Worksheet 4.

Mathematical Foundations of Quantum Mechanics

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Statement: Calculate the Clebsh-Gordan coefficients corresponding to the composition of the angular momenta $l_1 = 1$ and $l_2 = 2$ by using the step ladder angular momentum operators.

Solution: Given the step ladder angular momentum operators

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y, \qquad \hat{L}_- = \hat{L}_x - i\hat{L}_y, \tag{1}$$

that act on the eigenfunctions of $\hat{L}^2$, the spherical harmonics $|Y_{lm}\rangle \equiv |l, m\rangle$, as

$$\hat{L}_+ |l, m\rangle = \sqrt{l(l+1) - m(m+1)}\hbar\, |l, m+1\rangle\,, \tag{2}$$

$$\hat{L}_- |l, m\rangle = \sqrt{l(l+1) - m(m-1)}\hbar\, |l, m-1\rangle\,, \tag{3}$$

and can be expressed, for each electron, as

$$\hat{L}_+ = \hat{L}_{1-} + \hat{L}_{2-}, \quad \hat{L}_- = \hat{L}_{1-} + \hat{L}_{2-}. \tag{4}$$

The possible values of the composition of the angular momenta l and m_l for the first electron, $|l, m\rangle_1$, and for the second electron, $|l, m\rangle_2$, are

$$l_1 = 2 \begin{cases} m_{l_1} = +2 & |2,2\rangle_1 \\ m_{l_1} = +1 & |2,1\rangle_1 \\ m_{l_1} = 0 & |2,0\rangle_1 \\ m_{l_1} = -1 & |2,-1\rangle_1 \\ m_{l_1} = -2 & |2,-2\rangle_1 \end{cases} ; \quad l_2 = 1 \begin{cases} m_{l_2} = +1 & |1,1\rangle_2 \\ m_{l_2} = 0 & |1,0\rangle_2 \\ m_{l_2} = -1 & |1,-1\rangle_2 \end{cases} . \tag{5}$$

Then, the total angular momentum $L = l_1 + l_2, \ldots, |l_1 - l_2|$, is

$$L = 3, 2, 1. \tag{6}$$

Now, for each component of the total angular momentum, the posible combinations of $|L, M\rangle$ are

$$L=3\begin{cases} M=+3 & |3,3\rangle \\ M=+2 & |3,2\rangle \\ M=+1 & |3,1\rangle \\ M=0 & |3,0\rangle \\ M=-1 & |3,-1\rangle \\ M=-2 & |3,-2\rangle \\ M=-3 & |3,-3\rangle \end{cases} ; \quad L=2\begin{cases} M=+2 & |2,2\rangle \\ M=+1 & |2,1\rangle \\ M=0 & |2,0\rangle \\ M=-1 & |2,-1\rangle \\ M=-2 & |2,-2\rangle \end{cases} ; \quad L=1\begin{cases} M=+1 & |1,1\rangle \\ M=0 & |1,0\rangle \\ M=-1 & |1,-1\rangle \end{cases} . \tag{7}$$

The Clebsh-Gordan coefficients must be calculated for each of the combinations above.

1 $L = 3$

Starting with the $L = 3$, $M = 3$ state, there is only one possible combination of states for electron 1 and electron 2, and that is

$$\boxed{|3,3\rangle = |2,2\rangle_1 |1,1\rangle_2 .} \tag{8}$$

(Note: the results are summarized at the end of the document)

In order to construct the rest of combinations for $L = 3$, we can apply the lower operator $\hat{L}_-$ [eq. (3)] to the state $|3, 3\rangle$ to get $|3, 2\rangle$

$$\hat{L}_- |3,3\rangle = \sqrt{3\,(3+1) - 3\,(3-1)}\hbar\,|3,2\rangle = \sqrt{6}\hbar\,|3,2\rangle . \quad (= \text{LHS}) \tag{9}$$

The ladder operator must be also applied to the right hand side (RHS) of the eq. (8). Considering the separation of the ladder operator as [eq. (4)]:

$$\hat{L}_- = \hat{L}_{1-} + \hat{L}_{2-}, \tag{10}$$

then

$$\hat{L}_- |2,2\rangle_1 |1,1\rangle_2 = \left(\hat{L}_{1-} + \hat{L}_{2-}\right) |2,2\rangle_1 |1,1\rangle_2 = \hat{L}_{1-} |2,2\rangle_1 |1,1\rangle_2 + \hat{L}_{2-} |2,2\rangle_1 |1,1\rangle_2 , \tag{11}$$

and the fact that each component of the ladder operator acts only on the corresponding electron

$$\left(\hat{L}_{1-} |2,2\rangle_1\right) |1,1\rangle_2 = \left(\sqrt{2\,(2+1) - 2\,(2-1)}\hbar\,|2,1\rangle_1\right) |1,1\rangle_2 = 2\hbar\,|2,1\rangle_1 |1,1\rangle_2 , \tag{12}$$

$$\left(\hat{L}_{2-} |1,1\rangle_2\right) |2,2\rangle_1 = \left(\sqrt{1\,(1+1) - 1\,(1-1)}\hbar\,|1,0\rangle_2\right) |2,2\rangle_1 = \sqrt{2}\hbar\,|2,2\rangle_1 |1,0\rangle_2 , \tag{13}$$

then

$$\hat{L}_- |2,2\rangle_1 |1,1\rangle_2 = 2\hbar\,|2,1\rangle_1 |1,1\rangle_2 + \sqrt{2}\hbar\,|2,2\rangle_1 |1,0\rangle_2 . \quad (= \text{RHS}) \tag{14}$$

Therefore, equating LHS = RHS [equations eq. (9) and eq. (14)]

$$\sqrt{6}\hbar\,|3,2\rangle = 2\hbar\,|2,1\rangle_1 |1,1\rangle_2 + \sqrt{2}\hbar\,|2,2\rangle_1 |1,0\rangle_2 , \tag{15}$$

and factorizing $\sqrt{2}\hbar$ on the RHS

$$\sqrt{6}\hbar\,|3,2\rangle = \sqrt{2}\hbar\left(\sqrt{2}\,|2,1\rangle_1\,|1,1\rangle_2 + |2,2\rangle_1\,|1,0\rangle_2\right), \tag{16}$$

then

$$\boxed{|3,2\rangle = \frac{1}{\sqrt{3}}\left(|2,2\rangle_1\,|1,0\rangle_2 + \sqrt{2}\,|2,1\rangle_1\,|1,1\rangle_2\right).} \tag{17}$$

Just as before, the down operator $\hat{L}_-$ can be applied again over $|3,2\rangle$ to get $|3,1\rangle$

$$\hat{L}_-\,|3,2\rangle = \sqrt{3\,(3+1) - 2\,(2-1)}\hbar\,|3,1\rangle = \sqrt{10}\hbar\,|3,1\rangle\,. \quad (=\text{LHS}) \tag{18}$$

and to the RHS

$$\hat{L}_-\underbrace{\left\{\frac{1}{\sqrt{3}}\left(\sqrt{2}\,|2,1\rangle_1\,|1,1\rangle_2 + |2,2\rangle_1\,|1,0\rangle_2\right)\right\}}_{\{\text{RHS}\}} = \sqrt{\frac{1}{3}}\left(\sqrt{2}\,\underbrace{\hat{L}_-\,|2,1\rangle_1\,|1,1\rangle_2}_{(*)} + \underbrace{\hat{L}_-\,|2,2\rangle_1\,|1,0\rangle_2}_{(**)}\right). \tag{19}$$

