@R_Trotta

Bayesian Model Comparison & Bayesian Hierachical Models

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Imperial Centre for Inference & Cosmology

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Frequentist Hypothesis Testing

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- **Warning:** frequentist hypothesis testing (e.g., likelihood ratio test) cannot be interpreted as a statement about the probability of the hypothesis!
- **Example:** to test the null hypothesis H_0: $\theta = 0$, draw *n* normally distributed points (with known variance σ^2). The χ^2 is distributed as a chi-square distribution with *(n-1)* degrees of freedom (dof). Pick a significance level α (or p-value, e.g. $\alpha = 0.05$). If $P(\chi^2 > \chi^2_{obs}) < \alpha$ reject the null hypothesis.
- This is a statement about the likelihood of observing data as extreme or more extreme than have been measured *assuming the null hypothesis is correct*.
- **It is not a statement about the probability of the null hypothesis itself and cannot be interpreted as such! (or you'll make gross mistakes)**
- *The use of p-values implies that a hypothesis that may be true can be rejected because it has not predicted observable results that have not actually occurred.* (Jeffreys, 1961)

Exercise: Is the coin fair? [Stopping Rule Problem]

Divide students in team of 3-4 students each
Half of the teams are Red, half are Blue

Blue Team: $N=12$ is fixed, H the random variable

Red Team: $H=3$ is fixed, N the random variable

Data: T T H T H T T T T T T H

Question: What is the p-value for the null hypothesis?

The Significance of “Significance"

- **Important:** A 2-sigma result does not wrongly reject the null hypothesis 5% of the time: **at least 29% of 2-sigma results are wrong!**
    - Take an equal mixture of H_0, H_1
    - Simulate data, perform hypothesis testing for H_0
    - Select results rejecting H_0 at (or within a small range from) 1-α CL (this is the prescription by Fisher)
    - What fraction of those results did actually come from H_0 ("true nulls", should not have been rejected)?

p–value	sigma	fraction of true nulls	lower bound
0.05	1.96	0.51	0.29
0.01	2.58	0.20	0.11
0.001	3.29	0.024	0.018

Recommended reading:
Sellke, Bayarri & Berger, *The American Statistician*, 55, 1 (2001)

Bayesian Model Comparison

The 3 Levels of Inference

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LEVEL 1
I have selected a model M and prior $P(\theta|M)$

Parameter inference
What are the favourite values of the parameters?
(assumes M is true)

$$P(\theta|d,M) = \frac{P(d|\theta,M)P(\theta|M)}{P(d|M)}$$

LEVEL 2
Actually, there are several possible models: M_0, M_1,...

Model comparison
What is the relative plausibility of M_0, M_1,... in light of the data?

$$\text{odds} = \frac{P(M_0|d)}{P(M_1|d)}$$

LEVEL 3
None of the models is clearly the best

Model averaging
What is the inference on the parameters accounting for model uncertainty?

$$P(\theta|d) = \sum_i P(M_i|d)P(\theta|d,M_i)$$

Model Comparison Questions

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ASTROPARTICLE

Gravitational waves detection

Do cosmic rays correlate with AGNs?

Which SUSY model is ‘best’?

Is there evidence for DM modulation?

Is there a DM signal in gamma ray/ neutrino data?

COSMOLOGY

Is the Universe flat?

Does dark energy evolve?

Are there anomalies in the CMB?

Which inflationary model is ‘best’?

Is there evidence for modified gravity?

Are the initial conditions adiabatic?

Many scientific questions are of the model comparison type

ASTROPHYSICS

Exoplanets detection

Is there a line in this spectrum?

Is there a source in this image?

Level 2 Inference: Model Comparison

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$$P(\theta|d,M) = \frac{P(d|\theta,M)P(\theta|M)}{P(d|M)}$$

Bayesian evidence or model likelihood

The evidence is the integral of the likelihood over the prior:

$$P(d|M) = \int_\Omega d\theta P(d|\theta,M)P(\theta|M)$$

Bayes' Theorem delivers the model's posterior:

$$P(M|d) = \frac{P(d|M)P(M)}{P(d)}$$

When we are comparing two models:

$$\frac{P(M_0|d)}{P(M_1|d)} = \frac{P(d|M_0)}{P(d|M_1)}\frac{P(M_0)}{P(M_1)}$$

The Bayes factor:

$$B_{01} \equiv \frac{P(d|M_0)}{P(d|M_1)}$$

Posterior odds = Bayes factor × prior odds

Strength of Evidence

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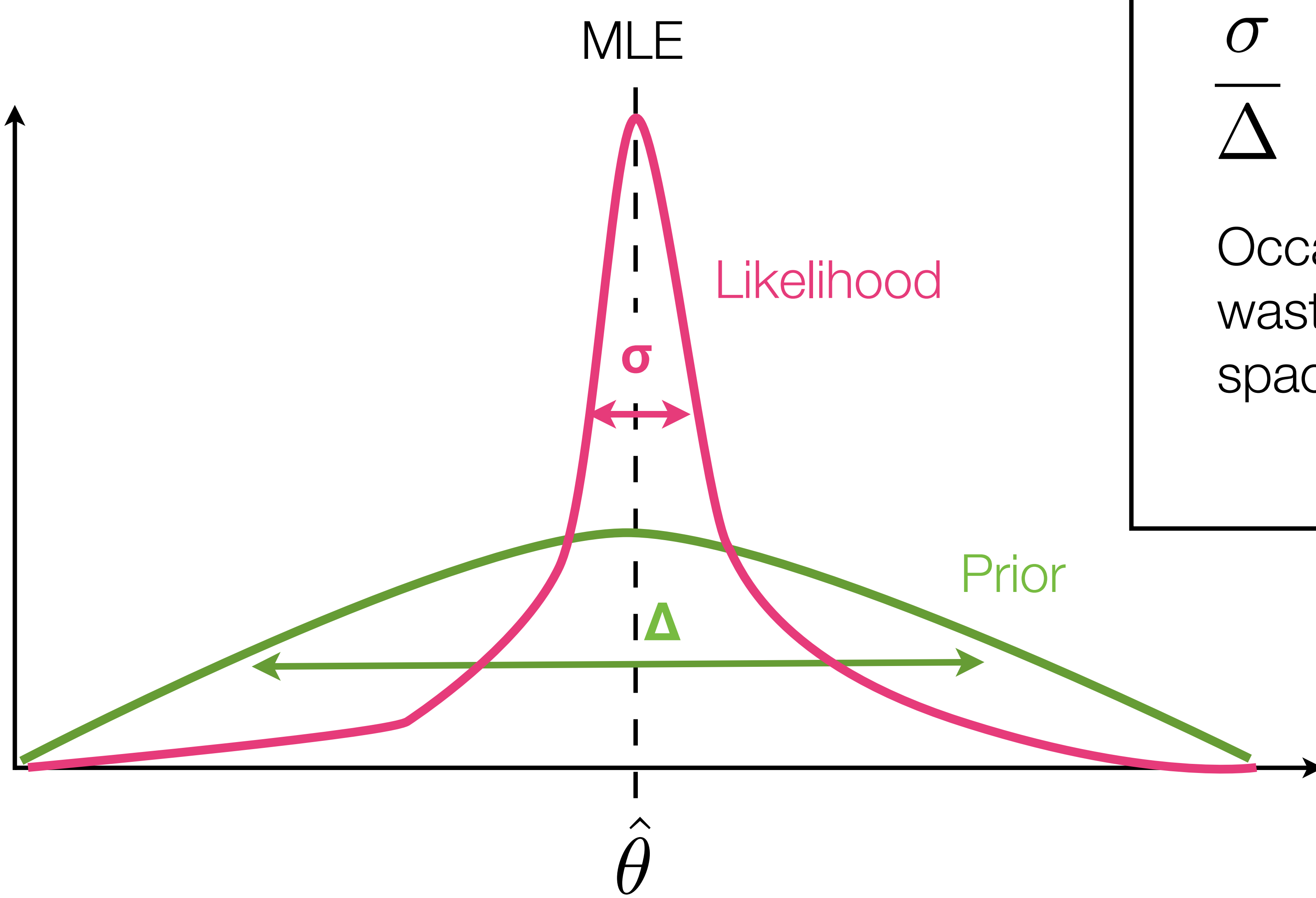
- A (slightly modified) Jeffreys' scale to assess the strength of evidence

$\|\ln B\|$	relative odds	favoured model's probability	Interpretation
< 1.0	< 3:1	< 0.750	not worth mentioning
< 2.5	< 12:1	0.923	weak
< 5.0	< 150:1	0.993	moderate
> 5.0	> 150:1	> 0.993	strong

An Automatic Occam's Razor

$$P(d|M) = \int \mathcal{L}(\theta)P(\theta|M)d\theta \approx \mathcal{L}(\hat{\theta})\sigma P(\hat{\theta}) \approx \mathcal{L}(\hat{\theta})\frac{\sigma}{\Delta}$$

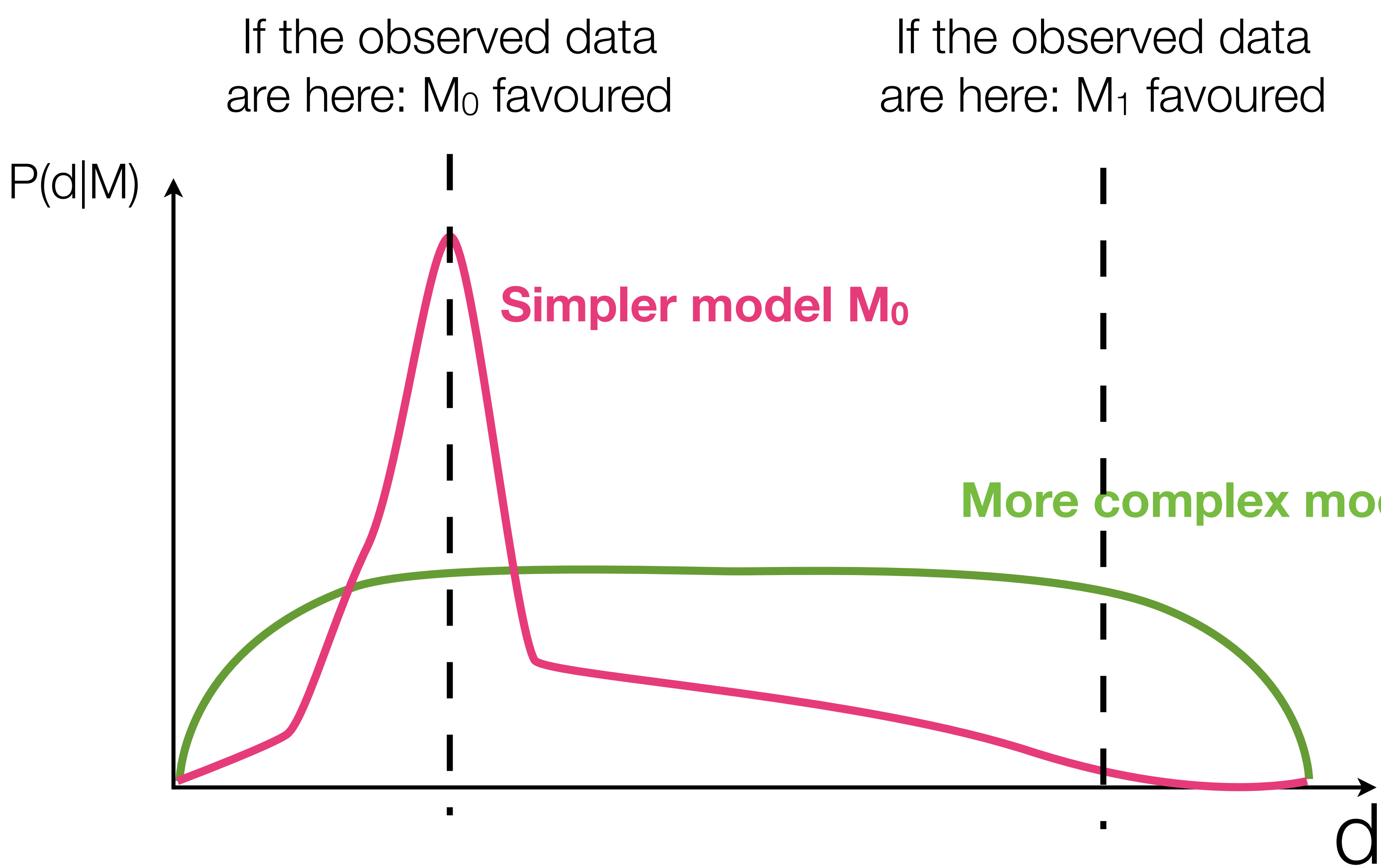


$$\frac{\sigma}{\Delta} \ll 1 \text{ for } \Delta \gg \sigma$$

Occam's factor:
wasted parameter
space under the prior

The Evidence as Predictive Probability

- The evidence can be understood as a function of d to give the predictive probability under the model M:



The Logic of Science

- Popper: “science proceeds by falsifying null hypotheses” (hypothesis testing, p-value)
- Jaynes: “a successful prediction [failure to falsify] ought to be rewarded” (model comparison, Bayes Factor)
- Einstein: “everything should be as simple as possible, but not simpler” (Occam’s razor)

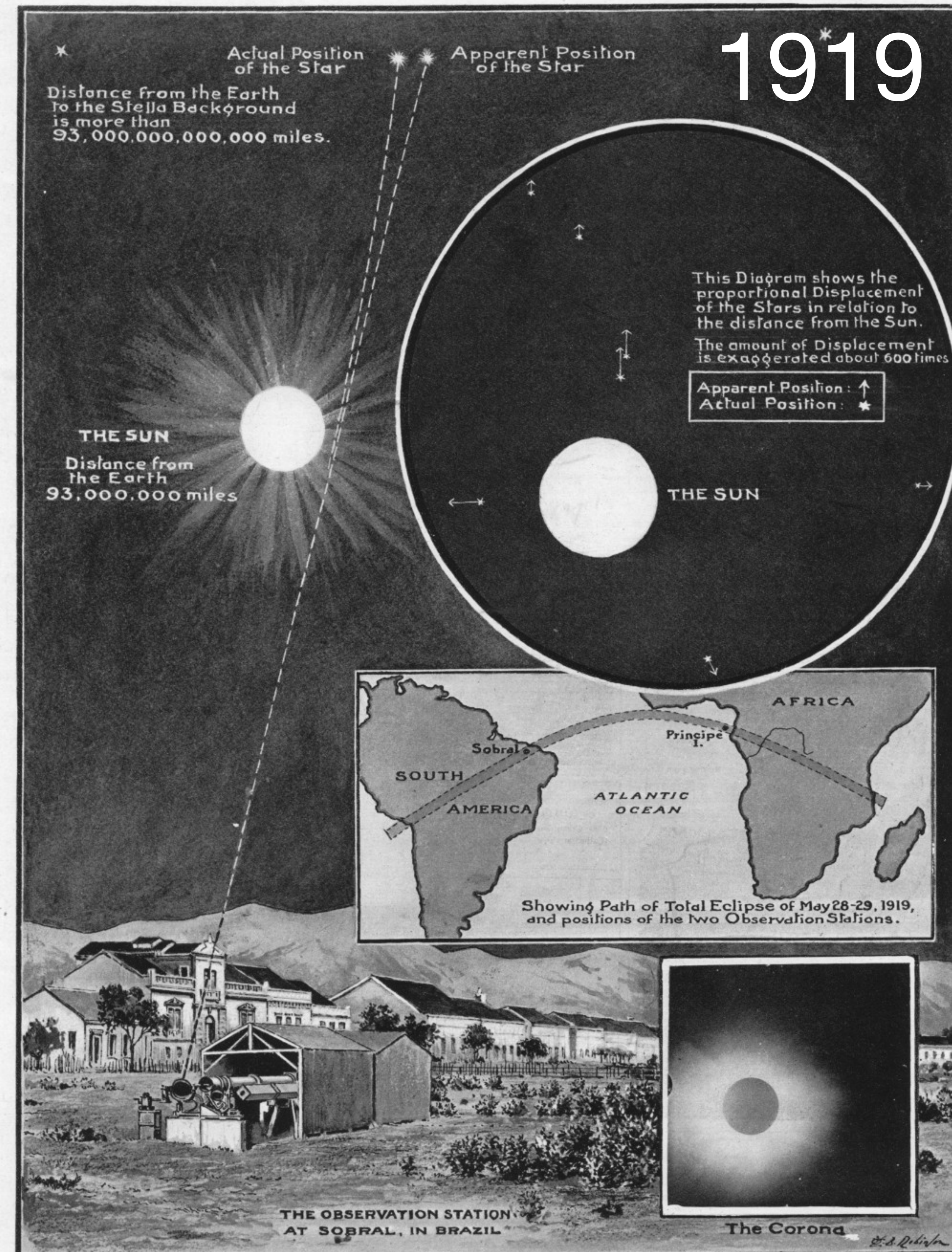
$$\text{Newton: } \alpha = 0.87''$$
$$\text{Einstein: } \alpha = 1.74''$$

Posterior Odds:

5 : 1 for Einstein

“STARLIGHT BENT BY THE SUN’S ATTRACTION”: THE EINSTEIN THEORY.

DRAWN BY W. B. ROBINSON, FROM MATERIAL SUPPLIED BY DR. CROMMELIN.

