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New parametrization of the Second-order Reduced Density Matrix in  
Reduced Density Matrix Functional Theory

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# 1 Introduction

In the document, the second quantization formalism is used. Consequently, some useful definitions [1] are recalled in this section. Additionally, atomic units are used all throughout the document.

## 1.1 Second quantization

The occupation-number (ON) operators are defined for the sets of creation and annihilation operators,  $\{a^\dagger\}$ ,  $\{a\}$ , as

$$n_p = a_p^\dagger a_p. \quad (1)$$

For a given ON vector  $|k\rangle$ , the occupation number  $k_p$  is obtained by counting the number of electrons in spin orbital  $p$

$$n_p |k\rangle = a_p^\dagger a_p |k\rangle = k_p |k\rangle. \quad (2)$$

The ON operators are Hermitian

$$n_p^\dagger = \left(a_p^\dagger a_p\right)^\dagger = a_p^\dagger a_p = n_p, \quad (3)$$

commute among themselves

$$n_p n_q |k\rangle = k_p k_q |k\rangle = k_q k_p |k\rangle = n_q n_p |k\rangle, \quad (4)$$

and, since in the spin-orbital basis the ON operators are projection operators, they are idempotent,

$$n_p^2 = n_p n_p = a_p^\dagger a_p a_p^\dagger a_p = a_p^\dagger \left(1 - a_p^\dagger a_p\right) a_p = a_p^\dagger a_p = n_p. \quad (5)$$

Therefore, its eigenvalues can only be 0 or 1.

The particle-number operator, or simply the number operator, is the Hermitian operator resulting from adding together all ON operators in the Fock space

$$\hat{N} = \sum_{p=1}^m a_p^\dagger a_p, \quad (6)$$

which returns the number of electrons in an ON vector

$$\hat{N} |k\rangle = \sum_{p=1}^m k_p |k\rangle = n |k\rangle. \quad (7)$$

Also, the  $\delta_{pq}$  operator is defined as

$$\delta_{pq} = a_p a_q^\dagger + a_q^\dagger a_p, \quad (8)$$

which is null if  $p \neq q$ .

## 1.2 Reduced Density Matrix Functional Theory

Many methods of electronic structure theory are based on variational optimization of an energy functional. An idempotent first-order reduced density matrix (1-RDM) is variationally optimized in the Hartree-Fock (HF) method [2].

The 1-RDM is determined by the natural orbitals and their occupation numbers [3], leading to denote the corresponding energy functional as density matrix functional (DMF) or natural orbital functional (NOF). This is explained in detail in the following sections.

In the coordinate representation of first quantization, the *first-order reduced density matrix* (1-RDM),  $\gamma$ , is defined for an  $n$ -electron wavefunction,  $\Psi$ , as

$$\gamma(\mathbf{x}, \mathbf{x}') = n \int \cdots \int \Psi(\mathbf{x}, \mathbf{x}_2, \dots, \mathbf{x}_n) \Psi^*(\mathbf{x}', \mathbf{x}_2, \dots, \mathbf{x}_n) d\mathbf{x}_2 \cdots d\mathbf{x}_n, \quad (9)$$

where  $\mathbf{x} = (\mathbf{r}, s)$  is a combined spatial and spin coordinate.

Employing the 1-RDM rather than the electron density,  $\rho$ , has the immediate advantage of making the kinetic energy an explicit functional of  $\gamma$  rather than  $\rho$ .

Then, introducing a fictitious noninteracting system is not necessary.

Furthermore, fractionally occupied orbitals are a feature of Reduced Density Matrix Functional Theory (RDMFT), which suggests that functionals of  $\gamma$  are better suited to explain static correlation.

Self-adjointness of  $\gamma$ , as defined in eq. (9), allows for its spectral representation [4]

$$\gamma(\mathbf{x}, \mathbf{x}') = \sum_p n_p \varphi_p(\mathbf{x}) \varphi_p^*(\mathbf{x}'). \quad (10)$$

The eigenvalues of 1-RDM are called natural occupation numbers,  $\{n_p\}$ , and the eigenfunctions are known as natural spin-orbitals,  $\{\varphi_p\}$ . By convention, natural spin-orbitals are denoted by the indices  $p, q, r$ , and  $s$ , and arbitrary one-electron functions by the indices  $a, b, c$ , and  $d$ .

