R05 - Multiple Regression

STAT 5870 (Engineering)
Iowa State University

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Multiple regression

Recall the simple linear regression model is

$$Y_i \overset{ind}{\sim} N(\mu_i, \sigma^2), \quad \mu_i = \beta_0 + \beta_1 X_i$$

The multiple regression model has mean

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}$$

where for observation i

- Y_i is the response and
- $X_{i,p}$ is the pth explanatory variable.

Explanatory variables

There is a lot of flexibility in the mean

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}$$

as there are many possibilities for the explanatory variables $X_{i,1}, \ldots, X_{i,p}$:

- Functions $(f(X))$
- Dummy variables for categorical variables $(X_1 = \mathrm{I}())$
- Higher order terms (X^2)
- Additional explanatory variables (X_1, X_2)
- Interactions $(X_1 X_2)$
    - Continuous-continuous
    - Continuous-categorical
    - Categorical-categorical

Parameter interpretation

Model:

$$Y_i \overset{ind}{\sim} N(\beta_0 + \beta_1 X_{i,1} + \cdots + \beta_p X_{i,p}, \sigma^2)$$

The interpretation is

- β_0 is the expected value of the response Y_i when all explanatory variables are zero.
- β_p, $p \neq 0$ is the expected increase in the response for a one-unit increase in the pth explanatory variable when all other explanatory variables are held constant.
- R^2 is the proportion of the variability in the response explained by the model

Parameter estimation and inference

Let

$$y = X\beta + \epsilon$$

where

- $y = (y_1, \ldots, y_n)^\top$
- X is $n \times p$ with ith row $X_i = (1, X_{i,1}, \ldots, X_{i,p})$
- $\beta = (\beta_0, \beta_1, \ldots, \beta_p)^\top$
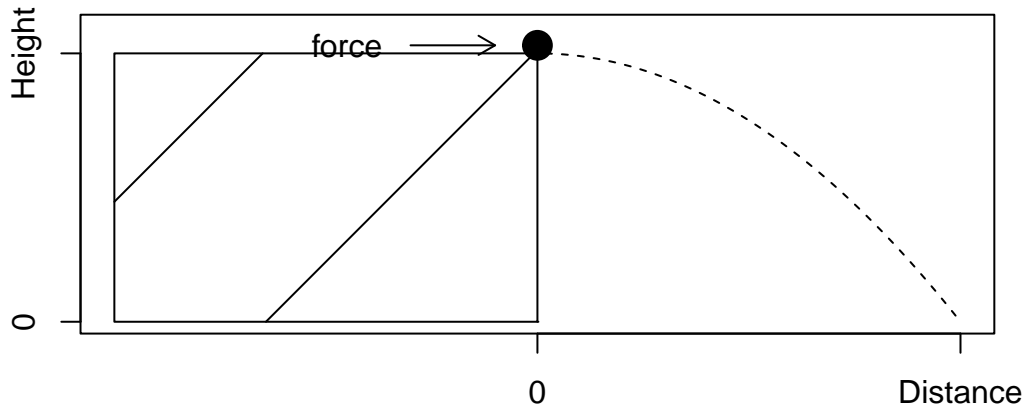- $\epsilon = (\epsilon_1, \ldots, \epsilon_n)^\top$

Then we have

$$
\begin{aligned}
\hat{\beta} &= (X^\top X)^{-1} X^\top y \\
Var(\hat{\beta}) &= \sigma^2 (X^\top X)^{-1} \\
r &= y - X\hat{\beta} \\
\hat{\sigma}^2 &= \frac{1}{n-(p+1)} r^\top r
\end{aligned}
$$