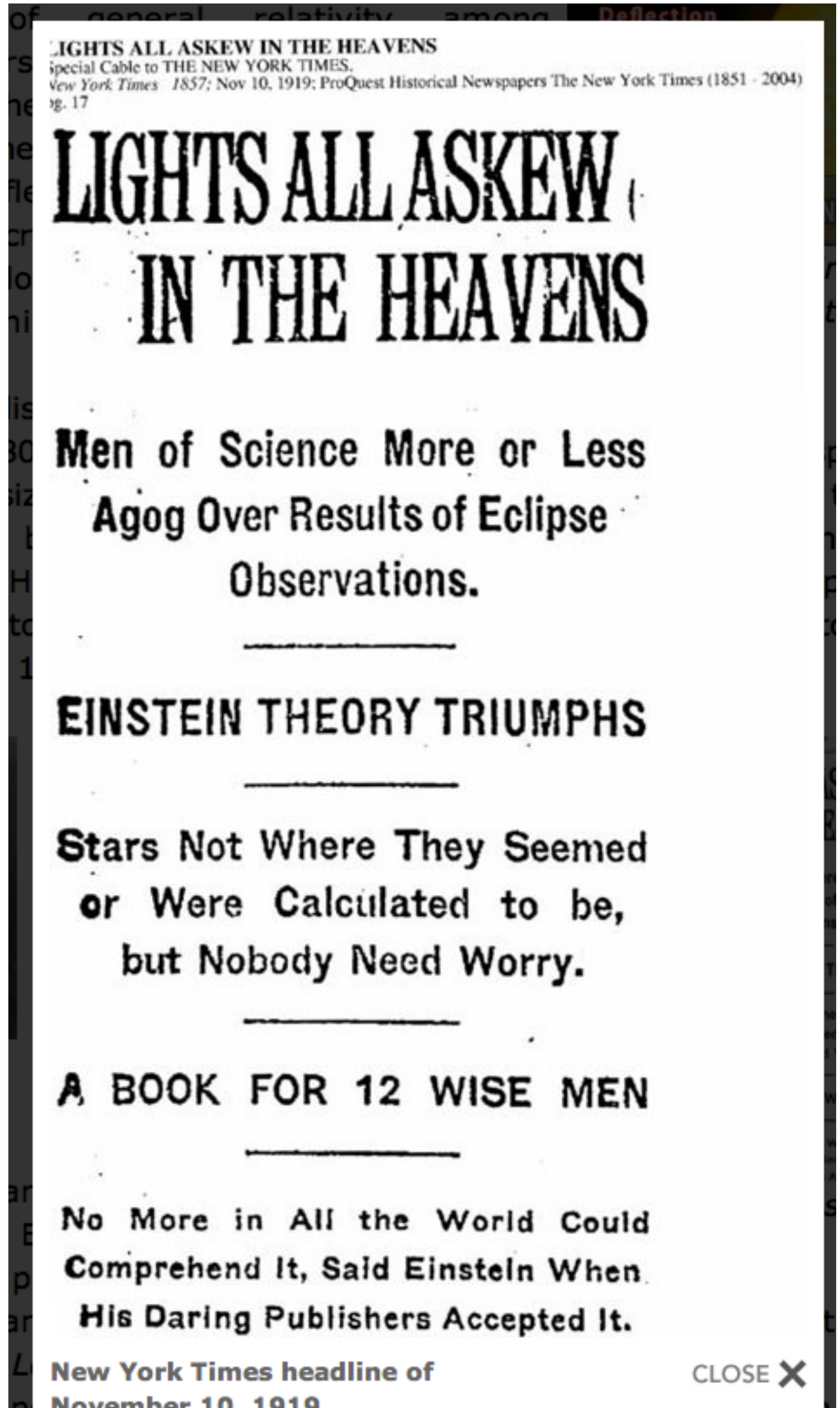


THE CURVATURE OF LIGHT: EVIDENCE FROM BRITISH OBSERVERS’ PHOTOGRAPHS AT THE ECLIPSE OF THE SUN.

The results obtained by the British expeditions to observe the total eclipse of the sun last May verified Professor Einstein’s theory that light is subject to gravitation. Writing in our issue of November 15, Dr. A. C. Crommelin, one of the British observers, said : “The eclipse was specially favourable for the purpose, there being no fewer than twelve fairly bright stars near the limb of the sun. The process of observation consisted in taking photographs of these stars during totality, and comparing them with other plates of the same region taken when the sun was not in the neighbourhood. Then if the starlight is bent by the sun’s attraction, the stars on the eclipse plates would seem to be pushed outward compared with those on the other plates. . . . The second Sobral camera and the one used at Principe agree in supporting (Einstein’s theory). . . . It is of profound philosophical interest. Straight lines in Einstein’s space cannot exist ; they are parts of gigantic curves.”—*[Drawing Copyrighted in the United States and Canada.]*

Posterior Odds:

5 : 1 for Einstein



LIGHTS ALL ASKEW IN THE HEAVENS
Special Cable to THE NEW YORK TIMES.
New York Times 1857; Nov 10, 1919; ProQuest Historical Newspapers The New York Times (1851 - 2004)
pg. 17

LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less Agog Over Results of Eclipse Observations.

EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

No More in All the World Could Comprehend It, Said Einstein When His Daring Publishers Accepted It.

New York Times headline of November 10, 1919.

CLOSE

Simple Example: Nested Models

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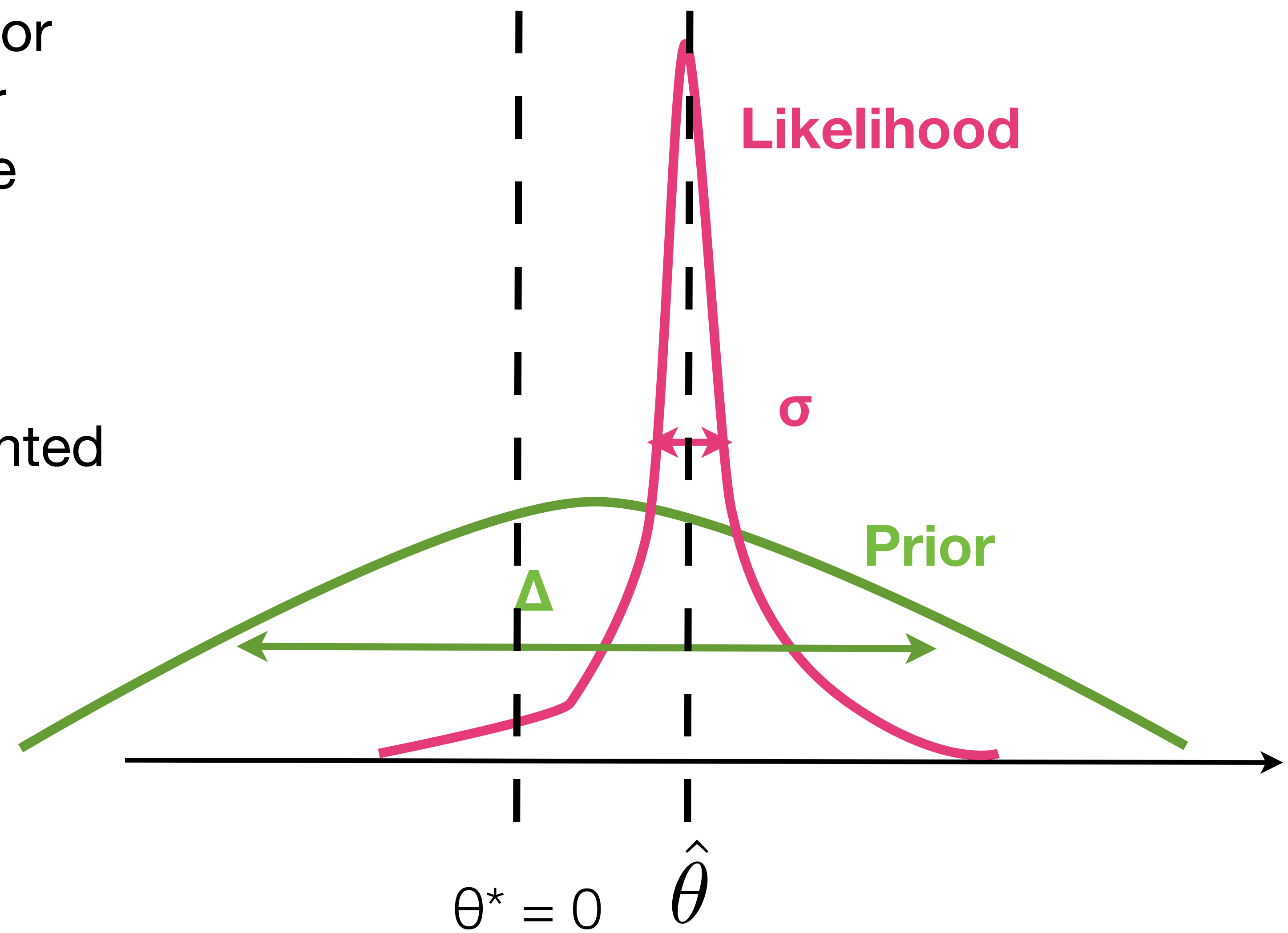
- This happens often in practice: we have a more complex model, M_1 with prior $P(\theta|M_1)$, which reduces to a simpler model (M_0) for a certain value of the parameter, e.g. $\theta = \theta^* = 0$ (**nested models**)

- Is the extra complexity of M_1 warranted by the data?



Simple Example: Nested Models

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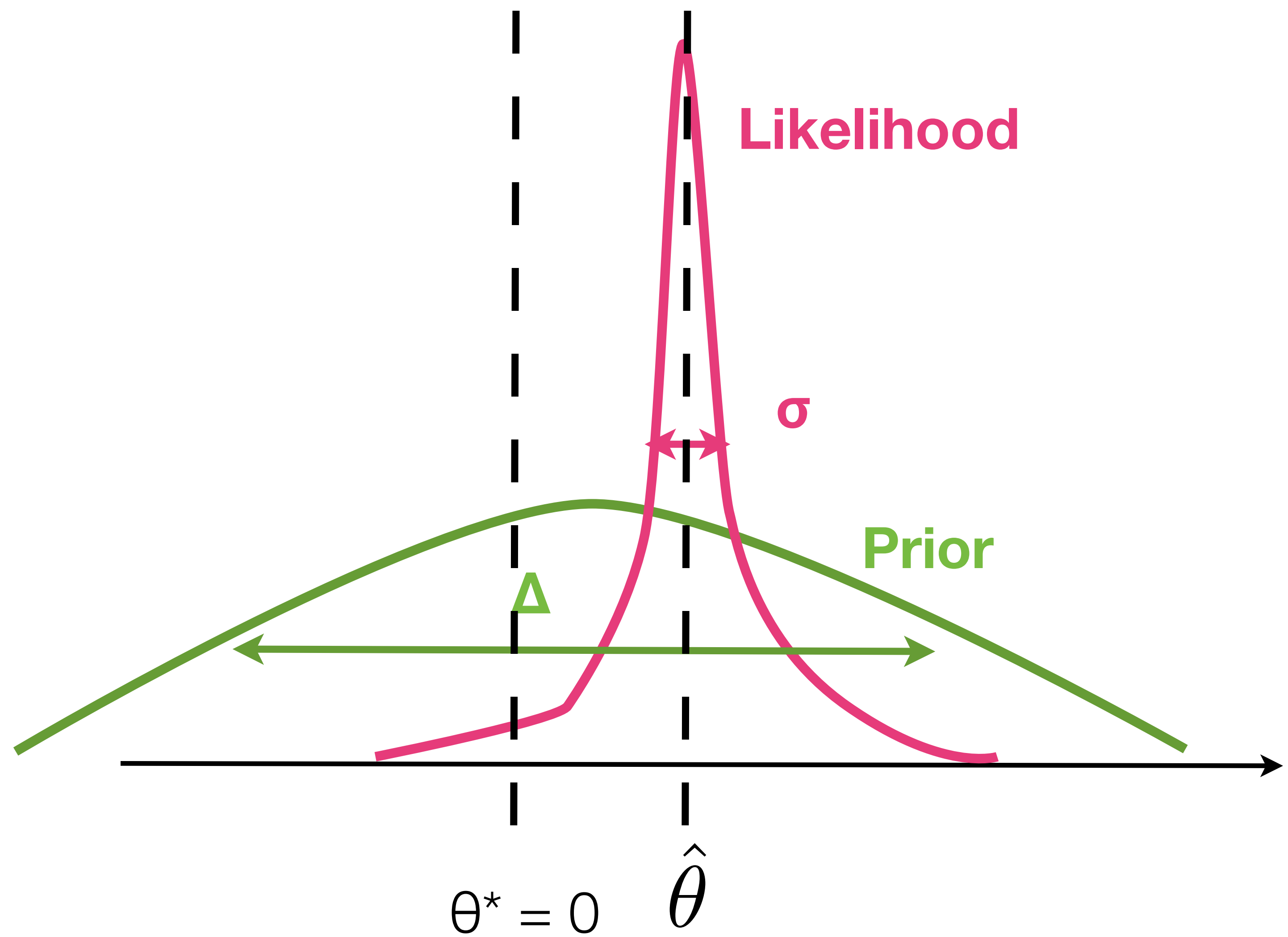
Define: $$\lambda \equiv \frac{\hat{\theta} - \theta^\star}{\sigma}$$

For “informative” data (i.e., Δ>> σ):

$$\ln B_{01} \approx \ln \frac{\Delta}{\sigma} - \frac{\lambda^2}{2}$$

wasted parameter space (favours simpler model)

mismatch of prediction with observed data (favours more complex model)



The Rough Guide to Model Comparison

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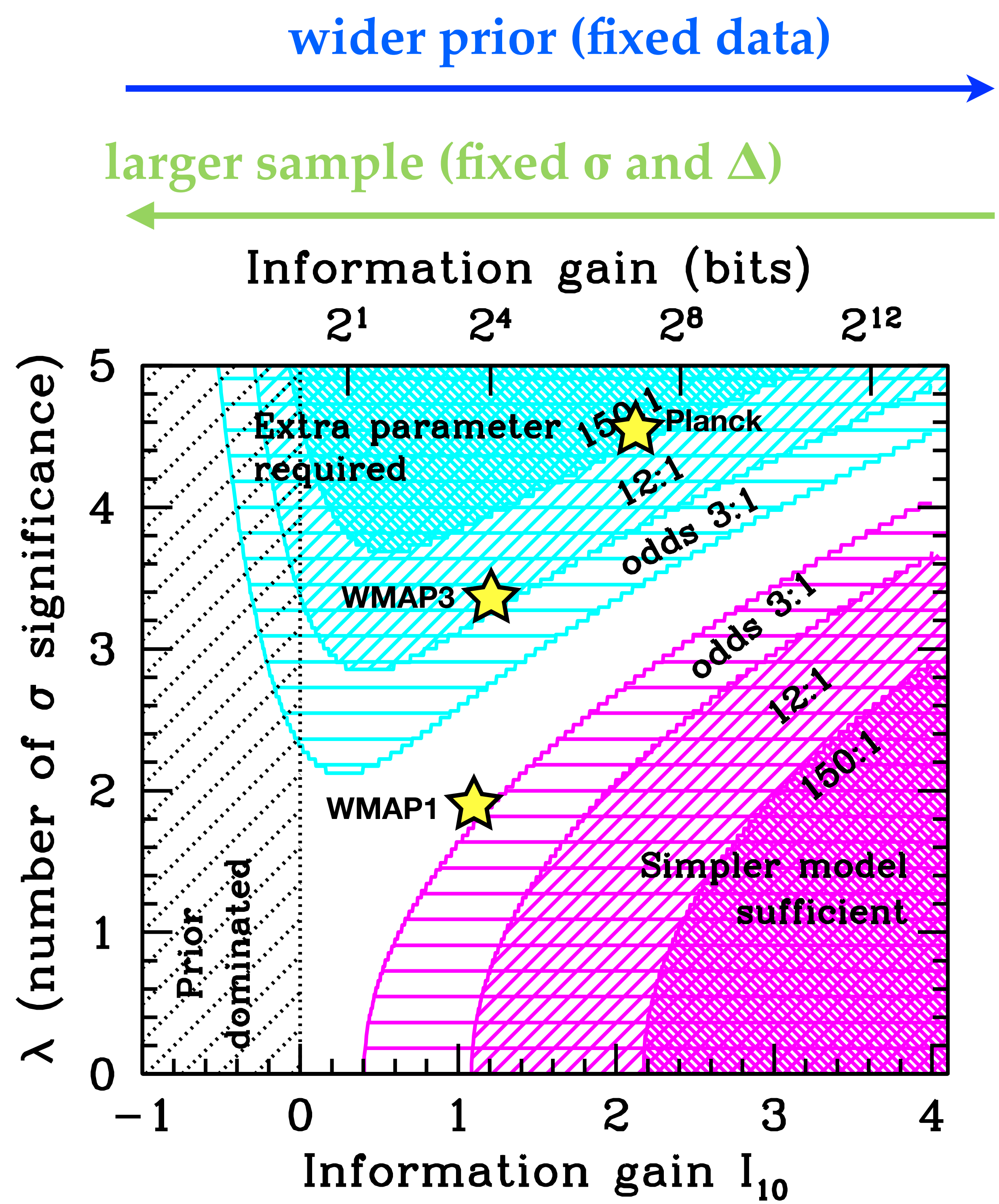


