I09 - Comparing means

STAT 587 (Engineering)
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March 30, 2021

One mean

Consider the model $Y_i \stackrel{ind}{\sim} N(\mu, \sigma^2)$. We have discussed a number of statistical procedures to draw inferences about μ:

- Frequentist: based on distribution of $\frac{\overline{Y}-\mu}{s/\sqrt{n}}$
  - p-value for a hypothesis test, e.g. $H_0 : \mu = \mu_0$,
  - confidence interval for μ,
- Bayesian: based on posterior for μ
  - credible interval for μ,
  - posterior model probability, e.g. $p(H_0|y)$, and
  - posterior probabilities, e.g. $P(\mu < \mu_0|y)$.

Now, we will consider what happens when you have multiple μs.

Two means

Consider the model

$$Y_{g,i} \stackrel{ind}{\sim} N(\mu_g, \sigma_g^2)$$

for $g = 1, 2$ and $i = 1, \ldots, n_g$. and you are interested in the relationship between μ_1 and μ_2.

- Frequentist: based on distribution of

$$\frac{\overline{Y}_1 - \overline{Y}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

    - p-value for a hypothesis test, e.g. $H_0 : \mu_1 = \mu_2$,
    - confidence interval for $\mu_1 - \mu_2$,

- Bayesian: posterior for μ_1, μ_2, i.e. $p(\mu_1, \mu_2|y)$
    - credible interval for $\mu_1 - \mu_2$,
    - posterior model probability, e.g. $p(H_0|y)$, and
    - probability statements, e.g. $P(\mu_1 < \mu_2|y)$.

where $y = (y_{1,1}, \ldots, y_{1,n_1}, y_{2,1}, \ldots, y_{2,n_2})$.

Data example

Suppose you have two manufacturing processes to produce sensors and you are interested in the average sensitivity of the sensors.

So you run the two processes and record the sensitivity of each sensor in units of mV/V/mm Hg (`http://www.ni.com/white-paper/14860/en/`).
And you have the following summary statistics:

```
# A tibble: 2 x 4
  process      n  mean    sd
  <chr>    <int> <dbl> <dbl>
1 P1          22  7.74  1.87
2 P2          34  9.24  2.26
```

p-values and confidence intervals

Because there is no indication that you have any expectation regarding the sensitivities of process 1 compared to process 2, we will conduct a two-sided two-sample t-test assuming the variances are not equal, i.e.

$$Y_{g,i} \stackrel{ind}{\sim} N(\mu_g, \sigma_g^2)$$

and

$$H_0 : \mu_1 = \mu_2 \quad \text{and} \quad H_A : \mu_1 \neq \mu_2$$

```
t.test(sensitivity ~ process, data = d2)

Welch Two Sample t-test

data:  sensitivity by process
t = -2.6932, df = 50.649, p-value = 0.009571
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -2.610398 -0.380530
sample estimates:
mean in group P1 mean in group P2
        7.743761         9.239224
```

Posterior for μ_1, μ_2

Assume

$$Y_{g,i} \stackrel{ind}{\sim} N(\mu_g, \sigma_g^2) \quad \text{and} \quad p(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2) \propto \frac{1}{\sigma_1^2}\frac{1}{\sigma_2^2}.$$

Then

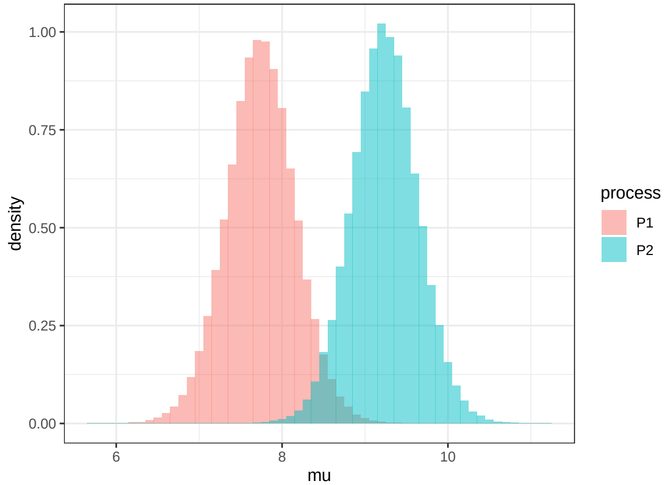
$$\mu_g|y \stackrel{ind}{\sim} t_{n_g-1}(\overline{y}_g, s_g^2/n_g)$$

and a draw for μ_g can be obtained by taking

$$\overline{y}_g + T_{n_g-1} s_g/\sqrt{n_g}, \quad T_{n_g-1} \stackrel{ind}{\sim} t_{n_g-1}(0,1).$$

Simulations:

We can use these draws to compare the posteriors



Credible interval for the difference

To obtain statistical inference on the difference, we use the samples and take the difference

```
d3 <- sims %>%
  spread(process, mu) %>%
  mutate(diff = P1-P2)

# Bayes estimate for the difference
mean(d3$diff)

[1] -1.493267

# Estimated 95% equal-tail credible interval
quantile(d3$diff, c(.025,.975))

      2.5%      97.5%
-2.6339752 -0.3483025

# Estimate of the probability that mu1 is larger than mu2
mean(d3$diff > 0)

[1] 0.00591
```

Three or more means

Now, let's consider the more general problem of

$$Y_{g,i} \stackrel{ind}{\sim} N(\mu_g, \sigma_g^2)$$

for $g = 1, 2, \ldots, G$ and $i = 1, \ldots, n_g$ and you are interested in the relationship amongst the μ_g.