The result of the operations are calculated separately

$$\begin{aligned}(*)\ \hat{L}_-\,|2,1\rangle_1\,|1,1\rangle_2 &= \left(\hat{L}_{1-}\,|2,1\rangle_1\right)|1,1\rangle_2 + \left(\hat{L}_{2-}\,|1,1\rangle_2\right)|2,1\rangle_1 = \\ &= \sqrt{2\,(2+1) - 1\,(1-1)}\hbar\,|2,0\rangle_1\,|1,1\rangle_2 + \sqrt{1\,(1+1) - 1\,(1-1)}\hbar\,|1,0\rangle_2\,|2,1\rangle_1 = \\ &= \sqrt{6}\hbar\,|2,0\rangle_1\,|1,1\rangle_2 + \sqrt{2}\hbar\,|1,0\rangle_2\,|2,1\rangle_1\,,\end{aligned} \tag{20}$$

and

$$\begin{aligned}(**)\ \hat{L}_-\,|2,2\rangle_1\,|1,0\rangle_2 &= \left(\hat{L}_{1-}\,|2,2\rangle_1\right)|1,0\rangle_2 + \left(\hat{L}_{2-}\,|1,0\rangle_2\right)|2,2\rangle_1 = \\ &= \sqrt{2\,(2+1) - 2\,(2-1)}\hbar\,|2,1\rangle_1\,|1,0\rangle_2 + \sqrt{1\,(1+1) - 0\,(0-1)}\hbar\,|1,-1\rangle_2\,|2,2\rangle_1 = \\ &= 2\hbar\,|2,1\rangle_1\,|1,0\rangle_2 + \sqrt{2}\hbar\,|1,-1\rangle_2\,|2,2\rangle_1\,.\end{aligned} \tag{21}$$

In eq. (19)

$$\begin{aligned}\hat{L}_-\,\{\text{RHS}\} = \frac{\hbar}{\sqrt{3}}\Big\{&\sqrt{2}\left(\sqrt{6}\,|2,0\rangle_1\,|1,1\rangle_2 + \sqrt{2}\,|1,0\rangle_2\,|2,1\rangle_1\right) + \\ &+ 2\,|2,1\rangle_1\,|1,0\rangle_2 + \sqrt{2}\hbar\,|1,-1\rangle_2\,|2,2\rangle_1\Big\}.\end{aligned} \tag{22}$$

It can be rewritten as

$$\begin{aligned}\hat{L}_-\,\{\text{RHS}\} &= \frac{\hbar}{\sqrt{3}}\left(2\sqrt{3}\,|2,0\rangle_1\,|1,1\rangle_2 + 2\,|2,1\rangle_1\,|1,0\rangle_2 + 2\,|2,1\rangle_1\,|1,0\rangle_2 + \sqrt{2}\,|2,2\rangle_1\,|1,-1\rangle_2\right) = \\ &= \frac{\hbar}{\sqrt{3}}\left(2\sqrt{3}\,|2,0\rangle_1\,|1,1\rangle_2 + 4\,|2,1\rangle_1\,|1,0\rangle_2 + \sqrt{2}\,|2,2\rangle_1\,|1,-1\rangle_2\right).\end{aligned} \tag{23}$$

Then, with eq. (18)

$$\sqrt{10}\hbar\,|3,1\rangle = \frac{\hbar}{\sqrt{3}}\left(2\sqrt{3}\,|2,0\rangle_1\,|1,1\rangle_2 + 4\,|2,1\rangle_1\,|1,0\rangle_2 + \sqrt{2}\,|2,2\rangle_1\,|1,-1\rangle_2\right), \tag{24}$$

therefore
$$\boxed{|3,1\rangle = \sqrt{\frac{1}{30}}\left(\sqrt{2}\,|2,2\rangle_1\,|1,-1\rangle_2 + 4\,|2,1\rangle_1\,|1,0\rangle_2 + 2\sqrt{3}\,|2,0\rangle_1\,|1,1\rangle_2\right).} \tag{25}$$

Finally, applying again $\hat{L}_-$ to $|3,1\rangle$ to obtain $|3,0\rangle$

$$\hat{L}_-\,|3,1\rangle = \sqrt{3\,(3+1)-1\,(1-1)}\hbar\,|3,0\rangle = \sqrt{12}\hbar\,|3,0\rangle = 2\sqrt{3}\hbar\,|3,0\rangle\,, \quad (=\text{LHS}) \tag{26}$$

and to the RHS of the eq. (25)

$$\hat{L}_-\,\{\text{RHS}\} = \sqrt{\frac{1}{30}}\left(2\sqrt{3}\,\underbrace{\hat{L}_-\,|2,0\rangle_1\,|1,1\rangle_2}_{(*)} + 4\,\underbrace{\hat{L}_-\,|2,1\rangle_1\,|1,0\rangle_2}_{(**)} + \sqrt{2}\,\underbrace{\hat{L}_-\,|2,2\rangle_1\,|1,-1\rangle_2}_{(***)}\right) \tag{27}$$

Calculating separately $(*)$

$$\begin{aligned}(*)\ \hat{L}_-\,|2,0\rangle_1\,|1,1\rangle_2 &= \hat{L}_{1-}\,|2,0\rangle_1\,|1,1\rangle_2 + \hat{L}_{2-}\,|1,1\rangle_2\,|2,0\rangle_1 = \\ &= \sqrt{6}\hbar\,|2,-1\rangle_1\,|1,1\rangle_2 + \sqrt{2}\hbar\,|2,0\rangle_1\,|1,0\rangle_2\,,\end{aligned} \tag{28}$$

and $(**)$

$$\begin{aligned}(**)\ \hat{L}_-\,|2,1\rangle_1\,|1,0\rangle_2 &= \hat{L}_{1-}\,|2,1\rangle_1\,|1,0\rangle_2 + \hat{L}_{2-}\,|1,0\rangle_2\,|2,1\rangle_1 = \\ &= \sqrt{6}\hbar\,|2,0\rangle_1\,|1,0\rangle_2 + \sqrt{2}\hbar\,|2,1\rangle_1\,|1,-1\rangle_2\,,\end{aligned} \tag{29}$$

and, lastly, $(***)$

$$\begin{aligned}(***)\ \hat{L}_-\,|2,2\rangle_1\,|1,-1\rangle_2 &= \hat{L}_{1-}\,|2,2\rangle_1\,|1,-1\rangle_2 + \hat{L}_{2-}\,|1,-1\rangle_2\,|2,2\rangle_1 = \\ &= 2\hbar\,|2,1\rangle_1\,|1,-1\rangle_2\,.\end{aligned} \tag{30}$$

Then, in eq. (27)

$$\begin{aligned}\hat{L}_-\,\{\text{RHS}\} = \sqrt{\frac{1}{30}}\Big\{&2\sqrt{3}\left(\sqrt{6}\hbar\,|2,-1\rangle_1\,|1,1\rangle_2 + \sqrt{2}\hbar\,|2,0\rangle_1\,|1,0\rangle_2\right) + \\ &+ 4\left(\sqrt{6}\hbar\,|2,0\rangle_1\,|1,0\rangle_2 + \sqrt{2}\hbar\,|2,1\rangle_1\,|1,-1\rangle_2\right) + \\ &+ \sqrt{2}\,(2\hbar\,|2,1\rangle_1\,|1,-1\rangle_2)\Big\}\,,\end{aligned} \tag{31}$$

that, factorizing, can be rewritten as

$$\hat{L}_-\,\{\text{RHS}\} = 6\hbar\sqrt{\frac{1}{15}}\left(|2,-1\rangle_1\,|1,1\rangle_2 + \sqrt{3}\,|2,0\rangle_1\,|1,0\rangle_2 + |2,1\rangle_1\,|1,-1\rangle_2\right). \tag{32}$$

Doing LHS = RHS with eq. (26)

$$2\sqrt{3}\hbar\,|3,0\rangle = 6\hbar\sqrt{\frac{1}{15}}\left(|2,-1\rangle_1\,|1,1\rangle_2 + \sqrt{3}\,|2,0\rangle_1\,|1,0\rangle_2 + |2,1\rangle_1\,|1,-1\rangle_2\right), \tag{33}$$

reads

$$\boxed{|3,0\rangle = \sqrt{\frac{1}{5}}\left(|2,1\rangle_1 |1,-1\rangle_2 + \sqrt{3}\,|2,0\rangle_1 |1,0\rangle_2 + |2,-1\rangle_1 |1,1\rangle_2\right).} \tag{34}$$

