Hierarchical models

Dr. Jarad Niemi

STAT 544 - Iowa State University

February 9, 2024

Outline

- Motivating example
  - Independent vs pooled estimates
- Hierarchical models
  - General structure
  - Posterior distribution
- Binomial hierarchial model
  - Posterior distribution
  - Prior distributions
- Stan analysis of binomial hierarchical model
  - informative prior
  - default prior
  - integrating out θ
  - across seasons

Andre Dawkin's three-point percentage

Suppose Y_i are the number 3-pointers Andre Dawkin's makes in season i, and assume

$$Y_i \stackrel{ind}{\sim} Bin(n_i, \theta_i)$$

where

- n_i are the number of 3-pointers attempted and
- θ_i is the probability of making a 3-pointer in season i.

Do these models make sense?

- The 3-point percentage every season is the same, i.e. $\theta_i = \theta$.
- The 3-point percentage every season is independent of other seasons.
- The 3-point percentage a season should be similar to other seasons.

Andre Dawkin's three-point percentage

Suppose Y_i are the number of 3-pointers Andre Dawkin's makes in game i, and assume

$$Y_i \stackrel{ind}{\sim} Bin(n_i, \theta_i)$$

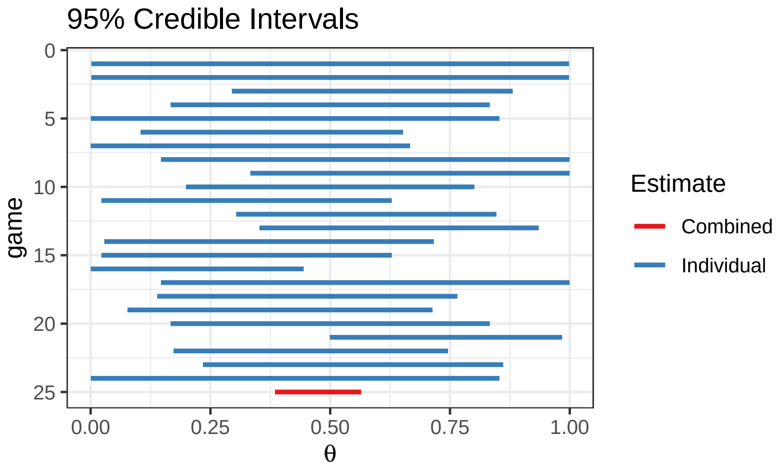
where

- n_i are the number of 3-pointers attempted in game i and
- θ_i is the probability of making a 3-pointer in game i.

Do these models make sense?

- The 3-point percentage every game is the same, i.e. $\theta_i = \theta$.
- The 3-point percentage every game is independent of other games.
- The 3-point percentage a game should be similar to other games.

Andre Dawkin's 3-point percentage



Andre Dawkin's 3-point percentage

date	opponent	made	attempts	game
2013-11-08	davidson	0	0	1
2013-11-12	kansas	0	0	2
2013-11-15	florida atlantic	5	8	3
2013-11-18	unc asheville	3	6	4
2013-11-19	east carolina	0	1	5
2013-11-24	vermont	3	9	6
2013-11-27	alabama	0	2	7
2013-11-29	arizona	1	1	8
2013-12-03	michigan	2	2	9
2013-12-16	gardner-webb	4	8	10
2013-12-19	ucla	1	5	11
2013-12-28	eastern michigan	6	10	12
2013-12-31	elon	5	7	13
2014-01-04	notre dame	1	4	14
2014-01-07	georgia tech	1	5	15
2014-01-11	clemson	0	4	16
2014-01-13	virginia	1	1	17
2014-01-18	nc state	3	7	18
2014-01-22	miami	2	6	19
2014-01-25	florida state	3	6	20
2014-01-27	pitt	6	7	21
2014-02-01	syracuse	4	9	22
2014-02-04	wake forest	4	7	23
2014-02-08	boston college	0	1	24

Hierarchical models

Consider the following model

$$\begin{aligned} y_i &\stackrel{ind}{\sim} p(y|\theta_i) \\ \theta_i &\stackrel{ind}{\sim} p(\theta|\phi) \\ \phi &\sim p(\phi) \end{aligned}$$

where

- y_i is observed,
- $\theta = (\theta_1, \ldots, \theta_n)$ and ϕ are parameters, and
- only ϕ has a prior that is set.

