

# P1 - Probability

STAT 5870 (Engineering)  
Iowa State University

August 28, 2024

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- **Relative frequency**: Probability is the proportion of times the event occurs as the number of times the event is attempted tends to infinity.

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Interpretations:

- **Relative frequency**: Probability is the proportion of times the event occurs as the number of times the event is attempted tends to infinity.
- **Personal belief**: Probability is a statement about your personal belief in the event occurring.

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Let  $C$  be a successful connection to the internet from a laptop event.



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We write  $P(C) = 0.9$ .

*To be able to work with probabilities, in particular, to be able to compute **probabilities of events**, a mathematical foundation is necessary.*

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The **sample space**,  $\Omega$ , is the set of all outcomes of an experiment.

## Set - examples

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- $2 \in \Omega$
- $1 \notin \Omega$
- $\{2, 3, 4\} \subset \Omega$

## Set comparison, operations, terminology

For the following  $A, B \subseteq \Omega$  where  $\Omega$  is the implied universe of all elements under study,

1. **Union** ( $\cup$ ): A union of events is an event consisting of all the outcomes in these events.

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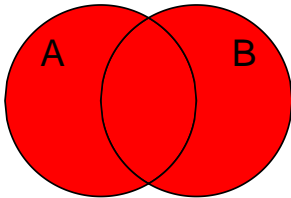
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4. **Set difference** ( $A \setminus B$ ): All elements in  $A$  that are not in  $B$ , i.e.

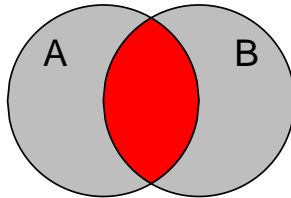
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# Venn diagrams

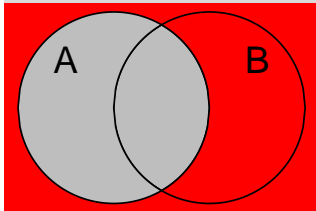
union



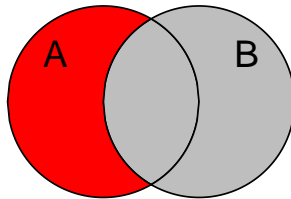
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- $B \setminus A = \{2, 4\}$

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8. **De Morgan's Laws**:

$$(A \cup B)^C = A^C \cap B^C \quad \text{and} \quad (A \cap B)^C = A^C \cup B^C$$



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$$\begin{aligned}A^C &= \{5, 6, 7, 8, 9, 10, 11, 12\} \\ B^C &= \{2, 3, 4, 8, 9, 10, 11, 12\} \\ A^C \cap B^C &= \{8, 9, 10, 11, 12\}\end{aligned}$$

so, by example,

$$(A \cup B)^C = A^C \cap B^C.$$



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- (iii) if  $A_1, A_2, \dots$  are pairwise disjoint events (i.e.  $A_i \cap A_j = \emptyset$  for all  $i \neq j$ ) then

$$\begin{aligned} P(A_1 \cup A_2 \cup \dots) &= P(A_1) + P(A_2) + \dots \\ &= \sum_i P(A_i). \end{aligned}$$

## Kolmogorov's Axioms (cont.)

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Mathematically, both schemes are equally valid.

But, of course, our real world experience would prefer model 1 over model 2.



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- Addition Rule of Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

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# Conditional probability - Definition

The **conditional probability** of an event  $A$  given an event  $B$  is

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Corrollary:

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A).$$

# Random CPUs

A box has 500 CPUs with a speed of 1.8 GHz and 500 with a speed of 2.0 GHz. The numbers of good (G) and defective (D) CPUs at the two different speeds are as shown below.

	1.8 GHz	2.0 GHz	Total
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Then

- $P(S) = 500/1000 = 0.5$
- $P(S \cap D) = 20/1000 = 0.02$ .
- $P(D|S) = P(S \cap D)/P(S) = 0.02/0.5 = 0.04$ .

# Statistical independence - Definition

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Example: In two tosses of a coin, the result of the first toss does not affect the probability of the second toss being heads.

## WiFi example

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Since we have independence, we know

$$P(A \cap B) = P(A) \times P(B) = 0.9 \times 0.6 = 0.54.$$

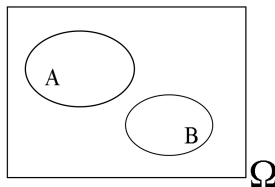
# Independence and disjoint

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Disjoint:



$A, B$  are disjoint

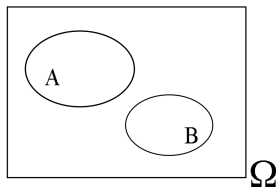
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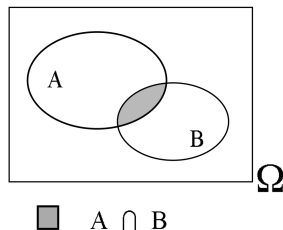


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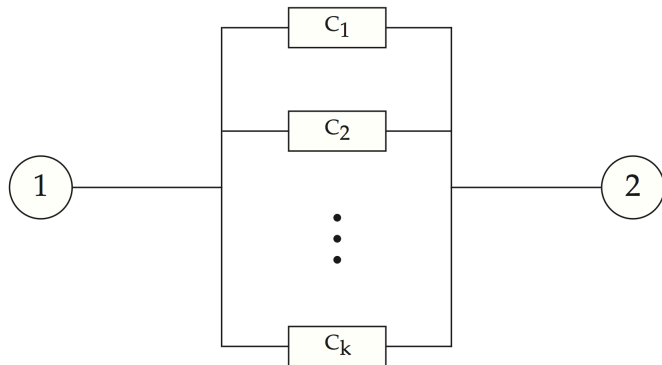
If  $A$  and  $B$  are independent events, the probability of their intersection can be computed as the product of their individual probabilities:

$$P(A \cap B) = P(A) \cdot P(B)$$



## Parallel system - Definition

A **parallel** system consists of  $K$  components  $c_1, \dots, c_K$  arranged in such a way that the system works if **at least one** of the  $K$  components functions properly.

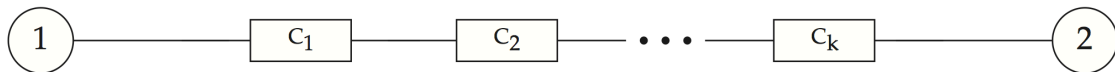


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Example: The reliability of the WiFi-ISP network (assuming independence) is 0.54.

# Reliability of parallel systems with independent components

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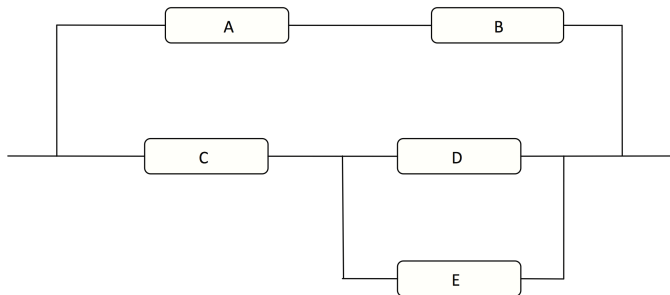
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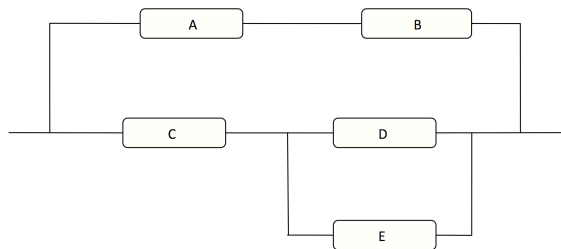
## Reliability example

Each component in the system shown below is operable with probability 0.92 independently of other components. Calculate the reliability.



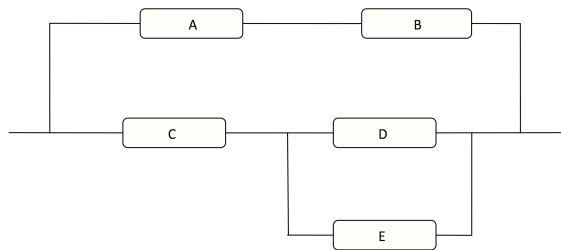
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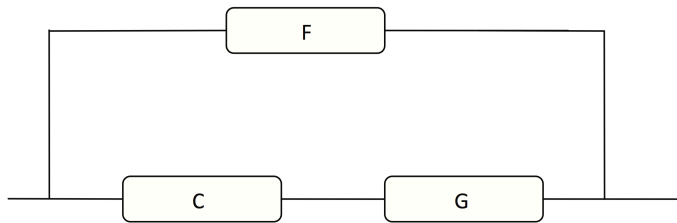
Each component in the system shown below is operable with probability 0.92 independently of other components. Calculate the reliability.



1. Serial components A and B can be replaced by a component F that operates with probability  $P(A \cap B) = (0.92)^2 = 0.8464$ .
2. Parallel components D and E can be replaced by component G that operates with probability  $P(D \cup E) = 1 - (1 - 0.92)^2 = 0.9936$ .

## Reliability example (cont.)

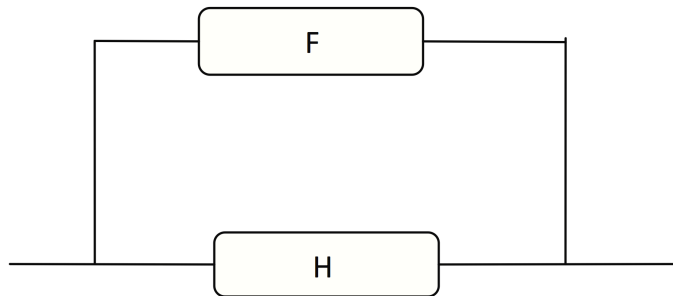
Updated circuit:



3. Serial components C and G can be replaced by a component H that operates with probability  $P(C \cap G) = (0.92)(0.9936) = 0.9141$ .