Now, the rest of combinations for $L = 3$ can be calculated analogously starting with $|3,-3\rangle$ and applying the rising ladder operator $\hat{L}_+$ [eq. (2)]. The only combination for $|3,-3\rangle$ is

$$\boxed{|3,-3\rangle = |2,-2\rangle_1 |1,-1\rangle_2 .} \tag{35}$$

Applying $\hat{L}_+$ to the LHS

$$\hat{L}_+ |3,-3\rangle = \sqrt{3\,(3+1) - (-3)\,((-3)+1)}\hbar\,|3,-2\rangle = \sqrt{6}\hbar\,|3,-2\rangle\,, \quad (= \text{LHS}) \tag{36}$$

and to the RHS

$$\hat{L}_+ |2,-2\rangle_1 |1,-1\rangle_2 = \hat{L}_{1-} |2,-2\rangle_1 |1,-1\rangle_2 + \hat{L}_{2-} |1,-1\rangle_2 |2,-2\rangle_1 . \tag{37}$$

As

$$\hat{L}_{1-} |2,-2\rangle_1 = \sqrt{2\,(2+1) - (-2)\,((-2)+1)}\hbar\,|2,-1\rangle_1 = 2\hbar\,|2,-1\rangle_1\,, \tag{38}$$

$$\hat{L}_{2-} |1,-1\rangle_2 = \sqrt{1\,(1+1) - (-1)\,((-1)+1)}\hbar\,|1,0\rangle_2 = \sqrt{2}\hbar\,|1,0\rangle_2\,, \tag{39}$$

then

$$\hat{L}_+ |2,-2\rangle_1 |1,-1\rangle_2 = 2\hbar\,|2,-1\rangle_1 |1,-1\rangle_2 + \sqrt{2}\hbar\,|2,-2\rangle_1 |1,0\rangle_2 . \quad (= \text{RHS}) \tag{40}$$

Equating LHS = RHS

$$\begin{aligned} \sqrt{6}\hbar\,|3,-2\rangle &= 2\hbar\,|2,-1\rangle_1 |1,-1\rangle_2 + \sqrt{2}\hbar\,|2,-2\rangle_1 |1,0\rangle_2 = \\ &= \sqrt{2}\hbar\left(\sqrt{2}\,|2,-1\rangle_1 |1,-1\rangle_2 + |2,-2\rangle_1 |1,0\rangle_2\right), \end{aligned} \tag{41}$$

$|3,-2\rangle$ is found and reads

$$\boxed{|3,-2\rangle = \sqrt{\frac{1}{3}}\left(\sqrt{2}\,|2,-1\rangle_1 |1,-1\rangle_2 + |2,-2\rangle_1 |1,0\rangle_2\right).} \tag{42}$$

Now, $\hat{L}_+$ can be applied again over $|3,-2\rangle$

$$\hat{L}_+ |3,-2\rangle = \sqrt{3\,(3+1) - (-2)\,((-2)+1)}\hbar\,|3,-1\rangle = \sqrt{10}\hbar\,|3,-1\rangle\,, \quad (= \text{LHS}) \tag{43}$$

and for the RHS

$$\hat{L}_+ \{\text{RHS}\} = \sqrt{\frac{1}{3}}\left(\sqrt{2}\hat{L}_+ |2,-1\rangle_1 |1,-1\rangle_2 + \hat{L}_+ |2,-2\rangle_1 |1,0\rangle_2\right), \tag{44}$$

as

$$\hat{L}_+ |2,-1\rangle_1 |1,-1\rangle_2 = \hat{L}_{1-} |2,-1\rangle_1 |1,-1\rangle_2 + \hat{L}_{2-} |1,-1\rangle_2 |2,-1\rangle_1 = \\ = \sqrt{6}\hbar |2,0\rangle_1 |1,-1\rangle_2 + \sqrt{2}\hbar |1,0\rangle_2 |2,-1\rangle_1 \\ = \sqrt{2}\hbar \left(\sqrt{3} |2,0\rangle_1 |1,-1\rangle_2 + |2,-1\rangle_1 |1,0\rangle_2\right), \tag{45}$$

and

$$\hat{L}_+ |2,-2\rangle_1 |1,0\rangle_2 = \hat{L}_{1-} |2,-2\rangle_1 |1,0\rangle_2 + \hat{L}_{2-} |1,0\rangle_2 |2,-2\rangle_1 = \\ = 2\hbar |2,-1\rangle_1 |1,0\rangle_2 + \sqrt{2}\hbar |1,1\rangle_2 |2,-2\rangle_1 \\ = \sqrt{2}\hbar \left(\sqrt{2} |2,-1\rangle_1 |1,0\rangle_2 + |2,-2\rangle_1 |1,1\rangle_2\right), \tag{46}$$

then

$$\hat{L}_+ \{\text{RHS}\} = \sqrt{\frac{1}{3}} \left\{\sqrt{2}\sqrt{2}\hbar \left(\sqrt{3} |2,0\rangle_1 |1,-1\rangle_2 + |2,-1\rangle_1 |1,0\rangle_2\right) + \right. \\ \left. + \sqrt{2}\hbar \left(\sqrt{2} |2,-1\rangle_1 |1,0\rangle_2 + |2,-2\rangle_1 |1,1\rangle_2\right)\right\} \tag{47}$$

$$= \sqrt{\frac{1}{3}}\hbar \left(2\sqrt{3} |2,0\rangle_1 |1,-1\rangle_2 + 4 |2,-1\rangle_1 |1,0\rangle_2 + \sqrt{2} |2,-2\rangle_1 |1,1\rangle_2\right). \quad (= \text{RHS}) \tag{48}$$

Doing LHS = RHS

$$\sqrt{10}\hbar |3,-1\rangle = \sqrt{\frac{1}{3}}\hbar \left(2\sqrt{3} |2,0\rangle_1 |1,-1\rangle_2 + 4 |2,-1\rangle_1 |1,0\rangle_2 + \sqrt{2} |2,-2\rangle_1 |1,1\rangle_2\right), \tag{49}$$

$|3,-1\rangle$ reads

$$|3,-1\rangle = \sqrt{\frac{1}{30}} \left(2\sqrt{3} |2,0\rangle_1 |1,-1\rangle_2 + 4 |2,-1\rangle_1 |1,0\rangle_2 + \sqrt{2} |2,-2\rangle_1 |1,1\rangle_2\right), \tag{50}$$

or

$$\boxed{|3,-1\rangle = \sqrt{\frac{1}{15}} \left(\sqrt{6} |2,0\rangle_1 |1,-1\rangle_2 + 2\sqrt{2} |2,-1\rangle_1 |1,0\rangle_2 + |2,-2\rangle_1 |1,1\rangle_2\right).} \tag{51}$$

2 $L = 2$

Now, the ladder operators can't be applied ver the already found combinations, since it is not the same value of L. Therefore, the new states for $L = 2$ are found imposing orthogonality conditions

$$\left\langle L, M_L \,\middle|\, L', M_{L'} \right\rangle = 0 \qquad L' \neq L. \tag{52}$$

Then, $|2,2\rangle$ can be found imposing

$$\langle 3,2 \,|\, 2,2\rangle = 0. \tag{53}$$

Rewriting $|3,2\rangle$ from eq. (17) as

$$|3,2\rangle = \sqrt{\frac{1}{3}}\left(\sqrt{2}\,|2,1\rangle_1\,|1,1\rangle_2 + |2,2\rangle_1\,|1,0\rangle_2\right) = \sqrt{\frac{1}{3}}\left(\sqrt{2}\,|\psi_1\rangle + |\psi_2\rangle\right), \tag{54}$$

$|2,2\rangle$ can be written as

$$|2,2\rangle = c_1\,|\psi_1\rangle + c_2\,|\psi_2\rangle\,. \tag{55}$$

Imposing now the orthogonality condition from eq. (53)

$$\langle 3,2\,|\,2,2\rangle = \sqrt{\frac{1}{3}}\left(\sqrt{2}\,\langle\psi_1| + \langle\psi_2|\right)(c_1\,|\psi_1\rangle + c_2\,|\psi_2\rangle) = 0, \tag{56}$$

and operating considering that $\langle\psi_i\,|\,\psi_j\rangle = \delta_{ij}$

$$\begin{aligned}\langle 3,2\,|\,2,2\rangle &= \sqrt{\frac{1}{3}}\left(\sqrt{2}c_1\,\langle\psi_1\,|\,\psi_1\rangle + c_1\,\langle\psi_2\,|\,\psi_1\rangle + \sqrt{2}c_2\,\langle\psi_1\,|\,\psi_2\rangle + c_2\,\langle\psi_2\,|\,\psi_2\rangle\right) = \\ &= \sqrt{\frac{1}{3}}\left(\sqrt{2}c_1 + c_2\right),\end{aligned} \tag{57}$$

then

$$\sqrt{\frac{1}{3}}\left(\sqrt{2}c_1 + c_2\right) = 0 \implies c_1 = -\frac{1}{\sqrt{2}}c_2. \tag{58}$$

