P1 - Probability

STAT 5870 (Engineering)
Iowa State University

August 28, 2024

Probability - Interpretation

What do we mean when we say the word probability/chance/likelihood/odds?

Probability - Interpretation

What do we mean when we say the word probability/chance/likelihood/odds? For example,

- The probability that Kamala Harris will becomes president in 2025 is 55%.

Probability - Interpretation

What do we mean when we say the word probability/chance/likelihood/odds? For example,

- The probability that Kamala Harris will becomes president in 2025 is 55%.
- The chance I will win a game of solitaire is 20%.

Probability - Interpretation

What do we mean when we say the word probability/chance/likelihood/odds? For example,

- The probability that Kamala Harris will becomes president in 2025 is 55%.
- The chance I will win a game of solitaire is 20%.
- The odds of there being a time capsule behind this wall is 100:1.

Probability - Interpretation

What do we mean when we say the word probability/chance/likelihood/odds? For example,

- The probability that Kamala Harris will becomes president in 2025 is 55%.
- The chance I will win a game of solitaire is 20%.
- The odds of there being a time capsule behind this wall is 100:1.

Interpretations:

- Relative frequency: Probability is the proportion of times the event occurs as the number of times the event is attempted tends to infinity.

Probability - Interpretation

What do we mean when we say the word probability/chance/likelihood/odds? For example,

- The probability that Kamala Harris will becomes president in 2025 is 55%.
- The chance I will win a game of solitaire is 20%.
- The odds of there being a time capsule behind this wall is 100:1.

Interpretations:

- Relative frequency: Probability is the proportion of times the event occurs as the number of times the event is attempted tends to infinity.
- Personal belief: Probability is a statement about your personal belief in the event occuring.

Probability - Example

Let C be a successful connection to the internet from a laptop event.

Probability - Example

Let C be a successful connection to the internet from a laptop event.

> *From our experience with the wireless network and our internet service provider, we believe the probability we successfully connect is 90 %.*

Probability - Example

Let C be a successful connection to the internet from a laptop event.

> *From our experience with the wireless network and our internet service provider, we believe the probability we successfully connect is 90 %.*

We write $P(C) = 0.9$.

> *To be able to work with probabilities, in particular, to be able to compute probabilities of events, a mathematical foundation is necessary.*

Sets - definition

A set is a collection of things.

Sets - definition

A set is a collection of things. We use the following notation

- $\omega \in A$ means ω is an element of the set A,

Sets - definition

A set is a collection of things. We use the following notation

- $\omega \in A$ means ω is an element of the set A,
- $\omega \notin A$ means ω is not an element of the set A,

Sets - definition

A set is a collection of things. We use the following notation

- $\omega \in A$ means ω is an element of the set A,
- $\omega \notin A$ means ω is not an element of the set A,
- $A \subseteq B$ (or $B \supseteq A$) means the set A is a subset of B (with the sets possibly being equal), and

Sets - definition

A set is a collection of things. We use the following notation

- $\omega \in A$ means ω is an element of the set A,
- $\omega \notin A$ means ω is not an element of the set A,
- $A \subseteq B$ (or $B \supseteq A$) means the set A is a subset of B (with the sets possibly being equal), and
- $A \subset B$ (or $B \supset A$) means the set A is a proper subset of B, i.e. there is at least one element in B that is not in A.

Sets - definition

A set is a collection of things. We use the following notation

- $\omega \in A$ means ω is an element of the set A,
- $\omega \notin A$ means ω is not an element of the set A,
- $A \subseteq B$ (or $B \supseteq A$) means the set A is a subset of B (with the sets possibly being equal), and
- $A \subset B$ (or $B \supset A$) means the set A is a proper subset of B, i.e. there is at least one element in B that is not in A.

Sets - definition

A set is a collection of things. We use the following notation

- $\omega \in A$ means ω is an element of the set A,
- $\omega \notin A$ means ω is not an element of the set A,
- $A \subseteq B$ (or $B \supseteq A$) means the set A is a subset of B (with the sets possibly being equal), and
- $A \subset B$ (or $B \supset A$) means the set A is a proper subset of B, i.e. there is at least one element in B that is not in A.

The sample space, Ω, is the set of all outcomes of an experiment.

Set - examples

The set of all possible sums of two 6-sided dice rolls is Ω

Set - examples

The set of all possible sums of two 6-sided dice rolls is $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Set - examples

The set of all possible sums of two 6-sided dice rolls is $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and

- $2 \in \Omega$

Set - examples

The set of all possible sums of two 6-sided dice rolls is $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and

- $2 \in \Omega$
- $1 \notin \Omega$

Set - examples

The set of all possible sums of two 6-sided dice rolls is $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and

- $2 \in \Omega$
- $1 \notin \Omega$
- $\{2, 3, 4\} \subset \Omega$

Set comparison, operations, terminology

For the following $A, B \subseteq \Omega$ where Ω is the implied universe of all elements under study,

1. Union ($\cup$): A union of events is an event consisting of all the outcomes in these events.

$$A \cup B = \{\omega \mid \omega \in A \text{ or } \omega \in B\}$$

Set comparison, operations, terminology

For the following $A, B \subseteq \Omega$ where Ω is the implied universe of all elements under study,

1. Union ($\cup$): A union of events is an event consisting of all the outcomes in these events.

$$A \cup B = \{\omega \mid \omega \in A \text{ or } \omega \in B\}$$

2. Intersection ($\cap$): An intersection of events is an event consisting of the common outcomes in these events.

$$A \cap B = \{\omega \mid \omega \in A \text{ and } \omega \in B\}$$

Set comparison, operations, terminology

For the following $A, B \subseteq \Omega$ where Ω is the implied universe of all elements under study,

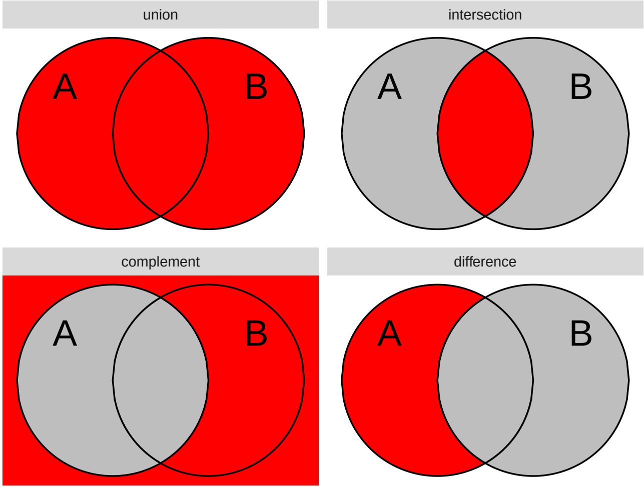
1. Union ($\cup$): A union of events is an event consisting of all the outcomes in these events.
$$A \cup B = \{\omega \mid \omega \in A \text{ or } \omega \in B\}$$
2. Intersection ($\cap$): An intersection of events is an event consisting of the common outcomes in these events.
$$A \cap B = \{\omega \mid \omega \in A \text{ and } \omega \in B\}$$
3. Complement (A^C): A complement of an event A is an event that occurs when event A does not happen.
$$A^C = \{\omega \mid \omega \notin A \text{ and } \omega \in \Omega\}$$

Set comparison, operations, terminology

For the following $A, B \subseteq \Omega$ where Ω is the implied universe of all elements under study,

