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Initial size estimation for the linear pure death process

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SUMMARY

Consider a linear pure death process in which the common death rate and the initial size n of the population are both unknown. An estimate of n is required, based on the observed times of the first j deaths, i.e. with censored sampling. It is shown that the only reliable non-Bayesian method is to use interval estimates, for all the standard point estimation techniques are liable to fail. Appropriate interval estimators are derived and discussed. The corresponding truncated sampling model is briefly considered.

Some key words: Censored sampling; Death process; Estimating population size; Interval estimate; Invariance; Maximum likelihood; Method of moments; Similar test; Sufficiency; Truncated sampling; Unbiased.

1. INTRODUCTION

The linear pure death process may be characterized as representing a dying population in which the lifetimes are independently and identically distributed, each with probability density $\lambda e^{-\lambda t}(t > 0)$, for some $\lambda > 0$. We assume that both the initial population size n and the rate parameter λ are unknown, and our aim is to estimate n.

For many years work on the estimation of population size, such as by Johnson (1962), Hoel (1968) and Marcus & Blumenthal (1974), assumed a lifetime distribution which was general, but completely specified. Recently interest has been shown in lifetime distributions with unknown parameters, in particular in papers by Sanathanan (1977), who gives asymptotic results for maximum likelihood estimation, and Blumenthal & Marcus (1975), who consider precisely the above problem.

We consider mainly the censored sampling model, under which an estimate of n is required immediately after the jth death, where j is a preassigned integer known not to exceed n. In § 2 we derive an invariantly sufficient statistic and its distribution. In § 3 we discuss point estimation of n: unbiased estimators do not exist, while the methods of moments and of maximum likelihood are both liable to fail in that there is positive probability, sometimes close to $\frac{1}{2}$, that the equation for the estimator has no finite solution. In § 4 we derive an interval estimator and discuss its properties. While an interval estimate is always available, it is in some circumstances quite vague. This brings out clearly that for the present problem it is more than usually inadvisable to rely on point estimates, even when such are available. Indeed the very distribution of the invariantly sufficient statistic shows vividly the weakness of the information provided by the experiment.

Blumenthal & Marcus (1975) consider censored sampling as above, along with truncated sampling, in which observation of the process ends after a fixed time T.

Truncated sampling is also discussed by Watson & Blumenthal (1980). These papers advocate certain quasi-Bayesian modal estimators, described in the later paper as maximum modified likelihood estimators. The earlier paper considers just one non-Bayesian method, namely maximum likelihood, which is found to have positive probability of failing to yield an estimate. Here we consider truncated sampling briefly in § 5.

The problem of estimating n has an obvious application in biology, when n is the size of a closed population. N. Starr, in a University of Michigan Technical Report, discusses estimating the number of fish in a lake, assuming that the time to catch any particular fish is exponentially distributed with parameter λ. Hoel (1968) considers a group of people who have been exposed at the same point in time to radiation or some disease. He wishes to estimate the size of the group, when the time until symptoms appear has an exponential distribution. Comparable examples by Starr (1974) include hunting for relics during archaeological excavations, and estimating numbers of atoms by observing radioactive decay. Further applications are given by Blumenthal & Marcus (1975).

2. Distributional results

We observe the first j deaths. The times $t_1 < \ldots < t_j$ at which they occur have joint density

$$f(t_1, \ldots, t_j \mid \lambda, n) = j!\binom{n}{j}\lambda^j \exp\left[-\lambda\left\{(n-j)\,t_j + \sum_{i=1}^{j} t_i\right\}\right] \quad (0 < t_1 < \ldots < t_j). \tag{1}$$

The notation here and elsewhere is to use f for any sampling density, and F for any distribution function. The same functional form is implied only when the arguments are the same.