Imposing now the normalisation condition

$$\sum_{i=1} |c_i|^2 = 1. \tag{59}$$

Then

$$|c_1|^2 + |c_2|^2 = 1, \tag{60}$$

substituting c_1 from eq. (58)

$$\left|-\frac{1}{\sqrt{2}}c_2\right|^2 + |c_2|^2 = \frac{3}{2}\,|c_2|^2 = 1 \implies c_2 = \sqrt{\frac{2}{3}}, \tag{61}$$

and

$$c_1 = -\frac{1}{\sqrt{2}}c_2 = -\frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}} = -\frac{1}{\sqrt{3}} \tag{62}$$

Then, eq. (55) can be written as

$$|2,2\rangle = \frac{1}{\sqrt{3}}\left(-\,|\psi_1\rangle + \sqrt{2}\,|\psi_2\rangle\right) = \frac{1}{\sqrt{3}}\left(-\,|2,1\rangle_1\,|1,1\rangle_2 + \sqrt{2}\,|2,2\rangle_1\,|1,0\rangle_2\right). \tag{63}$$

Therefore, $|2,2\rangle$ reads

$$\boxed{|2,2\rangle = \sqrt{\frac{1}{3}}\left(\sqrt{2}\,|2,2\rangle_1\,|1,0\rangle_2 - |2,1\rangle_1\,|1,1\rangle_2\right).} \tag{64}$$

Now, the ladder operator can be applied to get the rest of combinations for $L = 2$. Applying $\hat{L}_-$

over $|2,2\rangle$ [eq. (64)]

$$\hat{L}_{-}|2,2\rangle = \sqrt{2(2+1)-2(2-1)}\hbar|2,1\rangle = 2\hbar|2,1\rangle\,, \quad (=\text{LHS}) \tag{65}$$

and to the RHS

$$\hat{L}_{-}\{\text{RHS}\} = \sqrt{\frac{1}{3}}\left(\sqrt{2}\hat{L}_{-}|2,2\rangle_1|1,0\rangle_2 - \hat{L}_{-}|2,1\rangle_1|1,1\rangle_2\right). \tag{66}$$

Calculating separately $\hat{L}_{-}|2,2\rangle_1|1,0\rangle_2$

$$\begin{aligned}\hat{L}_{-}|2,2\rangle_1|1,0\rangle_2 &= \hat{L}_{1-}|2,2\rangle_1|1,0\rangle_2 + \hat{L}_{2-}|1,0\rangle_2|2,2\rangle_1 = \\ &= 2\hbar|2,1\rangle_1|1,0\rangle_2 + \sqrt{2}\hbar|1,-1\rangle_2|2,2\rangle_1 = \\ &= \sqrt{2}\hbar\left(\sqrt{2}|2,1\rangle_1|1,0\rangle_2 + |2,2\rangle_1|1,-1\rangle_2\right),\end{aligned} \tag{67}$$

and $\hat{L}_{-}|2,1\rangle_1|1,1\rangle_2$

$$\begin{aligned}\hat{L}_{-}|2,1\rangle_1|1,1\rangle_2 &= \hat{L}_{1-}|2,1\rangle_1|1,1\rangle_2 + \hat{L}_{2-}|1,1\rangle_2|2,1\rangle_1 = \\ &= \sqrt{6}\hbar|2,0\rangle_1|1,1\rangle_2 + \sqrt{2}\hbar|1,0\rangle_2|2,1\rangle_1 = \\ &= \sqrt{2}\hbar\left(\sqrt{3}|2,0\rangle_1|1,1\rangle_2 + |2,1\rangle_1|1,0\rangle_2\right),\end{aligned} \tag{68}$$

then, on eq. (66)

$$\begin{aligned}\hat{L}_{-}\{\text{RHS}\} = \sqrt{\frac{1}{3}}\Big\{&\sqrt{2}\left[\sqrt{2}\hbar\left(\sqrt{2}|2,1\rangle_1|1,0\rangle_2 + |2,2\rangle_1|1,-1\rangle_2\right)\right] \\ &-\sqrt{2}\hbar\left(\sqrt{3}|2,0\rangle_1|1,1\rangle_2 + |2,1\rangle_1|1,0\rangle_2\right)\Big\} = \\ = \sqrt{\frac{2}{3}}\hbar&\left\{|2,1\rangle_1|1,0\rangle_2 + \sqrt{2}|2,2\rangle_1|1,-1\rangle_2 - \sqrt{3}|2,0\rangle_1|1,1\rangle_2\right\}. \quad (=\text{RHS})\end{aligned} \tag{69}$$

Equating LHS = RHS with eq. (65) and eq. (69)

$$2\hbar|2,1\rangle = \sqrt{\frac{2}{3}}\hbar\left(|2,1\rangle_1|1,0\rangle_2 + \sqrt{2}|2,2\rangle_1|1,-1\rangle_2 - \sqrt{3}|2,0\rangle_1|1,1\rangle_2\right), \tag{70}$$

then

$$|2,1\rangle = \sqrt{\frac{1}{6}}\left(|2,1\rangle_1|1,0\rangle_2 + \sqrt{2}|2,2\rangle_1|1,-1\rangle_2 - \sqrt{3}|2,0\rangle_1|1,1\rangle_2\right). \tag{71}$$

Therefore, $|2,1\rangle$ reads

$$\boxed{|2,1\rangle = \sqrt{\frac{1}{6}}\left(\sqrt{2}|2,2\rangle_1|1,-1\rangle_2 + |2,1\rangle_1|1,0\rangle_2 - \sqrt{3}|2,0\rangle_1|1,1\rangle_2\right).} \tag{72}$$

As before, applying $\hat{L}_{-}$ over $|2,1\rangle$

$$\hat{L}_{-}|2,1\rangle = \sqrt{2(2+1)-1(1-1)}\hbar|2,0\rangle = \sqrt{6}\hbar|2,0\rangle\,, \quad (=\text{LHS}) \tag{73}$$

and to the RHS

$$\hat{L}_-\{\text{RHS}\} = \sqrt{\frac{1}{6}}\left(\sqrt{2}\hat{L}_-\left|2,2\right\rangle_1\left|1,-1\right\rangle_2 + \hat{L}_-\left|2,1\right\rangle_1\left|1,0\right\rangle_2 - \sqrt{3}\hat{L}_-\left|2,0\right\rangle_1\left|1,1\right\rangle_2\right). \tag{74}$$

Calculating separately $\hat{L}_-\left|2,2\right\rangle_1\left|1,-1\right\rangle_2$

$$\begin{aligned}\hat{L}_-\left|2,2\right\rangle_1\left|1,-1\right\rangle_2 &= \hat{L}_{1-}\left|2,2\right\rangle_1\left|1,-1\right\rangle_2 + \hat{L}_{2-}\left|1,-1\right\rangle_2\left|2,2\right\rangle_1 = \\ &= 2\hbar\left|2,1\right\rangle_1\left|1,-1\right\rangle_2,\end{aligned} \tag{75}$$

and $\hat{L}_-\left|2,1\right\rangle_1\left|1,0\right\rangle_2$

$$\begin{aligned}\hat{L}_-\left|2,1\right\rangle_1\left|1,0\right\rangle_2 &= \hat{L}_{1-}\left|2,1\right\rangle_1\left|1,0\right\rangle_2 + \hat{L}_{2-}\left|1,0\right\rangle_2\left|2,1\right\rangle_1 = \\ &= \sqrt{6}\hbar\left|2,0\right\rangle_1\left|1,0\right\rangle_2 + \sqrt{2}\hbar\left|1,-1\right\rangle_2\left|2,1\right\rangle_1 = \\ &= \sqrt{2}\hbar\left(\sqrt{3}\left|2,0\right\rangle_1\left|1,0\right\rangle_2 + \left|2,1\right\rangle_1\left|1,-1\right\rangle_2\right),\end{aligned} \tag{76}$$

and, lastly, $\hat{L}_-\left|2,0\right\rangle_1\left|1,1\right\rangle_2$

$$\begin{aligned}\hat{L}_-\left|2,0\right\rangle_1\left|1,1\right\rangle_2 &= \hat{L}_{1-}\left|2,0\right\rangle_1\left|1,1\right\rangle_2 + \hat{L}_{2-}\left|1,1\right\rangle_2\left|2,0\right\rangle_1 = \\ &= \sqrt{6}\hbar\left|2,-1\right\rangle_1\left|1,1\right\rangle_2 + \sqrt{2}\hbar\left|1,0\right\rangle_2\left|2,0\right\rangle_1 = \\ &= \sqrt{2}\hbar\left(\sqrt{3}\left|2,-1\right\rangle_1\left|1,1\right\rangle_2 + \left|2,0\right\rangle_1\left|1,0\right\rangle_2\right).\end{aligned} \tag{77}$$

