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Part 1

Prerequisites

Vectors

1

In this chapter we'll give an introduction to vectors, and some results in geometry that you may find useful. We'll first introduce them geometrically, then give a rigorous algebraic treatment of them. Vectors are very useful quantities, that will aid us all the way to Quantum Mechanics, so being familiar with them will aid you a lot.

1.1 Geometrical Vectors

We will somewhat loosely follow the vectors chapter from `https://www.damtp.cam.ac.uk/user/sjc1/teaching/VandM/notes.pdf`.

We first describe vectors in a geometrical fashion. This isn't in anyway rigorous, and we won't pretend it is. We'll give some proofs, but you may need to take a leap of faith in some places.

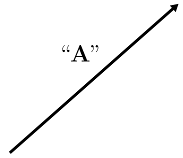


Figure 1.1: A directed line segment, the vector **A**.

Definition 1.1

A *Vector* is a quantity described by a magnitude and a direction in space. They're represented as line segments, as in fig. 1.1.

You can think of a vector as a *directed line segment*. Vectors are usually denoted using boldface, **A**.

Because they're directed line segments, only their length matters and we're not concerned with the location of their end points. This means that a vector can be freely translated, without altering it. An equivalent way to say this is that two vectors are equal iff their magnitude is equal, and they have the same direction (i.e. they're parallel).

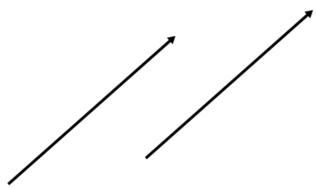


Figure 1.2: Two identical vectors.

Also, we define a vector field, which is a special sort of function, that'll be relevant for future discussions.

Definition 1.2

A *Vector Field* is a vector $\mathbf{B}$ which is a function of position, $\mathbf{B}(\mathbf{r}) = \mathbf{B}(x, y, z)$.

Where the position of a point P refers to the vector from the origin O to the point, P.

The magnitude of the vector is its length, which we call it's *norm*, and write as $\|\mathbf{A}\|$ or simply A. You may also see it being written as $|\mathbf{A}|$.

A unit vector, $\hat{\boldsymbol{\gamma}}$ ("gamma hat") is a vector whose magnitude is unity. We use the unit vector to often denote the direction of a vector by multiplying the unit vector with its magnitude. For instance, the unit vector that point is the direction of $\mathbf{A}$, $\hat{\mathbf{A}}$ can be calculated as,

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{A}.$$

Further, vectors have certain operations defined on them. The base operations are *Scalar Multiplication* and *Vector Addition*.

Scalar Multiplication

Scalar Multiplication refers to multiplying a vector by a *scalar*. A scalar for our purposes refers to a real number. Consider a vector $\mathbf{v}$. Multiplying it by a scalar, $\alpha \in \mathbb{R}$ produces a vector $\alpha\mathbf{v}$ parallel to the original vector and changes the magnitude of the vector so that $\|\alpha\mathbf{v}\|$ is $|\alpha|$ times greater than $\|\mathbf{v}\|$, thus $\|\alpha\mathbf{v}\| = |\alpha|\|\mathbf{v}\|$. $|\alpha|$ can be smaller or greater than 1 which accordingly increases/decreases the length of $\mathbf{v}$.

If $\alpha > 0$, the vector produced is in the same direction as the original vector. If $\alpha < 0$, the direction of the vector is reversed.

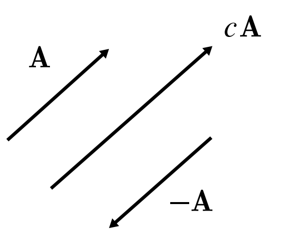


Figure 1.3: Scalar multiplication of a vector $\mathbf{A}$ by $c > 1$ and -1.

Vector Addition

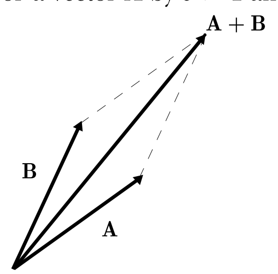
Adding two vectors, $\mathbf{A}$ and $\mathbf{B}$ produces another vector $\mathbf{A} + \mathbf{B}$. Geometrically, vector addition is done by placing the tail of $\mathbf{B}$ on the head of $\mathbf{A}$, and then the vector joining the tail of $\mathbf{A}$ and head of $\mathbf{B}$ is the vector $\mathbf{A} + \mathbf{B}$ as in fig. 1.3a.

