Nonparametric Bayesian Statistics

Tamara Broderick
ITT Career Development Assistant Professor
Electrical Engineering & Computer Science
MIT

Nonparametric Bayes

Nonparametric Bayes

- Bayesian statistics that is not parametric

Nonparametric Bayes

- Bayesian statistics that is not parametric (wait!)

Nonparametric Bayes

- Bayesian statistics that is not parametric
- Bayesian

Nonparametric Bayes

- Bayesian statistics that is not parametric
- Bayesian

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

Nonparametric Bayes

- Bayesian statistics that is not parametric
- Bayesian

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Not parametric (i.e. not finite parameter, unbounded/ growing/infinite number of parameters)

Nonparametric Bayes

- Bayesian statistics that is not parametric
- Bayesian

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Not parametric (i.e. not finite parameter, unbounded/ growing/infinite number of parameters)

[wikipedia.org]

Nonparametric Bayes

- Bayesian statistics that is not parametric
- Bayesian

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Not parametric (i.e. not finite parameter, unbounded/growing/infinite number of parameters)

[wikipedia.org]

"Wikipedia phenomenon"

Nonparametric Bayes

- Bayesian statistics that is not parametric
- Bayesian

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Not parametric (i.e. not finite parameter, unbounded/growing/infinite number of parameters)

[wikipedia.org]

Nonparametric Bayes

- Bayesian statistics that is not parametric
- Bayesian

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Not parametric (i.e. not finite parameter, unbounded/growing/infinite number of parameters)

[wikipedia.org]


[Ed Bowlby, NOAA]

Nonparametric Bayes

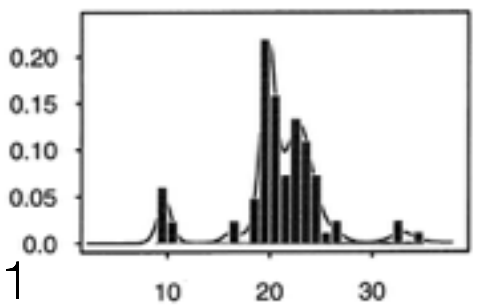
- Bayesian statistics that is not parametric
- Bayesian

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Not parametric (i.e. not finite parameter, unbounded/growing/infinite number of parameters)

[wikipedia.org]

[Ed Bowlby, NOAA]



[Escobar, West 1995; Ghosal et al 1999]

Nonparametric Bayes

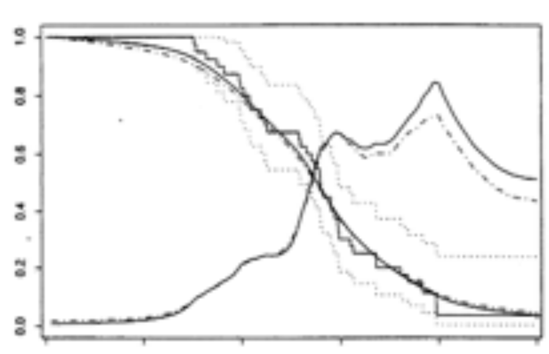
- Bayesian statistics that is not parametric
- Bayesian

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

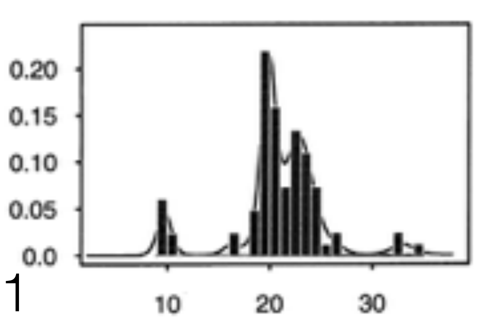
- Not parametric (i.e. not finite parameter, unbounded/growing/infinite number of parameters)

[wikipedia.org]

[Ed Bowlby, NOAA]



[Arjas, Gasbarra 1994]



[Escobar, West 1995; Ghosal et al 1999]

Nonparametric Bayes

- Bayesian statistics that is not parametric
- Bayesian

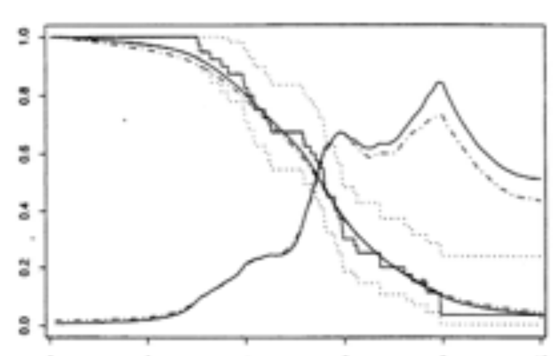
$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Not parametric (i.e. not finite parameter, unbounded/growing/infinite number of parameters)

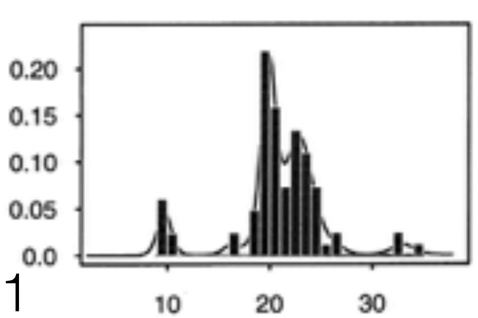
[wikipedia.org]

[Ed Bowlby, NOAA]

[Fox et al 2014]



[Arjas, Gasbarra 1994]



[Escobar, West 1995; Ghosal et al 1999]

Nonparametric Bayes

- Bayesian statistics that is not parametric
- Bayesian

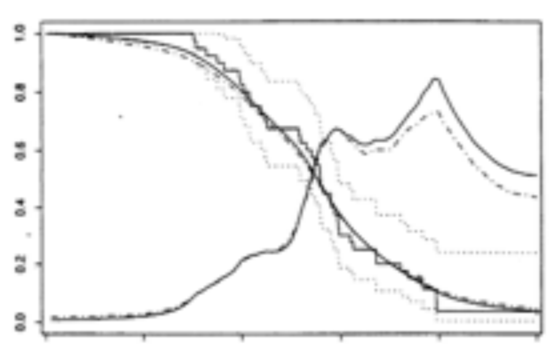
$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Not parametric (i.e. not finite parameter, unbounded/ growing/infinite number of parameters)

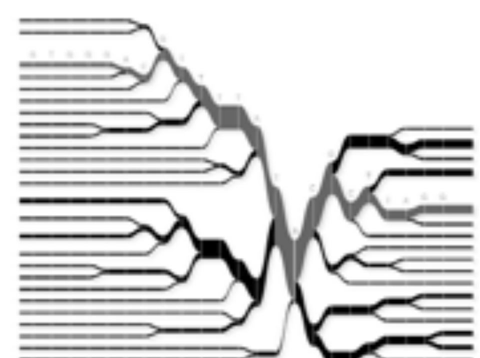
[wikipedia.org]

[Ed Bowlby, NOAA]

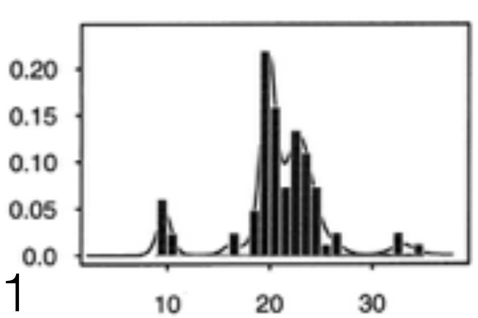
[Fox et al 2014]



[Arjas, Gasbarra 1994]



[Ewens, 1972; Hartl, Clark 2003]



[Escobar, West 1995; Ghosal et al 1999]

Nonparametric Bayes

- Bayesian statistics that is not parametric
- Bayesian

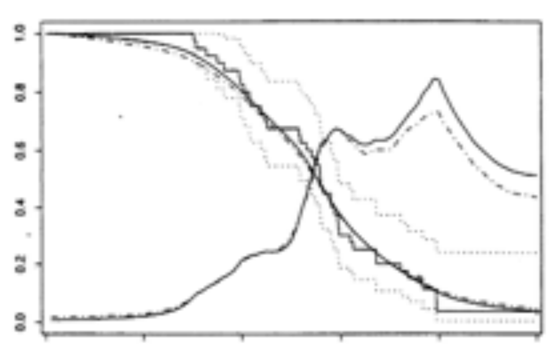
$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Not parametric (i.e. not finite parameter, unbounded/ growing/infinite number of parameters)

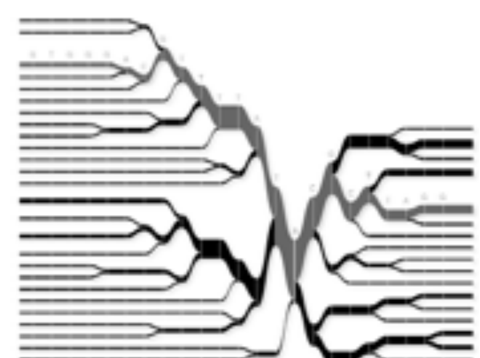
[wikipedia.org]

[Ed Bowlby, NOAA]

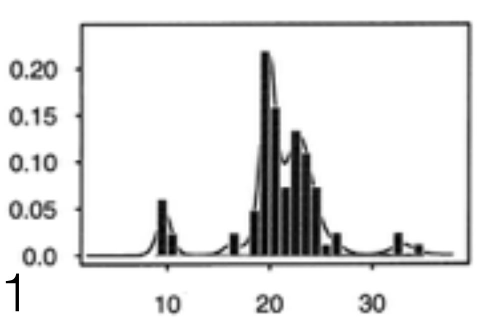
[Fox et al 2014]



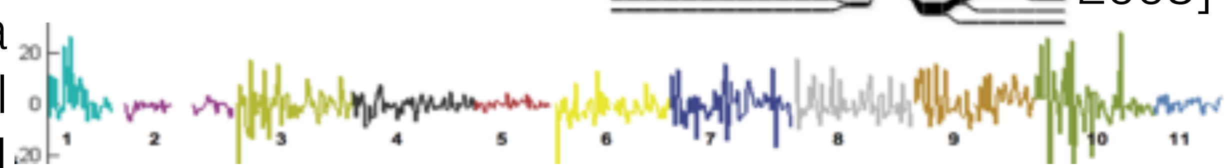
[Arjas, Gasbarra 1994]



[Ewens, 1972; Hartl, Clark 2003]



[Escobar, West 1995; Ghosal et al 1999]



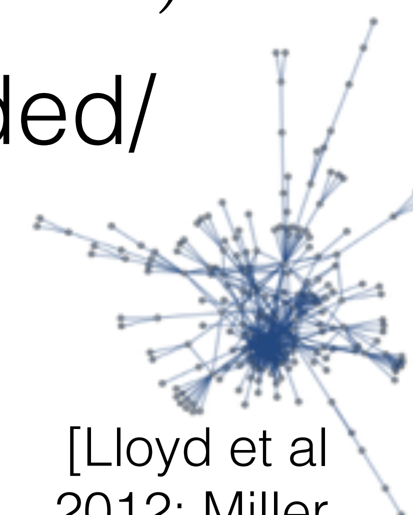
[Saria et al 2010]

Nonparametric Bayes

- Bayesian statistics that is not parametric
- Bayesian

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Not parametric (i.e. not finite parameter, unbounded/ growing/infinite number of parameters)

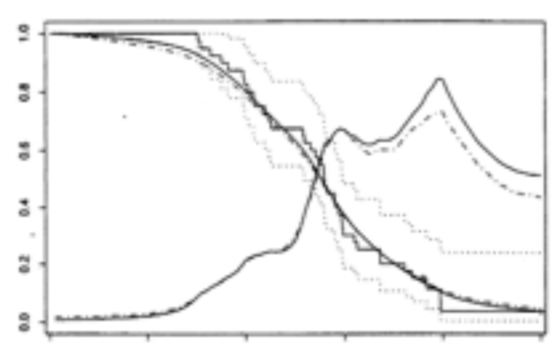


[Lloyd et al 2012; Miller et al 2010]

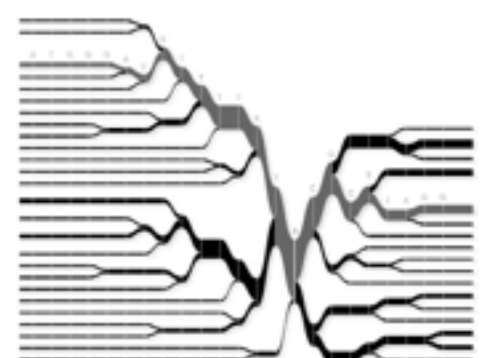
[wikipedia.org]

[Ed Bowlby, NOAA]

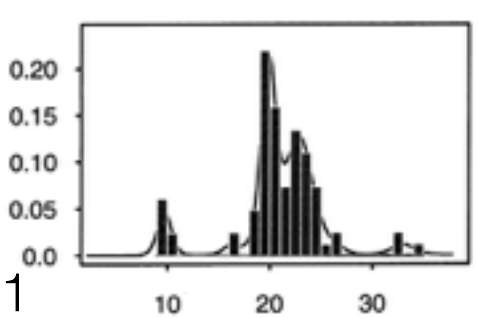
[Fox et al 2014]



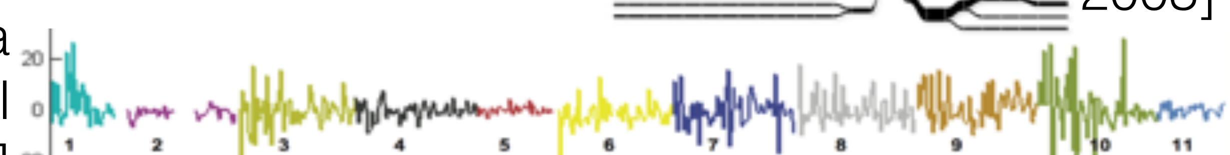
[Arjas, Gasbarra 1994]



[Ewens, 1972; Hartl, Clark 2003]



[Escobar, West 1995; Ghosal et al 1999]



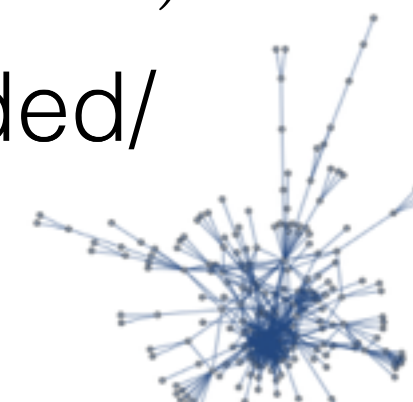
[Saria et al 2010]

Nonparametric Bayes

- Bayesian statistics that is not parametric
- Bayesian

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Not parametric (i.e. not finite parameter, unbounded/growing/infinite number of parameters)

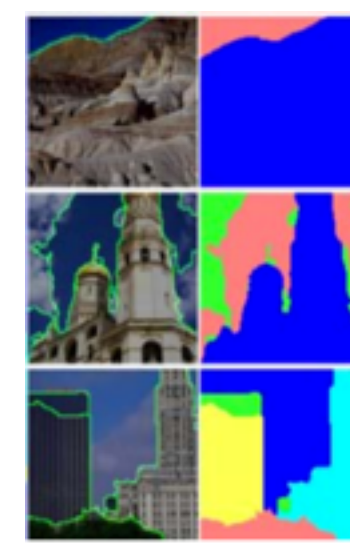


[Lloyd et al 2012; Miller et al 2010]

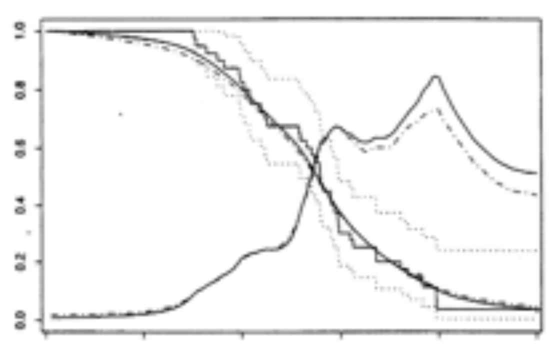
[wikipedia.org]

[Ed Bowlby, NOAA]

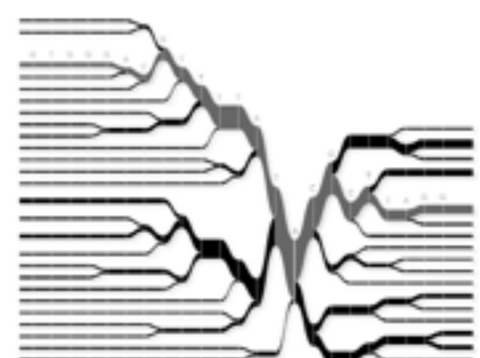
[Fox et al 2014]



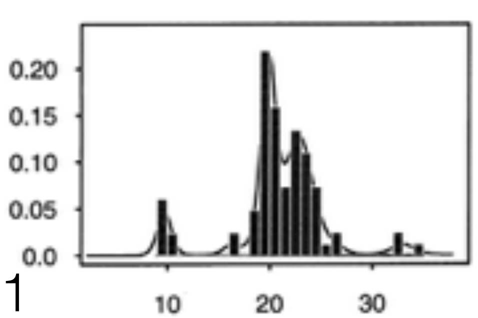
[Sudderth, Jordan 2009]



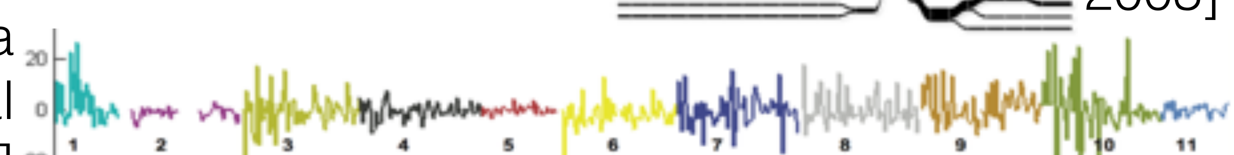
[Arjas, Gasbarra 1994]



[Ewens 1972; Hartl, Clark 2003]



[Escobar, West 1995; Ghosal et al 1999]



[Saria et al 2010]

Nonparametric Bayes

- A theoretical motivation: De Finetti's Theorem

Nonparametric Bayes

- A theoretical motivation: De Finetti's Theorem
- A data sequence is *infinitely exchangeable* if the distribution of any *N* data points doesn't change when permuted: $p(X_1, \ldots, X_N) = p(X_{\sigma(1)}, \ldots, X_{\sigma(N)})$

Nonparametric Bayes

- A theoretical motivation: De Finetti's Theorem
- A data sequence is *infinitely exchangeable* if the distribution of any *N* data points doesn't change when permuted: $p(X_1, \ldots, X_N) = p(X_{\sigma(1)}, \ldots, X_{\sigma(N)})$
- *De Finetti's Theorem* (roughly): A sequence $X_1, X_2, \ldots$ is infinitely exchangeable if and only if, for all *N* and some distribution *P*:

[Hewitt, Savage 1955; Aldous 1983]

Nonparametric Bayes

- A theoretical motivation: De Finetti's Theorem
- A data sequence is *infinitely exchangeable* if the distribution of any *N* data points doesn't change when permuted: $p(X_1, \ldots, X_N) = p(X_{\sigma(1)}, \ldots, X_{\sigma(N)})$
- *De Finetti's Theorem* (roughly): A sequence $X_1, X_2, \ldots$ is infinitely exchangeable if and only if, for all *N* and some distribution *P*:

$$p(X_1, \ldots, X_N) = \int_\theta \prod_{n=1}^{N} p(X_n|\theta) P(d\theta)$$

[Hewitt, Savage 1955; Aldous 1983]

Nonparametric Bayes

- A theoretical motivation: De Finetti's Theorem
- A data sequence is *infinitely exchangeable* if the distribution of any *N* data points doesn't change when permuted: $p(X_1, \ldots, X_N) = p(X_{\sigma(1)}, \ldots, X_{\sigma(N)})$
- *De Finetti's Theorem* (roughly): A sequence $X_1, X_2, \ldots$ is infinitely exchangeable if and only if, for all *N* and some distribution *P*:

$$p(X_1, \ldots, X_N) = \int_\theta \prod_{n=1}^N p(X_n|\theta)P(d\theta)$$

- Motivates:

[Hewitt, Savage 1955; Aldous 1983]

Nonparametric Bayes

- A theoretical motivation: De Finetti's Theorem
- A data sequence is *infinitely exchangeable* if the distribution of any *N* data points doesn't change when permuted: $p(X_1, \ldots, X_N) = p(X_{\sigma(1)}, \ldots, X_{\sigma(N)})$
- *De Finetti's Theorem* (roughly): A sequence $X_1, X_2, \ldots$ is infinitely exchangeable if and only if, for all *N* and some distribution *P*:

$$p(X_1, \ldots, X_N) = \int_\theta \prod_{n=1}^{N} p(X_n|\theta) P(d\theta)$$

- Motivates:
    - Parameters and likelihoods

[Hewitt, Savage 1955; Aldous 1983]

Nonparametric Bayes

- A theoretical motivation: De Finetti's Theorem
- A data sequence is *infinitely exchangeable* if the distribution of any *N* data points doesn't change when permuted: $p(X_1, \ldots, X_N) = p(X_{\sigma(1)}, \ldots, X_{\sigma(N)})$
- *De Finetti's Theorem* (roughly): A sequence $X_1, X_2, \ldots$ is infinitely exchangeable if and only if, for all *N* and some distribution *P*:

$$p(X_1, \ldots, X_N) = \int_\theta \prod_{n=1}^{N} p(X_n|\theta) P(d\theta)$$

- Motivates:
    - Parameters and likelihoods
    - Priors

[Hewitt, Savage 1955; Aldous 1983]

Nonparametric Bayes

- A theoretical motivation: De Finetti's Theorem
- A data sequence is *infinitely exchangeable* if the distribution of any *N* data points doesn't change when permuted: $p(X_1, \ldots, X_N) = p(X_{\sigma(1)}, \ldots, X_{\sigma(N)})$
- *De Finetti's Theorem* (roughly): A sequence $X_1, X_2, \ldots$ is infinitely exchangeable if and only if, for all *N* and some distribution *P*:

$$p(X_1, \ldots, X_N) = \int_\theta \prod_{n=1}^{N} p(X_n|\theta) P(d\theta)$$

- Motivates:
  - Parameters and likelihoods
  - Priors
  - "Nonparametric Bayesian" priors

[Hewitt, Savage 1955; Aldous 1983]

Outline

Outline

- Dirichlet process

Outline

- Dirichlet process
  - Background for intuition

Outline

- Dirichlet process
  - Background for intuition
  - Generative model

Outline

- Dirichlet process
  - Background for intuition
  - Generative model
  - What does a growing/infinite number of parameters really mean (in Nonparametric Bayes)?

Outline

- Dirichlet process
  - Background for intuition
  - Generative model
  - What does a growing/infinite number of parameters really mean (in Nonparametric Bayes)?
- Chinese restaurant process

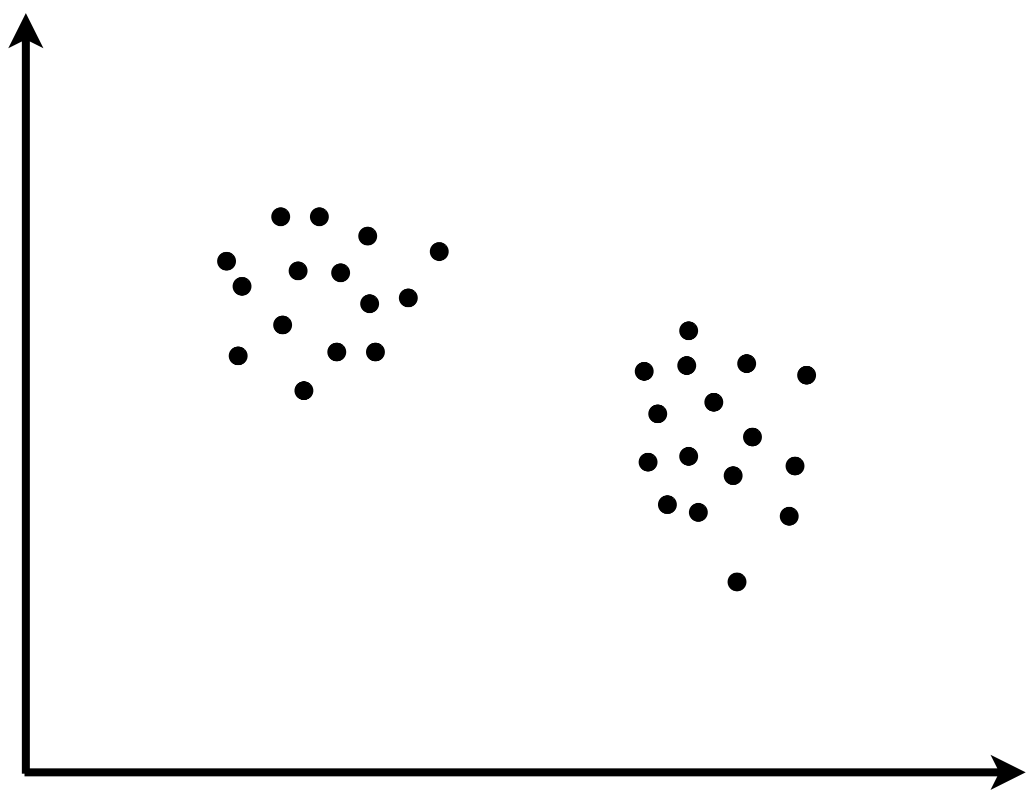
Outline

- Dirichlet process
  - Background for intuition
  - Generative model
  - What does a growing/infinite number of parameters really mean (in Nonparametric Bayes)?
- Chinese restaurant process
- Inference

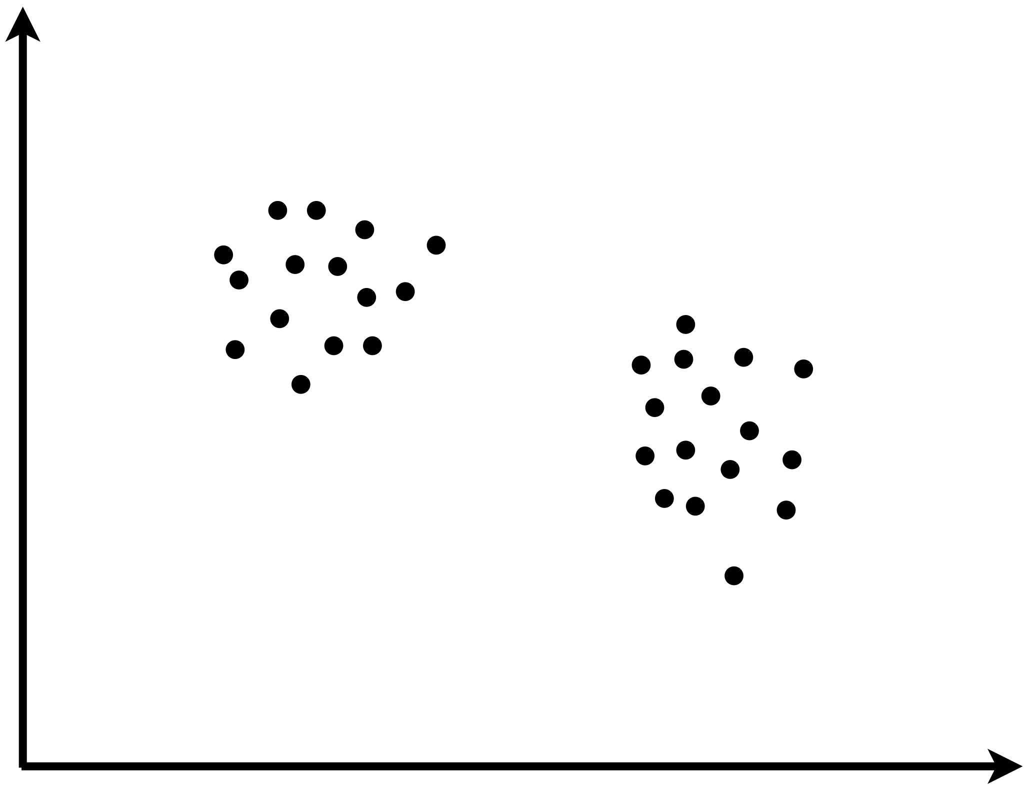
Outline

- Dirichlet process
    - Background for intuition
    - Generative model
    - What does a growing/infinite number of parameters really mean (in Nonparametric Bayes)?
- Chinese restaurant process
- Inference
- Venture further into the wild world of Nonparametric Bayesian statistics

Generative model



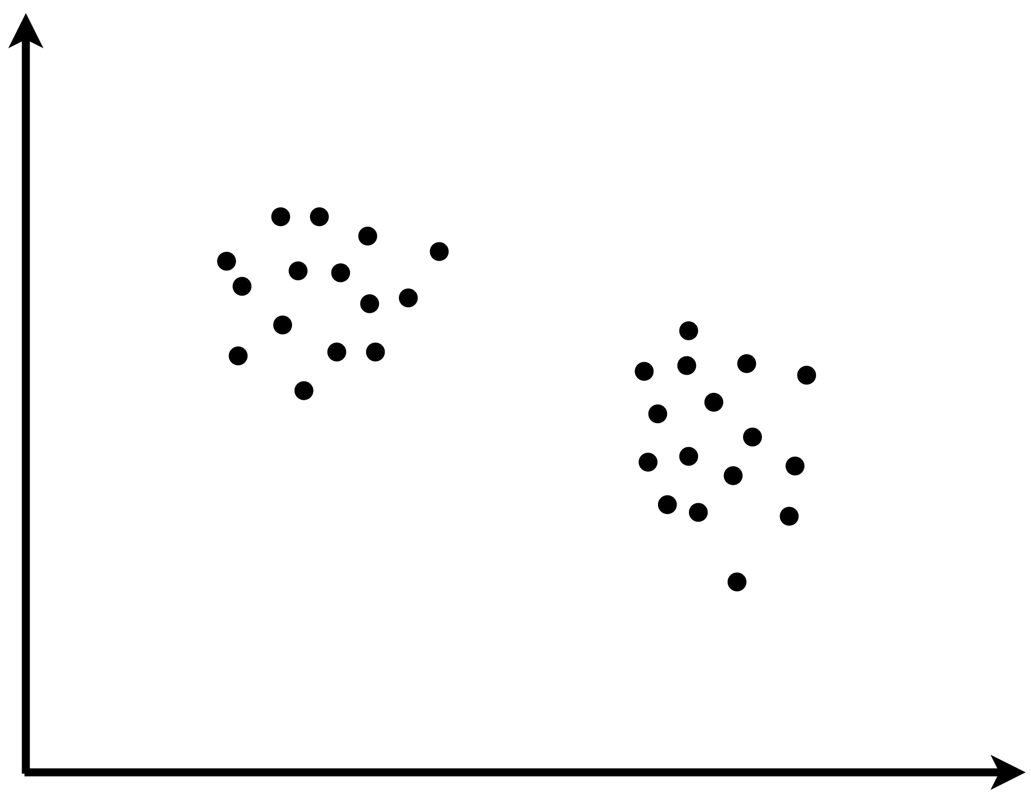
Generative model



- Finite Gaussian mixture model (K=2 clusters)