Then, in eq. (74)

$$\begin{aligned}\hat{L}_-\{\text{RHS}\} = \sqrt{\frac{1}{6}}\Big\{&\sqrt{2}\left(2\hbar\left|2,1\right\rangle_1\left|1,-1\right\rangle_2\right) \\ &+ \sqrt{2}\hbar\left(\sqrt{3}\left|2,0\right\rangle_1\left|1,0\right\rangle_2 + \left|2,1\right\rangle_1\left|1,-1\right\rangle_2\right) \\ &-\sqrt{3}\left[\sqrt{2}\hbar\left(\sqrt{3}\left|2,-1\right\rangle_1\left|1,1\right\rangle_2 + \left|2,0\right\rangle_1\left|1,0\right\rangle_2\right)\right]\Big\} = \\ = \sqrt{3}\hbar&\left(\left|2,1\right\rangle_1\left|1,-1\right\rangle_2 - \left|2,-1\right\rangle_1\left|1,1\right\rangle_2\right). \quad (=\text{RHS})\end{aligned} \tag{78}$$

Equating LHS = RHS with eq. (73) and eq. (78)

$$\sqrt{6}\hbar\left|2,0\right\rangle = \sqrt{3}\hbar\left(\left|2,1\right\rangle_1\left|1,-1\right\rangle_2 - \left|2,-1\right\rangle_1\left|1,1\right\rangle_2\right). \tag{79}$$

Then, $\left|2,0\right\rangle$ reads

$$\boxed{\left|2,0\right\rangle = \sqrt{\frac{1}{2}}\left(\left|2,1\right\rangle_1\left|1,-1\right\rangle_2 - \left|2,-1\right\rangle_1\left|1,1\right\rangle_2\right).} \tag{80}$$

Now, instead applying again $\hat{L}_-$ to $\left|2,0\right\rangle$, it is simpler to apply $\hat{L}_+$ to $\left|2,-2\right\rangle$. Imposing the orthogonality condition [eq. (52)] between $\left|3,-2\right\rangle$ and $\left|2,-2\right\rangle$

$$\left\langle 3,-2\,|\,2,-2\right\rangle = 0, \tag{81}$$

rewriting $|3,-2\rangle$ as

$$|3,-2\rangle = \sqrt{\frac{1}{3}}\left(\sqrt{2}\,|2,-1\rangle_1\,|1,-1\rangle_2 + |2,-2\rangle_1\,|1,0\rangle_2\right) = \sqrt{\frac{1}{3}}\left(\sqrt{2}\,|\psi_1\rangle + |\psi_2\rangle\right), \tag{82}$$

and writing $|2,-2\rangle$ as

$$|2,-2\rangle = c_1\,|\psi_1\rangle + c_2\,|\psi_2\rangle\,. \tag{83}$$

In eq. (81)

$$\begin{aligned}\langle 3,-2\,|\,2,-2\rangle &= \sqrt{\frac{1}{3}}\left[\left(\sqrt{2}\,\langle\psi_1| + \langle\psi_2|\right)(c_1\,|\psi_1\rangle + c_2\,|\psi_2\rangle)\right] = \\ &= \sqrt{\frac{1}{3}}\left[\sqrt{2}c_1\,\langle\psi_1\,|\,\psi_1\rangle + \sqrt{2}c_2\,\langle\psi_1\,|\,\psi_2\rangle + c_1\,\langle\psi_2\,|\,\psi_1\rangle + c_2\,\langle\psi_2\,|\,\psi_2\rangle\right] = \\ &= \sqrt{\frac{1}{3}}\left(\sqrt{2}c_1 + c_2\right),\end{aligned} \tag{84}$$

then

$$\sqrt{\frac{1}{3}}\left(\sqrt{2}c_1 + c_2\right) = 0 \implies \sqrt{2}c_1 + c_2 = 0 \implies c_1 = -\frac{1}{\sqrt{2}}c_2. \tag{85}$$

Imposing also the normalisation condition [eq. (59)]

$$|c_1|^2 + |c_2|^2 = 1, \tag{86}$$

and substituting c_1 from eq. (85)

$$\left|-\frac{1}{\sqrt{2}}c_2\right|^2 + |c_2|^2 = 1 \implies \left(\frac{1}{2}+1\right)|c_2|^2 = 1 \implies c_2 = \sqrt{\frac{2}{3}}. \tag{87}$$

Now, on eq. (85)

$$c_1 = -\frac{1}{\sqrt{2}}c_2 = -\frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}} = -\sqrt{\frac{1}{3}}. \tag{88}$$

Therefore, in eq. (83)

$$|2,-2\rangle = -\sqrt{\frac{1}{3}}\,|\psi_1\rangle + \sqrt{\frac{2}{3}}\,|\psi_2\rangle\,, \tag{89}$$

$|2,-2\rangle$ reads

$$\boxed{|2,-2\rangle = \sqrt{\frac{1}{3}}\left(-\,|2,-1\rangle_1\,|1,-1\rangle_2 + \sqrt{2}\,|2,-2\rangle_1\,|1,0\rangle_2\right).} \tag{90}$$

Applying now $\hat{L}_+$ to $|2,-2\rangle$

$$\hat{L}_+\,|2,-2\rangle = \sqrt{2\,(2+1) - (-2)\,((-2)+1)}\hbar\,|2,-1\rangle = 2\hbar\,|2,-1\rangle\,, \quad (=\text{LHS}) \tag{91}$$

and to the RHS

$$\hat{L}_+\,\{\text{RHS}\} = \sqrt{\frac{1}{3}}\left(-\hat{L}_+\,|2,-1\rangle_1\,|1,-1\rangle_2 + \sqrt{2}\hat{L}_+\,|2,-2\rangle_1\,|1,0\rangle_2\right). \tag{92}$$

Calculating separately $\hat{L}_+ |2,-1\rangle_1 |1,-1\rangle_2$

$$\begin{aligned}\hat{L}_+ |2,-1\rangle_1 |1,-1\rangle_2 &= \hat{L}_{1-} |2,-1\rangle_1 |1,-1\rangle_2 + \hat{L}_{2-} |1,-1\rangle_2 |2,-1\rangle_1 = \\ &= \sqrt{6}\hbar |2,0\rangle_1 |1,-1\rangle_2 + \sqrt{2}\hbar |1,0\rangle_2 |2,-1\rangle_1 = \\ &= \sqrt{2}\hbar \left(\sqrt{3} |2,0\rangle_1 |1,-1\rangle_2 + |2,-1\rangle_1 |1,0\rangle_2\right),\end{aligned} \tag{93}$$

and $\hat{L}_+ |2,-2\rangle_1 |1,0\rangle_2$

$$\begin{aligned}\hat{L}_+ |2,-2\rangle_1 |1,0\rangle_2 &= \hat{L}_{1-} |2,-2\rangle_1 |1,0\rangle_2 + \hat{L}_{2-} |1,0\rangle_2 |2,-2\rangle_1 = \\ &= 2\hbar |2,-1\rangle_1 |1,0\rangle_2 + \sqrt{2}\hbar |1,1\rangle_2 |2,-2\rangle_1 = \\ &= \sqrt{2}\hbar \left(\sqrt{2} |2,-1\rangle_1 |1,0\rangle_2 + |2,-2\rangle_1 |1,1\rangle_2\right).\end{aligned} \tag{94}$$

