Data Asymptotics

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Normal approximation to the posterior

Suppose $p(\theta|y)$ is unimodal and roughly symmetric, then a Taylor series expansion of the logarithm of the posterior around the posterior mode $\hat{\theta}$ is

$$\log p(\theta|y) = \log p(\hat{\theta}|y) - \frac{1}{2}(\theta - \hat{\theta})^\top \left[-\frac{d^2}{d\theta^2} \log p(\theta|y) \right]_{\theta=\hat{\theta}} (\theta - \hat{\theta}) + \cdots$$

where the linear term in the expansion is zero because the derivative of the log-posterior density is zero at its mode.

Discarding the higher order terms, this expansion provides a normal approximation to the posterior, i.e.

$$p(\theta|y) \overset{d}{\approx} N(\hat{\theta}, \mathrm{J}(\hat{\theta})^{-1})$$

where $\mathrm{J}(\hat{\theta})$ is the sum of the prior and observed information, i.e.

$$\mathrm{J}(\hat{\theta}) = -\frac{d^2}{d\theta^2} \log p(\theta)_{|_{\theta=\hat{\theta}}} - \frac{d^2}{d\theta^2} \log p(y|\theta)_{|_{\theta=\hat{\theta}}}.$$

Binomial probability

Let $y \sim Bin(n, \theta)$ and $\theta \sim Be(a, b)$, then $\theta|y \sim Be(a+y, b+n-y)$ and the posterior mode is

$$\hat{\theta} = \frac{y'}{n'} = \frac{a+y-1}{a+b+n-2}.$$

Thus

$$\mathrm{J}(\hat{\theta}) = \frac{n'}{\hat{\theta}(1-\hat{\theta})}.$$

Thus

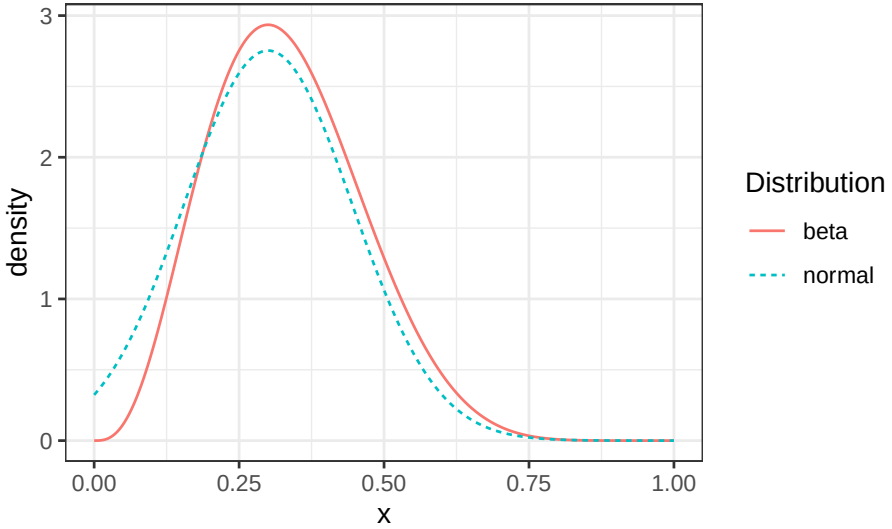
$$p(\theta|y) \overset{d}{\approx} N\left(\hat{\theta}, \frac{\hat{\theta}(1-\hat{\theta})}{n'}\right).$$

Binomial probability

```
a <- b <- 1      # Prior
n <- 10; y <- 3 # Data (attempts, successes)

yp <- a + y - 1; np <- a + b + n - 2
theta_hat = yp / np # mode of a beta

d <- data.frame(x = seq(0, 1, length = 1001)) |>
  mutate(beta   = dbeta(x,a+y,b+n-y),
         normal = dnorm(x, theta_hat, sqrt(theta_hat*(1-theta_hat)/np))) |>
  pivot_longer(beta:normal, names_to = "Distribution", values_to = "density")

ggplot(d, aes(x = x, y = density, color = Distribution, linetype = Distribution)) +
  geom_line()
```

https://youtu.be/cRhD9FbSb34

Binomial probability



Large-sample theory

Consider a model $y_i \overset{iid}{\sim} p(y|\theta_0)$ for some true value θ_0.