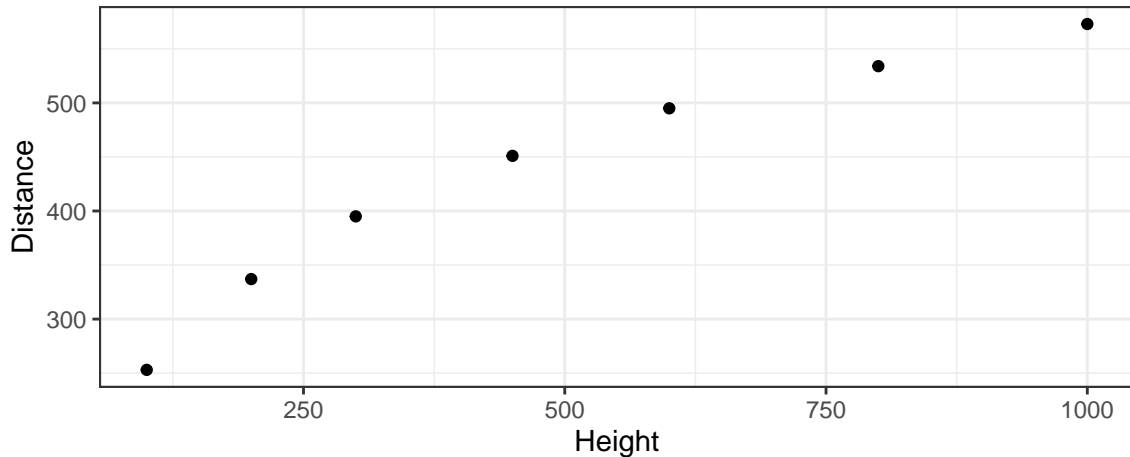Confidence/credible intervals and (two-sided) p-values are constructed using

$$\hat{\beta}_j \pm t_{n-(p+1),1-a/2} SE(\hat{\beta}_j) \quad \text{and} \quad \text{pvalue} = 2P\left(T_{n-(p+1)} > \left| \frac{\hat{\beta}_j - b_j}{SE(\hat{\beta}_j)} \right|\right)$$

where $T_{n-(p+1)} \sim t_{n-(p+1)}$ and $SE(\hat{\beta}_j)$ is the jth diagonal element of $\hat{\sigma}^2 (X^\top X)^{-1}$.

Galileo experiment

Galileo data (`Sleuth3::case1001`)

Higher order terms (X^2)

Let

- Y_i be the distance for the ith run of the experiment and
- H_i be the height for the ith run of the experiment.

Simple linear regression assumes

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 H_i \qquad\qquad , \sigma^2)$$

The quadratic multiple regression assumes

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 H_i + \beta_2 H_i^2 \qquad , \sigma^2)$$

The cubic multiple regression assumes

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 H_i + \beta_2 H_i^2 + \beta_3 H_i^3, \sigma^2)$$

R code and output

```
# Construct the variables by hand
m1 = lm(Distance ~ Height,                              case1001)
m2 = lm(Distance ~ Height + I(Height^2),                case1001)
m3 = lm(Distance ~ Height + I(Height^2) + I(Height^3), case1001)

coefficients(m1)

(Intercept)      Height
 269.712458    0.333337

coefficients(m2)

  (Intercept)        Height    I(Height^2)
 1.999128e+02  7.083225e-01  -3.436937e-04

coefficients(m3)

  (Intercept)        Height    I(Height^2)    I(Height^3)
 1.557755e+02  1.115298e+00  -1.244943e-03   5.477104e-07
```
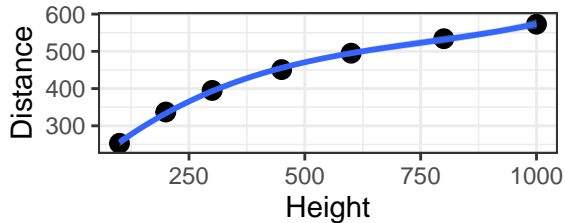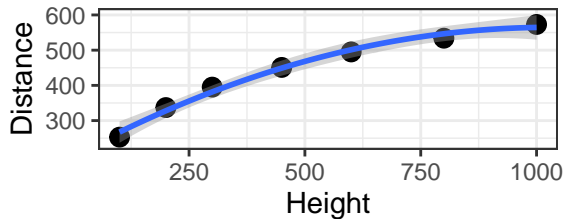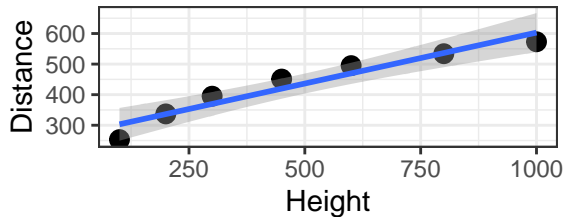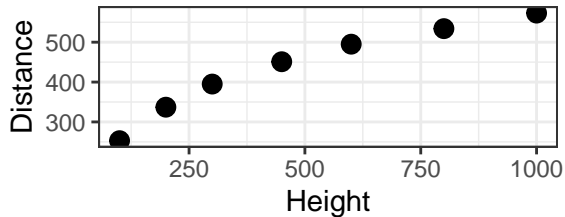
Galileo experiment (Sleuth3::case1001)