Then, in eq. (92)

$$\begin{aligned}\hat{L}_+ \{\text{RHS}\} = \sqrt{\frac{1}{3}} \Big\{ &-\sqrt{2}\hbar \left(\sqrt{3} |2,0\rangle_1 |1,-1\rangle_2 + |2,-1\rangle_1 |1,0\rangle_2\right) \\ &+\sqrt{2}\sqrt{2}\hbar \left(\sqrt{2} |2,-1\rangle_1 |1,0\rangle_2 + |2,-2\rangle_1 |1,1\rangle_2\right)\Big\} \\ = \sqrt{\frac{2}{3}}\hbar &\left(-\sqrt{3} |2,0\rangle_1 |1,-1\rangle_2 + |2,-1\rangle_1 |1,0\rangle_2 + \sqrt{2} |2,-2\rangle_1 |1,1\rangle_2\right). \quad (= \text{RHS})\end{aligned} \tag{95}$$

Doing LHS = RHS

$$2\hbar |2,-1\rangle = \sqrt{\frac{2}{3}}\hbar \left(-\sqrt{3} |2,0\rangle_1 |1,-1\rangle_2 + |2,-1\rangle_1 |1,0\rangle_2 + \sqrt{2} |2,-2\rangle_1 |1,1\rangle_2\right), \tag{96}$$

$|2,-1\rangle$ reads

$$\boxed{|2,-1\rangle = \sqrt{\frac{1}{6}} \left(-\sqrt{3} |2,0\rangle_1 |1,-1\rangle_2 + |2,-1\rangle_1 |1,0\rangle_2 + \sqrt{2} |2,-2\rangle_1 |1,1\rangle_2\right).} \tag{97}$$

3 $L = 1$

Now, for $L = 1$, $|1,1\rangle$ can be constructed imposing orthogonality conditions as before, but now two orthogonality conditions are needed

$$\langle 3,1 \,|\, 1,1\rangle = 0, \qquad \langle 2,1 \,|\, 1,1\rangle = 0. \tag{98}$$

Rewritting $|3,1\rangle$ as

$$\begin{aligned}|3,1\rangle &= \sqrt{\frac{1}{30}} \left(\sqrt{2} |2,2\rangle_1 |1,-1\rangle_2 + 4 |2,1\rangle_1 |1,0\rangle_2 + 2\sqrt{3} |2,0\rangle_1 |1,1\rangle_2\right) = \\ &= \sqrt{\frac{1}{30}} \left(\sqrt{2} |\psi_1\rangle + 4 |\psi_2\rangle + 2\sqrt{3} |\psi_3\rangle\right),\end{aligned} \tag{99}$$

and $|2,1\rangle$ as

$$|2,1\rangle = \sqrt{\frac{1}{6}}\left(\sqrt{2}\,|2,2\rangle_1\,|1,-1\rangle_2 + |2,1\rangle_1\,|1,0\rangle_2 - \sqrt{3}\,|2,0\rangle_1\,|1,1\rangle_2\right) = \\ = \sqrt{\frac{1}{6}}\left(\sqrt{2}\,|\psi_1\rangle + |\psi_2\rangle - \sqrt{3}\,|\psi_3\rangle\right). \tag{100}$$

$|1,1\rangle$ can be written as

$$|1,1\rangle = c_1\,|\psi_1\rangle + c_2\,|\psi_2\rangle + c_3\,|\psi_3\rangle\,. \tag{101}$$

Imposing the first condition from eq. (98) the first equation (I) from the system of equations is found

$$\langle 3,1\,|\,1,1\rangle = \sqrt{\frac{1}{30}}\left[\left(\sqrt{2}\,\langle\psi_1| + 4\,\langle\psi_2| + 2\sqrt{3}\,\langle\psi_3|\right)(c_1\,|\psi_1\rangle + c_2\,|\psi_2\rangle + c_3\,|\psi_3\rangle)\right] = \\ = \sqrt{\frac{1}{30}}\left(\sqrt{2}c_1 + 4c_2 + 2\sqrt{3}c_3\right) = 0, \tag{102}$$

then

$$\sqrt{2}c_1 + 4c_2 + 2\sqrt{3}c_3 = 0. \quad \text{(I)} \tag{103}$$

Repeating for the second condition from eq. (98)

$$\langle 2,1\,|\,1,1\rangle = \sqrt{\frac{1}{6}}\left[\left(\sqrt{2}\,\langle\psi_1| + \langle\psi_2| - \sqrt{3}\,\langle\psi_3|\right)(c_1\,|\psi_1\rangle + c_2\,|\psi_2\rangle + c_3\,|\psi_3\rangle)\right] = \\ = \sqrt{\frac{1}{6}}\left(\sqrt{2}c_1 + c_2 - \sqrt{3}c_3\right) = 0, \tag{104}$$

then

$$\sqrt{2}c_1 + c_2 - \sqrt{3}c_3 = 0. \quad \text{(II)} \tag{105}$$

The third equation is found from the normalisation condition eq. (59)

$$|c_1|^2 + |c_2|^2 + |c_3|^2 = 1. \quad \text{(III)} \tag{106}$$

Then, the system of equations to solve reads

$$\begin{cases} \sqrt{2}c_1 + 4c_2 + 2\sqrt{3}c_3 = 0 & \text{(I)} \\ \sqrt{2}c_1 + c_2 - \sqrt{3}c_3 = 0 & \text{(II)} \\ |c_1|^2 + |c_2|^2 + |c_3|^2 = 1 & \text{(III)} \end{cases} \tag{107}$$

Isolating c_1 from the first equation

$$c_1 = -\frac{1}{\sqrt{2}}\left(4c_2 + 2\sqrt{3}c_3\right), \tag{108}$$

and c_2 from the second equation

$$c_2 = \sqrt{3}c_3 - \sqrt{2}c_1, \tag{109}$$

substituting the expresion for c_1 given in eq. (108)

$$c_2 = \sqrt{3}c_3 - \sqrt{2}\left[-\frac{1}{\sqrt{2}}\left(4c_2 + 2\sqrt{3}c_3\right)\right] = 4c_2 + 3\sqrt{3}c_3 \implies c_2 = -\sqrt{3}c_3, \tag{110}$$

and substituting in eq. (108)

$$c_1 = -\frac{1}{\sqrt{2}}\left[4\left(-\sqrt{3}c_3\right) + 2\sqrt{3}c_3\right] = \sqrt{6}c_3. \tag{111}$$

Isolating now c_3 from the third equation

$$\left|\sqrt{6}c_3\right|^2 + \left|-\sqrt{3}c_3\right|^2 + |c_3|^2 = 1 \implies 10\,|c_3|^2 = 1 \implies c_3 = \sqrt{\frac{1}{10}}. \tag{112}$$

Then, this expression for c_3 can be substitued in eqs. (110) and (111), as

$$c_1 = \sqrt{6}\sqrt{\frac{1}{10}} = \sqrt{\frac{6}{10}} = \sqrt{\frac{3}{5}}, \tag{113}$$

$$c_2 = -\sqrt{3}\sqrt{\frac{1}{10}} = -\sqrt{\frac{3}{10}}. \tag{114}$$

Substituting in eq. (101) for $|1,1\rangle$

$$\begin{aligned} |1,1\rangle &= \sqrt{\frac{3}{5}}\,|\psi_1\rangle - \sqrt{\frac{3}{10}}\,|\psi_2\rangle + \sqrt{\frac{1}{10}}\,|\psi_3\rangle \\ &= \sqrt{\frac{3}{5}}\,|2,2\rangle_1\,|1,-1\rangle_2 - \sqrt{\frac{3}{10}}\,|2,1\rangle_1\,|1,0\rangle_2 + \sqrt{\frac{1}{10}}\,|2,0\rangle_1\,|1,1\rangle_2\,. \end{aligned} \tag{115}$$

Therefore, $|1,1\rangle$ reads

$$\boxed{|1,1\rangle = \sqrt{\frac{1}{10}}\left(\sqrt{6}\,|2,2\rangle_1\,|1,-1\rangle_2 - \sqrt{3}\,|2,1\rangle_1\,|1,0\rangle_2 + |2,0\rangle_1\,|1,1\rangle_2\right).} \tag{116}$$

