

R04 - Regression with Categorical Explanatory Variables

STAT 5870 (Engineering)
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Binary explanatory variable

Recall the simple linear regression model

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2).$$

If we have a binary explanatory variable, i.e. the explanatory variable only has two levels say level A and level B, we can code it as

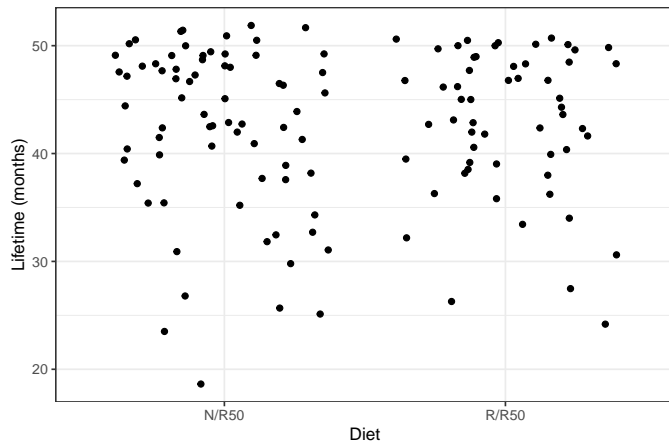
$$X_i = I(\text{observation } i \text{ is level A})$$

where $I(\text{statement})$ is an **indicator function** that is 1 when *statement* is true and 0 otherwise. Then

- β_0 is the expected response for level B,
- $\beta_0 + \beta_1$ is the expected response for level A, and
- β_1 is the expected difference in response (level A minus level B).

Mice lifetimes

```
Sleuth3::case0501
```



Regression model for mice lifetimes

Let

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_i, \sigma^2)$$

where Y_i is the lifetime of the i th mouse and

$$X_i = I(\text{Diet}_i = \text{N/R50})$$

then

$$\begin{aligned} E[\text{Lifetime} | \text{R/R50}] &= E[Y_i | X_i = 0] = \beta_0 \\ E[\text{Lifetime} | \text{N/R50}] &= E[Y_i | X_i = 1] = \beta_0 + \beta_1 \end{aligned}$$

and

$$\begin{aligned} E[\text{Lifetime difference}] &= E[\text{Lifetime} | \text{N/R50}] - E[\text{Lifetime} | \text{R/R50}] \\ &= (\beta_0 + \beta_1) - \beta_0 = \beta_1. \end{aligned}$$

R code

```

case0501$X <- ifelse(case0501$Diet == "N/R50", 1, 0)
(m <- lm(Lifetime ~ X, data = case0501, subset = Diet %in% c("R/R50", "N/R50")))

Call:
lm(formula = Lifetime ~ X, data = case0501, subset = Diet %in%
    c("R/R50", "N/R50"))

Coefficients:
(Intercept)          X
    42.8857      -0.5885

confint(m)

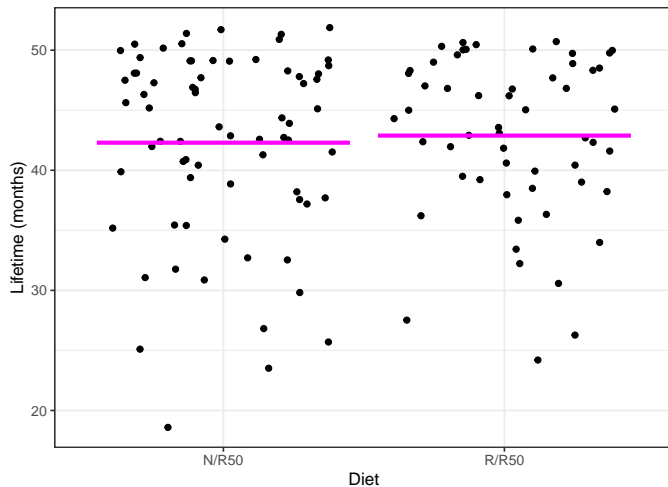
              2.5 %    97.5 %
(Intercept) 40.952257 44.819172
X           -3.174405  1.997342

predict(m, data.frame(X=1), interval = "confidence") # Expected lifetime on N/R50

      fit      lwr      upr
1 42.29718 40.58007 44.0143

```

Mice lifetimes



Equivalence to a two-sample t-test

Recall that our two-sample t-test had the model

$$Y_{ij} \overset{ind}{\sim} N(\mu_j, \sigma^2)$$

for groups $j = 0, 1$. This is equivalent to our current regression model where

$$\begin{aligned}\mu_0 &= \beta_0 \\ \mu_1 &= \beta_0 + \beta_1\end{aligned}$$

assuming

- μ_0 represents the mean for the R/R50 group and
- μ_1 represents the mean for N/R50 group.