Generative model

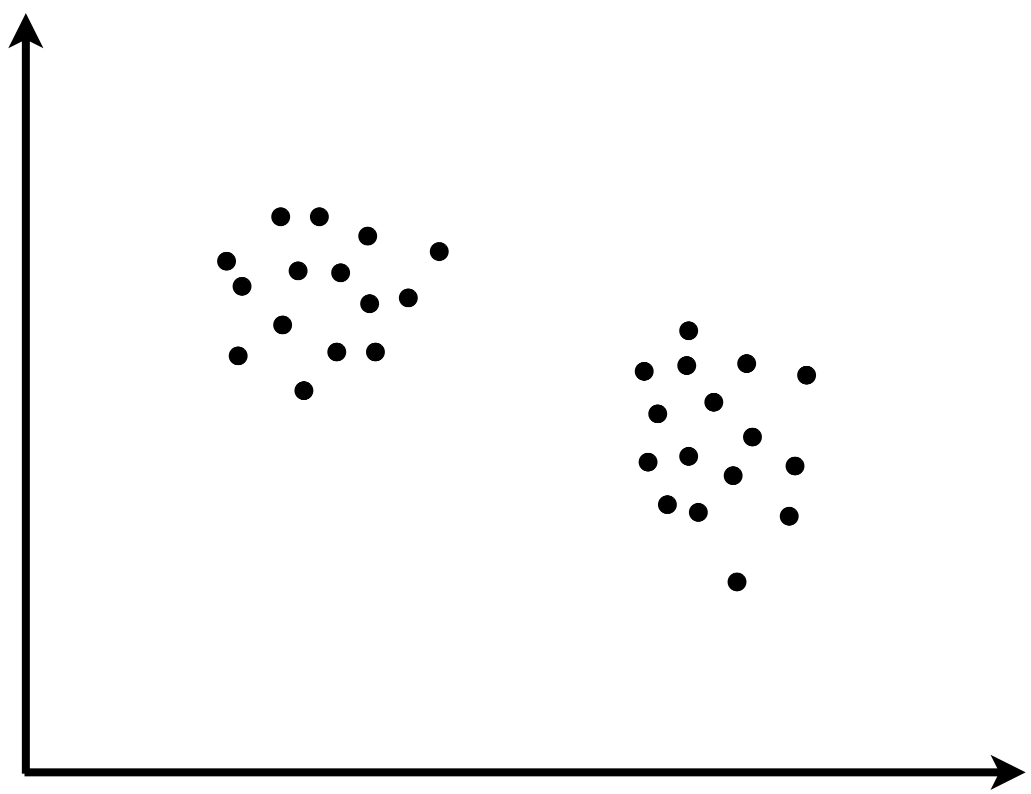
$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$



- Finite Gaussian mixture model (*K*=2 clusters)

Generative model

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$



- Finite Gaussian mixture model (K=2 clusters)

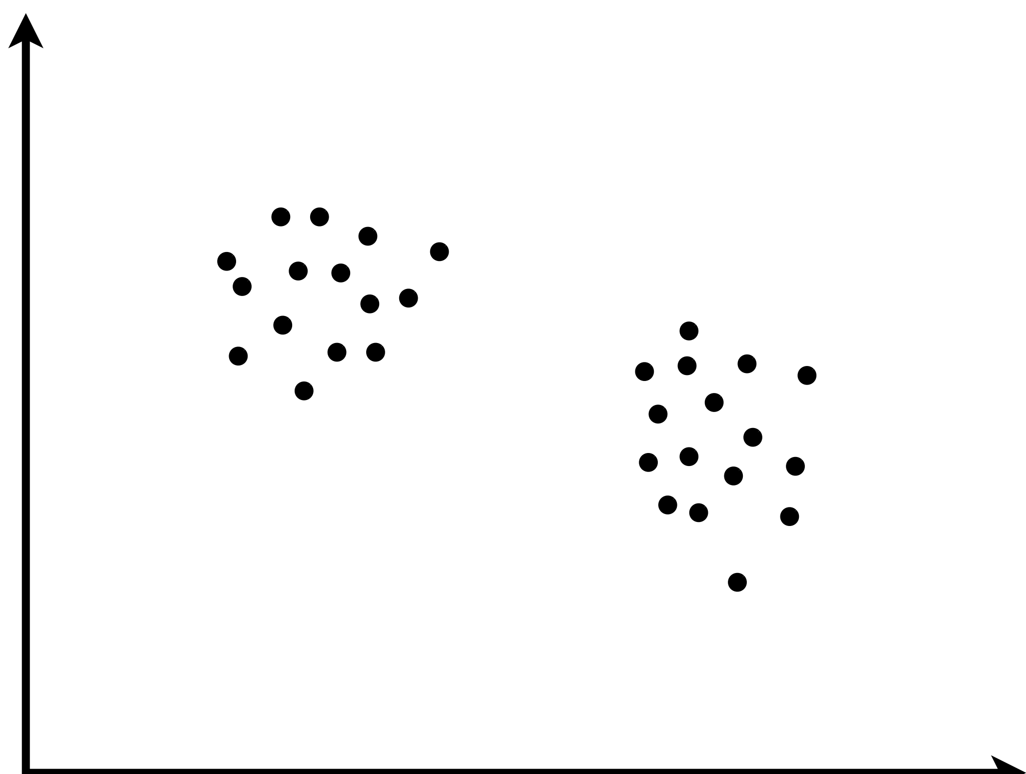
$$z_n \overset{iid}{\sim} \text{Categorical}(\rho_1, \rho_2)$$

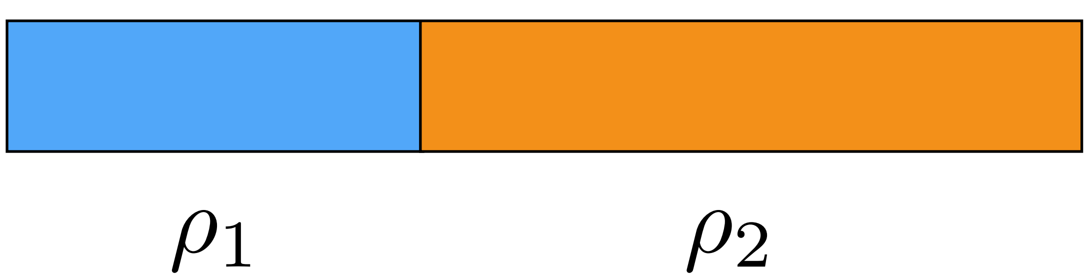
Generative model

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Finite Gaussian mixture model (K=2 clusters)

$$z_n \overset{iid}{\sim} \text{Categorical}(\rho_1, \rho_2)$$



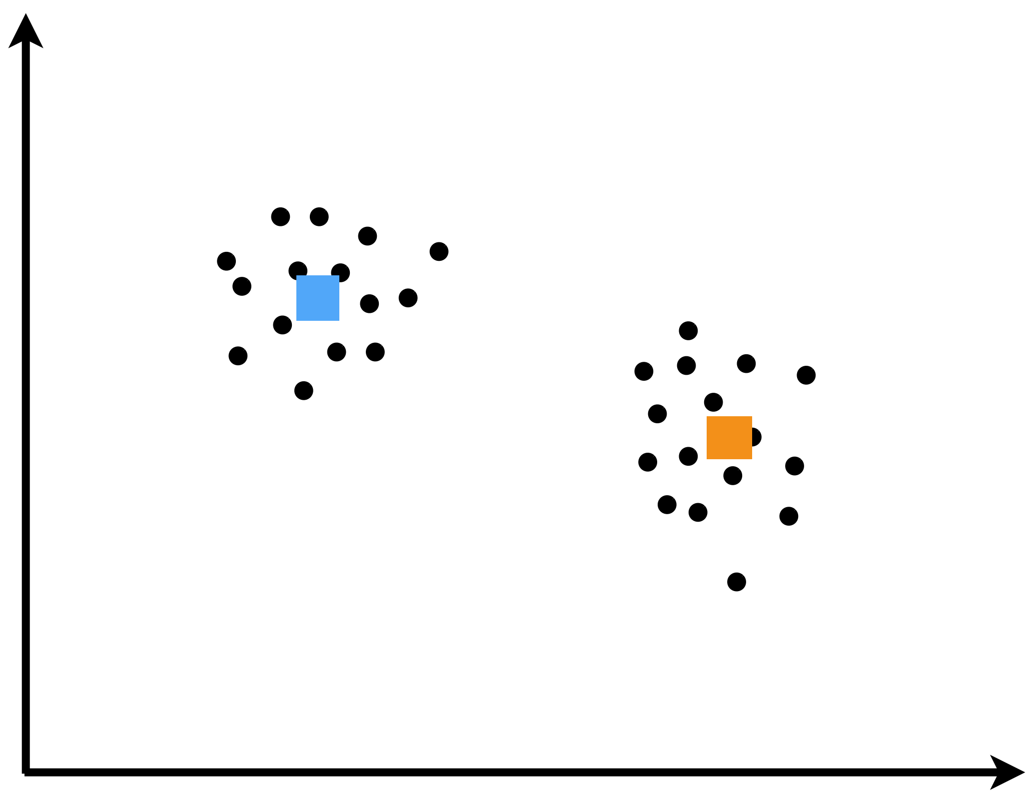


Generative model

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Finite Gaussian mixture model (*K*=2 clusters)

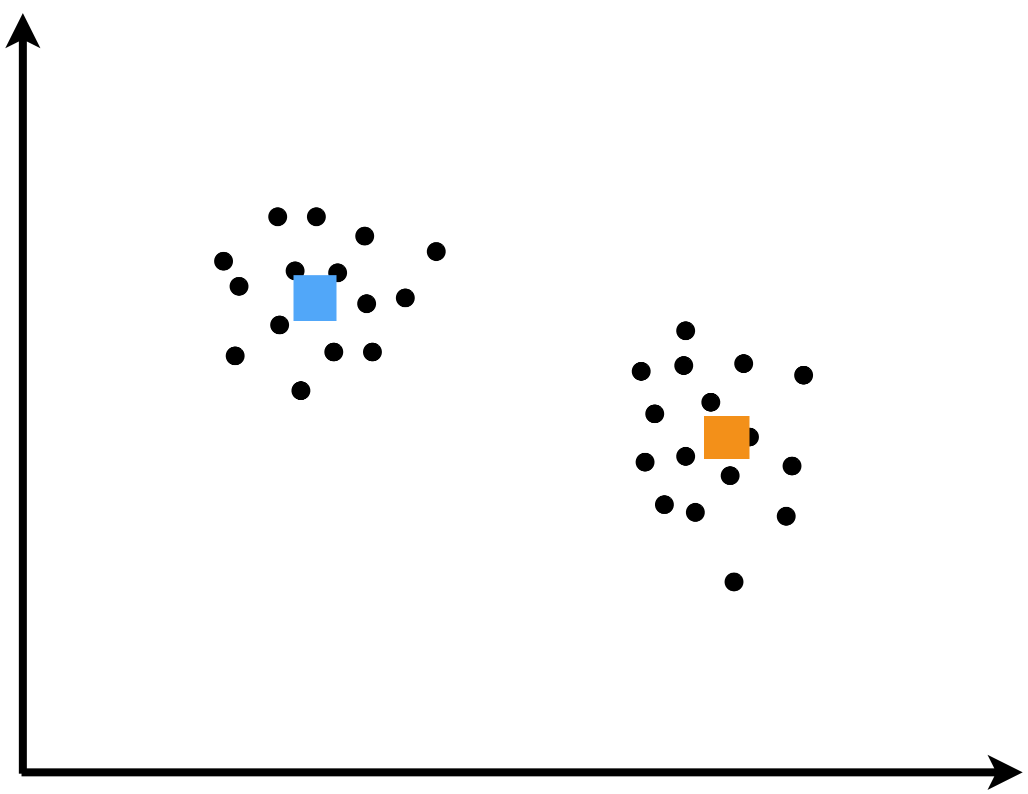
$$z_n \overset{iid}{\sim} \text{Categorical}(\rho_1, \rho_2)$$
$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_{z_n}, \Sigma)$$



ρ_1 ρ_2

Generative model

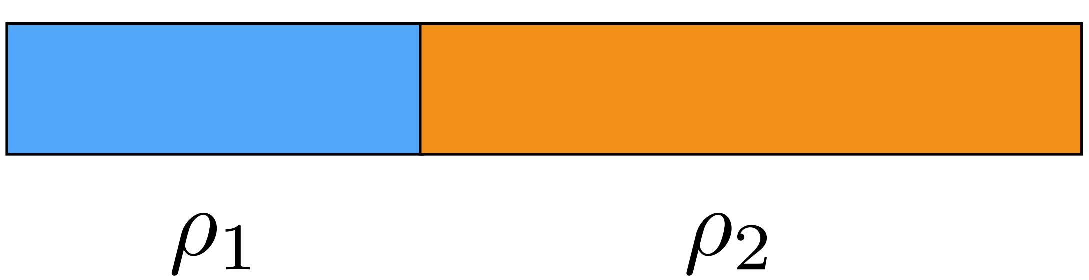
$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$



- Finite Gaussian mixture model (K=2 clusters)

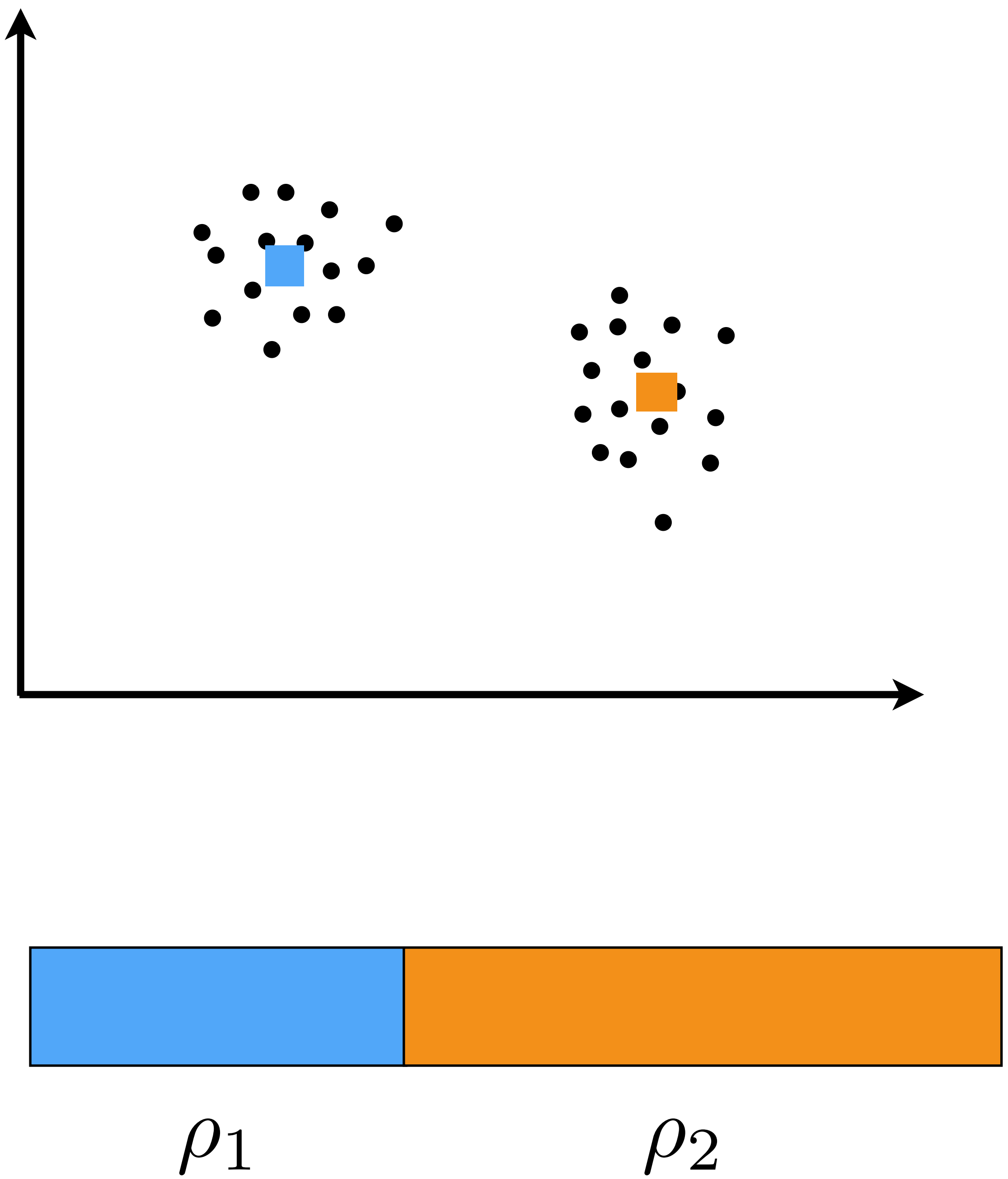
$$z_n \overset{iid}{\sim} \text{Categorical}(\rho_1, \rho_2)$$
$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_{z_n}, \Sigma)$$

- Don't know μ_1, μ_2



Generative model

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$



- Finite Gaussian mixture model (K=2 clusters)

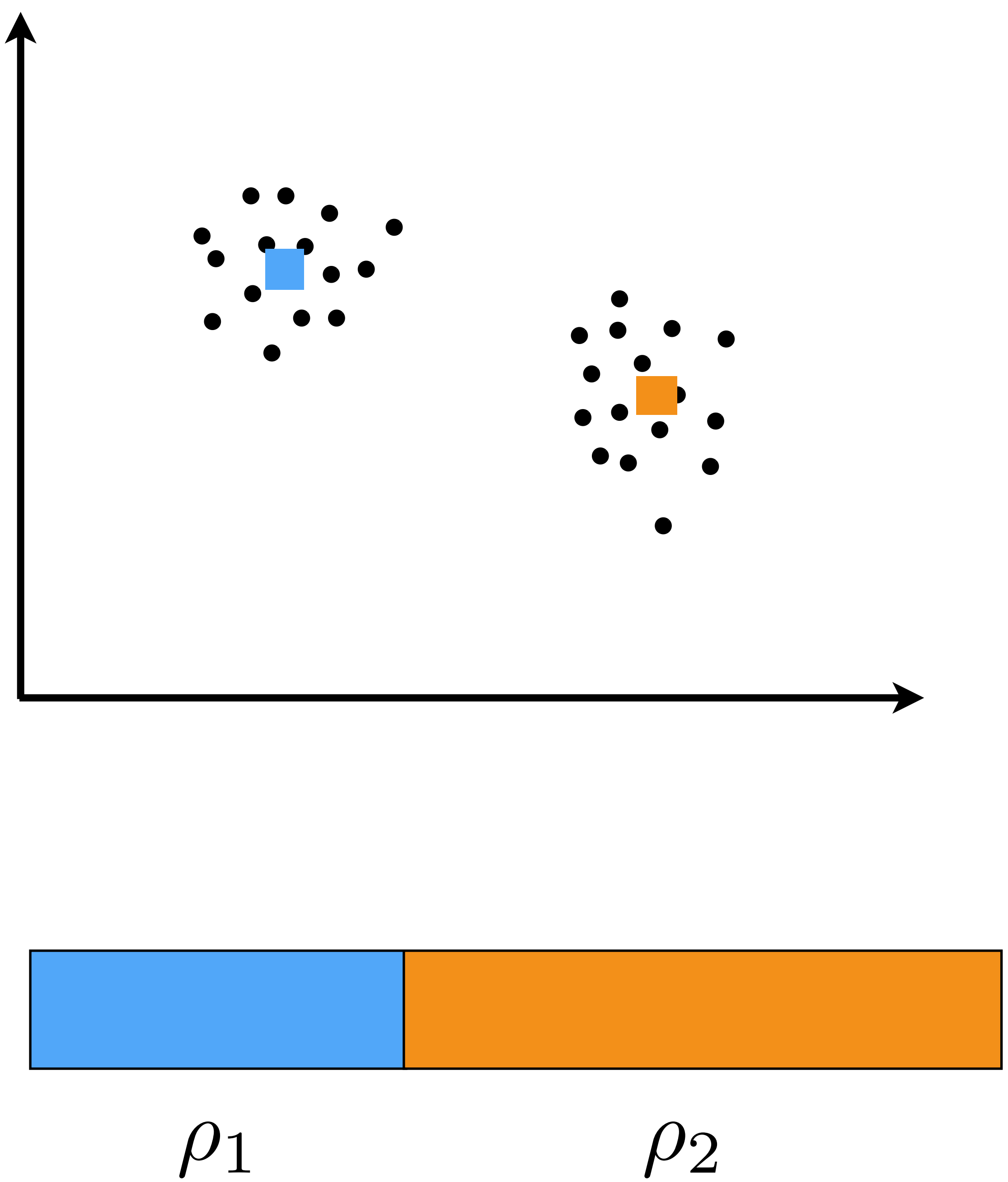
$$z_n \overset{iid}{\sim} \text{Categorical}(\rho_1, \rho_2)$$
$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_{z_n}, \Sigma)$$

- Don't know μ_1, μ_2

$$\mu_k \overset{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0)$$

Generative model

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$



- Finite Gaussian mixture model (*K*=2 clusters)

$$z_n \overset{iid}{\sim} \text{Categorical}(\rho_1, \rho_2)$$
$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_{z_n}, \Sigma)$$

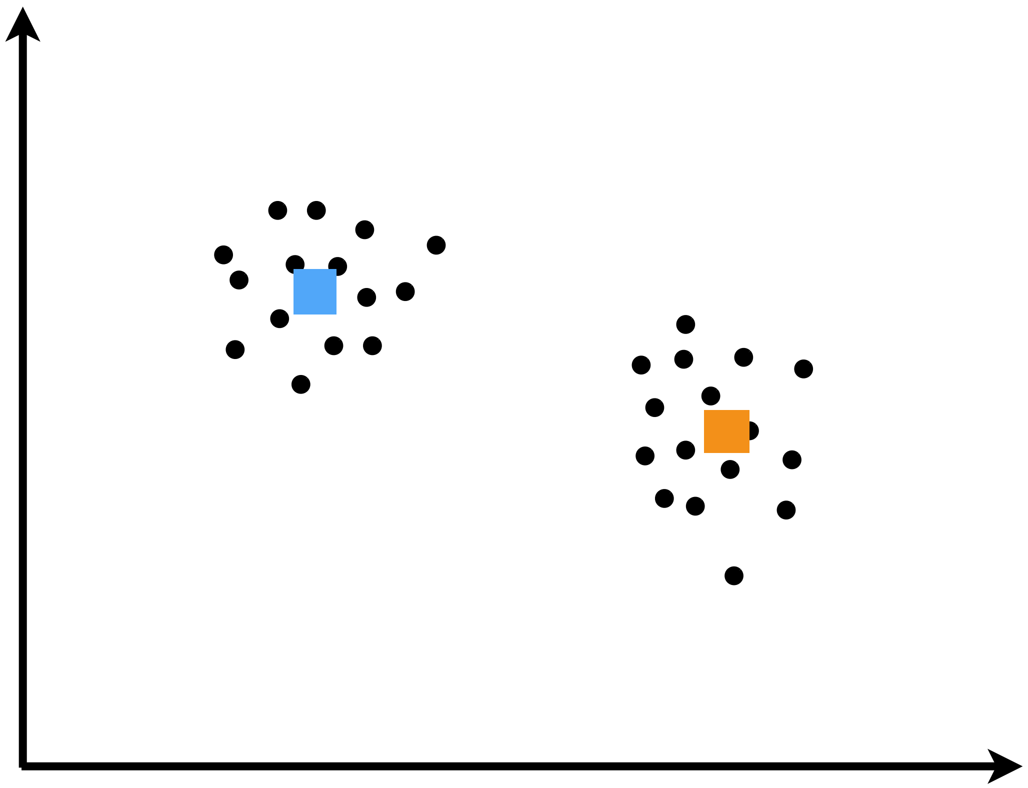
- Don't know μ_1, μ_2

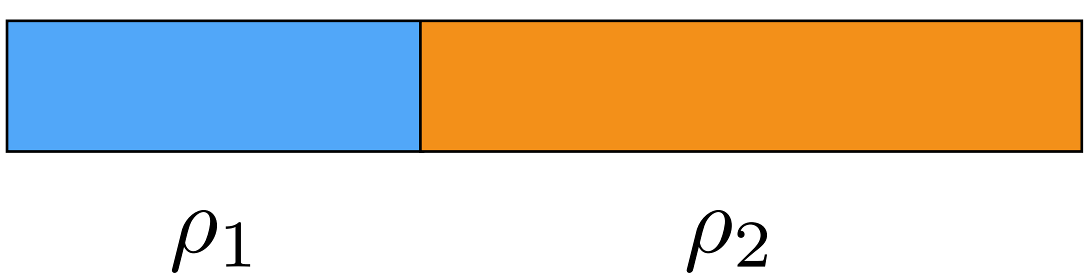
$$\mu_k \overset{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0)$$

- Don't know ρ_1, ρ_2

Generative model

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$





- Finite Gaussian mixture model (*K*=2 clusters)

$$z_n \overset{iid}{\sim} \text{Categorical}(\rho_1, \rho_2)$$
$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_{z_n}, \Sigma)$$

- Don't know μ_1, μ_2

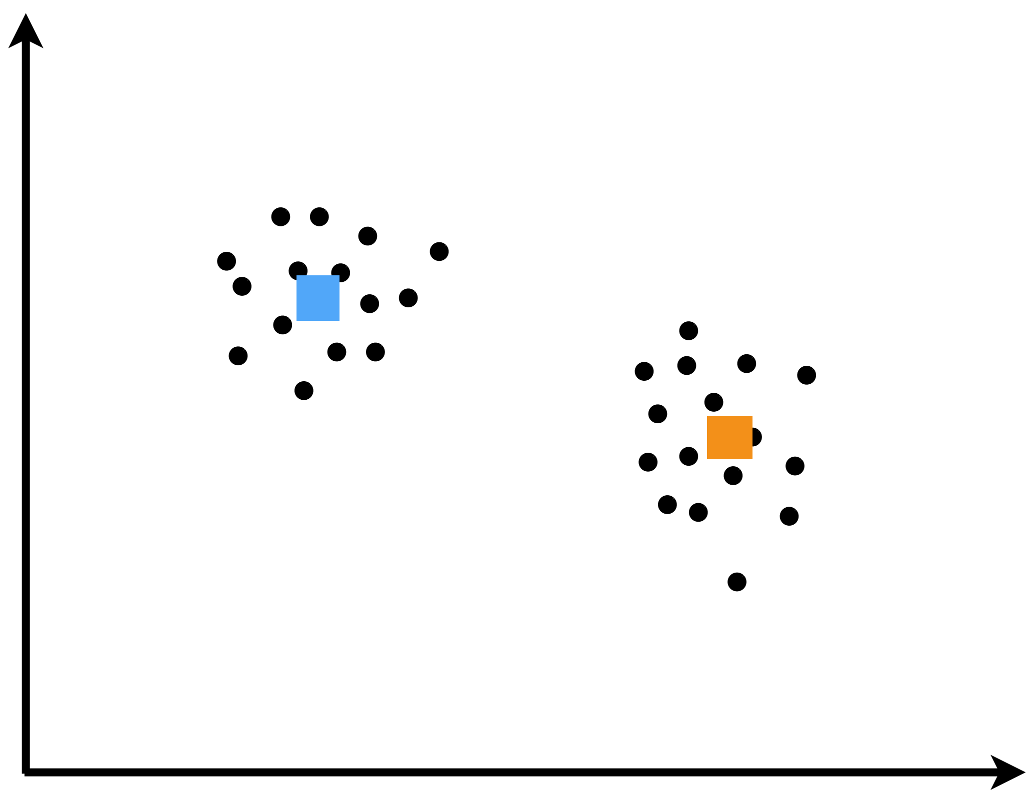
$$\mu_k \overset{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0)$$

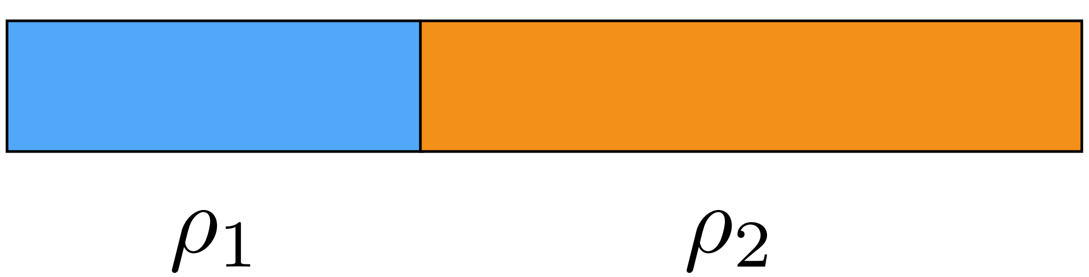
- Don't know ρ_1, ρ_2

$$\rho_1 \sim \text{Beta}(a_1, a_2)$$
$$\rho_2 = 1 - \rho_1$$

Generative model

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$





- Finite Gaussian mixture model (*K*=2 clusters)

$$z_n \overset{iid}{\sim} \text{Categorical}(\rho_1, \rho_2)$$
$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_{z_n}, \Sigma)$$

- Don't know μ_1, μ_2

$$\mu_k \overset{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0)$$

- Don't know ρ_1, ρ_2

$$\rho_1 \sim \text{Beta}(a_1, a_2)$$
$$\rho_2 = 1 - \rho_1$$

- Inference goal: assignments of data points to clusters, cluster parameters

Generative model

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Finite Gaussian mixture model (K=2 clusters)

$$z_n \overset{iid}{\sim} \text{Categorical}(\rho_1, \rho_2)$$
$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_{z_n}, \Sigma)$$

- Don't know μ_1, μ_2

$$\mu_k \overset{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0)$$

- Don't know ρ_1, ρ_2

$$\rho_1 \sim \text{Beta}(a_1, a_2)$$
$$\rho_2 = 1 - \rho_1$$

- Inference goal: assignments of data points to clusters, cluster parameters

Generative model

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

- Finite Gaussian mixture model (K=2 clusters)

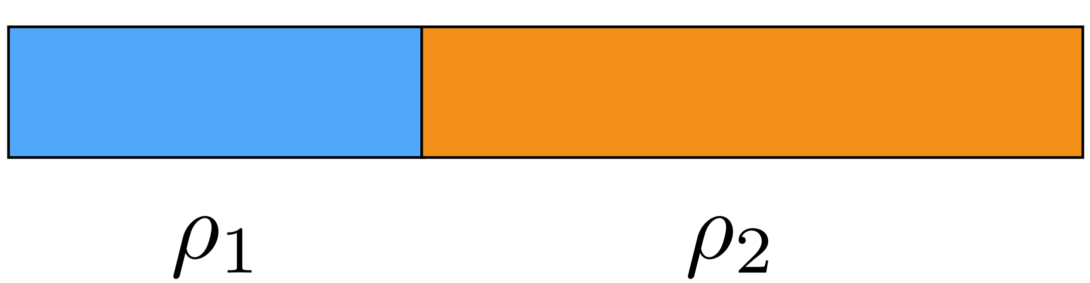
$$z_n \overset{iid}{\sim} \text{Categorical}(\rho_1, \rho_2)$$
$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_{z_n}, \Sigma)$$

