Parameter estimation

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STAT 544 - Iowa State University

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Outline

- Parameter estimation
  - Beta-binomial example
  - Point estimation
  - Interval estimation
  - Simulation from the posterior
- Priors
  - Subjective
  - Conjugate
  - Default
  - Improper

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The kernel of a probability density (mass) function is the form of the pdf (pmf) with any terms not involving the random variable omitted.

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These interpretations may aid in construction of this prior for a given application.

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Thus, the posterior mean is a weighted average of the prior mean $a/(a+b)$ and the MLE y/n with weights equal to the prior sample size $(a+b)$ and the data sample size (n).

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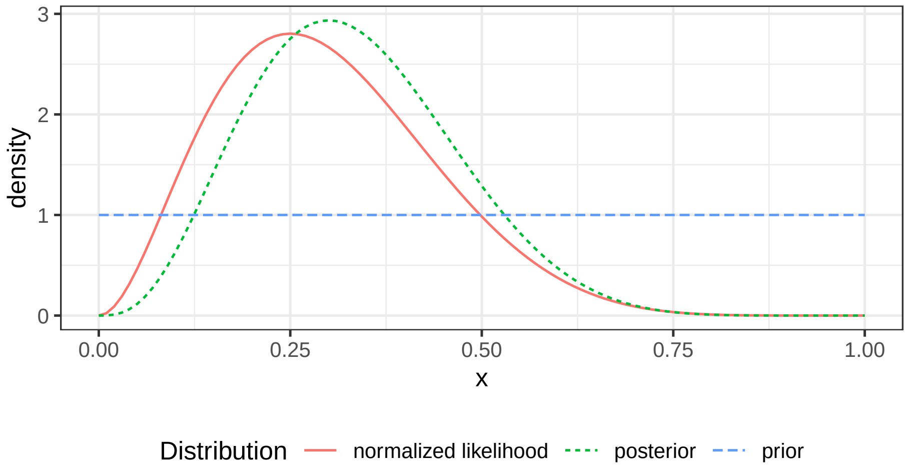
Posterior distribution



Try it yourself at `https://jaradniemi.shinyapps.io/one_parameter_conjugate/`.

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A $100(1-a)\%$ credible interval is any interval in the posterior that contains the parameter with probability $(1-a)$.

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Define a loss (or utility) function $L\left(\theta,\hat{\theta}\right) = -U\left(\theta,\hat{\theta}\right)$ where

- θ is the parameter of interest
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Find the estimator that minimizes the expected loss:

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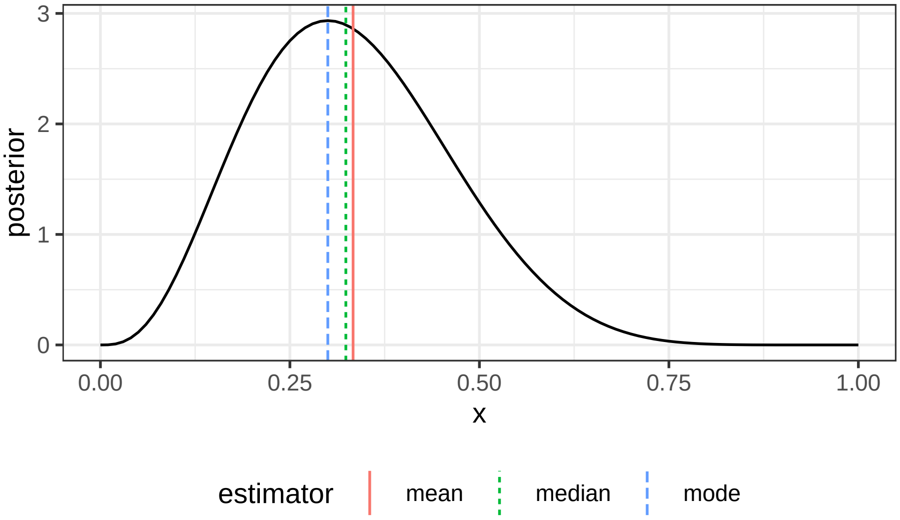
So $\hat{\theta} = E[\theta|y]$ minimizes expected squared-error loss. $\square$