This is a hierarchical or multilevel model.

Posterior distribution for hierarchical models

The joint posterior distribution of interest in hierarchical models is

$$p(\theta,\phi|y) \propto p(y|\theta,\phi)p(\theta,\phi) = p(y|\theta)p(\theta|\phi)p(\phi) = \left[\prod_{i=1}^n p(y_i|\theta_i)p(\theta_i|\phi)\right]p(\phi).$$

The joint posterior distribution can be decomposed via

$$p(\theta,\phi|y) = p(\theta|\phi,y)p(\phi|y)$$

where

$$\begin{array}{rl}
p(\theta|\phi,y) & \propto p(y|\theta)p(\theta|\phi) = \prod_{i=1}^n p(y_i|\theta_i)p(\theta_i|\phi) \propto \prod_{i=1}^n p(\theta_i|\phi,y_i) \\
p(\phi|y) & \propto p(y|\phi)p(\phi) \\
p(y|\phi) & = \int p(y|\theta)p(\theta|\phi)d\theta \\
& = \int \cdots \int \prod_{i=1}^n \left[p(y_i|\theta_i)p(\theta_i|\phi)\right] d\theta_1 \cdots d\theta_n \\
& = \prod_{i=1}^n \int p(y_i|\theta_i)p(\theta_i|\phi)d\theta_i \\
& = \prod_{i=1}^n p(y_i|\phi)
\end{array}$$

Three-pointer example

Our statistical model

$$
\begin{aligned}
Y_i &\stackrel{ind}{\sim} Bin(n_i, \theta_i) \\
\theta_i &\stackrel{ind}{\sim} Be(\alpha, \beta) \\
\alpha, \beta &\sim p(\alpha, \beta)
\end{aligned}
$$

In this example,

- $\phi = (\alpha, \beta)$
- $Be(\alpha, \beta)$ describes the variability in 3-point percentage across games, and
- we are going to learn about this variability.

Decomposed posterior

$$Y_i \stackrel{ind}{\sim} Bin(n_i, \theta_i) \quad \theta_i \stackrel{ind}{\sim} Be(\alpha, \beta) \quad \alpha, \beta \sim p(\alpha, \beta)$$

Conditional posterior for θ:

$$p(\theta|\alpha, \beta, y) = \prod_{i=1}^{n} p(\theta_i|\alpha, \beta, y_i) = \prod_{i=1}^{n} Be(\theta_i|\alpha + y_i, \beta + n_i - y_i)$$

Marginal posterior for (α, β):

$$\begin{array}{rl} p(\alpha, \beta|y) & \propto p(y|\alpha, \beta)p(\alpha, \beta) \\ p(y|\alpha, \beta) & = \prod_{i=1}^{n} p(y_i|\alpha, \beta) = \prod_{i=1}^{n} \int p(y_i|\theta_i)p(\theta_i|\alpha, \beta)d\theta_i \\ & = \prod_{i=1}^{n} \binom{n_i}{y_i} \frac{B(\alpha+y_i, \beta+n_i-y_i)}{B(\alpha, \beta)} \end{array}$$

Thus $y_i|\alpha, \beta \stackrel{ind}{\sim}$ Beta-binomial(n_i, α, β).

A prior distribution for α and β

Recall the interpretation:

- α: prior successes
- β: prior failures

A more natural parameterization is

- prior expectation: $\mu = \frac{\alpha}{\alpha+\beta}$
- prior sample size: $\eta = \alpha + \beta$

Place priors on these parameters or transformed to the real line:

- logit $\mu = \log(\mu/[1-\mu]) = \log(\alpha/\beta)$
- $\log \eta$

A prior distribution for α and β

It seems reasonable to assume the mean (μ) and size (η) are independent *a priori*:

$$p(\mu, \eta) = p(\mu)p(\eta)$$

Let's construct a prior that has

- $P(0.1 < \mu < 0.5) \approx 0.95$ since most college basketball players have a three-point percentage between 10% and 50% and
- is somewhat diffuse for η but has more mass for smaller values.