- Don't know μ_1, μ_2

$$\mu_k \overset{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0)$$

- Don't know ρ_1, ρ_2

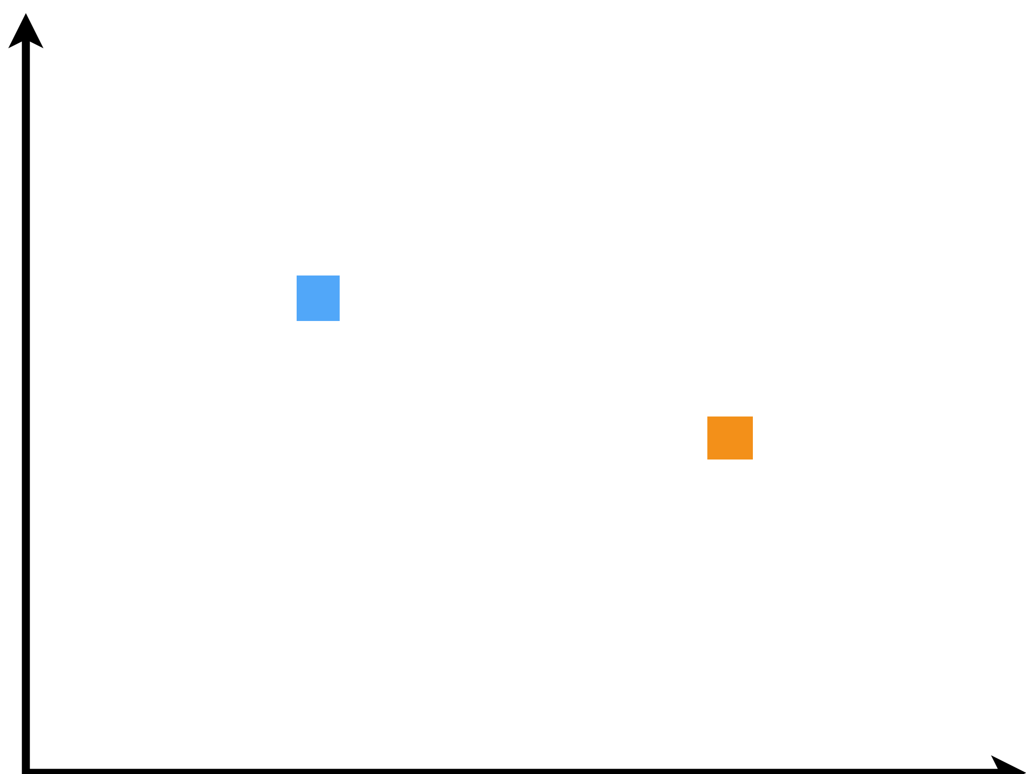
$$\rho_1 \sim \text{Beta}(a_1, a_2)$$
$$\rho_2 = 1 - \rho_1$$

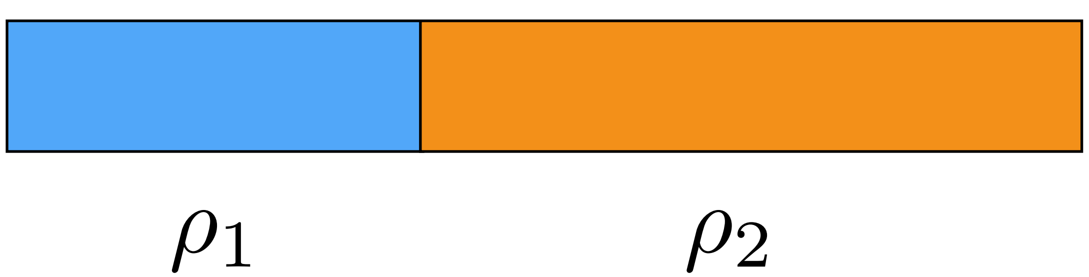


- Inference goal: assignments of data points to clusters, cluster parameters

Generative model

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$





- Finite Gaussian mixture model (*K*=2 clusters)

$$z_n \stackrel{iid}{\sim} \text{Categorical}(\rho_1, \rho_2)$$
$$x_n \stackrel{indep}{\sim} \mathcal{N}(\mu_{z_n}, \Sigma)$$

- Don't know μ_1, μ_2

$$\mu_k \stackrel{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0)$$

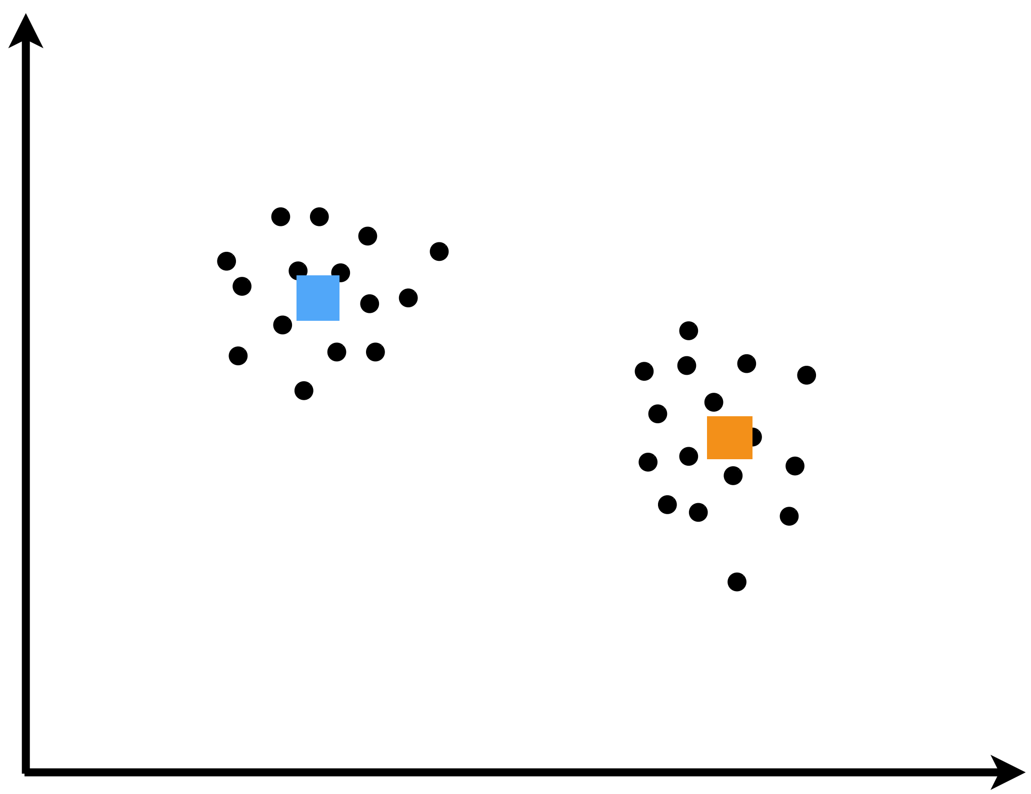
- Don't know ρ_1, ρ_2

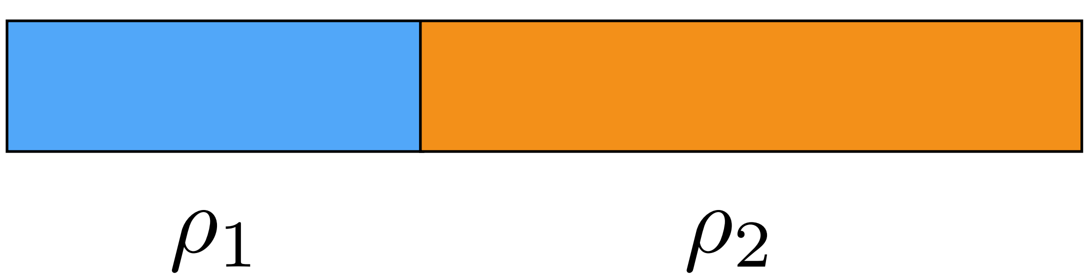
$$\rho_1 \sim \text{Beta}(a_1, a_2)$$
$$\rho_2 = 1 - \rho_1$$

- Inference goal: assignments of data points to clusters, cluster parameters

Generative model

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$





- Finite Gaussian mixture model (K=2 clusters)

$$z_n \overset{iid}{\sim} \text{Categorical}(\rho_1, \rho_2)$$
$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_{z_n}, \Sigma)$$

- Don't know μ_1, μ_2

$$\mu_k \overset{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0)$$

- Don't know ρ_1, ρ_2

$$\rho_1 \sim \text{Beta}(a_1, a_2)$$
$$\rho_2 = 1 - \rho_1$$

- Inference goal: assignments of data points to clusters, cluster parameters

Beta distribution review

$$\mathrm{Beta}(\rho_1 | a_1, a_2) = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} \rho_1^{a_1 - 1} (1 - \rho_1)^{a_2 - 1}$$

$$\rho_1 \in (0, 1)$$

$$a_1, a_2 > 0$$

Beta distribution review

$$\mathrm{Beta}(\rho_1|a_1,a_2) = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\rho_1^{a_1-1}(1-\rho_1)^{a_2-1}$$

$\rho_1 \in (0,1)$

$a_1, a_2 > 0$

- Gamma function Γ

Beta distribution review

$$\rho_1 \in (0,1)$$

$$\mathrm{Beta}(\rho_1 | a_1, a_2) = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} \rho_1^{a_1 - 1} (1 - \rho_1)^{a_2 - 1} \qquad a_1, a_2 > 0$$

- Gamma function Γ
  - integer *m*: $\Gamma(m+1) = m!$

Beta distribution review

$$\mathrm{Beta}(\rho_1|a_1, a_2) = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)}\rho_1^{a_1-1}(1-\rho_1)^{a_2-1}$$

$\rho_1 \in (0, 1)$

$a_1, a_2 > 0$

- Gamma function Γ
  - integer *m*: $\Gamma(m+1) = m!$
  - for *x* > 0: $\Gamma(x+1) = x\Gamma(x)$

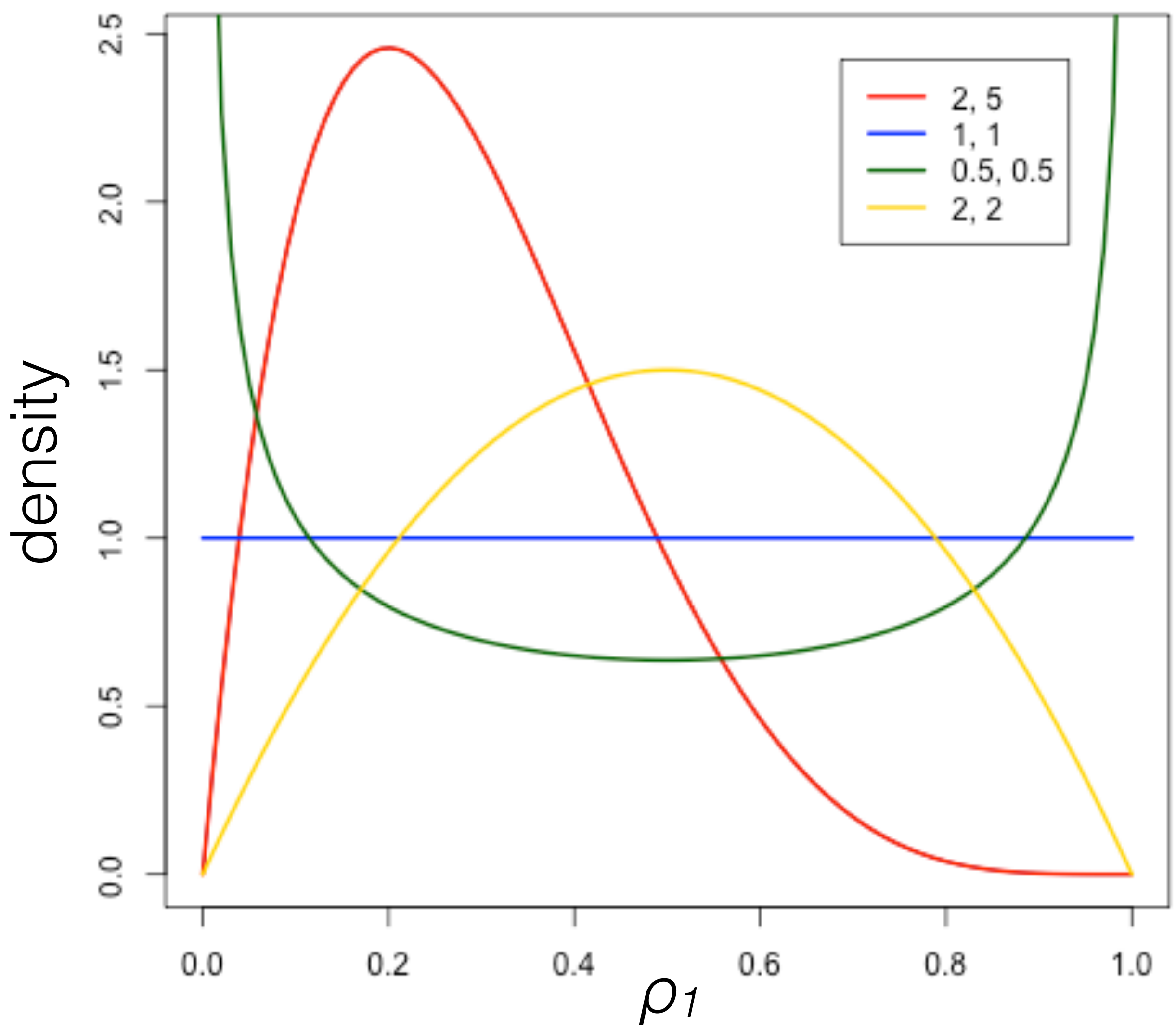
Beta distribution review

$$\mathrm{Beta}(\rho_1|a_1, a_2) = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)}\rho_1^{a_1-1}(1-\rho_1)^{a_2-1} \qquad \begin{array}{l}\rho_1 \in (0,1)\\ a_1, a_2 > 0\end{array}$$

- Gamma function Γ
  - integer *m*: $\Gamma(m+1) = m!$
  - for *x* > 0: $\Gamma(x+1) = x\Gamma(x)$
- What happens?

Beta distribution review

$$\mathrm{Beta}(\rho_1|a_1,a_2) = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\rho_1^{a_1-1}(1-\rho_1)^{a_2-1} \qquad \begin{array}{l}\rho_1 \in (0,1)\\ a_1,a_2>0\end{array}$$

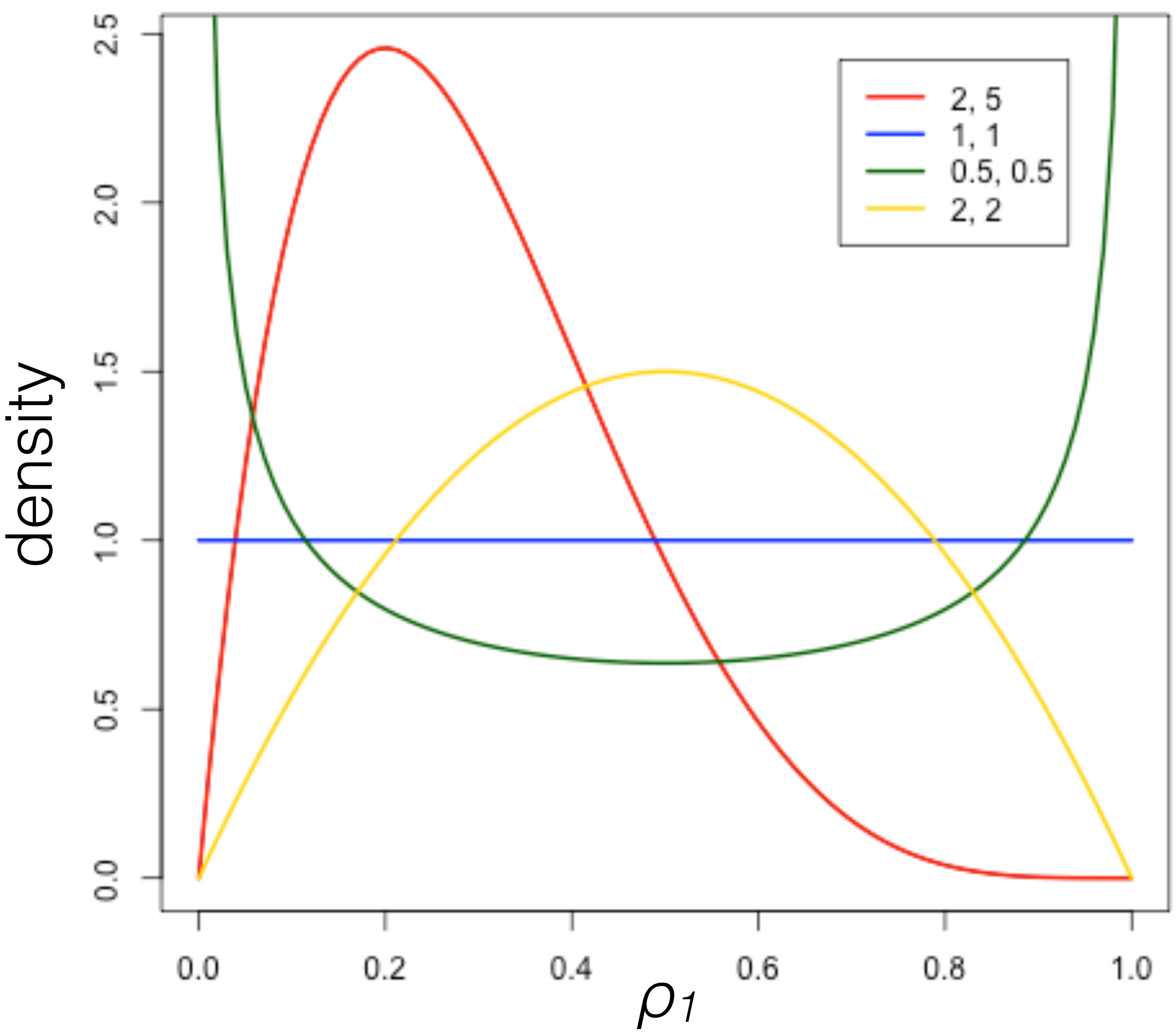


- Gamma function Γ
  - integer *m*: $\Gamma(m+1) = m!$
  - for *x* > 0: $\Gamma(x+1) = x\Gamma(x)$
- What happens?

Beta distribution review

$$\rho_1 \in (0, 1)$$

$$\text{Beta}(\rho_1 | a_1, a_2) = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} \rho_1^{a_1 - 1} (1 - \rho_1)^{a_2 - 1} \qquad a_1, a_2 > 0$$



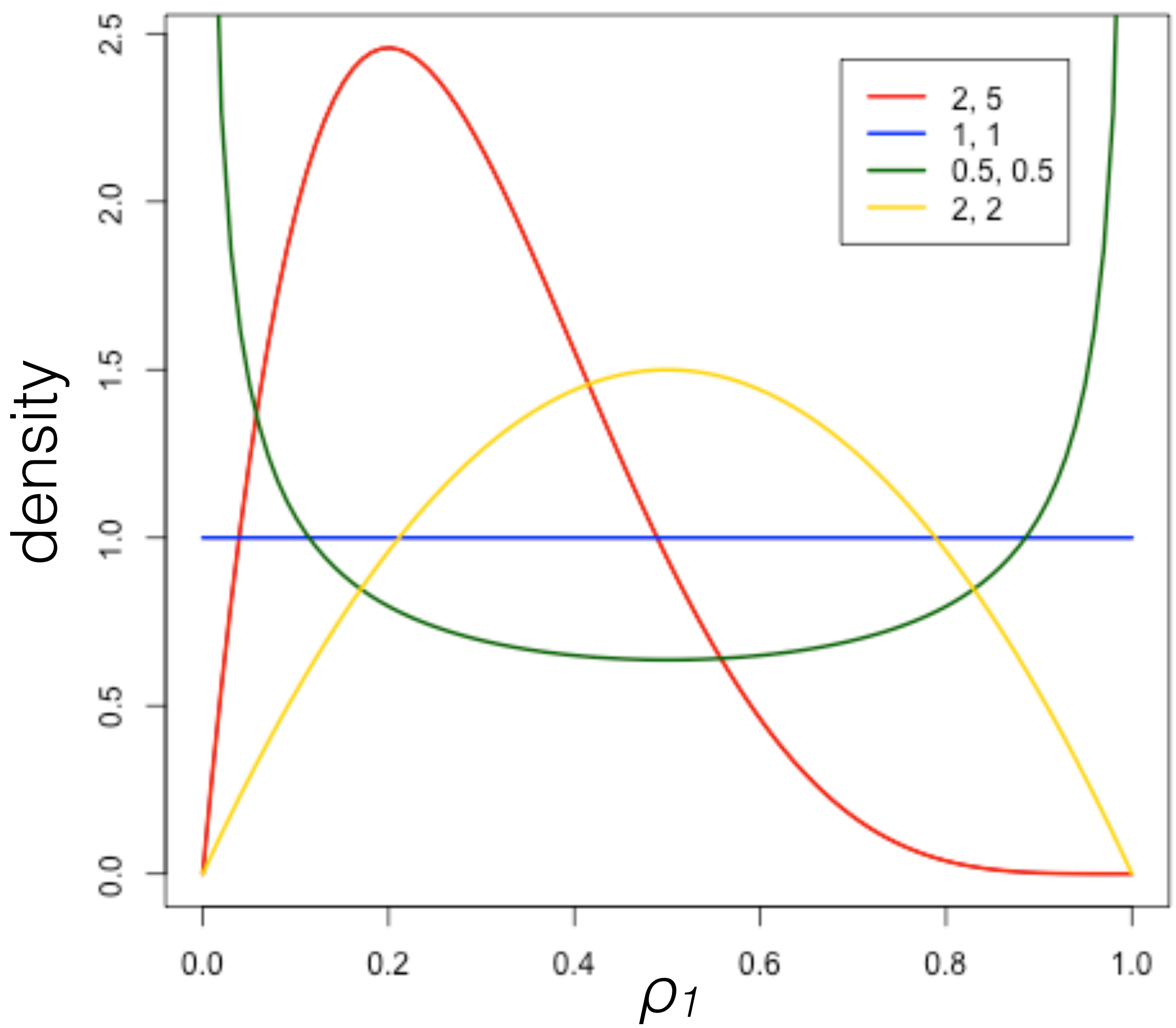
- Gamma function Γ
  - integer *m*: $\Gamma(m+1) = m!$
  - for *x* > 0: $\Gamma(x+1) = x\Gamma(x)$
- What happens?
  - $a = a_1 = a_2 \to 0$

Beta distribution review

$$\mathrm{Beta}(\rho_1|a_1,a_2) = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\rho_1^{a_1-1}(1-\rho_1)^{a_2-1}$$

$\rho_1 \in (0,1)$

$a_1, a_2 > 0$



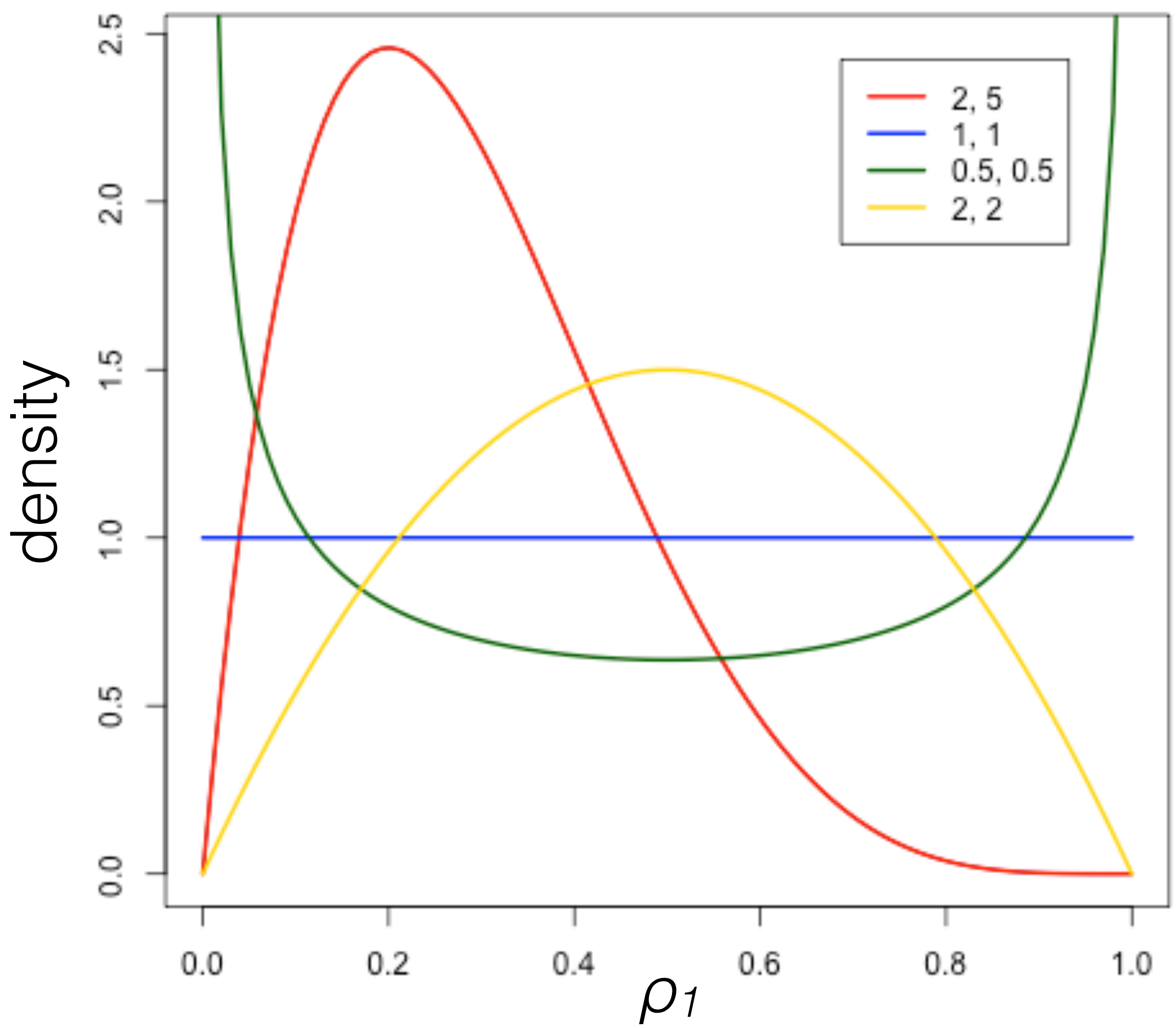
- Gamma function Γ
  - integer *m*: $\Gamma(m+1) = m!$
  - for *x* > 0: $\Gamma(x+1) = x\Gamma(x)$
- What happens?
  - $a = a_1 = a_2 \to 0$
  - $a = a_1 = a_2 \to \infty$

Beta distribution review

$$\mathrm{Beta}(\rho_1|a_1, a_2) = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} \rho_1^{a_1 - 1}(1 - \rho_1)^{a_2 - 1}$$

$\rho_1 \in (0, 1)$

$a_1, a_2 > 0$



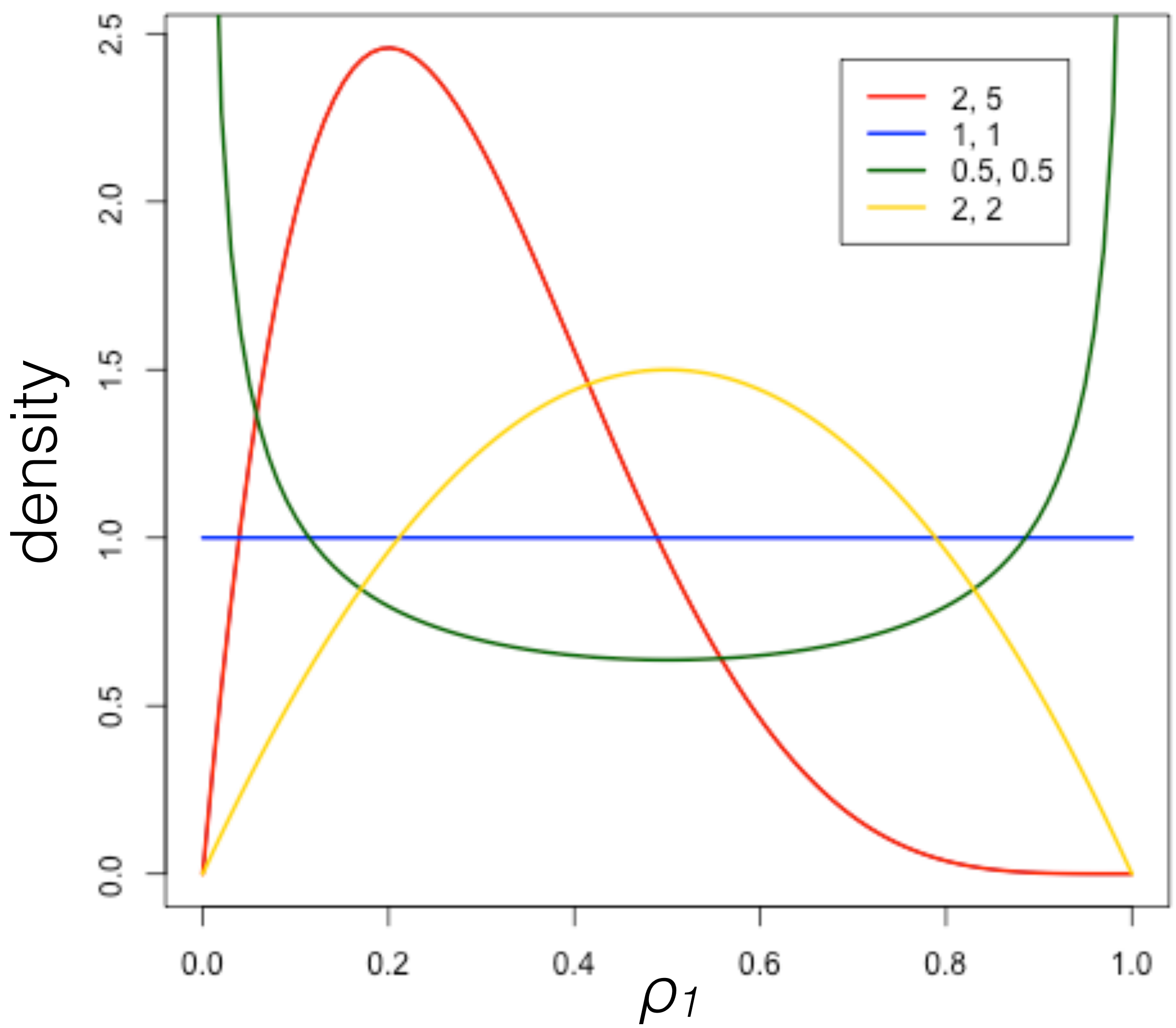
- Gamma function Γ
  - integer *m*: $\Gamma(m+1) = m!$
  - for *x* > 0: $\Gamma(x+1) = x\Gamma(x)$
- What happens?
  - $a = a_1 = a_2 \to 0$
  - $a = a_1 = a_2 \to \infty$
  - $a_1 > a_2$