- Does the posterior distribution converge to θ_0?
- Does a point estimator (mode) converge to θ_0?
- What is the limiting posterior distribution?

Convergence of the posterior distribution

Consider a model $y_i \overset{iid}{\sim} p(y|\theta_0)$ for some true value θ_0.

Theorem

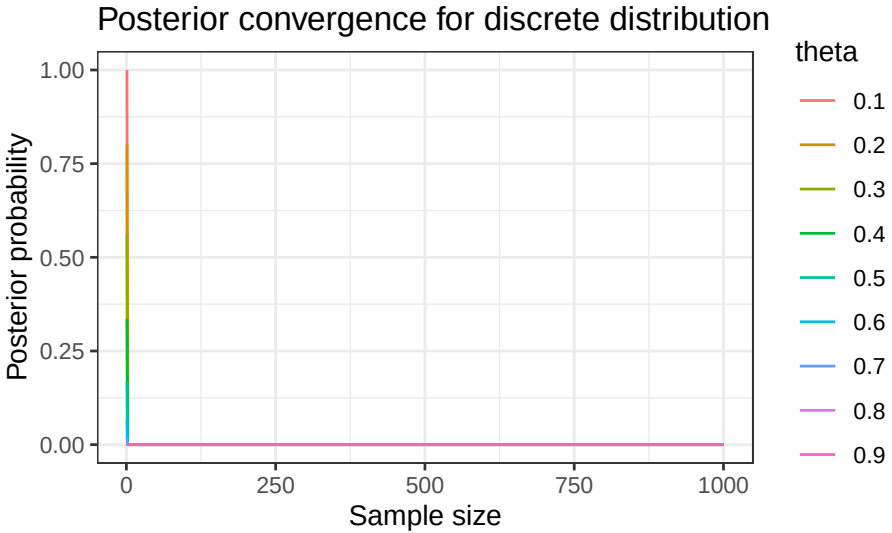
If the parameter space Θ is discrete and $Pr(\theta = \theta_0) > 0$, then $Pr(\theta = \theta_0|y) \to 1$ as $n \to \infty$.

Theorem

If the parameter space Θ is continuous and A is a neighborhood around θ_0 with $Pr(\theta \in A) > 0$, then $Pr(\theta \in A|y) \to 1$ as $n \to \infty$.