The ensemble  $n$ -representability conditions, which are some properties of the natural spin-orbitals and occupation numbers, have a particularly simple form [5] and read:

1. Self-adjointness of  $\gamma$ : implies orthonormality of the natural orbitals

$$\int \varphi_p^*(\mathbf{x}) \varphi_q(\mathbf{x}) = \delta_{pq} \quad \forall_{p,q}. \quad (11)$$

2.  $\gamma$  is assumed to be normalized to a number of electrons,  $n$ . Therefore, the natural occupancies sum up to  $n$

$$\sum_p n_p = n. \quad (12)$$

3. Occupation number,  $n_p$ , is nonnegative and not greater than 1

$$0 \leq n_p \leq 1 \quad \forall_p. \quad (13)$$

Gilbert proved that there is a 1-RDM functional [6, 7] such that the Hohenberg-Kohn theorems

extend to nonlocal potentials [6, 8]

$$E_{\nu}^{\text{HK}}[\gamma] = \text{tr} [\hat{h}\hat{\gamma}] + \left\langle \Psi[\gamma] \left| \hat{V}_{\text{ee}} \right| \Psi[\gamma] \right\rangle, \quad (14)$$

where a ground state wavefunction relevant to a  $\nu$ -representable  $\gamma$  is denoted by the symbol  $\Psi[\gamma]$ . The one-electron Hamiltonian  $\hat{h}$  is composed of up of the external potential  $\hat{V}_{\text{ext}}$  and the kinetic energy  $\hat{T}$

$$\hat{h} = \hat{T} + \hat{V}_{\text{ext}}, \quad (15)$$

and  $\hat{V}_{\text{ee}}$  is an electron interaction operator

$$\hat{V}_{\text{ee}} = \sum_{i>j}^n \frac{1}{r_{ij}}. \quad (16)$$

This functional follows the variational principle

$$E_{\nu}[\gamma] \geq E_0 \quad \forall \gamma \in \nu\text{-rep}, \quad (17)$$

where  $\nu$ -rep denotes a set of pure-state  $\nu$ -representable 1-RDMs.

Levy defined the electron repulsion functional [9, 10], extending the domain of a density matrix functional to all pure-state  $n$ -representable 1-RDMs

$$E_{\text{ee}}^{\text{L}}[\gamma] = \min_{\Psi \rightarrow \gamma} \left\langle \Psi \left| \hat{V}_{\text{ee}} \right| \Psi \right\rangle, \quad (18)$$

and further extended to ensemble  $n$ -representable 1-RDMs (belonging to a set “ $n$ -rep”) by Valone [11, 12], so the exact functional reads

$$E_{\text{ee}}[\gamma] = \min_{\Gamma^{(n)} \rightarrow \gamma} \text{tr} [\hat{\mathbf{H}}\hat{\Gamma}^{(n)}], \quad (19)$$

where the minimization is carried out with respect to  $n$ -electron density matrices  $\Gamma^{(n)}$  that yield  $\gamma$ .

The minima of the Levy functional given in eq. (18), the Hohenberg-Kohn functional given in eq. (14), and that defined in eq. (19), respectively for  $\nu$ -rep, pure-state  $n$ -representable, and ensemble  $n$ -representable ( $n$ -rep) 1-RDMs coincide [11, 13] for a given external potential,  $\hat{V}_{\text{ext}}$ . Thus, considering the variational principle given in eq. (17), it can be concluded that a ground state energy at minimum is produced by a functional defined for  $n$ -rep 1-RDMs

$$E_0 = \min_{\gamma \in n\text{-rep}} \left\{ \text{tr} [\hat{h}\hat{\gamma}] + E_{\text{ee}}[\gamma] \right\}. \quad (20)$$

Indeed, eqs. (11) to (13) and (20) are the foundation for RDMFT.

