P5 - Multiple random variables

STAT 5870 (Engineering)
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Multiple discrete random variables

If X and Y are two discrete variables, their joint probability mass function is defined as

$$p_{X,Y}(x,y) = P(X = x \cap Y = y) = P(X = x, Y = y).$$

CPU example

A box contains 5 PowerPC G4 processors of different speeds:

#	speed
2	400 mHz
1	450 mHz
2	500 mHz

Randomly select two processors out of the box (without replacement) and let

- X be speed of the first selected processor and
- Y be speed of the second selected processor.

CPU example - outcomes

		1st processor (X)				
	Ω	400_1	400_2	450	500_1	500_2
2nd processor (Y)	400_1	-	x	x	x	x
	400_2	x	-	x	x	x
	450	x	x	-	x	x
	500_1	x	x	x	-	x
	500_2	x	x	x	x	-

Reasonable to believe each outcome is equally probable.

CPU example - joint pmf

Joint probability mass function for X and Y:

	mHz	1st processor (X) 400	450	500
2nd processor (Y)	400	$2/20$	$2/20$	$4/20$
	450	$2/20$	$0/20$	$2/20$
	500	$4/20$	$2/20$	$2/20$

- What is $P(X = Y)$?
- What is $P(X > Y)$?

CPU example - probabilities

What is the probability that $X = Y$?

$$\begin{aligned} &P(X = Y) \\ &\quad = p_{X,Y}(400, 400) + p_{X,Y}(450, 450) + p_{X,Y}(500, 500) \\ &\quad = 2/20 + 0/20 + 2/20 = 4/20 = 0.2 \end{aligned}$$

What is the probability that $X > Y$?

$$\begin{aligned} &P(X > Y) \\ &\quad = p_{X,Y}(450, 400) + p_{X,Y}(500, 400) + p_{X,Y}(500, 450) \\ &\quad = 2/20 + 4/20 + 2/20 = 8/20 = 0.4 \end{aligned}$$

Marginal distribution

For discrete random variables X and Y, the marginal probability mass functions are

$$\begin{aligned} p_X(x) &= \textstyle\sum_y p_{X,Y}(x,y) \qquad \text{and} \\ p_Y(y) &= \textstyle\sum_x p_{X,Y}(x,y) \end{aligned}$$

Marginal distribution

Joint probability mass function for X and Y:

		1st processor (X)		
	mHz	400	450	500
	400	2/20	2/20	4/20
2nd processor (Y)	450	2/20	0/20	2/20
	500	4/20	2/20	2/20

Summing the rows within each column provides

x	400	450	500
$p_X(x)$	0.4	0.2	0.4

Summing the columns within each row provides

y	400	450	500
$p_Y(y)$	0.4	0.2	0.4

CPU example - independence

Are X and Y independent?

X and Y are independent if $p_{x,y}(x,y) = p_X(x)p_Y(y)$ for all x and y.

Since

$$p_{X,Y}(450,450) = 0 \neq 0.2 \cdot 0.2 = p_X(450) \cdot p_Y(450)$$

they are **not** independent.

Expectation

The expected value of a function $h(x, y)$ is

$$E[h(X, Y)] = \sum_{x,y} h(x, y)\, p_{X,Y}(x, y).$$

CPU example - expected absolute speed difference

What is $E[|X-Y|]$?

Here, we have the situation $E[|X-Y|] = E[h(X,Y)]$, with $h(X,Y) = |X-Y|$. Thus, we have

$$
\begin{aligned}
&E[|X-Y|] \\
&\quad = \textstyle\sum_{x,y} |x-y|\, p_{X,Y}(x,y) = \\
&\quad = |400-400|\cdot 0.1 + |400-450|\cdot 0.1 + |400-500|\cdot 0.2 \\
&\quad\quad + |450-400|\cdot 0.1 + |450-450|\cdot 0.0 + |450-500|\cdot 0.1 \\
&\quad\quad + |500-400|\cdot 0.2 + |500-450|\cdot 0.1 + |500-500|\cdot 0.1 \\
&\quad = 0+5+20+5+0+5+20+5+0 = 60.
\end{aligned}
$$

Covariance

The covariance between two random variables X and Y is

$$Cov[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$$

where

$$\mu_X = E[X] \qquad \text{and} \qquad \mu_Y = E[X].$$

If $Y = X$ in the above definition, then
$Cov[X,X] = Var[X]$.