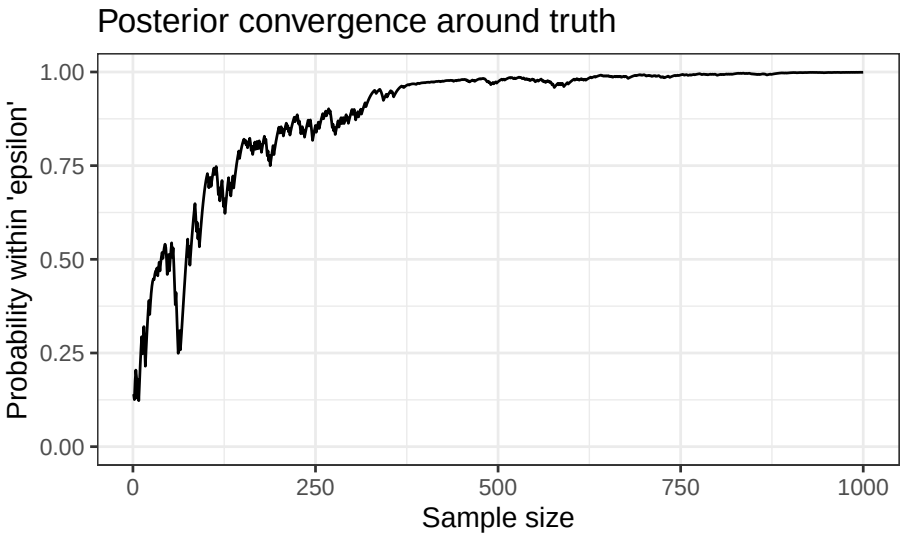
```r
n <- 1000
theta0 <- 0.3
d <- data.frame(
    n     = 1:n,
    y     = rbinom(n, 1, prob = 0.3))

dt <- expand_grid(d, theta = seq(0.1, 0.9, by = 0.1)) |>
  mutate(
    log_prob = dbinom(y, 1, prob = theta, log = TRUE),
  ) |>
  group_by(theta) |>
  arrange(n) |>
  mutate(
    log_prob = cumsum(log_prob)
  ) |>
  group_by(n) |>
  mutate(
    log_prob = log_prob - max(log_prob),
    prob     = exp(log_prob),
    prob     = prob / sum(prob),
    theta    = factor(theta)
  )

ggplot(dt, aes(x = n, y = prob,
               color = theta, group = theta)) +
  geom_line() +
  labs(
    x = "Sample size",
    y = "Posterior probability",
    title = "Posterior convergence for discrete distribution"
  )
```

Posterior distribution convergence of a discrete distribution



```r
a <- b <- 1 # prior
e <- 0.05   # window half-width

# Calculate P(theta0 - e < \theta < theta0 + e | y)
dc <- d |> mutate(y = cumsum(y),
                  prob = pbeta(theta0 + e, a + y, b + n - y) -
                         pbeta(theta0 - e, a + y, b + n - y))

# Plot calculated probability as a function of sample size
ggplot(dc, aes(x = n, y = prob)) +
  geom_line() +
  labs(
    x     = "Sample size",
    y     = "Probability within 'epsilon'",
    title = "Posterior convergence around truth"
  ) +
  ylim(0,1)
```

Posterior distribution convergence of a continuous distribution



Consistency of Bayesian point estimates

Suppose $y_i \overset{iid}{\sim} p(y|\theta_0)$ where θ_0 is a particular value for θ.

Recall that an estimator is consistent, i.e. $\hat{\theta} \xrightarrow{p} \theta_0$, if

$$\lim_{n\to\infty} P(|\hat{\theta} - \theta_0| < \epsilon) = 1.$$

Recall, under regularity conditions that $\hat{\theta}_{MLE} \xrightarrow{p} \theta_0$. If Bayesian estimators converge to the MLE, then they have the same properties.

Binomial example

Consider $y \sim Bin(n, \theta)$ with true value $\theta = \theta_0$ and prior $\theta \sim Be(a, b)$. Then $\theta|y \sim Be(a + y, b + n - y)$.

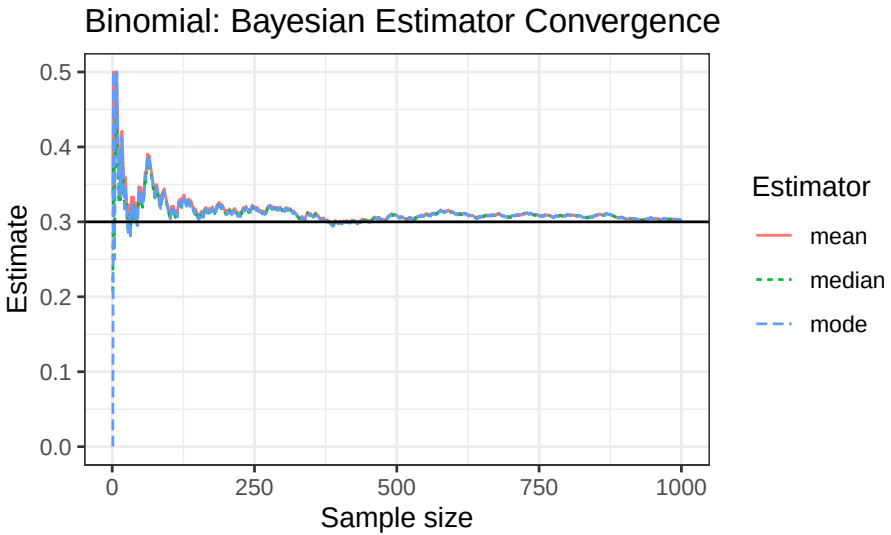
Recall that $\hat{\theta}_{MLE} = y/n$. The following estimators are all consistent

