Contents

Part 1

Prerequisites

1 Vectors

In this chapter we'll give an introduction to vectors, and some results in geometry that you may find useful. We'll first introduce them geometrically, then give a rigorous algebraic treatment of them. Vectors are very useful quantities, that will aid us all the way to Quantum Mechanics, so being familiar with them will aid you a lot.

1.1 Geometrical Vectors

We will somewhat loosely follow the vectors chapter from `https://www.damtp.cam.ac.uk/user/sjc1/teaching/VandM/notes.pdf`.

We first describe vectors in a geometrical fashion. This isn't in anyway rigorous, and we won't pretend it is. We'll give some proofs, but you may need to take a leap of faith in some places.

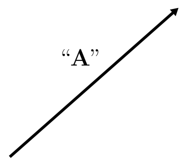


Figure 1.1: A directed line segment, the vector **A**.

Definition 1.1

A *Vector* is a quantity described by a magnitude and a direction in space. They're represented as line segments, as in fig. 1.1.

You can think of a vector as a *directed line segment*. Vectors are usually denoted using boldface, **A**.

Because they're directed line segments, only their length matters and we're not concerned with the location of their end points. This means that a vector can be freely translated, without altering it. An equivalent way to say this is that two vectors are equal iff their magnitude is equal, and they have the same direction (i.e. they're parallel).

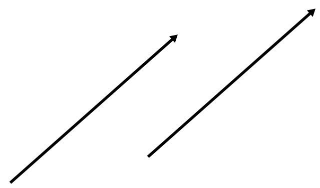


Figure 1.2: Two identical vectors.

Also, we define a vector field, which is a special sort of function, that'll be relevant for future discussions.

Definition 1.2

A *Vector Field* is a vector $\mathbf{B}$ which is a function of position, $\mathbf{B}(\mathbf{r}) = \mathbf{B}(x, y, z)$.

Where the position of a point P refers to the vector from the origin $\mathcal{O}$ to the point, P.

The magnitude of the vector is its length, which we call it's *norm*, and write as $\|\mathbf{A}\|$ or simply A. You may also see it being written as $|\mathbf{A}|$.

A unit vector, $\hat{\boldsymbol{\gamma}}$ ("gamma hat") is a vector whose magnitude is unity. We use the unit vector to often denote the direction of a vector by multiplying the unit vector with its magnitude. For instance, the unit vector that point is the direction of $\mathbf{A}$, $\hat{\mathbf{A}}$ can be calculated as,

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{A}.$$

Further, vectors have certain operations defined on them. The base operations are *Scalar Multiplication* and *Vector Addition*.

Scalar Multiplication

Scalar Multiplication refers to multiplying a vector by a *scalar*. A scalar for our purposes refers to a real number. Consider a vector $\mathbf{v}$. Multiplying it by a scalar, $\alpha \in \mathbb{R}$ produces a vector $\alpha\mathbf{v}$ parallel to the original vector and changes the magnitude of the vector so that $\|\alpha\mathbf{v}\|$ is $|\alpha|$ times greater than $\|\mathbf{v}\|$, thus $\|\alpha\mathbf{v}\| = |\alpha|\|\mathbf{v}\|$. $|\alpha|$ can be smaller or greater than 1 which accordingly increases/decreases the length of $\mathbf{v}$.

If $\alpha > 0$, the vector produced is in the same direction as the original vector. If $\alpha < 0$, the direction of the vector is reversed.

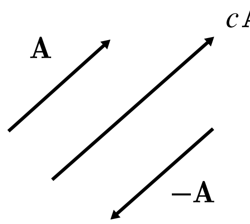
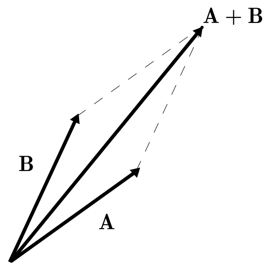


Figure 1.3: Scalar multiplication of a vector $\mathbf{A}$ by $c > 1$ and -1.

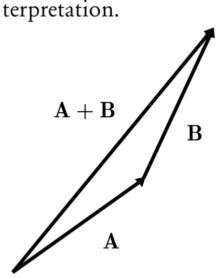
Vector Addition

Adding two vectors, $\mathbf{A}$ and $\mathbf{B}$ produces another vector $\mathbf{A} + \mathbf{B}$. Geometrically, vector addition is done by placing the tail of $\mathbf{B}$ on the head of $\mathbf{A}$, and then the vector joining the tail of $\mathbf{A}$ and head of $\mathbf{B}$ is the vector $\mathbf{A} + \mathbf{B}$ as in fig. 1.3a.

It can also be interpreted as the diagonal of the parallelogram made by placing the tails of the two vectors together, and producing two sides parallel to them as show in fig. 1.3b.



(a) Adding vectors, parallelogram interpretation.



(b) Adding vectors, triangle interpretation.

Figure 1.4: Addition of two vectors, $\mathbf{A}$ and $\mathbf{B}$ produces another vector, $\mathbf{A} + \mathbf{B}$.

From this we may interpret that vector addition is commutative, that is, $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$. We can also deduce that its associative, $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$. I'll recommend convincing yourself of this using geometrical constructions.

Subtraction of vectors, $\mathbf{A} - \mathbf{B}$ is equivalent to multiplying $\mathbf{B}$ by -1 and then adding it with $\mathbf{A}$.

If $\|\mathbf{v}\| = 0$, we write $\mathbf{v} = \mathbf{0}$.

Scalar Product

Part 2

Mechanics

Kinematics

2

Kinematics is the study of motion without any concern to its cause. It introduces us with the basic quantities that will used throughout physics. While there's not that much physical content in this topic, it stands as quite an important topic in mechanics. Once you've solved the dynamic equations, all that's left is kinematics.

2.1 The Physical Quantities

Frames and Particles

Motion of any object is considered to relative to an observer. The observer defines a particular co-ordinate system called the reference frame.

A *Reference Frame* is a frame with respect to which we measure the physical quantities. **Definition 2.1**

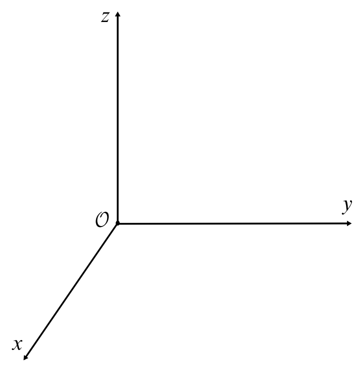


Figure 2.1: A reference frame with cartesian co-ordinates.

There is some distinction between the mathematics and physics to be made here. What we call a reference frame here is not purely a physical thing, it has co-ordinates. Some people prefer to distinguish between this and what we'll refer to as an *observational reference frame*, which is purely a physical concept which may be rotating, moving or be stationary.

An observer in this observational reference frame may choose to employ any set of co-ordinates for measurements, for instance cartesian or spherical co-ordinates. The physical quantities are not affected by changing our co-ordinate system in the same observational frame, but can change upon a change of the observational frame.

Remark 2.1.1. We have yet to discuss acceleration and velocity, and things like angular velocity, and energy won't be introduced for a few more chapters, so you might wish to return to this remark later.

In the spirit of the prior discussion, we can define an observational reference frame implicitly through its properties. We might say that any co-ordinate frame with the same *state of motion* is in the same observation frame of reference. By the state of motion we mean that it has the same velocity, acceleration, angular velocity, etc, and by extension if its inertial or non inertial. Any sort of quantity that we measure in this frame: velocity, acceleration, momentum, the rotational analogues, energy, etc.; does not depend on our choice of co-ordinates, as is obvious intuitively.