## Reliability example (cont.)

Updated circuit:



4. Parallel components F and H are in parallel, so the reliability of the system is
- $$P(F \cup H) = 1 - (1 - 0.8464)(1 - 0.9141) \approx 0.99.$$

# Partition

## Definition

A collection of events  $B_1, \dots, B_K$  is called a **partition** (or **cover**) of  $\Omega$  if

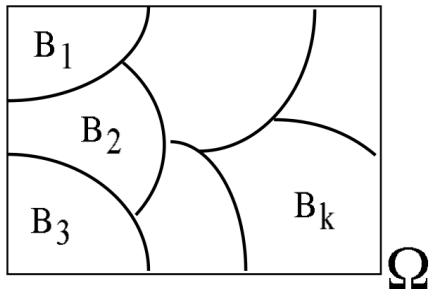
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- the events are pairwise disjoint (i.e.,  $B_i \cap B_j = \emptyset$  for  $i \neq j$ ), and
- the union of the events is  $\Omega$  (i.e.,  $\bigcup_{k=1}^K B_k = \Omega$ ).





## Example

Consider the sum of two 6-sided die, i.e.

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- $A_2, A_3, \dots, A_{12}$  where  $A_i = \{i\}$
- any  $A$  and  $A^C$  where  $A \subseteq \Omega$

# Law of Total Probability

## Law of Total Probability:

If the collection of events  $B_1, \dots, B_K$  is a partition of  $\Omega$ , and  $A$  is an event, then

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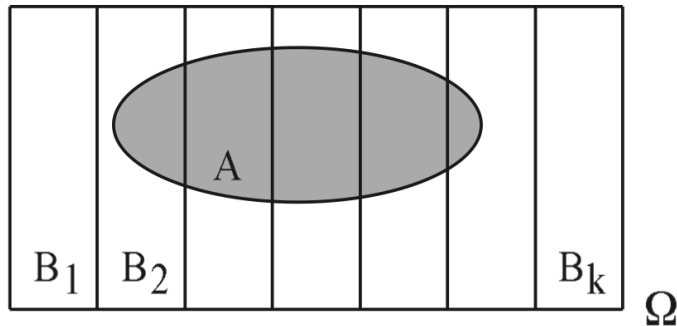
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# Law of Total Probability - Graphically



## Law of Total Probability - Example

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In the come out roll of craps, you win if the roll is a 7 or 11. By the law of total probability, the probability you win is

$$P(\text{Win}) = \sum_{i=2}^{12} P(\text{Win}|i)P(i) = P(7) + P(11)$$

since  $P(\text{Win}|i) = 1$  if  $i = 7, 11$  and 0 otherwise.

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$$P(\text{tester says CPU is defective} \mid \text{CPU is defective}) = 0.94$$

If the test device says the CPU is defective, what is the probability that the CPU is actually defective?

## CPU testing (cont.)

Let

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We know

- $0.02 = P(C_d) = 1 - P(C_g)$
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$$P(C_d|T_d) = \frac{P(T_d|C_d)P(C_d)}{P(T_d|C_d)P(C_d) + P(T_d|C_g)P(C_g)}$$

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 &= 0.28
 \end{aligned}$$



# Probability Summary

- Probability Interpretation
- Sets and set operations
- Kolmogorov's Axioms
- Conditional Probability
- Independence
- Reliability
- Law of Total Probability
- Bayes' Rule

## P2 - Discrete Random Variables

STAT 5870 (Engineering)  
Iowa State University

September 5, 2024

# Random variables

If  $\Omega$  is the sample space of an experiment, a **random variable**  $X$  is a function  $X(\omega) : \Omega \mapsto \mathbb{R}$ .

**Idea:** If the value of a numerical variable depends on the outcome of an experiment, we call the variable a *random variable*.

Examples of random variables from rolling two 6-sided dice:

- Sum of the two dice
- Indicator of the sum being greater than 5

We will use an upper case Roman letter (late in the alphabet) to indicate a random variable and a lower case Roman letter to indicate a realized value of the random variable.

## 8 bit example

Suppose, 8 bits are sent through a communication channel. Each bit has a certain probability to be received incorrectly. We are interested in the number of bits that are received incorrectly.

- Let  $X$  be the number of incorrect bits received.
- The possible values for  $X$  are  $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ .
- Example events:
  - No incorrect bits received:  $\{X = 0\}$ .
  - At least one incorrect bit received:  $\{X \geq 1\}$ .
  - Exactly two incorrect bits received:  $\{X = 2\}$ .
  - Between two and seven (inclusive) incorrect bits received:  $\{2 \leq X \leq 7\}$ .

# Range/image of random variables

The **range** (or **image**) of a random variable  $X$  is defined as

$$\text{Range}(X) := \{x : x = X(\omega) \text{ for some } \omega \in \Omega\}$$

If the range is finite or countably infinite, we have a **discrete** random variable. If the range is uncountably infinite, we have a **continuous** random variable.

Examples:

- Put a hard drive into service, measure  $Y$  = “time until the first major failure” and thus  $\text{Range}(Y) = (0, \infty)$ . Range of  $Y$  is an interval (uncountable range), so  $Y$  is a **continuous** random variable.
- Communication channel:  $X$  = “# of incorrectly received bits out of 8 bits sent” with  $\text{Range}(X) = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ . Range of  $X$  is a finite set, so  $X$  is a **discrete** random variable.
- Communication channel:  $Z$  = “# of incorrectly received bits in 10 minutes” with  $\text{Range}(Z) = \{0, 1, \dots\}$ . Range of  $Z$  is a countably infinite set, so  $Z$  is a **discrete** random variable.

# Distribution

The collection of all the probabilities related to  $X$  is the **distribution** of  $X$ .

For a discrete random variable, the function

$$p_X(x) = P(X = x)$$

is the **probability mass function** (pmf) and the **cumulative distribution function** (cdf) is

$$F_X(x) = P(X \leq x) = \sum_{y \leq x} p_X(y).$$

The set of non-zero probability values of  $X$  is called the **support** of the distribution  $f$ .

This is the same as the **range** of  $X$ .

# Examples

A probability mass function is valid if it defines a valid set of probabilities, i.e. they obey Kolmogorov's axioms.

Which of the following functions are a valid probability mass functions?

- |          |     |      |      |      |      |
|----------|-----|------|------|------|------|
| $x$      | -3  | -1   | 0    | 5    | 7    |
| $p_X(x)$ | 0.1 | 0.45 | 0.15 | 0.25 | 0.05 |
- |          |     |      |      |       |      |
|----------|-----|------|------|-------|------|
| $y$      | -1  | 0    | 1.5  | 3     | 4.5  |
| $p_Y(y)$ | 0.1 | 0.45 | 0.25 | -0.05 | 0.25 |
- |          |      |      |      |      |      |
|----------|------|------|------|------|------|
| $z$      | 0    | 1    | 3    | 5    | 7    |
| $p_Z(z)$ | 0.22 | 0.18 | 0.24 | 0.17 | 0.18 |

## Rolling a fair 6-sided die

Let  $Y$  be the number of pips on the upturned face of a die. The support of  $Y$  is  $\{1, 2, 3, 4, 5, 6\}$ . If we believe the die has equal probability for each face, then image, pmf, and cdf for  $Y$  are

$y$	1	2	3	4	5	6
$p_Y(y) = P(Y = y)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$F_Y(y) = P(Y \leq y)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$



# Dragonwood

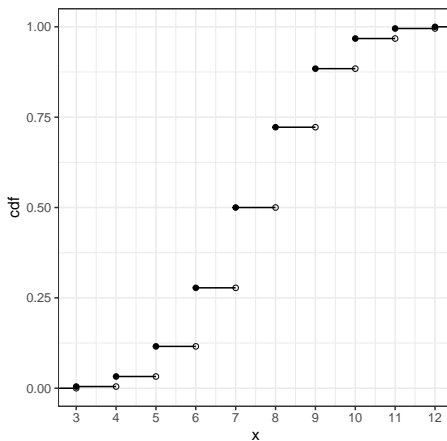
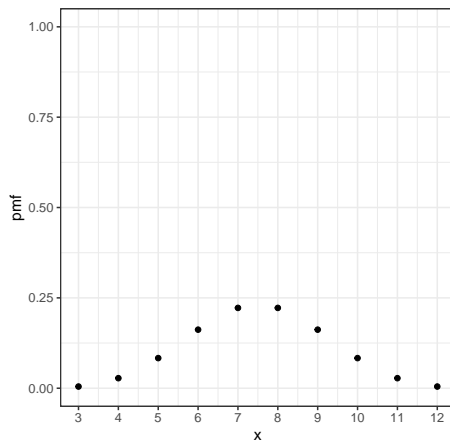
Dragonwood has 6-sided dice with the following # on the 6 sides:  $\{1, 2, 2, 3, 3, 4\}$ .

What is the support, pmf, and cdf for the sum of the upturned numbers when rolling 3 Dragonwood dice?

```
# Three dice
die  = c(1,2,2,3,3,4)
rolls = expand.grid(die1 = die, die2 = die, die3 = die)
sum   = rowSums(rolls); tsum = table(sum)
dragonwood3 = data.frame(x = as.numeric(names(tsum)),
                        pmf = as.numeric(table(sum)/length(sum))) %>%
  mutate(cdf = cumsum(pmf))
```

$x$	$P(X = x)$	$P(X \leq x)$
3	0.005	0.005
4	0.028	0.032
5	0.083	0.116
6	0.162	0.278
7	0.222	0.500
8	0.222	0.722
9	0.162	0.884
10	0.083	0.968
11	0.028	0.995
12	0.005	1.000

# Dragonwood - pmf and cdf



## Properties of pmf and cdf

Properties of probability mass function  $p_X(x) = P(X = x)$ :

- $0 \leq p_X(x) \leq 1$  for all  $x \in \mathbb{R}$ .
- $\sum_{x \in S} p_X(x) = 1$  where  $S$  is the support.