It can also be interpreted as the diagonal of the parallelogram made by placing the tails of the two vectors together, and producing two sides parallel to them as show in fig. 1.3b.

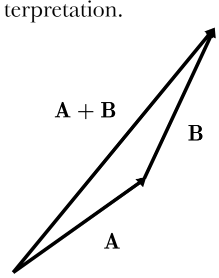
From this we may interpret that vector addition is commutative, that is, $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$. We can also deduce that its associative, $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$. I'll recommend convincing yourself of this using geometrical constructions.

Subtraction of vectors, $\mathbf{A} - \mathbf{B}$ is equivalent to multiplying $\mathbf{B}$ by -1 and then adding it with $\mathbf{A}$.

If $\|\mathbf{v}\| = 0$, we write $\mathbf{v} = \mathbf{0}$.



(a) Adding vectors, parallelogram interpretation.



(b) Adding vectors, triangle interpretation.

Figure 1.4: Addition of two vectors, $\mathbf{A}$ and $\mathbf{B}$ produces another vector, $\mathbf{A} + \mathbf{B}$.

Scalar Product

Mathematically Useful Things

2

2.1 Linear, Second Order, Homogenous, Time Translation Invariant ODEs

Second order means highest derivative in equation is d^2x/dt^2. An ODE is *linear* if it doesn't have any products of x, $\dot{x}$ and $\ddot{x}$ other than with a constant. A ODE is *homogeneous* if it only contains terms that are proportional to exactly one power of x or its derivatives. *Time translational invariant* just means coefficients are independent of time.

These are ODEs of the form

$$m\ddot{x} + b\dot{x} + kx = 0.$$

The general method to solve them is basically the same for all such equations. First, we elevate x to a complex variable, z. The motivation for this is basically thus. If $z(t)$ satisfies the complex differential equation,

$$\{m\,\mathrm{Re}(\ddot{z}) + b\,\mathrm{Re}(\dot{z}) + k\,\mathrm{Re}(z)\} + i\,\{m\,\mathrm{Im}(\ddot{z}) + b\,\mathrm{Im}(\dot{z}) + k\,\mathrm{Im}(z)\} = 0$$

where $\mathrm{Re}(x)$ is the real part of x and $\mathrm{Im}(x)$ is the imaginary part. Thus, we can write $z = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$. The above equation follows.

Now the idea is basically that if $a + bi = 0$, $a = b = 0$. This follows from equality of complex numbers (which is trivial to show). Thus, if we find $z(t)$, we can find $x = \mathrm{Re}(z)$ which satisfies our DE and is real. (Also $\mathrm{Im}(z)$, of course.)

Anyway, the other thing is just, from the equation we want to work with exponentials, but that can be messy if for instance we have

$$m\ddot{x} + kx = 0$$

with $k > 0$. A real exponential function $e^{\zeta t}$ gives you $\zeta^2 + k = 0$, which doesn't have a real solution. So working in complex numbers is neater.

For our equation, we can just plug in an ansatz (which is a fancy way to say guess) $z(t) = Ae^{i\omega t}$, to get,

$$-\omega^2 m + bi\omega + k = 0 \iff \omega = \frac{bi \pm \sqrt{4km - b^2}}{2m}$$

The most general solution is an arbritrary superposition of $Ae^{i\omega t}$ for our two values of ω. Thus,

$$z(t) = e^{-bt/2m}\{Ae^{i\Omega t} + Be^{-i\Omega t}\}$$

where $\Omega = \sqrt{4km - b^2}/2m$, and $A, B \in \mathbb{C}$.

1 $\Omega^2 > 0$

Since Ω is real, we can take the real part to get,

$$x(t) = e^{-bt/2m}\{C \sin \Omega t + D \cos \Omega t\}$$

where $C = \mathrm{Im}(B) - \mathrm{Im}(A)$ and $D = \mathrm{Re}(A) + \mathrm{Re}(B)$.

The amplitude roughly decreases like $e^{-bt/2m}$ (this is not exact because the factors affects the whole motion, not when we have our extremes, regardless the net change is described a constant times this).

This is called under-dampening in a case when m is the mass of the spring, k is its constant, and bv is a dampening frictional force.