Longnose Dace Abundance

From http://udel.edu/~mcdonald/statmultreg.html:

I extracted some data from the Maryland Biological Stream Survey. ... The [response] variable is the number of Longnose Dace ... per 75-meter section of [a] stream. The [explanatory] variables are ... the maximum depth (in cm) of the 75-meter segment of stream; nitrate concentration (mg/liter) ....
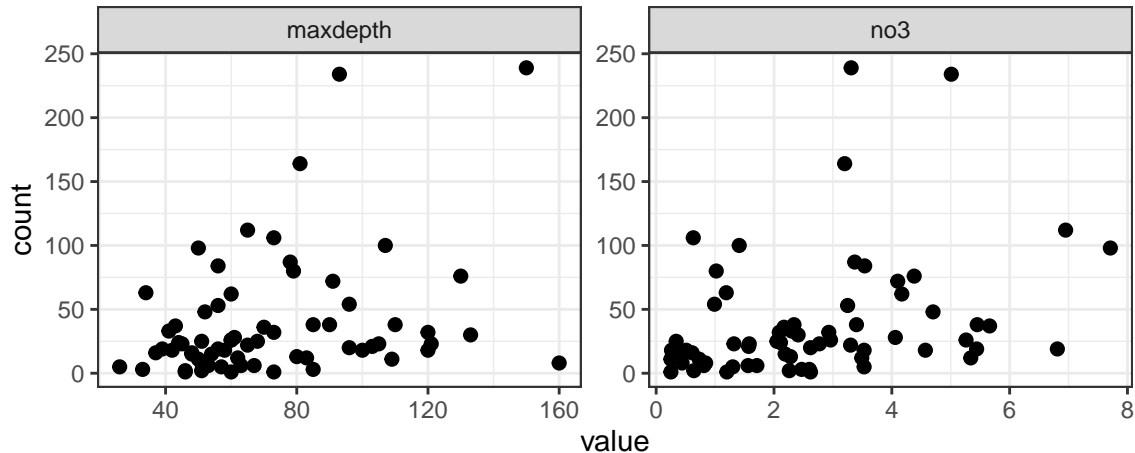
Consider the model

$$Y_i \stackrel{ind}{\sim} N(\beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2}, \sigma^2)$$

where

- Y_i: count of Longnose Dace in stream i
- $X_{i,1}$: maximum depth (in cm) of stream i
- $X_{i,2}$: nitrate concentration (mg/liter) of stream i

Exploratory

R code and output

```
m <- lm(count ~ maxdepth + no3, longnosedace)
summary(m)

Call:
lm(formula = count ~ maxdepth + no3, data = longnosedace)

Residuals:
    Min      1Q  Median      3Q     Max
-55.060 -27.704  -8.679  11.794 165.310

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -17.5550    15.9586  -1.100  0.27544
maxdepth      0.4811     0.1811   2.656  0.00997 **
no3           8.2847     2.9566   2.802  0.00671 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 43.39 on 64 degrees of freedom
Multiple R-squared:  0.1936,Adjusted R-squared:  0.1684
F-statistic: 7.682 on 2 and 64 DF,  p-value: 0.001022
```

Interpretation

- Intercept (β_0): The expected count of Longnose Dace when maximum depth and nitrate concentration are both zero is -18.

