Gamma distribution

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Gamma distribution

The random variable X has a gamma distribution with

- shape parameter $\alpha > 0$ and
- rate parameter $\lambda > 0$

if its probability density function is

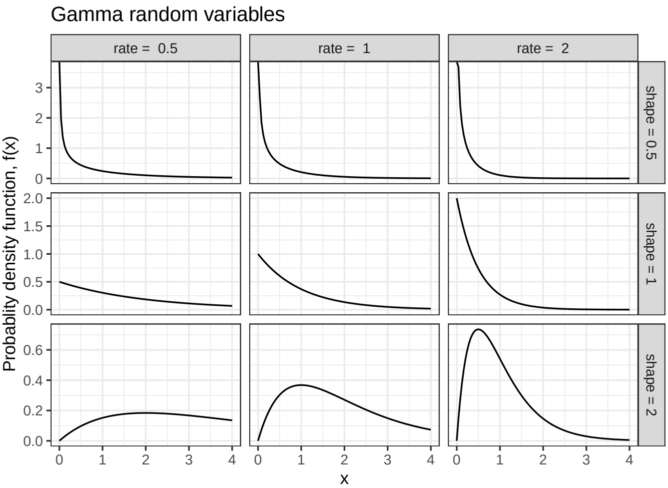
$$p(x|\alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \mathrm{I}(x > 0)$$

where $\Gamma(\alpha)$ is the gamma function,

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

We write $X \sim Ga(\alpha, \lambda)$.

Gamma probability density function



Gamma mean and variance

If $X \sim Ga(\alpha, \lambda)$, then

$$E[X] = \int_0^\infty x \, \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx = \cdots = \frac{\alpha}{\lambda}$$

and

$$Var[X] = \int_0^\infty \left(x - \frac{\alpha}{\lambda}\right)^2 \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx = \cdots = \frac{\alpha}{\lambda^2}.$$

Gamma cumulative distribution function

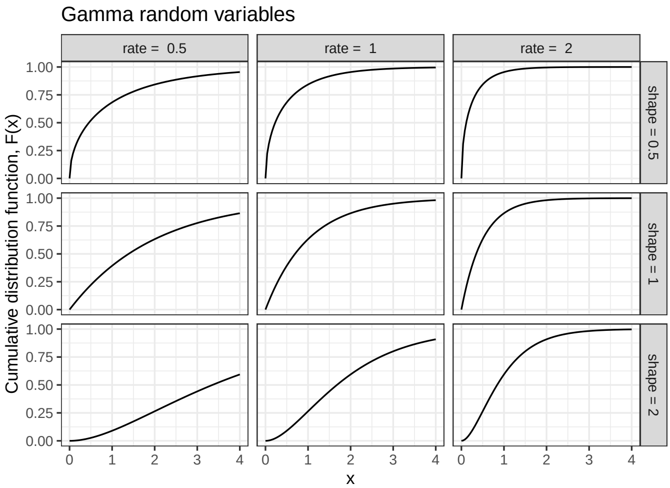
If $X \sim Ga(\alpha, \lambda)$, then its cumulative distribution function is

$$F(x) = \int_0^x \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t} dt = \cdots = \frac{\gamma(\alpha, \beta x)}{\Gamma(\alpha)}$$

where $\gamma(\alpha, \beta x)$ is the incomplete gamma function, i.e.

$$\gamma(\alpha, \beta x) = \int_0^{\beta x} t^{\alpha-1} e^{-t} dt.$$

Gamma cumulative distribution function - graphically



Relationship to exponential distribution

If $X_i \overset{iid}{\sim} Exp(\lambda)$, then

$$Y = \sum_{i=1}^{n} X_i \sim Ga(n, \lambda).$$

Thus, $Ga(1, \lambda) \overset{d}{=} Exp(\lambda)$.

Parameterization by the scale

A common alternative parameterization of the Gamma distribution uses the scale $\theta = \frac{1}{\lambda}$. In this parameterization, we have

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta} \, \mathrm{I}(x > 0)$$

and

$$E[X] = \alpha\theta \qquad \text{and} \qquad Var[X] = \alpha\theta^2.$$

Summary

Gamma random variable

- $X \sim Ga(\alpha, \lambda)$, $\alpha, \lambda > 0$
- $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$, $x > 0$
- $E[X] = \frac{\alpha}{\lambda}$
- $Var[X] = \frac{\alpha}{\lambda^2}$