Let's assume an informative prior for μ and η perhaps

- $\mu \sim Be(6, 14)$
- $\eta \sim Exp(0.05)$

```
a = 6
b = 14
e = 1/20
```

Prior draws

```
n <- 1e4

prior_draws <- data.frame(mu  = rbeta(n, a, b),
                          eta = rexp(n, e)) %>%
  mutate(alpha = eta*   mu,
         beta  = eta*(1-mu))

prior_draws %>%
  tidyr::gather(parameter, value) %>%
  group_by(parameter) %>%
  summarize(lower95 = quantile(value, prob = 0.025),
            median  = quantile(value, prob = 0.5),
            upper95 = quantile(value, prob = 0.975))

# A tibble: 4 x 4
  parameter lower95 median upper95
  <chr>       <dbl>  <dbl>   <dbl>
1 alpha       0.129  3.87   23.9
2 beta        0.359  9.61   51.4
3 eta         0.514 13.8    72.4
4 mu          0.124  0.292   0.511

cor(prior_draws$alpha, prior_draws$beta)

[1] 0.7951507
```

```
model_informative_prior = "
data {
  int<lower=0> N;    // data
  int<lower=0> n[N];
  int<lower=0> y[N];
  real<lower=0> a;   // prior
  real<lower=0> b;
  real<lower=0> e;
}
parameters {
  real<lower=0,upper=1> mu;
  real<lower=0> eta;
  real<lower=0,upper=1> theta[N];
}
transformed parameters {
  real<lower=0> alpha;
  real<lower=0> beta;

  alpha = eta*   mu ;
  beta  = eta*(1-mu);
}
model {
  mu    ~ beta(a,b);
  eta   ~ exponential(e);

  // implicit joint distributions
  theta ~ beta(alpha,beta);
  y     ~ binomial(n,theta);
}
"
```

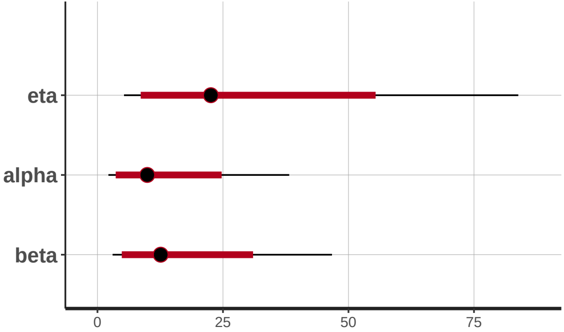
Stan

```
dat = list(y = d$made, n = d$attempts, N = nrow(d),a = a, b = b, e = e)
m = stan_model(model_code = model_informative_prior)
r = sampling(m, dat, c("mu","eta","alpha","beta","theta"),
             iter = 10000)
```

```
r

Inference for Stan model: anon_model.
4 chains, each with iter=10000; warmup=5000; thin=1;
post-warmup draws per chain=5000, total post-warmup draws=20000.

            mean se_mean    sd  2.5%   25%   50%   75% 97.5% n_eff Rhat
mu          0.44    0.00  0.05  0.34  0.41  0.44  0.47  0.53  5429    1
eta        28.37    0.44 21.02  5.28 13.63 22.59 36.84 83.82  2315    1
alpha      12.55    0.20  9.54  2.18  5.86  9.92 16.31 38.22  2318    1
beta       15.82    0.24 11.73  3.02  7.62 12.60 20.47 46.71  2356    1
theta[1]    0.44    0.00  0.12  0.19  0.36  0.44  0.52  0.70 14481    1
theta[2]    0.44    0.00  0.12  0.19  0.36  0.44  0.51  0.69 14333    1
theta[3]    0.49    0.00  0.10  0.31  0.43  0.49  0.56  0.70 14108    1
theta[4]    0.45    0.00  0.10  0.26  0.39  0.45  0.52  0.66 17874    1
theta[5]    0.42    0.00  0.12  0.17  0.34  0.42  0.49  0.65 12842    1
theta[6]    0.41    0.00  0.10  0.22  0.34  0.41  0.47  0.60 13657    1
theta[7]    0.40    0.00  0.12  0.15  0.32  0.40  0.47  0.62 10358    1
theta[8]    0.47    0.00  0.12  0.24  0.39  0.47  0.54  0.73 15136    1
theta[9]    0.49    0.00  0.12  0.28  0.41  0.49  0.57  0.76 11804    1
theta[10]   0.46    0.00  0.10  0.27  0.39  0.46  0.52  0.66 16617    1
theta[11]   0.39    0.00  0.11  0.17  0.32  0.39  0.46  0.59  9644    1
theta[12]   0.49    0.00  0.10  0.31  0.43  0.49  0.55  0.69 14221    1
theta[13]   0.51    0.00  0.11  0.32  0.44  0.51  0.58  0.74 11588    1
theta[14]   0.41    0.00  0.11  0.18  0.34  0.41  0.48  0.62 11585    1
theta[15]   0.39    0.00  0.11  0.17  0.32  0.39  0.46  0.59 10164    1
theta[16]   0.36    0.00  0.11  0.12  0.29  0.37  0.44  0.57  6682    1
theta[17]   0.47    0.00  0.12  0.24  0.39  0.47  0.54  0.73 15593    1
theta[18]   0.44    0.00  0.10  0.24  0.37  0.44  0.50  0.64 15963    1
theta[19]   0.41    0.00  0.10  0.21  0.35  0.42  0.48  0.61 14077    1
theta[20]   0.45    0.00  0.10  0.26  0.39  0.45  0.52  0.66 17013    1
theta[21]   0.55    0.00  0.11  0.35  0.47  0.54  0.62  0.79  7677    1
theta[22]   0.44    0.00  0.10  0.26  0.38  0.44  0.50  0.63 18378    1
```