CPU example - covariance

Use marginal pmfs to compute:

$$E[X] = E[Y] = 450 \qquad \text{and} \qquad Var[X] = Var[Y] = 2000.$$

The covariance between X and Y is:

$$\begin{aligned}
&Cov[X,Y] \\
&= \textstyle\sum_{x,y}(x - E[X])(y - E[Y])p_{X,Y}(x,y) = \\
&= (400 - 450)(400 - 450) \cdot 0.1 \\
&\quad +(450 - 450)(400 - 450) \cdot 0.1 \\
&\quad +\cdots \\
&\quad +(500 - 450)(500 - 450) \cdot 0.1 \\
&= 250 + 0 - 500 + 0 + 0 + 0 - 500 + 250 + 0 \\
&= -500.
\end{aligned}$$

Correlation

The correlation between two variables X and Y is

$$\rho[X,Y] = \frac{Cov[X,Y]}{\sqrt{Var[X] \cdot Var[Y]}} = \frac{Cov[X,Y]}{SD[X] \cdot SD[Y]}.$$

Correlation properties

- ρ is between -1 and 1
- if $\rho = 1$ or -1, Y is a linear function of X:
  - $\rho = 1 \implies Y = mX + b$ with $m > 0$,
  - $\rho = -1 \implies Y = mX + b$ with $m < 0$,
- ρ is a measure of linear association between X and Y
  - ρ near ± 1 indicates a strong linear relationship,
  - ρ near 0 indicates a lack of linear association.

CPU example - correlation

Recall

$$Cov[X,Y] = -500 \qquad \text{and} \qquad Var[X] = Var[Y] = 2000.$$

The correlation is

$$\rho[X,Y] = \frac{Cov[X,Y]}{\sqrt{Var[X] \cdot Var[Y]}} = \frac{-500}{\sqrt{2000 \cdot 2000}} = -0.25,$$

and thus there is a weak negative (linear) association.

Continuous random variables

Suppose X and Y are two continuous random variables with joint probability density function $p_{X,Y}(x, y)$. Probabilities are calculated by integrating this function. For example,

$$P(a < X < b, c < Y < d) = \int_c^d \int_a^b p_{X,Y}(x, y)\, dx\, dy.$$

Then the marginal probability density functions are

$$\begin{aligned} p_X(x) &= \int p_{X,Y}(x, y)\, dy \\ p_Y(y) &= \int p_{X,Y}(x, y)\, dx. \end{aligned}$$

Continuous random variables

Two continuous random variables are independent if

$$p_{X,Y}(x, y) = p_X(x)\, p_Y(y).$$

The expected value of $h(X, Y)$ is

$$E[h(X, Y)] = \int \int h(x, y)\, p_{X,Y}(x, y)\, dx\, dy.$$

Properties of variances and covariances

For any random variables X, Y, W and Z,

$$Var[aX + bY + c] = a^2 Var[X] + b^2 Var[Y] + 2ab Cov[X, Y]$$

$$\begin{aligned} Cov[aX + bY, cZ + dW] &= ac Cov[X, Z] + ad Cov[X, W] \\ &\quad + bc Cov[Y, Z] + bd Cov[Y, W] \end{aligned}$$

$$\begin{aligned} Cov[X, Y] &= Cov[Y, X] \\ \rho[X, Y] &= \rho[Y, X] \end{aligned}$$

If X and Y are independent, then

$$\begin{aligned} Cov[X, Y] &= 0 \\ Var[aX + bY + c] &= a^2 Var[X] + b^2 Var[Y]. \end{aligned}$$

Summary

- Multiple random variables
  - joint probability mass function
  - marginal probability mass function
  - joint probability density function
  - marginal probability density function
  - expected value
  - covariance
  - correlation