- Coefficient for maxdepth (β_1): Holding nitrate concentration constant, each cm increase in maximum depth is associated with an additional 0.48 Longnose Dace counted on average.

- Coefficient for no3 (β_2): Holding maximum depth constant, each mg/liter increase in nitrate concentration is associated with an addition 8.3 Longnose Dace counted on average.

- Coefficient of determination (R^2): The model explains 19% of the variability in the count of Longnose Dace.

Interactions

Why an interaction?

Two explanatory variables are said to interact if the effect that one of them has on the mean response depends on the value of the other.

For example,

- Longnose dace count: The effect of nitrate (no3) on longnose dace count depends on the maxdepth. (Continuous-continuous)

- Energy expenditure: The effect of mass depends on the species type. (Continuous-categorical)

- Crop yield: the effect of tillage method depends on the fertilizer brand (Categorical-categorical)

Continuous-continuous interaction

For observation i, let

- Y_i be the response
- $X_{i,1}$ be the first explanatory variable and
- $X_{i,2}$ be the second explanatory variable.

The mean containing only main effects is

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2}.$$

The mean with the interaction is

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2} + \beta_3 X_{i,1} X_{i,2}.$$

Intepretation - main effects only

Let $X_{i,1} = x_1$ and $X_{i,2} = x_2$, then we can rewrite the line (μ) as

$$\mu = (\beta_0 + \beta_2 x_2) + \beta_1 x_1$$

which indicates that the intercept of the line for x_1 depends on the value of x_2.

Similarly,

$$\mu = (\beta_0 + \beta_1 x_1) + \beta_2 x_2$$

which indicates that the intercept of the line for x_2 depends on the value of x_1.

Intepretation - with an interaction

Let $X_{i,1} = x_1$ and $X_{i,2} = x_2$, then we can rewrite the mean (μ) as

$$\mu = (\beta_0 + \beta_2 x_2) + (\beta_1 + \beta_3 x_2)x_1$$

which indicates that both the intercept and slope for x_1 depend on the value of x_2.

Similarly,

$$\mu = (\beta_0 + \beta_1 x_1) + (\beta_2 + \beta_3 x_1)x_2$$

which indicates that both the intercept and slope for x_2 depend on the value of x_1.

R code and output - main effects only

```
Call:
lm(formula = count ~ no3 + maxdepth, data = longnosedace)

Residuals:
    Min      1Q  Median      3Q     Max
-55.060 -27.704  -8.679  11.794 165.310

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -17.5550    15.9586  -1.100  0.27544
no3           8.2847     2.9566   2.802  0.00671 **
maxdepth      0.4811     0.1811   2.656  0.00997 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 43.39 on 64 degrees of freedom
Multiple R-squared:  0.1936,Adjusted R-squared:  0.1684
F-statistic: 7.682 on 2 and 64 DF,  p-value: 0.001022
```