We can perform the following statistical procedures:

- Frequentist:
  - p-value for test of $H_0 : \mu_g = \mu$ for all g,
  - confidence interval for $\mu_g - \mu_{g'}$,
- Bayesian: based on posterior for $\mu_1, \ldots, \mu_G$
  - credible interval for $\mu_g - \mu_{g'}$,
  - posterior model probability, e.g. $p(H_0|y)$, and
  - probability statements, e.g. $P(\mu_g < \mu_{g'}|y)$

  where g and g' are two different groups.

Data example

Suppose you have three manufacturing processes to produce sensors and you are interested in the average sensitivity of the sensors.

So you run the three processes and record the sensitivity of each sensor in units of mV/V/mm Hg (`http://www.ni.com/white-paper/14860/en/`). And you have the following summary statistics:

```
# A tibble: 3 x 4
  process      n  mean    sd
  <chr>    <int> <dbl> <dbl>
1 P1          22  7.74  1.87
2 P2          34  9.24  2.26
3 P3           7 10.8   1.96
```

p-values

When there are lots of means, the first null hypothesis is typically

$$H_0 : \mu_g = \mu \,\forall\, g$$

```
oneway.test(sensitivity ~ process, data = d)

One-way analysis of means (not assuming equal variances)

data:  sensitivity and process
F = 7.6287, num df = 2.000, denom df = 17.418, p-value = 0.004174
```

Pairwise differences

Then we typically look at pairwise differences:

```
pairwise.t.test(d$sensitivity,
                d$process,
                pool.sd = FALSE,
                p.adjust.method = "none")

Pairwise comparisons using t tests with non-pooled SD

data:  d$sensitivity and d$process

   P1     P2
P2 0.0096 -
P3 0.0045 0.0870

P value adjustment method: none
```

Posteriors for μ

When

$$Y_{g,i} \stackrel{ind}{\sim} N(\mu_g, \sigma_g^2),$$

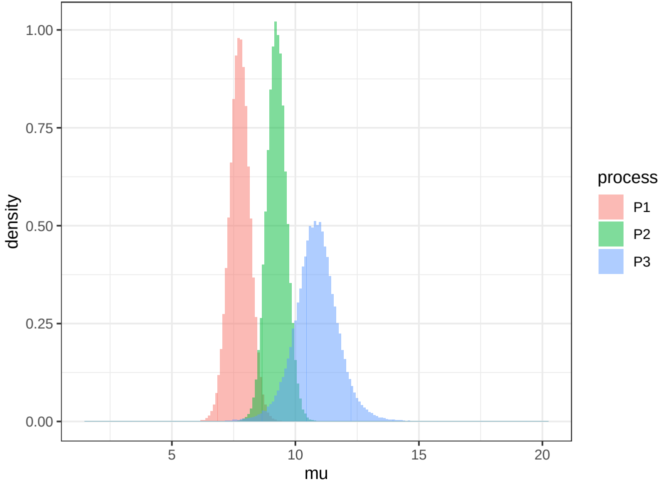
we have

$$\mu_g | y \stackrel{ind}{\sim} t_{n_g-1}(\overline{y}_g, s_g^2/n_g)$$

and that a draw for μ_g can be obtained by taking

$$\overline{y}_g + T_{n_g-1} s_g/\sqrt{n_g}, \quad T_{n_g-1} \stackrel{ind}{\sim} t_{n_g-1}(0,1).$$

Compare posteriors



Credible intervals for differences

Use the simulations to calculate posterior probabilities and credible intervals for differences.

```
# Estimate of the probability that one mean is larger than another
sims %>%
  spread(process, mu) %>%
  mutate(`mu1-mu2` = P1-P2,
          `mu1-mu3` = P1-P3,
          `mu2-mu3` = P2-P3) %>%
  select(`mu1-mu2`,`mu1-mu3`,`mu2-mu3`) %>%
  gather(comparison, diff) %>%
  group_by(comparison) %>%
  summarize(probability = mean(diff>0) %>% round(4),
            lower = quantile(diff, .025) %>% round(2),
            upper = quantile(diff, .975) %>% round(2)) %>%
  mutate(credible_interval = paste("(",lower,",",upper,")", sep="")) %>%
  select(comparison, probability, credible_interval)

# A tibble: 3 x 3
  comparison probability credible_interval
  <chr>            <dbl> <chr>
1 mu1-mu2         0.0059 (-2.63,-0.35)
2 mu1-mu3         0.0037 (-5.06,-1.11)
3 mu2-mu3         0.0493 (-3.56,0.37)
```