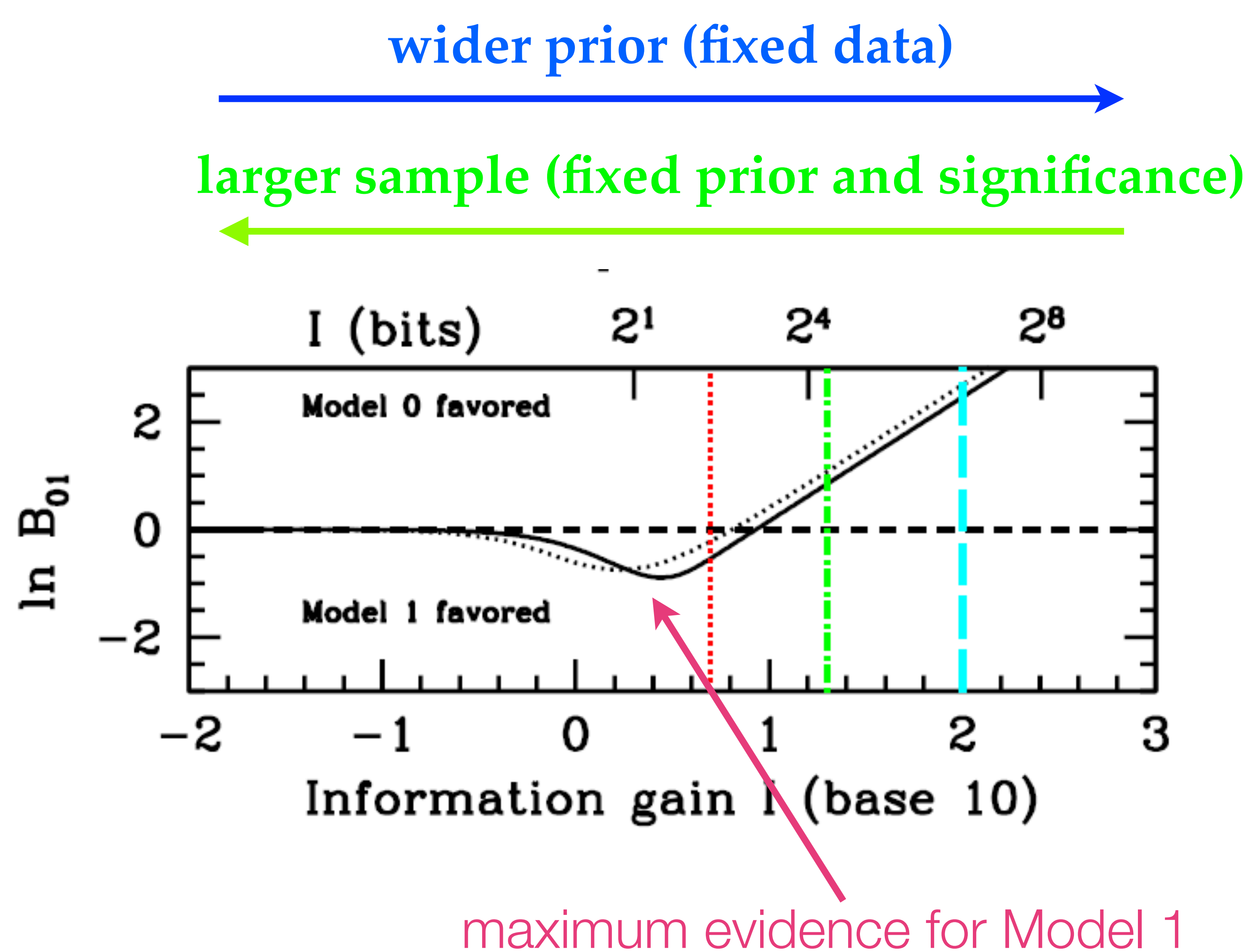
$$I_{10} \equiv \log_{10} \frac{\Delta}{\sigma}$$

Δ = Prior width
σ = Likelihood width

R. Trotta (2008), Contemporary Physics, 49, No. 2, March-April 2008, 71-104, e-print archive: 0803.4089,

“Prior-free” Evidence Bounds

- What if we do not know how to set the prior? For nested models, we can still choose a prior that will maximise the support for the more complex model:



Maximum Evidence for a Detection

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- **The absolute upper bound:** put all prior mass for the alternative onto the observed maximum likelihood value. Then

$$B < \exp(-\chi^2/2)$$

- **More reasonable class of priors:** symmetric and unimodal around Ψ=0, then (α = significance level)

$$B < \frac{-1}{\exp(1)\alpha \ln \alpha}$$

If the upper bound is small, no other choice of prior will make the extra parameter significant.

For details, see: Sellke, Bayarri & Berger, The American Statistician, 55, 1 (2001)

How to Interpret the “Number of σ”

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α	sigma	Absolute bound on lnB (B)	“Reasonable” bound on lnB (B)
0.05	2	2.0 (7:1) weak	0.9 (3:1) undecided
0.003	3	4.5 (90:1) moderate	3.0 (21:1) moderate
0.0003	3.6	6.48 (650:1) strong	5.0 (150:1) strong

How to Assess p-values

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Rule of thumb:
interpret a n-sigma result as a (n-1)-sigma result

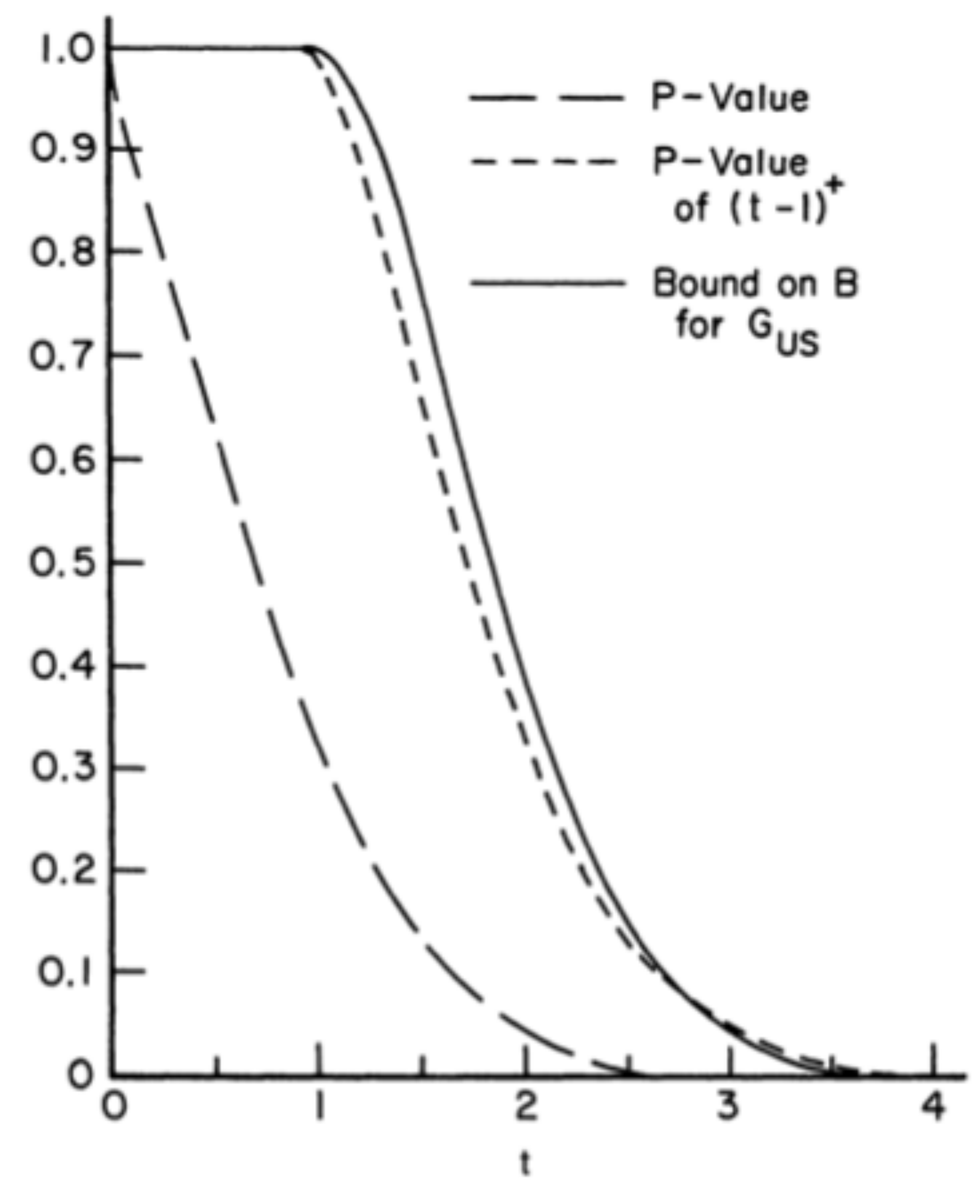

Figure 4. Comparison of $\underline{B}(x, G_{US})$ and P Values.

Sellke, Bayarri & Berger, The American Statistician, 55, 1 (2001)

Computational Aspects

Computing the Model Likelihood

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Model likelihood: $P(d|M) = \int_\Omega d\theta P(d|\theta, M)P(\theta|M)$

Bayes factor: $B_{01} \equiv \frac{P(d|M_0)}{P(d|M_1)}$

- Usually computational demanding: it's a multi-dimensional integral, averaging the likelihood over the (possibly much wider) prior
- I'll present two methods used by cosmologists:
    - **Savage-Dickey density ratio (Dickey 1971):** Gives the Bayes factor between *nested* models (under mild conditions). Can be usually derived from posterior samples of the larger (higher D) model.
    - **Nested sampling (Skilling 2004):** Transforms the D-dim integral in 1D integration. Can be used generally (within limitations of the efficiency of the sampling method adopted).
- There are many others. One recent clever idea to re-use MCMC samples to estimate the evidence is presented in Heavens et al, arXiv:1704.03472
- For a method that sidesteps the need to compute the evidence explicitly, see Hee et al, Mon.Not.Roy.Astron.Soc. 455 (2016) no.3, 2461-2473, arXiv:1506.09024

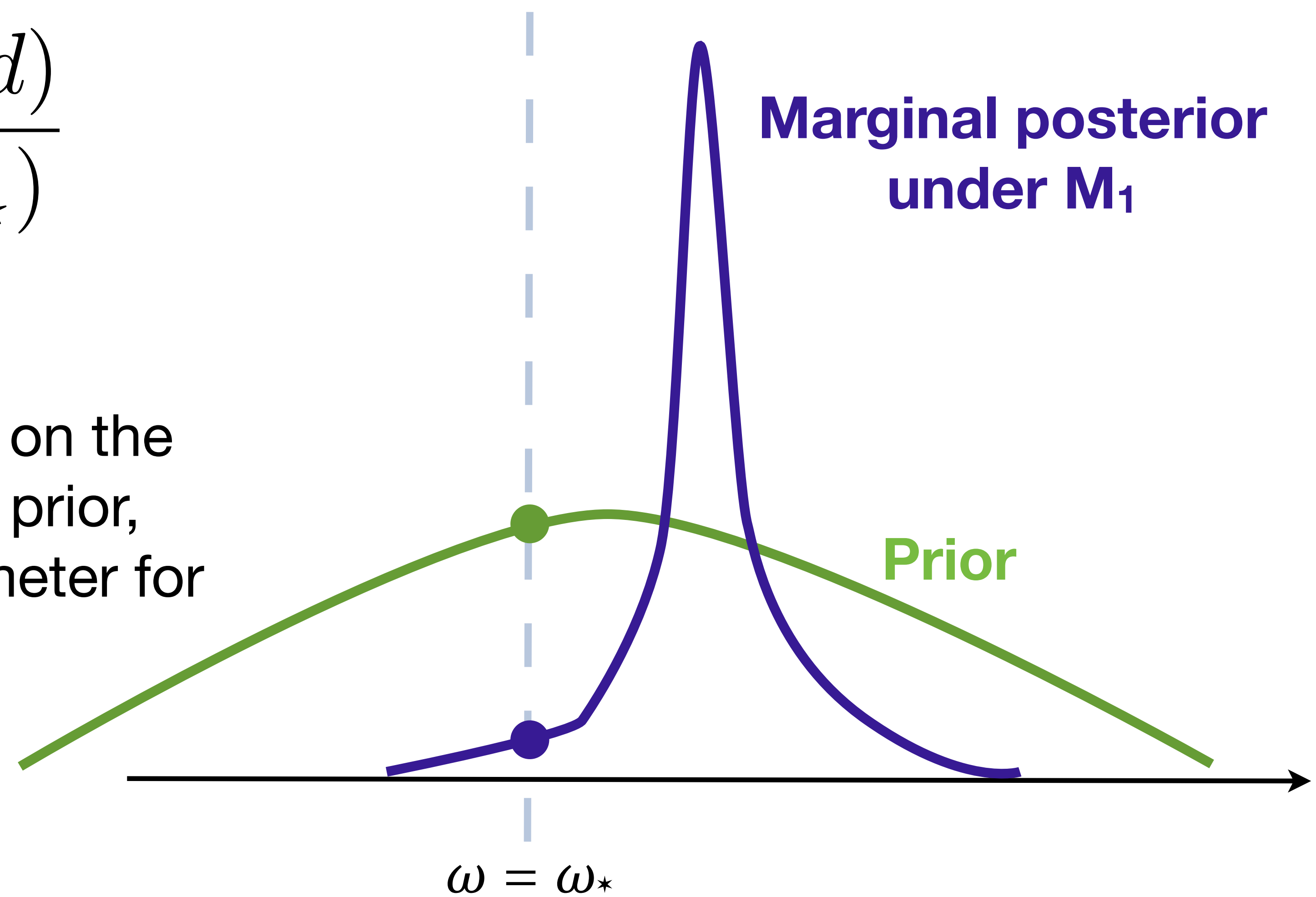
The Savage-Dickey Density Ratio

Dickey J. M., 1971, Ann. Math. Stat., 42, 204

- This method works for *nested models* and gives the Bayes factor analytically.
- **Assumptions:**
  - Nested models: M_1 with parameters (Ψ,ω) reduces to M_0 for e.g. $\omega=\omega_\star$
  - Separable priors: the prior $\pi_1(\Psi,\omega|M_1)$ is uncorrelated with $\pi_0(\Psi|M_0)$
- **Result:**

$$B_{01} = \frac{p(\omega_\star|d)}{\pi_1(\omega_\star)}$$

- The Bayes factor is the ratio of the normalised (1D) marginal posterior on the additional parameter in M_1 over its prior, evaluated at the value of the parameter for which M_1 reduces to M_0.



Derivation of the SDDR

RT, Mon.Not.Roy.Astron.Soc. 378 (2007) 72-82

$$P(d|M_0) = \int d\Psi \pi_0(\Psi) p(d|\Psi, \omega_\star) \quad P(d|M_1) = \int d\Psi d\omega \pi_1(\Psi, \omega) p(d|\Psi, \omega)$$

Divide and multiply B_{01} by:

$$p(\omega_\star|d) = \frac{p(\omega_\star, \Psi|d)}{p(\Psi|\omega_\star, d)}$$

$$B_{01} = p(\omega_\star|d) \int d\Psi \frac{\pi_0(\Psi) p(d|\Psi, \omega_\star)}{P(M_1|d)} \frac{p(\Psi|\omega_\star, d)}{p(\omega_\star, \Psi|d)}$$

Since:

$$p(\omega_\star, \Psi|d) = \frac{p(d|\omega_\star, \Psi)\pi_1(\omega_\star, \Psi)}{P(M_1|d)}$$

$$B_{01} = p(\omega_\star|d) \int d\Psi \frac{\pi_0(\Psi) p(\Psi|\omega_\star, d)}{\pi_1(\omega_\star, \Psi)}$$

Assuming separable priors:

$$\pi_1(\omega, \Psi) = \pi_1(\omega)\pi_0(\Psi)$$

$$B_{01} = \frac{p(\omega_\star|d)}{\pi_1(\omega_\star)} \int d\Psi p(\Psi|\omega_\star, d) = \frac{p(\omega_\star|d)}{\pi_1(\omega_\star)}$$

SDDR: Some Comments

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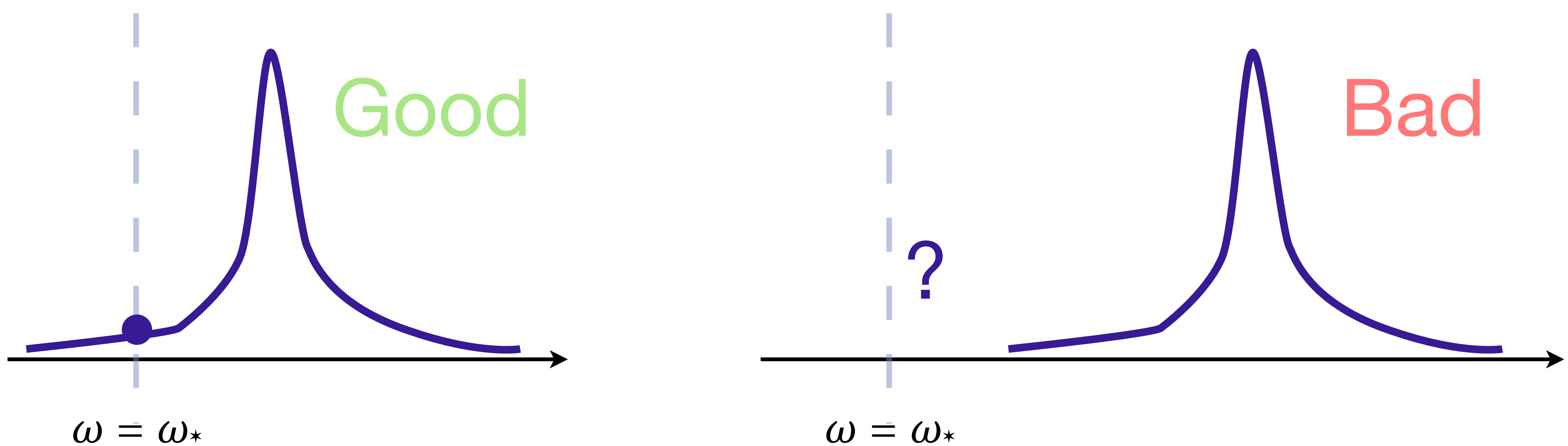
- For separable priors (and nested models), the common parameters do not matter for the value of the Bayes factor
- No need to spend time/resources to average the likelihoods over the common parameters
- Role of the prior on the additional parameter is clarified: the wider, the stronger the Occam's razor effect (due to dilution of the predictive power of model 1)
- Sensitivity analysis simplified: only the prior/scale on the additional parameter between the models needs to be considered.
- Notice: SDDR does not assume Gaussianity, but it does require sufficiently detailed sampling of the posterior to evaluate reliably its value at $\omega=\omega_{\star}$.