Beta distribution review

$$\mathrm{Beta}(\rho_1|a_1, a_2) = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} \rho_1^{a_1 - 1}(1 - \rho_1)^{a_2 - 1}$$

$\rho_1 \in (0, 1)$

$a_1, a_2 > 0$



- Gamma function Γ
  - integer *m*: $\Gamma(m+1) = m!$
  - for *x* > 0: $\Gamma(x+1) = x\Gamma(x)$
- What happens?
  - $a = a_1 = a_2 \to 0$
  - $a = a_1 = a_2 \to \infty$
  - $a_1 > a_2$

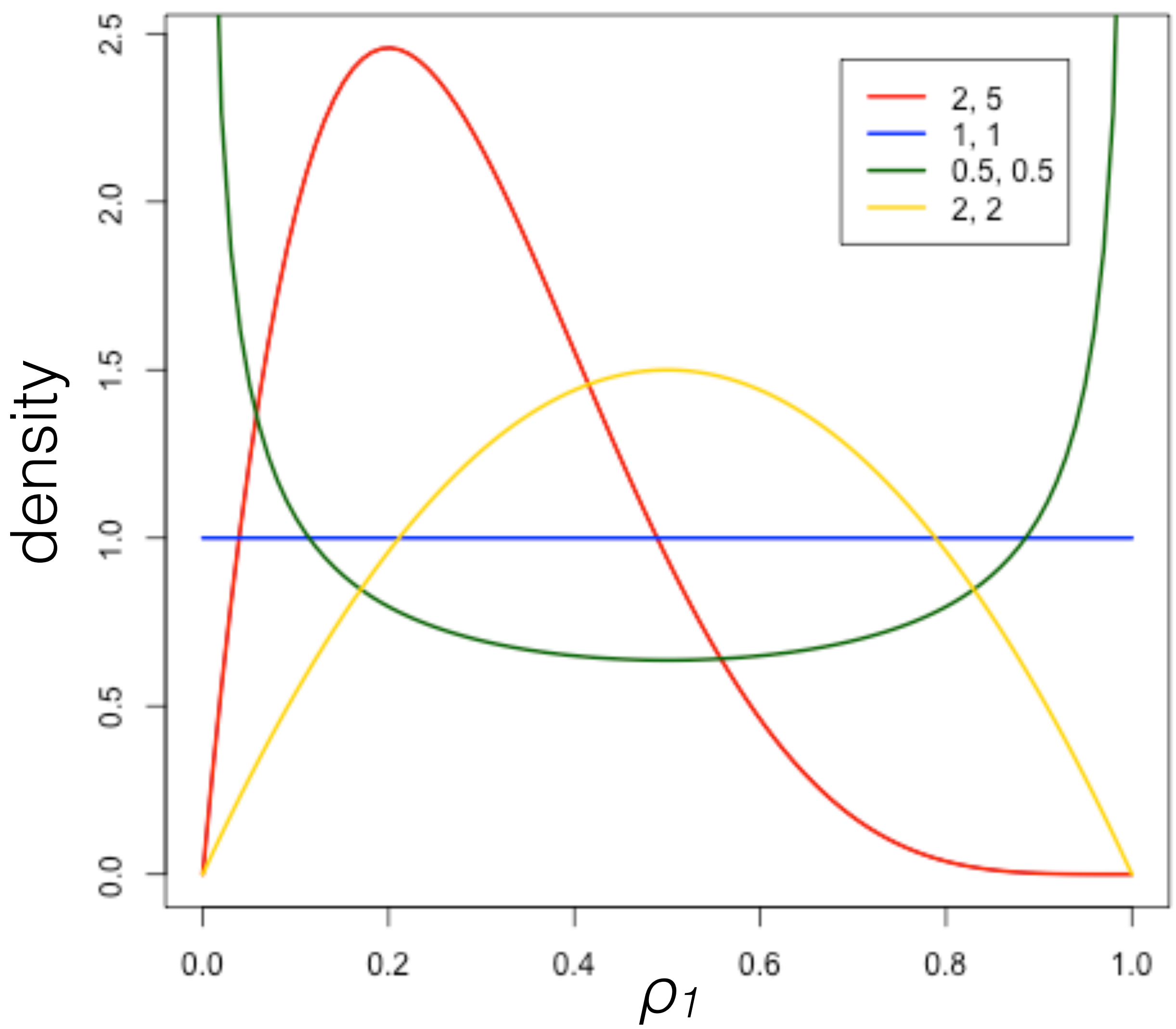
[demo]

Beta distribution review

$$\mathrm{Beta}(\rho_1|a_1,a_2) = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\rho_1^{a_1-1}(1-\rho_1)^{a_2-1}$$

$\rho_1 \in (0,1)$

$a_1, a_2 > 0$



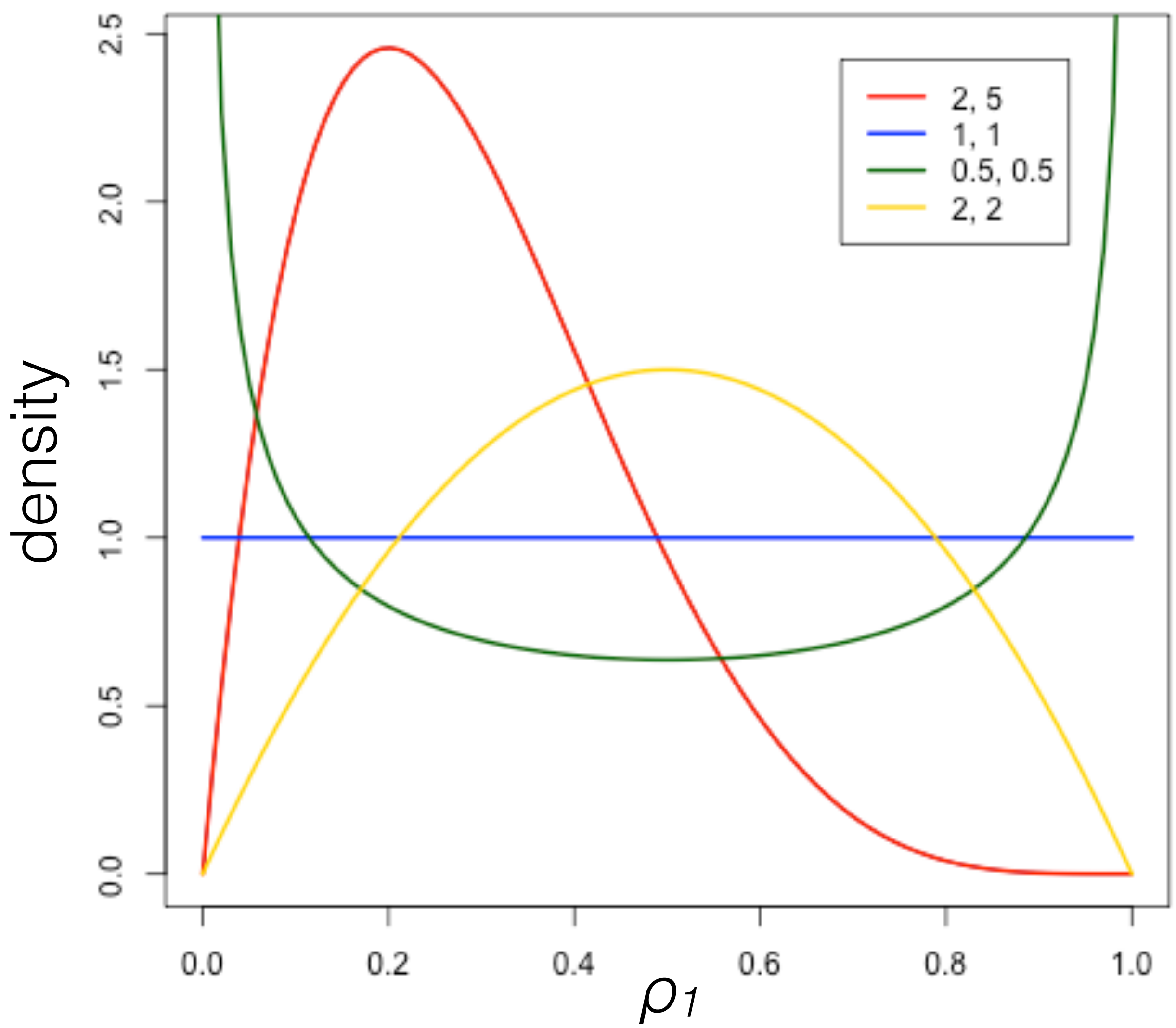
- Gamma function Γ
  - integer *m*: $\Gamma(m+1) = m!$
  - for *x* > 0: $\Gamma(x+1) = x\Gamma(x)$
- What happens?
  - $a = a_1 = a_2 \to 0$
  - $a = a_1 = a_2 \to \infty$
  - $a_1 > a_2$ [demo]
- Beta is conjugate to Cat

Beta distribution review

$$\mathrm{Beta}(\rho_1|a_1,a_2) = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\rho_1^{a_1-1}(1-\rho_1)^{a_2-1}$$

$\rho_1 \in (0,1)$

$a_1, a_2 > 0$



- Gamma function Γ
  - integer *m*: $\Gamma(m+1) = m!$
  - for *x* > 0: $\Gamma(x+1) = x\Gamma(x)$
- What happens?
  - $a = a_1 = a_2 \to 0$
  - $a = a_1 = a_2 \to \infty$
  - $a_1 > a_2$ [demo]
- Beta is conjugate to Cat

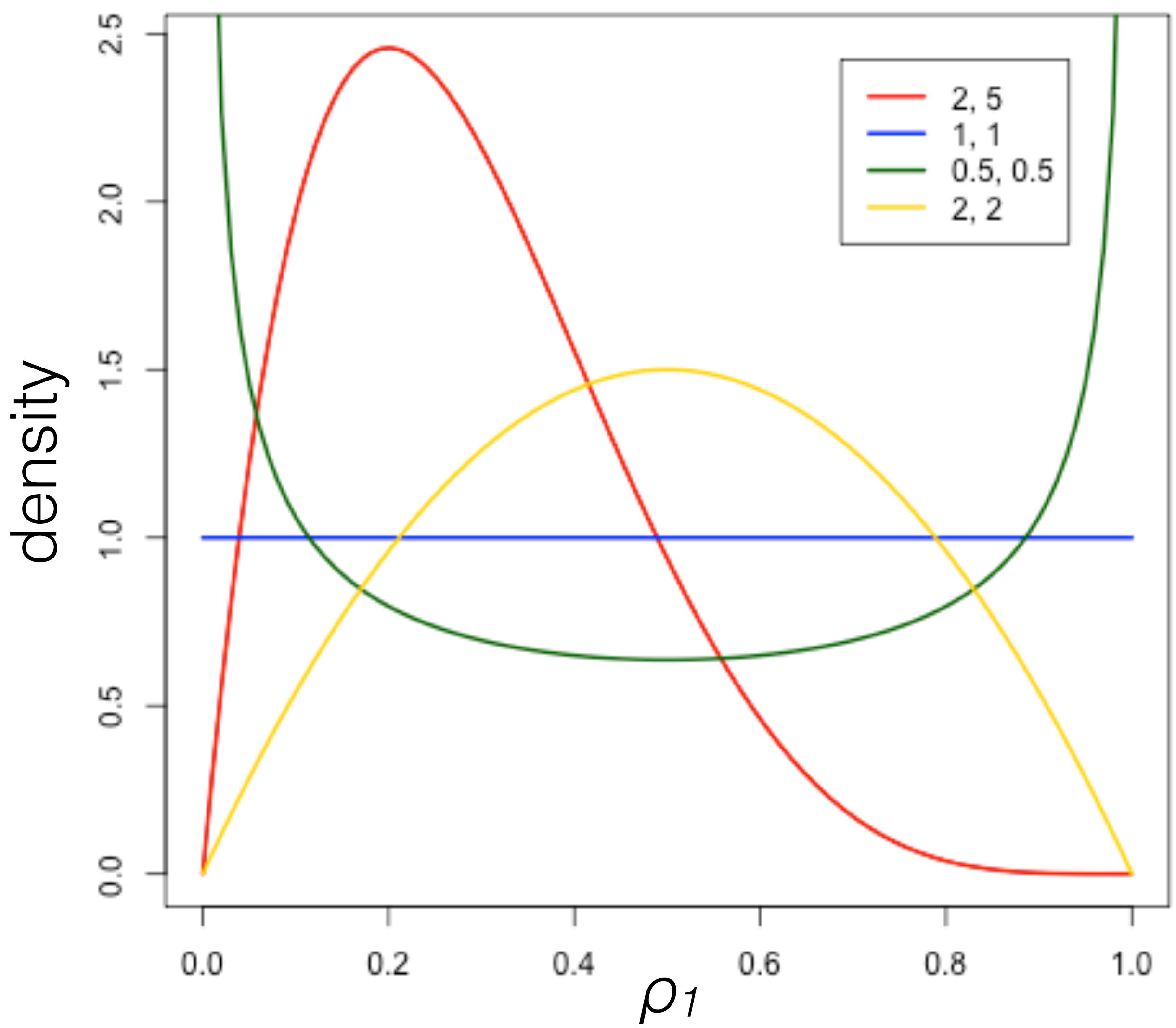
$$\rho_1 \sim \mathrm{Beta}(a_1,a_2),\, z \sim \mathrm{Cat}(\rho_1,\rho_2)$$

Beta distribution review

$$\mathrm{Beta}(\rho_1|a_1,a_2) = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\rho_1^{a_1-1}(1-\rho_1)^{a_2-1}$$

$\rho_1 \in (0,1)$

$a_1, a_2 > 0$



- Gamma function Γ
  - integer *m*: $\Gamma(m+1) = m!$
  - for *x* > 0: $\Gamma(x+1) = x\Gamma(x)$
- What happens?
  - $a = a_1 = a_2 \to 0$
  - $a = a_1 = a_2 \to \infty$
  - $a_1 > a_2$ [demo]
- Beta is conjugate to Cat

$\rho_1 \sim \mathrm{Beta}(a_1,a_2), z \sim \mathrm{Cat}(\rho_1,\rho_2)$

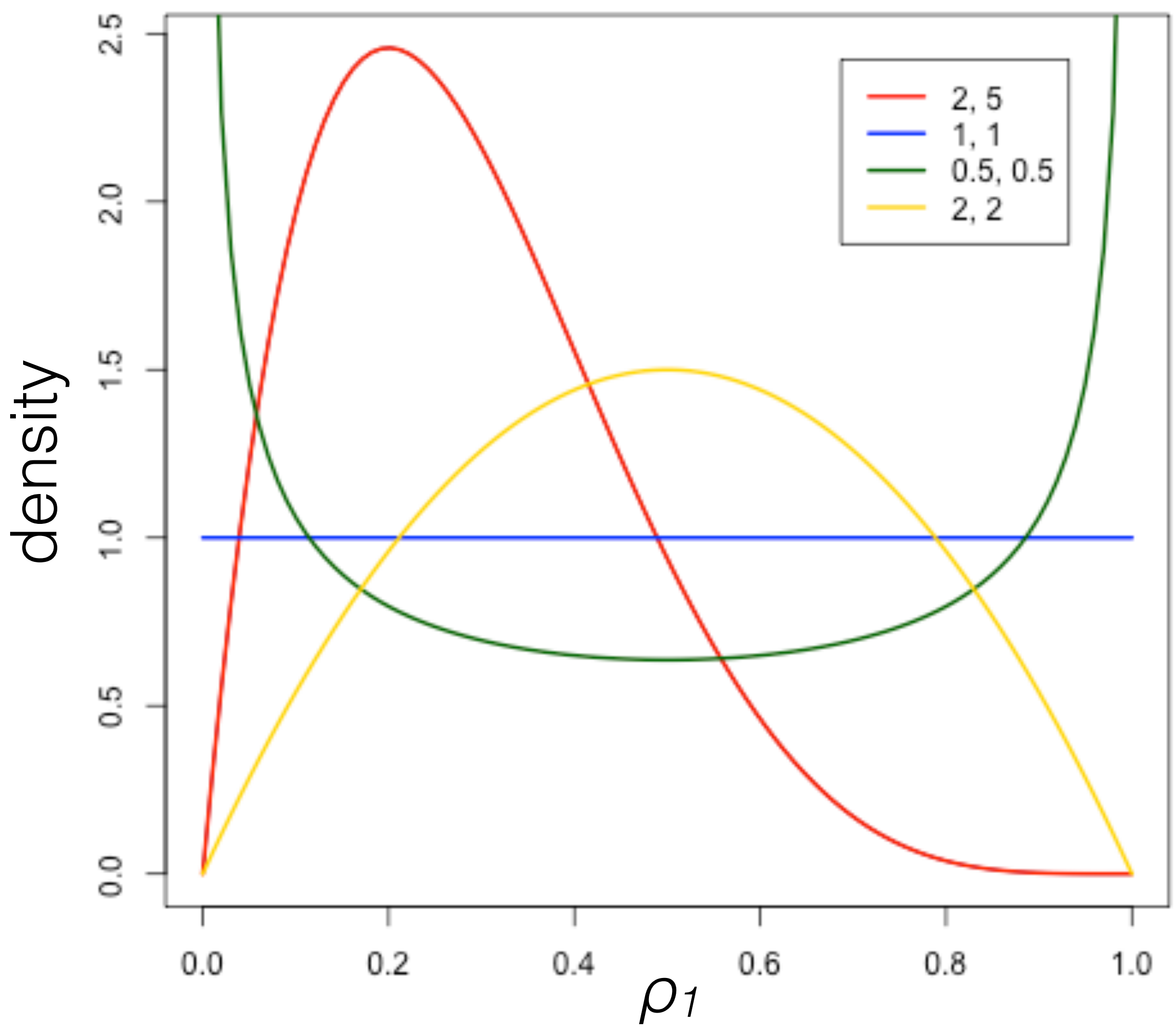
$p(\rho_1, z) \propto$

Beta distribution review

$$\mathrm{Beta}(\rho_1 | a_1, a_2) = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} \rho_1^{a_1 - 1} (1 - \rho_1)^{a_2 - 1}$$

$\rho_1 \in (0, 1)$

$a_1, a_2 > 0$



$$p(\rho_1, z) \propto \rho_1^{\mathbf{1}\{z=1\}} (1 - \rho_1)^{\mathbf{1}\{z=2\}}$$

- Gamma function Γ
  - integer *m*: $\Gamma(m + 1) = m!$
  - for *x* > 0: $\Gamma(x + 1) = x\Gamma(x)$
- What happens?
  - $a = a_1 = a_2 \to 0$
  - $a = a_1 = a_2 \to \infty$
  - $a_1 > a_2$ [demo]
- Beta is conjugate to Cat

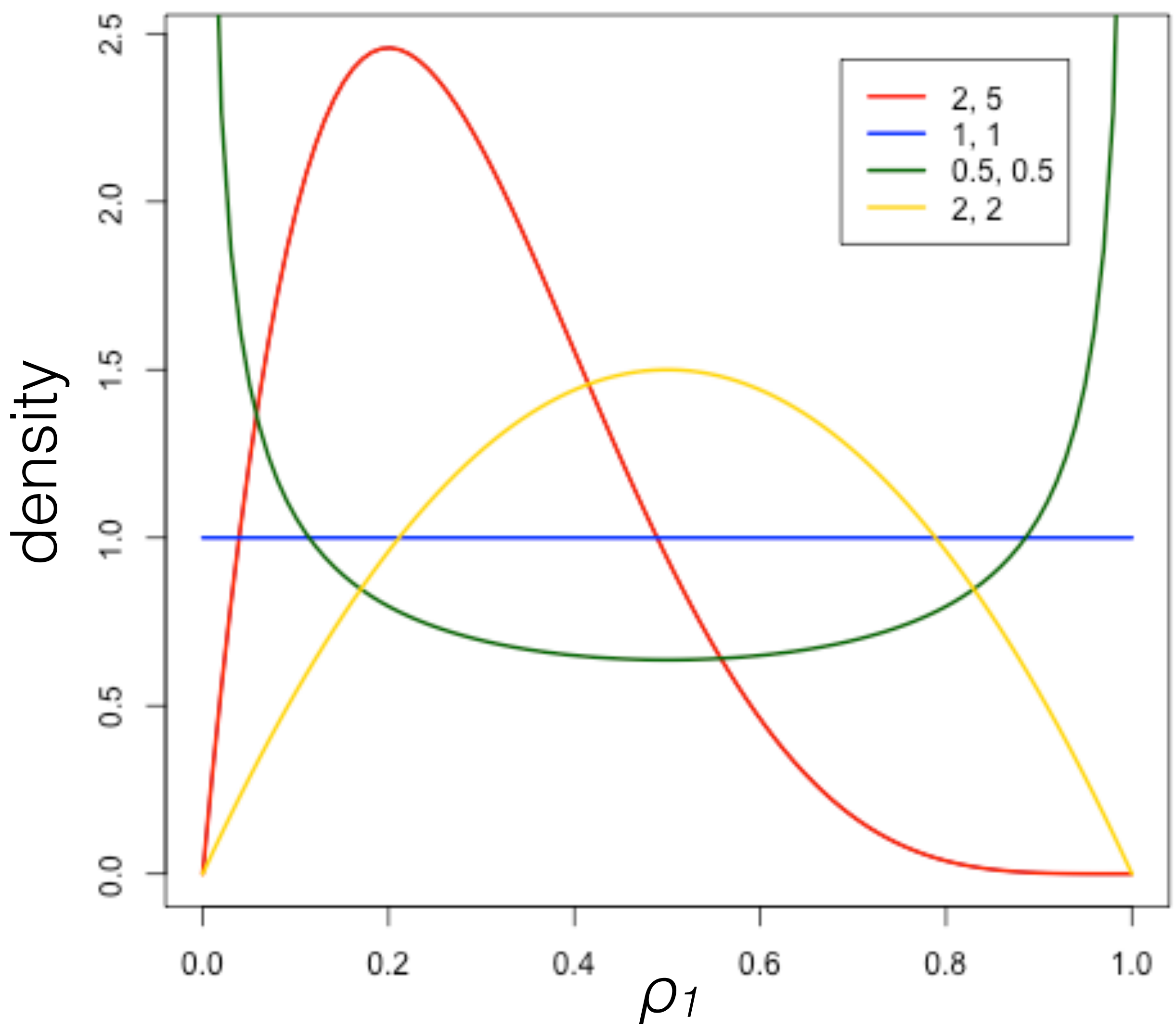
$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z \sim \mathrm{Cat}(\rho_1, \rho_2)$$

Beta distribution review

$$\mathrm{Beta}(\rho_1|a_1, a_2) = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} \rho_1^{a_1-1}(1-\rho_1)^{a_2-1}$$

$\rho_1 \in (0, 1)$

$a_1, a_2 > 0$



- Gamma function Γ
  - integer *m*: $\Gamma(m+1) = m!$
  - for *x* > 0: $\Gamma(x+1) = x\Gamma(x)$
- What happens?
  - $a = a_1 = a_2 \to 0$
  - $a = a_1 = a_2 \to \infty$
  - $a_1 > a_2$ [demo]
- Beta is conjugate to Cat

$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z \sim \mathrm{Cat}(\rho_1, \rho_2)$$

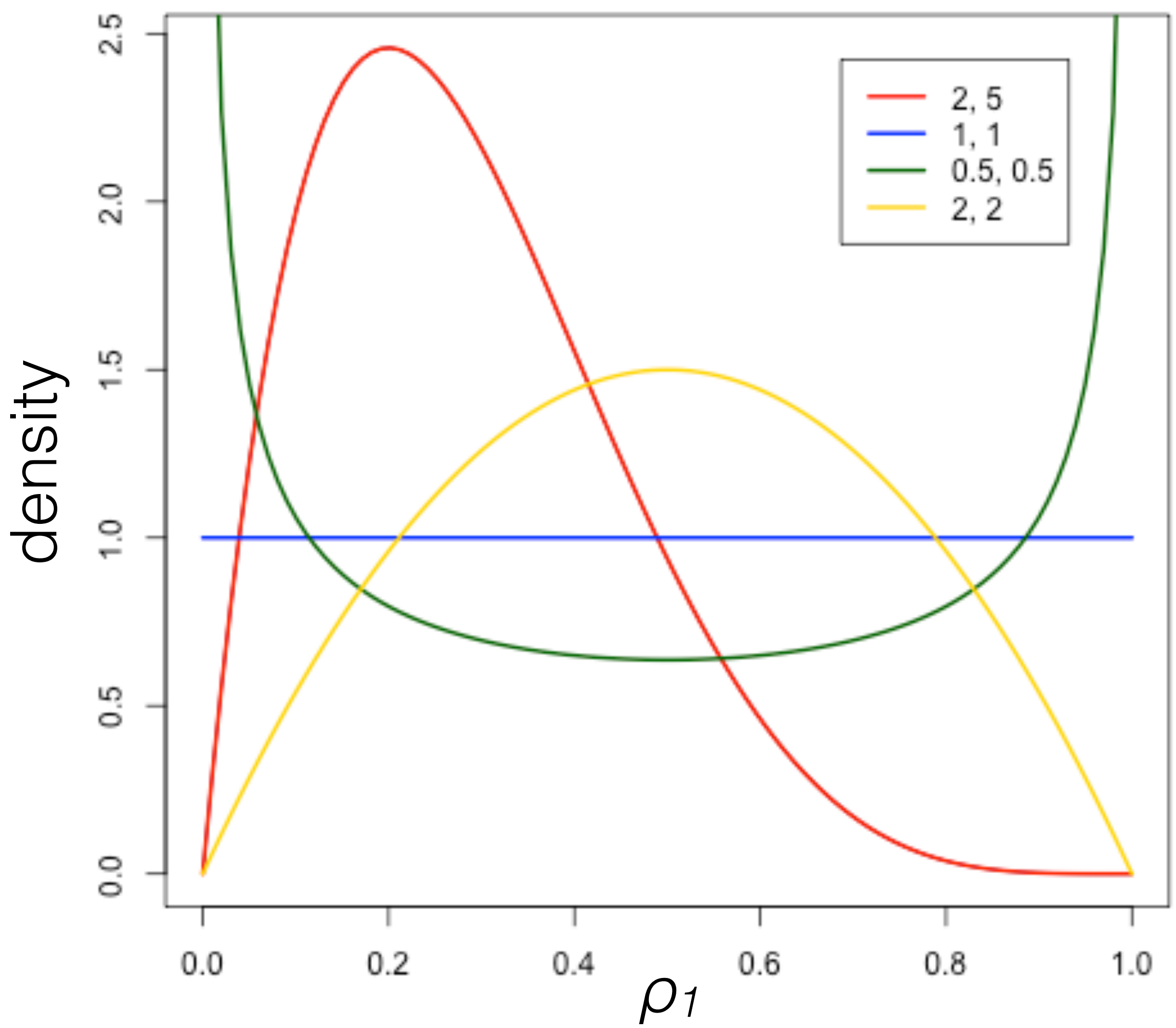
$$p(\rho_1, z) \propto \rho_1^{\mathbf{1}\{z=1\}}(1-\rho_1)^{\mathbf{1}\{z=2\}}\rho_1^{a_1-1}(1-\rho_1)^{a_2-1}$$

Beta distribution review

$$\mathrm{Beta}(\rho_1|a_1, a_2) = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)}\rho_1^{a_1-1}(1-\rho_1)^{a_2-1}$$

$\rho_1 \in (0, 1)$

$a_1, a_2 > 0$



- Gamma function Γ
  - integer *m*: $\Gamma(m+1) = m!$
  - for *x* > 0: $\Gamma(x+1) = x\Gamma(x)$
- What happens?
  - $a = a_1 = a_2 \to 0$
  - $a = a_1 = a_2 \to \infty$
  - $a_1 > a_2$

[demo]

- Beta is conjugate to Cat

$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z \sim \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(\rho_1, z) \propto \rho_1^{\mathbf{1}\{z=1\}}(1-\rho_1)^{\mathbf{1}\{z=2\}}\rho_1^{a_1-1}(1-\rho_1)^{a_2-1}$$

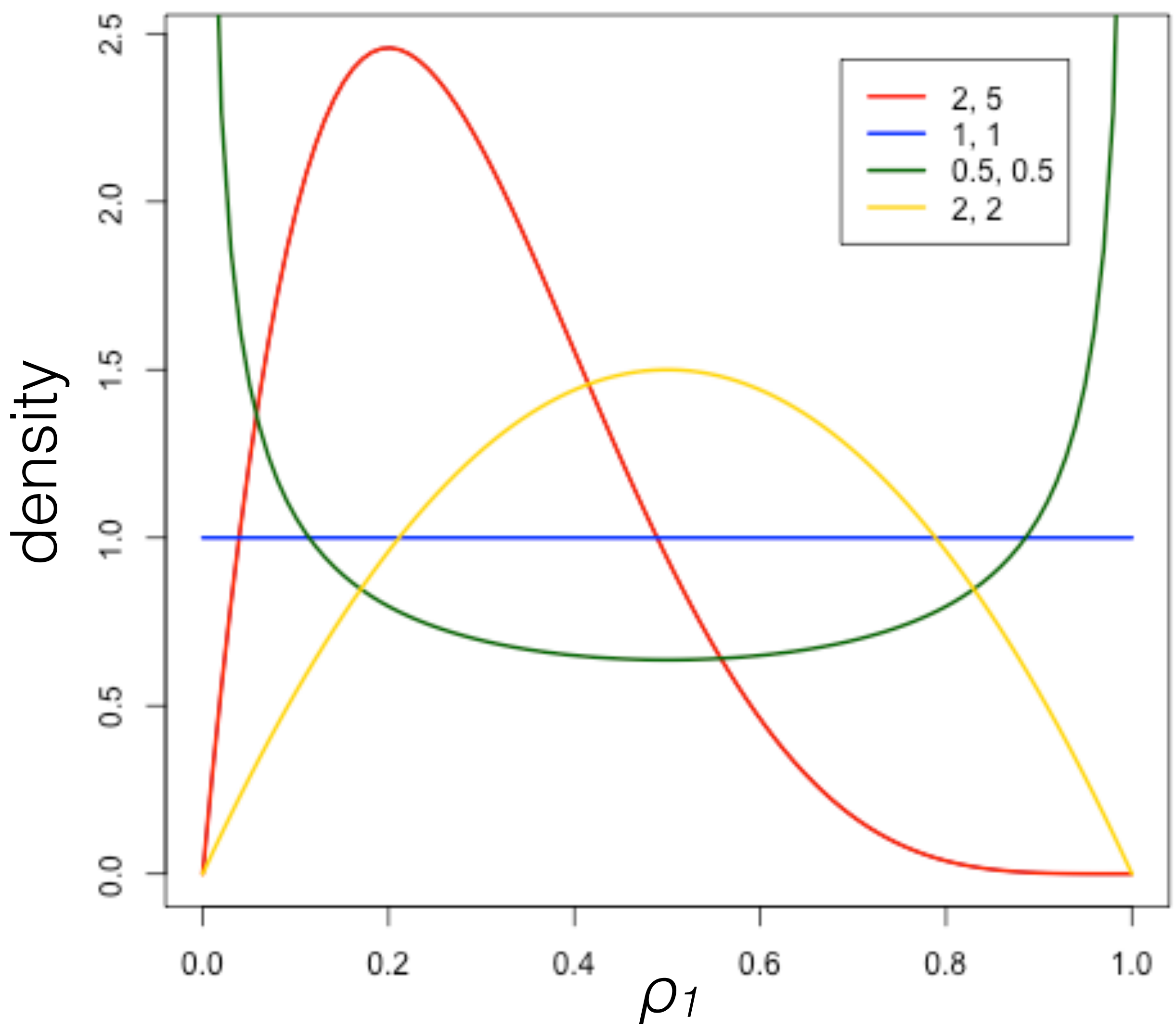
$$p(\rho_1|z) \propto$$

Beta distribution review

$$\mathrm{Beta}(\rho_1|a_1,a_2) = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\rho_1^{a_1-1}(1-\rho_1)^{a_2-1}$$

