Linear integer programming

The problem we're going to tackle is the graph partitioning problem. Given a weighted graph, separate the vertices into **K** subsets of vertices such that the sum of the edge weights in each subset is minimal. The aim of this tutorial is to propose a integer linear program for the graph partitioning problem. In the rest of the subject the graph by the tuple **G = (V, E)**.

Partition representation

To identify a partition, you can use one of the vertices it comprises. To do this, you can use a binary variable $\mathbf{x_i}$ for each vertex **i**, which indicates whether **i** represents a partition or not.

Objective function

Here we'll use the cut-based objective function described in the meta-heuristics topic. Show how to represent edges that are in that partition, using indexed variables on two vertices that indicate whether the two vertices are in the same part. What is the objective function?

Necessary constraints

The first classic constraint to define is the so-called triangular inequality constraint. This family of constraints ensures that if vertex **i** is in the same partition as two **j** and **k** then **j** and **k** must also be in the same partition.

We'll also need constraints that fix the representative of a part. If we assume that the vertices are numbered (this is a classic representation), we'll make sure that the vertex with the lowest index in a part will represent that part. To do this, propose a constraint that, for a vertex **i** prevents the variable $\mathbf{x_i}$ from taking the value 1 if a vertex with a smaller index is in the same partition.

We also need to propose constraints that prevent several variables from being chosen to represent the same part.

We also need a constraint on the number of parts. For the moment, we want the model to find **K** partitions. Give a constraint to ensure this.

Model reinforcement

It is possible to use stronger constraints than those defined above. For example, the choice of a partition representative can be made using the following constraints:

$$x_j + \sum_{i=1,j-1} x_i x_{ij} = 1 \ (j \in V)$$

This constraint indicates that a vertex **j** is either a representative ($\mathbf{x_j}$ = 1) or lies in the same part as exactly one vertex - smaller than it - that is a representative ($\mathbf{x_i x_{ij}}$ = 1). These constraints are not linear, but quadratic. Propose a linearization of these constraints and a model using them. Compare the results of the two models, in particular the value of the linear relaxation at the root of the search tree.

Other constraints on partition size or number of partitions

Propose constraints that modify the size or number of partitions in such a way that (we won't use more than one of these constraints or the one on the number of partitions at a time):

1. A partition cannot have more than **N** vertices.
2. The difference in the number of vertices between the largest and smallest partition cannot be greater than a constant **R**.

Assignment

You will be asked to submit a short report outlining the models you are proposing. Briefly describe the meaning of variables and constraints. You will also provide the AMPL code to define your models and resolve instances you define.