If Ω is small, then the oscillatory behaviour is not really visible, since the whole motion goes to 0 in about $t = 2m/b < 2\pi/\Omega$ which is the time it takes to complete one oscillation. Of course, it is apparent that $\Omega > b/2m$ using $\Omega^2 > 0 \iff 4km > b^2$.

2 $\Omega^2 < 0$

In this case we can write $\Omega = i\tilde{\omega}$, with $\tilde{\omega} = \sqrt{b^2 - 4km}/2m \in \mathbb{R}$. Thus,

$$x(t) = e^{-bt/2m}\{Ce^{\tilde{\omega}t} + De^{-\tilde{\omega}t}\}$$

with $C = \mathrm{Re}(B)$, D = $\mathrm{Re}(A)$. Also since $\Omega^2 < 0 \iff 4km < b^2$, this implies $0 > -b/2m + \tilde{\omega} > -b/2m - \tilde{\omega}$, so the motion is of the form of a decaying exponential one. This is called overdampening in the same case as the one for underdampening.

3 $\Omega^2 = 0$

This is called critical dampening. Now first of all this poses a problem. If $\Omega = 0$, our solutions are not very nice (well kinda). In fact, in that case we only have one $\omega = -bi/2m$.

But this doesn't actually work. The theory of differential equations tell us that our solution space is spanned by two basis vectors, $e^{i\omega_1 t}$ and $e^{i\omega_2 t}$ which are linearly independent. In this case our basis vectors are both the same, and not linearly independent!

Physically this can be understood as having two basis vectors gives us a solution that is their linear combination. If they're not linearly independent, our solution is of form $Ae^{i\omega t}$ which gives you only one free constant. But this means that for different velocities, the system behaves the same, which is unphysical (if we know $x(0)$, this solution determines $v(0)$, we aren't free to choose).

A good way to look at it like this, we will write out our solution in a little different form,

$$x(t) = Ae^{i\omega_1 t} + Be^{i\omega_2 t} = e^{i\omega_1 t}\{A + Be^{i(\omega_2 - \omega_1)t}\} = e^{i\omega_1 t}\{A + B\{\cos(\Delta\omega t) + i\sin(\Delta\omega t)\}\}$$

where $\Delta\omega = \omega_2 - \omega_1$. For $\Delta\omega t \ll 1$, we can use the taylor expansion to get $\cos\Delta\omega t \approx 1$ and $\sin\Delta\omega t \approx \Delta\omega t$. Write this out as

$$x(t) \approx e^{i\omega t}\{A + B + Bi\Delta\omega t\} = e^{i\omega t}\{C + Dt\}.$$

As $\Delta\omega \to 0$, that is we arrive at the same solutions, this behaviour persists for longer and longer, till we get the solution of the form,

$$x(t) = e^{-b/2m}\{A + Bt\}.$$

We can hand-wave the smallness of $\Delta\omega$ in the $B\Delta\omega t$ for now. This is just to give some form of intuition, and is not exactly what's going on, but this should allow for a fair idea.

This is called critical dampening in the case of springs.

2.2 nth order, homogeneous, time translational invariant, linear ODEs

These are ODEs of the form,

$$\left\{a_n \frac{d^n}{dt^n} + \cdots + a_1 \frac{d}{dt} + a_0\right\} x = 0$$

The solutions are of the form,

$$x(t) = A_1 e^{i\omega_1 t} + \cdots + A_n e^{i\omega_n t}$$

If $\omega_i = \cdots = \omega_j$ are repeated, we substitute them with $e^{i\omega_i t}, te^{i\omega_i t}, \ldots, t^{j-i-1}e^{i\omega_i t}$.

2.3 Non Time Translational Invariant, Second order, Linear ODEs

We'll look at some very special kind of ODEs here,

$$m\ddot{x} + b\dot{x} + kx = f(t).$$

Part 2

Mechanics

3 Kinematics

Kinematics is the study of motion without any concern to its cause. It introduces us with the basic quantities that will used throughout physics. While there's not that much physical content in this topic, it stands as quite an important topic in mechanics. Once you've solved the dynamic equations, all that's left is kinematics.

3.1 The Physical Quantities

Frames and Particles

Motion of any object is considered to relative to an observer. The observer defines a particular co-ordinate system called the reference frame.

Definition 3.1 A *Reference Frame* is a co-ordinate frame which can be moving or rotating, with respect to which we measure the physical quantities.