Properties of cumulative distribution function  $F_X(x)$ :

- $0 \leq F_X(x) \leq 1$  for all  $x \in \mathbb{R}$
- $F_X$  is nondecreasing, (i.e. if  $x_1 \leq x_2$  then  $F_X(x_1) \leq F_X(x_2)$ .)
- $\lim_{x \rightarrow -\infty} F_X(x) = 0$  and  $\lim_{x \rightarrow \infty} F_X(x) = 1$ .
- $F_X(x)$  is right continuous with respect to  $x$

## Dragonwood (cont.)

In Dragonwood, you capture monsters by rolling a sum equal to or greater than its defense. Suppose you can roll 3 dice and the following monsters are available to be captured:

- Spooky Spiders worth 1 victory point with a defense of 3.
- Hungry Bear worth 3 victory points with a defense of 7.
- Grumpy Troll worth 4 victory points with a defense of 9.

Which monster should you attack?

## Dragonwood (cont.)

Calculate the probability by computing one minus the cdf evaluated at “defense minus 1”. Let  $X$  be the sum of the number on 3 Dragonwood dice. Then

- $P(X \geq 3) = 1 - P(X \leq 2) = 1$
- $P(X \geq 7) = 1 - P(X \leq 6) = 0.722.$
- $P(X \geq 9) = 1 - P(X \leq 8) = 0.278.$

If we multiply the probability by the number of victory points, then we have the “expected points”:

- $1 \times P(X \geq 3) = 1$
- $3 \times P(X \geq 7) = 2.17.$
- $4 \times P(X \geq 9) = 1.11.$

# Expectation

Let  $X$  be a random variable and  $h$  be some function. The **expected value** of a function of a (discrete) random variable is

$$E[h(X)] = \sum_i h(x_i) \cdot p_X(x_i).$$

**Intuition:** Expected values are *weighted averages* of the possible values weighted by their probability.

If  $h(x) = x$ , then

$$E[X] = \sum_i x_i \cdot p_X(x_i)$$

and we call this the **expectation** of  $X$   
and commonly use the symbol  $\mu$  for the expectation.

## Dragonwood (cont.)

What is the expectation of the sum of 3 Dragonwood dice?

```
expectation = with(dragonwood3, sum(x*pmf))  
expectation
```

```
[1] 7.5
```

The expectation can be thought of as the **center of mass** if we place mass  $p_X(x)$  at corresponding points  $x$ .



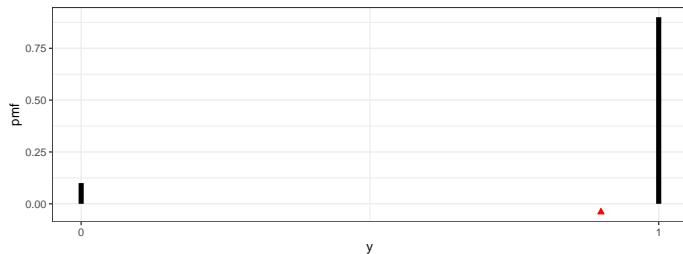
## Biased coin

Suppose we have a biased coin represented by the following pmf:

$y$	0	1
$p_Y(y)$	$1 - p$	$p$

What is the expected value?

If  $p = 0.9$ ,





# Properties of expectations

Let  $X$  and  $Y$  be random variables and  $a$ ,  $b$ , and  $c$  be constants. Then

$$E[aX + bY + c] = aE[X] + bE[Y] + c.$$

In particular

- $E[X + Y] = E[X] + E[Y]$ ,
- $E[aX] = aE[X]$ , and
- $E[c] = c$ .

## Dragonwood (cont.)

Enhancement cards in Dragonwood allow you to improve your rolls. Here are two enhancement cards:

- *Cloak of Darkness* adds 2 points to all capture attempts and
- *Friendly Bunny* allows you (once) to roll an extra die.

What is the expected attack roll total if you had 3 Dragonwood dice, the Cloak of Darkness, and are using the Friendly Bunny?

Let

- $X$  be the sum of 3 Dragonwood dice (we know  $E[X] = 7.5$ ),
- $Y$  be the sum of 1 Dragonwood die which has  $E[Y] = 2.5$ .

Then the attack roll total is  $X + Y + 2$  and the *expected* attack roll total is

$$E[X + Y + 2] = E[X] + E[Y] + 2 = 7.5 + 2.5 + 2 = 12.$$

# Variance

The **variance** of a random variable is defined as the expected squared deviation from the mean. For discrete random variables, variance is

$$\text{Var}[X] = E[(X - \mu)^2] = \sum_i (x_i - \mu)^2 \cdot p_X(x_i)$$

where  $\mu = E[X]$ . The symbol  $\sigma^2$  is commonly used for the variance. The variance is analogous to **moment of inertia** in classical mechanics.

The **standard deviation** (sd) is the positive square root of the variance:

$$SD[X] = \sqrt{\text{Var}[X]}.$$

The symbol  $\sigma$  is commonly used for sd.

## Properties of variance

Two discrete random variables  $X$  and  $Y$  are **independent** if

$$p_{X,Y}(x,y) = p_X(x)p_Y(y).$$

If  $X$  and  $Y$  are **independent**, and  $a$ ,  $b$ , and  $c$  are constants, then

$$\text{Var}[aX + bY + c] = a^2\text{Var}[X] + b^2\text{Var}[Y].$$

Special cases:

- $\text{Var}[c] = 0$
- $\text{Var}[aX] = a^2\text{Var}[X]$
- $\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$   
(if  $X$  and  $Y$  are independent)

## Dragonwood (cont.)

What is the variance for the sum of the 3 Dragonwood dice?

```
variance = with(dragonwood3, sum((x-expectation)^2*pmf))  
variance
```

```
[1] 2.75
```

What is the standard deviation for the sum of the pips on 3 Dragonwood dice?

```
sqrt(variance)
```

```
[1] 1.658312
```

# Biased coin

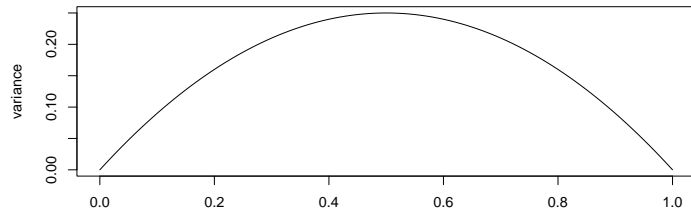
Suppose we have a biased coin represented by the following pmf:

$y$	0	1
$p_Y(y)$	$1 - p$	$p$

What is the variance?

1.  $E[Y] = p$
2.  $Var[y] = (0 - p)^2(1 - p) + (1 - p)^2 \times p = p - p^2 = p(1 - p)$

When is this variance maximized?



# Special discrete distributions

- Bernoulli
- Binomial
- Poisson

Note: The range is always finite or countable.

# Bernoulli random variables

A Bernoulli experiment has only two outcomes: success/failure.

Let

- $X = 1$  represent success and
- $X = 0$  represent failure.

The probability mass function  $p_X(x)$  is

$$p_X(0) = 1 - p \quad p_X(1) = p.$$

We use the notation  $X \sim Ber(p)$  to denote a random variable  $X$  that follows a Bernoulli distribution with success probability  $p$ , i.e.  $P(X = 1) = p$ .



## Bernoulli experiment examples

- Toss a coin:  $\Omega = \{Heads, Tails\}$
- Throw a fair die and ask if the face value is a six:  
 $\Omega = \{\text{face value is a six, face value is not a six}\}$
- Send a message through a network and record whether or not it is received:  
 $\Omega = \{\text{successful transmission, unsuccessful transmission}\}$
- Draw a part from an assembly line and record whether or not it is defective:  
 $\Omega = \{\text{defective, good}\}$
- Response to the question  
“Are you in favor of an increased in property tax  
xto pay for a new high school?”:  
 $\Omega = \{\text{yes, no}\}$

## Bernoulli random variable (cont.)

The cdf of the Bernoulli random variable is

$$F_X(x) = P(X \leq x) = \begin{cases} 0 & x < 0 \\ 1 - p & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$

The expected value is

$$E[X] = \sum_x p_X(x) = 0 \cdot (1 - p) + 1 \cdot p = p.$$

The variance is

$$\begin{aligned} Var[X] &= \sum_x (x - E[X])^2 p_X(x) \\ &= (0 - p)^2 \cdot (1 - p) + (1 - p)^2 \cdot p \\ &= p(1 - p). \end{aligned}$$

# Sequence of Bernoulli experiments

An experiment consisting of  $n$  independent and identically distributed Bernoulli experiments.

Examples:

- Toss a coin  $n$  times and record the number of heads.
- Send 23 identical messages through the network independently and record the number successfully received.
- Draw 5 cards from a standard deck with replacement (and reshuffling) and record whether or not the card is a king.

# Independent and identically distributed

Let  $X_i$  represent the  $i^{th}$  Bernoulli experiment.

**Independence** means

$$p_{X_1, \dots, X_n}(x_1, \dots, x_n) = \prod_{i=1}^n p_{X_i}(x_i),$$

i.e. the joint probability is the product of the individual probabilities.

**Identically distributed** (for Bernoulli random variables) means

$$P(X_i = 1) = p \quad \forall i,$$

and more generally, the distribution is the same for all the random variables.

- *iid*: independent and identically distributed
- *ind*: independent

## Sequences of Bernoulli experiments

Let  $X_i$  denote the outcome of the  $i^{th}$  Bernoulli experiment. We use the notation

$$X_i \stackrel{iid}{\sim} Ber(p), \quad \text{for } i = 1, \dots, n$$

to indicate a sequence of  $n$  independent and identically distributed Bernoulli experiments.