Clearly, the pair $(t_1 + \ldots + t_j, t_j)$ is minimal sufficient for (λ, n). For later convenience we use (α, τ) as minimal sufficient statistic, where $\alpha = (t_1 + \ldots + t_j)/t_j$, $\tau = t_j$. To find the joint distribution of α and τ we adapt Puri (1968, pp. 147–8). First, τ, as the jth order statistic, has marginal density

$$f(\tau \mid \lambda, n) = j\binom{n}{j}\lambda(1-e^{-\lambda\tau})^{j-1}\exp\{-\lambda\tau(n-j+1)\} \quad (\tau > 0).$$

Defining $s_i = t_i/\tau$, the conditional density of $s_1, \ldots, s_{j-1}$ given τ is

$$(j-1)!\,(\lambda\tau)^{j-1}(1-e^{-\lambda\tau})^{-j+1}\exp\left\{-\lambda\tau\sum_{i=1}^{j-1} s_i\right\} \quad (0 < s_1 < \ldots < s_{j-1} < 1).$$

Then

$$f(\alpha \mid \tau;\, \lambda, n) = (\lambda\tau)^{j-1}(1-e^{-\lambda\tau})^{-j+1}e^{-\lambda\tau(\alpha-1)}\int_{S(\alpha)}\cdots\int (j-1)!\,ds_1\ldots ds_{j-1}, \tag{2}$$

where

$$S(\alpha) = \{(s_1, \ldots, s_{j-1})\colon 0 < s_1 < \ldots < s_{j-1} < 1, \alpha = 1 + s_1 + \ldots + s_{j-1}\}.$$

The integral in (2) is the density $g_j(\alpha)$ of $1 + U_1 + \ldots + U_{j-1}$, where $U_1, \ldots, U_{j-1}$ are independent uniform random variables on $(0, 1)$. For, although in (2) the random variables s_i are ordered, the ordering clearly does not affect the sum. The density g_j may be evaluated (Kendall & Stuart, 1977, p. 275) as

$$g_j(\alpha) = \{1/(j-2)!\}\sum_{k=1}^{[\alpha]}(-1)^{k-1}\binom{j-1}{k-1}(\alpha-k)^{j-2} \quad (1 < \alpha < j), \tag{3}$$

where $[\alpha]$ is the largest integer not exceeding α. The density of (α, τ) is thus

$$f(\alpha, \tau \mid \lambda, n) = j\binom{n}{j}\lambda^j \tau^{j-1} \exp\{-\lambda\tau(\alpha+n-j)\}\, g_j(\alpha) \quad (1 < \alpha < j, \tau > 0). \tag{4}$$

This joint density may alternatively be derived by first finding the bivariate Laplace transform of $\alpha\tau$ and τ.

For any given $c > 0$, the map $(t_1, \dots, t_j) \to (ct_1, \dots, ct_j)$ induces a map $(\lambda, n) \to (\lambda/c, n)$ on the parameter, and so leaves the problem invariant. On the sufficient statistic the map becomes $(\alpha, \tau) \to (\alpha, c\tau)$, and so a maximal invariant is α. Applying minimal sufficiency and maximal invariance in reverse order yields the same result. Using (4), the marginal density of α is

$$f(\alpha \mid n) = j!\binom{n}{j}(\alpha+n-j)^{-j} g_j(\alpha) \quad (1 < \alpha < j). \tag{5}$$

Figure 1 illustrates this family of densities for the case $j = 10$. Obviously the estimation of n will be problematic unless α is small.

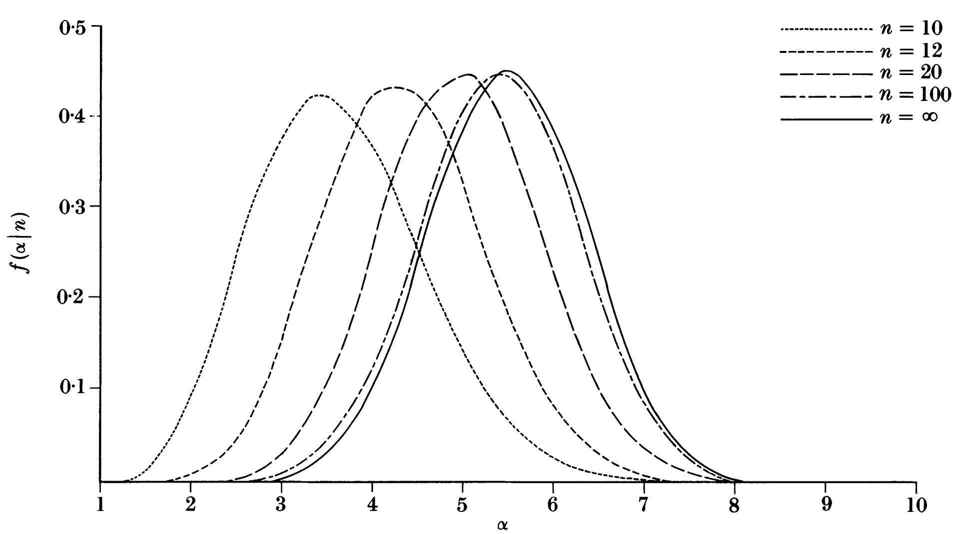

Fig. 1. Density functions of α for various values of n, when $j = 10$.