There are two cases where the exact forms of  $E_{\text{ee}}[\gamma]$  are known:

1.  **$n$ -electron noninteracting systems:** the 1-RDM corresponding to a single determinant wavefunction is idempotent, which implies integer (0 or 1) values of the natural occupation numbers

$$\hat{\gamma}^2 = \hat{\gamma} \Leftrightarrow n_p = 0 \vee n_p = 1 \quad \forall p. \quad (21)$$

In this case, a two-electron reduced density matrix (2-RDM),  $\Gamma$ , is defined for a general wavefunction,  $\Psi$ , as

$$\Gamma(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}'_1, \mathbf{x}'_2) = n(n-1) \int \cdots \int \Psi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n) \Psi^*(\mathbf{x}'_1, \mathbf{x}'_2, \mathbf{x}_3, \dots, \mathbf{x}_n) d\mathbf{x}_3 \cdots d\mathbf{x}_n, \quad (22)$$

which is explicitly expressible in terms of 1-RDM if the wavefunction takes the form of a Slater determinant

$$\Gamma(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}'_1, \mathbf{x}'_2) = \gamma(\mathbf{x}_1, \mathbf{x}'_1) \gamma(\mathbf{x}_2, \mathbf{x}'_2) - \gamma(\mathbf{x}_1, \mathbf{x}'_2) \gamma(\mathbf{x}_2, \mathbf{x}'_1). \quad (23)$$

The electron interaction functional corresponding to such a noninteracting 2-RDM reads

$$E_{\text{ee}}^{\text{HF}}[\gamma] = E_{\text{H}}[\gamma] + E_{\text{x}}[\gamma], \quad (24)$$

referred to as the Hartree-Fock functional as the HF equations' solution coincides with an idempotent density matrix that results from functional optimization [14]. The Hartree functional,  $E_{\text{H}}$ , describes the classical part of electron interaction

$$E_{\text{H}}[\gamma] = \frac{1}{2} \iint \frac{\gamma(\mathbf{x}, \mathbf{x}) \gamma(\mathbf{x}', \mathbf{x}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{x} d\mathbf{x}', \quad (25)$$

whereas the exchange functional,  $E_{\text{x}}$ , reads

$$E_{\text{x}}[\gamma] = -\frac{1}{2} \iint \frac{\gamma(\mathbf{x}, \mathbf{x}') \gamma(\mathbf{x}', \mathbf{x})}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{x} d\mathbf{x}'. \quad (26)$$

## 2. Two-electron closed-shell system: an interacting two-electron species.

The exact density matrix functional for a two-electron system is known [15, 16].

Based on the work of Löwdin and Shull (LS) [17], a Slater-determinant-expansion of a singlet wavefunction (assumed to be real-valued) in the natural spin-orbitals basis  $\{\varphi_p\}$  is entirely given by *diagonal* determinants composed of the spin-orbitals that share spatial parts

$$\psi^{\text{LS}} = \sum_p c_p |\varphi_p \varphi_{\bar{p}}|, \quad (27)$$

where  $|\varphi_p \varphi_{\bar{p}}|$  indicates a normalized Slater determinant and  $p$  and  $\bar{p}$  are spin-orbitals of opposite spin. The normalization of  $\psi^{\text{LS}}$  imposes that  $\sum_p c_p^2 = 1$ .

This wavefunction provides the 1-RDM directly in its spectral representation, implying that the squares of the expansion coefficients are just the natural occupation numbers

$$n_p = c_p^2 \quad \forall p. \quad (28)$$

The energy is given by

$$E = \left\langle \psi^{\text{LS}} \left| \hat{H} \right| \psi^{\text{LS}} \right\rangle = \sum_p c_p^2 h_{pp} + \frac{1}{2} \sum_{pq} c_p c_q \langle pp | qq \rangle, \quad (29)$$

assuming that the coefficients corresponding to spin-orbitals of opposite spins and same spatial parts are equal.