$\rho_1 \in (0,1)$

$a_1, a_2 > 0$



- Gamma function Γ
  - integer m: $\Gamma(m+1) = m!$
  - for $x > 0$: $\Gamma(x+1) = x\Gamma(x)$
- What happens?
  - $a = a_1 = a_2 \to 0$
  - $a = a_1 = a_2 \to \infty$
  - $a_1 > a_2$ [demo]
- Beta is conjugate to Cat

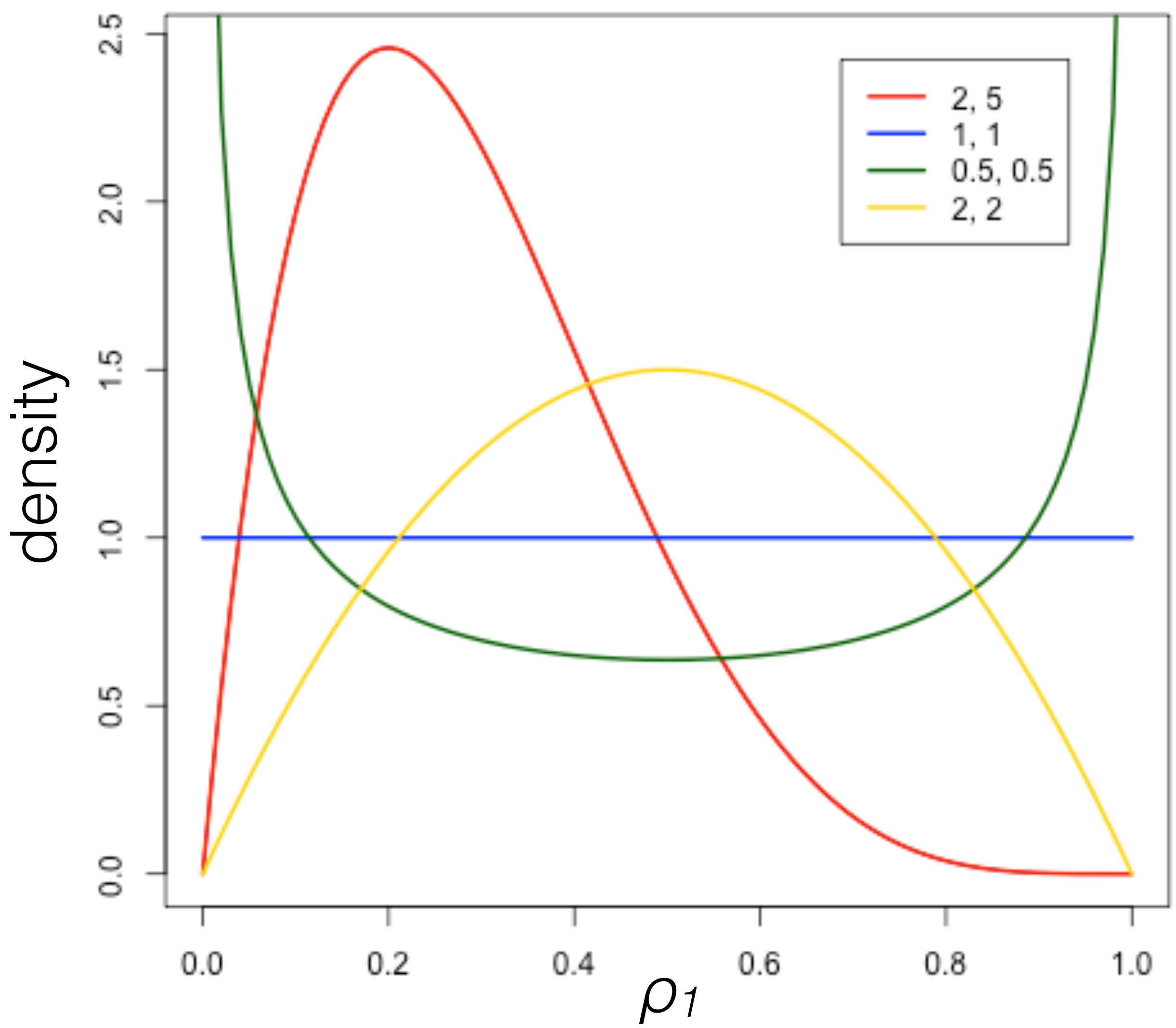
$$\rho_1 \sim \mathrm{Beta}(a_1,a_2),\, z \sim \mathrm{Cat}(\rho_1,\rho_2)$$

$$p(\rho_1, z) \propto \rho_1^{\mathbf{1}\{z=1\}}(1-\rho_1)^{\mathbf{1}\{z=2\}}\rho_1^{a_1-1}(1-\rho_1)^{a_2-1}$$

$$p(\rho_1|z) \propto \rho_1^{a_1+\mathbf{1}\{z=1\}-1}(1-\rho_1)^{a_2+\mathbf{1}\{z=2\}-1}$$

Beta distribution review

$$\text{Beta}(\rho_1|a_1,a_2) = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\rho_1^{a_1-1}(1-\rho_1)^{a_2-1} \qquad \begin{array}{r}\rho_1 \in (0,1)\\ a_1, a_2 > 0\end{array}$$



- Gamma function Γ
  - integer *m*: $\Gamma(m+1) = m!$
  - for *x* > 0: $\Gamma(x+1) = x\Gamma(x)$
- What happens?
  - $a = a_1 = a_2 \to 0$
  - $a = a_1 = a_2 \to \infty$
  - $a_1 > a_2$ [demo]
- Beta is conjugate to Cat

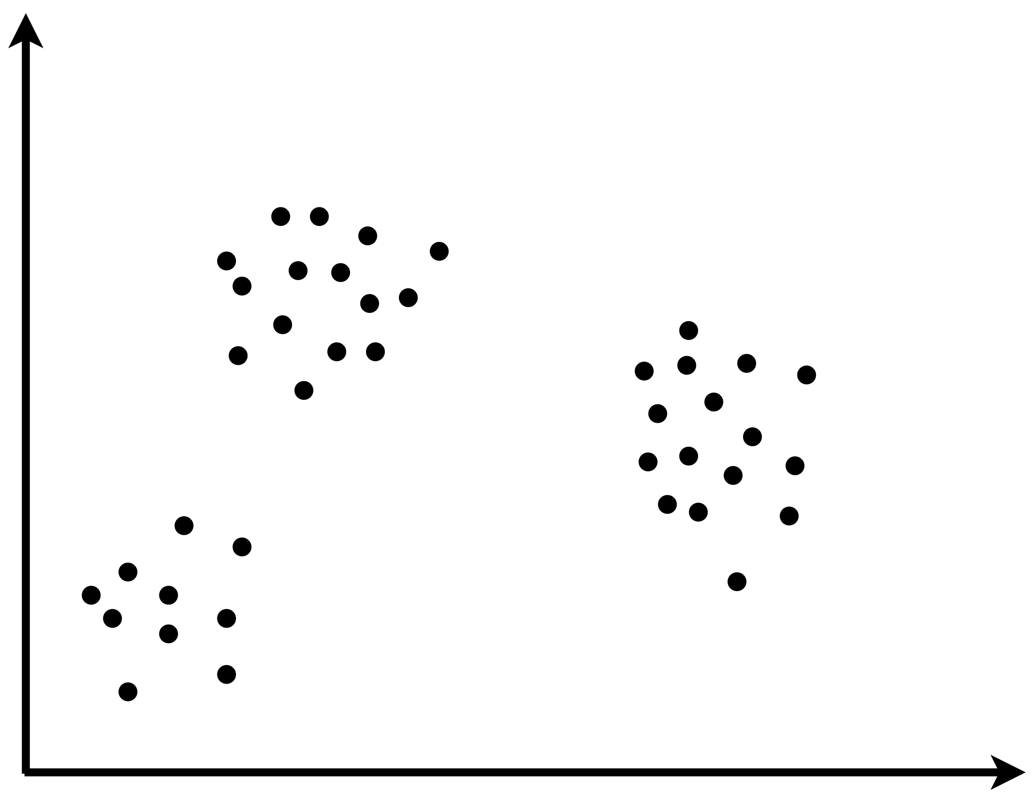
$$\rho_1 \sim \text{Beta}(a_1,a_2),\, z \sim \text{Cat}(\rho_1,\rho_2)$$

$$p(\rho_1,z) \propto \rho_1^{\mathbf{1}\{z=1\}}(1-\rho_1)^{\mathbf{1}\{z=2\}}\rho_1^{a_1-1}(1-\rho_1)^{a_2-1}$$

$$p(\rho_1|z) \propto \rho_1^{a_1+\mathbf{1}\{z=1\}-1}(1-\rho_1)^{a_2+\mathbf{1}\{z=2\}-1} \propto \text{Beta}(\rho_1|a_1+\mathbf{1}\{z=1\}, a_2+\mathbf{1}\{z=2\})$$

Generative model

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$



- Finite Gaussian mixture model (*K* clusters)

Generative model

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

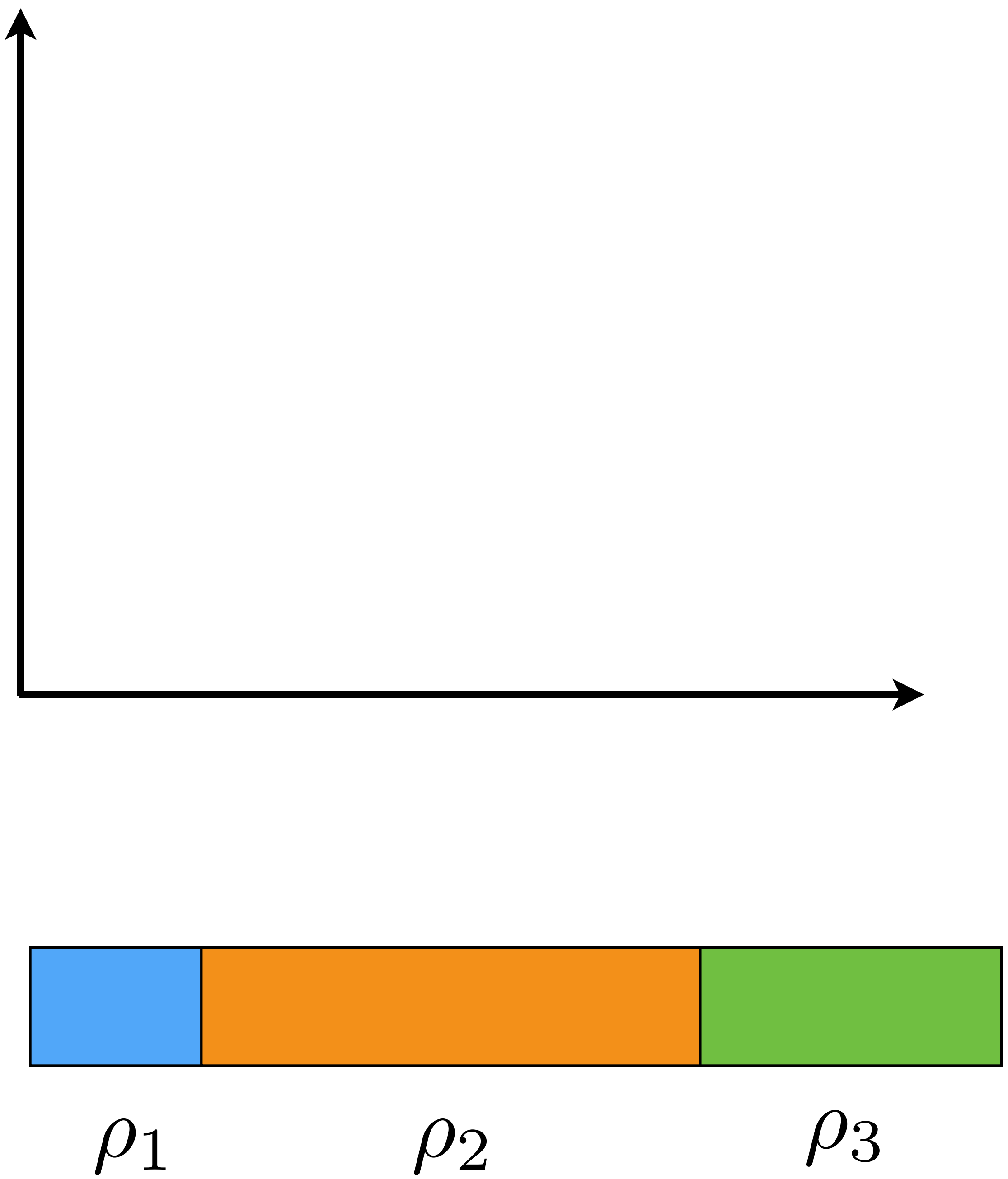
- Finite Gaussian mixture model (*K* clusters)

Generative model

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$

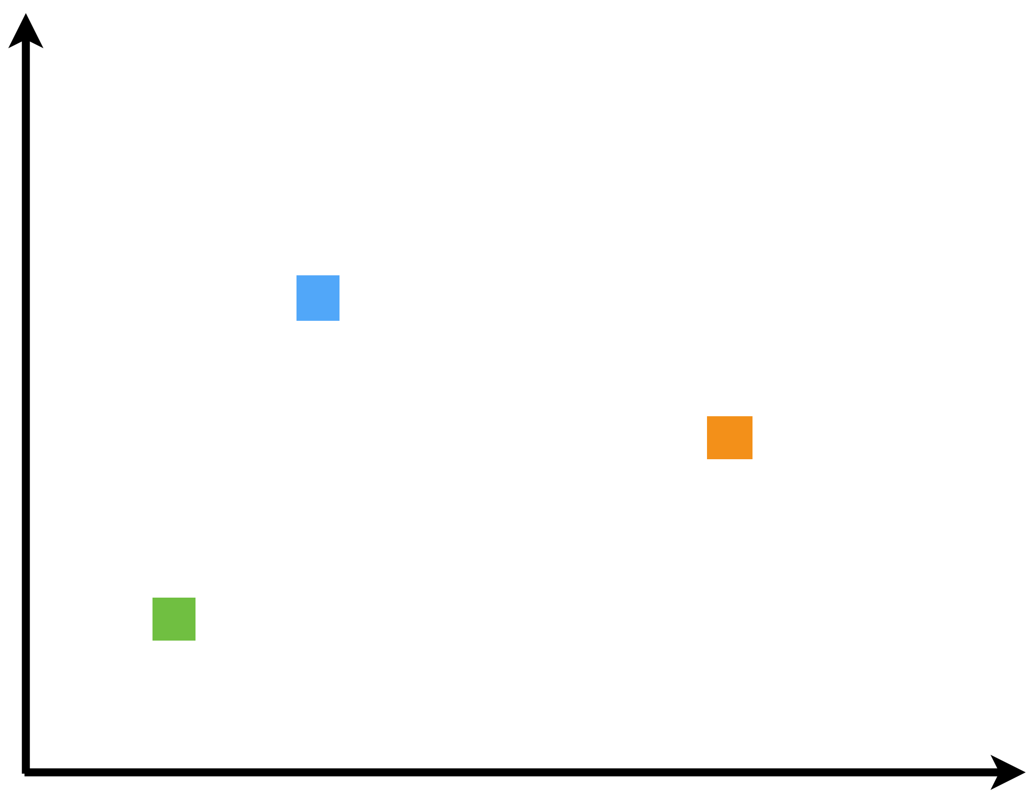
- Finite Gaussian mixture model (*K* clusters)

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K})$$



Generative model

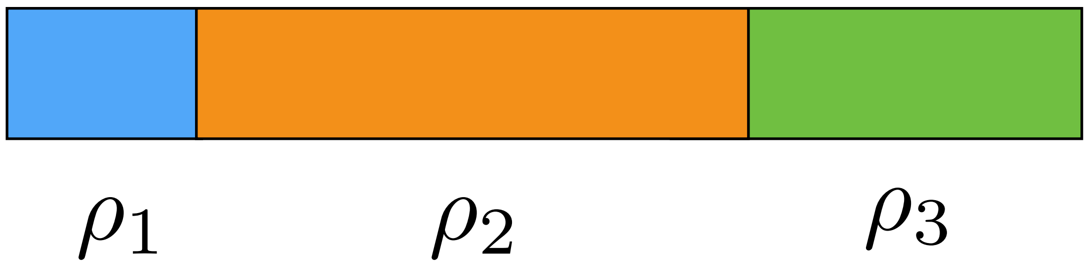
$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$



- Finite Gaussian mixture model (*K* clusters)

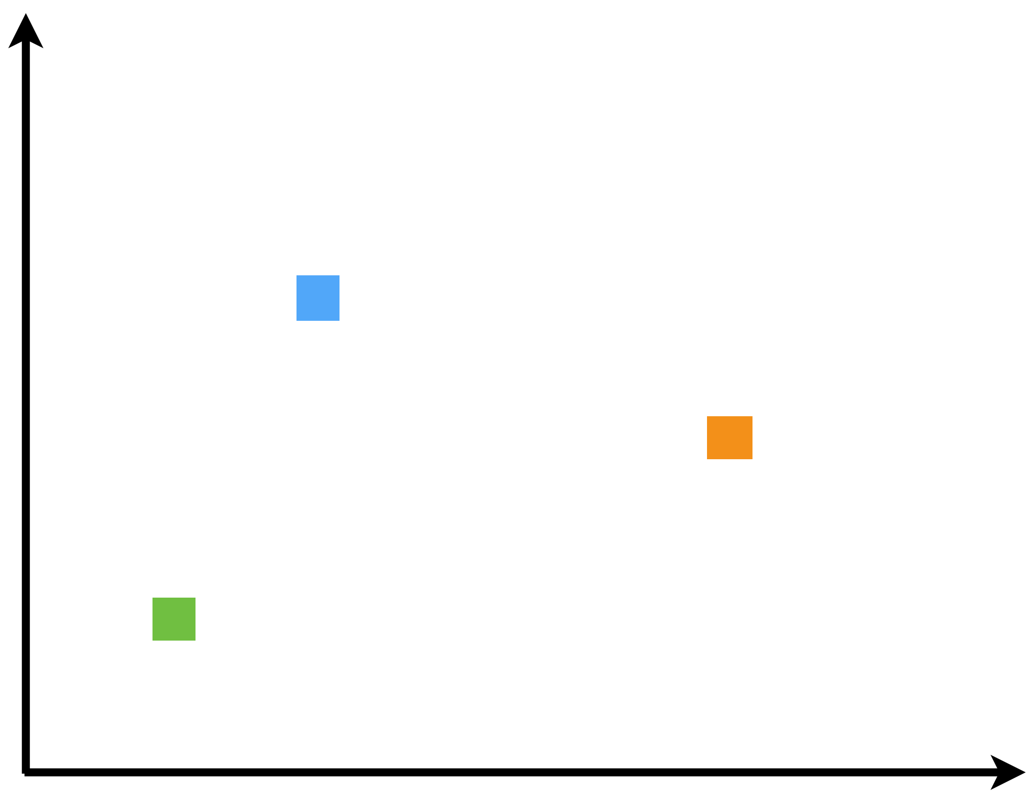
$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K})$$

$$\mu_k \overset{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0)$$



Generative model

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$



- Finite Gaussian mixture model (*K* clusters)

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K})$$

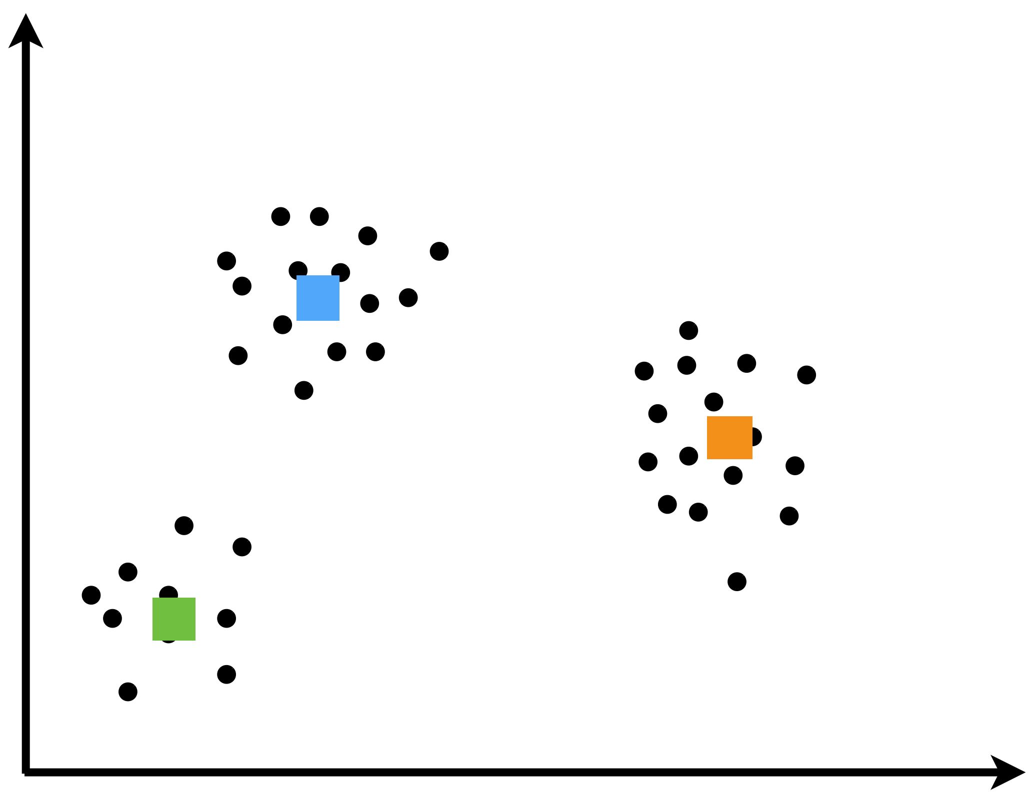
$$\mu_k \overset{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0)$$

$$z_n \overset{iid}{\sim} \text{Categorical}(\rho_{1:K})$$

ρ_1 ρ_2 ρ_3

Generative model

$$\mathbb{P}(\text{parameters}|\text{data}) \propto \mathbb{P}(\text{data}|\text{parameters})\mathbb{P}(\text{parameters})$$



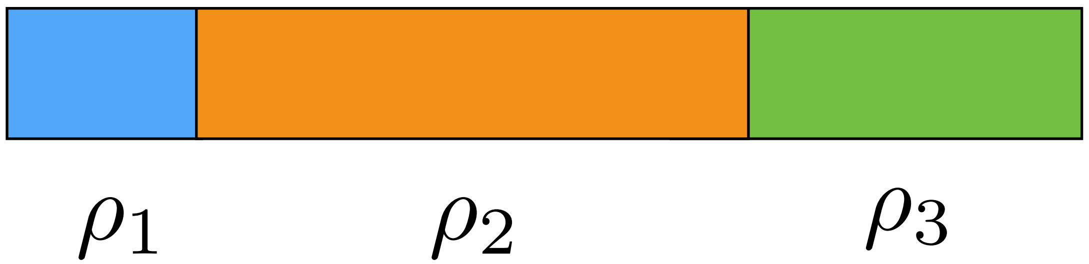
- Finite Gaussian mixture model (*K* clusters)

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K})$$

$$\mu_k \overset{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0)$$

$$z_n \overset{iid}{\sim} \text{Categorical}(\rho_{1:K})$$

$$x_n \overset{indep}{\sim} \mathcal{N}(\mu_{z_n}, \Sigma)$$



Dirichlet distribution review

$$\mathrm{Dirichlet}(\rho_{1:K}|a_{1:K}) = \frac{\Gamma(\sum_{k=1}^{K} a_k)}{\prod_{k=1}^{K} \Gamma(a_k)} \prod_{k=1}^{K} \rho_k^{a_k - 1} \qquad a_k > 0$$

Dirichlet distribution review

$$\mathrm{Dirichlet}(\rho_{1:K}|a_{1:K}) = \frac{\Gamma(\sum_{k=1}^K a_k)}{\prod_{k=1}^K \Gamma(a_k)} \prod_{k=1}^K \rho_k^{a_k-1}$$

$$a_k > 0$$

$$\rho_k \in (0,1)$$

$$\sum_k \rho_k = 1$$

Dirichlet distribution review

$$\mathrm{Dirichlet}(\rho_{1:K}|a_{1:K}) = \frac{\Gamma(\sum_{k=1}^{K} a_k)}{\prod_{k=1}^{K}\Gamma(a_k)}\prod_{k=1}^{K}\rho_k^{a_k-1} \qquad a_k > 0$$

Dirichlet distribution review

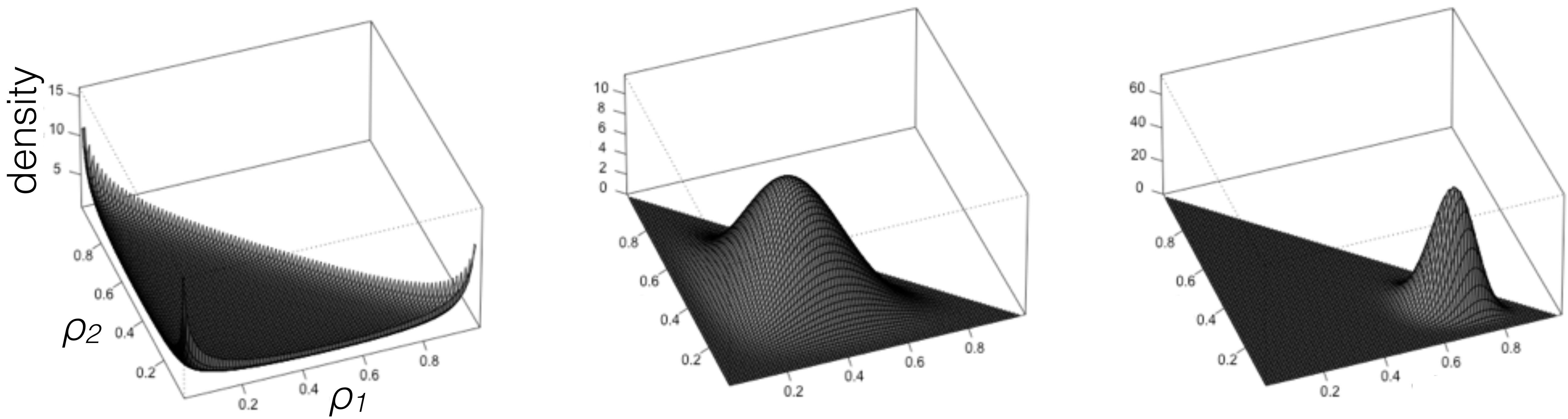
$$\mathrm{Dirichlet}(\rho_{1:K}|a_{1:K}) = \frac{\Gamma(\sum_{k=1}^{K} a_k)}{\prod_{k=1}^{K} \Gamma(a_k)} \prod_{k=1}^{K} \rho_k^{a_k - 1} \qquad a_k > 0$$

- What happens?

Dirichlet distribution review

$$\mathrm{Dirichlet}(\rho_{1:K}|a_{1:K}) = \frac{\Gamma(\sum_{k=1}^{K} a_k)}{\prod_{k=1}^{K}\Gamma(a_k)} \prod_{k=1}^{K} \rho_k^{a_k - 1} \qquad a_k > 0$$

a = (0.5,0.5,0.5) a = (5,5,5) a = (40,10,10)

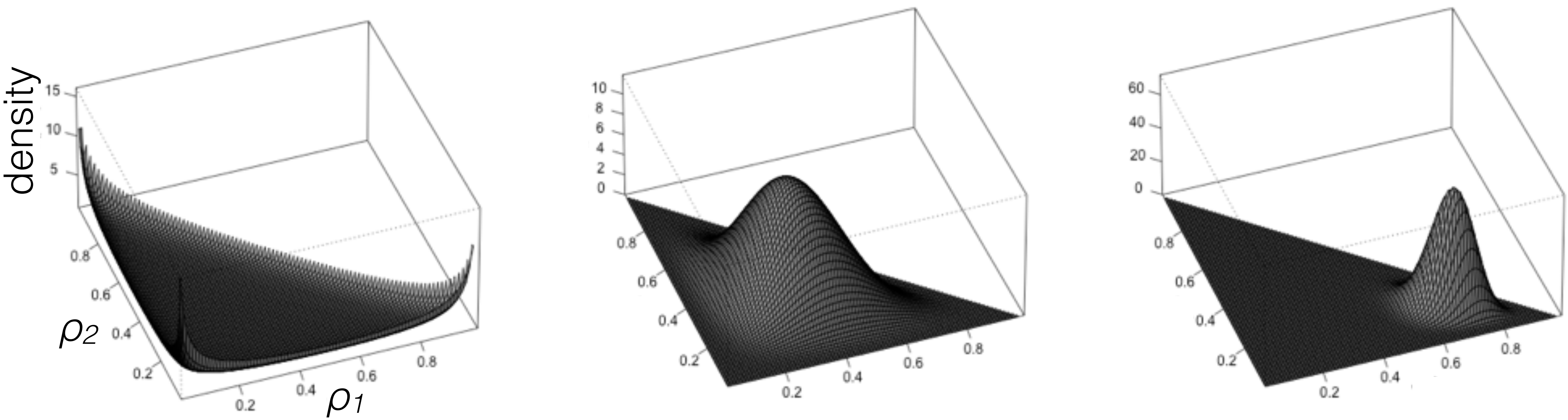


- What happens?