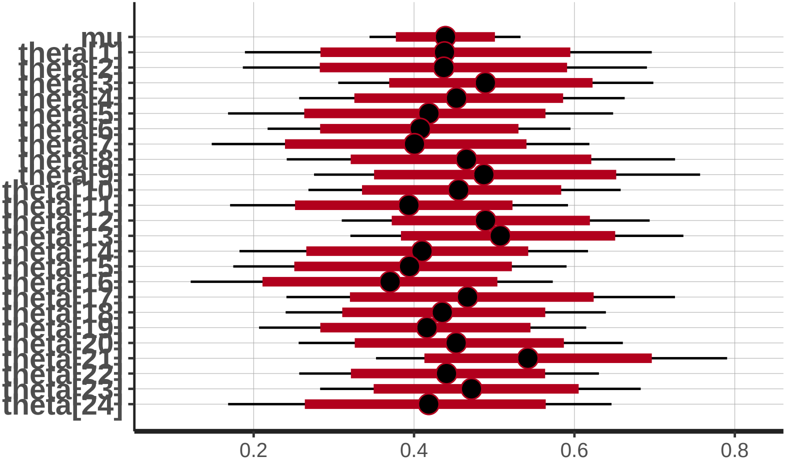
stan

```
plot(r, pars=c('eta','alpha','beta'))

ci_level: 0.8 (80% intervals)
outer_level: 0.95 (95% intervals)
```

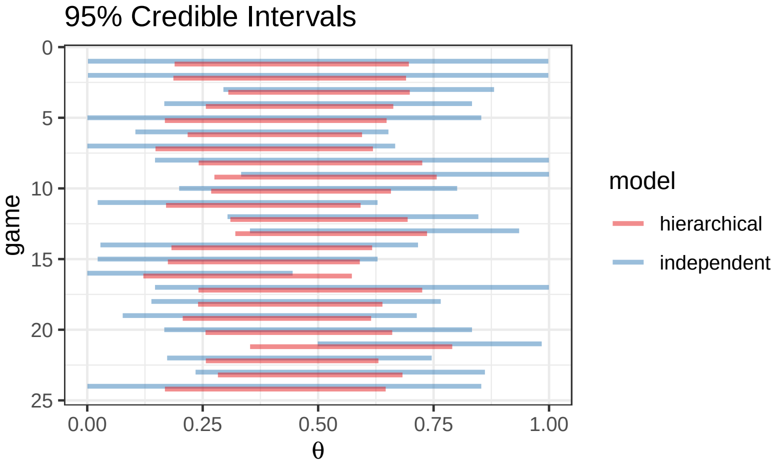


stan

```
plot(r, pars=c('mu','theta'))
```



Comparing independent and hierarchical models



A prior distribution for α and β

In Bayesian Data Analysis (3rd ed) page 110, several priors are discussed

- $(\log(\alpha/\beta), \log(\alpha+\beta)) \propto 1$ leads to an improper posterior.
- $(\log(\alpha/\beta), \log(\alpha+\beta)) \sim Unif([-10^{10}, 10^{10}] \times [-10^{10}, 10^{10}])$ while proper and seemingly vague is a very informative prior.
- $(\log(\alpha/\beta), \log(\alpha+\beta)) \propto \alpha\beta(\alpha+\beta)^{-5/2}$ which leads to a proper posterior and is equivalent to $p(\alpha, \beta) \propto (\alpha+\beta)^{-5/2}$.