By minimizing the energy with respect to  $\{c_p\}$  and  $\{\varphi_p\}$ , an exact electron interaction density matrix functional can be immediately written as

$$E_{\text{ee}}^{\text{LS}}[\gamma] = \frac{1}{2} \min_{\{f_q\}} \sum_{pq} f_p f_q \sqrt{n_p n_q} \langle pq | qp \rangle, \quad (30)$$

with

$$f_p = \pm 1 \quad \forall_p, \quad (31)$$

and  $\langle pp | qq \rangle = \langle pq | qp \rangle$ , as the orbitals are real.

For two-electron atoms and molecules in equilibrium geometry, it is known that the sign of the factor  $f_1$  corresponding to the highest occupation  $n_1$  is predominantly opposite to the signs of all other factors corresponding to weakly occupied ( $n_p < \frac{1}{2}$ ) orbitals [15].

There are known cases when this condition is violated [15, 18–20], in which a two-electron functional that depends explicitly on the occupation numbers is defined as

$$\tilde{E}_{\text{ee}}^{\text{LS}}[\gamma] = \frac{1}{2} \sum_{pq} G_{pq}^{\text{LS}} \langle pq | qp \rangle, \quad (32)$$

with

$$G_{pq}^{\text{LS}} = \begin{cases} n_p & p = q \\ -\sqrt{n_p n_q} & p = 1, q > 1 \text{ or } p > 1, q = 1, \\ \sqrt{n_p n_q} & \text{otherwise} \end{cases} \quad (33)$$

which, although not always fully equivalent to the exact LS functional given in eq. (30), is a good approximation.

Other functionals like the correction done to the Müller [21] or, as referred in this text, Buijse and Baerends functional [22, 23], denoted as the BB-corrected (BBC) functionals [24] have been developed.

Additionally, the LS functional has been extended to include all determinants resulting from diagonal double, diagonal quadruple, etc. excitations in the more general closed-shell  $n$  (even) electron ansatz. For systems for which a set of the natural spin-orbitals can be divided into *inner* orbitals, which are localized on atoms and have occupancies close to 1, and *outer* orbitals, which include a bonding orbital and all weakly occupied orbitals, the extended Löwdin- Shull (ELS) functional [25, 26] is applicable. In order to provide a balanced description of the dynamic and static correlation, it has been developed to treat only molecules with a single bond and no lone pairs.

Therefore, exact density matrix functionals for an uncorrelated and a strongly correlated electron pair are known.

### 1.2.1 2-RDM

In the coordinate representation, the *second-order reduced density matrix* (2-RDM),  $\Gamma$ , is defined as

$$\Gamma(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}'_1, \mathbf{x}'_2) = \frac{n(n-1)}{2} \int \cdots \int \Psi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n) \Psi^*(\mathbf{x}'_1, \mathbf{x}'_2, \mathbf{x}_3, \dots, \mathbf{x}_n) d\mathbf{x}_3 \cdots d\mathbf{x}_n. \quad (34)$$

Due to the two-particle nature of the electron interaction, the contraction of the two-electron reduced density matrix components,  $\Gamma_{abcd}$ , with two-electron integrals  $\langle ab | cd \rangle$ , yields the electronic repulsion energy  $E_{ee}$  given a system defined by the ground state wavefunction  $|0\rangle$

$$E_{ee} = \frac{1}{2} \sum_{abcd} \Gamma_{abcd} \langle cd | ab \rangle, \quad (35)$$

where the 2-RDM, for a given basis set  $\{\chi_a\}$ , is defined as

$$\Gamma_{abcd} = \langle 0 | \hat{c}^\dagger \hat{d}^\dagger \hat{b} \hat{a} | 0 \rangle. \quad (36)$$

Formally, the 2-RDM is a functional of the 1-RDM.

Using the Hartree-Fock approximation stated in eq. (23), where  $\Gamma$  in the representation of the natural spin-orbitals is given solely in terms of the occupation numbers

$$\Gamma_{pqrs}^{\text{HF}} = n_p n_q (\delta_{pr} \delta_{qs} - \delta_{ps} \delta_{qr}), \quad (37)$$

and assuming that the elements of  $\Gamma$  are functions of the natural occupation numbers,  $\{n_p\}$ , and the basis set is formed by natural spin-orbitals,  $\{\varphi_p\}$ , the entire dependence of  $E_{ee}$  on  $\varphi_p$  is included in two-electron integrals.

