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Mathematical and computer modeling of various technical and scientific problems – Documentation of the final laboratory project

Comparative analysis of methods for approximating functions

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1 Project objective

The goal of the project is to explain, present, and compare three methods of approximating functions either from a set of given points or by approximating the function itself. The methods that we chose are:

1. Interpolation using Lagrange polynomials
2. Fourier series
3. Linear regression

2 Description

2.1 Lagrange polynomials

Lagrange polynomials are polynomials $w(x)$ that interpolate a given set of points $P = \{(x_0, y_0),\ (x_1, y_1), \dots,\ (x_n, y_n)\}$ Which means that they satisfy the following condition:

$$\forall_{(x_i, y_i) \in P}\ w(x_i) = y_i$$

In other words, when a polynomial **interpolates** the dataset, it passes through all of the points in said dataset. The Lagrange interpolating polynomial is the unique polynomial of lowest degree that interpolates a given set of data

Langrage polynomials for a given set of data are calculated using the following formula

$$w(x) = \sum_{i=0}^{n} y_i \cdot \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}$$

In this section, I would like to demonstrate how this formula can be obtained and that it is actually quite straightforward and intuitive.

2.2 How to arrive at the formula for Lagrange polynomials?

This formula might seem daunting at first, but let us first consider a polynomial $w_i(x)$ that **goes through only one point** (x_i, y_i) **and is equal to 0 for the rest of the points**.

1. For a given pair (x_i, y_i), it has to be equal to y_i
2. For each pair (x_j, y_j) other than (x_i, y_i), it has to be equal to 0

Let's start with condition no. 2. It tells us that the polynomial $w(x)$ should be equal to 0 for all x_j inputs, that are not x_i. That is actually quite simple – all we have to do is construct a polynomial in factored form with the roots being all the x_j values other than x_i.

$$w_i(x) = \prod_{\substack{0 \leq j \leq n \\ j \neq i}} (x - x_j)$$

Let's now consider a simplified condition no. 1 – let's find a polynomial that is equal to 1 for (x_i, x_i), but also satisfies condition no. 2. That is actually also very simple – we just have to divide the polynomial that we have already constructed by $(x_i - x_j)$

$$w_i(x) = \prod_{\substack{0 \leq j \leq n \\ j \neq i}} \frac{(x - x_j)}{(x_i - x_j)}$$

Now,

$$w_i(x_i) = \prod_{\substack{0 \leq j \leq n \\ j \neq i}} \frac{(x_i - x_j)}{(x_i - x_j)} = 1$$

In order for the polynomial to satsify cond. no. 1, all we have to do is multiply it by y_i

$$w_i(x) = y_i \prod_{\substack{0 \leq j \leq n \\ j \neq i}} \frac{(x - x_j)}{(x_i - x_j)}$$

Condition no. 1 is satisfied,

$$w_i(x_i) = y_i \prod_{\substack{0 \leq j \leq n \\ j \neq i}} \frac{(x_i - x_j)}{(x_i - x_j)} = y_i \cdot 1 = y_i$$

Since both of the conditions are satisfied, $w_i(x)$ interpolates one point – (x_i, y_i) and is equal to 0 for all the other points in the dataset.

Below is a sample dataset comprising three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and the values that polynomials w_i take for each of them.

	x_2	x_2	x_3
w_1	y_2	0	0
w_2	0	y_2	0
w_3	0	0	y_3

To find a polynomial that interpolates all of the points, we simply create $w_i(x)$ for every point and add them together.

$$w(x) = \sum_{i=0}^{n} w_i(x)$$

Which expands to

$$w(x) = \sum_{i=0}^{n} y_i \cdot \prod_{\substack{0 \leq j \leq n \\ j \neq i}} \frac{x - x_j}{x_i - x_j}$$

We have arrived at the exact formula for Lagrange polynomials.

Enclosures

- File with the program (`Pietrasik_Lewandowicz_Hankus.nb`)