Point estimation



Interval estimation

Definition

A $100(1-a)\%$ credible interval is any interval (L,U) such that

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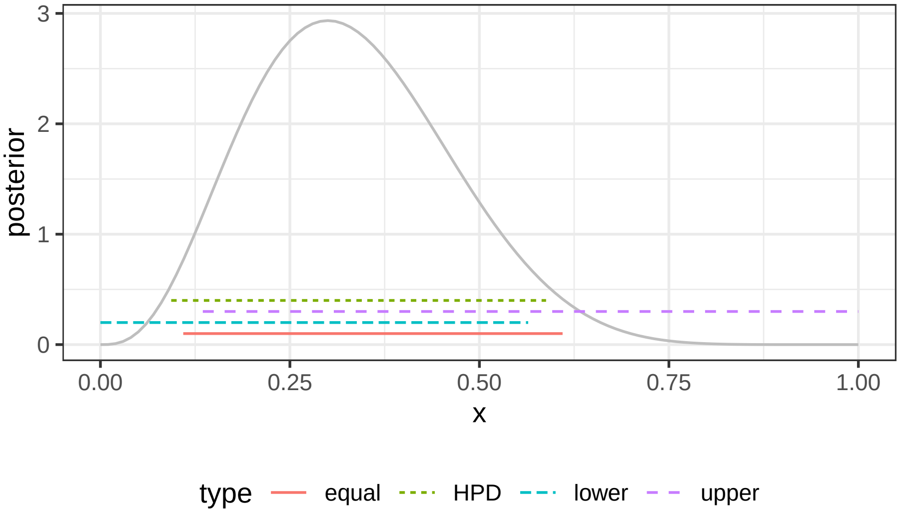
- One-sided: either $L = -\infty$ or $U = \infty$
- Highest posterior density (HPD): $p(L|y) = p(U|y)$ for a uni-modal posterior which is also the shortest interval

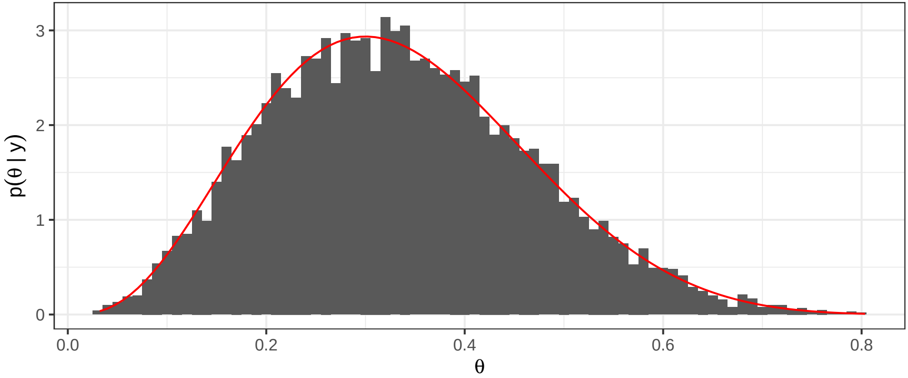
Interval estimation



Simulation from the posterior

An estimate of the full posterior can be obtained via simulation, i.e.

```
sim = data.frame(x = rbeta(10000, shape1 = a + y, shape2 = b + n - y))
```



Estimates via simulation

We can also obtain point and interval estimates using these simulations

```
round(c(mean = mean(sim$x), median = median(sim$x)),2)

  mean median
  0.34   0.33

round(quantile(sim$x, c(.025,.975)),2) # Equal-tail

 2.5% 97.5%
 0.11  0.61

round(c(quantile(sim$x, .05),1),2) # Upper

  5%
0.13 1.00

round(c(0,quantile(sim$x, .95)),2) # Lower

      95%
0.00 0.57
```

Guess the probability

What do you think the probability is?