Then, using the exploit provided in eq. (35), the electron-electron interaction functional given in eq. (19) may be approximated as

$$E_{ee}[\gamma] = \frac{1}{2} \sum_{pqrs} \Gamma_{pqrs}[\{n_t\}] \langle rs | pq \rangle. \quad (38)$$

In most approximations proposed so far,  $E_{ee}[\gamma]$  is an explicit function of the occupation numbers and the natural spin-orbitals.

## 1.3 Properties of density matrices

In the second quantization formalism, the electronic molecular Hamiltonian in the spin-orbital basis is given by

$$\hat{H} = \sum_{pq} h_{pq} a_p^\dagger a_q + \frac{1}{2} \sum_{pqrs} (pr | qs) a_p^\dagger a_q^\dagger a_s a_r. \quad (39)$$



Using restricted spin-orbitals, it is convenient to define the operators

$$E_{pq} = \sum_{\sigma} a_{p\sigma}^{\dagger} a_{q\sigma}, \quad (40)$$

such that

$$[E_{pq}, E_{rs}] = \delta_{qr} E_{ps} - \delta_{ps} E_{qr}. \quad (41)$$

Then, the Hamiltonian can be written as

$$\hat{H} = \sum_{pq} h_{pq} E_{pq} + \frac{1}{2} \sum_{pqrs} (pq | rs) (E_{pq} E_{rs} - \delta_{qr} E_{ps}). \quad (42)$$

Given a normalized reference state  $|0\rangle$  written as a linear combination of ON vectors

$$|0\rangle = \sum_k c_k |k\rangle, \quad \langle 0 | 0 \rangle = 1, \quad (43)$$

the expectation value of  $\hat{H}$  with respect to  $|0\rangle$  reads

$$E_0 = \langle 0 | \hat{H} | 0 \rangle = \sum_{pq} h_{pq} D_{pq} + \frac{1}{2} \sum_{pqrs} (pq | rs) (d_{pqrs} - \delta_{qr} D_{ps}), \quad (44)$$

where

$$d_{pqrs} = \langle 0 | E_{pq} E_{rs} | 0 \rangle, \quad (45)$$

and

$$D_{pq} = \langle 0 | E_{pq} | 0 \rangle \equiv D_{pq}^{(1)}, \quad (46)$$

are the elements (densities) of an  $m \times m$  Hermitian matrix, the *one-electron spin-orbital density matrix*,  $\mathbf{D}^{(1)}$ , since

$$D_{pq}^* = \langle 0 | a_p^{\dagger} a_q | 0 \rangle = \langle 0 | a_q^{\dagger} a_p | 0 \rangle = D_{qp}, \quad (47)$$

which is also symmetric for real wave functions.

Since the elements of the one-electron density matrix are inner products of states in the subspace  $F(m, n-1)$  or trivially equal to zero, the one-electron density matrix is positive semidefinite. The occupation numbers,  $\omega_p$ , of the electronic state are given by the diagonal elements of  $\mathbf{D}^{(1)}$ , i.e. the expectation values of the occupation-number operators,  $n_p$ , in  $F(m, n)$

$$\omega_p = D_{pp}^{(1)} = \langle 0 | n_p | 0 \rangle. \quad (48)$$

The diagonal elements of  $\mathbf{D}^{(1)}$  reduce to the usual occupation numbers,  $k_p$ , whenever the reference state is an eigenfunction of the ON operators, i.e. when the reference state is an ON vector

$$\langle k | n_p | k \rangle = k_p. \quad (49)$$

Considering that the ON operators are projectors,  $\omega_p$  can be written over the projected electronic

state

$$n_p |0\rangle = \sum_k k_p c_k |k\rangle, \quad (50)$$

as the squared norm of the part of the reference state where the spin orbital  $\varphi_p$  is occupied in each ON vector

$$\omega_p = \langle 0 | n_p n_p | 0 \rangle = \sum_k k_p |c_k|^2. \quad (51)$$

Recalling the  $n$ -representability conditions (eq. (13) and ??), the occupation numbers are real numbers between zero and one and its sum, the trace of the density matrix, is equal to the total number of electrons in the system

$$\text{tr } \mathbf{D}^{(1)} = \sum_p \omega_p = \sum_p \langle 0 | n_p | 0 \rangle = n. \quad (52)$$