Dirichlet distribution review

$$\mathrm{Dirichlet}(\rho_{1:K}|a_{1:K}) = \frac{\Gamma(\sum_{k=1}^{K} a_k)}{\prod_{k=1}^{K} \Gamma(a_k)} \prod_{k=1}^{K} \rho_k^{a_k - 1} \qquad a_k > 0$$

a = (0.5,0.5,0.5) a = (5,5,5) a = (40,10,10)

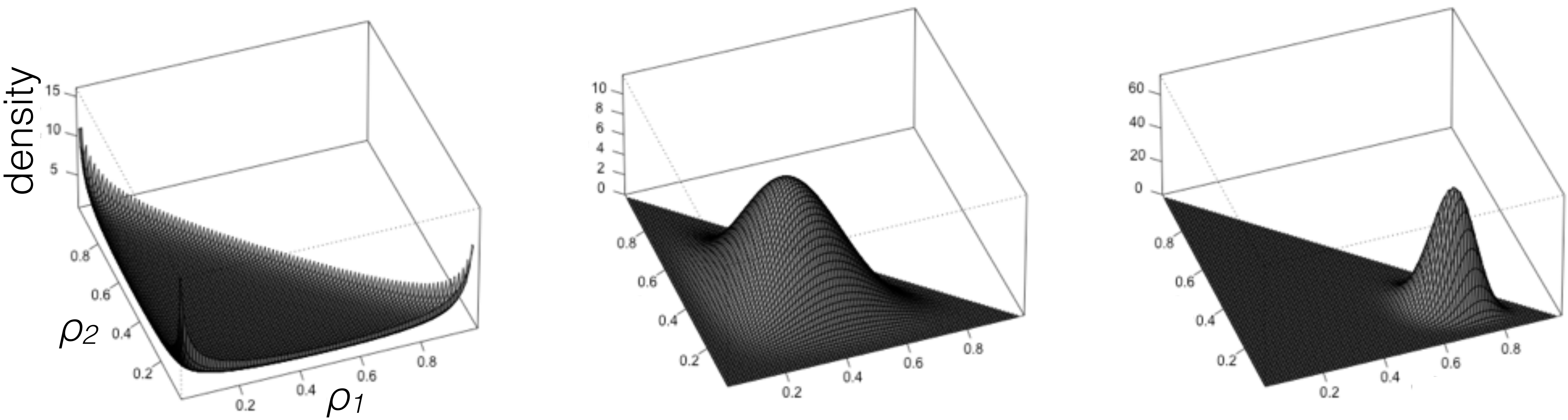


- What happens? $a = a_k = 1$

Dirichlet distribution review

$$\mathrm{Dirichlet}(\rho_{1:K}|a_{1:K}) = \frac{\Gamma(\sum_{k=1}^{K} a_k)}{\prod_{k=1}^{K}\Gamma(a_k)} \prod_{k=1}^{K} \rho_k^{a_k-1} \qquad a_k > 0$$

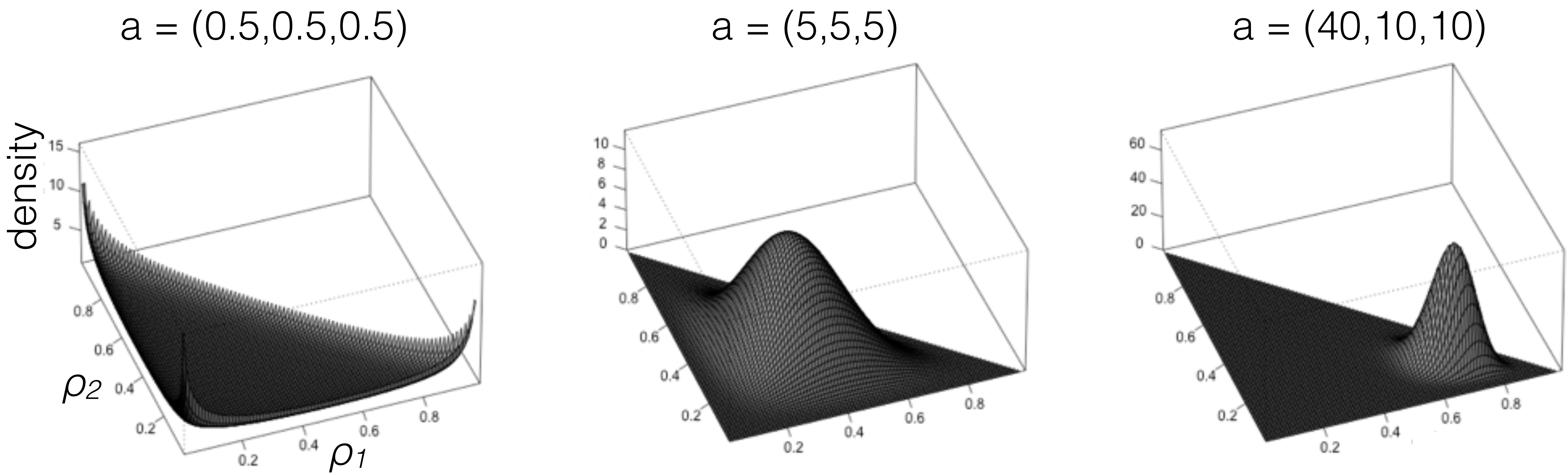
a = (0.5,0.5,0.5) a = (5,5,5) a = (40,10,10)



- What happens? $a = a_k = 1$ $a = a_k \to 0$

Dirichlet distribution review

$$\mathrm{Dirichlet}(\rho_{1:K}|a_{1:K}) = \frac{\Gamma(\sum_{k=1}^{K} a_k)}{\prod_{k=1}^{K}\Gamma(a_k)} \prod_{k=1}^{K} \rho_k^{a_k - 1} \qquad a_k > 0$$

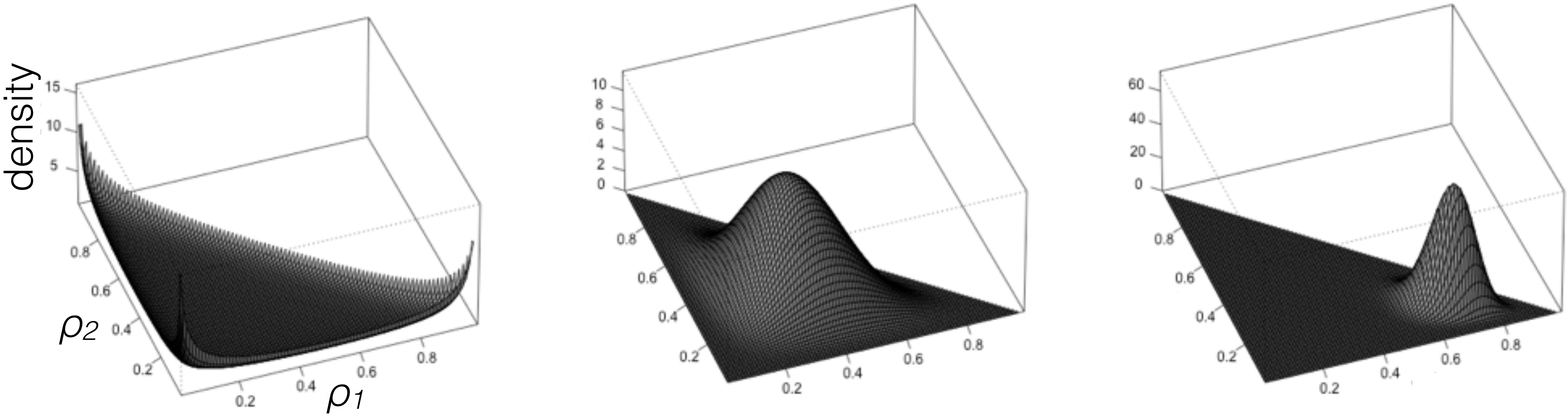


- What happens? $a = a_k = 1$ $\quad a = a_k \to 0$ $\quad a = a_k \to \infty$

Dirichlet distribution review

$$\text{Dirichlet}(\rho_{1:K} | a_{1:K}) = \frac{\Gamma(\sum_{k=1}^K a_k)}{\prod_{k=1}^K \Gamma(a_k)} \prod_{k=1}^K \rho_k^{a_k - 1} \qquad a_k > 0$$

a = (0.5,0.5,0.5) a = (5,5,5) a = (40,10,10)

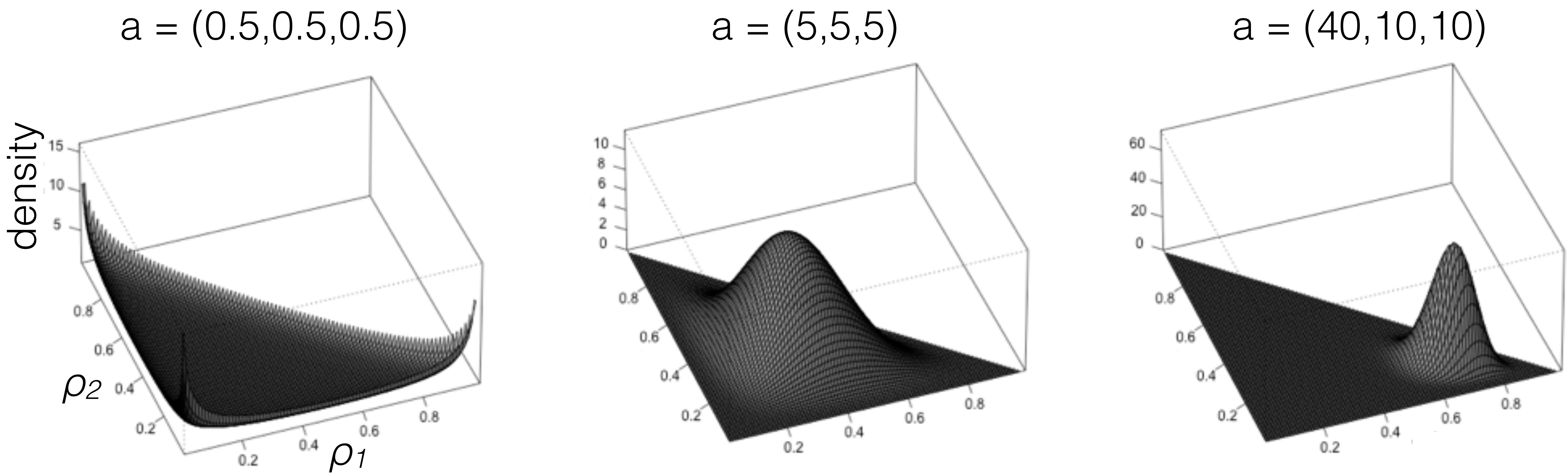


- What happens? $a = a_k = 1$ $\quad a = a_k \to 0$ $\quad a = a_k \to \infty$

[demo]

Dirichlet distribution review

$$\mathrm{Dirichlet}(\rho_{1:K}|a_{1:K}) = \frac{\Gamma(\sum_{k=1}^{K} a_k)}{\prod_{k=1}^{K}\Gamma(a_k)} \prod_{k=1}^{K} \rho_k^{a_k - 1} \qquad a_k > 0$$



- What happens? $a = a_k = 1$ $\quad a = a_k \to 0$ $\quad a = a_k \to \infty$

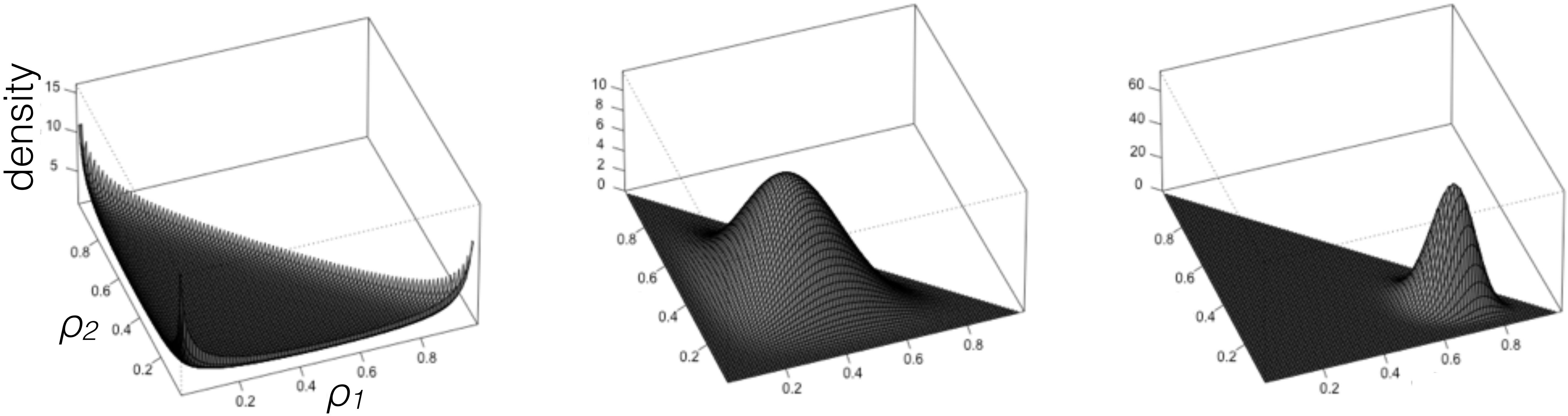
[demo]

- Dirichlet is conjugate to Categorical

Dirichlet distribution review

$$\mathrm{Dirichlet}(\rho_{1:K}|a_{1:K}) = \frac{\Gamma(\sum_{k=1}^{K} a_k)}{\prod_{k=1}^{K} \Gamma(a_k)} \prod_{k=1}^{K} \rho_k^{a_k - 1} \qquad a_k > 0$$

a = (0.5,0.5,0.5) a = (5,5,5) a = (40,10,10)



- What happens? $a = a_k = 1$ $\quad a = a_k \to 0$ $\quad a = a_k \to \infty$

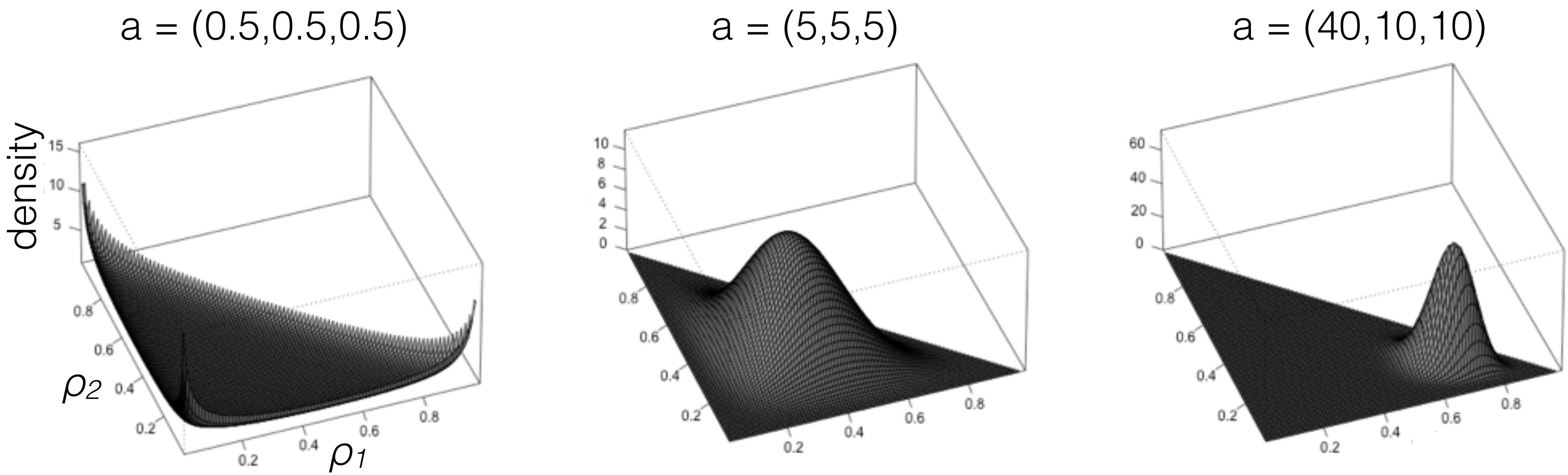
[demo]

- Dirichlet is conjugate to Categorical

$\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K}), z \sim \mathrm{Cat}(\rho_{1:K})$

Dirichlet distribution review

$$\mathrm{Dirichlet}(\rho_{1:K}|a_{1:K}) = \frac{\Gamma(\sum_{k=1}^{K} a_k)}{\prod_{k=1}^{K}\Gamma(a_k)} \prod_{k=1}^{K} \rho_k^{a_k - 1} \qquad a_k > 0$$



- What happens? $a = a_k = 1$ $\quad a = a_k \to 0$ $\quad a = a_k \to \infty$

[demo]

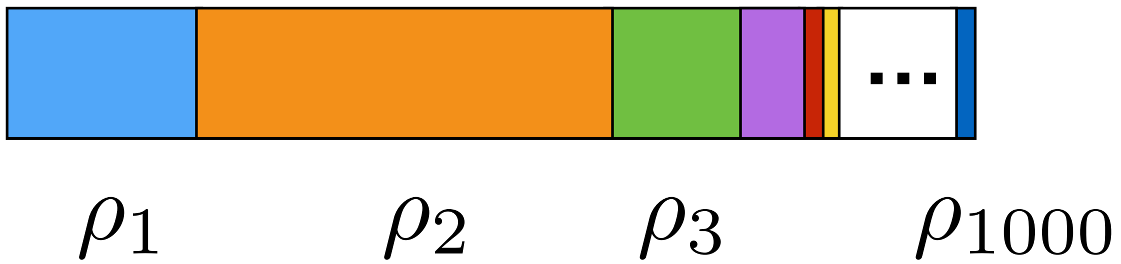
- Dirichlet is conjugate to Categorical

$$\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K}), z \sim \mathrm{Cat}(\rho_{1:K})$$

$$\rho_{1:K}|z \stackrel{d}{=} \mathrm{Dirichlet}(a'_{1:K}), a'_k = a_k + \mathbf{1}\{z = k\}$$

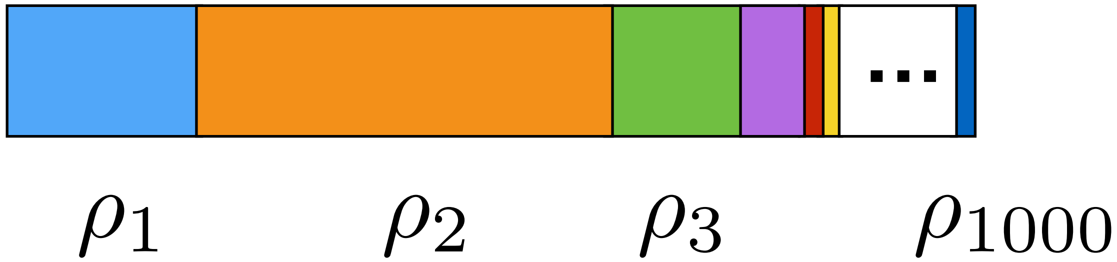
What if $K > N$?

What if $K > N$?



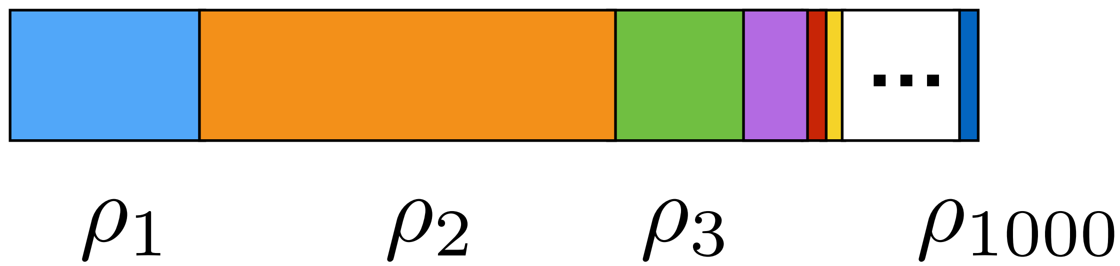
What if $K > N$?

- e.g. species sampling, topic modeling, groups on a social network, etc.



What if $K > N$?

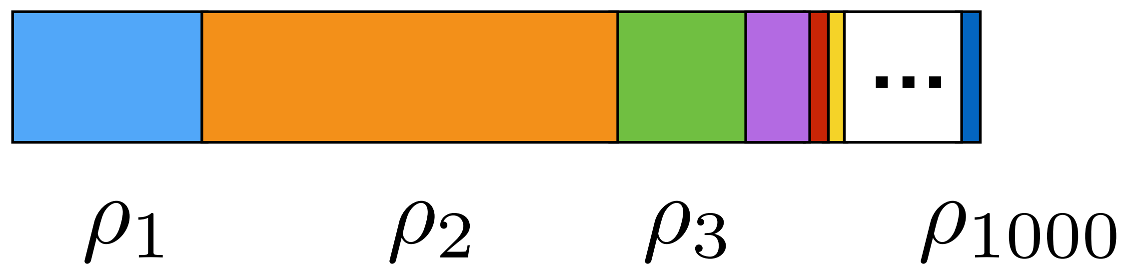
- e.g. species sampling, topic modeling, groups on a social network, etc.



- Components: number of latent groups

What if $K > N$?

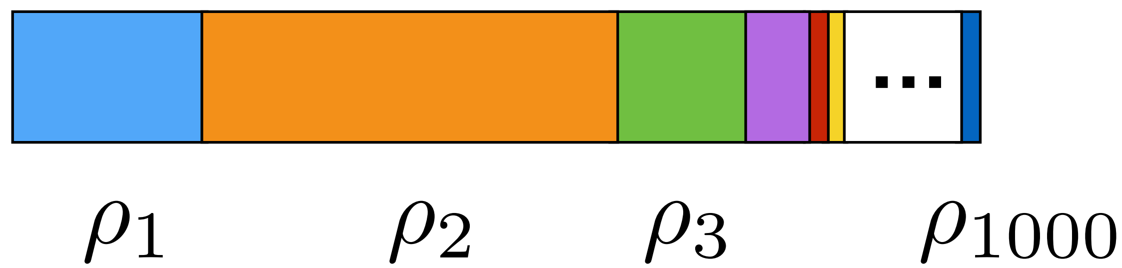
- e.g. species sampling, topic modeling, groups on a social network, etc.



- Components: number of latent groups
- Clusters: number of components represented in the data

What if $K > N$?

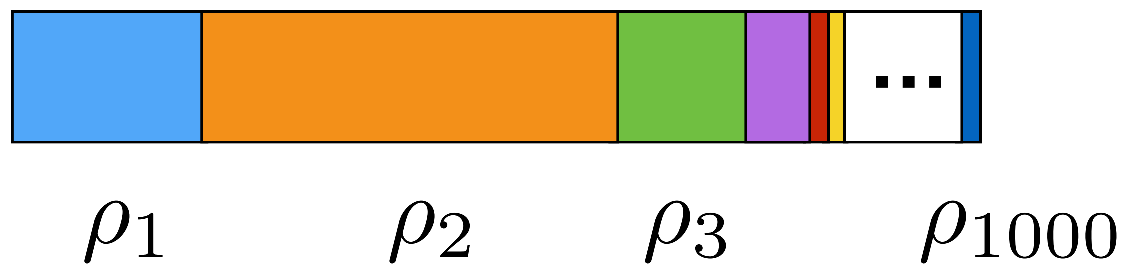
- e.g. species sampling, topic modeling, groups on a social network, etc.



- Components: number of latent groups
- Clusters: number of components represented in the data
- [demo 1, demo 2]

What if $K > N$?

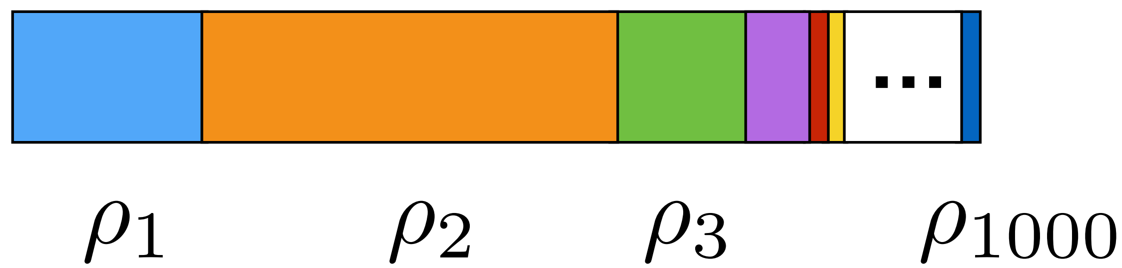
- e.g. species sampling, topic modeling, groups on a social network, etc.



- Components: number of latent groups
- Clusters: number of components represented in the data
- [demo 1, demo 2]
- Number of clusters for N data points is $< K$ and random

What if $K > N$?

- e.g. species sampling, topic modeling, groups on a social network, etc.



- Components: number of latent groups
- Clusters: number of components represented in the data
- [demo 1, demo 2]
- Number of clusters for N data points is < K and random
- Number of clusters grows with N

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?
    - Observation: $\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K})$

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?
  - Observation: $\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K})$

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?
  - Observation: $\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K})$

$$\Rightarrow \rho_1 \stackrel{d}{=} \mathrm{Beta}(a_1, \sum_{k=1}^{K} a_k - a_1)$$

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?
  - Observation: $\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K})$

$$\Rightarrow \rho_1 \stackrel{d}{=} \mathrm{Beta}(a_1, \sum_{k=1}^{K} a_k - a_1)$$

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?
  - Observation: $\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K})$

$$\Rightarrow \rho_1 \stackrel{d}{=} \mathrm{Beta}\left(a_1, \sum_{k=1}^{K} a_k - a_1\right) \perp\!\!\!\perp \tfrac{(\rho_2, \ldots, \rho_K)}{1-\rho_1} \stackrel{d}{=} \mathrm{Dirichlet}(a_2, \ldots, a_K)$$

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?
  - Observation: $\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K})$

$$\Rightarrow \rho_1 \stackrel{d}{=} \mathrm{Beta}\left(a_1, \sum_{k=1}^{K} a_k - a_1\right) \perp\!\!\!\perp \tfrac{(\rho_2, \ldots, \rho_K)}{1-\rho_1} \stackrel{d}{=} \mathrm{Dirichlet}(a_2, \ldots, a_K)$$

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?
  - Observation: $\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K})$

$$\Rightarrow \rho_1 \stackrel{d}{=} \mathrm{Beta}\left(a_1, \sum_{k=1}^{K} a_k - a_1\right) \perp\!\!\!\perp \tfrac{(\rho_2, \ldots, \rho_K)}{1-\rho_1} \stackrel{d}{=} \mathrm{Dirichlet}(a_2, \ldots, a_K)$$

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?
  - Observation: $\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K})$

$$\Rightarrow \rho_1 \stackrel{d}{=} \mathrm{Beta}(a_1, \sum_{k=1}^{K} a_k - a_1) \perp\!\!\!\perp \tfrac{(\rho_2, \ldots, \rho_K)}{1-\rho_1} \stackrel{d}{=} \mathrm{Dirichlet}(a_2, \ldots, a_K)$$

- "Stick breaking"

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?
  - Observation: $\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K})$

$$\Rightarrow \rho_1 \stackrel{d}{=} \mathrm{Beta}(a_1, \sum_{k=1}^{K} a_k - a_1) \perp\!\!\!\perp \tfrac{(\rho_2, \ldots, \rho_K)}{1 - \rho_1} \stackrel{d}{=} \mathrm{Dirichlet}(a_2, \ldots, a_K)$$

- "Stick breaking"

$$V_1 \sim \mathrm{Beta}(a_1, a_2 + a_3 + a_4)$$

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?
  - Observation: $\rho_{1:K} \sim \text{Dirichlet}(a_{1:K})$

$$\Rightarrow \rho_1 \stackrel{d}{=} \text{Beta}(a_1, \sum_{k=1}^{K} a_k - a_1) \perp\!\!\!\perp \tfrac{(\rho_2, \ldots, \rho_K)}{1-\rho_1} \stackrel{d}{=} \text{Dirichlet}(a_2, \ldots, a_K)$$

- "Stick breaking"

$V_1 \sim \text{Beta}(a_1, a_2 + a_3 + a_4)$ $\qquad \rho_1 = V_1$

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?
  - Observation: $\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K})$

$$\Rightarrow \rho_1 \stackrel{d}{=} \mathrm{Beta}\left(a_1, \sum_{k=1}^{K} a_k - a_1\right) \perp\!\!\!\perp \tfrac{(\rho_2, \ldots, \rho_K)}{1-\rho_1} \stackrel{d}{=} \mathrm{Dirichlet}(a_2, \ldots, a_K)$$

- "Stick breaking"

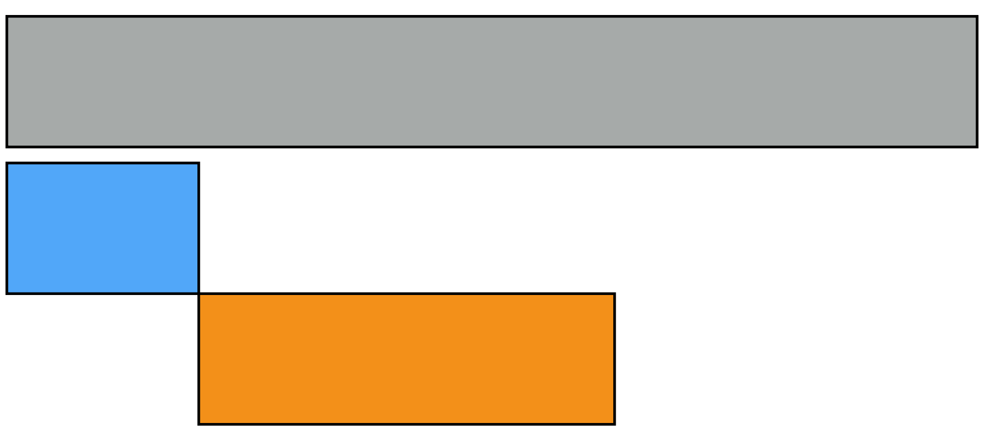
$V_1 \sim \mathrm{Beta}(a_1, a_2 + a_3 + a_4)$ $\qquad \rho_1 = V_1$

$V_2 \sim \mathrm{Beta}(a_2, a_3 + a_4)$

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?
  - Observation: $\rho_{1:K} \sim \text{Dirichlet}(a_{1:K})$

$$\Rightarrow \rho_1 \stackrel{d}{=} \text{Beta}\left(a_1, \sum_{k=1}^{K} a_k - a_1\right) \perp\!\!\!\perp \tfrac{(\rho_2, \ldots, \rho_K)}{1-\rho_1} \stackrel{d}{=} \text{Dirichlet}(a_2, \ldots, a_K)$$



- "Stick breaking"