1. Union ($\cup$): A union of events is an event consisting of all the outcomes in these events.
$$A \cup B = \{\omega \mid \omega \in A \text{ or } \omega \in B\}$$
2. Intersection ($\cap$): An intersection of events is an event consisting of the common outcomes in these events.
$$A \cap B = \{\omega \mid \omega \in A \text{ and } \omega \in B\}$$
3. Complement (A^C): A complement of an event A is an event that occurs when event A does not happen.
$$A^C = \{\omega \mid \omega \notin A \text{ and } \omega \in \Omega\}$$
4. Set difference ($A \setminus B$): All elements in A that are not in B, i.e.
$$A \setminus B = \{\omega | \omega \in A \text{ and } \omega \notin B\}$$

Venn diagrams



Example

Consider the set Ω equal to all possible sum of two 6-sided die rolls

Example

Consider the set Ω equal to all possible sum of two 6-sided die rolls i.e. $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and two subsets

- all odd rolls:

Example

Consider the set Ω equal to all possible sum of two 6-sided die rolls i.e. $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and two subsets

- all odd rolls: $A = \{3, 5, 7, 9, 11\}$

Example

Consider the set Ω equal to all possible sum of two 6-sided die rolls i.e. $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and two subsets

- all odd rolls: $A = \{3, 5, 7, 9, 11\}$
- all rolls below 6:

Example

Consider the set Ω equal to all possible sum of two 6-sided die rolls i.e. $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and two subsets

- all odd rolls: $A = \{3, 5, 7, 9, 11\}$
- all rolls below 6: $B = \{2, 3, 4, 5\}$

Example

Consider the set Ω equal to all possible sum of two 6-sided die rolls i.e. $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and two subsets

- all odd rolls: $A = \{3, 5, 7, 9, 11\}$
- all rolls below 6: $B = \{2, 3, 4, 5\}$

Then we have

- $A \cup B =$

Example

Consider the set Ω equal to all possible sum of two 6-sided die rolls i.e. $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and two subsets

- all odd rolls: $A = \{3, 5, 7, 9, 11\}$
- all rolls below 6: $B = \{2, 3, 4, 5\}$

Then we have

- $A \cup B = \{2, 3, 4, 5, 7, 9, 11\}$

Example

Consider the set Ω equal to all possible sum of two 6-sided die rolls i.e. $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and two subsets

- all odd rolls: $A = \{3, 5, 7, 9, 11\}$
- all rolls below 6: $B = \{2, 3, 4, 5\}$

Then we have

- $A \cup B = \{2, 3, 4, 5, 7, 9, 11\}$
- $A \cap B =$

Example

Consider the set Ω equal to all possible sum of two 6-sided die rolls i.e. $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and two subsets

- all odd rolls: $A = \{3, 5, 7, 9, 11\}$
- all rolls below 6: $B = \{2, 3, 4, 5\}$

Then we have

- $A \cup B = \{2, 3, 4, 5, 7, 9, 11\}$
- $A \cap B = \{3, 5\}$

Example

Consider the set Ω equal to all possible sum of two 6-sided die rolls i.e. $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and two subsets

- all odd rolls: $A = \{3, 5, 7, 9, 11\}$
- all rolls below 6: $B = \{2, 3, 4, 5\}$

Then we have

- $A \cup B = \{2, 3, 4, 5, 7, 9, 11\}$
- $A \cap B = \{3, 5\}$
- $A^C =$

Example

Consider the set Ω equal to all possible sum of two 6-sided die rolls i.e. $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and two subsets

- all odd rolls: $A = \{3, 5, 7, 9, 11\}$
- all rolls below 6: $B = \{2, 3, 4, 5\}$

Then we have

- $A \cup B = \{2, 3, 4, 5, 7, 9, 11\}$
- $A \cap B = \{3, 5\}$
- $A^C = \{2, 4, 6, 8, 10, 12\}$

Example

Consider the set Ω equal to all possible sum of two 6-sided die rolls i.e. $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and two subsets

- all odd rolls: $A = \{3, 5, 7, 9, 11\}$
- all rolls below 6: $B = \{2, 3, 4, 5\}$

Then we have

- $A \cup B = \{2, 3, 4, 5, 7, 9, 11\}$
- $A \cap B = \{3, 5\}$
- $A^C = \{2, 4, 6, 8, 10, 12\}$
- $B^C =$

Example

Consider the set Ω equal to all possible sum of two 6-sided die rolls i.e. $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and two subsets

- all odd rolls: $A = \{3, 5, 7, 9, 11\}$
- all rolls below 6: $B = \{2, 3, 4, 5\}$

Then we have

- $A \cup B = \{2, 3, 4, 5, 7, 9, 11\}$
- $A \cap B = \{3, 5\}$
- $A^C = \{2, 4, 6, 8, 10, 12\}$
- $B^C = \{6, 7, 8, 9, 10, 11, 12\}$

Example

Consider the set Ω equal to all possible sum of two 6-sided die rolls i.e. $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and two subsets

- all odd rolls: $A = \{3, 5, 7, 9, 11\}$
- all rolls below 6: $B = \{2, 3, 4, 5\}$

Then we have

- $A \cup B = \{2, 3, 4, 5, 7, 9, 11\}$
- $A \cap B = \{3, 5\}$
- $A^C = \{2, 4, 6, 8, 10, 12\}$
- $B^C = \{6, 7, 8, 9, 10, 11, 12\}$
- $A \setminus B =$

Example

Consider the set Ω equal to all possible sum of two 6-sided die rolls i.e. $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and two subsets

- all odd rolls: $A = \{3, 5, 7, 9, 11\}$
- all rolls below 6: $B = \{2, 3, 4, 5\}$

Then we have

- $A \cup B = \{2, 3, 4, 5, 7, 9, 11\}$
- $A \cap B = \{3, 5\}$
- $A^C = \{2, 4, 6, 8, 10, 12\}$
- $B^C = \{6, 7, 8, 9, 10, 11, 12\}$
- $A \setminus B = \{7, 9, 11\}$

Example

Consider the set Ω equal to all possible sum of two 6-sided die rolls i.e. $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and two subsets

- all odd rolls: $A = \{3, 5, 7, 9, 11\}$
- all rolls below 6: $B = \{2, 3, 4, 5\}$

Then we have

- $A \cup B = \{2, 3, 4, 5, 7, 9, 11\}$
- $A \cap B = \{3, 5\}$
- $A^C = \{2, 4, 6, 8, 10, 12\}$
- $B^C = \{6, 7, 8, 9, 10, 11, 12\}$
- $A \setminus B = \{7, 9, 11\}$
- $B \setminus A =$

Example

Consider the set Ω equal to all possible sum of two 6-sided die rolls i.e. $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and two subsets

- all odd rolls: $A = \{3, 5, 7, 9, 11\}$
- all rolls below 6: $B = \{2, 3, 4, 5\}$

Then we have

- $A \cup B = \{2, 3, 4, 5, 7, 9, 11\}$
- $A \cap B = \{3, 5\}$
- $A^C = \{2, 4, 6, 8, 10, 12\}$
- $B^C = \{6, 7, 8, 9, 10, 11, 12\}$
- $A \setminus B = \{7, 9, 11\}$
- $B \setminus A = \{2, 4\}$

Set comparison, operations, terminology (cont.)

5. Empty Set $\emptyset$ is a set having no elements, i.e. $\{\}$. The empty set is a subset of every set:

$$\emptyset \subseteq A$$

Set comparison, operations, terminology (cont.)