Applying $\hat{L}_-$ to $|1,1\rangle$

$$\hat{L}_-\,|1,1\rangle = \sqrt{1\,(1+1) - 1\,(1-1)}\hbar\,|1,0\rangle = \sqrt{2}\hbar\,|1,0\rangle\,, \quad (= \text{LHS}) \tag{117}$$

and to the RHS

$$\hat{L}_-\,\{\text{RHS}\} = \sqrt{\frac{1}{10}}\left(\sqrt{6}\hat{L}_-\,|2,2\rangle_1\,|1,-1\rangle_2 - \sqrt{3}\hat{L}_-\,|2,1\rangle_1\,|1,0\rangle_2 + \hat{L}_-\,|2,0\rangle_1\,|1,1\rangle_2\right). \tag{118}$$

Calculating separately $\hat{L}_-\,|2,2\rangle_1\,|1,-1\rangle_2$

$$\begin{aligned} \hat{L}_-\,|2,2\rangle_1\,|1,-1\rangle_2 &= \hat{L}_{1-}\,|2,2\rangle_1\,|1,-1\rangle_2 + \hat{L}_{2-}\,|1,-1\rangle_2\,|2,2\rangle_1 = \\ &= 2\hbar\,|2,1\rangle_1\,|1,-1\rangle_2\,, \end{aligned} \tag{119}$$

and $\hat{L}_- \left|2,1\right\rangle_1 \left|1,0\right\rangle_2$

$$\begin{aligned}\hat{L}_- \left|2,1\right\rangle_1 \left|1,0\right\rangle_2 &= \hat{L}_{1-} \left|2,1\right\rangle_1 \left|1,0\right\rangle_2 + \hat{L}_{2-} \left|1,0\right\rangle_2 \left|2,1\right\rangle_1 = \\ &= \sqrt{6}\hbar \left|2,0\right\rangle_1 \left|1,0\right\rangle_2 + \sqrt{2}\hbar \left|1,-1\right\rangle_2 \left|2,1\right\rangle_1 = \\ &= \sqrt{2}\hbar \left(\sqrt{3} \left|2,0\right\rangle_1 \left|1,0\right\rangle_2 + \left|2,1\right\rangle_1 \left|1,-1\right\rangle_2\right),\end{aligned} \tag{120}$$

and, lastly, $\hat{L}_- \left|2,0\right\rangle_1 \left|1,1\right\rangle_2$

$$\begin{aligned}\hat{L}_- \left|2,0\right\rangle_1 \left|1,1\right\rangle_2 &= \hat{L}_{1-} \left|2,0\right\rangle_1 \left|1,1\right\rangle_2 + \hat{L}_{2-} \left|1,1\right\rangle_2 \left|2,0\right\rangle_1 = \\ &= \sqrt{6}\hbar \left|2,-1\right\rangle_1 \left|1,1\right\rangle_2 + \sqrt{2}\hbar \left|1,0\right\rangle_2 \left|2,0\right\rangle_1 = \\ &= \sqrt{2}\hbar \left(\sqrt{3} \left|2,-1\right\rangle_1 \left|1,1\right\rangle_2 + \left|2,0\right\rangle_1 \left|1,0\right\rangle_2\right).\end{aligned} \tag{121}$$

Substituting in eq. (118)

$$\begin{aligned}\hat{L}_- \left\{\text{RHS}\right\} = \sqrt{\frac{1}{10}} &\left\{\sqrt{6}2\hbar \left|2,1\right\rangle_1 \left|1,-1\right\rangle_2\right. \\ &- \sqrt{3}\sqrt{2}\hbar \left(\sqrt{3} \left|2,0\right\rangle_1 \left|1,0\right\rangle_2 + \left|2,1\right\rangle_1 \left|1,-1\right\rangle_2\right) \\ &\left.+\sqrt{2}\hbar \left(\sqrt{3} \left|2,-1\right\rangle_1 \left|1,1\right\rangle_2 + \left|2,0\right\rangle_1 \left|1,0\right\rangle_2\right)\right\} = \\ = \sqrt{\frac{1}{5}}\hbar &\left(\sqrt{3} \left|2,1\right\rangle_1 \left|1,-1\right\rangle_2 - 2 \left|2,0\right\rangle_1 \left|1,0\right\rangle_2 + \sqrt{3} \left|2,-1\right\rangle_1 \left|1,1\right\rangle_2\right). \quad (= \text{RHS})\end{aligned} \tag{122}$$

Equating LHS = RHS with eqs. (117) and (122)

$$\sqrt{2}\hbar \left|1,0\right\rangle = \sqrt{\frac{1}{5}}\hbar \left(\sqrt{3} \left|2,1\right\rangle_1 \left|1,-1\right\rangle_2 - 2 \left|2,0\right\rangle_1 \left|1,0\right\rangle_2 + \sqrt{3} \left|2,-1\right\rangle_1 \left|1,1\right\rangle_2\right), \tag{123}$$

$\left|1,0\right\rangle$ reads

$$\boxed{\left|1,0\right\rangle = \sqrt{\frac{1}{10}} \left(\sqrt{3} \left|2,1\right\rangle_1 \left|1,-1\right\rangle_2 - 2 \left|2,0\right\rangle_1 \left|1,0\right\rangle_2 + \sqrt{3} \left|2,-1\right\rangle_1 \left|1,1\right\rangle_2\right).} \tag{124}$$

As for $\left|1,1\right\rangle$, the combination $\left|1,-1\right\rangle$ can be obtained imposing orthogonality conditions between

$$\left\langle 3,-1 \,|\, 1,-1\right\rangle = 0, \quad \left\langle 2,-1 \,|\, 1,-1\right\rangle = 0. \tag{125}$$

Rewriting $\left|3,-1\right\rangle$ as

$$\begin{aligned}\left|3,-1\right\rangle &= \sqrt{\frac{1}{15}} \left(\sqrt{6} \left|2,0\right\rangle_1 \left|1,-1\right\rangle_2 + 2\sqrt{2} \left|2,-1\right\rangle_1 \left|1,0\right\rangle_2 + \left|2,-2\right\rangle_1 \left|1,1\right\rangle_2\right) = \\ &= \sqrt{\frac{1}{15}} \left(\sqrt{6} \left|\psi_1\right\rangle + 2\sqrt{2} \left|\psi_2\right\rangle + \left|\psi_3\right\rangle\right),\end{aligned} \tag{126}$$

and $|2,-1\rangle$ as

$$|2,-1\rangle = \sqrt{\frac{1}{6}}\left(-\sqrt{3}\,|2,0\rangle_1\,|1,-1\rangle_2 + |2,-1\rangle_1\,|1,0\rangle_2 + \sqrt{2}\,|2,-2\rangle_1\,|1,1\rangle_2\right) =$$
$$= \sqrt{\frac{1}{6}}\left(-\sqrt{3}\,|\psi_1\rangle + |\psi_2\rangle + \sqrt{2}\,|\psi_3\rangle\right). \tag{127}$$

$|1,-1\rangle$ can be written as

$$|1,-1\rangle = c_1\,|\psi_1\rangle + c_2\,|\psi_2\rangle + c_3\,|\psi_3\rangle\,. \tag{128}$$

Imposing the orthogonality conditions from eq. (125)

$$\langle 3,-1\,|\,1,-1\rangle = \sqrt{\frac{1}{15}}\left\{\left(\sqrt{6}\,\langle\psi_1| + 2\sqrt{2}\,\langle\psi_2| + \langle\psi_3|\right)\left(c_1\,|\psi_1\rangle + c_2\,|\psi_2\rangle + c_3\,|\psi_3\rangle\right)\right\} =$$
$$= \sqrt{\frac{1}{15}}\left(\sqrt{6}c_1 + 2\sqrt{2}c_2 + c_3\right) = 0$$
$$\implies \sqrt{6}c_1 + 2\sqrt{2}c_2 + c_3 = 0, \quad \text{(I)} \tag{129}$$

and

$$\langle 2,-1\,|\,1,-1\rangle = \sqrt{\frac{1}{6}}\left\{\left(-\sqrt{3}\,|\psi_1\rangle + |\psi_2\rangle + \sqrt{2}\,|\psi_3\rangle\right)\left(c_1\,|\psi_1\rangle + c_2\,|\psi_2\rangle + c_3\,|\psi_3\rangle\right)\right\} =$$
$$= \sqrt{\frac{1}{6}}\left(-\sqrt{3}c_1 + c_2 + \sqrt{2}c_3\right) = 0$$
$$\implies -\sqrt{3}c_1 + c_2 + \sqrt{2}c_3 = 0. \quad \text{(II)} \tag{130}$$

