I01 - Statistics

STAT 5870 (Engineering)
Iowa State University

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Statistics

The field of statistics is the study of the collection, analysis, interpretation, presentation, and organization of data.

https://en.wikipedia.org/wiki/Statistics

There are two different phases of statistics:

- descriptive statistics
    - statistics
    - graphical statistics
- inferential statistics
    - uses a sample to make statements about a population.

Convenience sample

The population consists of all units of interest. Any numerical characteristic of a population is a parameter. The sample consists of observed units collected from the population. Any function of a sample is called a statistic.

Population: in-use routers by graduate students at Iowa State University.

Parameter: proportion of those routers that have Gigabit speed.

Sample: routers of students in STAT 5870-1/A

Statistics: proportion of routers that have gigabit speed

Simple random sampling

A simple random sample is a sample from the population where all subsets of the same size are equally likely to be sampled. Random samples ensure that statistical conclusions will be valid.

Population: in-use routers by graduate students at Iowa State University.

Parameter: proportion of those routers that have Gigabit speed.

Sample: a pseudo-random number generator gives each graduate student a Unif(0,1) number and the lowest 100 are contacted

Statistics: proportion of routers that have gigabit speed

Sampling and non-sampling errors

Sampling errors are caused by the mere fact that only a sample, a portion of a population, is observed. Fortunately,

$$\text{error} \downarrow \quad \text{as sample size } (n) \uparrow$$

Non-sampling errors are caused by inappropriate sampling schemes and wrong statistical techniques. Often, no statistical technique can rescue a poorly collected sample of data.

Sample: students in STAT 5870-1/A

Statistics and estimators

A statistic is any function of the data.

Descriptive statistics:

- Sample mean, median, mode
- Sample quantiles
- Sample variance, standard deviation

When a statistic is meant to estimate a corresponding population parameter, we call that statistic an estimator.

Sample mean

Let $X_1, \ldots, X_n$ be a random sample from a distribution with

$$E[X_i] = \mu \qquad \text{and} \qquad Var[X_i] = \sigma^2$$

where we assume independence between the X_i.

The sample mean is

$$\hat{\mu} = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and estimates the population mean μ.

Sample variance

Let $X_1, \ldots, X_n$ be a random sample from a distribution with

$$E[X_i] = \mu \qquad \text{and} \qquad Var[X_i] = \sigma^2$$

where we assume independence between the X_i.

The sample variance is

$$\hat{\sigma}^2 = S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 = \frac{\sum_{i=1}^{n} X_i^2 - n\overline{X}^2}{n-1}$$

and estimates the population variance σ^2.

The sample standard deviation is $\hat{\sigma} = \sqrt{\hat{\sigma}^2}$ and estimates the population standard deviation.

Quantiles

A p-quantile of a population is a number x that solves

$$P(X < x) \leq p \quad \text{and} \quad P(X > x) \leq 1 - p.$$

A sample p-quantile is any number that exceeds at most $100p\%$ of the sample, and is exceeded by at most $100(1-p)\%$ of the sample. A $100p$-percentile is a p-quantile. First, second, and third quartiles are the 25th, 50th, and 75th percentiles. They split a population or a sample into four equal parts. A median is a 0.5-quantile, 50th percentile, and 2nd quartile.