Common variance model

In the model

$$Y_{g,i} \stackrel{ind}{\sim} N(\mu_g, \sigma_g^2)$$

we can calculate a p-value for the following null hypothesis:

$$H_0 : \sigma_g = \sigma \quad \text{for all} \quad g$$

```
bartlett.test(sensitivity ~ process, data = d)


Bartlett test of homogeneity of variances

data:  sensitivity by process
Bartlett's K-squared = 0.90949, df = 2, p-value = 0.6346
```

This may give us reason to proceed as if the variances is the same in all groups, i.e.

$$Y_{g,i} \stackrel{ind}{\sim} N(\mu_g, \sigma^2).$$

This assumption is common when the number of observations in the groups is small.

Comparing means when the variances are equal

Assuming $Y_{g,i} \stackrel{ind}{\sim} N(\mu_g, \sigma^2)$, we can test

$$H_0 : \mu_g = \mu \,\forall\, g$$

```
oneway.test(sensitivity ~ process, data = d, var.equal = TRUE)

One-way analysis of means

data:  sensitivity and process
F = 6.7543, num df = 2, denom df = 60, p-value = 0.002261
```

Then we typically look at pairwise differences,
i.e. $H_0 : \mu_g = \mu_{g'}$.

```
pairwise.t.test(d$sensitivity, d$process, p.adjust.method = "none")

Pairwise comparisons using t tests with pooled SD

data:  d$sensitivity and d$process

   P1     P2
P2 0.0116 -
P3 0.0012 0.0720
```

Posteriors for μ

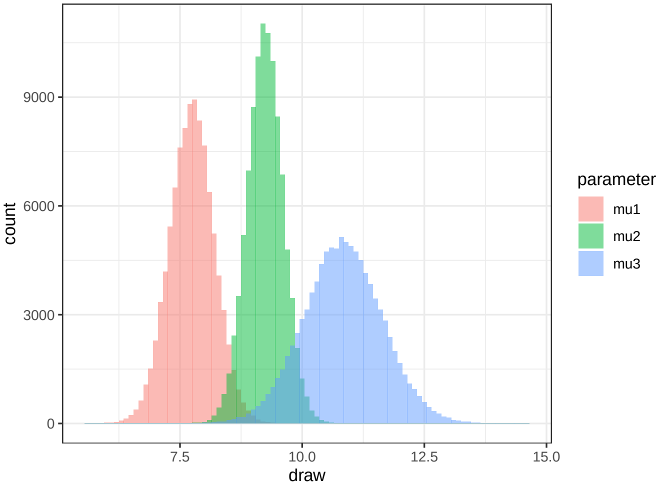
If $Y_{g,i} \stackrel{ind}{\sim} N(\mu_g, \sigma^2)$ and we use the prior $p(\mu_1, \ldots, \mu_G, \sigma^2) \propto 1/\sigma^2$, then

$$\mu_g|y, \sigma^2 \stackrel{ind}{\sim} N(\overline{y}_g, \sigma^2/n_g) \quad \sigma^2|y \sim IG\left(\frac{n-G}{2}, \frac{1}{2}\sum_{g=1}^{G}\sum_{i=1}^{n_g}(y_{g,i} - \overline{y}_g)^2\right)$$

where $n = \sum_{g=1}^{G} n_g$. and thus, we obtain joint samples for μ by performing the following

1. $\sigma^{2(m)} \sim p(\sigma^2|y)$
2. For $g = 1, \ldots, G$, $\mu_g \sim p(\mu_g|y, \sigma^{2(m)})$.

Compare posteriors



Credible interval for the differences

To compare the means, we compare the samples drawn from the posterior.

```
sims %>%
  mutate(`mu1-mu2` = mu1-mu2,
         `mu1-mu3` = mu1-mu3,
         `mu2-mu3` = mu2-mu3) %>%
  select(`mu1-mu2`,`mu1-mu3`,`mu2-mu3`) %>%
  gather(comparison, diff) %>%
  group_by(comparison) %>%
  summarize(probability = mean(diff>0) %>% round(4),
            lower = quantile(diff, .025) %>% round(2),
            upper = quantile(diff, .975) %>% round(2)) %>%
  mutate(credible_interval = paste("(",lower,",",upper,")", sep="")) %>%
  select(comparison, probability, credible_interval)

# A tibble: 3 x 3
  comparison probability credible_interval
  <chr>            <dbl> <chr>
1 mu1-mu2         0.0059 (-2.65,-0.35)
2 mu1-mu3         0.0007 (-4.92,-1.26)
3 mu2-mu3         0.036  (-3.34,0.15)
```

Summary

Multiple (independent) normal means

- p-values
- confidence intervals
- posterior densities
- credible intervals
- posterior probabilities