An example of what we mean by an observational frame is for instance one moving with some velocity with respect to the ground/lab frame. No matter which co-ordinates we use, the measurements of the quantities will remain invariant in this frame. However the velocity of objects will differ from what we had in the lab frame.

There are some things that depend on our choice of co-ordinates in our reference frame, and some that don't. We'll have a look at using reference frames to deal with some questions later.

Typically, we either use cartesian or curvilinear co-ordinates (spherical, cylindrical). We'll have a look at the 2d versions of both in this chapter, especially plane polar-coordinates (2d versions of cylindrical co-ordinates).

We usually deal with either point particles, or rigid bodies. You might think a point particle is not really a physically meaningful thing. However, we can approximate bodies, when their size is *not* meaningful to their motion, as particles to a very good precision.

A *Point Particle* is a particle whose size is negligible in the study of its motion. **Definition 2.2**

We'll later discuss extended bodies later, and in particular rigid ones in mechanics, but for now this idea of point particles will suffice for us. And anyway, even then we'll continue to use it as far as it proves to be useful.

Position and Displacement

To describe the position of the particle, we use a position vector.

The *position vector*, $\mathbf{r}(x, y, z)$, of the particle refers to the vector drawn from the origin to the particle. **Definition 2.3**

A position vector is an example of a *vector field*.

In Figure 2.2, the position vector of A is $\mathbf{r}_A = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $\mathbf{i}$ is the unit vector along the x axis, $\mathbf{j}$ the unit vector along the y axis and $\mathbf{k}$ is the one along the z axis.

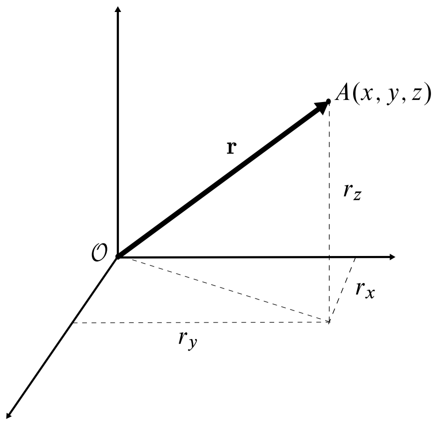


Figure 2.2: A position vector $\mathbf{r}_A = (x, y, z)$

As you may notice, this is dependent on our origin. The position vector is not quite a true vector, it's more so a relative quantity. Anyway, it is still plenty useful, if we just remember to adjust it when switching to a different reference frame.

The other thing of importance is *Displacement*. The *Displacement*, $\mathbf{s}_{AB}$ from A to B, is $\mathbf{s}_{AB} \equiv \mathbf{r}_B - \mathbf{r}_A$.

We need two points to define the displacement. Also, unlike position, displacement is not changed by using a rotated or translated reference frame.

Note that displacement is not the property of the path, which is the *distance*. For a path of the form $y(x)$, to calculate the distance between A and B, we need to calculate $d\ell = \sqrt{dx^2 + dy^2} = \sqrt{(dy/\,dx)^2 + 1}\,dx$ which is the *arc length*. $d\ell$ is just the infinitesmal length of the path, which by pythagoras is equal to $\sqrt{dy^2 + dx^2}$.

Doing this integral, we get,

$$\ell = \int_A^B \sqrt{(y'(x))^2 + 1}\,dx. \tag{2.1}$$

This is the distance. For euclidian geometry, it is also obvious that since displacement is the line joining A and B, the magnitude of displacement is lower than the length of any path joining A and B. Thus, $\|\mathbf{s}\| \leq \ell$.

Velocity And Acceleration

And now we can start by considering the quantities we have heard about frequently in our daily life, velocity and acceleration. Velocity refers to how the position vector changes with time, while acceleration refers to how velocity changes with time. So, acceleration is the second derivative of position.

You might wonder why we don't consider higher derivatives, and the answer to that lies in the fact that newton's laws give us a second order differential equation, so the only quantities of concern are till the second derivative of position.

Let's now formally define these quantities.

Velocity is defined as the time rate of change of position, i.e. $\mathbf{v} \equiv \dot{\mathbf{r}}$.

Acceleration is defined as the time rate of change of velocity, i.e. $\mathbf{a} \equiv \dot{\mathbf{v}} = \ddot{\mathbf{r}}$.

Definition 2.4

$\dot{x}$ refers to the derivative of x with respect to time. $\ddot{x}$ refers to the second derivative of x with respect to time. '$\equiv$' stands for defined as.

Remark 2.1.2. Some quick remarks. Position is a function of time, and to find said function is the aim of mechanics. We call the function the *trajectory* of the particle. There are some constraints on the function, for example it must have a continuous first, second derivative.

Similar, velocity and accelerations are also functions of time. Notably as you can see from their definitions, they're defined at a particular time, the derivative evaluated at some time, t. This is different from things like *average velocity*, which is a property of motion in a finite time interval, explicitly,

$$\mathbf{v}_{\text{avg}} = \bar{\mathbf{v}} \equiv \frac{\mathbf{r}(t_2) - \mathbf{r}(t_1)}{t_2 - t_1}. \tag{2.2}$$

And in the limit $t_2 \to t_1$, this becomes the velocity. The case for acceleration is the same.

Speed at a time t is the magnitude of velocity at that time. It is always greater than or equal to 0. In the case of average speed, it is defined as the *distance* the particle travels over the time interval in which it travels that distance.

When the time interval goes to 0, the distance becomes the displacement, or atleast the magnitude of it. Thus, speed is the magnitude of velocity at that point. Also, don't confuse between $\| d\mathbf{r}/\, dt \|$ and $d\|\mathbf{r}\|/\, dt$, the first is speed, the other is a quantity not at all useful.

The velocity is a map,

$$\mathbf{v} \colon \mathbb{R} \to \mathbb{R}^3$$

similar to position, which maps time to its three components,

$$t \mapsto (v_x(t), v_y(t), v_z(t)).$$

Based on the definition of acceleration,

$$\int d\mathbf{v} = \int \mathbf{a}\, dt \implies \mathbf{v} = \mathbf{v}_0 + \int \mathbf{a}\, dt. \tag{2.3}$$

Similarly,

$$\mathbf{r} = \mathbf{r}_0 + \int \mathbf{v}\, dt. \tag{2.4}$$

If acceleration is a function of position, which it often is, we can use a neat trick, at least when every thing is one dimension,

$$a = \frac{dv}{dt} = \frac{dv}{dt}\frac{dx}{dx} = \frac{dv}{dx}\frac{dx}{dt} \implies a = v\frac{dv}{dx}.$$

For constant acceleration, this gives us, $v^2 = v_0^2 + 2a\Delta x$, which is also the statement of work-energy theorem, which we'll encounter later.

You can generalise this to any function, f,

$$f''(x) = f'\frac{df'}{df}. \tag{2.5}$$

Example 2.1

Consider the motion of an object dropped from a height h, under drag. The acceleration due to gravity is $\mathbf{g} = -g\hat{\mathbf{j}}$ and due to drag is of the form $-\alpha v\hat{\mathbf{v}}$. This is called linear drag. Since the object is falling down, $\hat{\mathbf{v}} = -\hat{\mathbf{j}}$.

Based on the definition of acceleration,

$$\frac{dv}{dt} = -g + \alpha v \iff \int_0^{-v_f} \frac{dv}{\alpha v - g} = \int_0^t dt'$$

where we set the final velocity to $-v_f$ since its negative, and rename t to t' since we'll use it as our dummy variable, and t as the limit of integration. The final solution is,

$$v_f = \frac{g}{\alpha}(e^{\alpha t} - 1).$$

Example 2.2

A car is at distance d from a boy. It starts accelerating at constant acceleration a. What is the minimum velocity that the boy should have to catch up with the car?