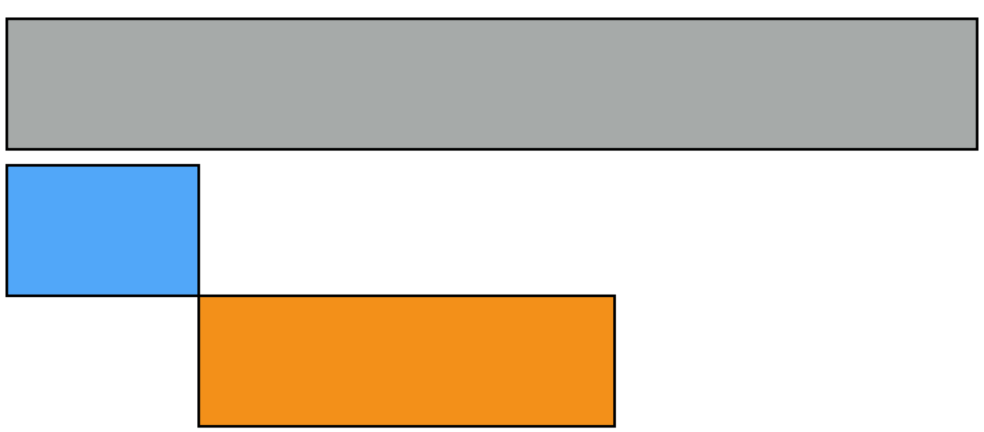
$$V_1 \sim \text{Beta}(a_1, a_2 + a_3 + a_4) \qquad \rho_1 = V_1$$

$$V_2 \sim \text{Beta}(a_2, a_3 + a_4) \qquad \rho_2 = (1 - V_1)V_2$$

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?
  - Observation: $\rho_{1:K} \sim \text{Dirichlet}(a_{1:K})$

$$\Rightarrow \rho_1 \overset{d}{=} \text{Beta}\left(a_1, \sum_{k=1}^{K} a_k - a_1\right) \perp\!\!\!\perp \tfrac{(\rho_2, \ldots, \rho_K)}{1 - \rho_1} \overset{d}{=} \text{Dirichlet}(a_2, \ldots, a_K)$$



- "Stick breaking"

$V_1 \sim \text{Beta}(a_1, a_2 + a_3 + a_4)$ $\qquad \rho_1 = V_1$

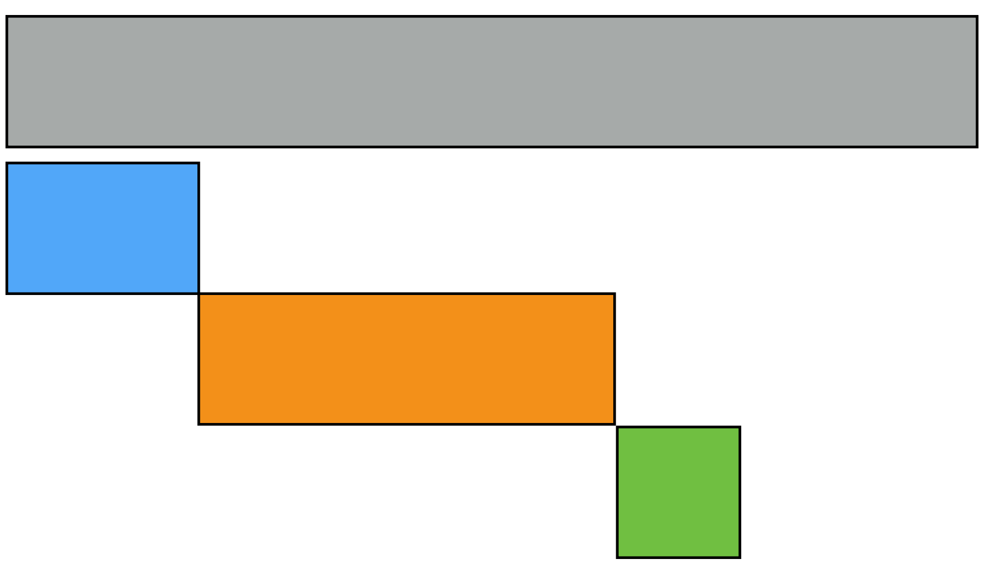
$V_2 \sim \text{Beta}(a_2, a_3 + a_4)$ $\qquad \rho_2 = (1 - V_1)V_2$

$V_3 \sim \text{Beta}(a_3, a_4)$

Choosing $K = \infty$

- Here, difficult to choose finite *K* in advance (contrast with small *K*): don't know *K*, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?
  - Observation: $\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K})$

$$\Rightarrow \rho_1 \stackrel{d}{=} \mathrm{Beta}(a_1, \sum_{k=1}^{K} a_k - a_1) \perp\!\!\!\perp \tfrac{(\rho_2,\ldots,\rho_K)}{1-\rho_1} \stackrel{d}{=} \mathrm{Dirichlet}(a_2,\ldots,a_K)$$



- "Stick breaking"

$$V_1 \sim \mathrm{Beta}(a_1, a_2 + a_3 + a_4) \qquad \rho_1 = V_1$$

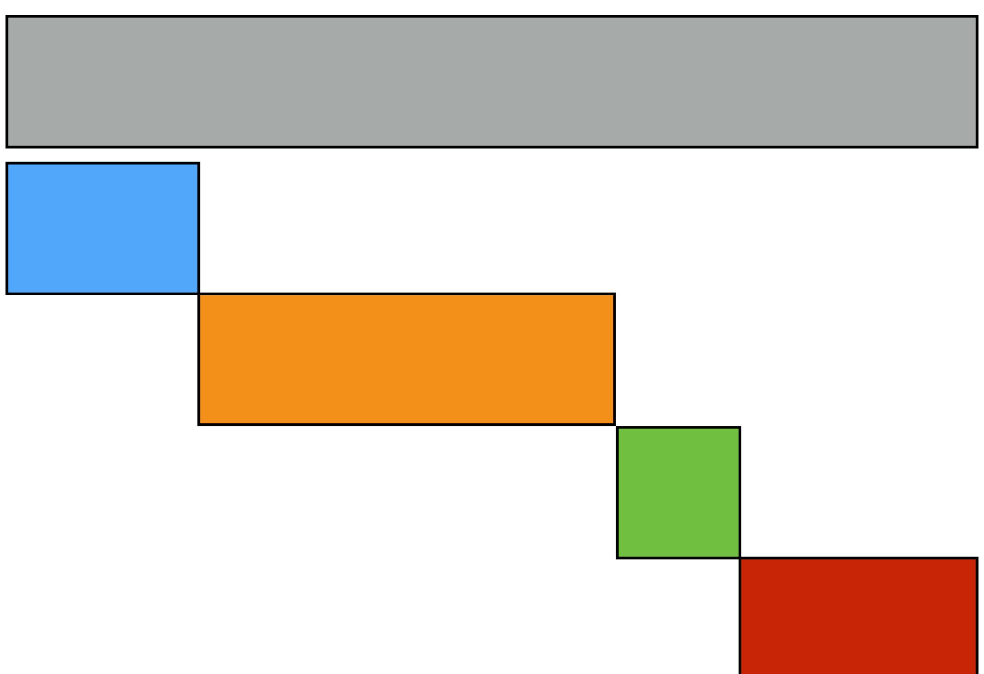
$$V_2 \sim \mathrm{Beta}(a_2, a_3 + a_4) \qquad \rho_2 = (1 - V_1)V_2$$

$$V_3 \sim \mathrm{Beta}(a_3, a_4) \qquad \rho_3 = (1 - V_1)(1 - V_2)V_3$$

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?
  - Observation: $\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K})$

$$\Rightarrow \rho_1 \stackrel{d}{=} \mathrm{Beta}\left(a_1, \sum_{k=1}^{K} a_k - a_1\right) \perp\!\!\!\perp \tfrac{(\rho_2, \ldots, \rho_K)}{1-\rho_1} \stackrel{d}{=} \mathrm{Dirichlet}(a_2, \ldots, a_K)$$



- "Stick breaking"

$$V_1 \sim \mathrm{Beta}(a_1, a_2 + a_3 + a_4) \qquad \rho_1 = V_1$$

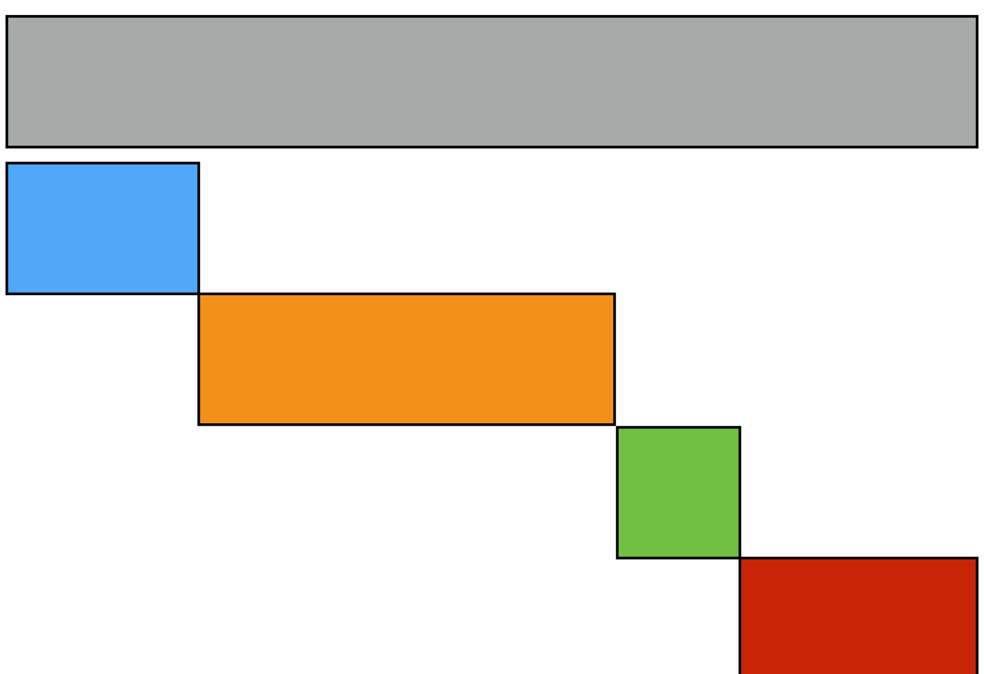
$$V_2 \sim \mathrm{Beta}(a_2, a_3 + a_4) \qquad \rho_2 = (1 - V_1)V_2$$

$$V_3 \sim \mathrm{Beta}(a_3, a_4) \qquad \rho_3 = (1 - V_1)(1 - V_2)V_3$$

$$\rho_4 = 1 - \sum_{k=1}^{3} \rho_k$$

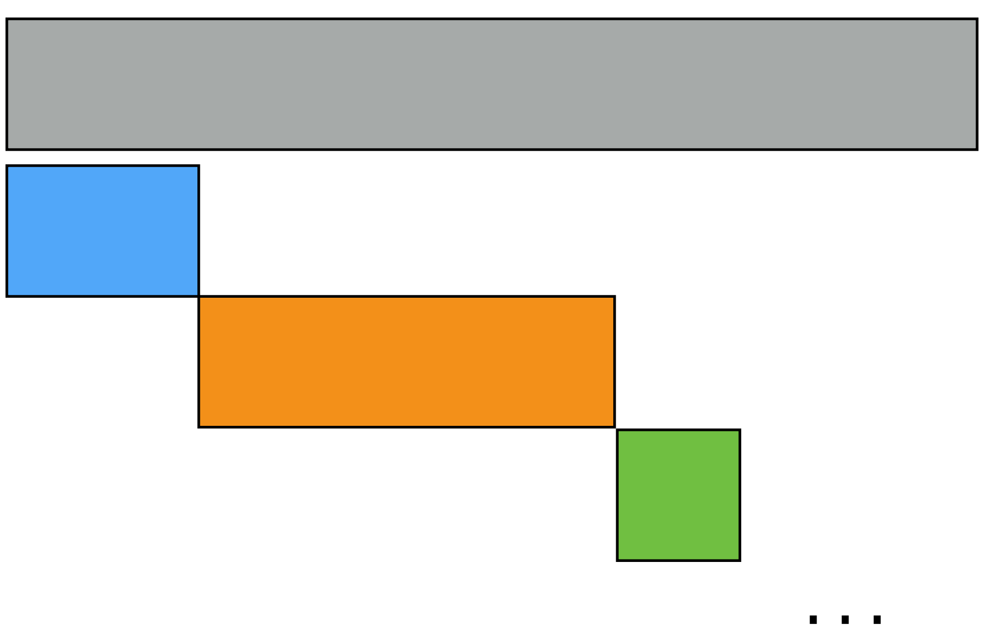
Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?



Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?



Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?

$$V_1 \sim \mathrm{Beta}(a_1, b_1)$$

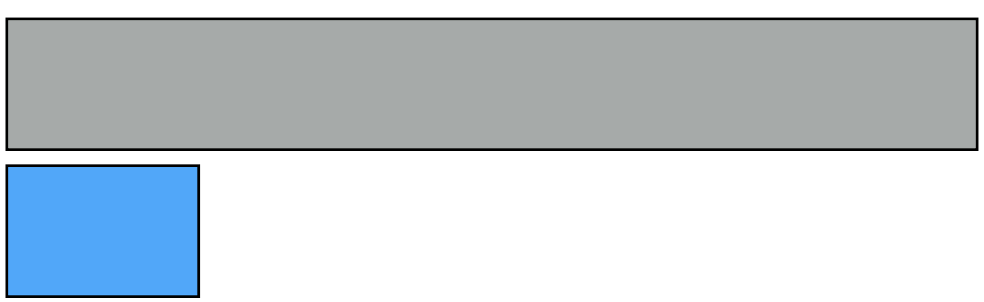
Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?

$$V_1 \sim \mathrm{Beta}(a_1, b_1) \qquad \rho_1 = V_1$$

Choosing $K = \infty$

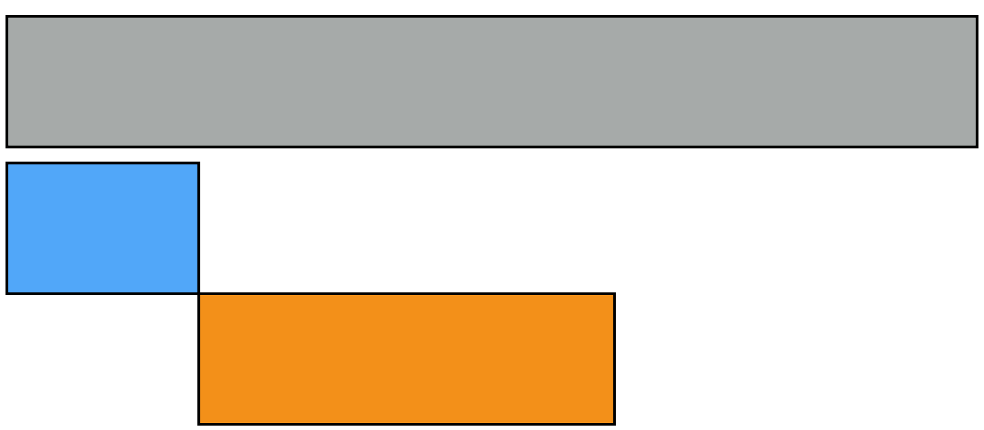
- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?



$$V_1 \sim \mathrm{Beta}(a_1, b_1) \qquad \rho_1 = V_1$$
$$V_2 \sim \mathrm{Beta}(a_2, b_2)$$

Choosing $K = \infty$

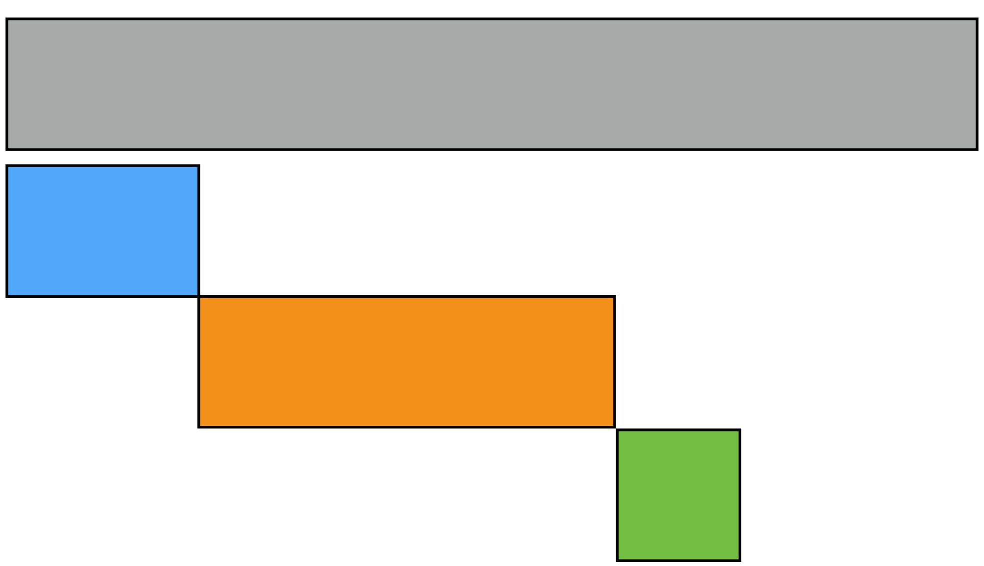
- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?



$$V_1 \sim \mathrm{Beta}(a_1, b_1) \qquad \rho_1 = V_1$$
$$V_2 \sim \mathrm{Beta}(a_2, b_2) \qquad \rho_2 = (1 - V_1)V_2$$

Choosing $K = \infty$

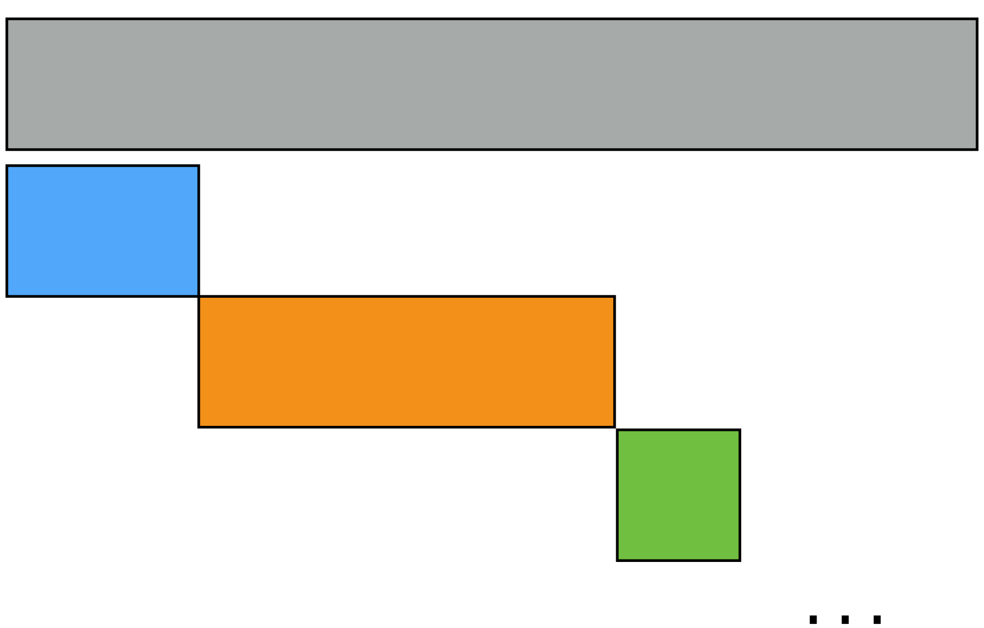
- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?



$$V_1 \sim \mathrm{Beta}(a_1, b_1) \qquad \rho_1 = V_1$$
$$V_2 \sim \mathrm{Beta}(a_2, b_2) \qquad \rho_2 = (1 - V_1)V_2$$

Choosing $K = \infty$

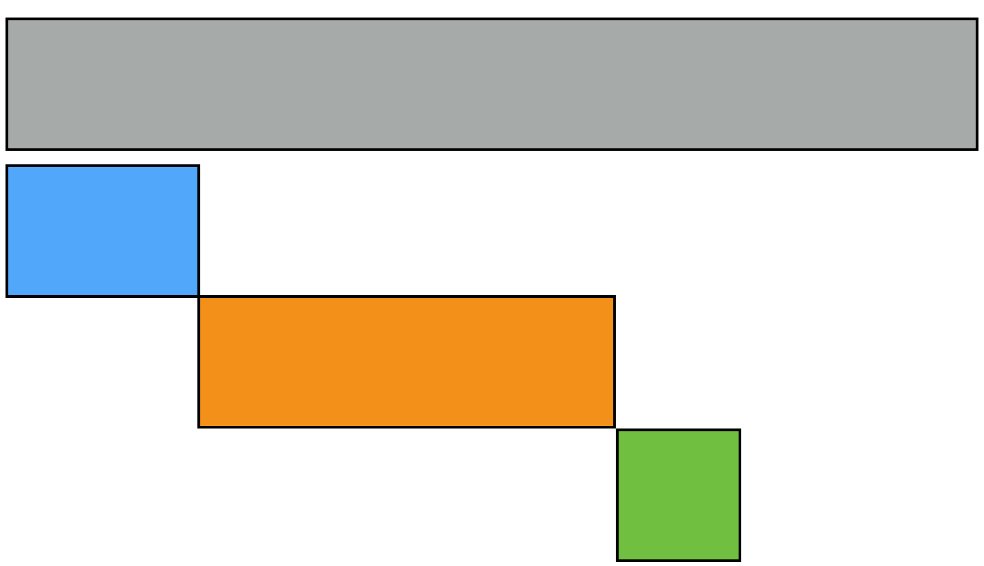
- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?



$$V_1 \sim \mathrm{Beta}(a_1, b_1) \qquad \rho_1 = V_1$$
$$V_2 \sim \mathrm{Beta}(a_2, b_2) \qquad \rho_2 = (1 - V_1)V_2$$

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?



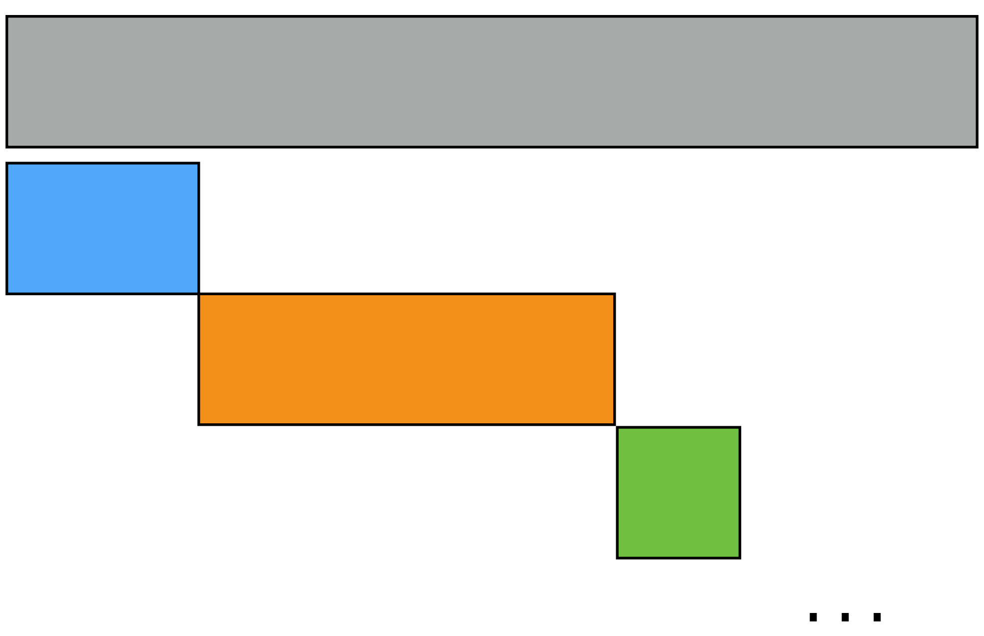
$$V_1 \sim \mathrm{Beta}(a_1, b_1) \qquad \rho_1 = V_1$$

$$V_2 \sim \mathrm{Beta}(a_2, b_2) \qquad \rho_2 = (1 - V_1)V_2$$

$$\cdots \quad V_k \sim \mathrm{Beta}(a_k, b_k)$$

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?



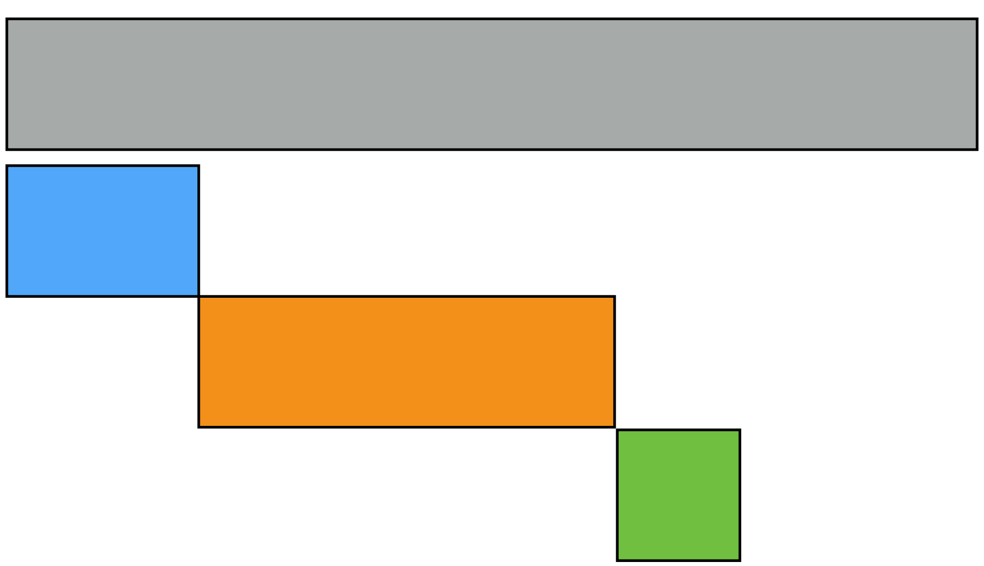
$$V_1 \sim \mathrm{Beta}(a_1, b_1) \qquad \rho_1 = V_1$$

$$V_2 \sim \mathrm{Beta}(a_2, b_2) \qquad \rho_2 = (1 - V_1)V_2$$

$$V_k \sim \mathrm{Beta}(a_k, b_k) \qquad \rho_k = \left[\prod_{j=1}^{k-1}(1 - V_j)\right] V_k$$

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?



$\cdots$

$$V_1 \sim \mathrm{Beta}(a_1, b_1) \qquad \rho_1 = V_1$$

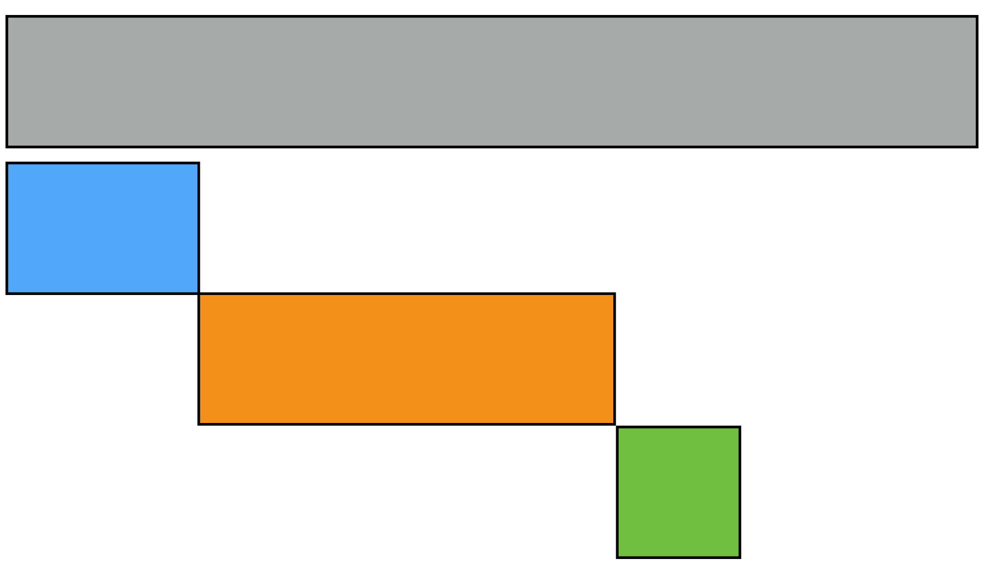
$$V_2 \sim \mathrm{Beta}(a_2, b_2) \qquad \rho_2 = (1 - V_1)V_2$$

$$V_k \sim \mathrm{Beta}(a_k, b_k) \qquad \rho_k = \left[\prod_{j=1}^{k-1}(1 - V_j)\right] V_k$$

[Ishwaran, James 2001]

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?
  - **Dirichlet process stick-breaking**: $a_k = 1, b_k = \alpha > 0$



$$V_1 \sim \mathrm{Beta}(a_1, b_1) \qquad \rho_1 = V_1$$

$$V_2 \sim \mathrm{Beta}(a_2, b_2) \qquad \rho_2 = (1 - V_1)V_2$$

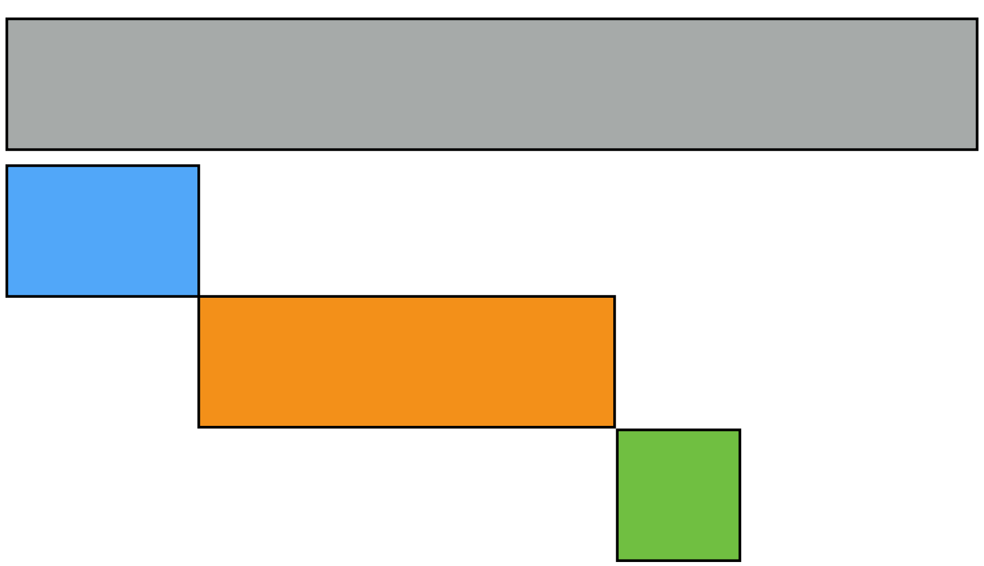
$$\cdots \qquad V_k \sim \mathrm{Beta}(a_k, b_k) \qquad \rho_k = \left[\prod_{j=1}^{k-1}(1 - V_j)\right] V_k$$

[Ishwaran, James 2001]

