered:

1. Point-Set Topology
2. Introduction to Topological Manifolds
3. Simplicial complexes & Introduction to Homology
4. Fundamental Groups & Covering Spaces
5. Classification of Surfaces

Basic Logistics for the Course

I shall leave this here for my own notes, in case something happens to my hard copy.

- OH: (Tue) 1630 - 1800, (Fri) 1245 - 1320
- OR: MC 6457
- EM: aaleyasin

Part I

Point-Set Topology

1 Lecture 1 Jan 07th

1.1 Euclidean Space

For any $(x_1, \ldots, x_m) \in \mathbb{R}^m$, we can measure its distance from the origin 0 using either

- $\|x\|_\infty = \max\{|x_i|\}$ (the supremum-norm);
- $\|x\|_2 = \sqrt{\sum (x_j)^2}$ (the 2-norm); or
- $\|x\|_p = \left(\sum |x_j|^p\right)^{\frac{1}{p}}$ (the p-norm),

where we may define a "distance" by

$$d_p(x, y) = \|x - y\|_p .$$

Definition 1 (Metric)

Let X be an arbitrary space. A function $d : X \times X \to \mathbb{R}$ is called a ***metric*** *if it satisfies*

1. *(symmetry) $d(x, y) = d(y, x)$ for any $x, y \in X$;*
2. *(positive definiteness) $d(x, y) \geq 0$ for any $x, y \in X$, and $d(x, y) = 0 \iff x = y$; and*
3. *(triangle inequality) $\forall x, y, z \in X$*

$$d(x, y) \leq d(x, z) + d(y, z).$$

Definition 2 (Open and Closed Sets)

Given a space X with a metric d, and $r > 0$, the set

$$B(x, r) := \{w \in X \mid d(x, w) < r\}$$

is called the ***open ball*** *of radius r centered at x. An* ***open set*** *A is such that $\forall a \in A$, $\exists r > 0$ such that*

$$B(a, r) \subseteq A.$$

We say that a set is ***closed*** *if its complement is open.*

Definition 3 (Continuous Map)

A function

$$f : (X, d_1) \to (Y, d_2)$$

is said to be ***continuous*** *if the preimage of an open set in Y is open in X.*

See notes on Real Analysis for why we defined a continuous map in such a way.

Warning

This definition does not imply that a continuous map f maps open sets to open sets.

Exercise 1.1.1

Contruct a function on $[0, 1]$ which assumes all values between its maximum and minimum, but is not continuous.

Solution

Consider the piecewise function

$$f(x) = \begin{cases} x & 0 \leq x < \frac{1}{2} \\ x - \frac{1}{2} & x \geq \frac{1}{2}. \end{cases}$$

It is clear that the maximum and minimum are $\frac{1}{2}$ and 0 respectively, and f assumes all values between 0 and $\frac{1}{2}$. However, a piecewise function is not continuous.

Definition 4 (Homeomorphism)

*A function f is a **homeomorphism** if it is a bijection and both f and f^{-1} are continuous.*

Example 1.1.1

The function

$$g : [0, 2\pi) \to \mathbb{R}^2 \text{ given by } \theta \mapsto (\cos\theta, \sin\theta)$$

is not homeomorphic, since if we consider an alternating series that converges to 0 on the unit circle on $\mathbb{R}^2$, we have that the preimage of the series does not converge and f^{-1} is in fact discontinuous.

Now, we want to talk about topologies without referring to a metric.

Definition 5 (Topology)

*Let X be a space. We say that the set $\mathcal{T} \subseteq \mathcal{P}(X)$ is a **topology** if*

1. $X, \emptyset \in \mathcal{T}$;
2. *if* $\{x_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ *for an arbitrary index set* A, *then*
$$\bigcup_{\alpha \in A} x_\alpha \in \mathcal{T}; \text{ and}$$
3. *If* $\{x_\beta\}_{\beta \in B} \subset \mathcal{T}$ *for some finite index set* B, *then*
$$\bigcap_{\beta \in B} x_\beta \in \mathcal{T}.$$

2 Lecture 2 Jan 09th

2.1 Euclidean Space (Continued)

In the last lecture, from metric topology, we generalized the notion to a more abstract one that is based solely on open sets.

Example 2.1.1

Let X be a set. The following two are uninteresting examples of topologies:

1. The **trivial topology** $\mathcal{T} = \{\emptyset, X\}$.
2. The **discrete topology** $\mathcal{T} = \mathcal{P}(X)$.

WE SHALL NOW continue with looking at more concepts that we shall need down the road.

Definition 6 (Closure of a Set)

*Let A be a set. Its **closure**, denoted as $\overline{A}$, is defined as*

$$\overline{A} = \bigcap_{C \supset A}^{C\text{: closed}} C.$$

It is the smallest closed set that contains A.

Note

In metric topology, one typically defines the closure of a set by taking the union of A and its limit points.

Definition 7 (Interior of a Set)

*Let A be a set. Its **interior**, denoted either as $\mathrm{Int}\,(A)$, A° or $\mathring{A}$, is defined as*

$$\mathring{A} = \bigcup_{G \subseteq A}^{G \colon \text{open}} G.$$

Definition 8 (Boundary of a Set)

*Let A be a set. Its **boundary**, denoted as ∂A, is defined as*

$$\partial A = \overline{A} \setminus \mathring{A}.$$

Exercise 2.1.1

Let A be a set. Prove that ∂A is closed.