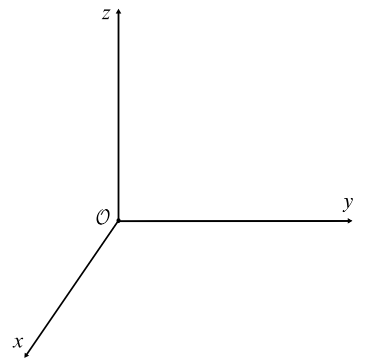


Figure 3.1: A cartesian reference frame

There are some things that depend on our choice of reference frame, and some that don't. We'll have a look at using reference frames to deal with some questions later.

Typically, we either use cartesian or curvilinear co-ordinates (spherical, cylindrical). We'll have a look at the 2d versions of both in this chapter, especially plane polar-coordinates (2d versions of cylindrical co-ordinates).

We usually deal with either point particles, or rigid bodies. You might think a point particle is not really a physically meaningful thing. However, we can approximate bodies, when their size is *not* meaningful to their motion, as particles to a very good precision.

Definition 3.2 A *Point Particle* is a particle whose size is negligible in the study of its motion.

We'll later discuss extended bodies, and in particular rigid ones in mechanics, but initially this idea of point particles will carry us far.

Position and Displacement

To describe the position of the particle, we use a position vector. A position vector is an example of a *vector field*, about which we'll have much to say in electromagnetism and gravitation.

Definition 3.3

The *position vector*, $\mathbf{r}(x, y, z)$, of the particle refers to the vector drawn from the origin to the particle.

In Figure 3.2, the position vector of A is $\mathbf{r}_A = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $\mathbf{i}$ is the unit vector along the x axis, $\mathbf{j}$ the unit vector along the y axis and $\mathbf{k}$ is the one along the z axis.

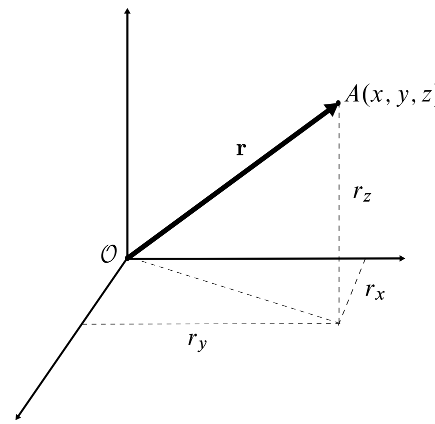


Figure 3.2: A position vector $\mathbf{r}_A = (x, y, z)$

As you may notice, this is dependent on our origin. The position vector is not quite a true vector, it's more so a relative quantity. Anyway, it is still plenty useful, if we just remember to adjust it when switching to a different reference frame.

The other thing of importance is *Displacement*. The *Displacement*, $\mathbf{s}_{AB}$ from A to B, is $\mathbf{s}_{AB} \equiv \mathbf{r}_B - \mathbf{r}_A$.

We need two points to define the displacement. Also, unlike position, displacement is not changed by using a rotated or translated reference frame.

Note that displacement is not the property of the path, which is the *distance*. For a path of the form $y(x)$, to calculate the distance between A and B, we need to calculate $d\ell = \sqrt{dx^2 + dy^2} = \sqrt{(dy/dx)^2 + 1}\, dx$ which is the *arc length*. $d\ell$ is just the infinitesmal length of the path, which by pythagoras is equal to $\sqrt{dy^2 + dx^2}$.

Doing this integral, we get,

$$\ell = \int_A^B \sqrt{(y'(x))^2 + 1}\, dx. \tag{3.1}$$

This is the distance. For euclidian geometry, it is also obvious that since displacement is the line joining A and B, the magnitude of displacement is lower than the length of any path joining A and B. Thus, $\|\mathbf{s}\| \leqslant \ell$.

Velocity And Acceleration

And now we can start by considering the quantities we have heard about frequently in our daily life, velocity and acceleration. Velocity refers to how the position vector changes with time, while acceleration refers to how velocity changes with time. So, acceleration is the second derivative of position.

You might wonder why we don't consider higher derivatives, and the answer to that lies in the fact that newton's laws give us a second order differential equation, so the only quantities of concern are till the second derivative of position.

Let's now formally define these quantities.