Since  $\mathbf{D}^{(1)}$  is Hermitian, it can be diagonalized by the spectral decomposition for a unitary matrix,  $\mathbf{U}$ , as

$$\mathbf{D}^{(1)} = \mathbf{U} \lambda \mathbf{U}^\dagger. \quad (53)$$

The eigenvalues,  $\lambda_p$ , are real numbers

$$0 \leq \lambda_p \leq 1, \quad (54)$$

known as the natural-orbital occupation numbers, which also fulfill

$$\sum_p \lambda_p = n. \quad (55)$$

A new set of spin orbitals, the natural spin orbitals, are obtained from the eigenvectors, the columns of  $\mathbf{U}$ , resulted from eq. (53).

With the definition of the one-electron density matrix given in eq. (46), eq. (44) can be further simplified introducing the *two-electron spin-orbital density matrix*

$$D_{pqrs}^{(2)} = d_{pqrs} - \delta_{qr} D_{ps}, \quad (56)$$

as

$$E_0 = \sum_{pq} h_{pq} D_{pq}^{(1)} + \frac{1}{2} \sum_{pqrs} (pq | rs) D_{pqrs}^{(2)}. \quad (57)$$

The elements of  $\mathbf{D}^{(2)}$  are not all independent because of the anticommutation relations between the creation and annihilation operators

$$D_{pqrs}^{(2)} = -D_{rqp s}^{(2)} = -D_{psrq}^{(2)} = D_{rspq}^{(2)}. \quad (58)$$

Also, in accordance with the Pauli principle

$$D_{pqps}^{(2)} = D_{pqrq}^{(2)} = D_{pqpq}^{(2)} = 0. \quad (59)$$

The two-electron density matrix can be rewritten as a  $m(m-1)/2 \times m(m-1)/2$  matrix,  $\mathbf{T}$

$$T_{pq,rs} = \langle 0 | a_p^\dagger a_q^\dagger a_s a_r | 0 \rangle, \quad p > q, \quad r > s, \quad (60)$$

with composite indices  $pq$ , composing a subset of  $\mathbf{D}^{(2)}$  by a reordering of the middle indices

$$T_{pq,rs} = D_{pqrs}^{(2)}, \quad p > q, \quad r > s. \quad (61)$$

The matrix  $\mathbf{T}$  is also Hermitian and, therefore, symmetric for real wave functions. Furthermore, it is positive semidefinite since its elements are either zero or inner products of states in  $F(m, n-2)$ .

The diagonal elements of  $\mathbf{T}$ , considering that  $p > q$  and introducing the ON operators, are given by

$$\omega_{pq} = T_{pq,pq} = \langle 0 | a_p^\dagger a_q^\dagger a_q a_p | 0 \rangle = \langle 0 | n_p n_q | 0 \rangle, \quad (62)$$

which can be interpreted as simultaneous occupations of pairs of spin orbitals from the part of the wave function where the spin orbitals  $\varphi_p$  and  $\varphi_q$  are simultaneously occupied. It can be simply denoted as pair occupations, and fulfill the following conditions:

1. The simultaneous occupation of a given spin-orbital cannot be greater than that of the individual spin orbitals

$$0 \leq \omega_{pq} \leq \min[\omega_p, \omega_q] \leq 1. \quad (63)$$

2. The sum of all pair occupations  $\omega_{pq}$  is equal to the number of electron pairs in the system

$$\text{tr } \mathbf{T} = \sum_{p>q} \langle 0 | n_p n_q | 0 \rangle = \frac{1}{2} \sum_{pq} \langle 0 | n_p n_q | 0 \rangle - \frac{1}{2} \sum_p \langle 0 | n_p | 0 \rangle = \frac{1}{2} n(n-1). \quad (64)$$

The  $\mathbf{T}$  matrix for a state with a single ON vector,  $|k\rangle$ , has a straightforward diagonal form and may be constructed directly from the one- electron density matrix as

$$T_{pq,rs}^{[k]} = D_{pq}^{[k]} D_{qs}^{[k]}. \quad (65)$$

Similarly, the one-electron density matrix can be used to obtain the expectation value of any one or two-electron operator.