3. Point estimation

3·1. *Maximum likelihood*

Invariance considerations suggest using the marginal likelihood, in the sense of Kalbfleisch & Sprott (1970), based on (5). It is, however, easy to see that this yields the same estimator as that obtained by Blumenthal & Marcus (1975). This estimator, they stated, has the major deficiency of failing to exist if α happens to fall in the upper half of its range, because there is then no finite n that maximizes the likelihood function.

One may readily prove this from (5). For, treating n as a continuous variable,

$$(\partial/\partial n) \log f(\alpha \mid n) = j\{H^{-1} - (\alpha+n-j)^{-1}\}, \tag{6}$$

where H is the harmonic mean of $n, \dots, n-j+1$. Since the harmonic mean cannot exceed the arithmetic mean, H^{-1} is at least $2/(2n+1-j)$. The conclusion follows since (6) is positive for all $n \geqslant j$ whenever $\alpha > \frac{1}{2}(j+1)$. The probability that this occurs may be proved to increase to the limit $\frac{1}{2}$ as $n \to \infty$; see § 3·2.

3·2. *Method of moments*

Lengthy calculations give

$$E(\alpha \mid n) = j-n+j(j-1)\binom{n}{j}\sum_{k=0}^{j-1}(-1)^{j-k}\binom{j-1}{k}\log(n-k).$$

One form of method of moments estimate is to take the value of n that makes $E(\alpha \mid n)$ nearest to the observed value of α. From (5) we see that, as $n \to \infty$, $f(\alpha \mid n)$ converges pointwise to $g_j(\alpha)$, the density of $U_1 + \ldots + U_{j-1}+1$, which has mean $\frac{1}{2}(j+1)$; see Fig. 1. The convergence of $E(\alpha \mid n)$ to this limit follows from the uniform boundedness of $\alpha f(\alpha \mid n)$ with respect to n.

Further, a consequence of the Neyman–Pearson lemma is that $E(\alpha \mid n)$ increases with n. For the family $\{f(\alpha \mid n)\}$ has a monotone likelihood ratio in α, so that for testing $n = N$ against $n > N$ on the basis of an observation of α, the test which rejects the null hypothesis $n = N$ when $\alpha > \alpha_0$ is best at some significance level, and its power function $1-F(\alpha_0 \mid n)$ is increasing. Hence the familiar device of writing

$$E(\alpha \mid n) = \int_0^\infty \{1-F(\alpha \mid n)\}\, d\alpha$$

establishes the claim.

This version of the method of moments has therefore the same defect as maximum likelihood: the estimator does not exist when $\alpha \geqslant \frac{1}{2}(j+1)$. If α takes a value below $E(\alpha \mid j)$, the method again technically fails, but here one would obviously take the estimator to be j.

Another version of the method of moments would be to use the relations

$$E(\tau) = \lambda^{-1}\sum_{s=1}^{j}(n-s+1)^{-1}, \quad E(\alpha\tau) = \lambda^{-1}\sum_{s=1}^{j}(j-s+1)/(n-s+1),$$

choosing λ and n jointly to equate the expected values of τ and $\alpha\tau$ to their observed values. However, one may show that $E(\alpha\tau)/E(\tau) < \frac{1}{2}(j+1)$ for all $n \geqslant j > 1$, so that again the method fails if α falls in the upper half of its range.

3·3. *Unbiased estimation*

Although the densities (1) form an exponential family, standard methods (Linnik, 1968; Washio *et al.*, 1956) of finding unbiased estimators are inapplicable when n is the estimand. In fact, as we now prove, there is no nonnegative statistic which is an unbiased estimator of n.