5. Empty Set $\emptyset$ is a set having no elements, i.e. $\{\}$. The empty set is a subset of every set:

$$\emptyset \subseteq A$$

6. Disjoint sets: Sets A, B are disjoint if their intersection is empty:

$$A \cap B = \emptyset$$

Set comparison, operations, terminology (cont.)

5. Empty Set $\emptyset$ is a set having no elements, i.e. $\{\}$. The empty set is a subset of every set:

$$\emptyset \subseteq A$$

6. Disjoint sets: Sets A, B are disjoint if their intersection is empty:

$$A \cap B = \emptyset$$

7. Pairwise disjoint sets: Sets $A_1, A_2, \ldots$ are pairwise disjoint if all pairs of these events are disjoint:

$$A_i \cap A_j = \emptyset \text{ for any } i \neq j$$

Set comparison, operations, terminology (cont.)

5. Empty Set $\emptyset$ is a set having no elements, i.e. $\{\}$. The empty set is a subset of every set:

$$\emptyset \subseteq A$$

6. Disjoint sets: Sets A, B are disjoint if their intersection is empty:

$$A \cap B = \emptyset$$

7. Pairwise disjoint sets: Sets $A_1, A_2, \ldots$ are pairwise disjoint if all pairs of these events are disjoint:

$$A_i \cap A_j = \emptyset \text{ for any } i \neq j$$

8. De Morgan's Laws:

$$(A \cup B)^C = A^C \cap B^C \qquad \text{and} \qquad (A \cap B)^C = A^C \cup B^C$$

Examples

Let $A = \{2, 3, 4\}, B = \{5, 6, 7\}, C = \{8, 9, 10\}, D = \{11, 12\}$.

Examples

Let $A = \{2, 3, 4\}, B = \{5, 6, 7\}, C = \{8, 9, 10\}, D = \{11, 12\}$. Then

- $A \cap B$

Examples

Let $A = \{2, 3, 4\}, B = \{5, 6, 7\}, C = \{8, 9, 10\}, D = \{11, 12\}$. Then

- $A \cap B = \emptyset$

Examples

Let $A = \{2, 3, 4\}, B = \{5, 6, 7\}, C = \{8, 9, 10\}, D = \{11, 12\}$. Then

- $A \cap B = \emptyset$
- A, B, C, D are

Examples

Let $A = \{2, 3, 4\}, B = \{5, 6, 7\}, C = \{8, 9, 10\}, D = \{11, 12\}$. Then

- $A \cap B = \emptyset$
- A, B, C, D are pairwise disjoint

Examples

Let $A = \{2, 3, 4\}, B = \{5, 6, 7\}, C = \{8, 9, 10\}, D = \{11, 12\}$. Then

- $A \cap B = \emptyset$
- A, B, C, D are pairwise disjoint
- De Morgan's:

$$(A \cup B)$$

Examples

Let $A = \{2, 3, 4\}, B = \{5, 6, 7\}, C = \{8, 9, 10\}, D = \{11, 12\}$. Then

- $A \cap B = \emptyset$
- A, B, C, D are pairwise disjoint
- De Morgan's:

$$(A \cup B) \quad = \{2, 3, 4, 5, 6, 7\}$$

Examples

Let $A = \{2,3,4\}, B = \{5,6,7\}, C = \{8,9,10\}, D = \{11,12\}$. Then

- $A \cap B = \emptyset$
- A, B, C, D are pairwise disjoint
- De Morgan's:

$$\begin{aligned} (A \cup B) &= \{2,3,4,5,6,7\} \\ (A \cup B)^C & \end{aligned}$$

Examples

Let $A = \{2,3,4\}, B = \{5,6,7\}, C = \{8,9,10\}, D = \{11,12\}$. Then

- $A \cap B = \emptyset$
- A, B, C, D are pairwise disjoint
- De Morgan's:

$$\begin{aligned} (A \cup B) &= \{2,3,4,5,6,7\} \\ (A \cup B)^C &= \{8,9,10,11,12\} \end{aligned}$$

Examples

Let $A = \{2,3,4\}, B = \{5,6,7\}, C = \{8,9,10\}, D = \{11,12\}$. Then

- $A \cap B = \emptyset$
- A, B, C, D are pairwise disjoint
- De Morgan's:

$$\begin{aligned} (A \cup B) &= \{2,3,4,5,6,7\} \\ (A \cup B)^C &= \{8,9,10,11,12\} \end{aligned}$$

$$A^C$$

Examples

Let $A = \{2,3,4\}, B = \{5,6,7\}, C = \{8,9,10\}, D = \{11,12\}$. Then

- $A \cap B = \emptyset$
- A, B, C, D are pairwise disjoint
- De Morgan's:

$$\begin{aligned} (A \cup B) &= \{2,3,4,5,6,7\} \\ (A \cup B)^C &= \{8,9,10,11,12\} \\ \\ A^C &= \{5,6,7,8,9,10,11,12\} \end{aligned}$$

Examples

Let $A = \{2,3,4\}, B = \{5,6,7\}, C = \{8,9,10\}, D = \{11,12\}$. Then

- $A \cap B = \emptyset$
- A, B, C, D are pairwise disjoint
- De Morgan's:

$$\begin{aligned}
(A \cup B) &= \{2,3,4,5,6,7\} \\
(A \cup B)^C &= \{8,9,10,11,12\} \\
\\
A^C &= \{5,6,7,8,9,10,11,12\} \\
B^C &
\end{aligned}$$

Examples

Let $A = \{2, 3, 4\}, B = \{5, 6, 7\}, C = \{8, 9, 10\}, D = \{11, 12\}$. Then

- $A \cap B = \emptyset$
- A, B, C, D are pairwise disjoint
- De Morgan's:

$$\begin{aligned}
(A \cup B) &= \{2, 3, 4, 5, 6, 7\} \\
(A \cup B)^C &= \{8, 9, 10, 11, 12\} \\
\\
A^C &= \{5, 6, 7, 8, 9, 10, 11, 12\} \\
B^C &= \{2, 3, 4, 8, 9, 10, 11, 12\}
\end{aligned}$$

Examples

Let $A = \{2,3,4\}, B = \{5,6,7\}, C = \{8,9,10\}, D = \{11,12\}$. Then

- $A \cap B = \emptyset$
- A, B, C, D are pairwise disjoint
- De Morgan's:

$$\begin{aligned}
(A \cup B) &= \{2,3,4,5,6,7\} \\
(A \cup B)^C &= \{8,9,10,11,12\} \\
\\
A^C &= \{5,6,7,8,9,10,11,12\} \\
B^C &= \{2,3,4,8,9,10,11,12\} \\
A^C \cap B^C &
\end{aligned}$$

Examples

Let $A = \{2, 3, 4\}, B = \{5, 6, 7\}, C = \{8, 9, 10\}, D = \{11, 12\}$. Then

- $A \cap B = \emptyset$
- A, B, C, D are pairwise disjoint
- De Morgan's:

$$\begin{aligned} (A \cup B) &= \{2, 3, 4, 5, 6, 7\} \\ (A \cup B)^C &= \{8, 9, 10, 11, 12\} \\ \\ A^C &= \{5, 6, 7, 8, 9, 10, 11, 12\} \\ B^C &= \{2, 3, 4, 8, 9, 10, 11, 12\} \\ A^C \cap B^C &= \{8, 9, 10, 11, 12\} \end{aligned}$$