When the models are effectively the same, but have different parameters we the model is **reparameterized**.

Equivalence

```
summary(m)$coefficients[2,4] # p-value
```

```
[1] 0.6531748
```

```
confint(m)
```

```

          2.5 %    97.5 %
(Intercept) 40.952257 44.819172
X           -3.174405  1.997342
```

```
t.test(Lifetime ~ Diet, data = case0501, subset = Diet %in% c("R/R50", "N/R50"), var.equal=TRUE)
```

Two Sample t-test

data: Lifetime by Diet

t = -0.45044, df = 125, p-value = 0.6532

alternative hypothesis: true difference in means between group N/R50 and group R/R50 is not equal to 0

95 percent confidence interval:

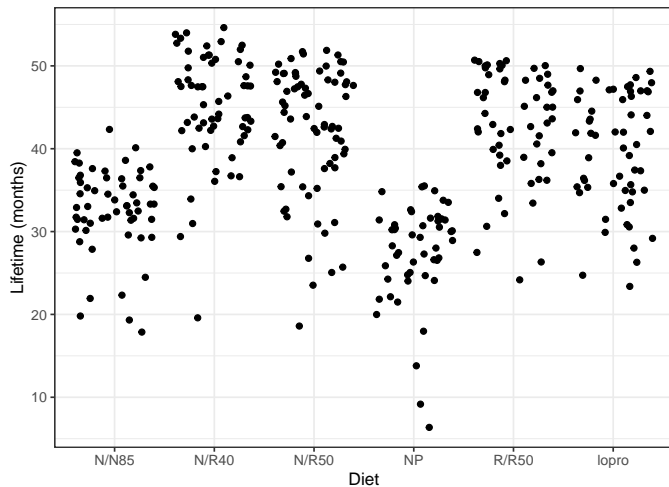
-3.174405 1.997342

sample estimates:

mean in group N/R50 mean in group R/R50

42.29718 42.88571

Using a categorical variable as an explanatory variable.



Regression with a categorical variable

1. Choose one of the levels as the **reference** level, e.g. N/N85
2. Construct dummy variables using indicator functions, i.e.

$$I(A) = \begin{cases} 1 & A \text{ is TRUE} \\ 0 & A \text{ is FALSE} \end{cases}$$

for the other levels, e.g.

$$X_{i,1} = I(\text{diet for observation } i \text{ is N/R40})$$

$$X_{i,2} = I(\text{diet for observation } i \text{ is N/R50})$$

$$X_{i,3} = I(\text{diet for observation } i \text{ is NP})$$

$$X_{i,4} = I(\text{diet for observation } i \text{ is R/R50})$$

$$X_{i,5} = I(\text{diet for observation } i \text{ is loopro})$$

3. Estimate the parameters of a multiple regression model using these dummy variables.

Regression model

Our regression model becomes

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2} + \beta_3 X_{i,3} + \beta_4 X_{i,4} + \beta_5 X_{i,5}, \sigma^2)$$

where

- β_0 is the expected lifetime for the N/N85 group
- $\beta_0 + \beta_1$ is the expected lifetime for the N/R40 group
- $\beta_0 + \beta_2$ is the expected lifetime for the N/R50 group
- $\beta_0 + \beta_3$ is the expected lifetime for the NP group
- $\beta_0 + \beta_4$ is the expected lifetime for the R/R50 group
- $\beta_0 + \beta_5$ is the expected lifetime for the lopro group

and thus β_p for $p > 0$ is the difference in expected lifetimes between one group and a **reference** group.