It suffices to consider an estimator $u(\alpha, \tau)$. Unbiasedness means that for all n, λ

$$\int_0^\infty \int_1^j u(\alpha,\tau)\binom{n-1}{j-1}\lambda^j \tau^{j-1} e^{-\lambda\tau(\alpha+n-j)} g_j(\alpha)\, d\alpha\, d\tau = 1.$$

Or, putting $v = (\alpha+n-j)\tau$,

$$\int_0^\infty e^{-\lambda v}\int_1^j u\{\alpha, v/(\alpha+n-j)\}\binom{n-1}{j-1}v^{j-1}(\alpha+n-j)^{-j}g_j(\alpha)\,d\alpha\,dv = \lambda^{-j}.$$

By the uniqueness of Laplace transforms, the inner integral equals $v^{j-1}/(j-1)!$ for all $v > 0$. Putting $s(x,y) = u(x,y)\,g_j(x)\,y^{j-1}$ gives

$$\int_1^j s\{\alpha, v/(\alpha+n-j)\}\,(\alpha+n-j)^{-j}\,d\alpha = v^{j-1}/\{(n-1)\ldots(n-j+1)\}.$$

Choosing $K > 0$, integrating out v over the interval $(0, (n-j+1)\,K)$, and then reverting to the original variables gives

$$\int_1^j\int_0^{K'} s(\alpha,\tau)\,d\tau\,d\alpha = (n-j+1)^j\,K^j/\{j(n-1)\ldots(n-j+1)\}, \tag{7}$$

where $K' = (n-j+1)\,K/(\alpha+n-j)$. The region of integration is a subset of the rectangle $R = \{(\alpha,\tau)\colon 1 < \alpha < j, 0 < \tau < K\}$. As we are assuming u is non-negative, so is s; hence the integral of s over R is at least the right-hand side of (7). As $n \to \infty$, the right-hand side of (7) tends to $+\infty$, so that the integral of s over R must be $+\infty$. However, replacing K by $2K$ and taking $n > 2j$ in (7), we get a region which includes R, and yet over which the integral is finite. The hypothesis that u is unbiased is therefore untenable.

Note that it is much easier to show that there is no nonnegative unbiased estimator based on α alone. But we know of no general result which would allow us to restrict attention to the narrower class *a priori*.

4. Significance tests and confidence intervals

4·1. *Significance tests*

The previous section has shown point estimators to be unsatisfactory so we turn to interval estimators. As the nuisance parameter must be discarded, we obtain these by inverting invariant tests or similar tests.

Consider the problem of testing H_0: $n = N$ against the alternative H_1: $n > N$, for some fixed integer N. The problem is invariant under the mapping given in § 2, and hence sufficiency and invariance, in either order, bring us to tests based on α alone. Since $\{f(\alpha\,|\,n)\}$ has a monotone likelihood ratio in α, the best, i.e. uniformly most powerful, invariant tests are those which reject H_0 when α is large.

To find the similar tests it will be useful to define $\beta = (\alpha+N-j)\,\tau$. Setting $n = N$ in (4), β is seen to be sufficient for H_0. As the density (4) belongs to the exponential family, the distribution of β under H_0 is complete, so the similar tests have Neyman structure, and we can find the best similar critical region by finding the best region in each line $\beta =$ constant. Whether $n = N$ or not, it follows from (4) that

$$f(\alpha\,|\,\beta;\,\lambda, n) = K_1(n, N, \lambda, \beta)\,(\alpha+N-j)^{-j}g_j(\alpha)\exp\left\{-\lambda\frac{(n-N)\,\beta}{\alpha+N-j}\right\} \quad (1 < \alpha < j) \tag{8}$$

for some positive function K_1. Fixing $\beta = \beta_0$, we apply the Neyman–Pearson lemma to the conditional densities under H_0 and under a specific alternative $n = n' > N$. This gives a best test (of size ε, say) in the line $\beta = \beta_0$ with critical region $\{(\alpha, \beta_0)\colon \alpha > K_2(n', N, \lambda, \beta_0)\}$

for some function K_2. Equivalently we may write this region as $\{(\alpha, \beta_0): \alpha > K_2(N)\}$, since, under H_0, we see from (8) that α and β are independent, and hence the conditional density of α depends only on N.

The critical region for the similar test of size ε is the union of the regions for the conditional tests, which is $\{(\alpha, \beta): \alpha > K_2(N)\}$. As this test does not depend on n', it is uniformly most powerful similar. It also coincides with the best invariant test.