We could write this equivalently as

$$X_i \stackrel{ind}{\sim} Ber(p), \quad \text{for } i = 1, \dots, n$$

but this is different than

$$X_i \stackrel{ind}{\sim} Ber(p_i), \quad \text{for } i = 1, \dots, n$$

as the latter has a different success probability for each experiment.

# Binomial random variable

Suppose we perform a sequence of  $n$  *iid* Bernoulli experiments and only record the number of successes, i.e.

$$Y = \sum_{i=1}^n X_i.$$

Then we use the notation  $Y \sim \text{Bin}(n, p)$  to indicate a binomial random variable with

- $n$  attempts and
- probability of success  $p$ .

# Binomial probability mass function

We need to obtain

$$p_Y(y) = P(Y = y) \quad \forall y \in \Omega = \{0, 1, 2, \dots, n\}.$$

The probability of obtaining a particular sequence of  $y$  success and  $n - y$  failures is

$$p^y(1 - p)^{n-y}$$

since the experiments are *iid* with success probability  $p$ . But there are

$$\binom{n}{y} = \frac{n!}{y!(n - y)!}$$

ways of obtaining a sequence of  $y$  success and  $n - y$  failures. Thus, the binomial pmf is

$$p_Y(y) = P(Y = y) = \binom{n}{y} p^y (1 - p)^{n-y}.$$

# Properties of binomial random variables

The expected value is

$$E[Y] = E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i] = \sum_{i=1}^n p = np.$$

The variance is

$$\text{Var}[Y] = \sum_{i=1}^n \text{Var}[X_i] = np(1-p)$$

since the  $X_i$  are independent.

The cumulative distribution function is

$$F_Y(y) = P(Y \leq y) = \sum_{x=0}^{\lfloor y \rfloor} \binom{n}{x} p^x (1-p)^{n-x}.$$



## Component failure rate

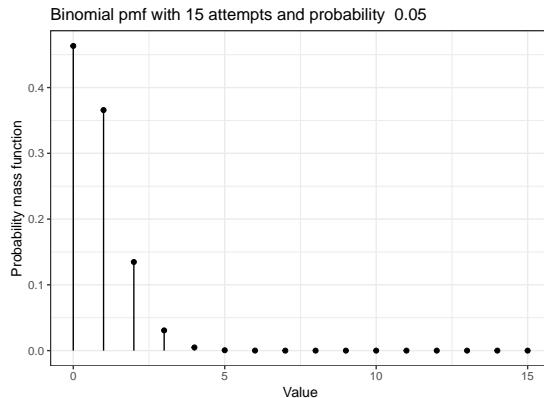
Suppose a box contains 15 components that each have a failure rate of 5%.

What is the probability that

1. exactly two out of the fifteen components are defective?
2. at most two components are defective?
3. more than three components are defective?
4. more than 1 but less than 4 are defective?

# Binomial pmf

Let  $Y$  be the number of defective components and assume  $Y \sim \text{Bin}(15, 0.05)$ .



## Component failure rate - solutions

Let  $Y$  be the number of defective components and assume  $Y \sim \text{Bin}(15, 0.05)$ .

1.  $P(Y = 2) = \binom{15}{2}(0.05)^2(1 - 0.05)^{15-2}$
2.  $P(Y \leq 2) = \sum_{x=0}^2 \binom{15}{x}(0.05)^x(1 - 0.05)^{15-x}$
3.  $P(Y > 3) = 1 - P(Y \leq 3) = 1 - \sum_{x=0}^3 \binom{15}{x}(0.05)^x(1 - 0.05)^{15-x}$
4.  $P(1 < Y < 4) = \sum_{x=2}^3 \binom{15}{x}(0.05)^x(1 - 0.05)^{15-x}$

# Component failure rate - solutions in R

```
n <- 15
p <- 0.05
choose(15,2)

[1] 105

dbinom(2,n,p)          #  $P(Y=2)$ 

[1] 0.1347523

pbinom(2,n,p)          #  $P(Y \leq 2)$ 

[1] 0.9637998

1-pbinom(3,n,p)        #  $P(Y > 3)$ 

[1] 0.005467259

sum(dbinom(c(2,3),n,p)) #  $P(1 < Y < 4) = P(Y=2) + P(Y=3)$ 

[1] 0.1654853
```

## Poisson experiments

Many experiments can be thought of as “how many *rare* events will occur in a certain amount of time or space?” For example,

- # of alpha particles emitted from a polonium bar in an 8 minute period
- # of flaws on a standard size piece of manufactured product, e.g., 100m coaxial cable, 100 sq.meter plastic sheeting
- # of hits on a web page in a 24h period

# Poisson random variable

A Poisson random variable has pmf

$$p(x) = \frac{e^{-\lambda} \lambda^x}{x!} \quad \text{for } x = 0, 1, 2, 3, \dots$$

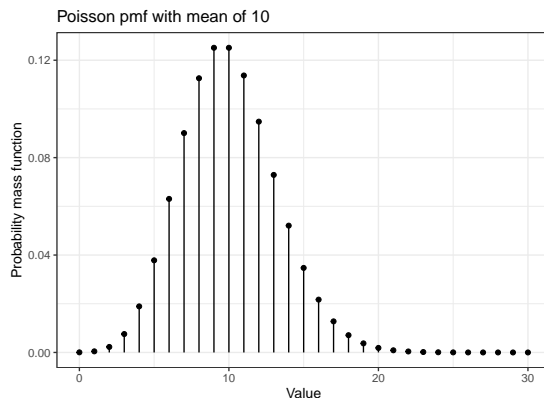
where  $\lambda$  is called the **rate parameter**.

We write  $X \sim Po(\lambda)$  to represent this random variable. We can show that

$$E[X] = Var[X] = \lambda.$$

# Poisson probability mass function

Customers of an internet service provider initiate new accounts at the average rate of 10 accounts per day. What is the probability that more than 8 new accounts will be initiated today?



## Poisson probability

Customers of an internet service provider initiate new accounts at the average rate of 10 accounts per day. What is the probability that more than 8 new accounts will be initiated today?

Let  $X$  be the number of accounts initiated today. Assume  $X \sim Po(10)$ .

$$P(X > 8) = 1 - P(X \leq 8) = 1 - \sum_{x=0}^8 \frac{\lambda^x e^{-\lambda}}{x!} \approx 1 - 0.333 = 0.667$$

In R,

```
# Using pmf  
1-sum(dpois(0:8, lambda=10))
```

```
[1] 0.6671803
```

```
# Using cdf  
1-ppois(8, lambda=10)
```

```
[1] 0.6671803
```



## Sum of Poisson random variables

Let  $X_i \stackrel{ind}{\sim} Po(\lambda_i)$  for  $i = 1, \dots, n$ . Then

$$Y = \sum_{i=1}^n X_i \sim Po\left(\sum_{i=1}^n \lambda_i\right).$$

Let  $X_i \stackrel{iid}{\sim} Po(\lambda)$  for  $i = 1, \dots, n$ . Then

$$Y = \sum_{i=1}^n X_i \sim Po(n\lambda).$$

## Poisson random variable - example

Customers of an internet service provider initiate new accounts at the average rate of 10 accounts per day. What is the probability that more than 16 new accounts will be initiated in the next two days?

Since the rate is 10/day, then for two days we expect, on average, to have 20. Let  $Y$  be the number initiated in a two-day period and assume  $Y \sim Po(20)$ . Then

$$\begin{aligned} P(Y > 16) &= 1 - P(Y \leq 16) \\ &= 1 - \sum_{x=0}^{16} \frac{\lambda^x e^{-\lambda}}{x!} \\ &= 1 - 0.221 = 0.779. \end{aligned}$$

In R,

```
# Using pmf  
1-sum(dpois(0:16, lambda=20))
```

```
[1] 0.7789258
```

```
# Using cdf  
1-ppois(16, lambda=20)
```

```
[1] 0.7789258
```

## Manufacturing example

A manufacturer produces 100 chips per day and, on average, 1% of these chips are defective. What is the probability that no defectives are found in a particular day?

Let  $X$  represent the number of defectives and assume  $X \sim \text{Bin}(100, 0.01)$ . Then

$$P(X = 0) = \binom{100}{0} (0.01)^0 (1 - 0.01)^{100} \approx 0.366.$$

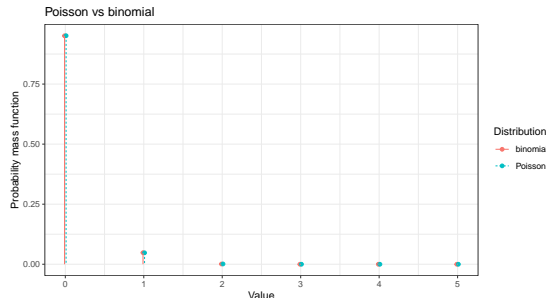
Alternatively, let  $Y$  represent the number of defectives and assume  $Y \sim \text{Po}(100 \times 0.01)$ . Then

$$P(Y = 0) = \frac{1^0 e^{-1}}{0!} \approx 0.368.$$

# Poisson approximation to the binomial

Suppose we have  $X \sim \text{Bin}(n, p)$  with  $n$  large (say  $\geq 20$ ) and  $p$  small (say  $\leq 0.05$ ). We can approximate  $X$  by  $Y \sim \text{Po}(np)$  because for large  $n$  and small  $p$

$$\binom{n}{k} p^k (1-p)^{n-k} \approx e^{-np} \frac{(np)^k}{k!}.$$



## Example

Imagine you are supposed to proofread a paper. Let us assume that there are on average 2 typos on a page and a page has 1000 words. This gives a probability of 0.002 for each word to contain a typo. What is the probability the page has no typos?