- A 6-sided die lands on 1.

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What do you think the probability is?

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- Kansas City Chiefs win 2023 Super Bowl.

What are priors?

Definition

A prior probability distribution, often called simply the prior, of an uncertain quantity θ is the probability distribution that would express one's uncertainty about θ before the "data" is taken into account.

http://en.wikipedia.org/wiki/Prior_distribution

Priors

Definition

A prior $p(\theta)$ is conjugate if for $p(\theta) \in \mathcal{P}$ and $p(y|\theta) \in \mathcal{F}$, $p(\theta|y) \in \mathcal{P}$ where $\mathcal{F}$ and $\mathcal{P}$ are families of distributions.

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Definition

A natural conjugate prior is a conjugate prior that has the same functional form as the likelihood.

For example, the beta distribution is a natural conjugate prior since

$$p(\theta) \propto \theta^{a-1}(1-\theta)^{b-1} \qquad \text{and} \qquad L(\theta) \propto \theta^{y}(1-\theta)^{n-y}.$$

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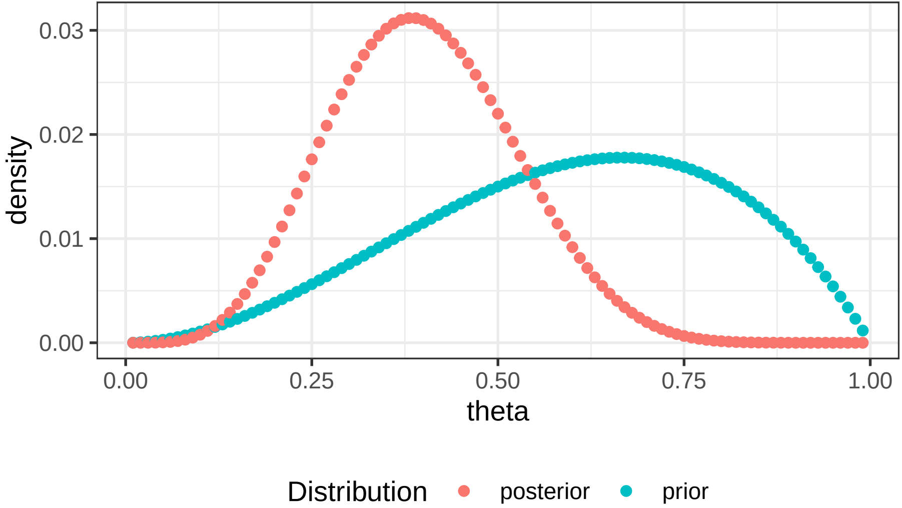
$$p_i' = \frac{p_i p(y|\theta_i)}{\sum_{j=1}^{\mathrm{I}} p_j p(y|\theta_j)} \propto p_i p(y|\theta_i).$$

□

Discrete prior



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Let $p_i = P(H_i)$ and $p_i(\theta) = p(\theta|H_i)$,

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where $p_i(\theta|y) = p(y|\theta)p_i(\theta)/p_i(y)$.

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which is called the beta-binomial distribution with parameters $a+y$ and $b+n-y$.