Accuracy Tests (Normal case)

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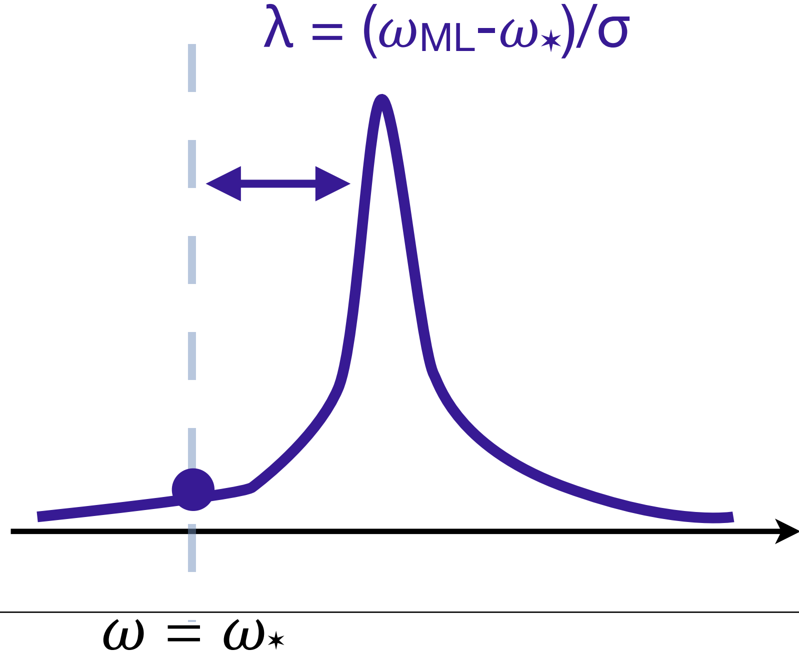
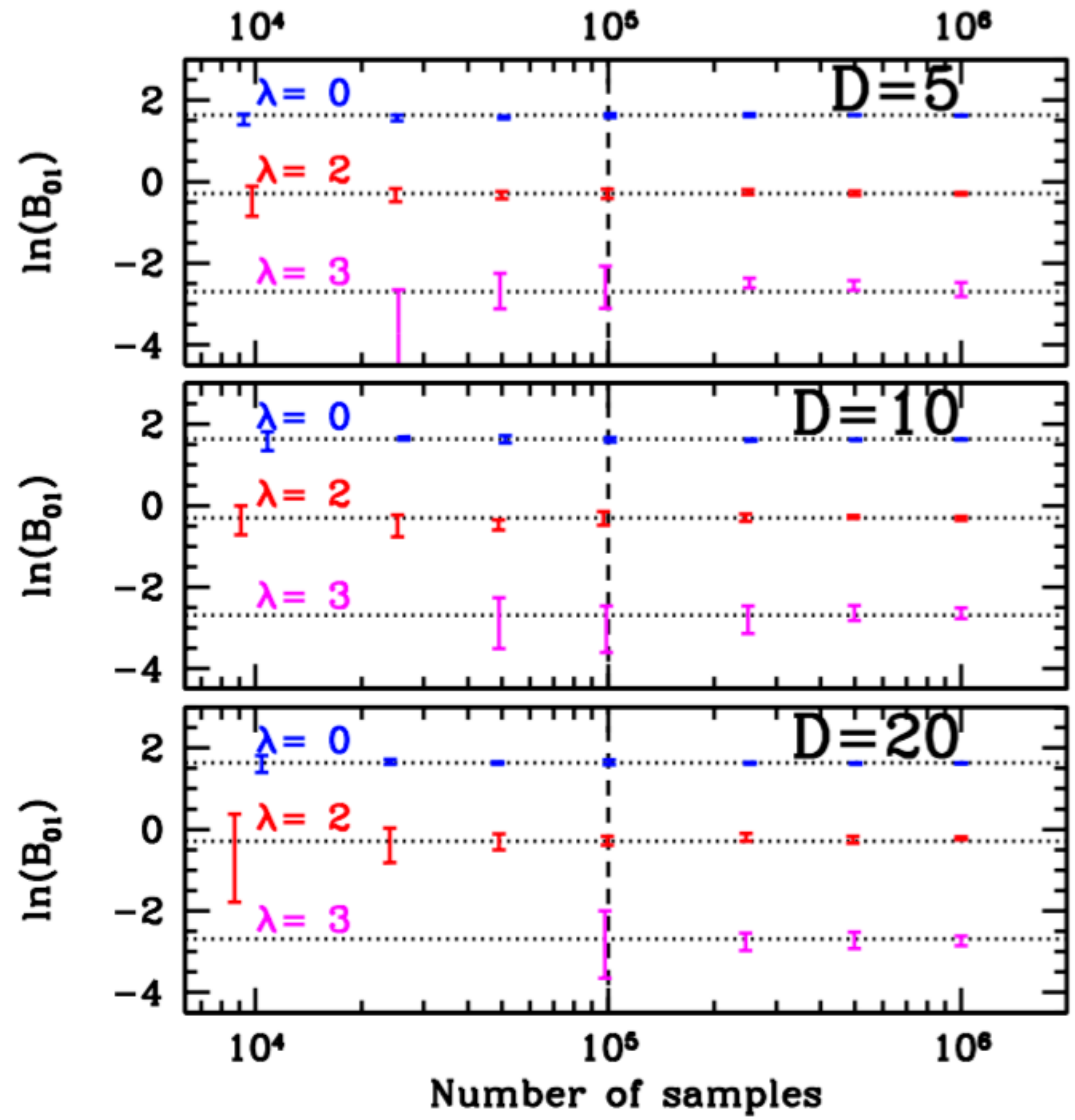
- Tests with variable dimensionality (D) and number of MCMC samples
- λ is the distance of peak posterior from ω_* in units of posterior std dev
- SDDR accurate with standard MCMC sampling up to 20-D and λ=3
- Accurate estimates further in the tails might required dedicated sampling schemes





RT, MNRAS, 378, 72-82 (2007)

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Nested Sampling

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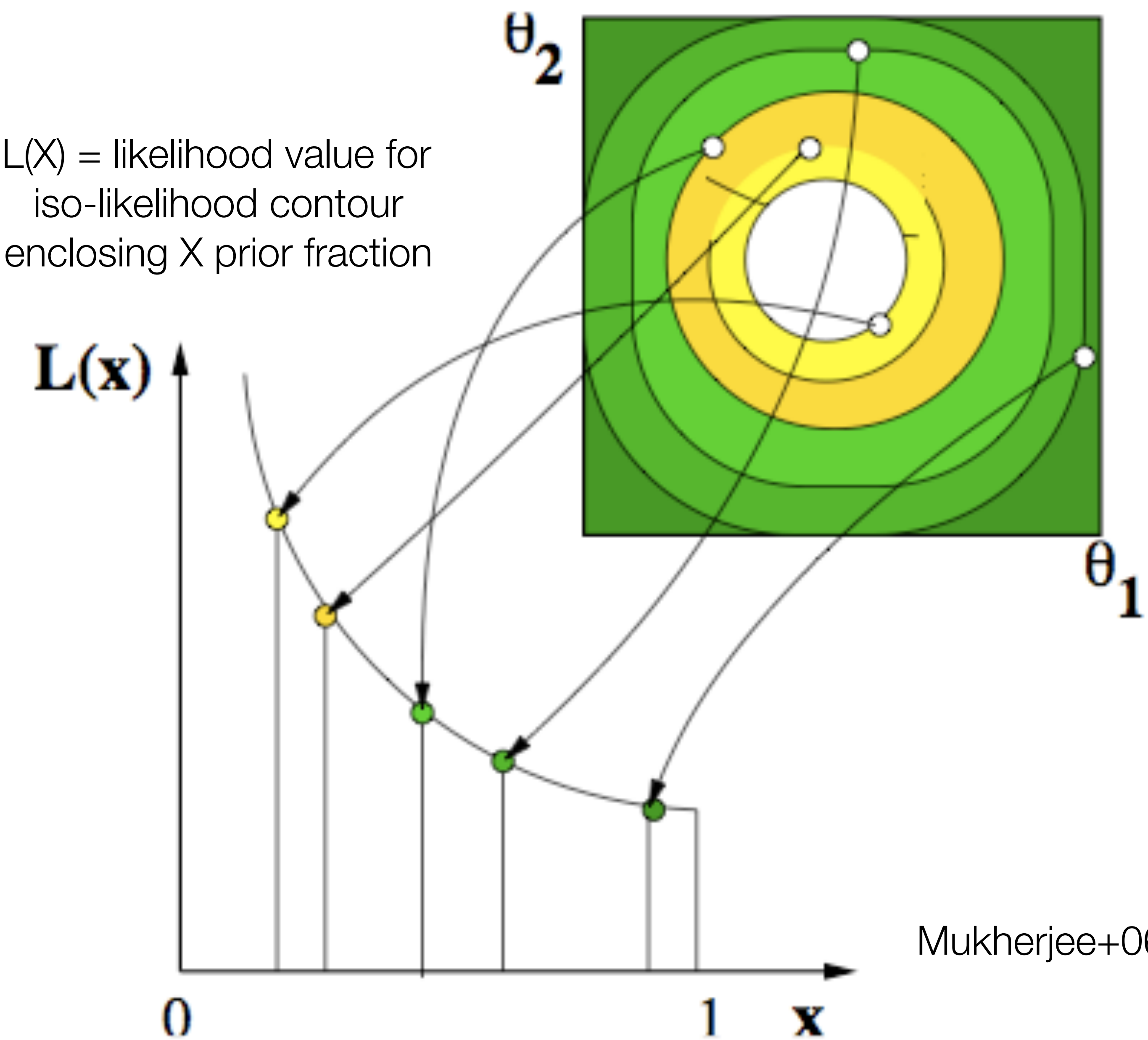
- Proposed by John Skilling in 2004: the idea is to convert a D-dimensional integral in a 1D integral that can be done easily.
- As a by-product, it also produces posterior samples: model likelihood and parameter inference obtained simultaneously

L(X) = likelihood value for iso-likelihood contour enclosing X prior fraction



Mukherjee+06

X = Prior fraction

Nested Sampling Basics

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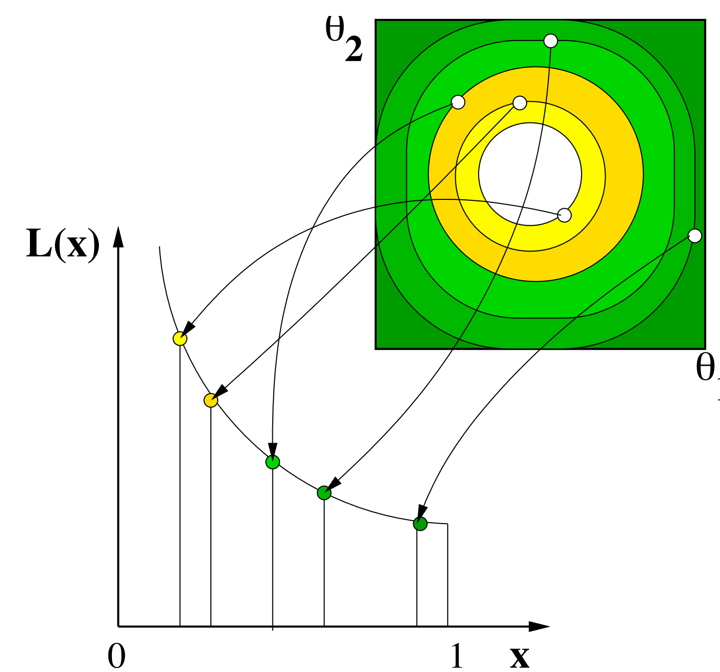
Skilling, AIP Conf.Proc. 735, 395 (2004); doi: 10.1063/1.1835238



Define X(λ) as the prior mass associated with likelihood values above λ

$$X(\lambda) = \int_{\mathcal{L}(\theta)>\lambda} P(\theta) d\theta$$

This is a decreasing function of λ:

$$X(0) = 1 \qquad X(\mathcal{L}_{\max}) = 0$$

dX is the prior mass associated with likelihoods [λ, λ+dλ]
An infinitesimal interval dX contributes λdx to the evidence, so that:

$$P(d) = \int d\theta L(\theta) P(\theta) = \int_0^1 L(X) dX$$

where L(X) is the inverse of X(λ).

Nested Sampling Basics

Suppose that we can evaluate $L_j = L(X_j)$, for a sequence:

$$0 < X_m < \cdots < X_2 < X_1 < 1$$

Then the model likelihood P(d) can be estimated numerically as:

$$P(d) = \sum_{j=1}^{m} w_j L_j$$

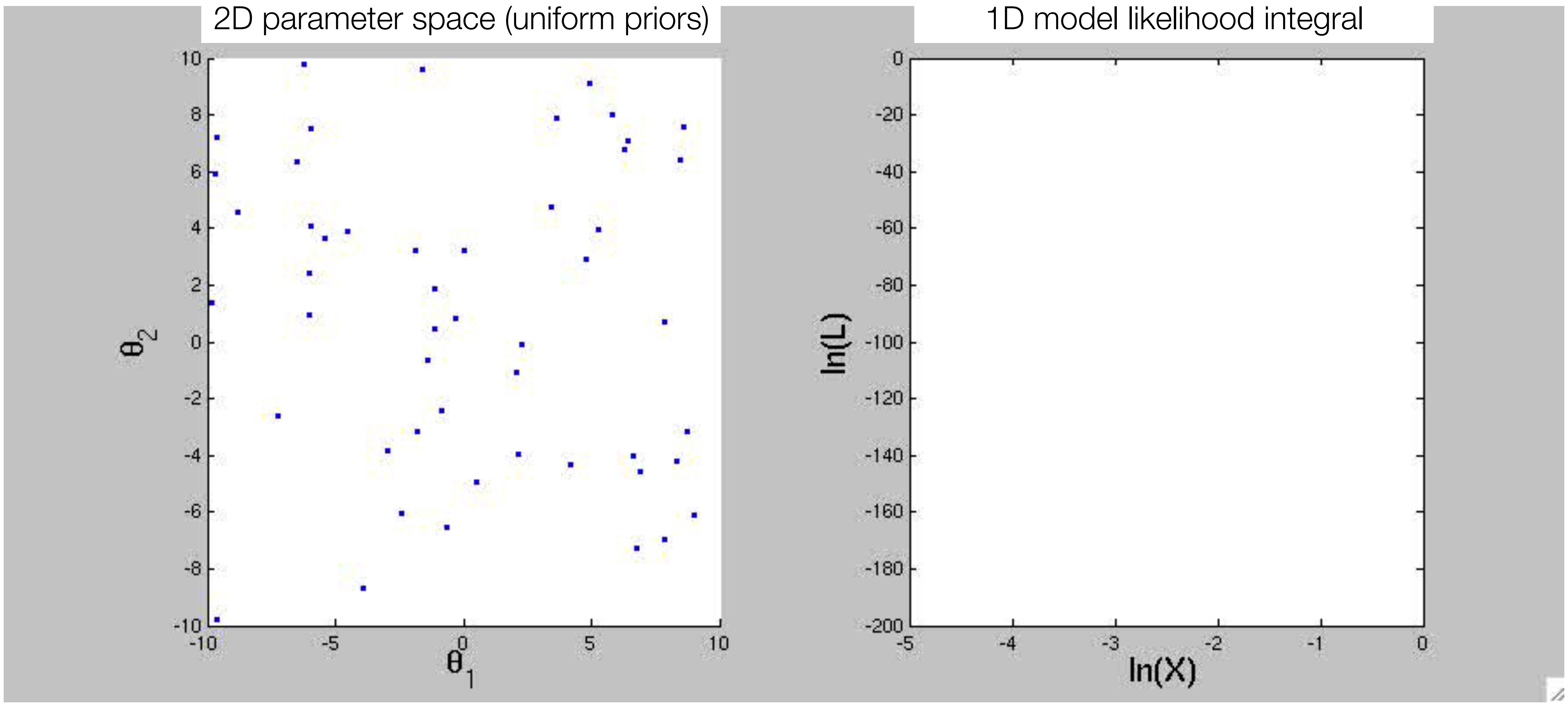
with a suitable set of weights, e.g. for the trapezium rule:

$$w_j = \frac{1}{2}(X_{j-1} - X_{j+1})$$

Nested Sampling in Action

(animation courtesy of David Parkinson)

