

# Hierarchical models (cont.)

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# Outline

- Theoretical justification for hierarchical models
  - Exchangeability
  - de Finetti's theorem
  - Application to hierarchical models
- Normal hierarchical model
  - Posterior
  - Simulation study
  - Shrinkage

# Exchangeability

## Definition

The set  $Y_1, Y_2, \dots, Y_n$  is **exchangeable** if the joint probability  $p(y_1, \dots, y_n)$  is invariant to permutation of the indices. That is, for any permutation  $\pi$ ,

$$p(y_1, \dots, y_n) = p(y_{\pi_1}, \dots, y_{\pi_n}).$$

An exchangeable but not iid example:

- Consider an urn with one red ball and one blue ball with probability  $1/2$  of drawing either.
- Draw without replacement from the urn.
- Let  $Y_i = 1$  if the  $i$ th ball is red and otherwise  $Y_i = 0$ .
- Since  $1/2 = P(Y_1 = 1, Y_2 = 0) = P(Y_1 = 0, Y_2 = 1) = 1/2$ ,  $Y_1$  and  $Y_2$  are exchangeable.
- But  $0 = P(Y_2 = 1 | Y_1 = 1) \neq P(Y_2 = 1) = 1/2$  and thus  $Y_1$  and  $Y_2$  are not independent.

# Exchangeability

## Theorem

*All independent and identically distributed random variables are exchangeable.*

## Proof.

Let  $y_i \stackrel{iid}{\sim} p(y)$ , then

$$p(y_1, \dots, y_n) = \prod_{i=1}^n p(y_i) = \prod_{i=1}^n p(y_{\pi_i}) = p(y_{\pi_1}, \dots, y_{\pi_n})$$



## Definition

The sequence  $Y_1, Y_2, \dots$  is **infinitely exchangeable** if, for any  $n$ ,  $Y_1, Y_2, \dots, Y_n$  are exchangeable.

# de Finetti's theorem

## Theorem

*A sequence of random variables  $(y_1, y_2, \dots)$  is infinitely exchangeable iff, for all  $n$ ,*

$$p(y_1, y_2, \dots, y_n) = \int \prod_{i=1}^n p(y_i | \theta) P(d\theta),$$

*for some measure  $P$  on  $\theta$ .*

If the distribution on  $\theta$  has a density, we can replace  $P(d\theta)$  with  $p(\theta)d\theta$ .

This means that there must exist

- a parameter  $\theta$ ,
- a likelihood  $p(y|\theta)$  such that  $y_i \stackrel{ind}{\sim} p(y|\theta)$ , and
- a distribution  $P$  on  $\theta$ .

## Application to hierarchical models

Assume  $(y_1, y_2, \dots)$  are infinitely exchangeable, then by de Finetti's theorem for the  $(y_1, \dots, y_n)$  that you actually observed, there exists

- a parameter  $\theta$ ,
- a distribution  $p(y|\theta)$  such that  $y_i \stackrel{ind}{\sim} p(y|\theta)$ , and
- a distribution  $P$  on  $\theta$ .

Assume  $\theta = (\theta_1, \theta_2, \dots)$  with  $\theta_i$  infinitely exchangeable. By de Finetti's theorem for  $(\theta_1, \dots, \theta_n)$ , there exists

- a parameter  $\phi$ ,
- a distribution  $p(\theta|\phi)$  such that  $\theta_i \stackrel{ind}{\sim} p(\theta|\phi)$ , and
- a distribution  $P$  on  $\phi$ .

Assume  $\phi = \phi$  with  $\phi \sim p(\phi)$ .

## Exchangeability with covariates

Suppose we observe  $y_i$  observations and  $x_i$  covariates for each unit  $i$ . Now we assume  $(y_1, y_2, \dots)$  are infinitely exchangeable given  $x_i$ , then by de Finetti's theorem for the  $(y_1, \dots, y_n)$ , there exists

- a parameter  $\theta$ ,
- a distribution  $p(y|\theta, \mathbf{x})$  such that  $y_i \stackrel{\text{ind}}{\sim} p(y|\theta, \mathbf{x}_i)$ , and
- a distribution  $P$  on  $\theta$  given  $\mathbf{x}$ .

Assume  $\theta = (\theta_1, \theta_2, \dots)$  with  $\theta_i$  infinitely exchangeable given  $\mathbf{x}$ . By de Finetti's theorem for  $(\theta_1, \dots, \theta_n)$ , there exists

- a parameter  $\phi$ ,
- a distribution  $p(\theta|\phi, \mathbf{x})$  such that  $\theta_i \stackrel{\text{ind}}{\sim} p(\theta|\phi, \mathbf{x}_i)$ , and
- a distribution  $P$  on  $\phi$  given  $\mathbf{x}$ .