- Posterior mean: $\frac{a+y}{a+b+n}$
- Posterior median: $\approx \frac{a+y-1/3}{a+b+n-2/3}$ for $\alpha, \beta > 1$
- Posterior mode: $\frac{a+y-1}{a+b+n-2}$

since as $n \to \infty$, these all converge to $\hat{\theta}_{MLE} = y/n$.

```r
# Calculate posterior mean, median, and mode
dbc <- dc |>
  mutate(
    mean   = (a + y) / (a + b + n),
    median = qbeta(0.5, a + y, a + b + n - y),
    mode   = (a + y - 1) / (a + b + n - 2)
  ) |>
  pivot_longer(mean:mode, names_to = "Estimator", values_to = "estimate")

# Plot estimates vs sample size
ggplot(dbc, aes(x = n, y = estimate,
                color = Estimator, linetype = Estimator, group = Estimator)) +
  geom_line() +
  geom_hline(yintercept = theta0) +
  labs(
    x = "Sample size",
    y = "Estimate",
    title = "Binomial: Bayesian Estimator Convergence"
  )
```

Binomial: Bayesian Estimator Convergence



Normal example

Consider $Y_i \overset{iid}{\sim} N(\theta, 1)$ with known and prior $\theta \sim N(c, 1)$. Then

$$\theta|y \sim N\left(\frac{1}{n+1}c + \frac{n}{n+1}\overline{y}, \frac{1}{n+1}\right)$$

Recall that $\hat{\theta}_{MLE} = \overline{y}$. Since the posterior mean converges to the MLE, then the posterior mean (as well as the median and mode) are consistent.

Asymptotic normality

Consider the Taylor series expansion of the log posterior

$$\log p(\theta|y) = \log p(\hat{\theta}|y) - \frac{1}{2}(\theta - \hat{\theta})^\top \left[-\frac{d^2}{d\theta^2} \log p(\theta|y) \right]_{\theta=\hat{\theta}} (\theta - \hat{\theta}) + R$$

where the linear term is zero because the derivative at the posterior mode $\hat{\theta}$ is zero and R represents all higher order terms.

With iid observations, the coefficient for the quadratic term can be written as

$$-\frac{d^2}{d\theta^2}[\log p(\theta|y)]_{\theta=\hat{\theta}} = -\frac{d^2}{d\theta^2} \log p(\theta)_{\theta=\hat{\theta}} - \sum_{i=1}^{n} \frac{d^2}{d\theta^2}[\log p(y_i|\theta)]_{\theta=\hat{\theta}}$$

where

$$E_y \left[-\frac{d^2}{d\theta^2}[\log p(y_i|\theta)]_{\theta=\hat{\theta}} \right] = \mathrm{I}(\theta_0)$$

where $\mathrm{I}(\theta_0)$ is the expected Fisher information and thus, by the LLN, the second term converges to $n\mathrm{I}(\theta_0)$.

Bernstein-von Mises Theorem

For large n, we have

$$\log p(\theta|y) \approx \log p(\hat{\theta}|y) - \frac{1}{2}(\theta - \hat{\theta})^\top \left[n\mathrm{I}(\theta_0)\right](\theta - \hat{\theta})$$

where $\hat{\theta}$ is the posterior mode.