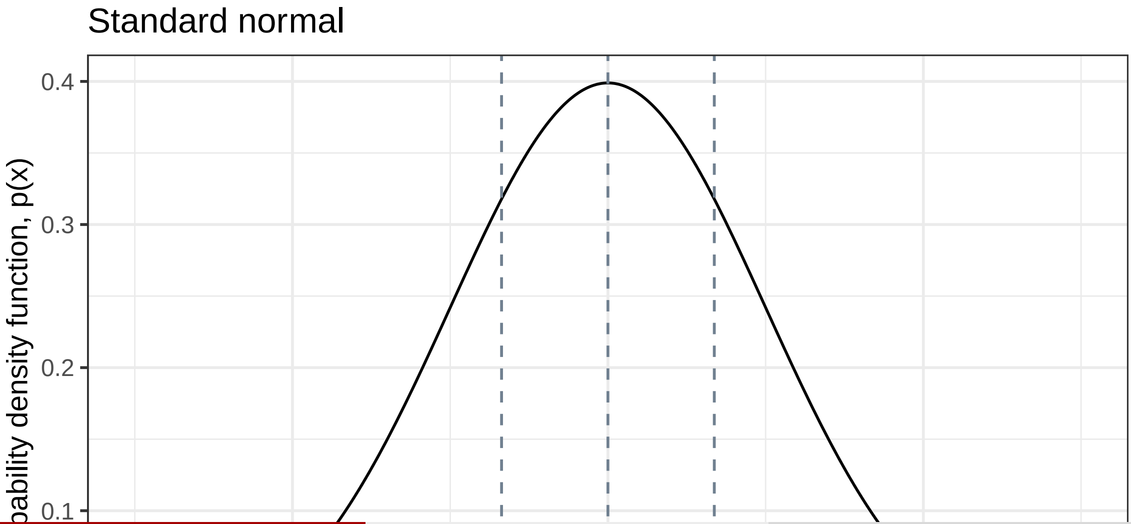
The interquartile range is the third quartile minus the first quartile, i.e.

$$IQR = Q_3 - Q_1$$

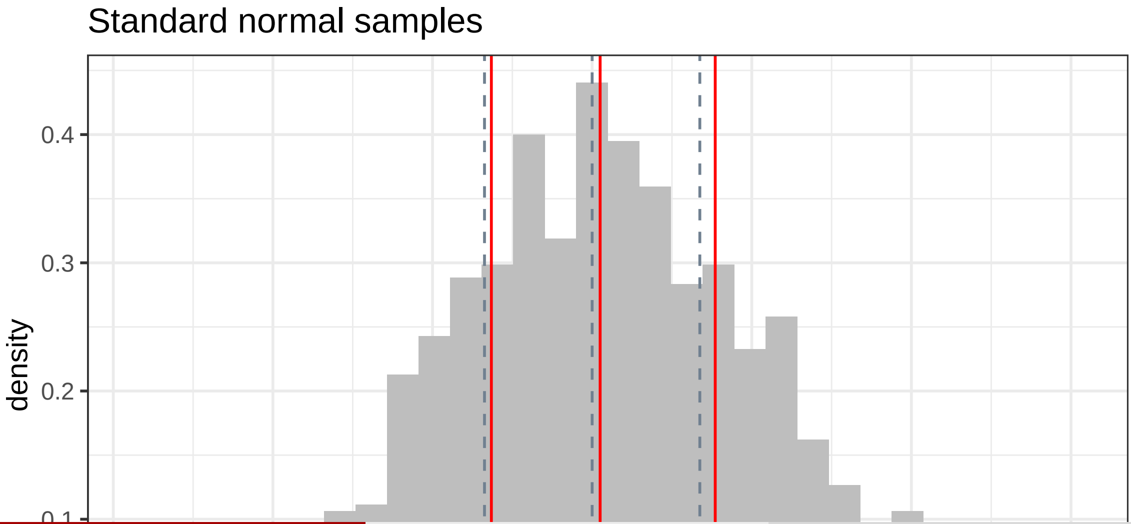
and the sample interquartile range is the third sample quartile minus the first sample quartile, i.e.

$$\widehat{IQR} = \hat{Q}_3 - \hat{Q}_1$$

Standard normal quartiles



Sample quartiles from a standard normal



Properties of statistics and estimators

Statistics can have properties, e.g.

- standard error

Estimators can have properties, e.g.

- unbiased
- consistent

Standard error

The standard error of a statistic $\hat{\theta}$ is the standard deviation of that statistic (when the data are considered random).