Assume  $\phi = \phi$  with  $\phi \sim p(\phi|\mathbf{x})$ .

# Summary

Hierarchical model:

$$y_i \overset{\text{ind}}{\sim} p(y|\theta_i), \quad \theta_i \overset{\text{ind}}{\sim} p(\theta|\phi), \quad \phi \sim p(\phi)$$

Hierarchical linear model:

$$y_i \overset{\text{ind}}{\sim} p(y|\theta_i, x_i), \quad \theta_i \overset{\text{ind}}{\sim} p(\theta|\phi, x_i), \quad \phi \sim p(\phi|x)$$

Although hierarchical models are typically written using the conditional independence notation above, the assumptions underlying the model are exchangeability and functional forms for the priors.



# Normal hierarchical models

Suppose we have the following model

$$\begin{aligned} y_{ij} &\stackrel{\text{ind}}{\sim} N(\theta_i, \sigma^2) \\ \theta_i &\stackrel{\text{iid}}{\sim} N(\mu, \tau^2) \end{aligned}$$

with  $j = 1, \dots, n_i$ ,  $i = 1, \dots, I$ , and  $n = \sum_{i=1}^I n_i$ . This is a normal hierarchical model.

Make the following assumptions for computational reasons:

- Let  $\sigma^2 = s^2$  be known.
- Assume  $p(\mu, \tau) \propto p(\mu|\tau)p(\tau) \propto p(\tau)$ , i.e. assume an improper uniform prior on  $\mu$ .

# Posterior distribution

The posterior is

$$p(\theta, \mu, \tau|y) \propto p(y|\theta)p(\theta|\mu, \tau)p(\mu|\tau)p(\tau)$$

but the decomposition

$$p(\theta, \mu, \tau|y) = p(\theta|\mu, \tau, y)p(\mu|\tau, y)p(\tau|y)$$

where

$$\begin{aligned} p(\theta|\mu, \tau, y) &\propto p(y|\theta)p(\theta|\mu, \tau) \\ p(\mu|\tau, y) &\propto \int p(y|\theta)p(\theta|\mu, \tau)d\theta p(\mu|\tau) \\ p(\tau|y) &\propto \int p(y|\theta)p(\theta|\mu, \tau)p(\mu|\tau)d\theta d\mu p(\tau) \end{aligned}$$

will aide computation via

1.  $\tau^{(k)} \sim p(\tau|y)$
2.  $\mu^{(k)} \sim p(\mu|\tau^{(k)}, y)$
3.  $\theta_i^{(k)} \sim p(\theta|\mu^{(k)}, \tau^{(k)}, y)$  for  $i = 1, \dots, I$ .

# Posterior distributions

The necessary conditional and marginal posteriors are presented in section 5.4 of BDA. Let

$$\bar{y}_{i\cdot} = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij} \quad \text{and} \quad s_i^2 = s^2/n_i$$

Then

$$\begin{aligned} p(\tau|y) &\propto p(\tau) V_\mu^{1/2} \prod_{i=1}^I (s_i^2 + \tau^2)^{-1/2} \exp\left(-\frac{(\bar{y}_{i\cdot} - \hat{\mu})^2}{2(s_i^2 + \tau^2)}\right) \\ \mu|\tau, y &\sim N(\hat{\mu}, V_\mu) \\ \theta_i|\mu, \tau, y &\sim N(\hat{\theta}_i, V_i) \end{aligned}$$

$$\begin{aligned} V_\mu^{-1} &= \sum_{i=1}^I \frac{1}{s_i^2 + \tau^2} & \hat{\mu} &= V_\mu \left( \sum_{i=1}^I \frac{\bar{y}_{i\cdot}}{s_i^2 + \tau^2} \right) \\ V_i^{-1} &= \frac{1}{s_i^2} + \frac{1}{\tau^2} & \hat{\theta}_i &= V_i \left( \frac{\bar{y}_{i\cdot}}{s_i^2} + \frac{\mu}{\tau^2} \right) \end{aligned}$$

# Simulation study

Common to both simulation scenarios:

- $I = 10$
- $n_i = 9$  for all  $i$
- $s = 1$  thus  $s_i = 1/3$  for all  $i$

Scenarios:

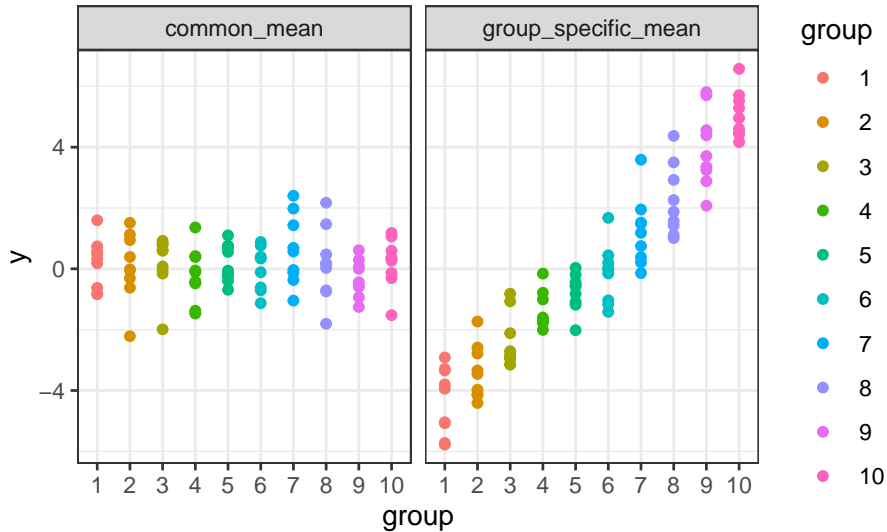
1. Common mean:  $\theta_i = 0$  for all  $i$
2. Group-specific means:  $\theta_i = i - (I/2 + .5)$

Use  $\tau \sim Ca^+(0, 1)$ .

# Simulation study

```
J <- 10
n_per_group <- 9
n <- rep(n_per_group, J)
sigma <- 1
N <- sum(n)
group <- rep(1:J, each=n_per_group)

set.seed(1)
df <- bind_rows(data.frame(group = factor(group),
                           simulation = "common_mean",
                           y = rnorm(N, 0, sigma)), # All means are the same
               data.frame(group = factor(group),
                           simulation = "group_specific_mean",
                           y = rnorm(N, group-(J/2+.5)))) # Each group has its own mean
```



# Summary statistics

simulation	group	n	mean	sd
common_mean	1	9	0.18	0.81
common_mean	2	9	0.09	1.11
common_mean	3	9	0.18	0.91
common_mean	4	9	-0.19	0.89
common_mean	5	9	0.17	0.62
common_mean	6	9	0.02	0.70
common_mean	7	9	0.61	1.14
common_mean	8	9	0.14	1.19
common_mean	9	9	-0.31	0.60
common_mean	10	9	0.20	0.81
group_specific_mean	1	9	-4.32	1.10
group_specific_mean	2	9	-3.40	0.88
group_specific_mean	3	9	-2.41	0.89
group_specific_mean	4	9	-1.38	0.60
group_specific_mean	5	9	-0.76	0.61
group_specific_mean	6	9	-0.16	0.95
group_specific_mean	7	9	1.21	1.12
group_specific_mean	8	9	2.23	1.15
group_specific_mean	9	9	3.97	1.26
group_specific_mean	10	9	5.08	0.77

# Sampling on a grid

Consider sampling from an arbitrary unnormalized density  $f(\tau) \propto p(\tau|y)$  using the following approach

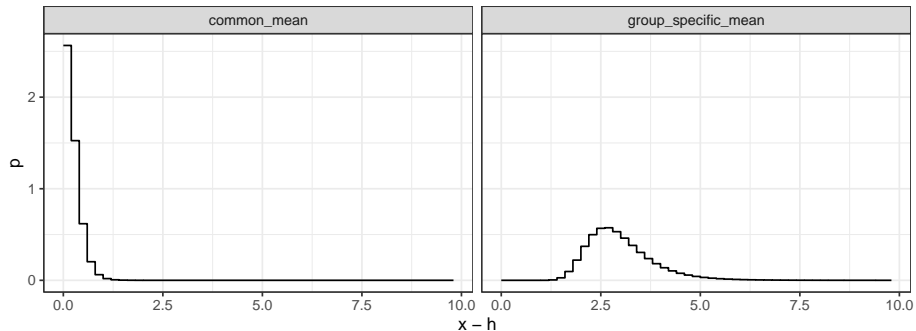
1. Construct a step-function approximation to this density:
  - a. Determine an interval  $[L, U]$  such that outside this interval  $f(\tau)$  is small.
  - b. Set an interval half-width  $h$  to generate a grid of  $M$  points  $(x_1, \dots, x_M)$  in this interval, i.e.

$$x_1 = L + h \text{ and } x_m = x_{m-1} + 2h \quad \forall 1 < m \leq M.$$

- c. Evaluate the density on this grid, i.e.  $f(x_m)$ .
  - d. Normalize interval weights, i.e.  $w_m = f(x_m) / \sum_{i=1}^M f(x_i)$   
(to constructed a normalized density, divide each  $w_m$  by  $2h$ .)
2. Sampling from this approximation:
  - a. Sample an interval  $m$  with probability  $w_m$ .
  - b. Sample uniformly within this interval, i.e.  $\tau \sim \text{Unif}(x_m - h, x_m + h)$ .