Examples

Let $A = \{2,3,4\}, B = \{5,6,7\}, C = \{8,9,10\}, D = \{11,12\}$. Then

- $A \cap B = \emptyset$
- A, B, C, D are pairwise disjoint
- De Morgan's:

$$
\begin{aligned}
(A \cup B) &= \{2,3,4,5,6,7\} \\
(A \cup B)^C &= \{8,9,10,11,12\} \\
\\
A^C &= \{5,6,7,8,9,10,11,12\} \\
B^C &= \{2,3,4,8,9,10,11,12\} \\
A^C \cap B^C &= \{8,9,10,11,12\}
\end{aligned}
$$

so, by example,

$$(A \cup B)^C = A^C \cap B^C.$$

Kolmogorov's Axioms

A system of probabilities (a probability model) is an assignment of numbers $P(A)$ to events $A \subseteq \Omega$

Kolmogorov's Axioms

A system of probabilities (a probability model) is an assignment of numbers $P(A)$ to events $A \subseteq \Omega$ such that

(i) $0 \leq P(A) \leq 1$ for all A

Kolmogorov's Axioms

A system of probabilities (a probability model) is an assignment of numbers $P(A)$ to events $A \subseteq \Omega$ such that

(i) $0 \leq P(A) \leq 1$ for all A

(ii) $P(\Omega) = 1$.

Kolmogorov's Axioms

A system of probabilities (a probability model) is an assignment of numbers $P(A)$ to events $A \subseteq \Omega$ such that

(i) $0 \leq P(A) \leq 1$ for all A

(ii) $P(\Omega) = 1$.

(iii) if $A_1, A_2, \ldots$ are pairwise disjoint events (i.e. $A_i \cap A_j = \emptyset$ for all $i \neq j$) then

$$\begin{aligned} P(A_1 \cup A_2 \cup \ldots) &= P(A_1) + P(A_2) + \ldots \\ &= \textstyle\sum_i P(A_i). \end{aligned}$$

Kolmogorov's Axioms (cont.)

These are the basic rules of operation of a probability model

- every valid model must obey these,
- any system that does, is a valid model.

Kolmogorov's Axioms (cont.)

These are the basic rules of operation of a probability model

- every valid model must obey these,
- any system that does, is a valid model.

Whether or not a particular model is realistic is different question.

Kolmogorov's Axioms (cont.)

These are the basic rules of operation of a probability model

- every valid model must obey these,
- any system that does, is a valid model.

Whether or not a particular model is realistic is different question.

Example: Draw a single card from a standard deck of playing cards: $\Omega = \{red, black\}$ Two different, equally valid probability models are:

Model 1	Model 2
$P(\Omega) = 1$	$P(\Omega) = 1$
$P(red) = 0.5$	$P(red) = 0.3$
$P(black) = 0.5$	$P(black) = 0.7$

Kolmogorov's Axioms (cont.)

These are the basic rules of operation of a probability model

- every valid model must obey these,
- any system that does, is a valid model.

Whether or not a particular model is realistic is different question.

Example: Draw a single card from a standard deck of playing cards: $\Omega = \{red, black\}$ Two different, equally valid probability models are:

Model 1	Model 2
$P(\Omega) = 1$	$P(\Omega) = 1$
$P(red) = 0.5$	$P(red) = 0.3$
$P(black) = 0.5$	$P(black) = 0.7$

Mathematically, both schemes are equally valid.
But, of course, our real world experience would prefer model 1 over model 2.

Useful Consequences of Kolmogorov's Axioms

Let A, $B \subseteq \Omega$.

Useful Consequences of Kolmogorov's Axioms

Let A, $B \subseteq \Omega$.

- Probability of the Complementary Event: $P\left(A^C\right) = 1 - P(A)$

Useful Consequences of Kolmogorov's Axioms

Let A, $B \subseteq \Omega$.

- Probability of the Complementary Event: $P\left(A^C\right) = 1 - P(A)$
  Corollary: $P(\emptyset) = 0$

Useful Consequences of Kolmogorov's Axioms

Let A, $B \subseteq \Omega$.

- Probability of the Complementary Event: $P\left(A^C\right) = 1 - P(A)$
  Corollary: $P(\emptyset) = 0$
- Addition Rule of Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- If $A \subseteq B$, then $P(A) \leq P(B)$.

Example: Using Kolmogorov's Axioms

We attempt to access the internet from a laptop at home.

Example: Using Kolmogorov's Axioms

We attempt to access the internet from a laptop at home. We connect successfully if and only if the wireless (WiFi) network works *and* the internet service provider (ISP) network works.

Example: Using Kolmogorov's Axioms

We attempt to access the internet from a laptop at home. We connect successfully if and only if the wireless (WiFi) network works *and* the internet service provider (ISP) network works. Assume

$$\begin{aligned} P(\text{ WiFi up }) &= .9 \\ P(\text{ ISP up }) &= .6, \text{ and} \\ P(\text{ WiFi up and ISP up }) &= .55. \end{aligned}$$

Example: Using Kolmogorov's Axioms

We attempt to access the internet from a laptop at home. We connect successfully if and only if the wireless (WiFi) network works *and* the internet service provider (ISP) network works. Assume

$$\begin{aligned} P(\text{ WiFi up }) &= .9 \\ P(\text{ ISP up }) &= .6, \text{ and} \\ P(\text{ WiFi up and ISP up }) &= .55. \end{aligned}$$

1. What is the probability that the WiFi is up or the ISP is up?

Example: Using Kolmogorov's Axioms

We attempt to access the internet from a laptop at home. We connect successfully if and only if the wireless (WiFi) network works *and* the internet service provider (ISP) network works. Assume

$$
\begin{aligned}
P(\text{ WiFi up }) &= .9 \\
P(\text{ ISP up }) &= .6, \text{ and} \\
P(\text{ WiFi up and ISP up }) &= .55.
\end{aligned}
$$

1. What is the probability that the WiFi is up or the ISP is up?
2. What is the probability that both the WiFi and the ISP are down?

Example: Using Kolmogorov's Axioms

We attempt to access the internet from a laptop at home. We connect successfully if and only if the wireless (WiFi) network works *and* the internet service provider (ISP) network works. Assume

$$\begin{aligned} P(\text{ WiFi up }) &= .9 \\ P(\text{ ISP up }) &= .6, \text{ and} \\ P(\text{ WiFi up and ISP up }) &= .55. \end{aligned}$$

1. What is the probability that the WiFi is up or the ISP is up?
2. What is the probability that both the WiFi and the ISP are down?
3. What is the probability that we fail to connect?