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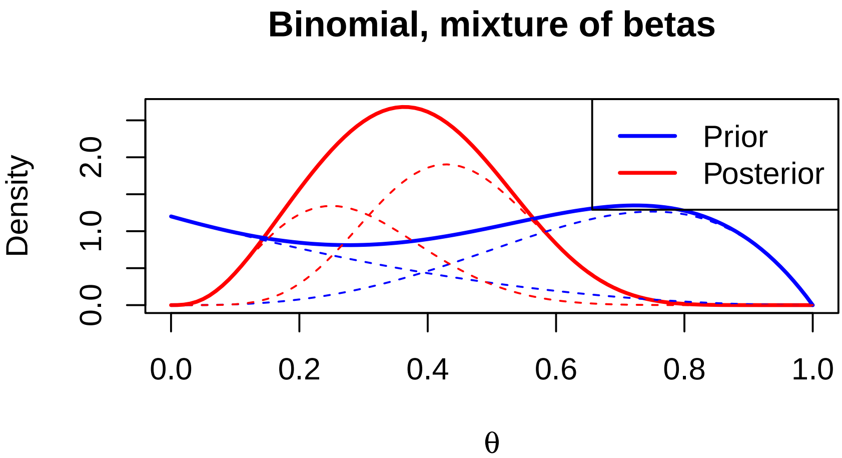

Default priors

Definition

A default prior is used when a data analyst is unable or unwilling to specify an informative prior distribution.

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a Be(0,0), if that were a proper distribution, and is different from setting $p(\theta) \propto 1$ which results in the Be(1,1) prior. Thus, the constant prior is not invariant to the parameterization used.

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For exponential families,

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If $\theta = (\theta_1, \ldots, \theta_n)$, then the Fisher information is the expectation of the Hessian matrix, which has the ith row and jth column that is the partial derivative with respect to θ_i followed by the partial derivative with respect to θ_j, of the log-likelihood.

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Theorem

The Fisher information for $Y \sim Bin(n,\theta)$ *is* $\mathcal{I}(\theta) = \frac{n}{\theta(1-\theta)}$.

Proof.

Since the binomial is an exponential family,

$$\mathcal{I}(\theta) \quad = -E_{y|\theta}\left[\frac{\partial^2}{\partial\theta^2}\log p(y|\theta)\right]$$

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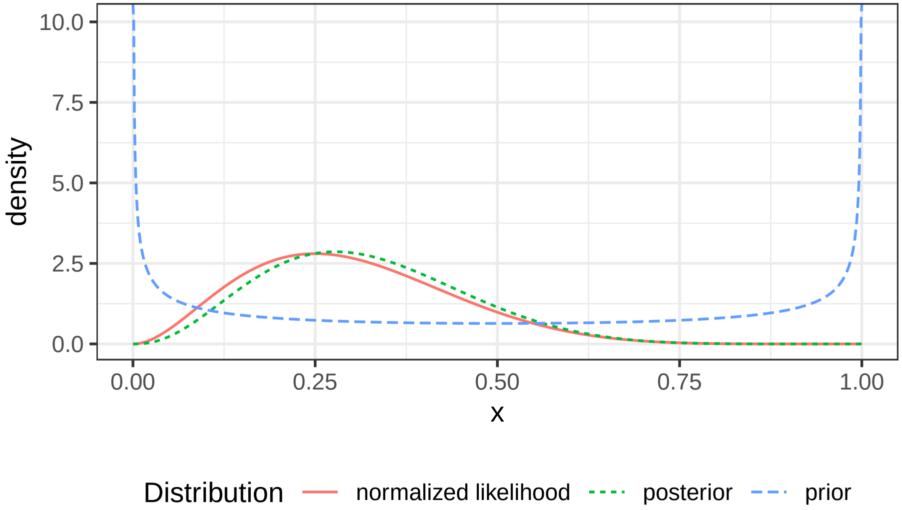
&= \frac{n}{\theta(1-\theta)}
\end{aligned}
$$

$\square$


10.0
7.5
5.0
2.5
0.0
density
0.00
0.25
0.50
0.75
1.00
x
Distribution
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Non-conjugate priors