R code and output - with an interaction

```
Call:
lm(formula = count ~ no3 * maxdepth, data = longnosedace)

Residuals:
    Min     1Q  Median     3Q     Max
-65.111 -21.399  -9.562   5.953 151.071

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  13.321043  23.455710   0.568   0.5721
no3          -4.646272   7.856932  -0.591   0.5564
maxdepth     -0.009338   0.329180  -0.028   0.9775
no3:maxdepth  0.201219   0.113576   1.772   0.0813 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 42.68 on 63 degrees of freedom
Multiple R-squared:  0.2319,Adjusted R-squared:  0.1953
F-statistic: 6.339 on 3 and 63 DF,  p-value: 0.0007966
```
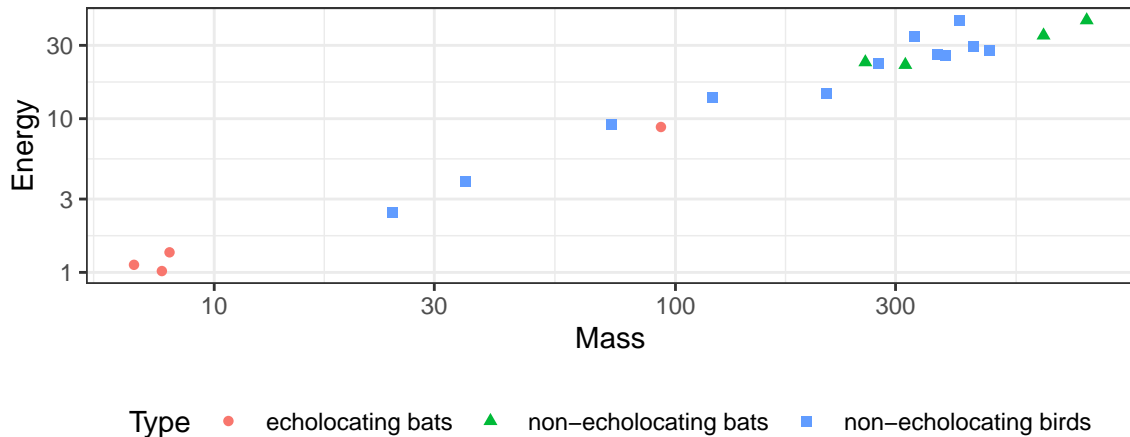
Visualizing the model
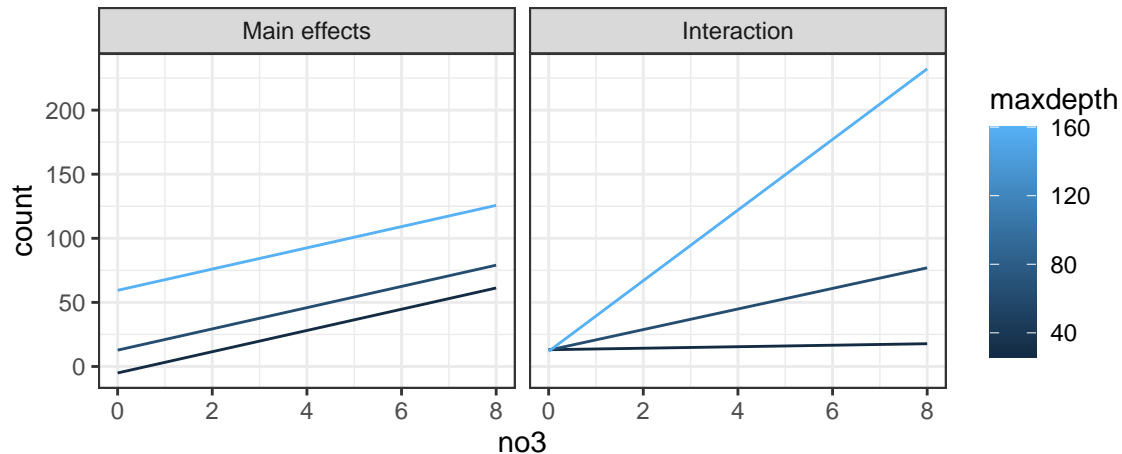
In-flight energy expenditure (Sleuth3::case1002)

Continuous-categorical interaction

Let category A be the reference level. For observation i, let

- Y_i be the response
- $X_{i,1}$ be the continuous explanatory variable,
- B_i be a dummy variable for category B, and
- C_i be a dummy variable for category C.

The mean containing only main effects is

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 B_i + \beta_3 C_i.$$

The mean with the interaction is

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 B_i + \beta_3 C_i + \beta_4 X_{i,1} B_i + \beta_5 X_{i,1} C_i.$$

Interpretation for the main effect model

The mean containing only main effects is

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 B_i + \beta_3 C_i.$$

For each category, the line is

Category	Line (μ)		
A	β_0	$+$	$\beta_1 X$
B	$(\beta_0 + \beta_2)$	$+$	$\beta_1 X$
C	$(\beta_0 + \beta_3)$	$+$	$\beta_1 X$

Each category has a different intercept, but a common slope.