Solution

Let $A \equiv$ WiFi up; $B \equiv$ ISP up

Solution

Let $A \equiv$ WiFi up; $B \equiv$ ISP up

1. What is the probability that the WiFi is up or the ISP is up?

$$P(\text{ WiFi up or ISP up}) = P(A \cup B) = 0.9 + 0.6 - 0.55 = 0.95$$

Solution

Let $A \equiv$ WiFi up; $B \equiv$ ISP up

1. What is the probability that the WiFi is up or the ISP is up?

$$P(\text{ WiFi up or ISP up}) = P(A \cup B) = 0.9 + 0.6 - 0.55 = 0.95$$

2. What is the probability that both the WiFi and the ISP are down?

$$\begin{aligned} P(\text{ WiFi down and ISP down}) &= P\left(A^C \cap B^C\right) = P\left([A \cup B]^C\right) \\ &= 1 - .95 = .05 \end{aligned}$$

Solution

Let $A \equiv$ WiFi up; $B \equiv$ ISP up

1. What is the probability that the WiFi is up or the ISP is up?

$$P(\text{ WiFi up or ISP up}) = P(A \cup B) = 0.9 + 0.6 - 0.55 = 0.95$$

2. What is the probability that both the WiFi and the ISP are down?

$$\begin{aligned} P(\text{ WiFi down and ISP down}) &= P\left(A^C \cap B^C\right) = P\left([A \cup B]^C\right) \\ &= 1 - .95 = .05 \end{aligned}$$

3. What is the probability that we fail to connect?

$$\begin{aligned} &P(\text{ WiFi down or ISP down}) \\ &\quad = P\left(A^C \cup B^C\right) = P\left(A^C\right) + P\left(B^C\right) - P\left(A^C \cap B^C\right) \\ &\quad = P\left(A^C \cup B^C\right) = (1 - .9) + (1 - .6) - .05 = .1 + .4 - .05 = .45 \end{aligned}$$

Conditional probability - Definition

The conditional probability of an event A given an event B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

if $P(B) > 0$.

Conditional probability - Definition

The conditional probability of an event A given an event B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

if $P(B) > 0$.

Intuitively, the fraction of outcomes in B that are also in A.

Conditional probability - Definition

The conditional probability of an event A given an event B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

if $P(B) > 0$.

Intuitively, the fraction of outcomes in B that are also in A.

Corrollary:

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A).$$

Random CPUs

A box has 500 CPUs with a speed of 1.8 GHz and 500 with a speed of 2.0 GHz. The numbers of good (G) and defective (D) CPUs at the two different speeds are as shown below.

	1.8 GHz	2.0 GHz	Total
G	480	490	970
D	20	10	30
Total	500	500	1000

Random CPUs

A box has 500 CPUs with a speed of 1.8 GHz and 500 with a speed of 2.0 GHz. The numbers of good (G) and defective (D) CPUs at the two different speeds are as shown below.

	1.8 GHz	2.0 GHz	Total
G	480	490	970
D	20	10	30
Total	500	500	1000

We select a CPU at random and observe its speed. What is the probability that the CPU is defective given that its speed is 1.8 GHz?

Random CPUs

A box has 500 CPUs with a speed of 1.8 GHz and 500 with a speed of 2.0 GHz. The numbers of good (G) and defective (D) CPUs at the two different speeds are as shown below.

	1.8 GHz	2.0 GHz	Total
G	480	490	970
D	20	10	30
Total	500	500	1000

We select a CPU at random and observe its speed. What is the probability that the CPU is defective given that its speed is 1.8 GHz?
Let

- D be the event the CPU is defective and
- S be the event the CPU speed is 1.8 GHz.

Random CPUs

A box has 500 CPUs with a speed of 1.8 GHz and 500 with a speed of 2.0 GHz. The numbers of good (G) and defective (D) CPUs at the two different speeds are as shown below.

	1.8 GHz	2.0 GHz	Total
G	480	490	970
D	20	10	30
Total	500	500	1000

We select a CPU at random and observe its speed. What is the probability that the CPU is defective given that its speed is 1.8 GHz?
Let

- D be the event the CPU is defective and
- S be the event the CPU speed is 1.8 GHz.

Then

- $P(S) = 500/1000 = 0.5$
- $P(S \cap D) = 20/1000 = 0.02.$
- $P(D|S) = P(S \cap D)/P(S) = 0.02/0.5 = 0.04.$

Statistical independence - Definition

Events A and B are statistically independent if

$$P(A \cap B) = P(A) \times P(B)$$

Statistical independence - Definition

Events A and B are statistically independent if

$$P(A \cap B) = P(A) \times P(B)$$

or, equivalently,

$$P(A|B) = P(A).$$

Statistical independence - Definition

Events A and B are statistically independent if

$$P(A \cap B) = P(A) \times P(B)$$

or, equivalently,

$$P(A|B) = P(A).$$

Intuition: the occurrence of one event does not affect the probability of the other.

Statistical independence - Definition

Events A and B are statistically independent if

$$P(A \cap B) = P(A) \times P(B)$$

or, equivalently,

$$P(A|B) = P(A).$$

Intuition: the occurrence of one event does not affect the probability of the other.

Example: In two tosses of a coin, the result of the first toss does not affect the probability of the second toss being heads.

WiFi example

In trying to connect my laptop to the internet, I need

- my WiFi network to be up (event A) and
- the ISP network to be up (event B).

WiFi example

In trying to connect my laptop to the internet, I need

- my WiFi network to be up (event A) and
- the ISP network to be up (event B).

Assume the probability the WiFi network is up is 0.9 and the ISP network is up is 0.6.

WiFi example

In trying to connect my laptop to the internet, I need

- my WiFi network to be up (event A) and
- the ISP network to be up (event B).

Assume the probability the WiFi network is up is 0.9 and the ISP network is up is 0.6. If the two events are independent, what is the probability we can connect to the internet?

WiFi example

In trying to connect my laptop to the internet, I need

- my WiFi network to be up (event A) and
- the ISP network to be up (event B).

Assume the probability the WiFi network is up is 0.9 and the ISP network is up is 0.6. If the two events are independent, what is the probability we can connect to the internet?

Since we have independence, we know

$$P(A \cap B) = P(A) \times P(B) = 0.9 \times 0.6 = 0.54.$$

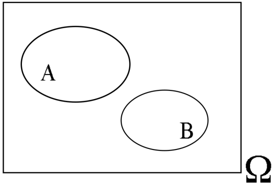
Independence and disjoint

Warning: Independence and disjointedness are two very different concepts!

Independence and disjoint

Warning: Independence and disjointedness are two very different concepts!

Disjoint:



A, B are disjoint

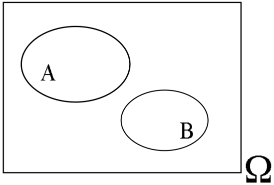
If A and B are disjoint, their intersection is empty and therefore has probability 0:

$$P(A \cap B) = P(\emptyset) = 0.$$

Independence and disjoint

Warning: Independence and disjointedness are two very different concepts!

Disjoint:

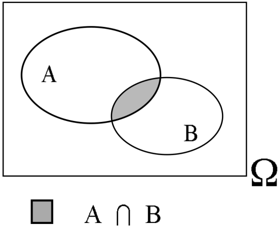


A, B are disjoint

If A and B are disjoint, their intersection is empty and therefore has probability 0:

$$P(A \cap B) = P(\emptyset) = 0.$$

Independence:

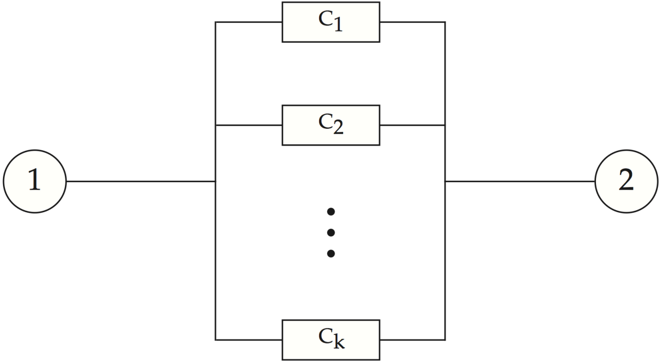


If A and B are independent events, the probability of their intersection can be computed as the product of their individual probabilities:

$$P(A \cap B) = P(A) \cdot P(B)$$

Parallel system - Definition

A parallel system consists of K components $c_1, \ldots, c_K$ arranged in such a way that the system works if **at least one** of the K components functions properly.