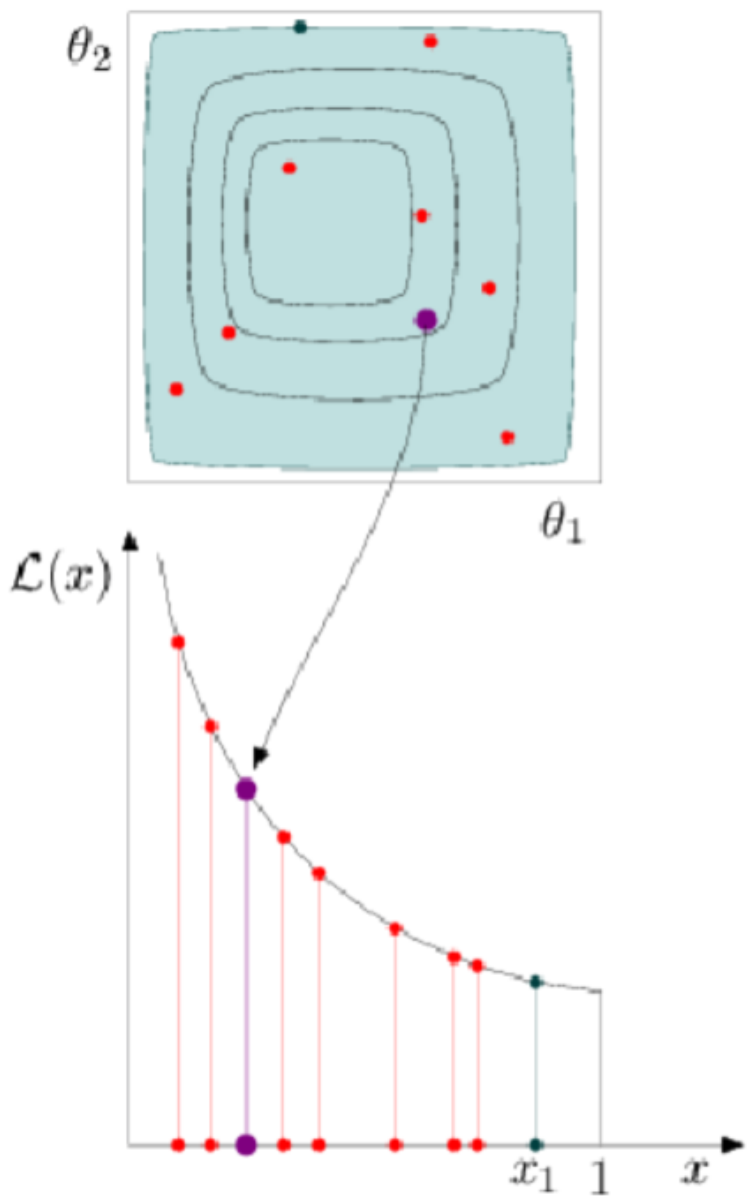
$$P(d) = \int d\theta L(\theta) P(\theta) = \int_0^1 L(X) dX$$



X = Prior fraction

MultiNest Sampling Approach

(Slide courtesy of Mike Hobson)



Nested sampling approach to summation:

1. Set $i = 0$; initially $X_0 = 1$, $E = 0$
2. Sample N points $\{\boldsymbol{\theta}_j\}$ randomly from $\pi(\boldsymbol{\theta})$ and calculate their likelihoods
3. Set $i \to i + 1$
4. Find point with lowest likelihood value (L_i)
5. Remaining prior volume $X_i = t_i X_{i-1}$ where $\Pr(t_i|N) = N t_i^{N-1}$; or just use $\langle t_i \rangle = N/(N+1)$
6. Increment evidence $E \to E + L_i w_i$
7. Remove lowest point from active set
8. Replace with new point sampled from $\pi(\boldsymbol{\theta})$ within hard-edged region $L(\boldsymbol{\theta}) > L_i$

Hard!

9. If $L_{\max} X_i < \alpha E$ (where some tolerance) $\Rightarrow E \to E + X_i \sum_{j=1}^{N} L(\boldsymbol{\theta}_j)/N$; stop else goto 3

Nested Sampling: Sampling Step

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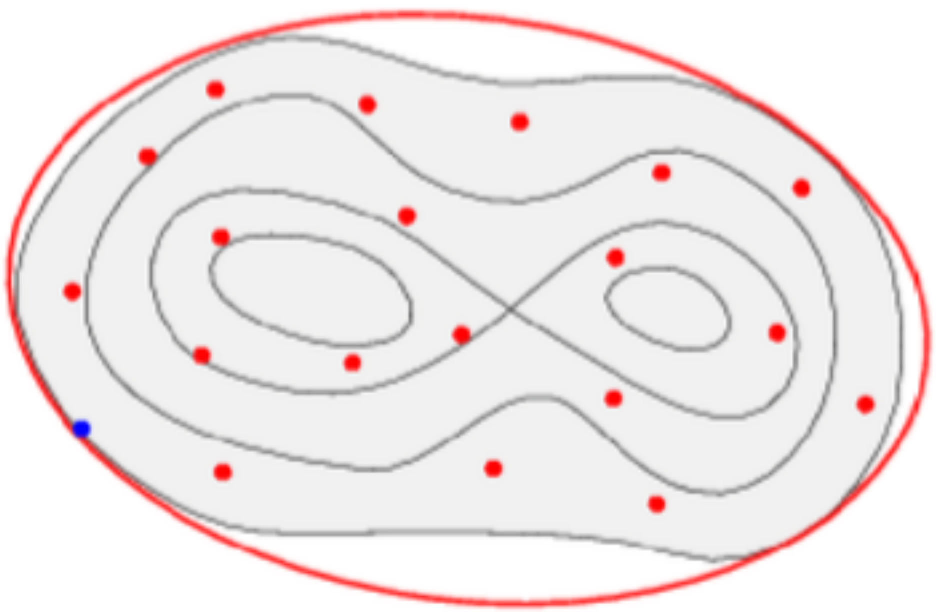
- The hardest part is to sample uniformly from the prior subject to the hard constraint that the likelihood needs to be above a certain level.
- Many specific implementations of this sampling step:
  - Single ellipsoidal sampling (Mukherjee+06)
  - Metropolis nested sampling (Sivia&Skilling06)
  - Clustered and simultaneous ellipsoidal sampling (Shaw+07)
  - Ellipsoidal sampling with k-means (Feroz&Hobson08)
  - Rejection sampling (MultiNest, Feroz&Hobson09)
  - Diffusion nested sampling (Brewer+09)
  - Artificial neural networks (Graff+12)
  - Galilean Sampling (Betancourt11; Feroz&Skilling13)
  - Simultaneous ellipsoidal sampling with X-means (DIAMONDS, Corsaro&deRidder14)
  - Slice Sampling Nested Sampling (PolyChord, Handley+15)
  - … there will be others, no doubt.

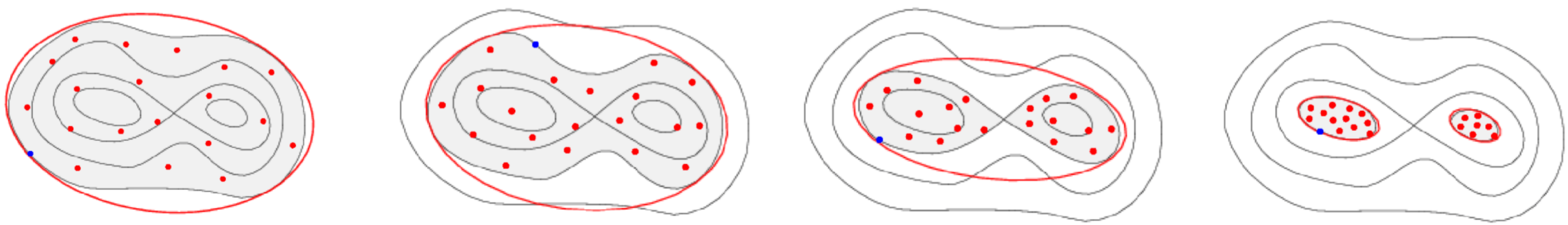
Sampling Step: Ellipsoid Fit

- Simple MCMC (e.g. Metropolis-Hastings) works but can be inefficient
- Mukherjee+06: Take advantage of the existing live points. Fit an ellipsoid to the live point, enlarge it sufficiently (to account for non-ellipsoidal shape), then sample from it using an exact method:



- This works, but is problematic/inefficient for multi-modal likelihoods and/or strong, non-linear degeneracies between parameters.

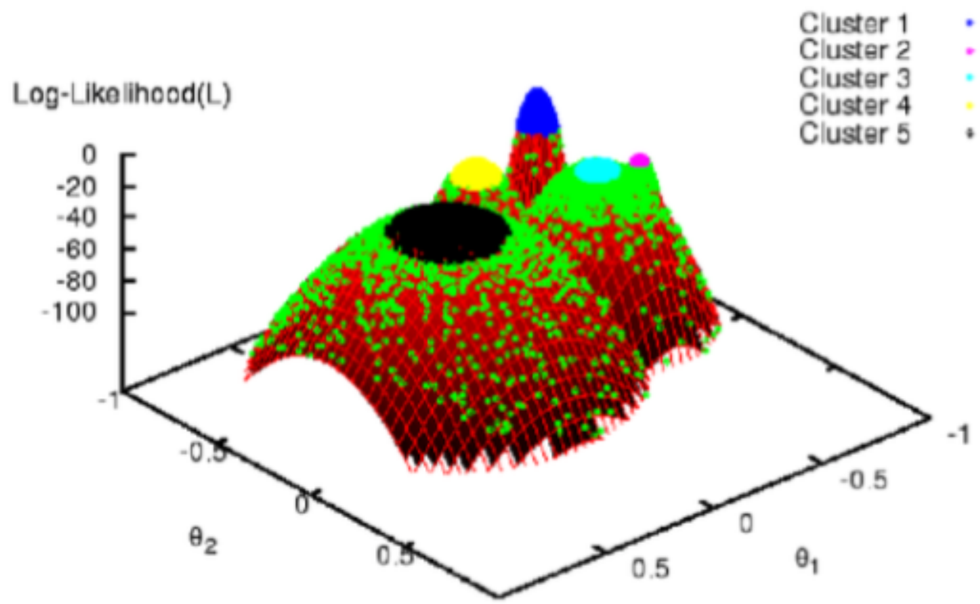
Sampling Step: Multimodal Sampling



- Feroz&Hobson08; Feroz+08: At each nested sampling iteration
  - Partition active points into clusters
  - Construct ellipsoidal bounds to each cluster
  - Determine ellipsoid overlap
  - Remove point with lowest L_i from active points; increment evidence.
  - Pick ellipsoid randomly and sample new point with $L > L_i$ accounting for overlaps
- Each isolated cluster gives local evidence
- Global evidence is the sum of the local evidences

Test: Gaussian Mixture Model

(Slide courtesy of Mike Hobson)



- Likelihood = five 2-D Gaussians of varying widths and amplitudes; prior = uniform
- Analytic evidence integral $\log E = -5.27$
- Multimodal ellipsoidal nested sampling: $\log E = -5.33 \pm 0.11$, $N_{\rm like} \approx 10^4$
- Metropolis nested sampling: $\log E = -5.22 \pm 0.11$, $N_{\rm like} \approx 10^5$
- Thermodynamic integration (+ error): $\log E = -5.24 \pm 0.12$, $N_{\rm like} \approx 4 \times 10^6$

Test: Egg-Box Likelihood

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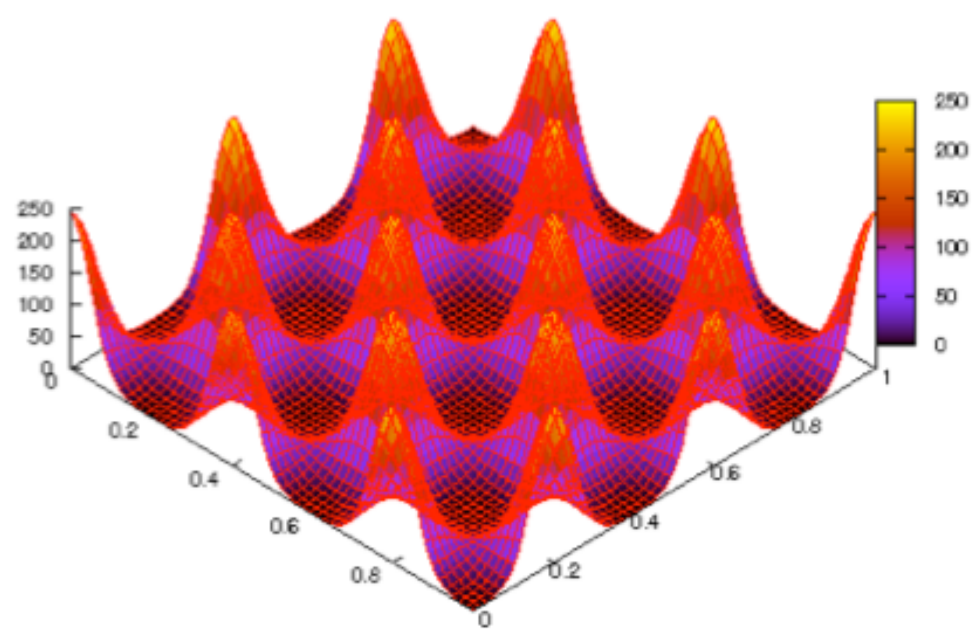
(Animation: Farhan Feroz)

- A more challenging example is the egg-box likelihood:

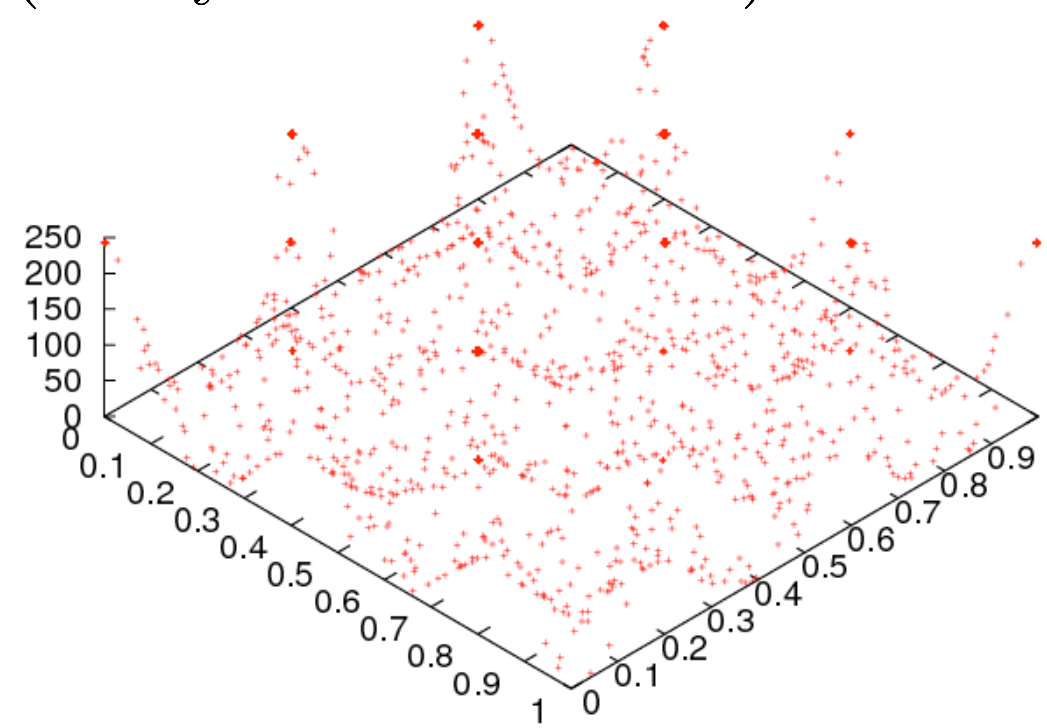
$$\mathcal{L}(\theta_1, \theta_2) = \exp\left(2 + \cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)\right)^5$$

- Prior: $\theta_i \sim U(0, 10\pi) \quad (i = 1, 2)$

$$\log P(d) = 235.86 \pm 0.06 \quad (\text{analytical} = 235.88)$$



Likelihood



Sampling (30k likelihood evaluations)

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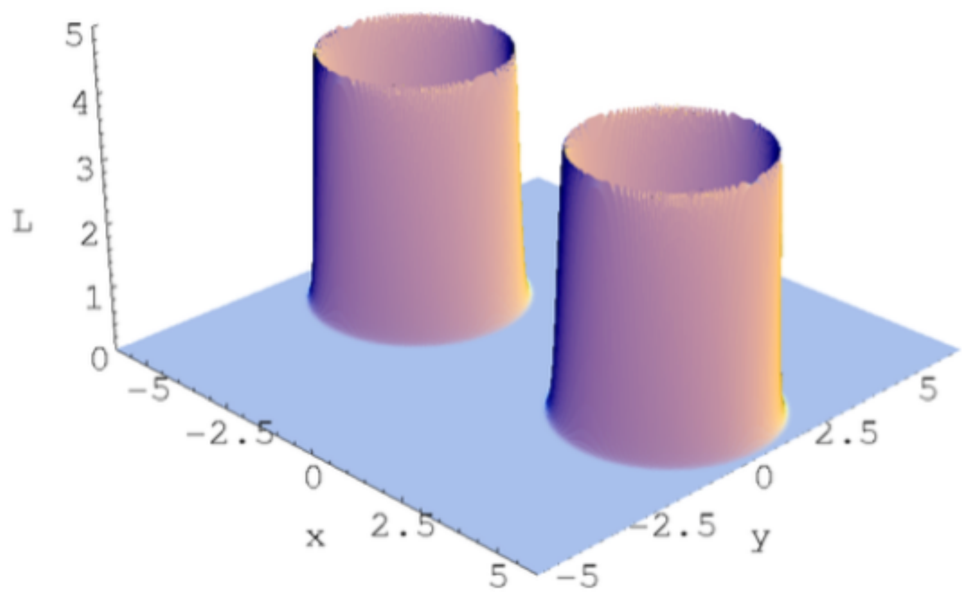
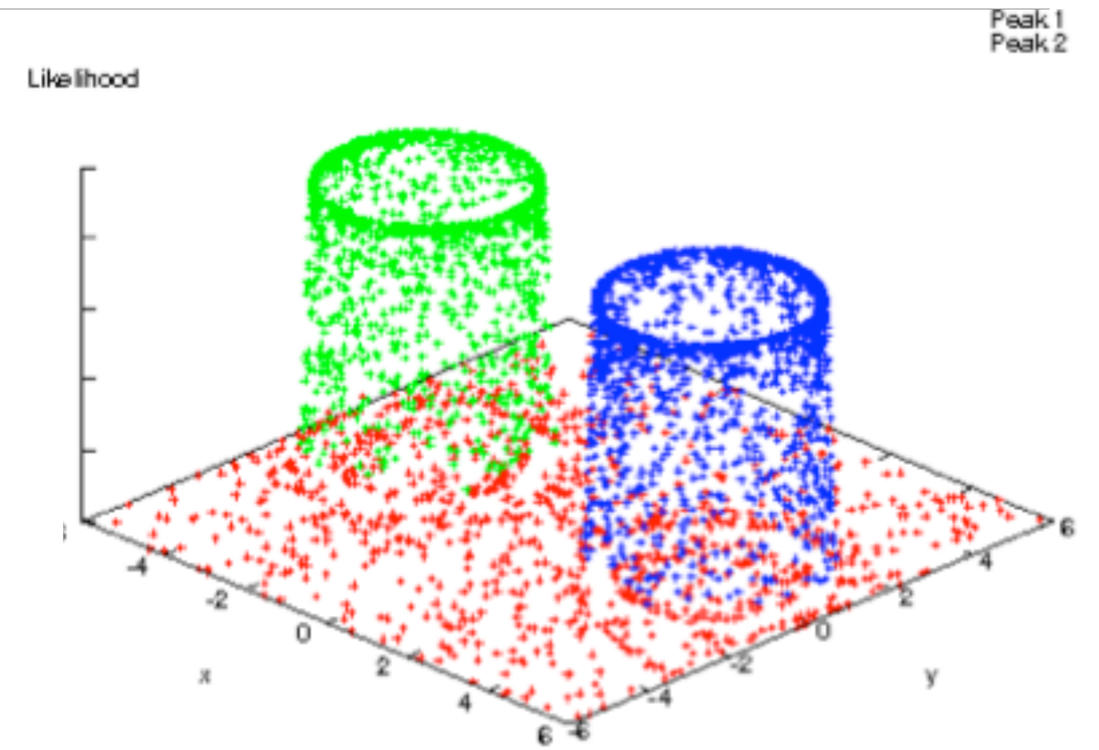
Test: Multiple Gaussian Shells