The picture of an uncorrelated description of the electronic system, in which the simultaneous occupations of pairs of spin orbitals are merely the products of the individual occupations, is compatible with this result. In the case of a general electronic state containing several ON vectors,  $\mathbf{T}$  is generally non-diagonal and cannot be produced directly from the elements of the one electron density matrix. As demonstrated for the one-electron density matrix in eq. (53),  $\mathbf{D}^{(2)}$  can be diagonalized in practice using the spectral decomposition.

Therefore, it can be stated that the one-electron density matrix probes the individual occupancies of the spin orbitals and describes how the  $n$  electrons are distributed among the  $m$  spin orbitals, whereas the two-electron density matrix probes the simultaneous occupations of the spin orbitals and describes how the  $n(n-1)/2$  electron pairs are distributed among the  $m(m-1)/2$  spin-orbital pairs [1].

### 1.3.1 Equivalence between spin-orbital and coordinate representations

In this section, the equivalence relations between first and second quantization formalisms for the density matrices are established.

The one-electron density matrix in the spin-orbital representation was introduced in second quantization for the evaluation of the expectation values of one-electron operators in the form

$$\langle 0 | \hat{g} | 0 \rangle = \sum_{pq} D_{pq}^{(1)} g_{pq}, \quad (66)$$

with the  $g_{pq}$  integrals

$$g_{pq} = \int \varphi_p(\mathbf{x}_1)^* g(\mathbf{x}_1) \varphi_q(\mathbf{x}_1) d\mathbf{x}_1. \quad (67)$$

Combining both, the 1-RDM in the coordinates representation given in eq. (9) can be written in terms of the one-electron density matrix in the spin-orbital representation given in eq. (46), as

$$\gamma(\mathbf{x}_1, \mathbf{x}'_1) = \sum_{pq} D_{pq}^{(1)} \varphi_p^*(\mathbf{x}'_1) \varphi_q(\mathbf{x}_1). \quad (68)$$

Similarly, the following relationship between the two-electron density matrix in the coordinate representation (eq. (34)) and that in the spin-orbital representation (eq. (56)) may be established

$$\Gamma(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}'_1, \mathbf{x}'_2) = \frac{1}{2} \sum_{pqrs} D_{pqrs}^{(2)} \varphi_p^*(\mathbf{x}'_1) \varphi_q(\mathbf{x}_1) \varphi_r^*(\mathbf{x}'_2) \varphi_s(\mathbf{x}_2). \quad (69)$$

## 1.4 Construction of 2-RDM in terms of 1-RDM

Assuming the cumulant expansion of 2-RDM [27] which consists of writing  $\Gamma$  as the antisymmetrized product of  $\gamma$  and the cumulant part,  $\lambda$ , being a functional of  $\gamma$

$$\Gamma_{pqrs} = n_p n_q (\delta_{pr} \delta_{qs} - \delta_{ps} \delta_{qr}) + \lambda_{pqrs}[\gamma], \quad (70)$$

novel one-electron density matrix functionals can be developed by finding approximations for the cumulant part imposing known conditions which the exact cumulant satisfies.

(...)

## 1.5 Approximate functionals

In the development of approximate functionals, it is convenient to search for approximations to the exchange-correlation term,  $E_{xc}$ , defined as

$$E_{xc}[\gamma] = E_{ee}[\gamma] - E_H[\gamma], \quad (71)$$

which can be decomposed into an exchange part (given in eq. (71)) and the correlation functional,  $E_c$

$$E_c[\gamma] = E_{xc}[\gamma] - E_x[\gamma]. \quad (72)$$

(...)

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