Definition 3.4

Velocity is defined as the time rate of change of position, i.e. $\mathbf{v} \equiv \dot{\mathbf{r}}$.

Acceleration is defined as the time rate of change of velocity, i.e. $\mathbf{a} \equiv \dot{\mathbf{v}} = \ddot{\mathbf{r}}$.

$\dot{x}$ refers to the derivative of x with respect to time. $\ddot{x}$ refers to the second derivative of x with respect to time. '≡' stands for defined as.

Remark 3.1.1. Some quick remarks. Position is a function of time, and to find said function is the aim of mechanics. We call the function the *trajectory* of the particle. There are some constraints on the function, for example it must have a continuous first, second derivative.

Similar, velocity and accelerations are also functions of time. Notably as you can see from their definitions, they're defined at a particular time, the derivative evaluated at some time, t. This is different from things like *average velocity*, which is a property of motion in a finite time interval, explicitly,

$$\mathbf{v}_{\text{avg}} = \bar{\mathbf{v}} \equiv \frac{\mathbf{r}(t_2) - \mathbf{r}(t_1)}{t_2 - t_1}. \tag{3.2}$$

And in the limit $t_2 \to t_1$, this becomes the velocity. The case for acceleration is the same.

Speed at a time t is the magnitude of velocity at that time. It is always greater than or equal to 0. In the case of average speed, it is defined as the *distance* the particle travels over the time interval in which it travels that distance.

When the time interval goes to 0, the distance becomes the displacement, or atleast the magnitude of it. Thus, speed is the magnitude of velocity at that point. Also, don't confuse between $\| d\mathbf{r}/\,dt \,\|$ and $d\|\mathbf{r}\|/\,dt$, the first is speed, the other is a quantity not at all useful.

The velocity is a map,

$$\mathbf{v} \colon \mathbb{R} \to \mathbb{R}^3$$

similar to position, which maps time to its three components,

$$t \mapsto (v_x(t), v_y(t), v_z(t)).$$

Based on the definition of acceleration,

$$\int d\mathbf{v} = \int \mathbf{a}\, dt \implies \mathbf{v} = \mathbf{v}_0 + \int \mathbf{a}\, dt. \tag{3.3}$$

Similarly,

$$\mathbf{r} = \mathbf{r}_0 + \int \mathbf{v}\, dt. \tag{3.4}$$

If acceleration is a function of position, which it often is, we can use a neat trick, at least when every thing is one dimension,

$$a = \frac{dv}{dt} = \frac{dv}{dt}\frac{dx}{dx} = \frac{dv}{dx}\frac{dx}{dt} \implies a = v\frac{dv}{dx}.$$

For constant acceleration, this gives us, $v^2 = v_0^2 + 2a\Delta x$, which is also the statement of work-energy theorem, which we'll encounter later.

You can generalise this to any function, f,

$$f''(x) = f'\frac{df'}{df}. \tag{3.5}$$

Example 3.1

Consider the motion of an object dropped from a height h, under drag. The acceleration due to gravity is $\mathbf{g} = -g\hat{\mathbf{j}}$ and due to drag is of the form $-\alpha v\hat{\mathbf{v}}$. This is called linear drag. Since the object is falling down, $\hat{\mathbf{v}} = -\hat{\mathbf{j}}$.

Based on the definition of acceleration,

$$\frac{dv}{dt} = -g + \alpha v \iff \int_0^{-v_f} \frac{dv}{\alpha v - g} = \int_0^t dt'$$

where we set the final velocity to $-v_f$ since its negative, and rename t to t' since we'll use it as our dummy variable, and t as the limit of integration.

This can be solved using a simple u-sub, and is left as an exercise to the reader. The final solution is,

$$v_f = \frac{g}{\alpha}(e^{\alpha t} - 1).$$

Example 3.2

A car is at distance d from a boy. It starts accelerating at constant acceleration a. What is the minimum velocity that the boy should have to catch up with the car?

Solution. Consider separation of boy and car, Δs. Integrating twice for constant acceleration, we have $s_b = vt$, $s_c = \frac{1}{2}at^2$. Thus,

$$\Delta s = d + \frac{1}{2}at^2 - vt$$

So when they meet, we have a quadratic in t,

$$at^2 - 2vt + 2d = 0$$

From an inspection of the co-efficients of the t, t^2 terms and the constant, we see that if a real solution to this exists, it must be positive (if this may not be apparent, recall Vieta's relation and note a, v, d are all positive).