Courtesy Mike Hobson



Likelihood



Sampling

D	N_{like}	Efficiency
2	7000	70%
5	18000	51%
10	53000	34%
20	255000	15%
30	753000	8%

Parallelisation and Efficiency

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- Sampling efficiency is less than unity since ellipsoidal approximation to the iso-likelihood contour is imperfect and ellipsoids may overlap
- **Parallel solution:**
  - At each attempt to draw a replacement point, drawn N_{CPU} candidates, with optimal number of CPUs given by $1/N_{CPU}$ = efficiency
- **Limitations:**
  - Performance improvement plateaus for $N_{CPU} >> 1$/efficiency
  - For D>>30, small error in the ellipsoidal decomposition entails large drop in efficiency as most of the volume is near the surface
  - MultiNest thus (fundamentally) limited to D <= 30 dimensions

Neural Network Acceleration

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Graff+12 (BAMBI) and Graff+14 (SkyNet); Johannesson,RT+16

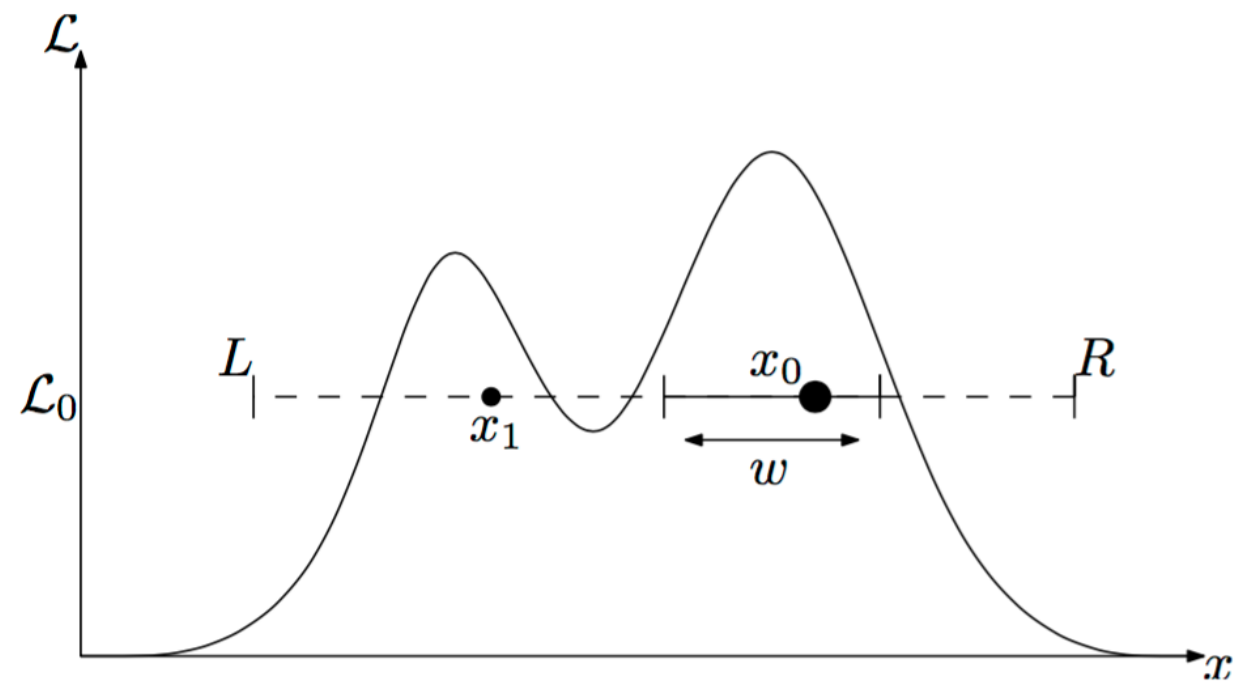
- A relatively straightforward idea: Use MultiNest discarded samples to train on-line a multi-layer Neural Network (NN) to learn the likelihood function.
- Periodically test the accuracy of predictions: when the NN is ready, replace (possibly expensive) likelihood calls with (fast) NN prediction.
- **SkyNet:** a feed-forward NN with N hidden layers, each with M_n nodes.
- **BAMBI** (Blind Accelerated Multimodal Bayesian Inference): SkyNet integration with MultiNest
- In cosmological applications, BAMBI typically accelerates the model likelihood computation by ~30% — useful, but not a game-changer.
- Further usage of the resulted trained network (e.g. with different priors) delivers speed increases of a factor 4 to 50 (limited by error prediction calculation time).

PolyChord: Nested Sampling in high-D

Handley et al, Mon.Not.Roy.Astron.Soc. 450 (2015)1, L61-L65

- A new sampling step scheme is required to beat the limitations of the ellipsoidal decomposition at the heart of MultiNest

- **Slice Sampling (Neal00) in 1D:**
  - Slice: All points with $L(x)>L_0$
  - From starting point x_0, set initial bounds L/R by expanding from a parameter w
  - Draw x_1 randomly from within L/R
  - If x_1 not in the slice, contract bound down to x_1 and re-sample x_1



High-D Slice Sampling

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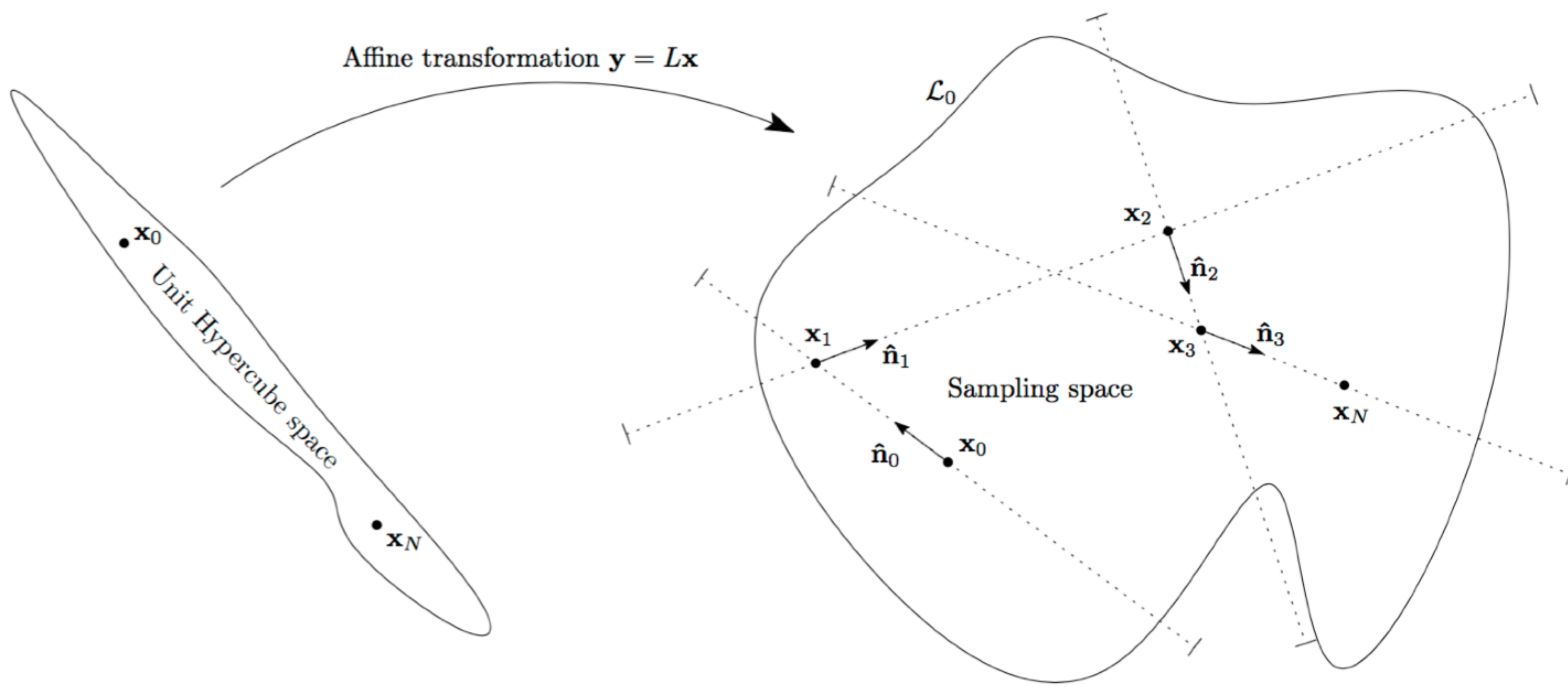
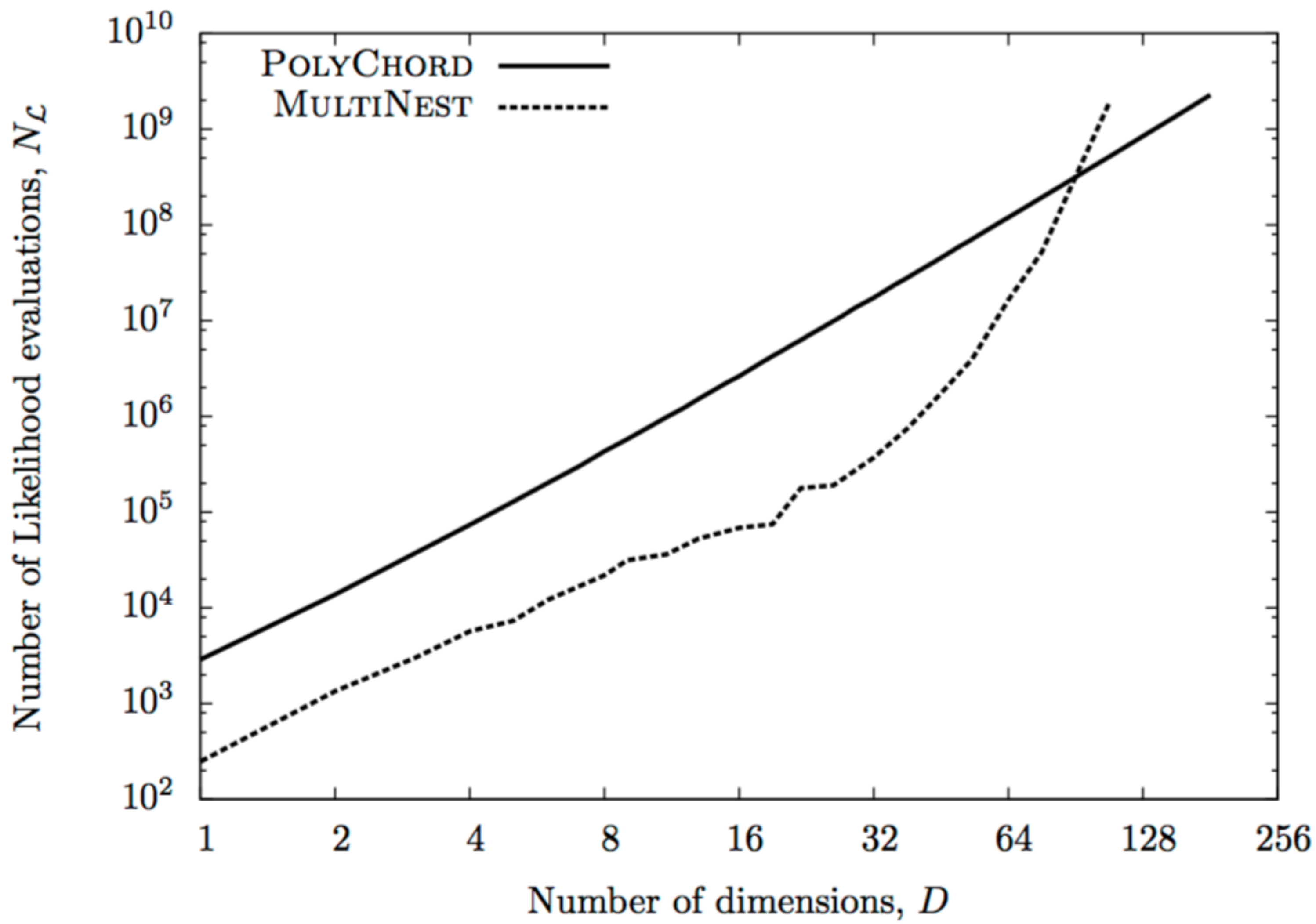
- A degenerate contour is transformed into a contour with dimensions of order O(1) in all directions (“whitening”)
- Linear skew transform defined by the inverse of the Cholesky decomposition of the live points’ covariance matrix
- Direction selected at random, then slice sampling in 1D performed (w=1)
- Repeat N times, with N of order O(D), generating a new point x_N decorrelated from x_0



Handley+15

PolyChord: Performance

- PolyChord number of likelihood evaluations scales at worst as $O(D^3)$ as opposed to exponential for MultiNest in high-D



Information Criteria

- Several "information criteria" exist for approximate (and quick) model comparison
- Define:
  k = number of fitted parameters
  N = number of data points
  -2 ln(L_{max}) = best-fit chi-squared
  D_{KL} = Kullback-Leibler divergence

- **Akaike Information Criterium (AIC):** $\mathrm{AIC} \equiv -2 \ln \mathcal{L}_{\mathrm{max}} + 2k$

- **Bayesian Information Criterium (BIC):** $\mathrm{BIC} \equiv -2 \ln \mathcal{L}_{\mathrm{max}} + k \ln N$

- **Deviance Information Criterium (DIC):** $\mathrm{DIC} \equiv -2\widehat{D_{\mathrm{KL}}} + 2\mathcal{C}_b.$

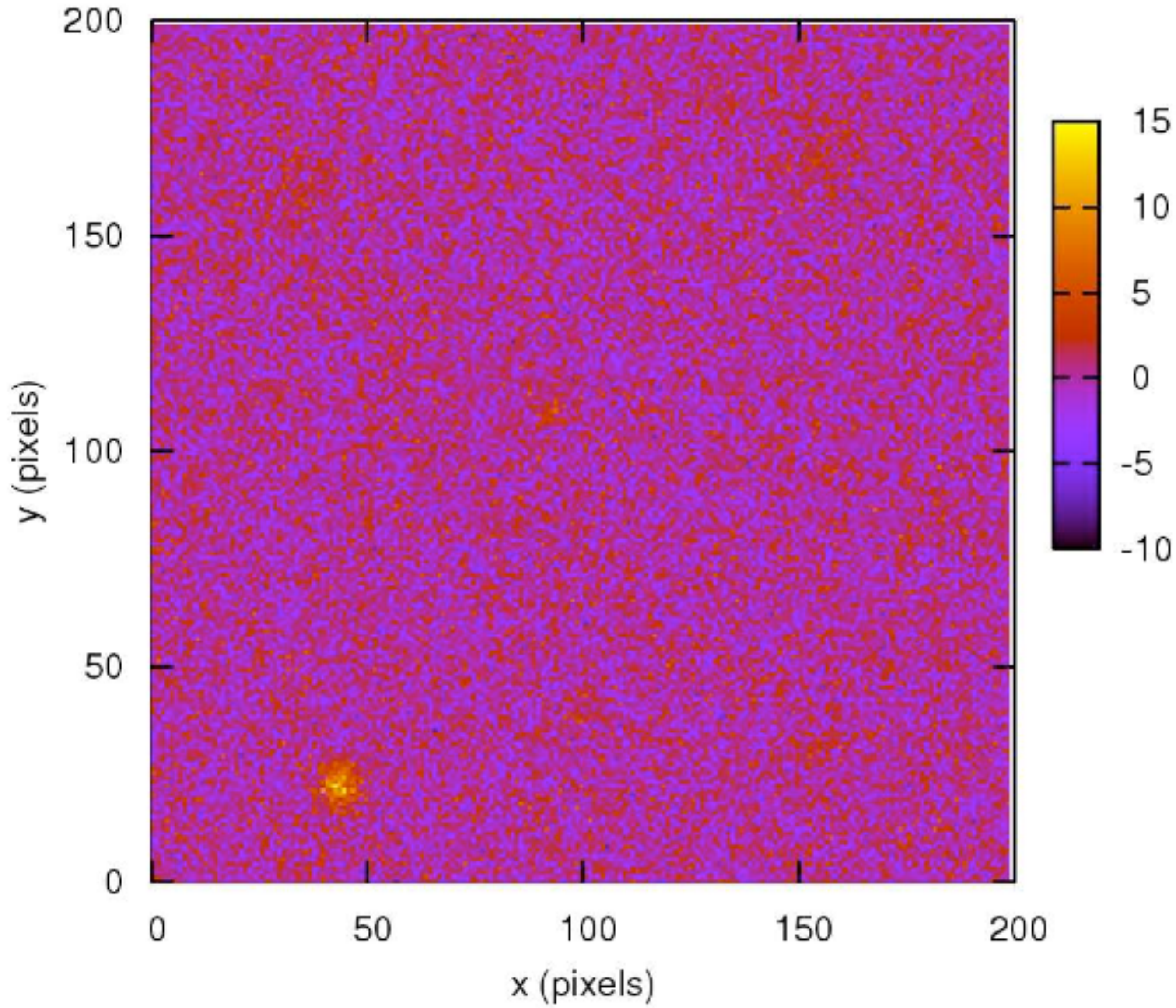
Notes on Information Criteria

- The “best” model is the one which minimizes the AIC/BIC/DIC
- **Warning:** AIC and BIC penalize models differently as a function of the number of data points, N:
  For N>7, BIC has a more strong penalty for models with a larger number of free parameters k.
- BIC is an approximation to the full Bayesian evidence with a default Gaussian prior equivalent to 1/N-th of the data in the large N limit.
- DIC takes into account whether parameters are measured or not (via the Bayesian complexity, see e.g M. Kunz, R. Trotta and D.R. Parkinson (2006), Phys. Rev. D 74, 023503 (2006), e-print archive: astro-ph/0602378).
- Whenever possible, computation of the Bayesian evidence is preferable (with explicit prior specification).