Solution. Consider separation of boy and car, Δs. Integrating twice for constant acceleration, we have $s_b = vt$, $s_c = \frac{1}{2}at^2$. Thus,

$$\Delta s = d + \frac{1}{2}at^2 - vt$$

So when they meet, we have a quadratic in t,

$$at^2 - 2vt + 2d = 0$$

From an inspection of the co-efficients of the t, t^2 terms and the constant, we see that if a real solution to this exists, it must be positive (if this may not be apparent, recall Vieta's relation and note a, v, d are all positive).

So they must always meet if this has a real solution. Therefore, $b^2 - 4ac \geq 0$ for the equation $at^2 + bt + c$. Solving for this, we have:

$$v \geq \sqrt{2ad}$$

Alternatively, moving into the frame of the boy (we discuss the later, so you can skip it for now), $v_f^2 = v^2 - 2ad$. Since $v_f^2 \geq 0$, $v^2 \geq 2ad$. □

Example 2.3

A body is dropped at $t = 0$, after time $t = t_0$, another body is thrown downwards with velocity u. Assuming first body reaches ground first, plot graph of separation.

Solution. At instant t_0, displacement of first particle = $gt_0^2/2$. Note that here we set up co-ordinates such that positive y is downwards from point of drop. The displacement of first body at time t after t_0 but before *reaching ground* is

$$s_1 = \frac{1}{2}gt_0^2 + \frac{1}{2}g(t - t_0)^2$$

While for second body is,

$$s_2 = ut + \frac{1}{2}g(t - t_0)^2$$

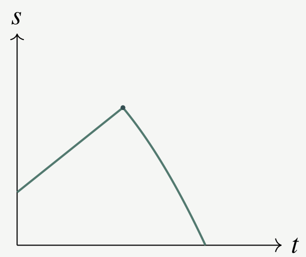
Thus,

$$s_{1,2} = \frac{1}{2}g(t_0)^2 + t(gt_0 - u)$$

However, after first body reaches ground,

$$s_{1,2} = ut + \frac{1}{2}g(t)^2$$

which is a parabola. The overall graph is:



□

2.2 Projectile Motion

Projectile Motion is a very famous setup in physics, and especially kinematic problems. Though the setup is not extremely physically interesting, there's a number of things you can learn from it. We'll cover a lot of cool tricks through problems, but let's first explore what this setup even is.

When an object, called a *projectile*, is thrown with some velocity v making some angle with the horizontal, θ, from co-ordinates (X, Y), under the effect of gravity, the motion is called projectile motion.

There are a no. of things in this setup that can teach you a fair bit. For one, this is our first actual two dimensional problem. We have two free co-ordinates, x and y. First of all, Let's choose our co-ordinate system so that $X = 0$. We won't choose $Y = 0$ because that corresponds to a real thing, namely the ground. It is more convenient to just leave it be.

First, let's discuss the special case $Y = 0$, that is, projectile is released from ground. This setup should be enough for you to be able to generalise it to $Y = h$. Anyway, gravity acts in the negative y direction, $\mathbf{g} = -g\mathbf{j}$, where $g = 9.81$ m/s. How do we proceed now? Well, we know $\mathbf{a}(t)$ so we should probably try to find $\mathbf{v}(t)$. Let's try that,

$$\dot{\mathbf{v}} = \mathbf{g} \implies \mathbf{v}(t) = \mathbf{v}_0 + \mathbf{g}t,$$

Where we use $\mathbf{v}(0) = \mathbf{v}_0$, the initial velocity as the limit of integration.

$$\dot{\mathbf{r}} = \mathbf{v}_0 + \mathbf{g}t \implies \mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2.$$

Since $\mathbf{r}_0 = (X, Y) = (0, 0) = \mathbf{0}$, we just have,

$$\mathbf{r}(t) = \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2.$$

Now here's where the neat part begins. As a general rule of thumb, it's always harder to deal with vectors than it is with scalars, so we will see various methods of trying to convert a vector problem into a scalar problem, and then re-converting if needed. In this case, we make use of a very important, but almost trivial looking identity,

$$\text{if } a\mathbf{i} + b\mathbf{j} = c\mathbf{i} + d\mathbf{j}, \text{ then } a = c, b = d.$$

How do we prove this? Well lets move things around a little,

$$(a - c)\mathbf{i} = (d - b)\mathbf{j}$$

Oh, but this cannot be true since $\mathbf{i}, \mathbf{j}$ are the basis for $\mathbb{R}^2$ so they're linearly independent. Thus, either $a = b$, or $c = d$. Now you can complete the proof by using the uniqueness of 0. A fancier way to do this is just dot both sides with $\mathbf{i}$ and then $\mathbf{j}$, I'll leave that proof to you.

Now, if $\mathbf{v}_0$ makes an angle θ with horizontal,

$$\mathbf{v}_0 = v_0 \cos\theta \mathbf{i} + v_0 \sin\theta \mathbf{j},$$

And plugging this into the vector equation for $\mathbf{v}(t)$, and using $\mathbf{v}(t) = v_x\mathbf{i} + v_y\mathbf{j}$ we get,

$$v_x = v_0 \cos\theta, \quad v_y = v_0 \sin\theta - gt \tag{2.6}$$

And using the vector equation for $\mathbf{r}(t) = x\mathbf{i} + y\mathbf{j}$,

$$x = v_0 t \cos\theta, \quad y = v_0 t \sin\theta - \frac{1}{2} g t^2. \tag{2.7}$$

We have reduced the vector equations to scalar equations! Now this is very easy to deal with, you can derive almost everything by just bashing using these equations. That's of course not always the nicest idea, and we'll learn some tricks to deal with them. Also, you might have notice that for each vector equation, we get two scalar equations. We actually get three, since we live in 3d space, but the components of the vector along $\mathbf{k}$ are just 0, so it doesn't matter.

The last fundamental equation of importance is the *trajectory* equation, how does the motion actually look in 2d space? That is, we need to find $y(x)$. You can do it by using $t = x/v_0 \cos\theta$, and substituting this in the $y(t)$ equation, which you should verify, gives you,

$$y(x) = x \tan\theta - \frac{x^2 g}{2 v_0^2 \cos^2\theta}. \tag{2.8}$$

Now we can use this for a bunch of problems. Now that we know the general shape is a parabola, let's try to find some properties of it. As you can see, the parabola opens downwards, so it achieves a maximum. What is that maximum? Well, you could try to find where dy/dx is 0, and then see what happens.

A nicer way is to realise that v_y must be 0 at max height. If this were not so, we could have moved to $H_{\text{max}} + \epsilon$ in time $t = \epsilon/v_y$. If the velocity were negative at, and thus just before H_{max}, we could have never reached the max height.

Using this, we get that,

$$H_{\text{max}} = \frac{v_{0y}^2}{2g} = \boxed{\frac{v_0^2 \sin^2\theta}{2g}.} \tag{2.9}$$

What about the range(the max value of x)? We have a maximum, since after $y = 0$ (after $t > 0$ that is), the projectile cannot move further, because it has encountered the ground.