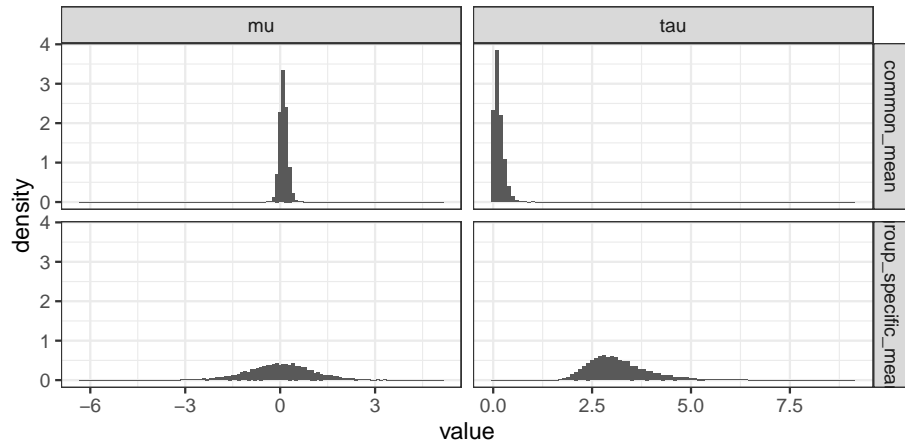


# Approximation to $p(\tau|y)$ when $\tau \sim Ca^+(0, 1)$



# Hyperparameters: group-to-group mean variability

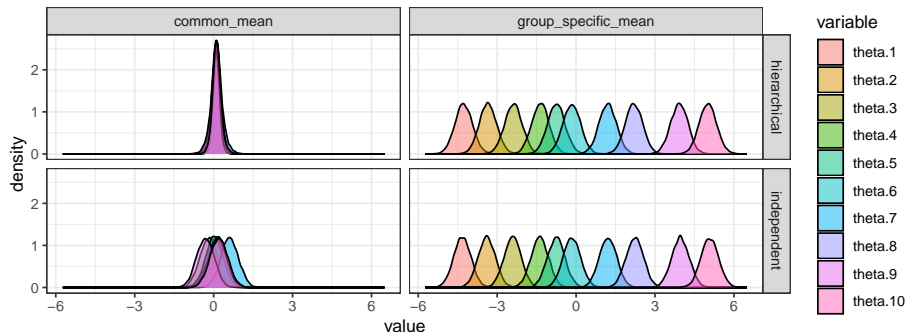
Recall  $\theta_i \stackrel{\text{ind}}{\sim} N(\mu, \tau^2)$ :



# Group-specific means

## Recall

- Common mean:  $E[Y_{ij}] = \mu$
- Group-specific mean:  $E[Y_{ij}] = i - 10/2 + 0.5$



# Extensions

- Unknown data variance:

$$y_{ij} \sim N(\theta_i, \sigma^2), \theta_i \sim N(\mu, \tau^2)$$

or

$$y_{ij} \sim N(\theta_i, \sigma^2), \theta_i \sim N(\mu, \sigma^2 \tau^2)$$

- Alternative distributions:

- Heavy-tailed:

$$y_{ij} \sim N(\theta_i, \sigma^2), \theta_i \sim t_\nu(\mu, \tau^2)$$

- Peak at zero:

$$y_{ij} \sim N(\theta_i, \sigma^2), \theta_i \sim \text{Laplace}(\mu, \tau^2)$$

- Point mass at zero:

$$y_{ij} \sim N(\theta_i, \sigma^2), \theta_i \sim \pi \delta_0 + (1 - \pi)N(\mu, \tau^2)$$

# Summary

## Hierarchical models

- allow the data to inform us about similarities across groups
- provide data driven shrinkage toward a grand mean
  - lots of shrinkage when means are similar
  - little shrinkage when means are different

Computation used the decomposition

$$p(\theta, \mu, \tau | y) = p(\theta | \mu, \tau, y) p(\mu | \tau, y) p(\tau | y)$$

which allowed for simulation from  $\tau$  then  $\mu$  and then  $\theta$  to obtain samples from the posterior.