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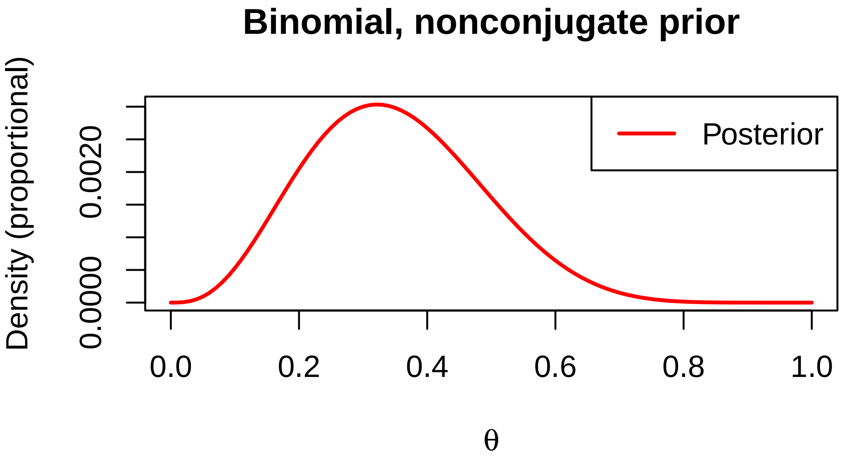
- Evaluate $f(\theta)$ on a grid and normalize by the grid spacing.

Plot of $f(\theta)$



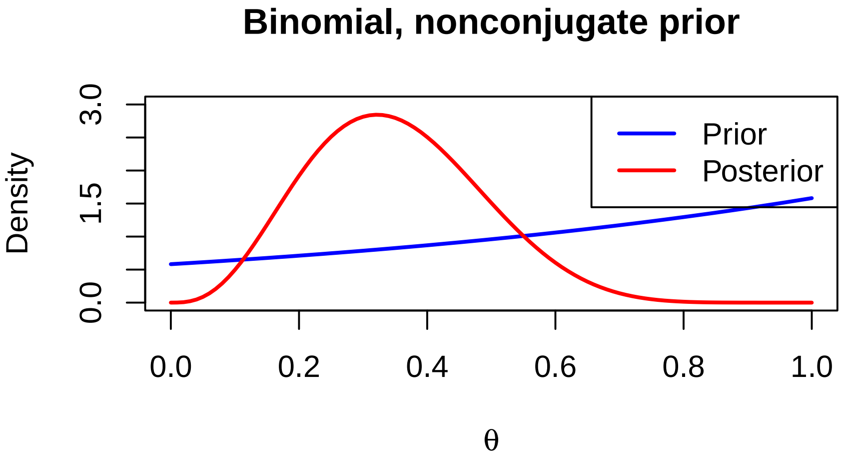
Numerical integration

Find $i = \int f(\theta)d\theta$, so that $p(\theta|y) = f(\theta)/i$.

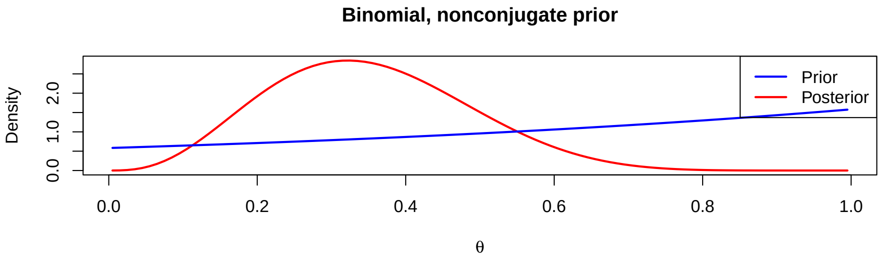
```
(i = integrate(f, 0, 1))

0.001066499 with absolute error < 1.2e-17
```

Nonconjugate prior, numerical integration



Nonconjugate prior, evaluated on a grid




```
theta[c(which(cumsum(d)*w>0.025)[1]-1, which(cumsum(d)*w>0.975)[1])] # 95\% CI

[1] 0.105 0.625
```

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Be(0,0) prior

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The posterior, $\theta|y \sim Be(y, n-y)$, is proper if $0 < y < n$.