Well, we can use the equation for y to find t, or just note that the parabola is symmetric about $y = H_{\text{max}}$, so $t_{\text{total}} = t_H$ where t_H is the time to reach H_{max}, or equivalently, when $v_y = 0$, which you can find easily. All of them give the same time, which when plug into the equation for x, we get

$$\boxed{R = \frac{v_0^2 \sin 2\theta}{g}}. \tag{2.10}$$

Using this, we can also alternatively write the equation of trajectory as,

$$y = x \tan\theta \left(1 - \frac{x}{R}\right) \tag{2.11}$$

What is the maximum value of R, the range, for a fixed velocity? Think about what we can vary. Well, θ of course, and its easy to see the maximum occurs for $\theta = \pi/4$, so we *optimise* the range, and say its maximum at $\theta = \pi/4$.

Example 2.4

A projectile is thrown from one vertex of a wedge grazes the other vertex, and ends up on the third vertex. It is thrown with an initial velocity v_0 making an angle θ with the

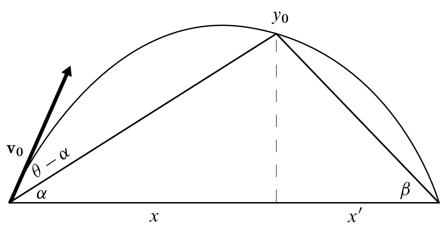


Figure 2.3: Projectile grazing a wedge

horizontal. If the angles of the vertices are α and β, figure out a relation between $\tan\theta$, $\tan\alpha$ and $\tan\beta$.

Solution. Divide the range of the projectile into two lengths, x and x' such that $x + x' = R$. Now, clearly,

$$\tan\alpha = \frac{y_0}{x} \qquad \tan\beta = \frac{y_0}{x'}$$

Also, by the alternate equation of trajectory,

$$y_0 = x\tan\theta\left(1 - \frac{x}{R}\right) = x\tan\theta\left(1 - \frac{x}{x + x'}\right)$$

Which gives us,

$$\tan\theta = y_0\left(\frac{x + x'}{xx'}\right) = \frac{y_0}{x} + \frac{y_0}{x'}$$

Finally substituting the values, we get,

$$\tan\theta = \tan\alpha + \tan\beta$$

□

There are two important properties of projectile motion. One is more general, and that is that for classical mechanics, motion is time-reversible. That is, if we do the transformation $t \to -t$, the motion doesn't change. The idea is that the acceleration remains invariant, and the velocity reverses, so the trajectory is the same, for forces of the form $F(x)$. We will say more about this later, when newton's laws are introduced, but till then you can use it as a fact.

The second is that the time to attain some vertical displacement can be found in terms of the maximum height alone, in fact the motion is independent of range.

Consider the vertical position of the particle to be s at some time t. Then,

$$s = u_y t - \frac{1}{2}gt^2$$

Also, note that if the maximum height is h,

$$u_y = \pm\sqrt{2gh}$$

Where the positive value is during the ascent and the negative is during the descent. Then,

$$s = \pm\sqrt{2gh}t - \frac{1}{2}gt^2$$

Solving for t gives us,

$$t = \pm\sqrt{\frac{2h}{g}} \pm \sqrt{\frac{2(h-s)}{g}} \tag{2.12}$$

If the initial velocity is positive and $s > 0$,

$$t = \sqrt{\frac{2h}{g}} \pm \sqrt{\frac{2(h-s)}{g}} \tag{2.13}$$

the two values correspond to the fact that the particle will have the same vertical displacement at two instances of time.

If the initial velocity is positive but $s < 0$, i.e., the particle starts at some height but then transverses below it,

$$t = \sqrt{\frac{2h}{g}} + \sqrt{\frac{2(h-s)}{g}} \tag{2.14}$$

is the only reasonable solution.

If the initial velocity is negative, then clearly the particle will attain a lower position. The maximum height in this case is simply the initial height of the particle.

$$t = -\sqrt{\frac{2h}{g}} + \sqrt{\frac{2(h-s)}{g}} \tag{2.15}$$

Thus qualitatively, projectile motion is independent of its horizontal range. If some projectiles have the same maximum height, their time of flight *will be the same*.

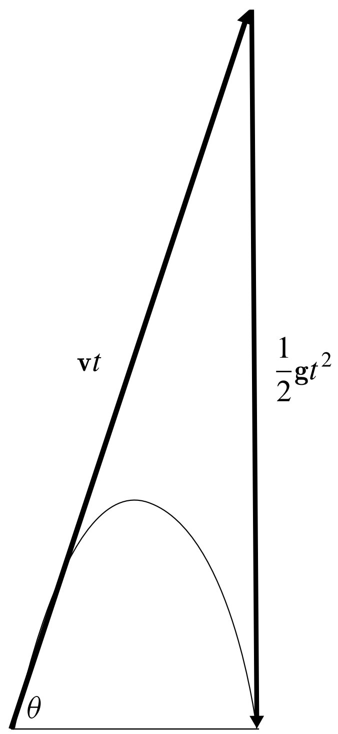


Figure 2.4: Projectile with Vectors

Exercises

2.2.1. *Projectile with Vectors.* [2]

We can integrate the kinematic equations vectorially, to get

$$\mathbf{v}(t) = \mathbf{v_0} + \mathbf{g}t,$$

and for the displacement,

$$\mathbf{s}(t) = \mathbf{v_0}t + \frac{1}{2}\mathbf{g}t^2.$$

(a) Show that these two equations are, in fact true. Justify our integration of a vector when discussing velocity and acceleration. *Hint: How can we integrate a vector? Try to use its components.*

(b) Note that the vectors $\mathbf{v_0}t$, $\mathbf{g}t^2/2$ are parallel to vector $\mathbf{v_0}$ and $\mathbf{g}$ respectively. Use this to find the range and time of flight. *Hint: You will find Figure 2.4 useful.*

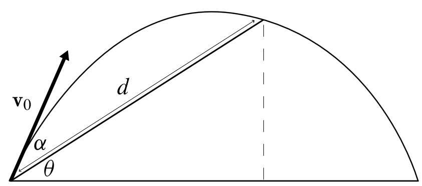


Figure 2.5: Projectile on tilted ground

2.2.2. *Projectile Motion in tilted axes.* [2]

Consider the case of a projectile launched along tilted ground (or a wedge), as in, fig. 3.7. If we were to figure out d, we could use a number of ways. The equation of the projectile is,

$$y = x\tan(\theta + \alpha) - \frac{gx^2}{2v_0^2\cos^2(\theta+\alpha)}$$

and of the wedge is,

$$y = x \tan\theta$$

We could simply equate them to figure out the distance d, which is the range of the projectile. However, let us look at alternate way of doing this, using tilted axis. This is the first time we're using the *transformation* of our reference frame. There are many more to come after this.

(a) Rotate the axes by the angle of inclination of the wedge, θ, so that the x axis lies along the plane, and y axis perpendicular to it. Now, find the kinematic equations along the two axes.

(b) Solve these to find the time at which the projectile hits the wedge, and the distance it travels.

2.2.3. *Projectile with drag, Morin* [4]

A ball is thrown with speed v_0 at an angle θ. Let the drag acceleration from the air take the form $a = -\alpha \mathbf{v}$.

(a) Find $x(t)$ and $y(t)$.

(b) Assume that the drag coefficient takes the value that makes the magnitude of the initial drag acceleration equal to g. If your goal is to have x be as large as possible when y achieves its maximum value (you don't care what this maximum value actually is), show that θ should satisfy $\sin\theta = (\sqrt{5}-1)/2$, which just happens to be the inverse of the golden ratio.