Proof

Notice that

$$(\partial A)^C = (\overline{A} \setminus \mathring{A})^C = X \setminus \overline{A} \cup \mathring{A} = X \cap \overline{A}^C \cup \mathring{A}$$

which is open. □

Exercise 2.1.2

Let A be a set. Show that

$$\partial(\partial A) = \partial A.$$

Proof

First, notice that $\mathring{\partial A} = \emptyset$. Since ∂A is closed, $\overline{A} = \partial A$. Then

$$\partial(\partial A) = \overline{\partial A} \setminus \mathring{\partial A} = \partial A \setminus \emptyset = \partial A$$

Example 2.1.2

We know that $\mathbb{Q} \subseteq \mathbb{R}$, and $\overline{\mathbb{Q}} = \mathbb{R}$. We say that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 9 (Dense)

We say that a subset A of a set X is dense if

$$\overline{A} = X.$$

Example 2.1.3

From the last example, we have that $\mathring{\mathbb{Q}} = \emptyset$.

Definition 10 (Limit Point)

We say that $p \in X \supseteq A$ is a limit point of A if any neighbourhood of p has a nontrivial intersection with A.

Example 2.1.4 (A Topologist's Circle)

Consider the function

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

on the interval $\left[-\frac{1}{2\pi}, \frac{1}{2\pi} \right]$. Extend the function on both ends such that we obtain Figure 2.1 (See also: Desmos).

The limit points of the graph includes all the points on the straight line from $(0, -1)$ to $(0, 1)$, including the endpoints. This is the case because for any of the points on this line, for any neighbourhood around the point, the neighbourhood intersects the graph f infinitely many times.

Going back to continuity, given a function f, how do we know if f^{-1} maps an open set to an open set?

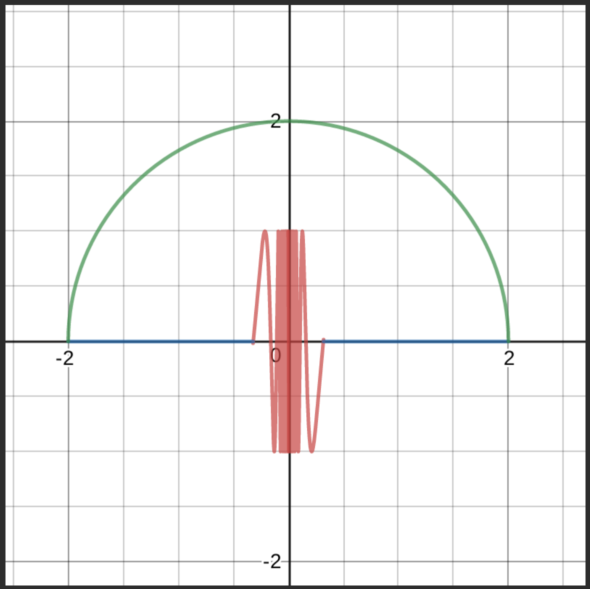

Figure 2.1: A Topologist's Circle

We can actually reduce the problem to only looking at open balls. But why are we allowed to do that?

Definition 11 (Basis of a Topology)

Given a topology $\mathcal{T}$, we say that $\mathcal{B} = \{B_\alpha\}_{\alpha \in I}$ is a ***basis*** *if $\forall T \in \mathcal{T}$, there exists $J \subset I$ such that*

$$T = \bigcup_{\alpha \in J} B_\alpha.$$

Note that while the definition is similar to that of a cover, we are now "covering" over sets and not points.

Example 2.1.5

Let $\mathcal{T}$ be the Euclidean topology on $\mathbb{R}$. Then we can take

$$\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}, a \leq b\}.$$

Note that $\mathcal{B}$ is **uncountable**. We can, in fact, have [1]

$$\mathcal{B}_1 = \{(a, b) \mid a, b \in \mathbb{Q}, a \leq b\},$$

which is countable, as a basis for $\mathbb{R}$. Furthermore, we can consider the set

$$\mathcal{B}_2 = \left\{(a, b) \;\middle|\; a \leq b,\, a = \frac{m}{2^p},\, b = \frac{n}{2^q},\, m, n, p, q \in \mathbb{Z}\right\},$$

[1] Recall from PMATH 351 that we can write $\mathbb{R}$ as a disjoint union of open intervals with rational endpoints.

which is also a countable basis for $\mathbb{R}$. Notice that

$$\mathcal{B}_2 \subseteq \mathcal{B}_1 \subseteq \mathcal{B}.$$

Example 2.1.6

In $\mathbb{R}^2$, we can do a similar construction of $\mathcal{B}$, $\mathcal{B}_1$, and $\mathcal{B}_2$ as in the last example and use them as a basis for $\mathbb{R}^2$. In particular, we would have

$$\mathcal{B} = \{(a_1, b_1) \times (a_2, b_2) \mid a_1, a_2, b_1, b_2 \in \mathbb{R}\}.$$

This is called a **dyadic partitioning** of $\mathbb{R}^2$.

Example 2.1.7

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be two topological spaces. Then the Cartesian product $X_1 \times X_2$ has topology induced from $\mathcal{T}_1$ and $\mathcal{T}_2$ by taking the set

$$\mathcal{B} = \{\beta_1 \times \beta_2 \mid \beta_1 \in \mathcal{T}_1,\ \beta_2 \in \mathcal{T}_2\}$$

as the basis.

Exercise 2.1.3

Prove that

1. *β_1 and β_2 can be taken to be elements of bases $\mathcal{B}_1 \subset \mathcal{T}_1$ and $\mathcal{B}_2 \subset \mathcal{T}_2$, respectively.*
2. *the product topology on $\mathbb{R}^2$ is the same as the Euclidean topology.*

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