

I05 - Confidence intervals

STAT 5870 (Engineering)
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Exact confidence intervals

The **coverage** of an interval estimator is the probability the interval will contain the true value of the parameter *when the data are considered to be random*. If an interval estimator has $100(1 - \alpha)\%$ coverage, then we call it a $100(1 - \alpha)\%$ **confidence interval** and $1 - \alpha$ is the **confidence level**.

That is, we calculate

$$1 - \alpha = P(L < \theta < U)$$

where L and U are random because they depend on the data. Thus **confidence** is a statement about the **procedure**.

Normal model

If $Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$ and we assume the default prior $p(\mu, \sigma^2) \propto 1/\sigma^2$, then a $100(1 - a)\%$ credible interval for μ is given by

$$\bar{y} \pm t_{n-1, a/2} s / \sqrt{n}.$$

When the data are considered random

$$T_{n-1} = \frac{\bar{Y} - \mu}{S/\sqrt{n}} \sim t_{n-1}(0, 1)$$

thus the probability μ is within our credible interval is

$$\begin{aligned} &P\left(\bar{Y} - t_{n-1, a/2} S / \sqrt{n} < \mu < \bar{Y} + t_{n-1, a/2} S / \sqrt{n}\right) \\ &= P\left(-t_{n-1, a/2} < \frac{\bar{Y} - \mu}{S/\sqrt{n}} < t_{n-1, a/2}\right) \\ &= P\left(-t_{n-1, a/2} < T_{n-1} < t_{n-1, a/2}\right) \\ &= 1 - a. \end{aligned}$$

Thus, this $100(1 - a)\%$ credible interval is also a $100(1 - a)\%$ confidence interval.

Yield data example

Recall the corn yield example from I04 with 9 randomly selected fields in Iowa whose sample average yield is 186 and sample standard deviation is 22. Then a 95% confidence interval for the mean corn yield on Iowa farms is

$$186 \pm 2.31 \times 22/\sqrt{9} = (169, 202).$$

Standard error

The **standard error of an estimator** is an *estimate* of the standard deviation of the estimator (when the data are considered random).

If $Y \sim \text{Bin}(n, \theta)$, then

$$\hat{\theta} = \frac{Y}{n} \quad \text{has} \quad SE[\hat{\theta}] = \sqrt{\frac{\hat{\theta}(1 - \hat{\theta})}{n}}.$$

If $Y_i \stackrel{\text{ind}}{\sim} N(\mu, \sigma^2)$, then

$$\hat{\mu} = \bar{Y} \quad \text{has} \quad SE[\hat{\mu}] = S/\sqrt{n}.$$

Approximate confidence intervals

If an **unbiased** estimator has an asymptotic normal distribution, then we can construct an **approximate** $100(1 - \alpha)\%$ confidence interval for $E[\hat{\theta}] = \theta$ using

$$\hat{\theta} \pm z_{\alpha/2} SE[\hat{\theta}].$$

where $SE[\hat{\theta}]$ is the **standard error** of the estimator and $P(Z > z_{\alpha/2}) = \alpha/2$.

This comes from the fact that if $\hat{\theta} \sim N(\theta, SE[\hat{\theta}]^2)$, then

$$\begin{aligned} P\left(\hat{\theta} - z_{\alpha/2} SE(\hat{\theta}) < \theta < \hat{\theta} + z_{\alpha/2} SE(\hat{\theta})\right) \\ &= P\left(-z_{\alpha/2} < \frac{\hat{\theta} - \theta}{SE(\hat{\theta})} < z_{\alpha/2}\right) \\ &\approx P\left(-z_{\alpha/2} < Z < z_{\alpha/2}\right) \\ &= 1 - \alpha. \end{aligned}$$