2.3 Polar Co-ordinates

The polar co-ordinate axes are the two-dimensional subset of the 3-d cylindrical co-ordinates. Any point in polar co-ordinates is depicted by a system of two unit vectors, $\mathbf{e}_r$ and $\mathbf{e}_\theta$. However, each of these are dependent on the position of the particle and may be written as $\mathbf{e}_r(\theta)$ and $\mathbf{e}_\theta(\theta)$.

Here, $\mathbf{e}_r$ points in the direction of increasing radius (along the radial vector), and $\mathbf{e}_\theta$ points in the direction of increasing θ (tangent to the radial vector). These two unit vectors are *orthogonal* at any point.

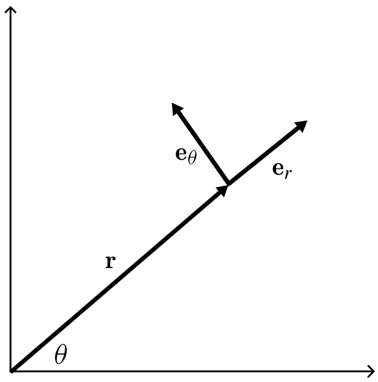


Figure 2.6: Polar-coordinates

In Figure 2.6, if $\mathbf{e}_r$ makes an angle θ with the horizontal, then,

$$\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \tag{2.16a}$$
$$\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \tag{2.16b}$$

Which follows by adding projections of $\mathbf{i}$ and $\mathbf{j}$ along the radial and tangential directions.

Now let us formulate kinematics in polar co-ordinates. The position vector $\mathbf{r}$ can be written as

$$\mathbf{r} = r\mathbf{e}_r \tag{2.17}$$

Velocity follows normally,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt}$$

Or does it? What even is $d\mathbf{e}_r / dt$? The answer lies in the time-derivative of the unit vectors. Using the definitions in Equation (2.16), we get:

$$\frac{d\mathbf{e}_r}{dt} = \dot{\theta}\mathbf{e}_\theta \tag{2.18a}$$

$$\frac{d\mathbf{e}_\theta}{dt} = -\dot{\theta}\mathbf{e}_r \tag{2.18b}$$

Using these results, we get that

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \tag{2.19}$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta \tag{2.20}$$

The most interesting result is the various terms of acceleration in Equation (2.20). First, let us consider the terms along $\mathbf{e}_r$. The term $\ddot{r}$ is the *radial acceleration*. It is the change in the radial speed (and thus the radius). The second term $-r\dot{\theta}^2$ is the *centripetal acceleration*. This term accounts for change in direction of tangential velocity and is responsible for the particle's curved path.

Now consider the terms along $\mathbf{e}_\theta$. The term $r\ddot{\theta}$ is the *tangential acceleration* due to change in tangential speed:

$$a_t = r\ddot{\theta} = \dot{v}$$

The other term $2\dot{r}\dot{\theta}$ is the *coriolis acceleration*, which we'll discuss with rotating frames.

Example 2.5

The movement of a particle in a circle about some point is called circular motion. If we move to the origin, r does not change with time. Thus, $\dot{r} = \ddot{r} = 0$. This gives us $v = \dot{\theta}r$. $\dot{\theta}$ is often refered to as the *Angular Speed*, ω.

We also find that $a_r = -m\omega^2 r$ and $a_\theta = r\ddot{\theta}$. If $\ddot{\theta} = 0$, then the motion is called uniform circular motion, and the acceleration is perpendicular to the velocity.

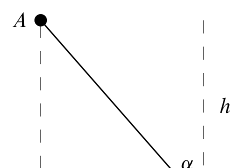
2.4 Optimisation Problems

These are a class of math problems with almost no physics involved. The following problem, for instance, is a purely geometrical one. These problems rely on tricks, and can be fun puzzles.

A man who can run at a constant speed stands at a point with coordinates $(4, 5)$ *and has to walk to the x-axis, then to the point* $(3, 7)$. *What path must the man walk along?*

Question

A large subset of these questions can be done purely by calculus, but that is not always the most elegant way to go about it. We'll cover questions of both kinds, sometimes you do need to do the effort and the elegant solution may not be so elegant.



Example 2.6

A man can run at a speed u and swim at a speed v. In the following diagram, the man starts at point A and has to save his friend from drowning at point B. Assuming he likes his friend, what should he do?

Solution. The man should call a lifeguard. Now, let's figure out how the lifeguard should run from point A to B. The time taken to reach the man is:

$$t = \frac{\sqrt{x^2 + h^2}}{u} + \frac{\sqrt{(d-x)^2 + h'^2}}{v}$$

where d is the horizontal distance between A, B, and h, h' is vertical distance of A and B to the shore, and hence independent of the path taken.

The derivative of this with respect to the x has to be 0 for the best path of the lifeguard:

$$\frac{x}{u\sqrt{x^2 + h^2}} = \frac{(d-x)}{v\sqrt{(d-x)^2 + h'^2}}$$

But $\sin\alpha = x/\sqrt{x^2 + h^2}$ and $\sin\beta = (d-x)/\sqrt{(d-x)^2 + h'^2}$, so we have,

$$\boxed{\frac{u}{\sin\alpha} = \frac{v}{\sin\beta}} \tag{2.21}$$

□

This path follows the laws of refraction. In fact, this hints at a more general result for light, that being that it follows the path of least time. This is called Fermat's principle. Of course, we don't care about what light does at this point, but it is a very cool result. Note for one that we paid no attention to which reference frame we were talking about, and it doesn't matter (unless its rotation).

As you may notice, the principle doesn't really depend on what the media or distances are, in fact this is a general result.

A beautiful application of this principle is the Brachistochrone problem, which is given at the end of this section.

Example 2.7

Consider a projectile thrown with a velocity v_0 making an angle α with horizontal, on wedge of inclination θ, which slopes downwards. Find out the angle α such that the projectile has maximum range.

Solution. The locus of the projectile is,

$$y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$$

and of the wedge is,

$$y = -x \tan\theta$$

At the range, they will be equal so,

$$x \tan\alpha - \frac{gx^2}{2v_0^2 \cos^2\alpha} = -x \tan\theta \implies \tan\alpha + \tan\theta = \frac{gx}{2v_0^2 \cos^2\alpha}$$

which gives us the desired range,

$$x = \frac{2v_0^2(\tan\alpha + \tan\theta)\cos^2\alpha}{g}$$

So the condition for maximum range is simply,

$$\frac{dx}{d\alpha} = 0.$$

This also kind of illustrates a nice idea, since we need to get the derivative 0, we can safely discard any multiplicative or additive constants right away which will either have a zero derivative, or can be divided by since the other side of the equation is simply 0.

We get that,

$$\begin{aligned}
&\frac{d}{d\alpha}\left\{(\tan\alpha + \tan\theta)\cos^2\alpha\right\} = 0 \\
&\frac{d}{d\alpha}\left\{\frac{\sin 2\alpha}{2} + \tan\theta\cos^2\alpha\right\} = 0 \\
&\cos 2\alpha - \tan\theta \sin 2\alpha = 0 \\
&\tan 2\alpha = \cot\theta \\
&\tan 2\alpha = \tan\left(\frac{\pi}{2} - \theta\right) \\
\implies &\alpha = \frac{\pi}{4} - \frac{\theta}{2}
\end{aligned}$$

□

An easier way to find x is to use Exercise 2.2.2., and then we can continue in the same manner.

Example 2.8

A bug wishes to jump over a cylindrical log of radius R lying on the ground, so that it just grazes the top of the log horizontally as it passes by. What is the minimum launch speed v required to do this?

Solution. First of all, since the the bug grazes the top of the log horziontally, v_y at the top is just 0. Since $v^2 = v_0^2 - 2gh$, this implies that $v_{0,y} = \sqrt{2gh}$. Now we just need to find v_x.