Interpretation for the model with an interaction

The model with an interaction is

$$\mu_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 B_i + \beta_3 C_i + \beta_4 X_{i,1} B_i + \beta_5 X_{i,1} C_i$$

For each category, the line is

Category	Line (μ)		
A	β_0	$+\ \beta_1$	X
B	$(\beta_0 + \beta_2)$	$+(\beta_1 + \beta_4)X$	
C	$(\beta_0 + \beta_3)$	$+(\beta_1 + \beta_5)X$	

Each category has its own intercept and its own slope.

R code and output - main effects only

```
summary(mM <- lm(log(Energy) ~ log(Mass) + Type, case1002))

Call:
lm(formula = log(Energy) ~ log(Mass) + Type, data = case1002)

Residuals:
     Min      1Q   Median      3Q     Max
-0.23224 -0.12199 -0.03637  0.12574  0.34457

Coefficients:
                          Estimate Std. Error t value Pr(>|t|)
(Intercept)               -1.49770    0.14987  -9.993 2.77e-08 ***
log(Mass)                  0.81496    0.04454  18.297 3.76e-12 ***
Typenon-echolocating bats -0.07866    0.20268  -0.388    0.703
Typenon-echolocating birds 0.02360    0.15760   0.150    0.883
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.186 on 16 degrees of freedom
Multiple R-squared:  0.9815,Adjusted R-squared:  0.9781
F-statistic: 283.6 on 3 and 16 DF,  p-value: 4.464e-14
```

R code and output - with an interaction

```
summary(mI <- lm(log(Energy) ~ log(Mass) * Type, case1002))

Call:
lm(formula = log(Energy) ~ log(Mass) * Type, data = case1002)

Residuals:
     Min      1Q   Median      3Q      Max
-0.25152 -0.12643 -0.00954  0.08124  0.32840

Coefficients:
                                  Estimate Std. Error t value Pr(>|t|)
(Intercept)                       -1.47052    0.24767  -5.937 3.63e-05 ***
log(Mass)                          0.80466    0.08668   9.283 2.33e-07 ***
Typenon-echolocating bats          1.26807    1.28542   0.987    0.341
Typenon-echolocating birds        -0.11032    0.38474  -0.287    0.779
log(Mass):Typenon-echolocating bats  -0.21487  0.22362  -0.961    0.353
log(Mass):Typenon-echolocating birds  0.03071  0.10283   0.299    0.770
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1899 on 14 degrees of freedom
Multiple R-squared:  0.9832,Adjusted R-squared:  0.9771
F-statistic: 163.4 on 5 and 14 DF,  p-value: 6.696e-12
```
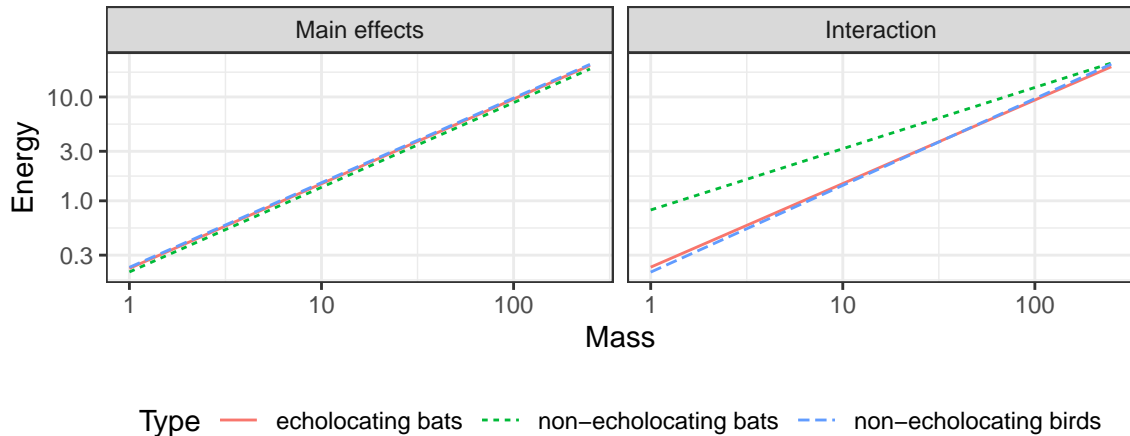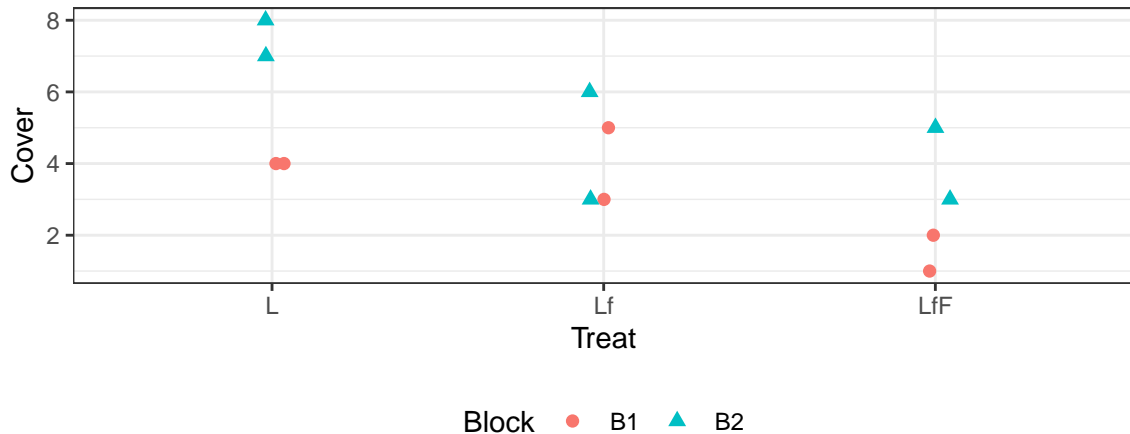
Visualizing the models