How Many Sources?

Feroz and Hobson (2007)

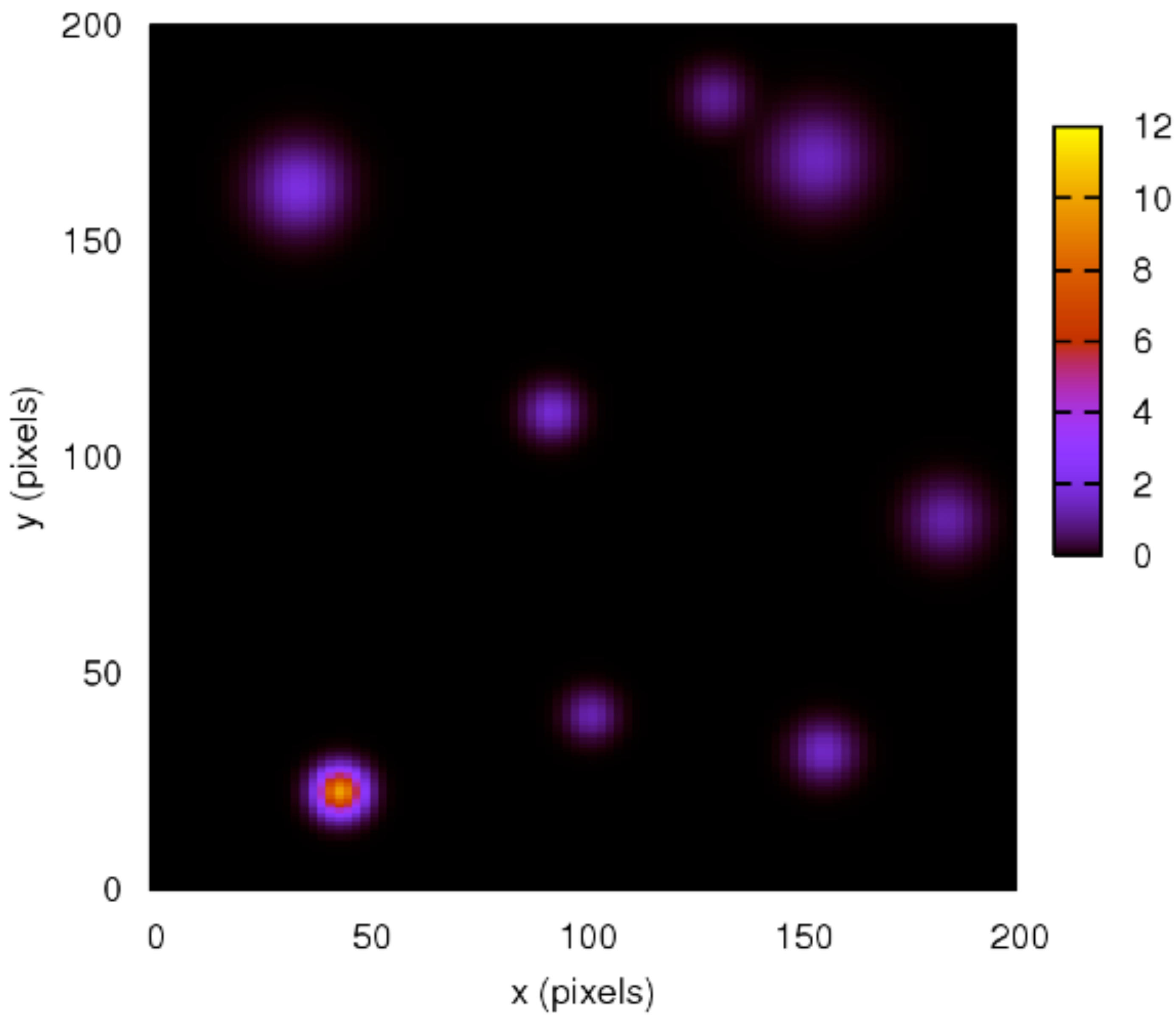
Signal + Noise



How Many Sources?

Feroz and Hobson (2007)

Signal: 8 sources



How Many Sources?

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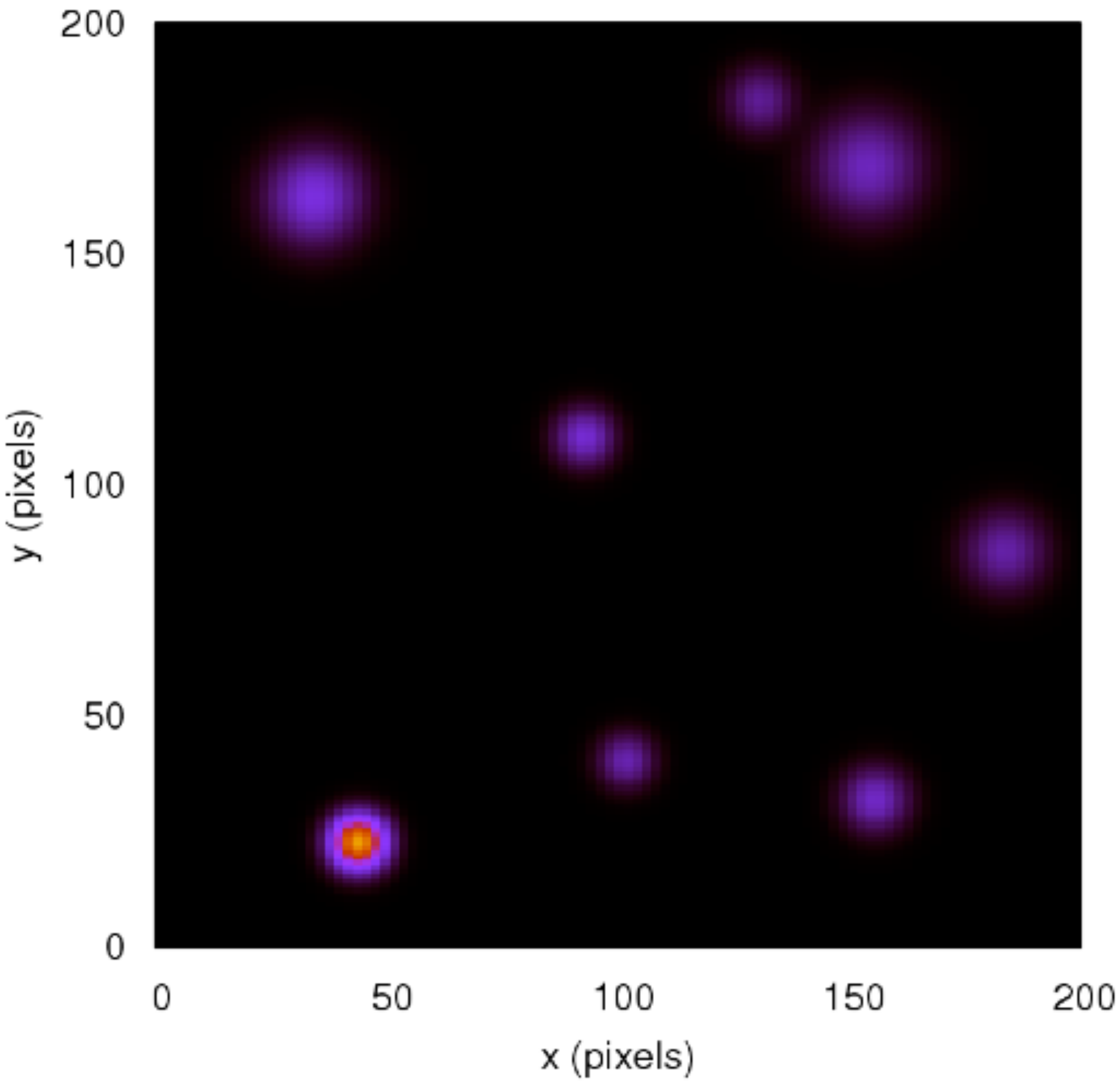
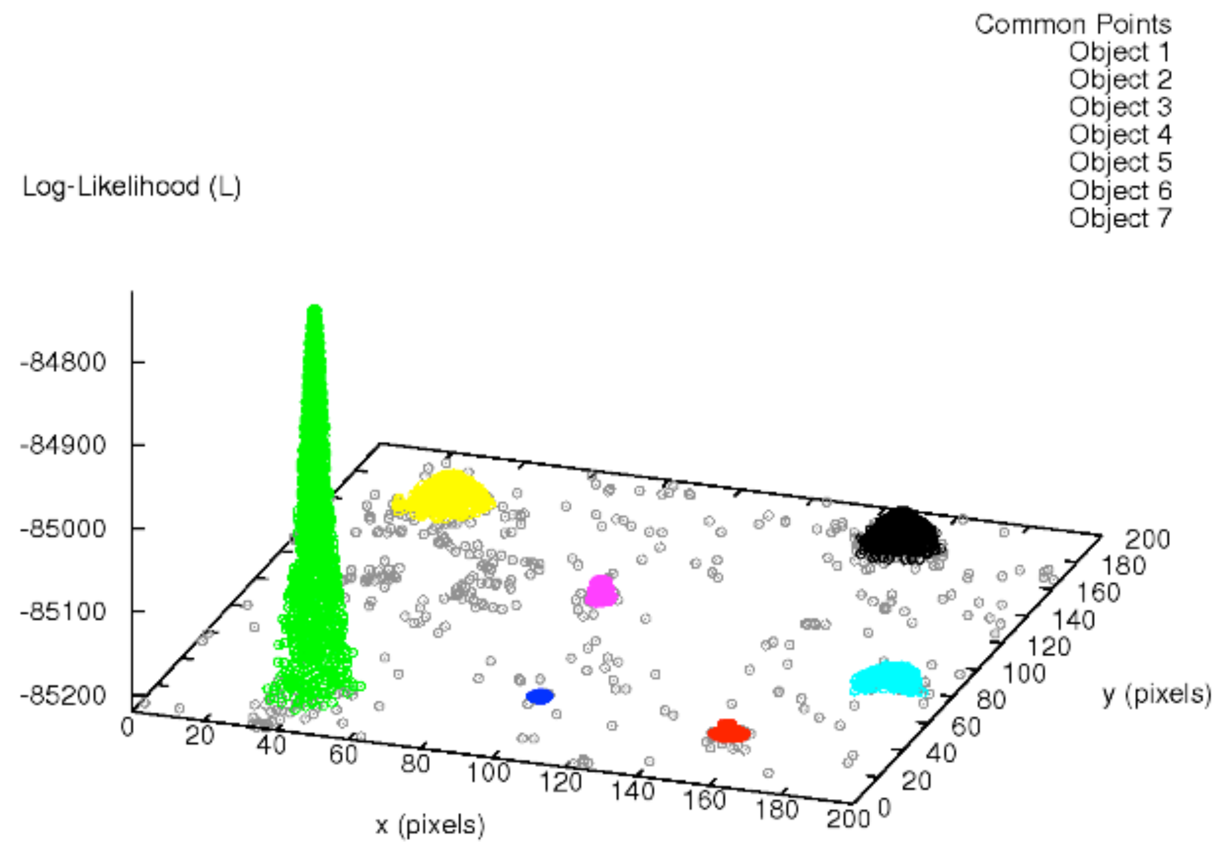
Feroz and Hobson (2007)



Bayesian reconstruction

7 out of 8 objects correctly identified.
Mistake happens because 2 objects very close.



The Cosmological Concordance Model

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Competing model	$\Delta N_{\rm par}$	$\ln B$	Ref	Data	Outcome
Initial conditions					
Isocurvature modes					
CDM isocurvature	+1	-7.6	[58]	WMAP3+, LSS	Strong evidence for adiabaticity
+ arbitrary correlations	+4	-1.0	[46]	WMAP1+, LSS, SN Ia	Undecided
Neutrino entropy	+1	$[-2.5, -6.5]^p$	[60]	WMAP3+, LSS	Moderate to strong evidence for adiabaticity
+ arbitrary correlations	+4	-1.0	[46]	WMAP1+, LSS, SN Ia	Undecided
Neutrino velocity	+1	$[-2.5, -6.5]^p$	[60]	WMAP3+, LSS	Moderate to strong evidence for adiabaticity
+ arbitrary correlations	+4	-1.0	[46]	WMAP1+, LSS, SN Ia	Undecided
Primordial power spectrum					
No tilt ($n_s = 1$)	-1	$+0.4$	[47]	WMAP1+, LSS	Undecided
		$[-1.1, -0.6]^p$	[51]	WMAP1+, LSS	Undecided
		-0.7	[58]	WMAP1+, LSS	Undecided
		-0.9	[70]	WMAP1+	Undecided
		$[-0.7, -1.7]^{p,d}$	[186]	WMAP3+	$n_s = 1$ weakly disfavoured
		-2.0	[185]	WMAP3+, LSS	$n_s = 1$ weakly disfavoured
		-2.6	[70]	WMAP3+	$n_s = 1$ moderately disfavoured
		-2.9	[58]	WMAP3+, LSS	$n_s = 1$ moderately disfavoured
		$< -3.9^c$	[65]	WMAP3+, LSS	Moderate evidence at best against $n_s \neq 1$
Running	+1	$[-0.6, 1.0]^{p,d}$	[186]	WMAP3+, LSS	No evidence for running
		$< 0.2^c$	[166]	WMAP3+, LSS	Running not required
Running of running	+2	$< 0.4^c$	[166]	WMAP3+, LSS	Not required
Large scales cut–off	+2	$[1.3, 2.2]^{p,d}$	[186]	WMAP3+, LSS	Weak support for a cut–off
Matter–energy content					
Non–flat Universe	+1	-3.8	[70]	WMAP3+, HST	Flat Universe moderately favoured
		-3.4	[58]	WMAP3+, LSS, HST	Flat Universe moderately favoured
Coupled neutrinos	+1	-0.7	[193]	WMAP3+, LSS	No evidence for non–SM neutrinos
Dark energy sector					
$w(z) = w_{\rm eff} \neq -1$	+1	$[-1.3, -2.7]^p$	[187]	SN Ia	Weak to moderate support for Λ
		-3.0	[50]	SN Ia	Moderate support for Λ
		-1.1	[51]	WMAP1+, LSS, SN Ia	Weak support for Λ
		$[-0.2, -1]^p$	[188]	SN Ia, BAO, WMAP3	Undecided
		$[-1.6, -2.3]^d$	[189]	SN Ia, GRB	Weak support for Λ
$w(z) = w_0 + w_1 z$	+2	$[-1.5, -3.4]^p$	[187]	SN Ia	Weak to moderate support for Λ
		-6.0	[50]	SN Ia	Strong support for Λ
		-1.8	[188]	SN Ia, BAO, WMAP3	Weak support for Λ
$w(z) = w_0 + w_a(1-a)$	+2	-1.1	[188]	SN Ia, BAO, WMAP3	Weak support for Λ
		$[-1.2, -2.6]^d$	[189]	SN Ia, GRB	Weak to moderate support for Λ
Reionization history					
No reionization ($\tau = 0$)	-1	-2.6	[70]	WMAP3+, HST	$\tau \neq 0$ moderately favoured
No reionization and no tilt	-2	-10.3	[70]	WMAP3+, HST	Strongly disfavoured