R code

```
case0501 <- case0501 %>%
  mutate(X1 = Diet == "N/R40",
         X2 = Diet == "N/R50",
         X3 = Diet == "NP",
         X4 = Diet == "R/R50",
         X5 = Diet == "lopro")

m <- lm(Lifetime ~ X1 + X2 + X3 + X4 + X5, data = case0501)
m
```

Call:

```
lm(formula = Lifetime ~ X1 + X2 + X3 + X4 + X5, data = case0501)
```

Coefficients:

(Intercept)	X1TRUE	X2TRUE	X3TRUE	X4TRUE	X5TRUE
32.691	12.425	9.606	-5.289	10.194	6.994

```
confint(m)
```

	2.5 %	97.5 %
(Intercept)	30.951394	34.431062
X1TRUE	9.995893	14.854984
X2TRUE	7.269897	11.942013
X3TRUE	-7.848142	-2.730232
X4TRUE	7.723030	12.665943
X5TRUE	4.523030	9.465943

R code (cont.)

```
summary(m)

Call:
lm(formula = Lifetime ~ X1 + X2 + X3 + X4 + X5, data = case0501)

Residuals:
    Min       1Q   Median       3Q      Max
-25.5167  -3.3857   0.8143   5.1833  10.0143

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  32.6912     0.8846  36.958 < 2e-16 ***
X1TRUE       12.4254     1.2352  10.059 < 2e-16 ***
X2TRUE        9.6060     1.1877   8.088 1.06e-14 ***
X3TRUE       -5.2892     1.3010  -4.065 5.95e-05 ***
X4TRUE       10.1945     1.2565   8.113 8.88e-15 ***
X5TRUE        6.9945     1.2565   5.567 5.25e-08 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 6.678 on 343 degrees of freedom
Multiple R-squared:  0.4543, Adjusted R-squared:  0.4463
F-statistic: 57.1 on 5 and 343 DF, p-value: < 2.2e-16
```

Interpretation

- $\beta_0 = E[Y_i | \text{reference level}]$, i.e. expected response for the reference level

Note: the only way $X_{i,1} = \dots = X_{i,p} = 0$ is if all indicators are zero, i.e. at the reference level.

- $\beta_p, p > 0$: expected change in the response moving from the reference level to the level associated with the p^{th} dummy variable

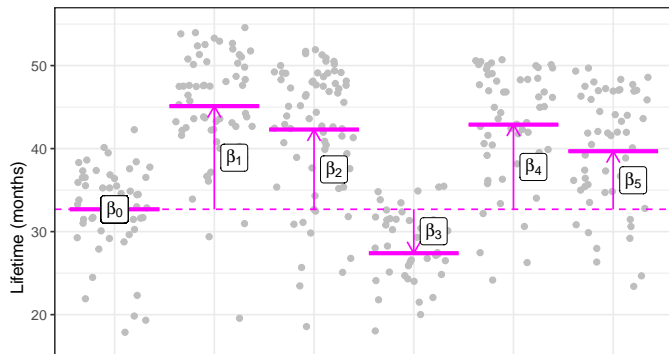
Note: the only way for $X_{i,p}$ to increase by one is if initially $X_{i,1} = \dots = X_{i,p} = 0$ and now $X_{i,p} = 1$

For example,

- The expected lifetime for mice on the N/N85 diet is 32.7 (31.0,34.4) months.
- The expected increase in lifetime for mice on the N/R40 diet compared to the N/N85 diet is 12.4 (10.0,14.9) months.
- The model explains 45% of the variability in mice lifetimes.

Using a categorical variable as an explanatory variable.

```
Warning in geom_label(aes(x = "N/N85", y = coef(m)[1], label = "beta[0]"), :  
All aesthetics have length 1, but the data has 349 rows.  
i Please consider using 'annotate()' or provide this layer with data containing  
a single row.
```



Equivalence to multiple group model

Recall that we had a multiple group model

$$Y_{ij} \stackrel{ind}{\sim} N(\mu_j, \sigma^2)$$

for groups $j = 0, 1, 2, \dots, 5$.

Our regression model is a **reparameterization** of the multiple group model:

$$\begin{aligned} N/N85 : \quad \mu_0 &= \beta_0 \\ N/R40 : \quad \mu_1 &= \beta_0 + \beta_1 \\ N/R50 : \quad \mu_2 &= \beta_0 + \beta_2 \\ NP : \quad \mu_3 &= \beta_0 + \beta_3 \\ R/R50 : \quad \mu_4 &= \beta_0 + \beta_4 \\ lopro : \quad \mu_5 &= \beta_0 + \beta_5 \end{aligned}$$

assuming the groups are labeled appropriately.

Summary

1. Choose one of the levels as the **reference** level.
2. Construct dummy variables using indicator functions for all other levels, e.g.

$$X_i = I(\text{observation } i \text{ is } <\text{some non-reference level}>).$$

3. Estimate the parameters of a multiple regression model using these dummy variables.