Seaweed regeneration (Sleuth3::case1301 subset)

Categorical-categorical

Let category A and type 0 be the reference level. For observation i, let

- Y_i be the response,
- 1_i be a dummy variable for type 1,
- B_i be a dummy variable for category B, and
- C_i be a dummy variable for category C.

The mean containing only main effects is

$$\mu_i = \beta_0 + \beta_1 1_i + \beta_2 B_i + \beta_3 C_i.$$

The mean with an interaction is

$$\mu_i = \beta_0 + \beta_1 1_i + \beta_2 B_i + \beta_3 C_i + \beta_4 1_i B_i + \beta_5 1_i C_i.$$

Interpretation for the main effects model

The mean containing only main effects is

$$\mu_i = \beta_0 + \beta_1 1_i + \beta_2 B_i + \beta_3 C_i.$$

The means in the main effect model are

Type	Category		
	A	B	C
0	β_0	$\beta_0 + \beta_2$	$\beta_0 + \beta_3$
1	$\beta_0 + \beta_1$	$\beta_0 + \beta_1 + \beta_2$	$\beta_0 + \beta_1 + \beta_3$

Interpretation for the model with an interaction

The mean with an interaction is

$$\mu_i = \beta_0 + \beta_1 1_i + \beta_2 B_i + \beta_3 C_i + \beta_4 1_i B_i + \beta_5 1_i C_i.$$

The means are

| | Category | | |
Type	A	B	C
0	β_0	$\beta_0 + \beta_2$	$\beta_0 + \beta_3$
1	$\beta_0 + \beta_1$	$\beta_0 + \beta_1 + \beta_2 + \beta_4$	$\beta_0 + \beta_1 + \beta_3 + \beta_5$

This is equivalent to a cell-means model where each combination has its own mean.