Let  $X$  represent the number of typos on the page and assume  $X \sim \text{Bin}(1000, 0.002)$ .  $P(X = 0)$  using R is

```
n = 1000; p = 0.002  
dbinom(0, size=n, prob=p)
```

```
[1] 0.1350645
```

Alternatively, let  $Y$  represent the number of defectives and assume  $Y \sim \text{Po}(1000 \times 0.002)$ .  $P(Y = 0)$  using R is

```
dpois(0, lambda = n*p)
```

```
[1] 0.1353353
```

# Summary

- General discrete random variables
  - Probability mass function (pmf)
  - Cumulative distribution function (cdf)
  - Expected value
  - Variance
  - Standard deviation
- Specific discrete random variables
  - Bernoulli
  - Binomial
  - Poisson

## P3 - Continuous random variables

STAT 5870 (Engineering)  
Iowa State University

August 28, 2024

# Continuous vs discrete random variables

Discrete random variables have

- finite or countable support and
- pmf:  $P(X = x)$ .

Continuous random variables have

- uncountable support and
- $P(X = x) = 0$  for all  $x$ .



# Cumulative distribution function

The **cumulative distribution function** for a continuous random variable is

$$F_X(x) = P(X \leq x) = P(X < x)$$

since  $P(X = x) = 0$  for any  $x$ .

The cdf still has the properties

- $0 \leq F_X(x) \leq 1$  for all  $x \in \mathbb{R}$ ,
- $F_X$  is monotone increasing,  
i.e. if  $x_1 \leq x_2$  then  $F_X(x_1) \leq F_X(x_2)$ , and
- $\lim_{x \rightarrow -\infty} F_X(x) = 0$  and  $\lim_{x \rightarrow \infty} F_X(x) = 1$ .

# Probability density function

The **probability density function** (pdf) for a continuous random variable is

$$f_X(x) = \frac{d}{dx}F_X(x)$$

and

$$F_X(x) = \int_{-\infty}^x f_X(t)dt.$$

Thus, the pdf has the following properties

- $f_X(x) \geq 0$  for all  $x$  and
- $\int_{-\infty}^{\infty} f(x)dx = 1.$

## Example

Let  $X$  be a random variable with probability density function

$$f_X(x) = \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

$f_X(x)$  defines a valid pdf because  $f_X(x) \geq 0$  for all  $x$  and

$$\int_{-\infty}^{\infty} f_X(x) dx = \int_0^1 3x^2 dx = x^3 \Big|_0^1 = 1.$$

The cdf is

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x^3 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}.$$

## Expected value

Let  $X$  be a continuous random variable and  $h$  be some function. The **expected value** of a function of a continuous random variable is

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) \cdot f_X(x) dx.$$

If  $h(x) = x$ , then

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) dx.$$

and we call this the **expectation** of  $X$ . We commonly use the symbol  $\mu$  for this expectation.

## Example (cont.)

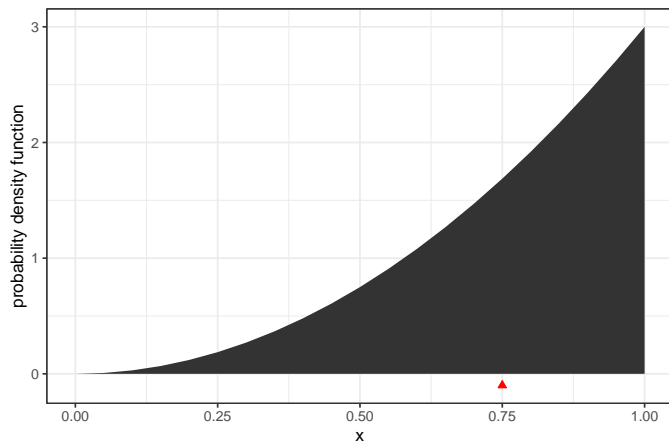
Let  $X$  be a random variable with probability density function

$$f_X(x) = \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The expected value is

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x \cdot f_X(x) dx \\ &= \int_0^1 3x^3 dx \\ &= 3 \frac{x^4}{4} \Big|_0^1 = \frac{3}{4}. \end{aligned}$$

## Example - Center of mass



# Variance

The **variance** of a random variable is defined as the expected squared deviation from the mean. For continuous random variables, variance is

$$Var[X] = E[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x) dx$$

where  $\mu = E[X]$ . The symbol  $\sigma^2$  is commonly used for the variance.

The **standard deviation** is the positive square root of the variance

$$SD[X] = \sqrt{Var[X]}.$$

The symbol  $\sigma$  is commonly used for the standard deviation.

## Example (cont.)

Let  $X$  be a random variable with probability density function

$$f_X(x) = \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The variance is

$$\begin{aligned} \text{Var}[X] &= \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x) dx \\ &= \int_0^1 \left(x - \frac{3}{4}\right)^2 3x^2 dx \\ &= \int_0^1 \left[x^2 - \frac{3}{2}x + \frac{9}{16}\right] 3x^2 dx \\ &= \int_0^1 3x^4 - \frac{9}{2}x^3 + \frac{27}{16}x^2 dx \\ &= \left[\frac{3}{5}x^5 - \frac{9}{8}x^4 + \frac{9}{16}x^3\right] \Big|_0^1 dx \\ &= \frac{3}{5} - \frac{9}{8} + \frac{9}{16} \\ &= \frac{3}{80}. \end{aligned}$$



# Comparison of discrete and continuous random variables

For simplicity here and later, we drop the subscript  $X$ .

	discrete	continuous
support ( $\mathcal{X}$ )	finite or countable	uncountable
pmf	$p(x) = P(X = x)$	
pdf		$p(x) = f(x) = F'(x)$
cdf	$F(x) = P(X \leq x)$ $= \sum_{t \leq x} p(t)$	$F(x) = P(X \leq x) = P(X < x)$ $= \int_{-\infty}^x p(t) dt$
expected value	$E[h(X)] = \sum_{x \in \mathcal{X}} h(x)p(x)$	$E[h(X)] = \int_{\mathcal{X}} h(x)p(x) dx$
expectation	$\mu = E[X] = \sum_{x \in \mathcal{X}} x p(x)$	$\mu = E[X] = \int_{\mathcal{X}} x p(x) dx$
variance	$\sigma^2 = Var[X] = E[(X - \mu)^2]$ $= \sum_{x \in \mathcal{X}} (x - \mu)^2 p(x)$	$\sigma^2 = Var[X] = E[(X - \mu)^2]$ $= \int_{\mathcal{X}} (x - \mu)^2 p(x) dx$

Note: we replace summations with integrals when using continuous as opposed to discrete random

# Uniform

A **uniform** random variable on the interval  $(a, b)$  has equal probability for any value in that interval and we denote this  $X \sim Unif(a, b)$ . The pdf for a uniform random variable is

$$f(x) = \frac{1}{b-a} \mathbf{I}(a < x < b)$$

where  $\mathbf{I}(A)$  is in indicator function that is 1 if  $A$  is true and 0 otherwise, i.e.

$$\mathbf{I}(A) = \begin{cases} 1 & A \text{ is true} \\ 0 & \text{otherwise.} \end{cases}$$

The expectation is

$$E[X] = \int_a^b x \frac{1}{b-a} dx = \frac{a+b}{2}$$

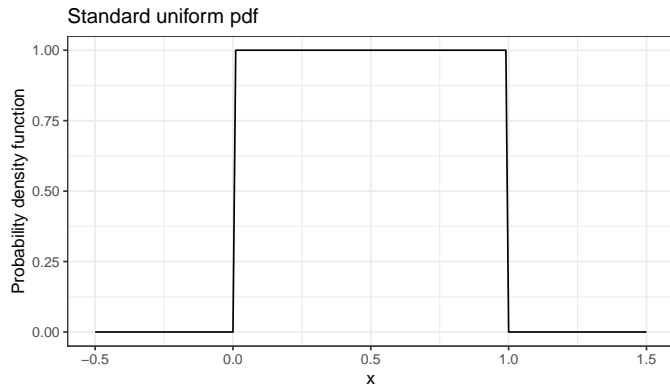
and the variance is

$$Var[X] = \int_a^b \frac{1}{b-a} \left( x - \frac{a+b}{2} \right)^2 dx = \frac{1}{12} (b-a)^2.$$

# Standard uniform

A **standard uniform** random variable is  $X \sim Unif(0, 1)$ . This random variable has

$$E[X] = \frac{1}{2} \quad \text{and} \quad Var[X] = \frac{1}{12}.$$



## Example (cont.)

Pseudo-random number generators generate pseudo uniform values on (0,1). These values can be used in conjunction with the inverse of the cumulative distribution function to generate pseudo-random numbers from any distribution.