When testing H_0: $n \leqslant N$ against H_1: $n > N$, there are no non-trivial similar tests in the above sense, as (α, β) is sufficient under both hypotheses. Our test, however, remains the best invariant test, and is also the best boundary-form similar test (Cox & Hinkley, 1974, p. 150). The form of the best similar and invariant tests for the problem H_0: $n \geqslant N$ against H_1: $n < N$ is obvious. For testing $n = N$ against two-sided alternatives, a two-tailed test based on α would be appropriate.

4·2. *Confidence intervals*

Starting with the acceptance regions $A^*(n) = \{\alpha: \alpha \leqslant K_2(n)\}$ of the above-mentioned size ε family of tests, we obtain one-sided confidence intervals $C(\alpha)$ with confidence coefficient $1-\varepsilon$ by the usual inversion

$$C(\alpha) = \{n: \alpha \in A^*(n)\}. \tag{9}$$

These give lower confidence bounds for n. Upper bounds come from inverting the other one-tailed acceptance regions.

'Two-sided' confidence intervals are obtained by inverting any family of two-tailed tests, using (9). The acceptance intervals $A(n)$ are long if they are chosen to produce equal-probability tails, and so the corresponding confidence intervals are longer than needed. Instead we choose $A(n)$ so that $f(\alpha \mid n) \geqslant f(\alpha' \mid n)$ whenever $\alpha \in A(n), \alpha' \notin A(n)$. Table 1 gives confidence intervals at several levels of confidence, using this method, for the case $j = 10$. Each entry of the table is two integers n_0, n_1 with $10 \leqslant n_0 \leqslant n_1 \leqslant \infty$, and the confidence interval is all integer values of n satisfying $n < \infty$, $n_0 \leqslant n \leqslant n_1$.

Table 1. *Confidence bounds for n when j = 10*

		Confidence coefficient, %									
		20	50	70	90	95	99	99·9	99·99	99·999	99·9999
	2	—	—	—	—	10, 10	10, 10	10, 13	10, 18	10, 71	10, ∞
	3	—	10, 10	10, 10	10, 12	10, 14	10, 26	10, ∞	10, ∞	10, ∞	10, ∞
	4	11, 11	10, 14	10, 18	10, 108	10, ∞	10, ∞	10, ∞	10, ∞	10, ∞	10, ∞
	5	17, 33	13, ∞	12, ∞	10, ∞	10, ∞	10, ∞	10, ∞	10, ∞	10, ∞	10, ∞
α	6	—	91, ∞	24, ∞	14, ∞	12, ∞	10, ∞	10, ∞	10, ∞	10, ∞	10, ∞
	7	—	—	—	—	44, ∞	15, ∞	11, ∞	10, ∞	10, ∞	10, ∞
	8	—	—	—	—	—	—	34, ∞	14, ∞	11, ∞	10, ∞
	9	—	—	—	—	—	—	—	—	—	28, ∞

Families of confidence regions are customarily labelled with optimality properties to the same extent as their corresponding families of tests. The justification of the tests in § 4·1 should therefore be read as applying to the confidence intervals.

It is illuminating to reconsider point estimation in the light of the table. The interval estimates are very vague if α takes a value just below its midrange, and therefore so are the point estimates, but this becomes obvious only in the context of interval estimation. When the maximum likelihood estimator is $+\infty$, for values of α greater than $5\frac{1}{2}$, the confidence intervals are necessarily unbounded above. Null confidence regions, signified by a blank, occur when all the acceptance regions $A(n)$ are, at the given confidence level, too narrow to catch the observed value of α. One may then question the validity of the model, or else increase the confidence level, sometimes very substantially, to find values of the parameter consistent with the data.

These interval estimates therefore suggest that only fairly weak inference statements about the value of n are justified by the data, and that the apparent precision of any point estimate will often be largely spurious.