R code and output - main effects only

```
Call:
lm(formula = Cover ~ Block + Treat, data = case1301_subset)

Residuals:
    Min      1Q  Median      3Q     Max
-2.3333 -0.6667  0.0000  0.7917  1.8333

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   4.6667     0.7683   6.074 0.000298 ***
BlockB2       2.1667     0.7683   2.820 0.022491 *
TreatLf      -1.5000     0.9410  -1.594 0.149578
TreatLfF     -3.0000     0.9410  -3.188 0.012838 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.331 on 8 degrees of freedom
Multiple R-squared:  0.6937,Adjusted R-squared:  0.5788
F-statistic: 6.039 on 3 and 8 DF,  p-value: 0.01881
```

R code and output - with an interaction

```
Call:
lm(formula = Cover ~ Block * Treat, data = case1301_subset)

Residuals:
   Min     1Q Median     3Q    Max
-1.500 -0.625  0.000  0.625  1.500

Coefficients:
                   Estimate Std. Error t value Pr(>|t|)
(Intercept)       4.000e+00  8.898e-01   4.496  0.00412 **
BlockB2           3.500e+00  1.258e+00   2.782  0.03193 *
TreatLf          -4.441e-16  1.258e+00   0.000  1.00000
TreatLfF         -2.500e+00  1.258e+00  -1.987  0.09413 .
BlockB2:TreatLf  -3.000e+00  1.780e+00  -1.686  0.14280
BlockB2:TreatLfF -1.000e+00  1.780e+00  -0.562  0.59450
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.258 on 6 degrees of freedom
Multiple R-squared:  0.7946,Adjusted R-squared:  0.6234
F-statistic: 4.642 on 5 and 6 DF,  p-value: 0.04429
```

Visualizing the models
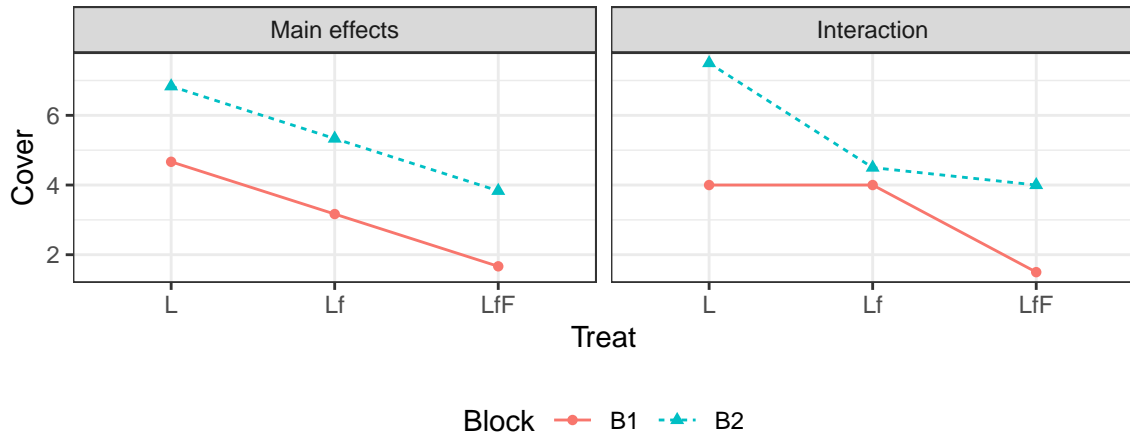
When to include interaction terms

From The Statistical Sleuth (3rd ed) page 250:

- when a question of interest pertains to an interaction
- when good reason exists to suspect an interaction or
- when interactions are proposed as a more general model for the purpose of examining the goodness of fit of a model without interaction.

Multiple regression explanatory variables

The possibilities for explanatory variables are

- Higher order terms (X^2)
- Additional explanatory variables (X_1 and X_2)
- Dummy variables for categorical variables ($X_1 = \text{I}()$)
- Interactions ($X_1 X_2$)
  - Continuous-continuous
  - Continuous-categorical
  - Categorical-categorical

We can also combine these explanatory variables, e.g.

- including higher order terms for continuous variables along with dummy variables for categorical variables and
- including higher order interactions ($X_1 X_2 X_3$).