The inverse of the cdf  $F_X(x) = x^3$  is

$$F_X^{-1}(u) = u^{1/3}.$$

A uniform random number on the interval (0,1) generated using the inverse cdf produces a random draw of  $X$ .

```
inverse_cdf = function(u) u^(1/3)
x = inverse_cdf(runif(1e6))
mean(x)
```

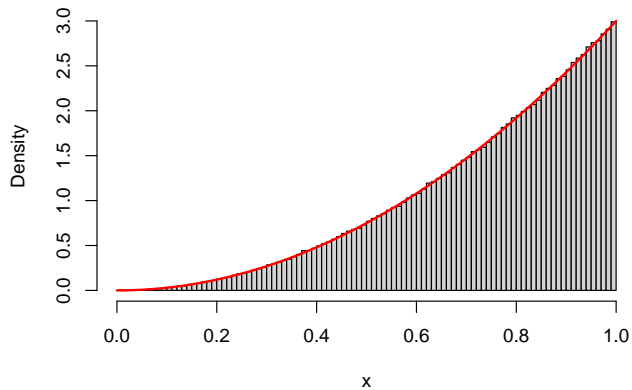
```
[1] 0.7502002
```

```
var(x); 3/80
```

```
[1] 0.03752111
```

```
[1] 0.0375
```

Histogram of x



# Normal random variable

The **normal (or Gaussian) density** is a “bell-shaped” curve. The density has two parameters: **mean**  $\mu$  and **variance**  $\sigma^2$  and is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \quad \text{for } -\infty < x < \infty$$

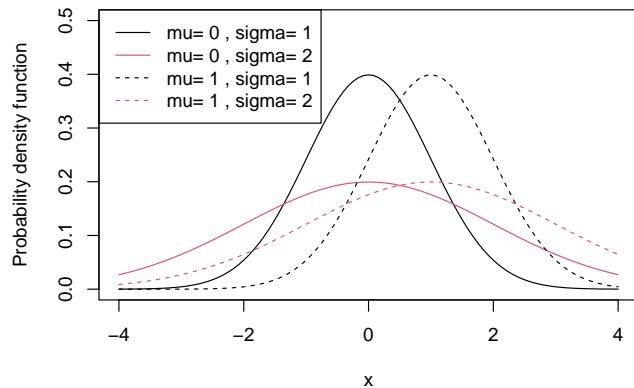
If  $X \sim N(\mu, \sigma^2)$ , then

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f(x) dx = \dots = \mu \\ \text{Var}[X] &= \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx = \dots = \sigma^2. \end{aligned}$$

Thus, the parameters  $\mu$  and  $\sigma^2$  are actually the mean and the variance of the  $N(\mu, \sigma^2)$  distribution.

There is no closed form cumulative distribution function for a normal random variable.

# Example normal probability density functions



# Properties of normal random variables

Let  $Z \sim N(0, 1)$ , i.e. a **standard normal** random variable. Then for constants  $\mu$  and  $\sigma$

$$X = \mu + \sigma Z \sim N(\mu, \sigma^2)$$

and

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

which is called **standardizing**.

Let  $X_i \stackrel{ind}{\sim} N(\mu_i, \sigma_i^2)$ . Then

$$Z_i = \frac{X_i - \mu_i}{\sigma_i} \stackrel{iid}{\sim} N(0, 1) \quad \text{for all } i$$

and

$$Y = \sum_{i=1}^n X_i \sim N\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\right).$$



## Calculating the standard normal cdf

If  $Z \sim N(0, 1)$ , what is  $P(Z \leq 1.5)$ ? Although the cdf does not have a closed form, very good approximations exist and are available as tables or in software, e.g.

```
pnorm(1.5) # default is mean=0, sd=1
```

```
[1] 0.9331928
```

If  $Z \sim N(0, 1)$ , then

- $P(Z \leq z) = \Phi(z)$
- $\Phi(z) = 1 - \Phi(-z)$  since a normal pdf is **symmetric** around its mean.

# Calculating any normal cumulative distribution function

If  $X \sim N(15, 4)$  what is  $P(X > 18)$ ?

$$\begin{aligned} P(X > 18) &= 1 - P(X \leq 18) \\ &= 1 - P\left(\frac{X-15}{2} \leq \frac{18-15}{2}\right) \\ &= 1 - P(Z \leq 1.5) \\ &\approx 1 - 0.933 = 0.067 \end{aligned}$$

```
1-pnorm((18-15)/2)      # by standardizing
```

```
[1] 0.0668072
```

```
1-pnorm(18, mean = 15, sd = 2) # using the mean and sd arguments
```

```
[1] 0.0668072
```

# Manufacturing

Suppose you are producing nails that must be within 5 and 6 centimeters in length. If the average length of nails the process produces is 5.3 cm and the standard deviation is 0.1 cm. What is the probability the next nail produced is outside of the specification?

Let  $X \sim N(5.3, 0.1^2)$  be the length (cm) of the next nail produced. We need to calculate

$$P(X < 5 \text{ or } X > 6) = 1 - P(5 < X < 6).$$

```
mu = 5.3
sigma = 0.1

1-diff(pnorm(c(5,6), mean = mu, sd = sigma))

[1] 0.001349898
```

# Summary

- Continuous random variables
  - Probability density function
  - Cumulative distribution function
  - Expectation
  - Variance
- Specific distributions
  - Uniform
  - Normal (or Gaussian)

## P4 - Central Limit Theorem

STAT 5870 (Engineering)  
Iowa State University

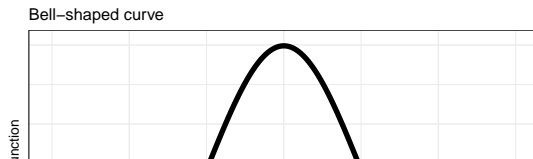
August 28, 2024

**Main Idea:** Sums and averages of iid random variables from **any distribution** have approximate normal distributions for sufficiently large sample sizes.

# Bell-shaped curve

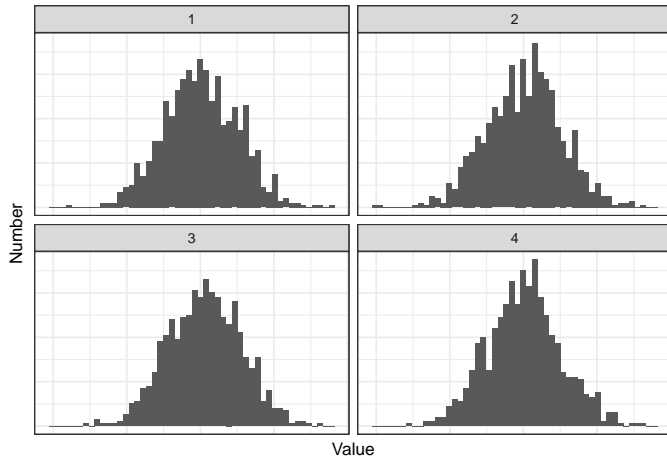
The term **bell-shaped curve** typically refers to the probability density function for a normal random variable:

```
Warning: Using 'size' aesthetic for lines was
deprecated in ggplot2 3.4.0.
i Please use 'linewidth' instead.
This warning is displayed once every 8 hours.
Call 'lifecycle::last_lifecycle_warnings()' to
see where this warning was generated.
```



# Histograms of samples from bell-shaped curves

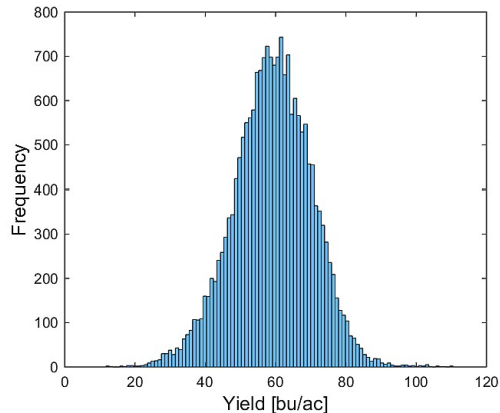
Histograms of 1,000 standard normal random variables



# Yield

<https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0184198>

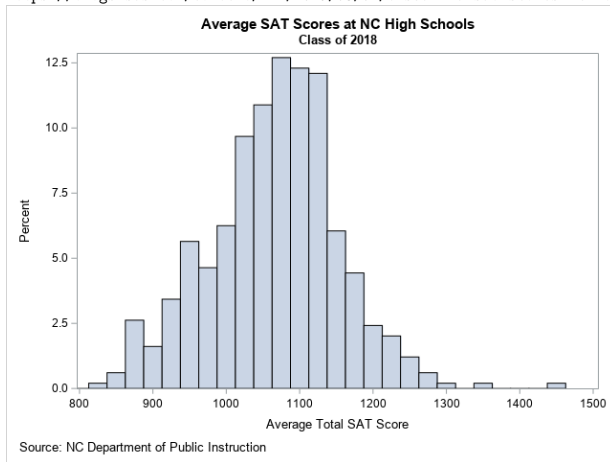
**S1 Fig. Histogram of yield.**





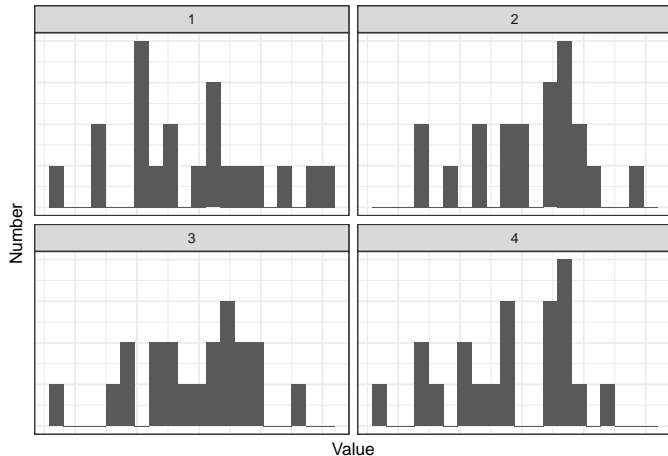
# SAT scores

<https://blogs.sas.com/content/iml/2019/03/04/visualize-sat-scores-nc.html>



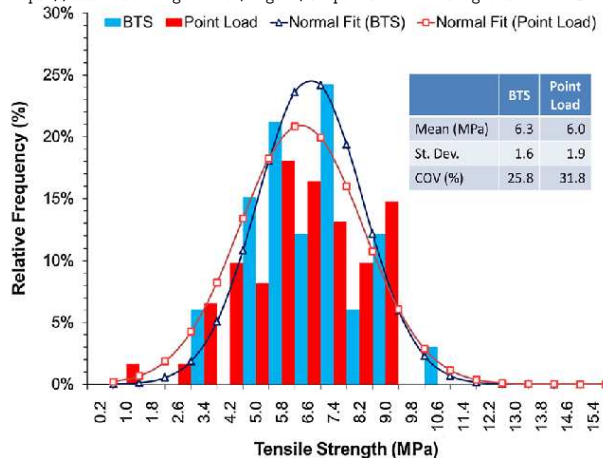
# Histograms of samples from bell-shaped curves