The idea here is of radius of curvature. If v_x is too low, the trajectory of the bug will just pass through the log. If it's too high, our velocity is not minimal. At the most optimal trajectory, the *curvature* of the trajectory is the same as that of the log, so the bug is doing circular motion at the top of the log with radius r. Using polar-cordinates, and $\dot{r} = 0$, we find that $v_x = gR$, since $\mathbf{g}$ points radially at top.

Thus, $v_{\min} = \sqrt{v_y^2 + v_x^2} = \sqrt{5gR}$. □

Exercises

2.4.4. *Projectile from a height* [3]

Consider a projectile thrown from height h with velocity v. What is its maxium horizontal range? There's two ways we will tackle this problem:

(a) Write down the equation of trajectory, but shifted so that $y \to y - h$. At $x = R$, the range, we should have $y = 0$ (or don't change y, but note that projectile motion is time reversible, so $y = h$ at $x = R$). Then take the derivative of it with respect to θ, while noting that for optimal trajectory $dR/d\theta = 0$. Find θ and then R.

(b) Again write down the same shifted equation of trajectory. Now, write down every trignometric function in terms of $\tan\theta$. You should get a quadratic. For such a trajectory to exist, $\tan\theta$ must have real roots. Use this to get R.

2.4.5. *River and Drift* [2]

A swimmer can swim with velocity u in still water. If the river has a velocity v, what is the minimum distance through which she can cross the river? We'll do this in two ways as well.

(a) Let the velocity of the swimmer make an angle θ with the vertical. The distance perpendicular to stream does not change with θ, so we can ignore it. Calculate and minimise then, the distance parallel to the flow of the river.

(b) Draw the *velocity* space of the resultant velocity of u and v. That is, by varying θ, consider how the resultant changes geometrically. Do this for both $v > u$ and $u \geq v$. When the accelerations or velocities are complicated, but it is often helpful to consider the *velocity space* or the position space, which is just the vector space of all velocities, instead of our usual position space.

2.4.6. *Brachistochrone* [4]

Suppose we drop a ball from a point A to point B, vertically h and horizontally d apart along a curve. Find the curve which minimises the time of descent. You might find $v^2 = v_0^2 + 2g\Delta h$ helpful.

2.4.7. *Jumping over Roofs, Kalda* [3]

Two fences of heights h_1 and h_2 are erected on a horizontal plain, so that the tops of the fences are separated by a distance d. Show that the minimum speed needed to throw a projectile over both fences is $\sqrt{g(h_1 + h_2 + d)}$.

2.4.8. *Optimal Projectile* [2]

Find the minimum speed required to the reach the point (X, Y), by a projectile thrown from $(0, 0)$.

2.4.9. *Ping Pong 1, OPhO* [3]

A legal serve in ping-pong requires that the ball bounces on one side of the table and that the ball goes over the net. A certain world-class Olympic ping-pong player does the serve at the level of the table at a distance d from the net of height h.

The Olympic player can give the ball such a spin that the translational speed of the ball is conserved after a bounce but the direction of velocity can be controlled freely. What is the minimal serving speed v?

2.4.10. *Perpendicular Velocities, Morin* [3]

In the maximum-distance case of Example 2.7 show that the initial and final velocities are perpendicular to each other. *Hint: Use the fact that projectile motion is time reversible.*

2.4.11. *Parabolic Envelope, Knzhou* [5]

The boundary of the set of all points a projectile with fixed velocity can reach by varying θ is a parabola, with the focus at launch point. We can use this to geometrically solve many problems.

(a) Show that the trajectory that touches any point on the parabola must be tangent to it.

(b) Show that the velocity at the point of tangency must be perpendicular to the initial one, if we reach the point on the parabola with minimum velocity. Use this to solve the previous problem.

(c) Find the optimal angle in Example 2.7 and solve Exercise 2.4.7. using the envelope and (a), (b).

2.4.12. *Parabolic Envelope* [3]

Justify the claim that the boundary of boundary of the set of all points a projectile with fixed velocity can reach by varying θ is a parabola using the the second method of Exercise 2.4.4..

2.4.13. *Projectiles with Vector, V2.* [3]

When a projectile thrown along a wedge hits it, the sum of the scaled velocity and acceleration vector must equal the vector along the slope with the magnitude equal to the distance the projectile travels along the wedge.

Using this, the vector equations for $\mathbf{v}$, $\mathbf{r}$ and part (b) of the previous problem, show that $\mathbf{v_0}$ lies along the line joining focus and projection of point at which it hits the wedge on the directrix, and find the angle using some angle chase.

Do not use the reflexive property of parabola, in fact, this gives you a proof for it.

2.4.14. *Tired Flappy Bird, OPhO* [4]

A flappy bird can jump multiple times in the air. Each time it jumps mid-air, it can suddenly change its speed and direction. For every jump, the bird can decide when to jump and in which direction. Between jumps, the bird falls freely under gravity, which pulls it down at the acceleration g.

Say, our tired flappy bird starts off the cliff of height H with the jumping velocity $V[1] = V_0$. Subsequent jumps in mid-air have decreasing velocities, i.e. the n-th jump has speed $V[n] = V_0/n$, $(n > 1)$. This majestic Vietnamese animal wants to travel as far as possible horizontally before it lands on the ground. Find the maximum horizontal distance the bird can travel.

2.5 Frames and Symmetry

Often in a mechanics problem, switching frames proves to be very beneficial. It is especially useful if we have two moving objects, and we move to the frame of one, or move to a frame that offers high symmetry. Suppose we move to a frame with position vector $\mathbf{R}$ with respect to our original frame. Call the original frame S and the current one S'. If the position vector of an object in S os $\mathbf{r}$ and in S' is $\mathbf{r}'$, then,

$$\mathbf{r} = \mathbf{r}' + \mathbf{R} \tag{2.22}$$

The derivative of this gives

$$\mathbf{v} = \mathbf{V} + \mathbf{v}' \tag{2.23}$$

where $\dot{\mathbf{R}} = \mathbf{V}$ is the velocity of S' with respect to S.

We can go further, describing how accelerations are related, but that's not quite useful for kinematics, and is a whole chapter in itself. So we'll leave that for now. Also note that S' is not rotating.

If we wish to switch to a frame rotating with angular speed, ω then if everything is nice and planar, we can use the fact that the angular speeds also add in a vectorial manner, i.e. if $\omega = \dot{\theta}$ is in the direction of increasing theta, switching to a frame rotating with this ω would increase the angular speed of all points by $-\omega$ which is in the direction of decreasing theta.

Exercises

2.5.15. *Symmetrical Circles, Kalda* [2]

Two intersecting circles of radius r have centers a distance a apart. If one circle moves towards the other with speed v, what is the speed of one of the points of intersection?

2.5.16. *Rotating Circles, Kalda* [3]

Two circles of radius r intersect at the point O. One of the circles rotates about the point O with constant angular speed ω. The other point of intersection O' is originally a distance d from O. Find the speed of O' as a function of time.

A point rotating with angular speed ω about O just means in the frame in which O is the origin, and in polar co-ordinates, $\dot{\theta} = \omega$.

2.6 Invariants

Invariants are universal across physics. These give us *conservation laws*, many of which we'll encounter in the later chapters.

An *Invariant* is a quantity that remains constant throughout time. Identifying an invariant in a problem is very helpful, and can help us solve some seemingly impossible problems.