The last equation can be obtained from the normalisation condition eq. (59)

$$|c_1|^2 + |c_2|^2 + |c_3|^2 = 1. \quad \text{(III)} \tag{131}$$

Then, the system of equations to solve reads

$$\begin{cases} \sqrt{6}c_1 + 2\sqrt{2}c_2 + c_3 = 0 & \text{(I)} \\ -\sqrt{3}c_1 + c_2 + \sqrt{2}c_3 = 0 & \text{(II)} \\ |c_1|^2 + |c_2|^2 + |c_3|^2 = 1 & \text{(III)} \end{cases} \tag{132}$$

Isolating c_1 from the first equation

$$c_1 = -\frac{1}{\sqrt{6}}\left(2\sqrt{2}c_2 + c_3\right), \tag{133}$$

and c_2 from the second equation

$$c_2 = \sqrt{3}c_1 - \sqrt{2}c_3. \tag{134}$$

The expresion for c_1 given in eq. (137) can be substitued in eq. (136), as

$$c_2 = \sqrt{3}\left[-\frac{1}{\sqrt{6}}\left(2\sqrt{2}c_2 + c_3\right)\right] - \sqrt{2}c_3 = -2c_2 - \frac{3}{\sqrt{2}}c_3, \tag{135}$$

so

$$3c_2 = -\frac{3}{\sqrt{2}}c_3 \implies c_2 = -\frac{1}{\sqrt{2}}c_3. \tag{136}$$

Substituting now this expresion for c_2 in eq. (137)

$$c_1 = -\frac{1}{\sqrt{6}}\left[2\sqrt{2}\left(-\frac{1}{\sqrt{2}}c_3\right) + c_3\right] = -\frac{1}{\sqrt{6}}\left(-2c_3 + c_3\right) \implies c_1 = \frac{1}{\sqrt{6}}c_3. \tag{137}$$

And substituting this expresions for c_2 and c_3 in the third equation of the system

$$\left|\frac{1}{\sqrt{6}}c_3\right|^2 + \left|-\frac{1}{\sqrt{2}}c_3\right|^2 + |c_3|^2 = 1 \implies \left(\frac{1}{6} + \frac{1}{2} + 1\right)|c_3|^2 = 1 \implies c_3 = \sqrt{\frac{3}{5}}. \tag{138}$$

Then, in eqs. (133) and (134), the coefficients read

$$c_1 = \frac{1}{\sqrt{6}}c_3 = \frac{1}{\sqrt{6}}\sqrt{\frac{3}{5}} = \sqrt{\frac{1}{10}}, \tag{139}$$

$$c_2 = -\frac{1}{\sqrt{2}}c_3 = -\frac{1}{\sqrt{2}}\sqrt{\frac{3}{5}} = -\sqrt{\frac{3}{10}}, \tag{140}$$

$$c_3 = \sqrt{\frac{3}{5}} = \sqrt{\frac{6}{10}}. \tag{141}$$

Therefore, $|1,-1\rangle$ from eq. (128) reads

$$|1,-1\rangle = \sqrt{\frac{1}{10}}\,|\psi_1\rangle - \sqrt{\frac{3}{10}}\,|\psi_2\rangle + \sqrt{\frac{6}{10}}\,|\psi_3\rangle = \sqrt{\frac{1}{10}}\left(|\psi_1\rangle - \sqrt{3}\,|\psi_2\rangle + \sqrt{6}\,|\psi_3\rangle\right), \tag{142}$$

then

$$\boxed{|1,-1\rangle = \sqrt{\frac{1}{10}}\left(|2,0\rangle_1|1,-1\rangle_2 - \sqrt{3}\,|2,-1\rangle_1|1,0\rangle_2 + \sqrt{6}\,|2,-2\rangle_1|1,1\rangle_2\right).} \tag{143}$$

4 Summary

1. $\boldsymbol{L=3}$

   - $|3,3\rangle = |2,2\rangle_1 \, |1,1\rangle_2$
   - $|3,2\rangle = \sqrt{\frac{1}{3}} \left(|2,2\rangle_1 \, |1,0\rangle_2 + \sqrt{2} \, |2,1\rangle_1 \, |1,1\rangle_2 \right)$
   - $|3,1\rangle = \sqrt{\frac{1}{30}} \left(\sqrt{2} \, |2,2\rangle_1 \, |1,-1\rangle_2 + 4 \, |2,1\rangle_1 \, |1,0\rangle_2 + 2\sqrt{3} \, |2,0\rangle_1 \, |1,1\rangle_2 \right)$
   - $|3,0\rangle = \sqrt{\frac{1}{5}} \left(|2,1\rangle_1 \, |1,-1\rangle_2 + \sqrt{3} \, |2,0\rangle_1 \, |1,0\rangle_2 + |2,-1\rangle_1 \, |1,1\rangle_2 \right)$
   - $|3,-1\rangle = \sqrt{\frac{1}{15}} \left(\sqrt{6} \, |2,0\rangle_1 \, |1,-1\rangle_2 + 2\sqrt{2} \, |2,-1\rangle_1 \, |1,0\rangle_2 + |2,-2\rangle_1 \, |1,1\rangle_2 \right)$
   - $|3,-2\rangle = \sqrt{\frac{1}{3}} \left(\sqrt{2} \, |2,-1\rangle_1 \, |1,-1\rangle_2 + |2,-2\rangle_1 \, |1,0\rangle_2 \right)$
   - $|3,-3\rangle = |2,-2\rangle_1 \, |1,-1\rangle_2$

2. $\boldsymbol{L=2}$

   - $|2,2\rangle = \sqrt{\frac{1}{3}} \left(\sqrt{2} \, |2,2\rangle_1 \, |1,0\rangle_2 - |2,1\rangle_1 \, |1,1\rangle_2 \right)$
   - $|2,1\rangle = \sqrt{\frac{1}{6}} \left(\sqrt{2} \, |2,2\rangle_1 \, |1,-1\rangle_2 + |2,1\rangle_1 \, |1,0\rangle_2 - \sqrt{3} \, |2,0\rangle_1 \, |1,1\rangle_2 \right)$
   - $|2,0\rangle = \sqrt{\frac{1}{2}} \left(|2,1\rangle_1 \, |1,-1\rangle_2 - |2,-1\rangle_1 \, |1,1\rangle_2 \right)$
   - $|2,-1\rangle = \sqrt{\frac{1}{6}} \left(-\sqrt{3} \, |2,0\rangle_1 \, |1,-1\rangle_2 + |2,-1\rangle_1 \, |1,0\rangle_2 + \sqrt{2} \, |2,-2\rangle_1 \, |1,1\rangle_2 \right)$
   - $|2,-2\rangle = \sqrt{\frac{1}{3}} \left(- |2,-1\rangle_1 \, |1,-1\rangle_2 + \sqrt{2} \, |2,-2\rangle_1 \, |1,0\rangle_2 \right)$

3. $\boldsymbol{L=1}$

   - $|1,1\rangle = \sqrt{\frac{1}{10}} \left(\sqrt{6} \, |2,2\rangle_1 \, |1,-1\rangle_2 - \sqrt{3} \, |2,1\rangle_1 \, |1,0\rangle_2 + |2,0\rangle_1 \, |1,1\rangle_2 \right)$
   - $|1,0\rangle = \sqrt{\frac{1}{10}} \left(\sqrt{3} \, |2,1\rangle_1 \, |1,-1\rangle_2 - 2 \, |2,0\rangle_1 \, |1,0\rangle_2 + \sqrt{3} \, |2,-1\rangle_1 \, |1,1\rangle_2 \right)$
   - $|1,-1\rangle = \sqrt{\frac{1}{10}} \left(|2,0\rangle_1 \, |1,-1\rangle_2 - \sqrt{3} \, |2,-1\rangle_1 \, |1,0\rangle_2 + \sqrt{6} \, |2,-2\rangle_1 \, |1,1\rangle_2 \right)$