Histograms of 20 standard normal random variables



# Tensile strength

[https://www.researchgate.net/figure/Comparison-of-histograms-for-BTS-and-tensile-strength-estimated-from-point-load\\_fig5\\_260617256](https://www.researchgate.net/figure/Comparison-of-histograms-for-BTS-and-tensile-strength-estimated-from-point-load_fig5_260617256)



# Sums and averages of iid random variables

Suppose  $X_1, X_2, \dots$  are iid random variables with

$$E[X_i] = \mu \quad \text{Var}[X_i] = \sigma^2.$$

Define

$$\begin{aligned} \text{Sample Sum: } S_n &= X_1 + X_2 + \dots + X_n \\ \text{Sample Average: } \bar{X}_n &= S_n/n. \end{aligned}$$

For  $S_n$ , we know

$$E[S_n] = n\mu, \quad \text{Var}[S_n] = n\sigma^2, \quad \text{and} \quad SD[S_n] = \sqrt{n}\sigma.$$

For  $\bar{X}_n$ , we know

$$E[\bar{X}_n] = \mu, \quad \text{Var}[\bar{X}_n] = \sigma^2/n, \quad \text{and} \quad SD[\bar{X}_n] = \sigma/\sqrt{n}.$$

# Central Limit Theorem (CLT)

Suppose  $X_1, X_2, \dots$  are iid random variables with

$$E[X_i] = \mu \quad \text{Var}[X_i] = \sigma^2.$$

Define

$$\begin{aligned} \text{Sample Sum: } S_n &= X_1 + X_2 + \dots + X_n \\ \text{Sample Average: } \bar{X}_n &= S_n/n. \end{aligned}$$

Then the **Central Limit Theorem** says

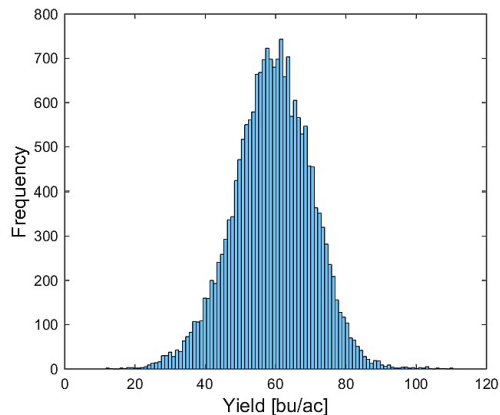
$$\lim_{n \rightarrow \infty} \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0, 1) \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{S_n - n\mu}{\sqrt{n}\sigma} \xrightarrow{d} N(0, 1).$$

**Main Idea:** Sums and averages of iid random variables from **any distribution** have approximate normal distributions for sufficiently large sample sizes.

# Yield

<https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0184198>

**S1 Fig. Histogram of yield.**



# Approximating distributions

Rather than considering the limit, I typically think of the following approximations as  $n$  gets large.

For the sample average,

$$\overline{X}_n \dot{\sim} N(\mu, \sigma^2/n).$$

where  $\dot{\sim}$  indicates *approximately distributed* because

$$E[\overline{X}_n] = \mu \quad \text{and} \quad \text{Var}[\overline{X}_n] = \sigma^2/n.$$

For the sample sum,

$$S_n \dot{\sim} N(n\mu, n\sigma^2)$$

because

$$\begin{aligned} E[S_n] &= n\mu \\ \text{Var}[S_n] &= n\sigma^2. \end{aligned}$$

# Averages and sums of uniforms

Let  $X_i \stackrel{ind}{\sim} Unif(0, 1)$ . Then

$$\mu = E[X_i] = \frac{1}{2} \quad \text{and} \quad \sigma^2 = Var[X_i] = \frac{1}{12}.$$

Thus

$$\overline{X}_n \dot{\sim} N\left(\frac{1}{2}, \frac{1}{12n}\right)$$

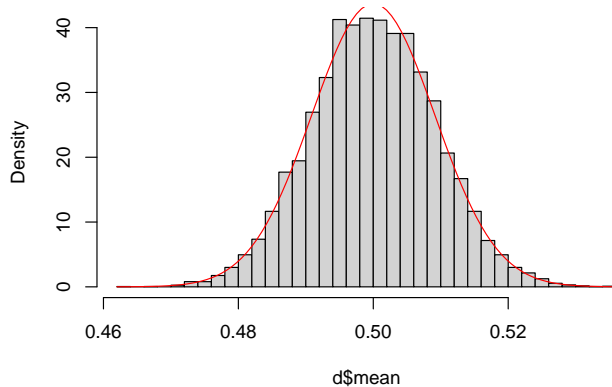
and

$$S_n \dot{\sim} N\left(\frac{n}{2}, \frac{n}{12}\right).$$



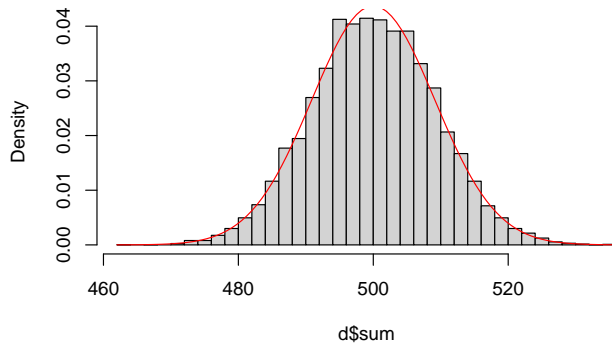
# Averages of uniforms

Histogram of d\$mean



# Sums of uniforms

Histogram of d\$sum



# Normal approximation to a binomial

Recall if  $Y_n = \sum_{i=1}^n X_i$  where  $X_i \stackrel{ind}{\sim} \text{Ber}(p)$ , then

$$Y_n \sim \text{Bin}(n, p).$$

For a binomial random variable, we have

$$E[Y_n] = np \quad \text{and} \quad \text{Var}[Y_n] = np(1 - p).$$

By the CLT,

$$\lim_{n \rightarrow \infty} \frac{Y_n - np}{\sqrt{np(1 - p)}} \rightarrow N(0, 1),$$

if  $n$  is large,

$$Y_n \dot{\sim} N(np, np[1 - p]).$$

# Roulette example

A European roulette wheel has 39 slots: one green, 19 black, and 19 red. If I play black every time, what is the probability that I will have won more than I lost after 99 spins of the wheel?

<https://isorepublic.com/photo/roulette-wheel/>



## Roulette example

A European roulette wheel has 39 slots: one green, 19 black, and 19 red. If I play black every time, what is the probability that I will have won more than I lost after 99 spins of the wheel?

Let  $Y$  indicate the total number of wins and assume  $Y \sim \text{Bin}(n, p)$  with  $n = 99$  and  $p = 19/39$ . The desired probability is  $P(Y \geq 50)$ . Then

$$P(Y \geq 50) = 1 - P(Y < 50) = 1 - P(Y \leq 49)$$

```
n = 99
p = 19/39
1-pbinom(49, n, p)

[1] 0.399048
```

## Roulette example

A European roulette wheel has 39 slots: one green, 19 black, and 19 red. If I play black every time, what is the probability that I will have won more than I lost after 99 spins of the wheel?

Let  $Y$  indicate the total number of wins. We can approximate  $Y$  using  $X \sim N(np, np(1-p))$ .

$$P(Y \geq 50) \approx 1 - P(X < 50)$$

```
1-pnorm(50, n*p, sqrt(n*p*(1-p)))
```

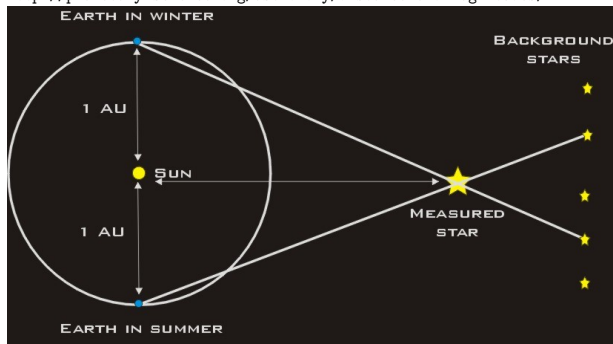
```
[1] 0.3610155
```

A better approximation can be found using a continuity correction.

## Astronomy example

An astronomer wants to measure the distance,  $d$ , from Earth to a star. Suppose the procedure has a known standard deviation of 2 parsecs. The astronomer takes 30 iid measurements and finds the average of these measurements to be 29.4 parsecs. What is the probability the average is within 0.5 parsecs?

<http://planetary-science.org/astrometry/distance-and-magnitudes/>



## Astronomy example

Let  $X_i$  be the  $i^{th}$  measurement. The astronomer assumes that  $X_1, X_2, \dots, X_n$  are iid with  $E[X_i] = d$  and  $Var[X_i] = \sigma^2 = 2^2$ . The estimate of  $d$  is

$$\bar{X}_n = \frac{(X_1 + X_2 + \dots + X_n)}{n} = 29.4.$$

and, by the Central Limit Theorem,  $\bar{X}_n \sim N(d, \sigma^2/n)$  where  $n = 30$ . We want to find

$$\begin{aligned} P(|\bar{X}_n - d| < 0.5) &= P(-0.5 < \bar{X}_n - d < 0.5) \\ &= P\left(\frac{-0.5}{2/\sqrt{30}} < \frac{\bar{X}_n - d}{\sigma/\sqrt{n}} < \frac{0.5}{2/\sqrt{30}}\right) \\ &\approx P(-1.37 < Z < 1.37) \end{aligned}$$

```
diff(pnorm(c(-1.37,1.37)))
```

```
[1] 0.8293131
```



## Astronomy example - sample size

Suppose the astronomer wants to be within 0.5 parsecs with at least 95% probability. How many more samples would she need to take?