Example 2.9

A cat moves with speed v always directed towards a mouse moving rectilinearly with speed v. They are initially at a distance ℓ from each other, and their initial velocities are perpendicular. After a long time they move along the same line, seperated by a distance d. Find d

Solution. Solving this problem using the techniques we have already discussed is incredibly hard. Whenever you do no know what to do, writing all the equations governing the system isn't a bad idea.

As I mentioned in the previous section, it might be good to move into the frame of one of them. Let's move into the frame of the cat. Let's orient our axis so that the seperation initially was along the y axis, and thus the mouse moves with $v\mathbf{i}$.

Consider the system after time t, as in Figure 2.8. We can't really say anything nice in cartesian co-ordinates alone. But note that the velocity of the mouse (aside from the horizontal one) because of switching the frames must point radially, and is the same as the horizontal velocity, so let's try out polar-coordinates.

The velocity along the x axis is $v - v\cos\theta$. The velocity along the radial direction is $v\cos\theta - v$. Thus,

$$\frac{dx}{dt} + \frac{dr}{dt} = v - v\cos\theta + v\cos\theta - v = 0 \implies \frac{d(x+r)}{dt} = 0.$$

So $x + r$ is a constant, independent of time. At $t = 0$, r is just ℓ and $x = 0$. After a long time, $x = r$. So, $d = r = \ell/2$. □

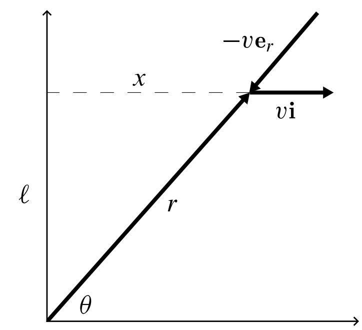


Figure 2.8: Velocities and distances in the frame of cat

It seems almost impossible to solve the previous problem without invariants, and in fact trying to solve without invariants is very difficult. You can look at this paper for an extensive discussion on the problem.

We'll only do one problem for this here, since most would require some knowledge of dynamics. But the idea is very important, and I urge you to keep it in mind.

Exercises

2.6.17. *Chasing Problem* [2]

Suppose the cat in the previous example has velocity $u > v$. How long does it take to catch the rabbit?

3 Laws of Motion

Newton's laws lay out the foundation of mechanics, and present us with the laws of physics that govern the everyday world. We'll later see that they aren't exact, but in the regime of common every day tasks, they present us with the tools to analyse any mechanical phenomenon. We'll also go through various applications of these laws along with the general discussion.

3.1 Newton's Laws

Classical mechanics is a deterministic theory, having known the state of the physical system at one instant, it wishes to determine the evolution of system as time goes on. You may have heard of indeterminance in quantum mechanics through pop science, but remember that it doesn't invalidate classical mechanics.

In general even a less correct theory is useful if we use it in the right regime. As long as we don't try to work in classical mechanics while our cars are moving at the speed of light or our balls are comparable to the size of atoms, we'll be fine.

Classical mechanics is almost entirely governed by Newton's laws. Using merely a few axioms, we can understand alot of the daily phenomenons that we see. Let's start with the first law.

The First Law

N1 Left alone, a particle moves with constant velocity.

There are various different ways to write it of course, in terms of zero net force and whatever, but to get the main physical insight from this we have to look a little deeper. First of all, what does it mean for a particle to be left alone?

Well let's first take about things that are not left alone, for instance the planets. We know that each of them interact physically with the sun and rotate around it. And there are other examples, a ball being juggled, a cart being pushed, a nail being attracted by a magnet, etc.

The common theme here is that each of the objects experiences a *physical interaction* of some manner. Gravity, magnetic forces, the force we exert, all are results of a physical interaction between two bodies.

And thus we expect that a body that if left alone does not experience any sort of physical interaction: it is *isolated* from all other objects.

This law doesn't really hold in all reference frames, but defines an inertial frame, which provide the setting in which newton's laws hold. We'll later see that a little modification of these laws will help us use them in any sort of frame, but we'll restrict ourself to inertial frames for now.

Definition 3.1

> An *Inertial Frame* is an observational frame of reference in which an isolated body moves with constant velocity.

The first laws then boils down to an axiom of existence, and that is the physical content of the law:

N1 Inertial frames exists.

Thus the first law is part definition and part experimental fact. From experiments, we know that our axiom is in fact valid. But there is one issue here, can we always isolate a body? For instance suppose that the interaction between it and some object increased as distance increases: then we could never isolate it, no matter how far we went (going too close would give us all sort of other forces, like gravity).

Fortunately, from what we know experimentally, this is not the case. All interactions eventually become negligible as we increase the distance, so we can get an isolated body. Even without going to very large distances, we can manually cancel out the other forces to get such a body (for instance by using an air track).

In any case, the idea of an inertial frame is extremely important for classical mechanics, and we'll see it pop up quite a few times.

The Second Law

N2 The force on a body is proportial to the time rate of change of its momentum.

Before talking about what forces are, let's define momentum:

Definition 3.2

> The *Linear Momentum* of a particle is:
>
> $$\mathbf{p} \equiv m\mathbf{v} \tag{3.1}$$

Thus it follows from the second law that the force, $\mathbf{F}$ is:

$$\mathbf{F} = \dot{\mathbf{p}} = m\dot{\mathbf{v}} + \dot{m}\mathbf{v}. \tag{3.2}$$

We choose our units so that the constant of proportionality is 1.

For constant mass $\dot{m} = 0$ so that,

$$\mathbf{F} = m\mathbf{a}. \tag{3.3}$$

which is the form of the second law you are probably already acquainted with. There are a couple of problems here: what is m, and what do we really mean by a force?

Part 3

Electromagnetism

Electrostatics

4

Electrostatics is the study of fixed charges. In this chapter we'll cover one of maxwell's equation and just a whole lot about how electric forces interact with each other, even though we won't quite talk about the motion of a charge particle.

4.1 Coloumb's Law

The fundamental law governing interactions between two charged particles is *Coloumb's Law*. The force on a particle of charge q_1 due to a particle of charge q_2, at positions $\mathbf{r}_1$ and $\mathbf{r}_2$ is:

We write $\mathbf{F}_{12}$ for the force on particle 2 due to interaction between particle 1 and 2 and $\boldsymbol{\mathscr{r}} \equiv \mathbf{r}_2 - \mathbf{r}_1$, the vector from 1 to 2.

$$\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}.} \tag{4.1}$$

The coloumb forces similar to other forces we have seen now, follows the law of superposition. The force is parallel to the line joining both the charges, and from a simple symmetry argument we can see that it can be parallel to no other direction, atleast for stationary charges, since the only unique direction in this case is the line joining both the charges. Space is isotropic, so if it points in any other direction, we can simply rotate our whole setup with out altering anything to see that it must now point in a different direction.

If for instance the charges were moving, the above argument would not hold— we could have some force in the direction perpendicular to acceleration and $\boldsymbol{\mathscr{r}}$ for instance. This would also be true if the charges had any structure, and so another intrinsic direction. But then we would need more quantities to talk about the charge than just the scalar q. This is true of the elementary particles (they have a property called the *spin*), including electrons and leads to magnetic phenomena, and so does the movement of charges. In both cases the magnetic force does not in general point along the direction joining the two charges.