5. Truncated sampling

Deaths are observed until a preassigned time T. The number of deaths observed is a random variable J with the binomial distribution

$$\operatorname{pr}(J = k) = \binom{n}{k}(1-e^{-\lambda T})^k e^{-\lambda T(n-k)} \quad (k = 0, 1, \ldots, n).$$

For $J = 1, \ldots, N$, the likelihood of $J, t_1, \ldots, t_J$ is

$$J!\binom{n}{J}\lambda^J \exp\left[-\lambda\left\{(n-J)\,T + \sum_{i=1}^{J} t_i\right\}\right] \quad (0 < t_1 < \ldots < t_J < T). \tag{10}$$

A suitable sufficient statistic is $(J, \tilde{\alpha})$ where $\tilde{\alpha} = (t_1 + \ldots + t_J)/T$ if $J > 0$, and $\tilde{\alpha} = 0$ if $J = 0$. For positive J, the conditional distribution of $\tilde{\alpha}$ given J, which can also be derived by Puri's (1968) method, is

$$f(\tilde{\alpha}\,|\,J;\lambda) = (\lambda T)^J \exp(-\lambda T\tilde{\alpha})\,(1-e^{-\lambda T})^{-J} g_{J+1}(\tilde{\alpha}+1) \quad (0 < \tilde{\alpha} < J),$$

where $g_J(\alpha)$ is defined by (3). The joint distribution of J and $\tilde{\alpha}$ follows. This result was essentially given by Hoem (1969) and Bain & Weeks (1964).

Regarding point estimation, the maximum likelihood method has a similar defect (Blumenthal and Marcus, 1975) to that for censored sampling, failing this time if $\tilde{\alpha} > \frac{1}{2}(J+1)$. This follows readily by comparing the forms of (4) and (10) as functions of n and λ. Note that this difficulty is avoided by a large subclass of the quasi-Bayesian estimators investigated by these authors, and by Watson & Blumenthal (1980). As for the method of moments, it cannot be used to estimate n alone, as there is no real-valued function of the sufficient statistic whose distribution does not involve λ. Further, if two moments are used to estimate λ and n together, a similar difficulty arises to that noted above. For instance, one may show that

$$E(\tilde{\alpha})/E(J) = (\lambda T)^{-1} - (e^{\lambda T} - 1)^{-1},$$

which never exceeds $\frac{1}{2}$. Hence it is impossible to equate both $\tilde{\alpha}$ and J to their expected values if $\tilde{\alpha} > \frac{1}{2}J$.

Again, comparably to censored sampling, there is no nonnegative function $w(J, \tilde{\alpha})$ which is unbiased for n. For, if there were, then, for all n and λ, we would have

$$n = w_0\, e^{-\lambda nT} + \sum_{J=1}^{n}\binom{n}{J}(\lambda T)^J e^{-\lambda T(n-J)}\int_0^\infty e^{-\lambda T\tilde{\alpha}}\bar{w}(J,\tilde{\alpha})\,d\tilde{\alpha},$$

where $w_0 = w(0,0)$ and $\bar{w}$ is given by

$$\bar{w}(J,\tilde{\alpha}) = \bar{w}(J,\tilde{\alpha})\,g_{J+1}(\tilde{\alpha}+1)\,I_{[0,J]}(\tilde{\alpha}).$$

Writing $w_J^*(\,.\,)$ for the Laplace transform of $\bar{w}(J, \tilde{\alpha})$, we obtain that, for all n,

$$n = w_0 \, e^{-\lambda n T} + \sum_{J=1}^{n} \binom{n}{J} (\lambda T)^J \, e^{-\lambda T(n-J)} \, w_J^*(\lambda T). \tag{11}$$

Putting $n = 1$, and inverting the transform, we have that

$$\bar{w}(1, \tilde{\alpha}) = \begin{cases} 1 & (0 < \tilde{\alpha} \leqslant 1), \\ 1 - w_0 & (\tilde{\alpha} > 1), \end{cases}$$

where, by definition of $\bar{w}$, we must have $w_0 = 1$. Putting $n = 2$, $w_0 = 1$ in (11), and inverting the transform for $\bar{w}(2, \tilde{\alpha})$, we get a solution which is nonzero for $\tilde{\alpha} > 2$, contradicting the definition of $\bar{w}(2, \tilde{\alpha})$.

For truncated sampling even interval estimation is unsatisfactory, and ad hoc methods seem to be necessary within a non-Bayesian framework. The difficulty is in disposing of the nuisance parameter λ. There are no invariances to be had from the expression (10). Moreover, methods for finding similar tests, in the sense of Cox & Hinkley (1974), also fail, since the minimal sufficient statistic for λ, given a hypothesized value N of n, is of the same dimension as $(J, \tilde{\alpha}_J)$. Hence the distribution of the data conditional on the sufficient statistic given N is independent of both parameters. Furthermore anyone wishing to use likelihood ratio methods is faced with an intractable ratio statistic.

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