Stan - default prior

```
model_default_prior <- "
data {
  int<lower=0> N;
  int<lower=0> n[N];
  int<lower=0> y[N];
}
parameters {
  real<lower=0> alpha;
  real<lower=0> beta;
  real<lower=0,upper=1> theta[N];
}

model {
  // default prior
  target += -5*log(alpha+beta)/2;

  // implicit joint distributions
  theta ~ beta(alpha,beta);
  y     ~ binomial(n,theta);
}
"

m2 <- stan_model(model_code = model_default_prior)
r2 <- sampling(m2, dat, c("alpha","beta","theta"), iter = 10000,
               control = list(adapt_delta = 0.9))

Warning:  There were 738 divergent transitions after warmup.  See
https://mc-stan.org/misc/warnings.html#divergent-transitions-after-warmup
```

Marginal posterior for α, β

An alternative to jointly sampling θ, α, β is to

1. sample $\alpha, \beta \sim p(\alpha, \beta|y)$, and then
2. sample $\theta_i \stackrel{ind}{\sim} p(\theta_i|\alpha, \beta, y_i) \stackrel{d}{=} Be(\alpha + y_i, \beta + n_i - y_i)$.

The maginal posterior for α, β is

$$p(\alpha, \beta|y) \propto p(y|\alpha, \beta)p(\alpha, \beta) = \left[\prod_{i=1}^{n} \text{Beta-binomial}(y_i|n_i, \alpha, \beta)\right] p(\alpha, \beta)$$

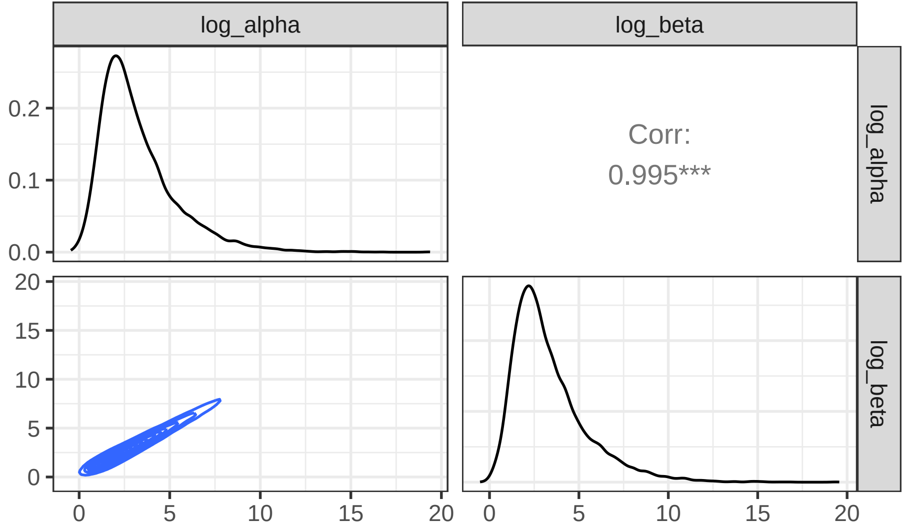
Stan - beta-binomial

```
# Marginalized (integrated) theta out of the model
model_marginalized <- "
data {
  int<lower=0> N;
  int<lower=0> n[N];
  int<lower=0> y[N];
}
parameters {
  real<lower=0> alpha;
  real<lower=0> beta;
}
model {
  target += -5*log(alpha+beta)/2;
  y     ~ beta_binomial(n,alpha,beta);
}
generated quantities {
  real<lower=0,upper=1> theta[N];
  for (i in 1:N)
    theta[i] = beta_rng(alpha+y[i],beta+n[i]-y[i]);
}
"

m3 <- stan_model(model_code = model_marginalized)
r3 <- sampling(m3, dat, iter = 10000)
```