The constant for the coloumbic force, $1/4\pi\epsilon_0$ is introduced to historical reasons, where:

$$\frac{1}{4\pi\epsilon_0} = 8.988\cdot 10^9\,\mathrm{N\,m^2\,C^{-2}} \implies \epsilon_0 = 8.854\cdot 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}$$

Consider the forces due to charges $q_1, q_2, \ldots, q_n$ on a test charge Q:

$$\mathbf{F} = \sum_{i=1}^{N}\frac{1}{4\pi\epsilon_0}\frac{q_i Q}{\mathscr{r}_i^{\,2}}\hat{\boldsymbol{\mathscr{r}}}_i$$

To simplify matters, because we wish to find the force due to the system on any such charge, it seems quite fair to definine a quantity independent of the test charge. Thus we define the electric field, which is the property of the *system* of charges.

Definition 4.1

The *Electric Field* due to a charge q at a point having position vector $\boldsymbol{\mathscr{r}}$ with respect to q is:

$$\mathbf{E} \equiv \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} \tag{4.2}$$

So that the force on a charge Q is simply:

$$\mathbf{F} = Q\mathbf{E}$$

where $\mathbf{E}$ is the electric field at the charge. We can see that $\mathbf{E}$ also follows the law of superposition. Thus for a system of charges:

$$\mathbf{E}(r) = \sum_{i=1}^{N}\frac{1}{4\pi\epsilon_0}\frac{q_i}{\mathscr{r}_i{}^2}\hat{\boldsymbol{\mathscr{r}}}_i$$

where $\boldsymbol{\mathscr{r}}_i = \mathbf{r} - \mathbf{r}_i$, the vector from charge q_i to $\mathbf{r}$.

For a continous charge distribution the sum becomes an integral:

$$\mathbf{E}(r) = \int_{\mathcal{V}}\frac{1}{4\pi\epsilon_0}\frac{\rho\, d^3r}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$$

Add field lines, quantisation/conservation of charge, how dq is fine even though quantisation.

Volume density for discrete charges

A more general way to look at discrete, line and surface charges is in terms of the delta function. We can extend volume density to such places (which is much more physical) by using the delta function. For discrete charges:

Motivate this properly

$$\rho(\mathbf{r}) = \sum q_i\delta(\mathbf{r} - \mathbf{r}_i) = \sum q_i\delta(\boldsymbol{\mathscr{r}}_i).$$

The extension to surface charges and line charges is similar, for instance for a spherical shell with surface charge σ and radius R centered at the origin is:

$$\rho(\mathbf{r}) = \sigma\delta(\|\mathbf{r}\| - R).$$

4.2 Gauss's Law

Suppose a charge is enclosed by a surface— for a concrete example consider a charged particle inside it. Then the amount of charge inside should be measure of the field lines, and thus the field.

$$\Phi = \int \mathbf{E}\cdot\hat{\mathbf{n}}\, da$$

The force perpendicular to a surface charge, if the flux through it is Φ is:

$$\mathbf{F}\cdot\hat{\mathbf{n}} = \sigma\Phi$$

Solid Angle

4.3 Symmetries

4.4 Scalar Potential

Since the curl of $\mathbf{E}$ is 0, it follows that:

$$\int_S (\nabla\times\mathbf{E})\cdot\hat{\mathbf{n}}\,d^2r = \int_{\partial S}\mathbf{E}\cdot d\boldsymbol{\ell} \implies \int_{\partial S}\mathbf{E}\cdot d\boldsymbol{\ell} = 0.$$

From this it follows that the line integral between any two points is independent of the path, so we define a fuction using it.

We define the scalar potential ϕ as:

$$\phi(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}}\mathbf{E}\cdot d\boldsymbol{\ell} \tag{4.3}$$

where $\mathcal{O}$ is the origin.

Definition 4.2

Thus,

$$\phi(\mathbf{b}) - \phi(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}}\mathbf{E}\cdot d\boldsymbol{\ell}$$

But from the gradient theorem,

$$\phi(\mathbf{b}) - \phi(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}}\nabla\phi\cdot d\boldsymbol{\ell}$$

So it follows that:

$$\mathbf{E} = -\nabla\phi. \tag{4.4}$$

From the definition it follows that the potential due to a charge distribution is simply:

$$\phi(\mathbf{r}) = \int_{\mathcal{V}}\frac{1}{4\pi\epsilon_0}\frac{\rho\,d^3r}{\mathscr{r}^2}\mathscr{r}$$

4.5 Electrostatic Potential Energy

Suppose we have a charge distribution $\rho(\mathbf{r})$ in a potential $\phi'(\mathbf{r})$. We define the electrostatic potential energy as:

The electrostatic potential energy of a charge distribution $\rho(\mathbf{r})$ in a potential $\phi'(\mathbf{r})$ is:

$$V(\mathbf{r}) \equiv \int_{\mathcal{V}} d^3r\,\rho(\mathbf{r})\phi'(\mathbf{r}) \tag{4.5}$$

where $\mathcal{V}$ is the region containing the charge distribution.

Definition 4.3

For point charges $V = q\phi'$. This immediately leads us to Green's Reciprocity Theorem:

Green's Reciprocity Theorem **Theorem 4.1**

The potential energy of ρ_1 in a field produced by ρ_2 is the same as the potential energy of ρ_2 in a field produced by ρ_1.

Proof. The proof of it follows from the definition of potential. The potential due to ρ_2 at $\mathbf{r}$ is:

$$\phi_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\rho_2(\mathbf{r}')}{\mathscr{r}'}$$

where $\boldsymbol{\mathscr{r}}' = \mathbf{r} - \mathbf{r}'$ and the intergration is carried over all of space because ρ_2 is simply zero outside of the region containing charge. So the potential energy of ρ_1 is:

$$V_1 = \int d^3r\rho_1(\mathbf{r}) \int \frac{1}{4\pi\epsilon_0} d^3r' \frac{\rho_2(\mathbf{r}')}{\mathscr{r}}$$

Since $\rho(\mathbf{r})$ is not a function of $\mathbf{r}'$ we can write this as:

$$V_1 = \frac{1}{4\pi\epsilon_0}\int d^3r \int d^3r' \frac{\rho_1(\mathbf{r})\rho_2(\mathbf{r}')}{\mathscr{r}}$$

But note that $\boldsymbol{\mathscr{r}}' = \mathbf{r}' - \mathbf{r} \implies \mathscr{r}' = \mathscr{r}$. So that we can rewrite this as:

$$V_1 = \frac{1}{4\pi\epsilon_0}\int d^3r \int d^3r' \frac{\rho_1(\mathbf{r})\rho_2(\mathbf{r}')}{\mathscr{r}'}$$

which is exactly the potential energy of ρ_2 due to ρ_1! So we have

$$V_1 = V_2 \tag{4.6}$$

□

The potential energy of a point charge seperated by a distance r_{12} from another point charge is

$$V = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r_{12}}$$

Total Electrostatic Energy

The total electrostatic energy of the system is the energy required to assemble the system. That is, if

basically this is equal to the potential energy of a particle if the reference point is infty which you can explicitly show.

It doesn't matter how each particle gets to its final position atleast, so consider them travelling on a line. Then consider infinitesmal displacements dr, dr'. Then $d\mathscr{r} = dr + dr' \implies dr = d\mathscr{r} - dr'$. So potential is:

$$\delta V = F_{12}\, dr + F_{12}\, dr' = F_{12}\, d\mathscr{r}.$$

(this is the potential energy of each particle just use the dot product to see). Anyway so we're done.

In general for n particles, the forces are independent of each other, so the potential energy just adds up in superposition for each particle and like yeah the potential energy caused by having any two pair of charges at distance r_{ij} is simply:

$$V_{ij} = \frac{kq_iq_j}{\mathscr{r}_{ij}}$$

Anyway like you can say that this is equivalent to bringing q_1 to a distance r_{12} from q_2 or whatever and compute.