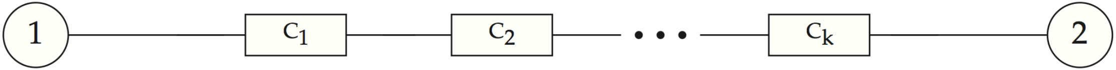


Serial system - Definition

A serial system consists of K components $c_1, \ldots, c_K$ arranged in such a way that the system works if and only if **all** of the components function properly.

Serial system - Definition

A serial system consists of K components $c_1, \ldots, c_K$ arranged in such a way that the system works if and only if **all** of the components function properly.



Reliability - Definition

The reliability of a system is the probability the system works.

Reliability - Definition

The reliability of a system is the probability the system works.

Example: The reliability of the WiFi-ISP network (assuming independence) is 0.54.

Reliability of parallel systems with independent components

Let $c_1, \ldots, c_K$ denote the K components in a parallel system.

Reliability of parallel systems with independent components

Let $c_1, \ldots, c_K$ denote the K components in a parallel system. Assume the K components operate independently and $P(c_k \text{ works}) = p_k$.

Reliability of parallel systems with independent components

Let $c_1, \ldots, c_K$ denote the K components in a parallel system. Assume the K components operate independently and $P(c_k \text{ works}) = p_k$. What is the reliability of the system?

Reliability of parallel systems with independent components

Let $c_1, \ldots, c_K$ denote the K components in a parallel system. Assume the K components operate independently and $P(c_k \text{ works}) = p_k$. What is the reliability of the system?

$$P(\text{ system works }) \; = P(\text{ at least one component works })$$

Reliability of parallel systems with independent components

Let $c_1, \ldots, c_K$ denote the K components in a parallel system. Assume the K components operate independently and $P(c_k \text{ works}) = p_k$. What is the reliability of the system?

$$\begin{aligned} P(\text{ system works }) &= P(\text{ at least one component works }) \\ &= 1 - P(\text{ all components fail }) \end{aligned}$$

Reliability of parallel systems with independent components

Let $c_1, \ldots, c_K$ denote the K components in a parallel system. Assume the K components operate independently and $P(c_k \text{ works}) = p_k$. What is the reliability of the system?

$$\begin{aligned}
P(\text{ system works }) &= P(\text{ at least one component works }) \\
&= 1 - P(\text{ all components fail }) \\
&= 1 - P(c_1 \text{ fails and } c_2 \text{ fails } \ldots \text{ and } c_k \text{ fails })
\end{aligned}$$

Reliability of parallel systems with independent components

Let $c_1, \ldots, c_K$ denote the K components in a parallel system. Assume the K components operate independently and $P(c_k \text{ works}) = p_k$. What is the reliability of the system?

$$\begin{aligned} P(\text{ system works }) &= P(\text{ at least one component works }) \\ &= 1 - P(\text{ all components fail }) \\ &= 1 - P(c_1 \text{ fails and } c_2 \text{ fails } \ldots \text{ and } c_k \text{ fails }) \\ &= 1 - \textstyle\prod_{k=1}^{K} P(c_k \text{ fails}) \end{aligned}$$

Reliability of parallel systems with independent components

Let $c_1, \ldots, c_K$ denote the K components in a parallel system. Assume the K components operate independently and $P(c_k \text{ works}) = p_k$. What is the reliability of the system?

$$
\begin{aligned}
P(\text{ system works }) &= P(\text{ at least one component works }) \\
&= 1 - P(\text{ all components fail }) \\
&= 1 - P(c_1 \text{ fails and } c_2 \text{ fails } \ldots \text{ and } c_k \text{ fails }) \\
&= 1 - \textstyle\prod_{k=1}^{K} P(c_k \text{ fails}) \\
&= 1 - \textstyle\prod_{k=1}^{K} (1 - p_k).
\end{aligned}
$$

Reliability of serial systems with independent components

Let $c_1, \ldots, c_K$ denote the K components in a serial system.

Reliability of serial systems with independent components

Let $c_1, \ldots, c_K$ denote the K components in a serial system. Assume the K components operate independently and $P(c_k \text{ works }) = p_k$.

Reliability of serial systems with independent components

Let $c_1, \ldots, c_K$ denote the K components in a serial system. Assume the K components operate independently and $P(c_k \text{ works }) = p_k$. What is the reliability of the system?

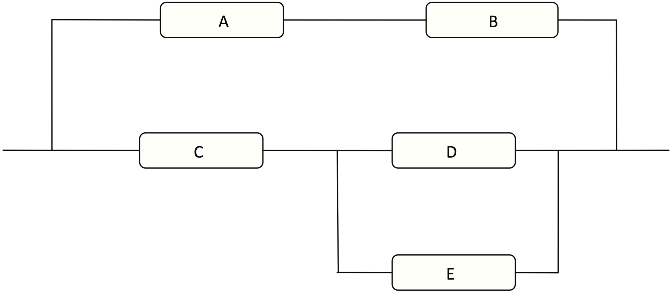
Reliability of serial systems with independent components

Let $c_1, \ldots, c_K$ denote the K components in a serial system. Assume the K components operate independently and $P(c_k \text{ works }) = p_k$. What is the reliability of the system?

$$\begin{aligned} P(\text{ system works }) &= P(\text{ all components work}) \\ &= \prod_{k=1}^{K} P(c_k \text{ works}) \\ &= \prod_{k=1}^{K} p_k. \end{aligned}$$

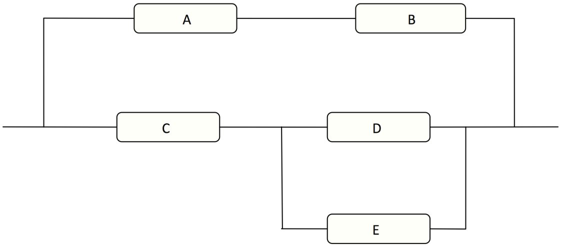
Reliability example

Each component in the system shown below is opearable with probability 0.92 independently of other components. Calculate the reliability.



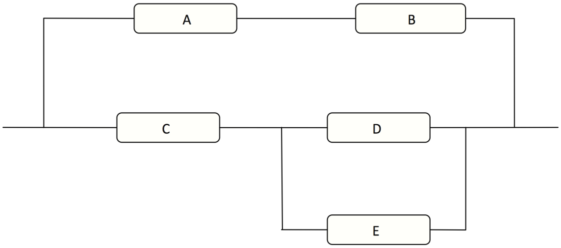
Reliability example

Each component in the system shown below is opearable with probability 0.92 independently of other components. Calculate the reliability.



Reliability example

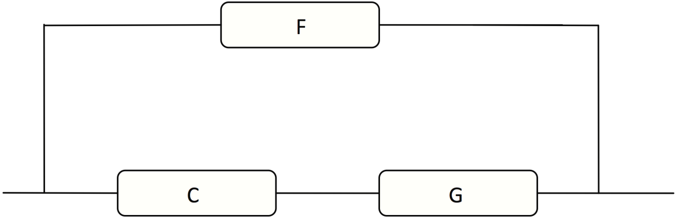
Each component in the system shown below is opearable with probability 0.92 independently of other components. Calculate the reliability.