Normal example

If $Y_i \stackrel{\text{ind}}{\sim} N(\mu, \sigma^2)$ and we have the estimator $\hat{\mu} = \bar{Y}$, then

$$E[\hat{\mu}] = \mu \quad \text{and} \quad SE[\hat{\mu}] = S/\sqrt{n}$$

Thus an **approximate** $100(1 - \alpha)\%$ confidence interval for $\mu = E[\hat{\mu}]$ is

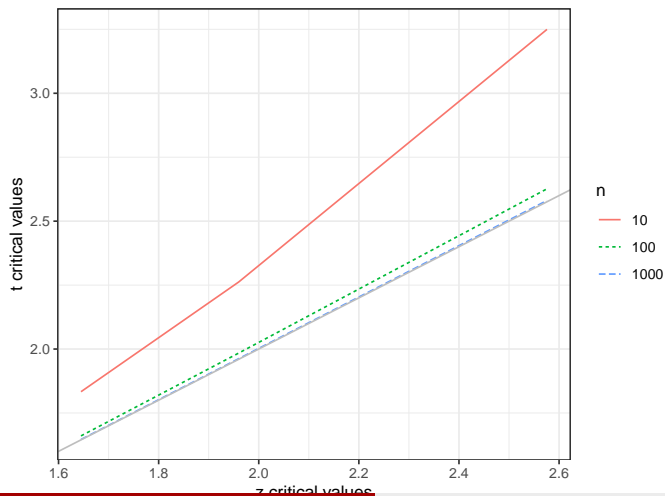
$$\hat{\mu} \pm z_{\alpha/2} SE[\hat{\mu}] = \bar{Y} \pm z_{\alpha/2} S/\sqrt{n}.$$

Note that this is almost identical to the **exact** $100(1 - \alpha)\%$ confidence interval for μ ,

$$\bar{Y} \pm t_{n-1, \alpha/2} S/\sqrt{n}$$

and when n is large $z_{\alpha/2} \approx t_{n-1, \alpha/2}$.

T critical values vs Z critical values



Approximate confidence interval for binomial proportion

If $Y \sim \text{Bin}(n, \theta)$, then an **approximate** $100(1 - \alpha)\%$ confidence interval for θ is

$$\hat{\theta} \pm z_{\alpha/2} \sqrt{\frac{\hat{\theta}(1 - \hat{\theta})}{n}}.$$

where $\hat{\theta} = Y/n$ since

$$E[\hat{\theta}] = E\left[\frac{Y}{n}\right] = \theta$$

and

$$SE[\hat{\theta}] = \sqrt{\frac{\hat{\theta}(1 - \hat{\theta})}{n}}.$$

Gallup poll example

In a Gallup poll dated 2017/02/19, 32.1% of respondents of the 1,500 randomly selected U.S. adults indicated that they were “engaged at work”. Thus an approximate 95% confidence interval for the proportion of all U.S. adults is

$$0.321 \pm 1.96 \times \sqrt{\frac{.321(1 - .321)}{1500}} = (0.30, 0.34).$$

Confidence interval summary

Model	Parameter	Estimator	Confidence Interval	Type
$Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$	μ	$\hat{\mu} = \bar{y}$	$\hat{\mu} \pm t_{n-1, a/2} s / \sqrt{n}$	exact
$Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$	μ	$\hat{\mu} = \bar{y}$	$\hat{\mu} \pm z_{a/2} s / \sqrt{n}$	approximate
$Y \sim Bin(n, \theta)$	θ	$\hat{\theta} = y/n$	$\hat{\theta} \pm z_{a/2} \sqrt{\hat{\theta}(1 - \hat{\theta})/n}$	approximate
$Y_i \stackrel{ind}{\sim} Ber(\theta)$	θ	$\hat{\theta} = \bar{y}$	$\hat{\theta} \pm z_{a/2} \sqrt{\hat{\theta}(1 - \hat{\theta})/n}$	approximate

Bayesian credible intervals generally provide approximate confidence intervals.

Approximate means that the coverage will get closer to the desired probability, i.e. $100(1 - a)\%$, as the sample size gets larger.