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?
    - **Dirichlet process stick-breaking**: $a_k = 1, b_k = \alpha > 0$
    - Griffiths-Engen-McCloskey (**GEM**) distribution:

$$\rho = (\rho_1, \rho_2, \ldots) \sim \mathrm{GEM}(\alpha)$$



$$V_1 \sim \mathrm{Beta}(a_1, b_1) \qquad \rho_1 = V_1$$

$$V_2 \sim \mathrm{Beta}(a_2, b_2) \qquad \rho_2 = (1 - V_1)V_2$$

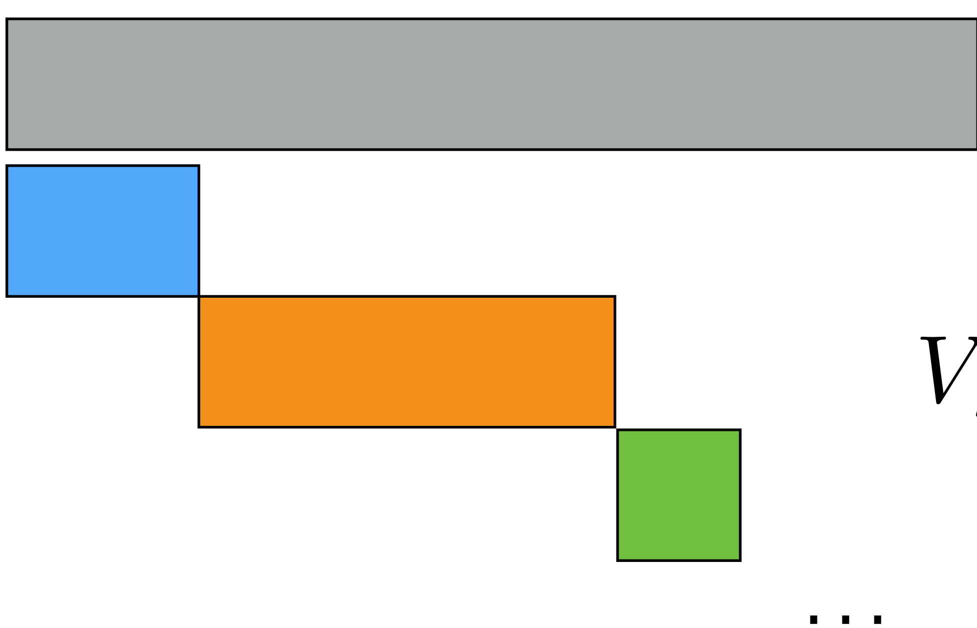
$$\cdots \qquad V_k \sim \mathrm{Beta}(a_k, b_k) \qquad \rho_k = \left[\prod_{j=1}^{k-1}(1 - V_j)\right] V_k$$

 [McCloskey 1965; Engen 1975; Patil and Taillie 1977; Ewens 1987; Sethuraman 1994; Ishwaran, James 2001]

Choosing $K = \infty$

- Here, difficult to choose finite K in advance (contrast with small K): don't know K, difficult to infer, streaming data
- How to generate $K = \infty$ strictly positive frequencies that sum to one?
  - **Dirichlet process stick-breaking**: $a_k = 1, b_k = \alpha > 0$
  - Griffiths-Engen-McCloskey (**GEM**) distribution:

$$\rho = (\rho_1, \rho_2, \ldots) \sim \mathrm{GEM}(\alpha)$$



$$V_k \overset{iid}{\sim} \mathrm{Beta}(1, \alpha) \qquad \rho_k = \left[\prod_{j=1}^{k-1} (1 - V_j)\right] V_k$$

[McCloskey 1965; Engen 1975; Patil and Taillie 1977; Ewens 1987; Sethuraman 1994; Ishwaran, James 2001]

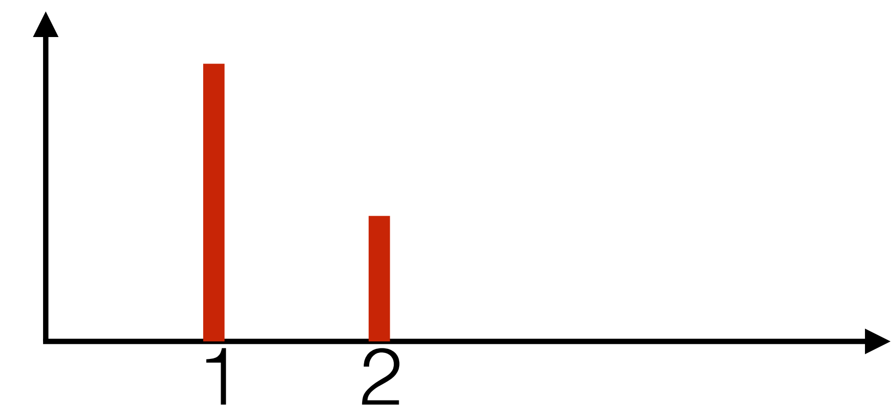
Distributions

Distributions

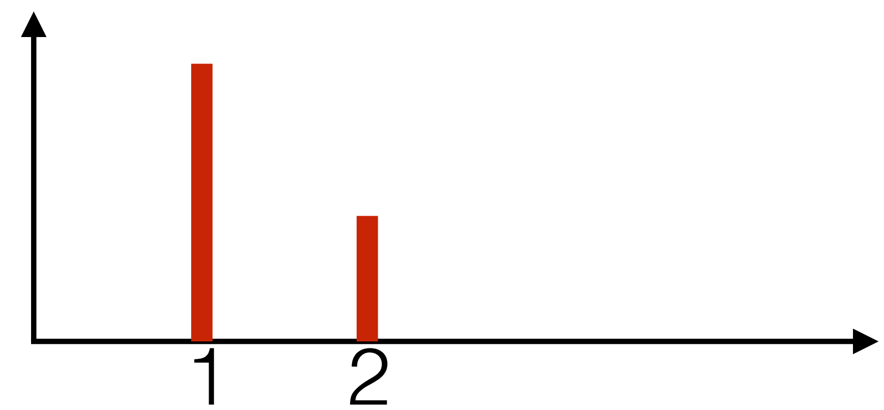
- Beta → random distribution over $1, 2$

Distributions

- Beta → random distribution over $1, 2$



Distributions



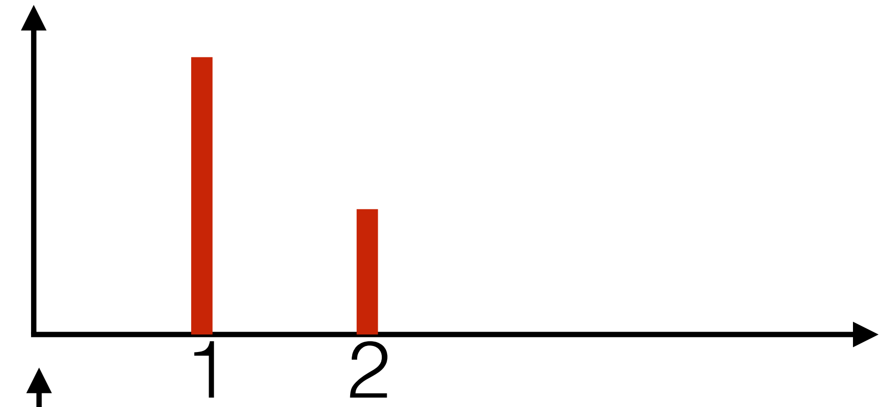
- Beta → random distribution over $1, 2$

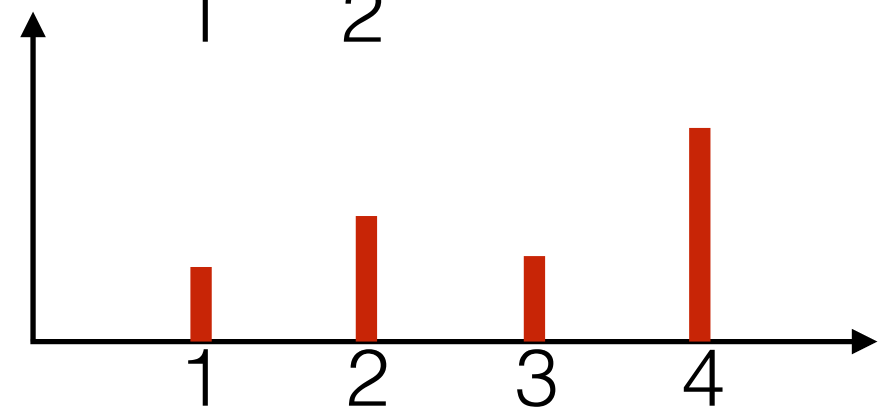
- Dirichlet → random distribution over $1, 2, \ldots, K$

Distributions

- Beta → random distribution over $1, 2$

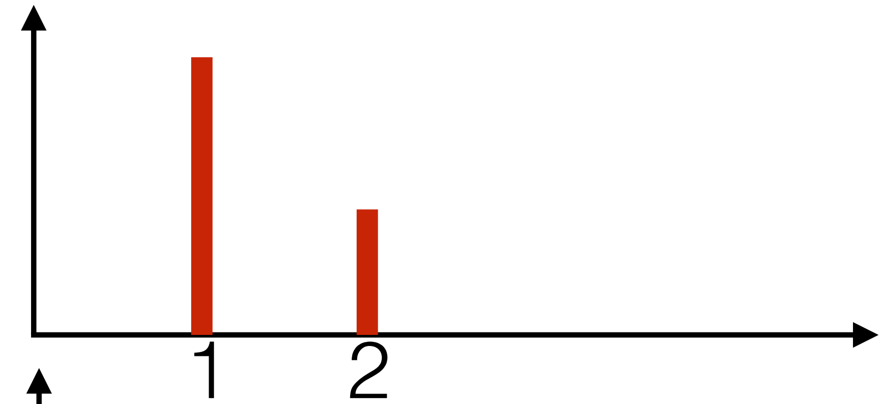
- Dirichlet → random distribution over $1, 2, \ldots, K$

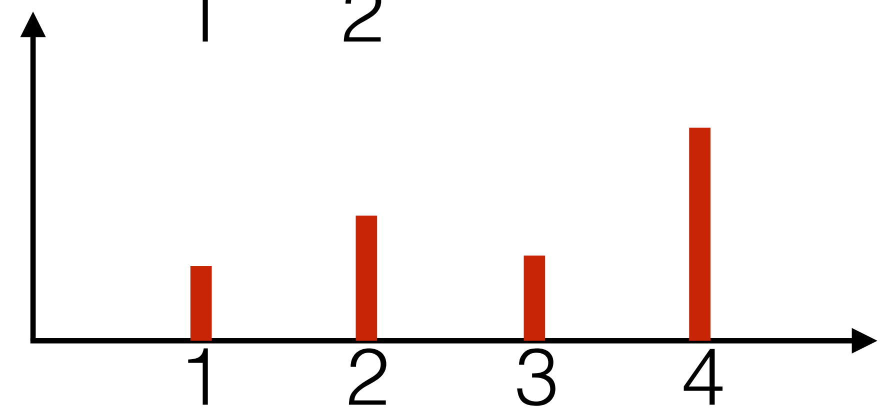




Distributions

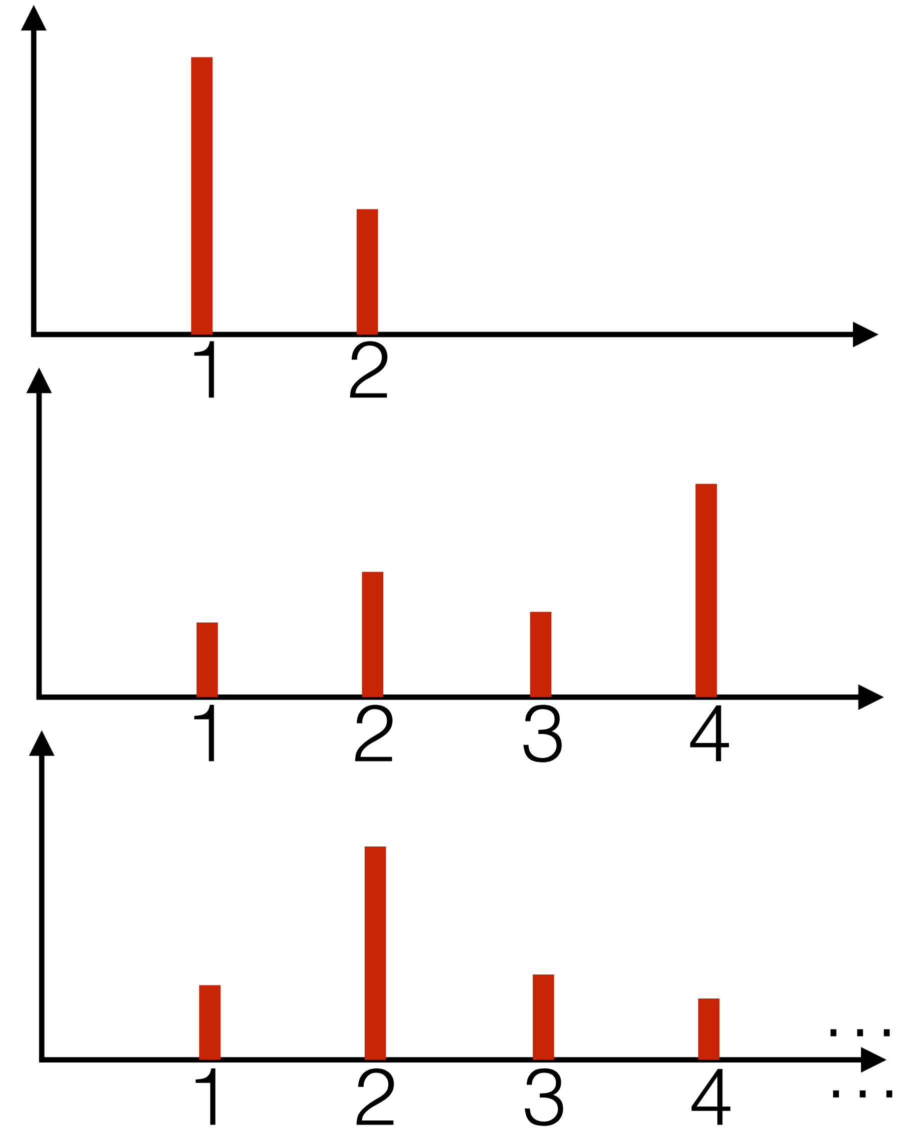
- Beta → random distribution over $1, 2$
- Dirichlet → random distribution over $1, 2, \ldots, K$
- GEM / Dirichlet process stick-breaking → random distribution over $1, 2, \ldots$





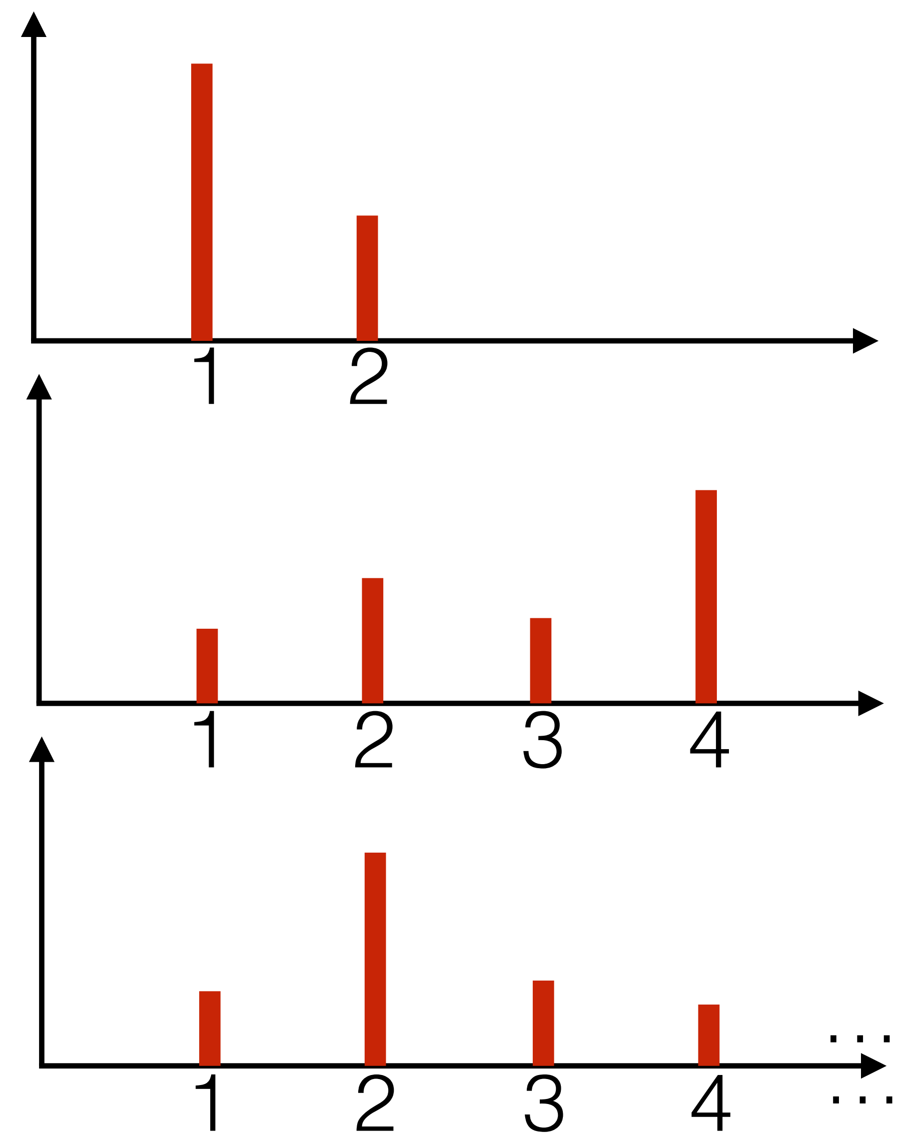
Distributions

- Beta → random distribution over $1, 2$
- Dirichlet → random distribution over $1, 2, \ldots, K$
- GEM / Dirichlet process stick-breaking → random distribution over $1, 2, \ldots$



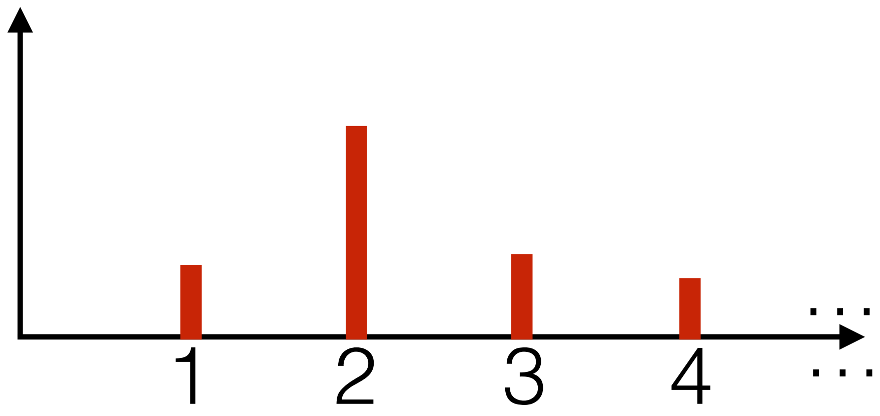
Distributions

- Beta → random distribution over $1, 2$
- Dirichlet → random distribution over $1, 2, \ldots, K$
- GEM / Dirichlet process stick-breaking → random distribution over $1, 2, \ldots$



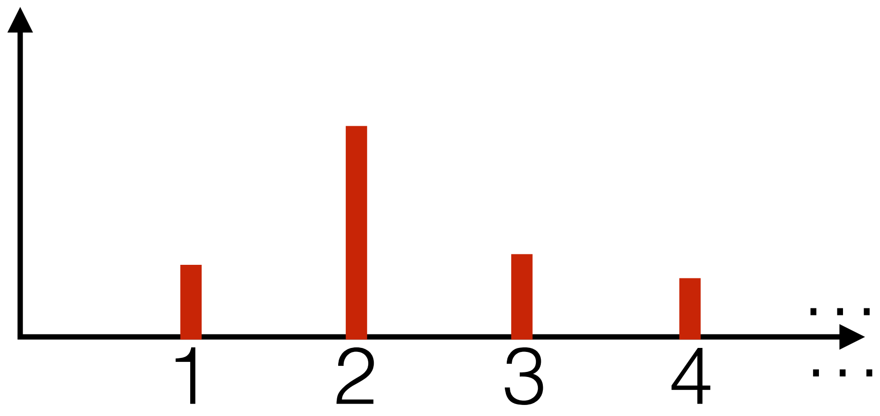
- Infinity of parameters: components
- Growing number of parameters: clusters

Exercises



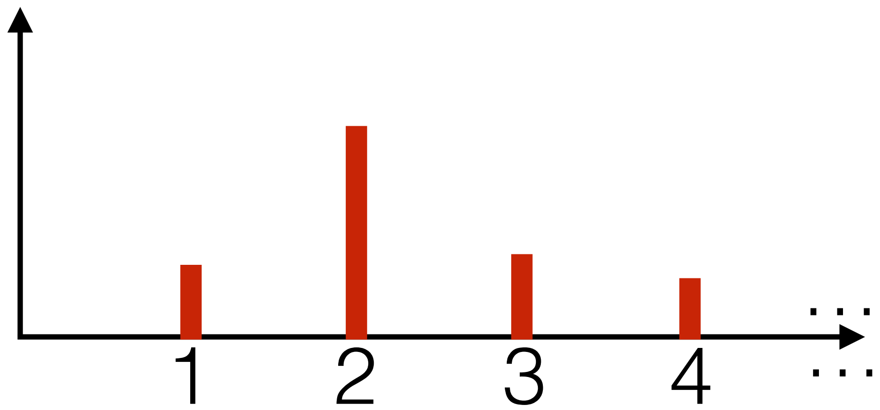
Exercises

- Prove the beta (Dirichlet) is conjugate to the categorical



Exercises

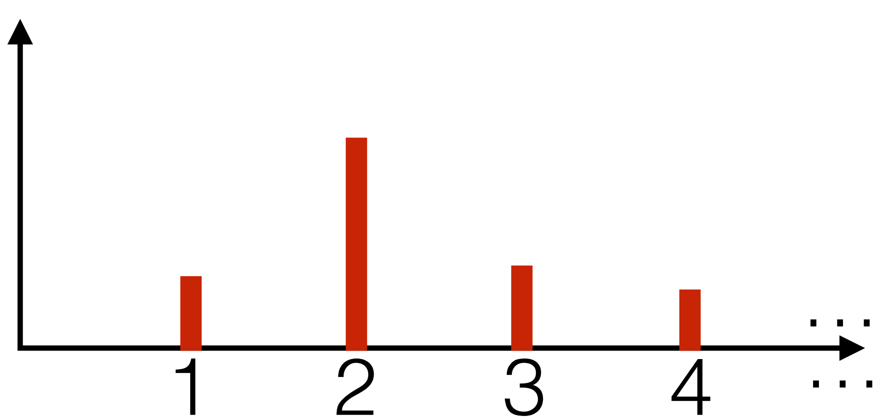
- Prove the beta (Dirichlet) is conjugate to the categorical
  - What is the posterior after N data points?



Exercises

- Prove the beta (Dirichlet) is conjugate to the categorical
  - What is the posterior after *N* data points?
- Suppose $\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K})$; prove that

$$\rho_1 \overset{d}{=} \mathrm{Beta}(a_1, \sum_{k=1}^{K} a_k - a_1) \perp\!\!\!\perp \frac{(\rho_2, \ldots, \rho_K)}{1-\rho_1} \overset{d}{=} \mathrm{Dirichlet}(a_2, \ldots, a_K)$$

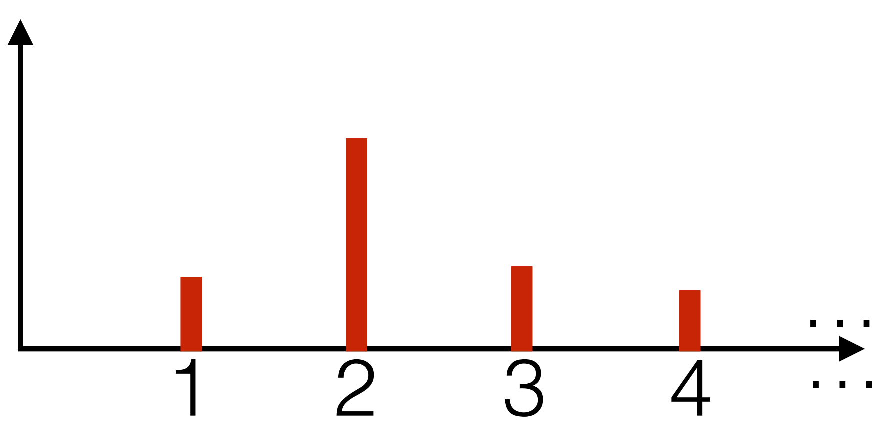


Exercises

- Prove the beta (Dirichlet) is conjugate to the categorical
  - What is the posterior after *N* data points?
- Suppose $\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K})$; prove that

$$\rho_1 \stackrel{d}{=} \mathrm{Beta}(a_1, \sum_{k=1}^{K} a_k - a_1) \perp\!\!\!\perp \tfrac{(\rho_2,\ldots,\rho_K)}{1-\rho_1} \stackrel{d}{=} \mathrm{Dirichlet}(a_2,\ldots,a_K)$$

- Code your own GEM simulator for ρ; why is this hard?

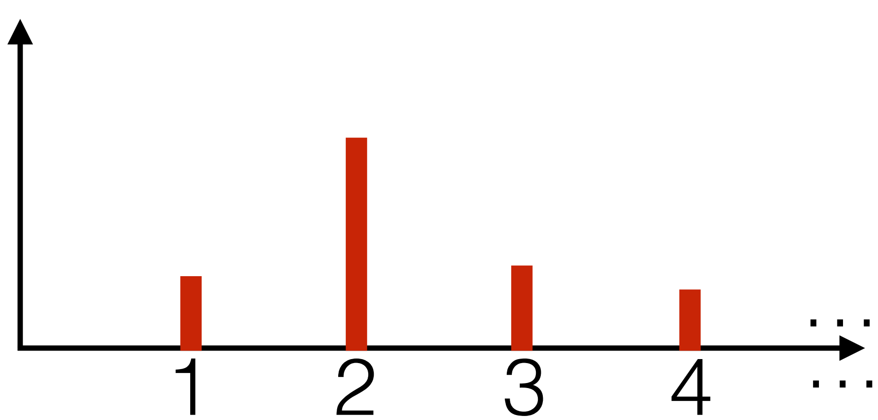


Exercises

- Prove the beta (Dirichlet) is conjugate to the categorical
  - What is the posterior after *N* data points?
- Suppose $\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K})$; prove that

$$\rho_1 \stackrel{d}{=} \mathrm{Beta}(a_1, \sum_{k=1}^{K} a_k - a_1) \perp\!\!\!\perp \frac{(\rho_2, \ldots, \rho_K)}{1-\rho_1} \stackrel{d}{=} \mathrm{Dirichlet}(a_2, \ldots, a_K)$$

- Code your own GEM simulator for ρ; why is this hard?
- Simulate drawing cluster indicators (*z*) from your ρ

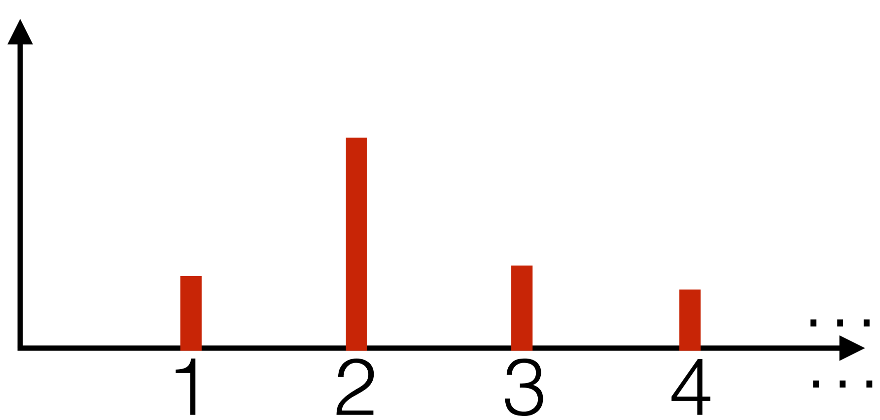


Exercises

- Prove the beta (Dirichlet) is conjugate to the categorical
  - What is the posterior after *N* data points?
- Suppose $\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K})$; prove that

$$\rho_1 \stackrel{d}{=} \mathrm{Beta}(a_1, \sum_{k=1}^{K} a_k - a_1) \perp\!\!\!\perp \frac{(\rho_2, \ldots, \rho_K)}{1-\rho_1} \stackrel{d}{=} \mathrm{Dirichlet}(a_2, \ldots, a_K)$$

- Code your own GEM simulator for ρ; why is this hard?
- Simulate drawing cluster indicators (*z*) from your ρ



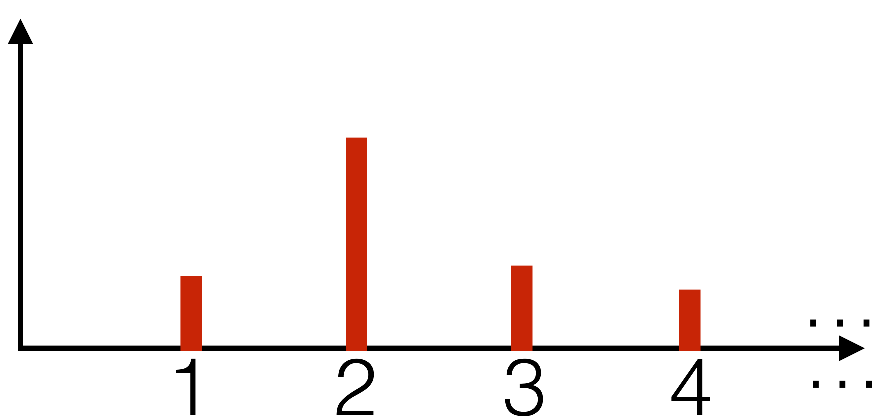
- Compare the number of clusters as *N* changes in the GEM case with the growth in the *K*=1000 case

Exercises

- Prove the beta (Dirichlet) is conjugate to the categorical
  - What is the posterior after *N* data points?
- Suppose $\rho_{1:K} \sim \text{Dirichlet}(a_{1:K})$; prove that

$$\rho_1 \stackrel{d}{=} \text{Beta}(a_1, \sum_{k=1}^{K} a_k - a_1) \perp\!\!\!\perp \tfrac{(\rho_2,\ldots,\rho_K)}{1-\rho_1} \stackrel{d}{=} \text{Dirichlet}(a_2, \ldots, a_K)$$

- Code your own GEM simulator for ρ; why is this hard?
- Simulate drawing cluster indicators (*z*) from your ρ



- Compare the number of clusters as *N* changes in the GEM case with the growth in the *K*=1000 case
- How does the growth in *N* change when you change α?

References

A full reference list is provided at the end of the “Part III” slides.