1. Serial components A and B can be replaced by a component F that operates with probability $P(A \cap B) = (0.92)^2 = 0.8464$.
2. Parallel components D and E can be replaced by component G that operates with probability $P(D \cup E) = 1 - (1 - 0.92)^2 = 0.9936$.

Reliability example (cont.)

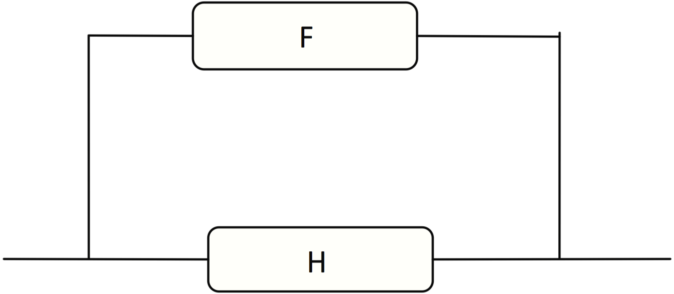
Updated circuit:



3. Serial components C and G can be replaced by a component H that operates with probability $P(C \cap G) = (0.92)(0.9936) = 0.9141.$

Reliability example (cont.)

Updated circuit:



4. Parallel componenents F and H are in parallel, so the reliability of the system is $P(F \cup H) = 1 - (1 - 0.8464)(1 - 0.9141) \approx 0.99.$

Partition

Definition

A collection of events $B_1, \ldots B_K$ is called a partition (or cover) of Ω if

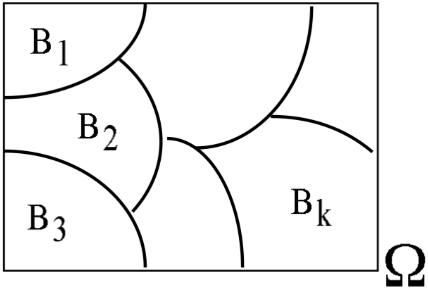
- the events are pairwise disjoint (i.e., $B_i \cap B_j = \emptyset$ for $i \neq j$)

Partition

Definition

A collection of events $B_1, \dots B_K$ is called a partition (or cover) of Ω if

- the events are pairwise disjoint (i.e., $B_i \cap B_j = \emptyset$ for $i \neq j$), and
- the union of the events is Ω (i.e., $\bigcup_{k=1}^{K} B_k = \Omega$).



Example

Consider the sum of two 6-sided die, i.e.

$$\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

Example

Consider the sum of two 6-sided die, i.e.

$$\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

Here are some covers:

- $\{2, 3, 4\}, \{5, 6, 7, 8, 9, 10, 11, 12\}$

Example

Consider the sum of two 6-sided die, i.e.

$$\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

Here are some covers:

- $\{2, 3, 4\}, \{5, 6, 7, 8, 9, 10, 11, 12\}$
- $\{2, 3, 4\}, \{5, 6, 7\}, \{8, 9, 10\}, \{11, 12\}$

Example

Consider the sum of two 6-sided die, i.e.

$$\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

Here are some covers:

- $\{2, 3, 4\}, \{5, 6, 7, 8, 9, 10, 11, 12\}$
- $\{2, 3, 4\}, \{5, 6, 7\}, \{8, 9, 10\}, \{11, 12\}$
- $A_2, A_3, \ldots, A_{12}$ where $A_i = \{i\}$

Example

Consider the sum of two 6-sided die, i.e.

$$\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

Here are some covers:

- $\{2, 3, 4\}, \{5, 6, 7, 8, 9, 10, 11, 12\}$
- $\{2, 3, 4\}, \{5, 6, 7\}, \{8, 9, 10\}, \{11, 12\}$
- $A_2, A_3, \ldots, A_{12}$ where $A_i = \{i\}$
- any A and A^C where $A \subseteq \Omega$

Law of Total Probability

Law of Total Probability:
If the collection of events $B_1, \ldots, B_K$ is a partition of Ω, and A is an event, then

$$P(A) = \sum_{k=1}^{K} P(A|B_k)P(B_k).$$

Law of Total Probability

Law of Total Probability:
If the collection of events $B_1, \ldots, B_K$ is a partition of Ω, and A is an event, then

$$P(A) = \sum_{k=1}^{K} P(A|B_k)P(B_k).$$

Proof:

$$P(A) \quad = P\left(\bigcup_{k=1}^{K} A \cap B_k\right) \qquad \text{partition}$$

Law of Total Probability

Law of Total Probability:
If the collection of events $B_1, \ldots, B_K$ is a partition of Ω, and A is an event, then

$$P(A) = \sum_{k=1}^{K} P(A|B_k)P(B_k).$$

Proof:

$$\begin{aligned} P(A) &= P\left(\bigcup_{k=1}^{K} A \cap B_k\right) && \text{partition} \\ &= \sum_{k=1}^{K} P(A \cap B_k) && \text{pairwise disjoint} \end{aligned}$$

Law of Total Probability

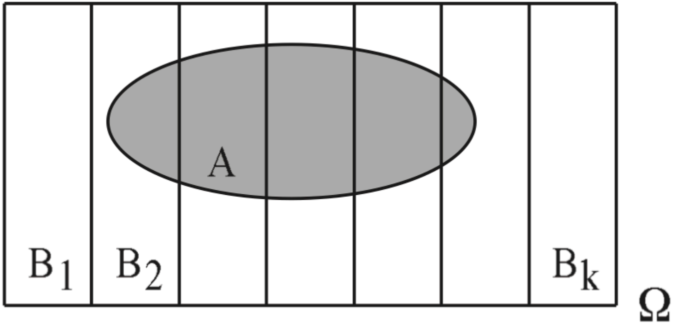
Law of Total Probability:
If the collection of events $B_1, \ldots, B_K$ is a partition of Ω, and A is an event, then

$$P(A) = \sum_{k=1}^{K} P(A|B_k)P(B_k).$$

Proof:

$$\begin{aligned}
P(A) &= P\left(\bigcup_{k=1}^{K} A \cap B_k\right) && \text{partition} \\
&= \sum_{k=1}^{K} P(A \cap B_k) && \text{pairwise disjoint} \\
&= \sum_{k=1}^{K} P(A|B_k)P(B_k) && \text{conditional probability}
\end{aligned}$$

Law of Total Probability - Graphically



Law of Total Probability - Example

In the come out roll of craps, you win if the roll is a 7 or 11.

Law of Total Probability - Example

In the come out roll of craps, you win if the roll is a 7 or 11. By the law of total probability, the probability you win is

$$P(\text{Win}) = \sum_{i=2}^{12} P(\text{Win}|i)P(i) = P(7) + P(11)$$

since $P(\text{Win}|i) = 1$ if $i = 7, 11$ and 0 otherwise.