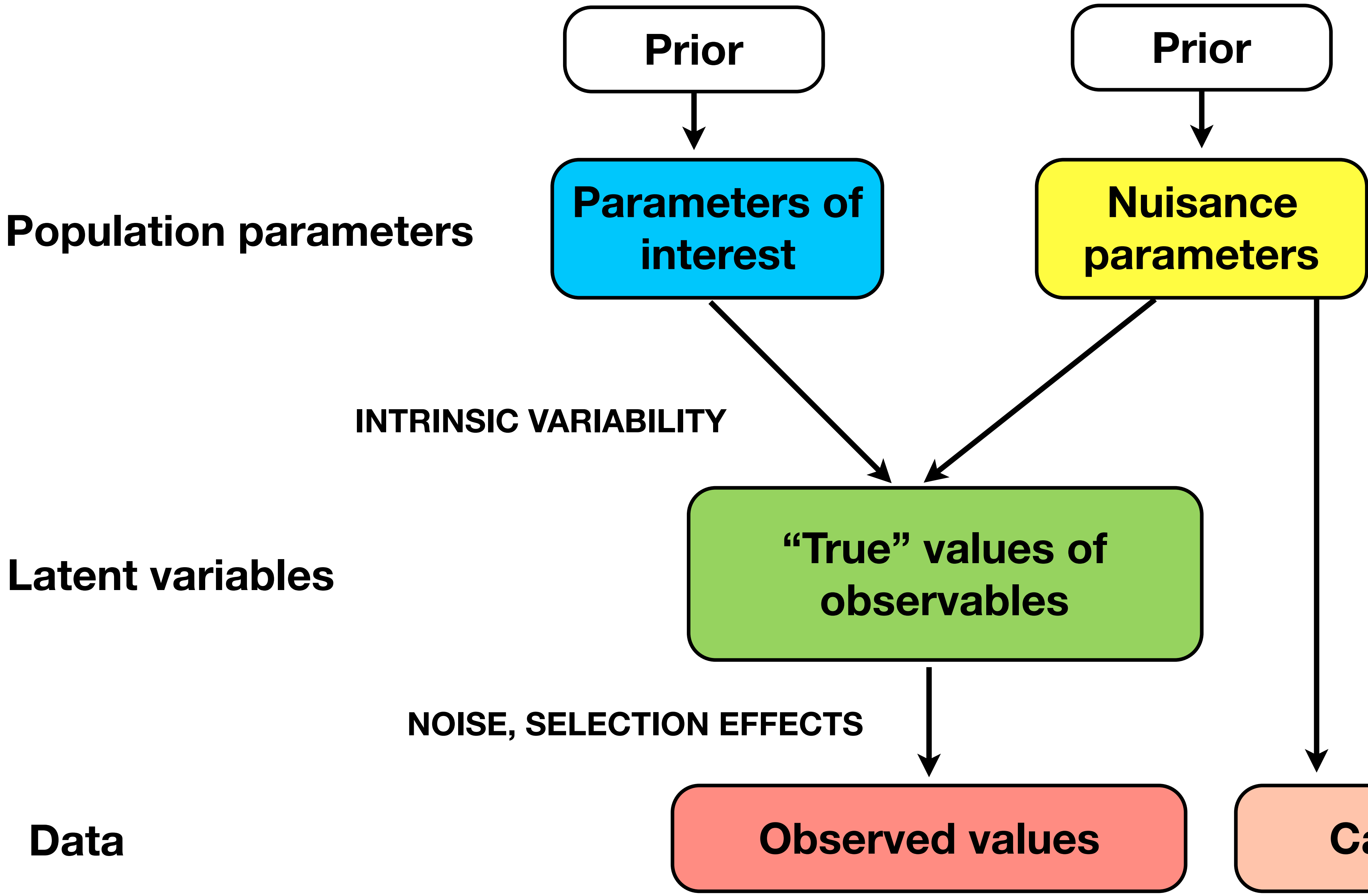
from Trotta (2008)

ICIC

lnB < 0: favours ΛCDM

Bayesian Hierarchical Models

Bayesian Hierarchical Models



Mathematical Formulation

The posterior distribution can be expanded in the usual Bayesian way:

$$\text{p(params | data)} \propto \text{p(data | params)p(params)}$$

$$\text{p(data|params)} \propto \int \text{p(data, true, pop | params) dtrue dpop}$$

$$= \int \boxed{\text{p(data | true)}}\,\boxed{\text{p(true | pop)}}\,\boxed{\text{p(pop)}}\ \text{dtrue drop}$$

Measurement errors

Intrinsic variability

Population-level priors

The Gaussian Linear Model

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The “simple” problem of **linear regression** in the presence of measurement errors on both the dependent and independent variable and intrinsic scatter in the relationship (e.g., Gull 1989, Gelman et al 2004, Kelly 2007):

$$y_i = b + a x_i$$

Model: unknown parameters of interest (a,b)

$$x_i \sim p(x|\Psi) = \mathcal{N}_{x_i}(x_\star, R_x)$$

POPULATION DISTRIBUTION

$$y_i|x_i \sim \mathcal{N}_{y_i}(b + a x_i, \sigma^2)$$

INTRINSIC VARIABILITY

$$\hat{x}_i, \hat{y}_i | x_i, y_i \sim \mathcal{N}_{\hat{x}_i, \hat{y}_i}([x_i, y_i], \Sigma^2)$$

MEASUREMENT ERROR

Bayesian Hierarchical vs MLE

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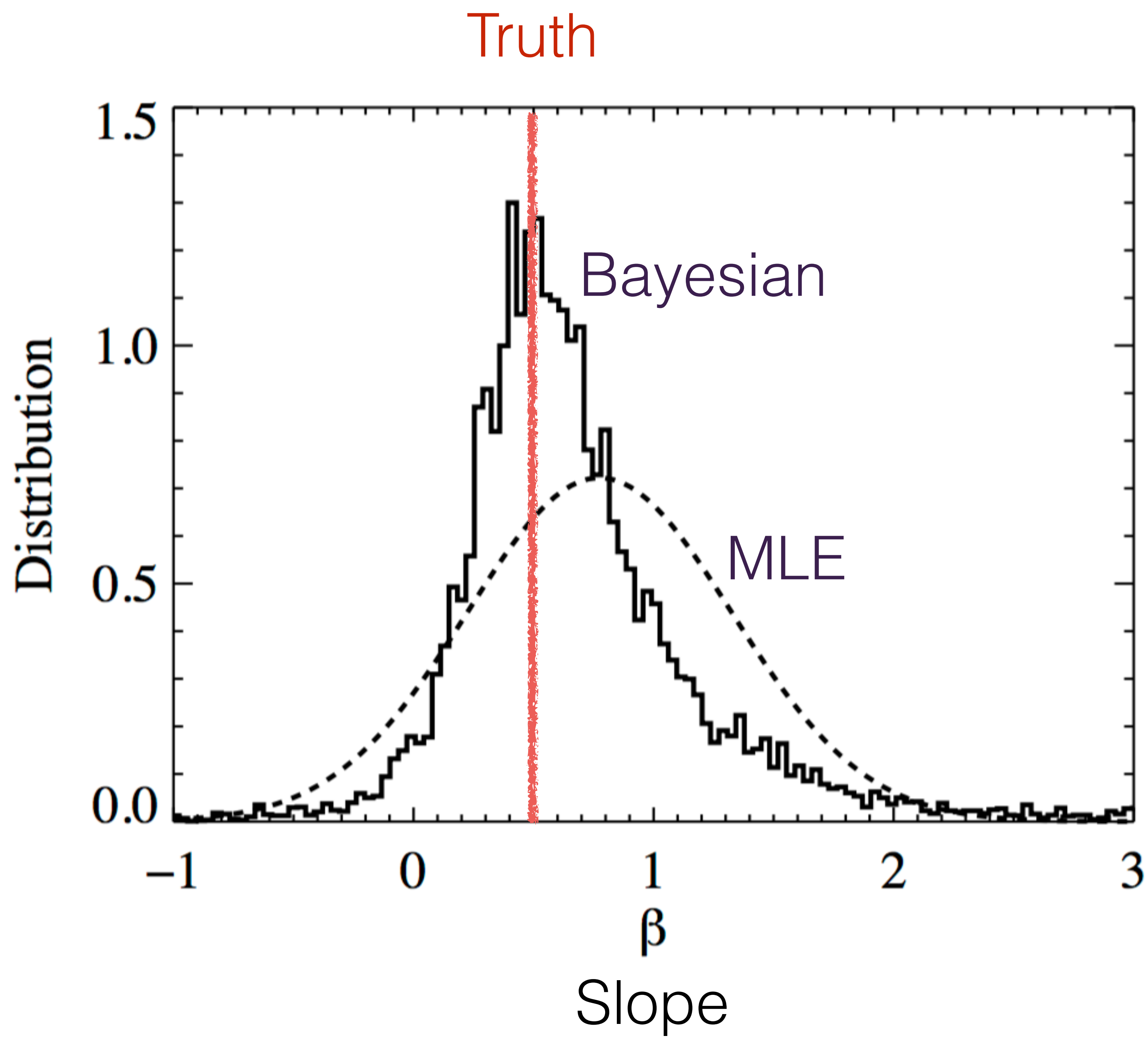
$R_x = \sigma_x^2/Var(x) = 1$ in this example: Comparing the MLE (dashed) with the Bayesian Hierarchical Model Posterior (histogram)



Standard MLE (or Least Squares/Chi-Squared) fits are biased

(even if you artificially inflate the errors to get Chi-Squared/dof ~ 1)

Kelly, Astr. J., 665, 1489-1506 (2007)

Borrowing of Strength

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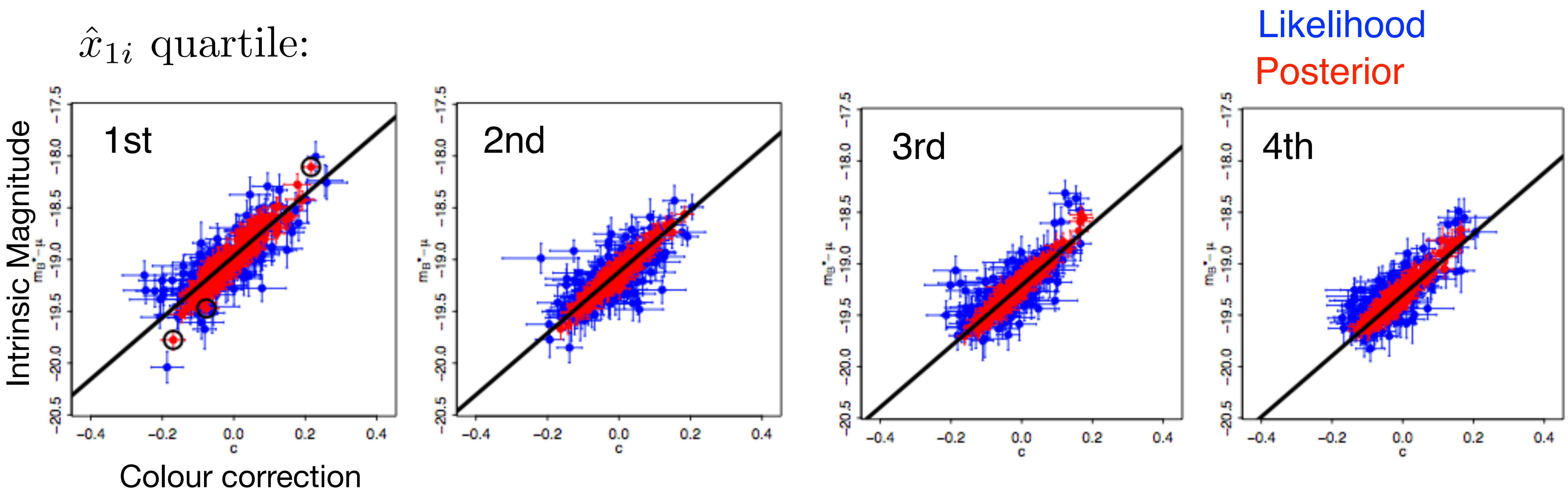
$\hat{x}_{1i}$ quartile:



SNIa cosmology application:

Posterior estimates (red) exhibit smaller residual scatter when compared to the likelihood (blue), around the regression line: "borrowing of strength" (shrinkage) from the structure of the hierarchical model.

Shariff, RT+15, arxiv: 1510.05954

Cosmic Shear Reconstruction

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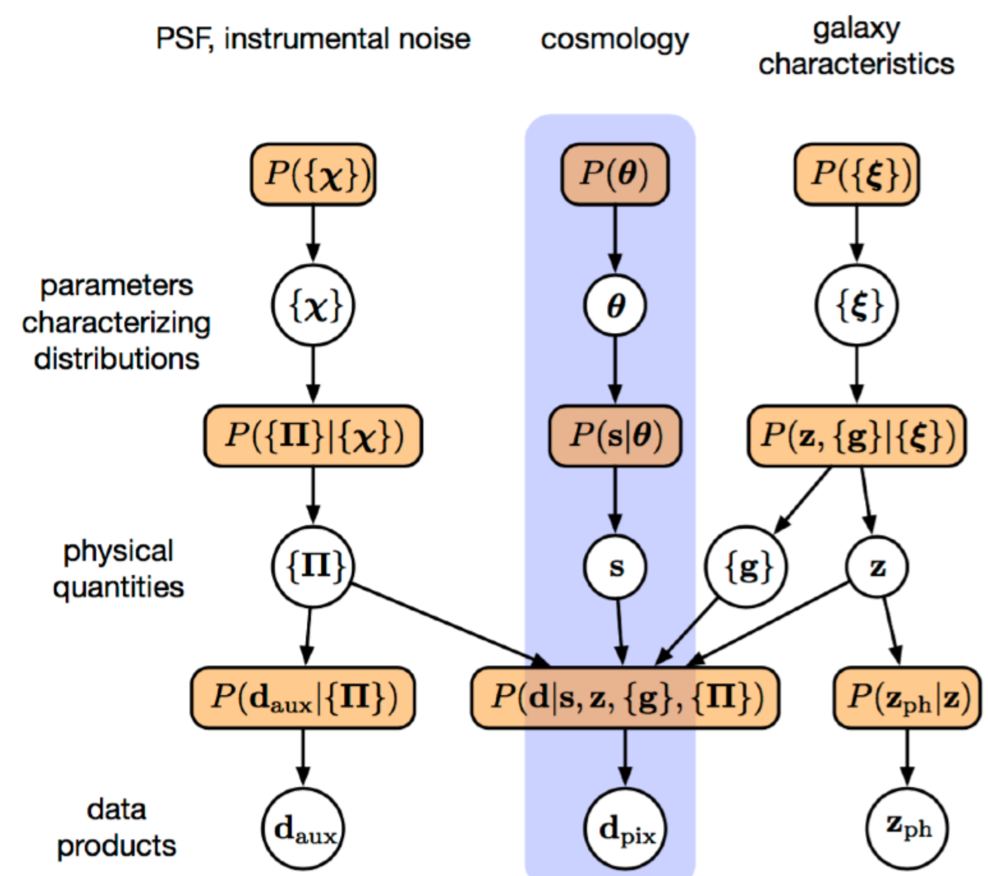
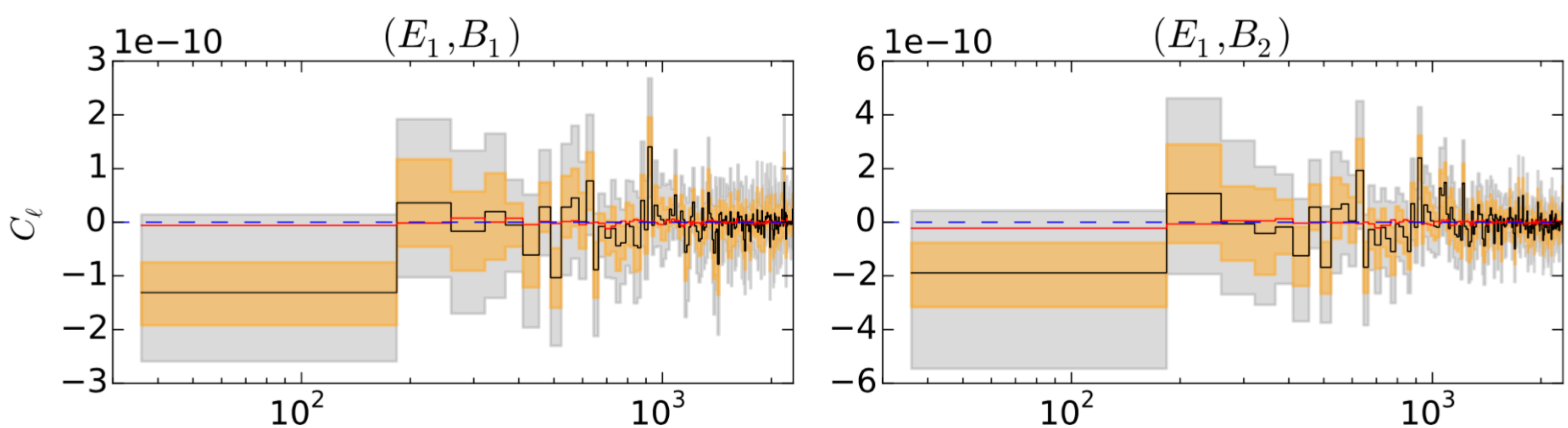
- Alsing, Heavens+(16) used a hierarchical Bayesian model to perform simultaneous reconstruction of the tomographic cosmic shear maps and its power spectra

- Includes realistic (and complex) measurements effects like masking, PSF, pixelization, photo-z, etc.

- Uses Gibbs+Messenger field as a sampling scheme for 67,000 params





Key Points

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- Bayesian model comparison extends parameter inference to the space of models
- The Bayesian evidence (model likelihood) represents the change in the degree of belief in the model after we have seen the data
- Models are rewarded for their predictivity (automatic Occam's razor)
- Prior specification is for model comparison a key ingredient of the model building step. If the prior cannot be meaningfully set, then the physics in the model is probably not good enough.
- Bayesian model complexity can help (together with the Bayesian evidence) in assessing model performance.