We solve

$$\begin{aligned}
 0.95 \leq P(|\bar{X}_n - d| < .5) &= P(-0.5 < \bar{X}_n - d < 0.5) \\
 &= P\left(\frac{-0.5}{2/\sqrt{n}} < \frac{\bar{X}_n - d}{\sigma/\sqrt{n}} < \frac{0.5}{2/\sqrt{n}}\right) \\
 &= P(-z < Z < z) & z = 0.5/(2/\sqrt{n}) \\
 &= 1 - [P(Z < -z) + P(Z > z)] \\
 &= 1 - 2P(Z < -z)
 \end{aligned}$$

where  $z = 1.96$  since

```
1-2*pnorm(-1.96)
```

```
[1] 0.9500042
```

# Summary

- Central Limit Theorem
  - Sums
  - Averages
- Examples
  - Uniforms
  - Binomial
    - Roulette
- Sample size
  - Astronomy

# P5 - Multiple random variables

STAT 5870 (Engineering)  
Iowa State University

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## Multiple discrete random variables

If  $X$  and  $Y$  are two discrete variables, their **joint probability mass function** is defined as

$$p_{X,Y}(x, y) = P(X = x \cap Y = y) = P(X = x, Y = y).$$

## CPU example

A box contains 5 PowerPC G4 processors of different speeds:

#	speed
2	400 mHz
1	450 mHz
2	500 mHz

Randomly select two processors out of the box (without replacement) and let

- $X$  be speed of the first selected processor and
- $Y$  be speed of the second selected processor.

## CPU example - outcomes

	$\Omega$	1st processor ( $X$ )				
		$400_1$	$400_2$	$450$	$500_1$	$500_2$
2nd processor ( $Y$ )	$400_1$	-	x	x	x	x
	$400_2$	x	-	x	x	x
	$450$	x	x	-	x	x
	$500_1$	x	x	x	-	x
	$500_2$	x	x	x	x	-

Reasonable to believe each outcome is equally probable.

# CPU example - joint pmf

Joint probability mass function for  $X$  and  $Y$ :

		1st processor ( $X$ )			
		mHz	400	450	500
2nd processor ( $Y$ )	400	2/20	2/20	4/20	
	450	2/20	0/20	2/20	
	500	4/20	2/20	2/20	

- What is  $P(X = Y)$ ?
- What is  $P(X > Y)$ ?

## CPU example - probabilities

What is the probability that  $X = Y$ ?

$$\begin{aligned}P(X = Y) &= p_{X,Y}(400, 400) + p_{X,Y}(450, 450) + p_{X,Y}(500, 500) \\&= 2/20 + 0/20 + 2/20 = 4/20 = 0.2\end{aligned}$$

What is the probability that  $X > Y$ ?

$$\begin{aligned}P(X > Y) &= p_{X,Y}(450, 400) + p_{X,Y}(500, 400) + p_{X,Y}(500, 450) \\&= 2/20 + 4/20 + 2/20 = 8/20 = 0.4\end{aligned}$$



# Expectation

The **expected value** of a function  $h(x, y)$  is

$$E[h(X, Y)] = \sum_{x,y} h(x, y) p_{X,Y}(x, y).$$

## CPU example - expected absolute speed difference

What is  $E[|X - Y|]$ ?

Here, we have the situation  $E[|X - Y|] = E[h(X, Y)]$ , with  $h(X, Y) = |X - Y|$ . Thus, we have

$$\begin{aligned} E[|X - Y|] &= \sum_{x,y} |x - y| p_{X,Y}(x, y) = \\ &= |400 - 400| \cdot 0.1 + |400 - 450| \cdot 0.1 + |400 - 500| \cdot 0.2 \\ &\quad + |450 - 400| \cdot 0.1 + |450 - 450| \cdot 0.0 + |450 - 500| \cdot 0.1 \\ &\quad + |500 - 400| \cdot 0.2 + |500 - 450| \cdot 0.1 + |500 - 500| \cdot 0.1 \\ &= 0 + 5 + 20 + 5 + 0 + 5 + 20 + 5 + 0 = 60. \end{aligned}$$

# Marginal distribution

For discrete random variables  $X$  and  $Y$ , the **marginal probability mass functions** are

$$\begin{aligned} p_X(x) &= \sum_y p_{X,Y}(x, y) & \text{and} \\ p_Y(y) &= \sum_x p_{X,Y}(x, y) \end{aligned}$$

# Marginal distribution

Joint probability mass function for  $X$  and  $Y$ :

	mHz	1st processor ( $X$ )		
		400	450	500
2nd processor ( $Y$ )	400	2/20	2/20	4/20
	450	2/20	0/20	2/20
	500	4/20	2/20	2/20

Summing the rows within each column provides

$x$	400	450	500
$p_X(x)$	0.4	0.2	0.4

Summing the columns within each row provides

$y$	400	450	500
$p_Y(y)$	0.4	0.2	0.4

## CPU example - independence

Are  $X$  and  $Y$  independent?

$X$  and  $Y$  are **independent** if  $p_{x,y}(x, y) = p_X(x)p_Y(y)$  for all  $x$  and  $y$ .

Since

$$p_{X,Y}(450, 450) = 0 \neq 0.2 \cdot 0.2 = p_X(450) \cdot p_Y(450)$$

they are **not** independent.

# Covariance

The **covariance** between two random variables  $X$  and  $Y$  is

$$\text{Cov}[X, Y] = E[(X - \mu_X)(Y - \mu_Y)]$$

where

$$\mu_X = E[X] \quad \text{and} \quad \mu_Y = E[Y].$$

If  $Y = X$  in the above definition, then

$$\text{Cov}[X, X] = \text{Var}[X].$$

## CPU example - covariance

Use marginal pmfs to compute:

$$E[X] = E[Y] = 450 \quad \text{and} \quad \text{Var}[X] = \text{Var}[Y] = 2000.$$

The covariance between  $X$  and  $Y$  is:

$$\begin{aligned} \text{Cov}[X, Y] &= \sum_{x,y} (x - E[X])(y - E[Y])p_{X,Y}(x, y) = \\ &= (400 - 450)(400 - 450) \cdot 0.1 \\ &\quad + (450 - 450)(400 - 450) \cdot 0.1 \\ &\quad + \dots \\ &\quad + (500 - 450)(500 - 450) \cdot 0.1 \\ &= 250 + 0 - 500 + 0 + 0 + 0 - 500 + 250 + 0 \\ &= -500. \end{aligned}$$

# Correlation

The **correlation** between two variables  $X$  and  $Y$  is

$$\rho[X, Y] = \frac{Cov[X, Y]}{\sqrt{Var[X] \cdot Var[Y]}} = \frac{Cov[X, Y]}{SD[X] \cdot SD[Y]}.$$



# Correlation properties

- $\rho$  is between -1 and 1
- if  $\rho = 1$  or  $-1$ ,  $Y$  is a linear function of  $X$ :
  - $\rho = 1 \implies Y = mX + b$  with  $m > 0$ ,
  - $\rho = -1 \implies Y = mX + b$  with  $m < 0$ ,
- $\rho$  is a measure of linear association between  $X$  and  $Y$ 
  - $\rho$  near  $\pm 1$  indicates a strong linear relationship,
  - $\rho$  near 0 indicates a lack of linear association.

## CPU example - correlation

Recall

$$\text{Cov}[X, Y] = -500 \quad \text{and} \quad \text{Var}[X] = \text{Var}[Y] = 2000.$$

The correlation is

$$\rho[X, Y] = \frac{\text{Cov}[X, Y]}{\sqrt{\text{Var}[X] \cdot \text{Var}[Y]}} = \frac{-500}{\sqrt{2000 \cdot 2000}} = -0.25,$$

and thus there is a weak negative (linear) association.

# Continuous random variables

Suppose  $X$  and  $Y$  are two continuous random variables with **joint probability density function**  $p_{X,Y}(x, y)$ . Probabilities are calculated by integrating this function. For example,

$$P(a < X < b, c < Y < d) = \int_c^d \int_a^b p_{X,Y}(x, y) dx dy.$$

Then the **marginal probability density functions** are

$$\begin{aligned} p_X(x) &= \int p_{X,Y}(x, y) dy \\ p_Y(y) &= \int p_{X,Y}(x, y) dx. \end{aligned}$$

# Continuous random variables

Two continuous random variables are **independent** if

$$p_{X,Y}(x, y) = p_X(x) p_Y(y).$$

The expected value is

$$E[h(X, Y)] = \int \int h(x, y) p_{X,Y}(x, y) dx dy.$$

# Properties of variances and covariances

For any random variables  $X$ ,  $Y$ ,  $W$  and  $Z$ ,

$$\text{Var}[aX + bY + c] = a^2\text{Var}[X] + b^2\text{Var}[Y] + 2ab\text{Cov}[X, Y]$$

$$\begin{aligned}\text{Cov}[aX + bY, cZ + dW] &= ac\text{Cov}[X, Z] + ad\text{Cov}[X, W] \\ &\quad + bc\text{Cov}[Y, Z] + bd\text{Cov}[Y, W]\end{aligned}$$

$$\begin{aligned}\text{Cov}[X, Y] &= \text{Cov}[Y, X] \\ \rho[X, Y] &= \rho[Y, X]\end{aligned}$$

If  $X$  and  $Y$  are independent, then

$$\begin{aligned}\text{Cov}[X, Y] &= 0 \\ \text{Var}[aX + bY + c] &= a^2\text{Var}[X] + b^2\text{Var}[Y].\end{aligned}$$

# Summary

- Multiple random variables
  - joint probability mass function
  - marginal probability mass function
  - joint probability density function
  - marginal probability density function
  - expected value
  - covariance
  - correlation