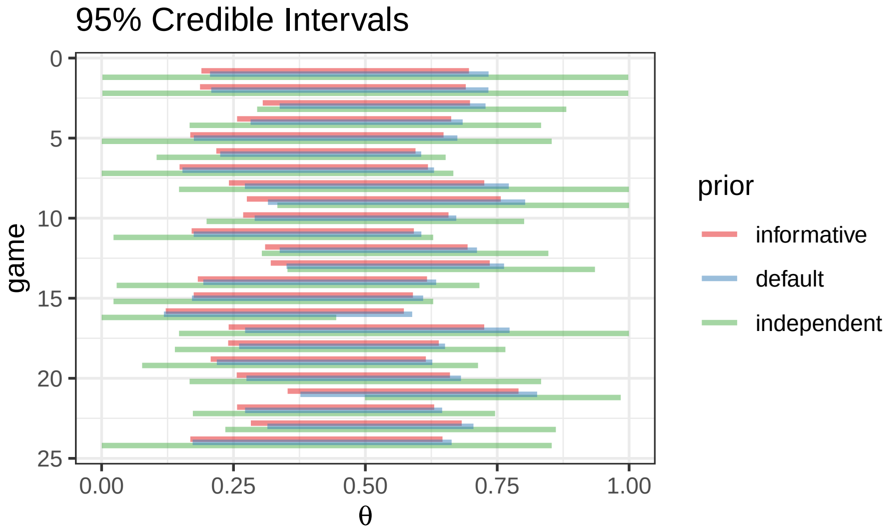
Stan - beta-binomial

```
Inference for Stan model: anon_model.
4 chains, each with iter=10000; warmup=5000; thin=1;
post-warmup draws per chain=5000, total post-warmup draws=20000.

              mean  se_mean         sd  2.5%   25%   50%   75%   97.5% n_eff Rhat
alpha     58295.72 51485.17 3477234.28  1.78  5.99 15.38 65.33 6133.35  4561    1
beta      62847.25 55417.85 3738352.85  2.09  6.83 17.14 72.76 6733.73  4551    1
theta[1]      0.47     0.00       0.12  0.21  0.41  0.47  0.53    0.73 19529    1
theta[2]      0.47     0.00       0.12  0.21  0.41  0.47  0.53    0.73 19621    1
theta[3]      0.51     0.00       0.10  0.34  0.45  0.50  0.56    0.73 13439    1
theta[4]      0.48     0.00       0.10  0.28  0.42  0.48  0.53    0.68 20449    1
theta[5]      0.45     0.00       0.12  0.17  0.39  0.46  0.52    0.67 14666    1
theta[6]      0.44     0.00       0.09  0.22  0.38  0.45  0.50    0.61 10707    1
theta[7]      0.43     0.00       0.12  0.15  0.37  0.44  0.50    0.63 10231    1
theta[8]      0.50     0.00       0.12  0.27  0.43  0.49  0.55    0.77 17006    1
theta[9]      0.52     0.00       0.12  0.32  0.44  0.50  0.57    0.80 10585    1
theta[10]     0.48     0.00       0.09  0.29  0.42  0.48  0.53    0.67 18896    1
theta[11]     0.42     0.00       0.11  0.17  0.36  0.44  0.50    0.61  8847    1
theta[12]     0.51     0.00       0.09  0.34  0.45  0.50  0.56    0.71 13045    1
theta[13]     0.52     0.00       0.10  0.35  0.46  0.51  0.58    0.76  9459    1
theta[14]     0.44     0.00       0.11  0.19  0.38  0.45  0.50    0.63 11839    1
theta[15]     0.42     0.00       0.11  0.17  0.36  0.44  0.50    0.61  8515    1
theta[16]     0.40     0.00       0.12  0.12  0.33  0.42  0.48    0.59  6249    1
theta[17]     0.50     0.00       0.12  0.27  0.43  0.49  0.55    0.77 15434    1
theta[18]     0.46     0.00       0.09  0.26  0.41  0.47  0.52    0.65 18879    1
theta[19]     0.44     0.00       0.10  0.22  0.39  0.45  0.51    0.63 13001    1
theta[20]     0.48     0.00       0.10  0.27  0.42  0.47  0.53    0.68 19991    1
theta[21]     0.56     0.00       0.12  0.38  0.47  0.53  0.62    0.83  5628    1
theta[22]     0.46     0.00       0.09  0.27  0.41  0.47  0.52    0.65 18833    1
```