If X_i are independent and have $Var[X_i] = \sigma^2$, then

$$\begin{aligned} Var\left[\overline{X}\right] &= Var\left[\tfrac{1}{n}\textstyle\sum_{i=1}^n X_i\right] \\ &= \tfrac{1}{n^2}\textstyle\sum_{i=1}^n Var[X_i] = \tfrac{1}{n^2}\sum_{i=1}^n \sigma^2 = \frac{\sigma^2}{n} \end{aligned}$$

and thus

$$SD\left[\overline{X}\right] = \sqrt{Var\left[\overline{X}\right]} = \sigma/\sqrt{n}.$$

Thus the standard error of the sample mean is $\sigma/\sqrt{n}$.

Unbiased

An estimator $\hat{\theta}$ is unbiased for a parameter θ if its expectation (when the data are considered random) equals the parameter, i.e.

$$E[\hat{\theta}] = \theta.$$

The sample mean is unbiased for the population mean μ since

$$E\left[\,\overline{X}\,\right] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \mu.$$

and the sample variance is unbiased for the population variance σ^2.

Consistent

An estimator $\hat{\theta}$, or $\hat{\theta}_n(x)$, is consistent for a parameter θ if the probability of its sampling error of any magnitude converges to 0 as the sample size n increases to infinity, i.e.

$$P\left(\left|\hat{\theta}_n(X) - \theta\right| > \epsilon\right) \to 0 \text{ as } n \to \infty$$

for any $\epsilon > 0$.

The sample mean is consistent for μ since $Var\left[\overline{X}\right] = \sigma^2/n$ and

$$P\left(\left|\overline{X} - \mu\right| > \epsilon\right) \le \frac{Var\left[\overline{X}\right]}{\epsilon^2} = \frac{\sigma^2/n}{\epsilon^2} \to 0$$

where the inequality is from Chebyshev's inequality.

Binomial example

Suppose $Y \sim Bin(n, \theta)$ where θ is the probability of success. The statistic $\hat{\theta} = Y/n$ is an estimator of θ.

Since

$$E\left[\hat{\theta}\right] = E\left[\frac{Y}{n}\right] = \frac{1}{n}E[Y] = \frac{1}{n}n\theta = \theta$$

the estimator is unbiased.

Binomial example

Suppose $Y \sim Bin(n, \theta)$ where θ is the probability of success. The statistic $\hat{\theta} = Y/n$ is an estimator of θ.

The variance of the estimator is

$$Var\left[\hat{\theta}\right] = Var\left[\frac{Y}{n}\right] = \frac{1}{n^2}Var[Y] = \frac{1}{n^2}n\theta(1-\theta) = \frac{\theta(1-\theta)}{n}.$$

Thus the standard error is

$$SE(\hat{\theta}) = \sqrt{Var[\hat{\theta}]} = \sqrt{\frac{\theta(1-\theta)}{n}}.$$

By Chebychev's inequality, this estimator is consistent for θ.

Summary

- Statistics are functions of data.
- Statistics have some properties:
  - Standard error
- Estimators are statistics that estimate population parameters.
- Estimators may have properties:
  - Unbiased
  - Consistent

Look at it!

Before you do anything with a data set, LOOK AT IT!

Why should you look at your data?

1. Find errors
   - Do variables have the correct range, e.g. positive?
   - How are Not Available encoded?
   - Are there outliers?
2. Do known or suspected relationships exist?
   - Is X linearly associated with Y?
   - Is X quadratically associated with Y?
3. Are there new relationships?
   - What is associated with Y and how?
4. Do variables adhere to distributional assumptions?
   - Does Y have an approximately normal distribution?
   - Right/left skew
   - Heavy tails

Principles of professional statistical graphics

https://moz.com/blog/data-visualization-principles-lessons-from-tufte

- Show the data
  - Avoid distorting the data, e.g. pie charts, 3d pie charts, exploding wedge 3d pie charts, bar charts that do not start at zero
- Plots should be self-explanatory
  - Use informative caption, legend
  - Use normative colors, shapes, etc
- Have a high information to ink ratio
  - Avoid bar charts
- Encourage eyes to compare
  - Use size, shape, and color to highlight differences

Stock market return