Bayes' Rule

Bayes' Rule:
If $B_1, \ldots, B_K$ is a partition of Ω, and A is an event in Ω,

Bayes' Rule

Bayes' Rule:
If $B_1, \ldots, B_K$ is a partition of Ω, and A is an event in Ω, then

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{\sum_{k=1}^{K} P(A|B_k)P(B_k)}.$$

Bayes' Rule

Bayes' Rule:
If $B_1, \ldots, B_K$ is a partition of Ω, and A is an event in Ω, then

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{\sum_{k=1}^K P(A|B_k)P(B_k)}.$$

Proof:

$$P(B_k|A) \quad = \frac{P(A \cap B_k)}{P(A)} \qquad \text{conditional probability}$$

Bayes' Rule

Bayes' Rule:
If $B_1, \ldots, B_K$ is a partition of Ω, and A is an event in Ω, then

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{\sum_{k=1}^{K} P(A|B_k)P(B_k)}.$$

Proof:

$$\begin{aligned} P(B_k|A) &= \tfrac{P(A \cap B_k)}{P(A)} && \text{conditional probability} \\ &= \tfrac{P(A|B_k)P(B_k)}{P(A)} && \text{conditional probability} \end{aligned}$$

Bayes' Rule

Bayes' Rule:
If $B_1, \ldots, B_K$ is a partition of Ω, and A is an event in Ω, then

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{\sum_{k=1}^K P(A|B_k)P(B_k)}.$$

Proof:

$$\begin{aligned} P(B_k|A) &= \tfrac{P(A\cap B_k)}{P(A)} && \text{conditional probability} \\ &= \tfrac{P(A|B_k)P(B_k)}{P(A)} && \text{conditional probability} \\ &= \tfrac{P(A|B_k)P(B_k)}{\sum_{k=1}^K P(A|B_k)P(B_k)} && \text{Law of Total Probability} \end{aligned}$$

Bayes' Rule: Craps example

If you win on a come-out roll in craps, what is the probability you rolled a 7?

Bayes' Rule: Craps example

If you win on a come-out roll in craps, what is the probability you rolled a 7?

$$
\begin{aligned}
P(7|\text{Win}) &= \frac{P(\text{Win}|7)P(7)}{\sum_{i=2}^{12} P(\text{Win}|i)P(i)} \\
&= \frac{P(7)}{P(7)+P(11)}.
\end{aligned}
$$

Bayes' Rule: CPU testing example

A given lot of CPUs contains 2% defective CPUs.

Bayes' Rule: CPU testing example

A given lot of CPUs contains 2% defective CPUs. Each CPU is tested before delivery.

Bayes' Rule: CPU testing example

A given lot of CPUs contains 2% defective CPUs. Each CPU is tested before delivery. However, the tester is not wholly reliable:

Bayes' Rule: CPU testing example

A given lot of CPUs contains 2% defective CPUs. Each CPU is tested before delivery. However, the tester is not wholly reliable:

$$P(\text{ tester says CPU is good} \mid \text{CPU is good }) \quad = 0.95$$

Bayes' Rule: CPU testing example

A given lot of CPUs contains 2% defective CPUs. Each CPU is tested before delivery. However, the tester is not wholly reliable:

$$\begin{aligned} P(\text{ tester says CPU is good } \mid \text{ CPU is good }) &= 0.95 \\ P(\text{ tester says CPU is defective } \mid \text{ CPU is defective }) &= 0.94 \end{aligned}$$

Bayes' Rule: CPU testing example

A given lot of CPUs contains 2% defective CPUs. Each CPU is tested before delivery. However, the tester is not wholly reliable:

$$\begin{aligned} P(\text{ tester says CPU is good} \mid \text{CPU is good }) &= 0.95 \\ P(\text{ tester says CPU is defective} \mid \text{CPU is defective }) &= 0.94 \end{aligned}$$

If the test device says the CPU is defective, what is the probability that the CPU is actually defective?

CPU testing (cont.)

Let

- C_g (C_d) be the event the CPU is good (defective)

CPU testing (cont.)

Let

- C_g (C_d) be the event the CPU is good (defective)
- T_g (T_d) be the event the tester says the CPU is good (defective)

We know

CPU testing (cont.)

Let

- C_g (C_d) be the event the CPU is good (defective)
- T_g (T_d) be the event the tester says the CPU is good (defective)

We know

- $0.02 = P(C_d) \qquad = 1 - P(C_g)$
- $0.95 = P(T_g|C_g) = 1 - P(T_d|C_g)$
- $0.94 = P(T_d|C_d) = 1 - P(T_g|C_d)$

CPU testing (cont.)

Let

- C_g (C_d) be the event the CPU is good (defective)
- T_g (T_d) be the event the tester says the CPU is good (defective)

We know

- $0.02 = P(C_d) \quad = 1 - P(C_g)$
- $0.95 = P(T_g|C_g) = 1 - P(T_d|C_g)$
- $0.94 = P(T_d|C_d) = 1 - P(T_g|C_d)$

Using Bayes' Rule, we have

$$P(C_d|T_d) = \frac{P(T_d|C_d)P(C_d)}{P(T_d|C_d)P(C_d)+P(T_d|C_g)P(C_g)}$$

CPU testing (cont.)

Let

- C_g (C_d) be the event the CPU is good (defective)
- T_g (T_d) be the event the tester says the CPU is good (defective)

We know

- $0.02 = P(C_d) \quad = 1 - P(C_g)$
- $0.95 = P(T_g|C_g) = 1 - P(T_d|C_g)$
- $0.94 = P(T_d|C_d) = 1 - P(T_g|C_d)$

Using Bayes' Rule, we have

$$\begin{aligned} P(C_d|T_d) &= \frac{P(T_d|C_d)P(C_d)}{P(T_d|C_d)P(C_d)+P(T_d|C_g)P(C_g)} \\ &= \frac{P(T_d|C_d)P(C_d)}{P(T_d|C_d)P(C_d)+[1-P(T_g|C_g)][1-P(C_d)]} \end{aligned}$$

CPU testing (cont.)

Let

- C_g (C_d) be the event the CPU is good (defective)
- T_g (T_d) be the event the tester says the CPU is good (defective)

We know

- $0.02 = P(C_d) \quad = 1 - P(C_g)$
- $0.95 = P(T_g|C_g) = 1 - P(T_d|C_g)$
- $0.94 = P(T_d|C_d) = 1 - P(T_g|C_d)$

Using Bayes' Rule, we have

$$
\begin{aligned}
P(C_d|T_d) &= \frac{P(T_d|C_d)P(C_d)}{P(T_d|C_d)P(C_d)+P(T_d|C_g)P(C_g)} \\
&= \frac{P(T_d|C_d)P(C_d)}{P(T_d|C_d)P(C_d)+[1-P(T_g|C_g)][1-P(C_d)]} \\
&= \frac{0.94\times 0.02}{0.94\times 0.02+[1-0.95]\times[1-0.02]}
\end{aligned}
$$

CPU testing (cont.)

Let

- C_g (C_d) be the event the CPU is good (defective)
- T_g (T_d) be the event the tester says the CPU is good (defective)

We know

- $0.02 = P(C_d) \quad = 1 - P(C_g)$
- $0.95 = P(T_g|C_g) = 1 - P(T_d|C_g)$
- $0.94 = P(T_d|C_d) = 1 - P(T_g|C_d)$

Using Bayes' Rule, we have

$$\begin{aligned} P(C_d|T_d) &= \frac{P(T_d|C_d)P(C_d)}{P(T_d|C_d)P(C_d)+P(T_d|C_g)P(C_g)} \\ &= \frac{P(T_d|C_d)P(C_d)}{P(T_d|C_d)P(C_d)+[1-P(T_g|C_g)][1-P(C_d)]} \\ &= \frac{0.94\times 0.02}{0.94\times 0.02+[1-0.95]\times[1-0.02]} \\ &= 0.28 \end{aligned}$$

Probability Summary

- Probability Interpretation
- Sets and set operations
- Kolmogorov's Axioms
- Conditional Probability
- Independence
- Reliability
- Law of Total Probability
- Bayes' Rule