Posterior samples for α and β

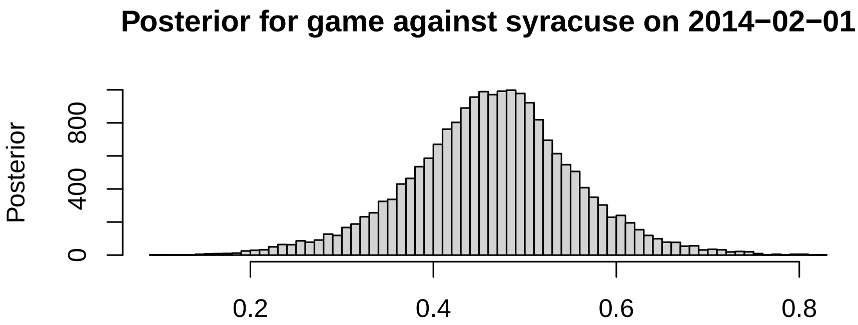


Comparing all models

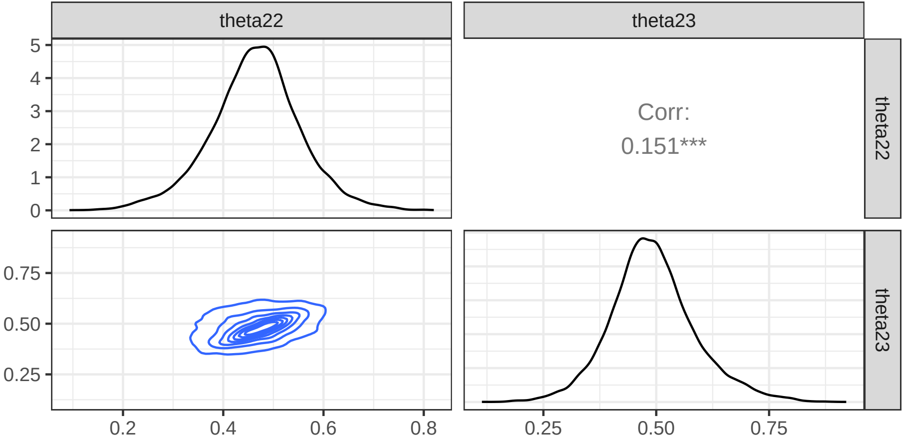


Posterior sample for θ_{22}

```
game <- 22
theta22 <- extract(r3, "theta")$theta[,game]

hist(theta22, 100,
     main=paste("Posterior for game against", d$opponent[game], "on", d$date[game]),
     xlab="3-point probability",
     ylab="Posterior")
```



θs are not independent in the posterior



3-point percentage across seasons

An alternative to modeling game-specific 3-point percentage is to model 3-point percentage in a season. The model is exactly the same, but the data changes.

season	y	n
1	36	95
2	64	150
3	67	171
4	64	152

Due to the low number of seasons (observations), we will use an informative prior for α and β.

Stan - beta-binomial

```
model_seasons <- "
data {
  int<lower=0> N; int<lower=0> n[N]; int<lower=0> y[N];
  real<lower=0> a; real<lower=0> b; real<lower=0> e;
}
parameters {
  real<lower=0,upper=1> mu;
  real<lower=0> eta;
}
transformed parameters {
  real<lower=0> alpha;
  real<lower=0> beta;
  alpha = eta *    mu;
  beta  = eta * (1-mu);
}
model {
  mu  ~ beta(a,b);
  eta ~ exponential(e);
  y   ~ beta_binomial(n,alpha,beta);
}
generated quantities {
  real<lower=0,upper=1> theta[N];
  for (i in 1:N) theta[i] = beta_rng(alpha+y[i], beta+n[i]-y[i]);
}
"
```