If $\hat{\theta} \to \theta_0$ as $n \to \infty$, $\mathrm{I}(\hat{\theta}) \to \mathrm{I}(\theta_0)$ as $n \to \infty$ and we have

$$p(\theta|y) \propto \exp\left(-\frac{1}{2}(\theta - \hat{\theta})^\top \left[n\mathrm{I}(\hat{\theta})\right](\theta - \hat{\theta})\right).$$

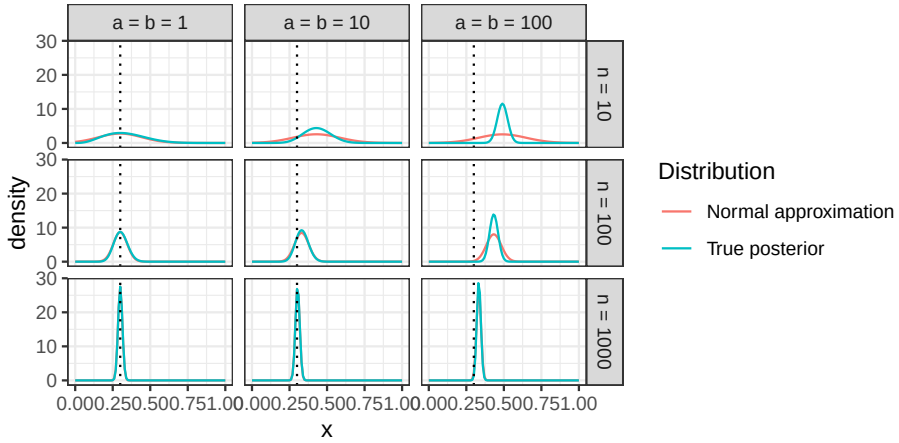
Thus, as $n \to \infty$

$$\theta|y \xrightarrow{d} N\left(\hat{\theta}, \frac{1}{n}\mathrm{I}(\hat{\theta})^{-1}\right)$$

Thus, the posterior distribution is asymptotically normal.

Binomial example

Suppose $y \sim Bin(n, \theta)$ and $\theta \sim Be(a, b)$.



What can go wrong?

- Not unique to Bayesian statistics
  - Unidentified parameters
  - Number of parameters increase with sample size
  - Aliasing
  - Unbounded likelihoods
  - Tails of the distribution
  - True sampling distribution is not $p(y|\theta)$
- Unique to Bayesian statistics
  - Improper posterior
  - Prior distributions that exclude the point of convergence
  - Convergence to the edge of the (prior) parameter space

Truncated priors

Suppose

$$Y \sim Bin(n, \theta)$$

and the true value for θ is

$$\theta_0 = 0.3.$$

Your belief is that there is no way θ is less than 0.5 and thus you assign a truncated beta distribution for a prior, i.e.

$$\theta \sim Be(a, b)\mathrm{I}(\theta > 0.5).$$

The posterior is then

$$\theta|y \sim Be(a + y, b + n - y)\mathrm{I}(\theta > 0.5).$$

The following occurs:

- the posterior will not converge to a neighborhood around θ_0,
- no Bayesian estimators will converge to θ_0, and
- the posterior will not converge to a normal distribution.

True sampling distribution is not $p(y|\theta)$

Suppose that $f(y)$, the true sampling distribution, does not correspond to $p(y|\theta)$ for any $\theta = \theta_0$.

Then the posterior $p(\theta|y)$ converges to a θ_0 that is the smallest in Kullback-Leibler divergence to the true $f(y)$ where

$$KL(f(y)||p(y|\theta)) = E\left[\log\left(\frac{f(y)}{p(y|\theta)}\right)\right] = \int \log\left(\frac{f(y)}{p(y|\theta)}\right) f(y)dy.$$

That is, we do about the best that we can given that we have assumed the wrong sampling distribution $p(y|\theta)$.