So they must always meet if this has a real solution. Therefore, $b^2 - 4ac \geq 0$ for the equation $at^2 + bt + c$. Solving for this, we have:

$$v \geq \sqrt{2ad}$$

Alternatively, moving into the frame of the boy (we discuss the later, so you can skip it for now), $v_f^2 = v^2 - 2ad$. Since $v_f^2 \geqslant 0$, $v^2 \geqslant 2ad$. □

Example 3.3

A body is dropped at $t = 0$, after time $t = t_0$, another body is thrown downwards with velocity u. Assuming first body reaches ground first, plot graph of separation.

Solution. At instant t_0, displacement of first particle = $gt_0^2/2$. Note that here we set up co-ordinates such that positive y is downwards from point of drop. The displacement of first body at time t after t_0 but before *reaching ground* is

$$s_1 = \frac{1}{2}gt_0^2 + \frac{1}{2}g(t - t_0)^2$$

While for second body is,

$$s_2 = ut + \frac{1}{2}g(t - t_0)^2$$

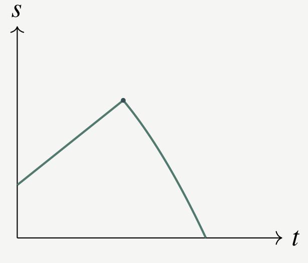
Thus,

$$s_{1,2} = \frac{1}{2}g(t_0)^2 + t(gt_0 - u)$$

However, after first body reaches ground,

$$s_{1,2} = ut + \frac{1}{2}g(t)^2$$

which is a parabola. The overall graph is:



□

3.2 Projectile Motion

Projectile Motion is a very famous setup in physics, and especially kinematic problems. Though the setup is not extremely physically interesting, there's a number of things you can learn from it. We'll cover a lot of cool tricks through problems, but let's first explore what this setup even is.

When an object, called a *projectile*, is thrown with some velocity v making some angle with the horizontal, θ, from co-ordinates (X, Y), under the effect of gravity, the motion is called projectile motion.

There are a no. of things in this setup that can teach you a fair bit. For one, this is our first actual two dimensional problem. We have two free co-ordinates, x and y. First of all, Let's choose our co-ordinate system so that $X = 0$. We won't choose $Y = 0$ because that corresponds to a real thing, namely the ground. It is more convenient to just leave it be.

First, let's discuss the special case $Y = 0$, that is, projectile is released from ground. This setup should be enough for you to be able to generalise it to $Y = h$. Anyway, gravity acts in the negative y direction, $\mathbf{g} = -g\mathbf{j}$, where $g = 9.81$ m/s. How do we proceed now? Well, we know $\mathbf{a}(t)$ so we should probably try to find $\mathbf{v}(t)$. Let's try that,

$$\dot{\mathbf{v}} = \mathbf{g} \implies \mathbf{v}(t) = \mathbf{v}_0 + \mathbf{g}t,$$

Where we use $\mathbf{v}(0) = \mathbf{v}_0$, the initial velocity as the limit of integration.

$$\dot{\mathbf{r}} = \mathbf{v}_0 + \mathbf{g}t \implies \mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2.$$

Since $\mathbf{r}_0 = (X, Y) = (0, 0) = \mathbf{0}$, we just have,

$$\mathbf{r}(t) = \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2.$$

Now here's where the neat part begins. As a general rule of thumb, it's always harder to deal with vectors than it is with scalars, so we will see various methods of trying to convert a vector problem into a scalar problem, and then re-converting if needed. In this case, we make use of a very important, but almost trivial looking identity,

$$\text{if } a\mathbf{i} + b\mathbf{j} = c\mathbf{i} + d\mathbf{j}, \text{then } a = c, b = d.$$

How do we prove this? Well lets move things around a little,

$$(a - c)\mathbf{i} = (d - b)\mathbf{j}$$

Oh, but this cannot be true since $\mathbf{i}, \mathbf{j}$ are the basis for $\mathbb{R}^2$ so they're linearly independent. Thus, either $a = b$, or $c = d$. Now you can complete the proof by using the uniqueness of 0. A fancier way to do this is just dot both sides with $\mathbf{i}$ and then $\mathbf{j}$, I'll leave that proof to you.