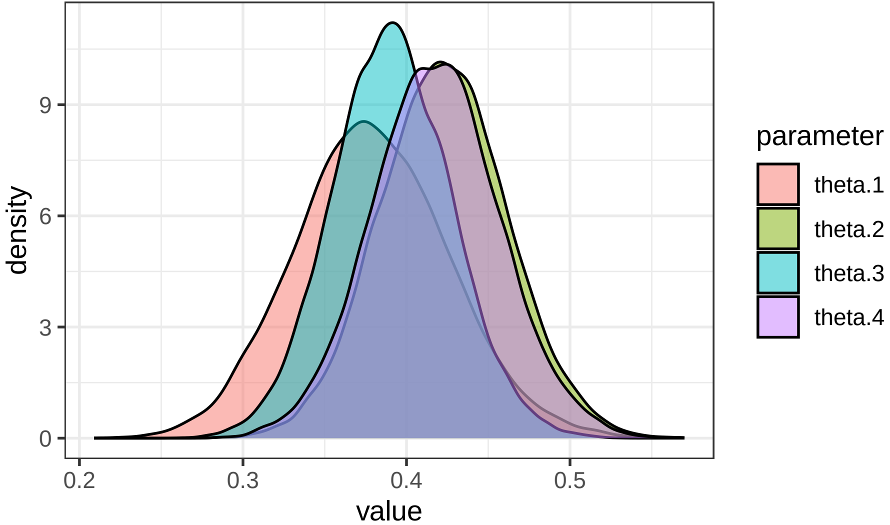
Run stan

```
dat       <- list(N = nrow(d), y = d$y, n = d$n, a = a, b = b, e = e)
m4        <- stan_model(model_code = model_seasons)
r_seasons <- sampling(m4, dat, iter = 10000,
                      c("alpha","beta","mu","eta","theta"))
```

Stan - hierarchical model for seasons

```
Inference for Stan model: anon_model.
4 chains, each with iter=10000; warmup=5000; thin=1;
post-warmup draws per chain=5000, total post-warmup draws=20000.

            mean se_mean   sd    2.5%     25%     50%     75%   97.5% n_eff Rhat
alpha       4.90    0.03 3.10    0.93    2.63    4.23    6.45   12.73 11470    1
beta        7.99    0.04 4.69    1.77    4.59    7.01   10.37   19.53 12285    1
mu          0.38    0.00 0.06    0.25    0.33    0.38    0.42    0.50 11466    1
eta        12.90    0.07 7.62    2.82    7.34   11.25   16.80   31.75 11838    1
theta[1]    0.38    0.00 0.05    0.29    0.35    0.38    0.41    0.47 19470    1
theta[2]    0.42    0.00 0.04    0.35    0.40    0.42    0.45    0.50 18697    1
theta[3]    0.39    0.00 0.04    0.32    0.37    0.39    0.42    0.46 19297    1
theta[4]    0.42    0.00 0.04    0.34    0.39    0.42    0.44    0.50 20269    1
lp__     -402.07    0.01 1.05 -404.91 -402.49 -401.76 -401.32 -401.02  7091    1

Samples were drawn using NUTS(diag_e) at Fri Feb  9 15:23:14 2024.
For each parameter, n_eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor on split chains (at
convergence, Rhat=1).
```

Stan - hierarchical model for seasons



Stan - hierarchical model for seasons

Probabilities that 3-point percentage is greater in season 4 than in the other seasons:

```
theta = extract(r_seasons, "theta")[[1]]
mean(theta[,4] > theta[,1])

[1] 0.73465

mean(theta[,4] > theta[,2])

[1] 0.45475

mean(theta[,4] > theta[,3])

[1] 0.699
```

Summary - hierarchical models

Two-level hierarchical model:

$$y_i \stackrel{ind}{\sim} p(y|\theta_i) \qquad \theta_i \stackrel{ind}{\sim} p(\theta|\phi) \qquad \phi \sim p(\phi)$$

Conditional independencies:

- $y_i \perp\!\!\!\perp y_j|\theta$ for $i \neq j$
- $\theta_i \perp\!\!\!\perp \theta_j|\phi$ for $i \neq j$
- $y \perp\!\!\!\perp \phi|\theta$
- $y_i \perp\!\!\!\perp y_j|\phi$ for $i \neq j$
- $\theta_i \perp\!\!\!\perp \theta_j|\phi, y$ for $i \neq j$

Summary - extension to more levels

Three-level hierarchical model:

$$y \sim p(y|\theta) \qquad \theta \sim p(\theta|\phi) \qquad \phi \sim p(\phi|\psi) \qquad \psi \sim p(\psi)$$

When deriving posteriors, remember the conditional independence structure, e.g.

$$p(\theta, \phi, \psi|y) \propto p(y|\theta)p(\theta|\phi)p(\phi|\psi)p(\psi)$$