Now, if $\mathbf{v}_0$ makes an angle θ with horizontal,

$$\mathbf{v}_0 = v_0 \cos\theta \mathbf{i} + v_0 \sin\theta \mathbf{j},$$

And plugging this into the vector equation for $\mathbf{v}(t)$, and using $\mathbf{v}(t) = v_x\mathbf{i} + v_y\mathbf{j}$ we get,

$$v_x = v_0 \cos\theta, \quad v_y = v_0 \sin\theta - gt \tag{3.6}$$

And using the vector equation for $\mathbf{r}(t) = x\mathbf{i} + y\mathbf{j}$,

$$x = v_0 t \cos\theta, \quad y = v_0 t \sin\theta - \frac{1}{2}gt^2. \tag{3.7}$$

We have reduced the vector equations to scalar equations! Now this is very easy to deal with, you can derive almost everything by just bashing using these equations. That's of course not always the nicest idea, and we'll learn some tricks to deal with them. Also, you might have notice that for each vector equation, we get two scalar equations. We actually get three, since we live in 3d space, but the components of the vector along $\mathbf{k}$ are just 0, so it doesn't matter.

The last fundamental equation of importance is the *trajectory* equation, how does the motion actually look in 2d space? That is, we need to find $y(x)$. You can do it by using $t = x/v_0 \cos\theta$, and substituting this in the $y(t)$ equation, which you should verify, gives you,

$$y(x) = x \tan\theta - \frac{x^2 g}{2v_0^2 \cos^2\theta}. \tag{3.8}$$

Now we can use this for a bunch of problems. Now that we know the general shape is a parabola, let's try to find some properties of it. As you can see, the parabola opens downards, so it achieves a maximum. What is that maximum? Well, you could try to find where dy/dx is 0, and then see what happens.

A nicer way is to realise that v_y must be 0 at max height. If this were not so, we could have moved to $H_{\max} + \epsilon$ in time $t = \epsilon/v_y$. If the velocity were negative at, and thus just before $H_{\max}$, we could have never reached the max height.

Using this, we get that,

$$H_{\max} = \frac{v_{0y}^2}{2g} = \boxed{\frac{v_0^2 \sin^2\theta}{2g}.} \tag{3.9}$$

What about the range(the max value of x)? We have a maximum, since after $y = 0$ (after $t > 0$ that is), the projectile cannot move further, because it has encountered the ground.

Well, we can use the equation for y to find t, or just note that the parabola is symmetric about $y = H_{\max}$, so $t_{\text{total}} = t_H$ where t_H is the time to reach $H_{\max}$, or equivalently, when $v_y = 0$, which you can find easily. All of them give the same time, which when plug into the equation for x, we get

$$\boxed{R = \frac{v_0^2 \sin 2\theta}{g}}. \tag{3.10}$$

Using this, we can also alternatively write the equation of trajectory as,

$$y = x \tan\theta \left(1 - \frac{x}{R}\right) \tag{3.11}$$

What is the maximum value of R, the range, for a fixed velocity? Think about what we can vary. Well, θ of course, and its easy to see the maximum occurs for $\theta = \pi/4$, so we *optimise* the range, and say its maximum at $\theta = \pi/4$.

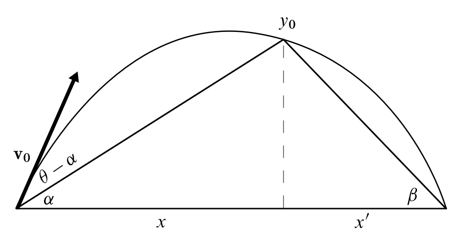


Figure 3.3: Projectile grazing a wedge

Example 3.4

A projectile is thrown from one vertex of a wedge grazes the other vertex, and ends up on the third vertex. It is thrown with an initial velocity v_0 making an angle θ with the horizontal. If the angles of the vertices are α and β, figure out a relation between $\tan\theta$, $\tan\alpha$ and $\tan\beta$.

Solution. Divide the range of the projectile into two lengths, x and x' such that $x + x' = R$. Now, clearly,

$$\tan\alpha = \frac{y_0}{x} \qquad \tan\beta = \frac{y_0}{x'}$$

Also, by the alternate equation of trajectory,

$$y_0 = x \tan\theta \left(1 - \frac{x}{R}\right) = x \tan\theta \left(1 - \frac{x}{x + x'}\right)$$

Which gives us,

$$\tan\theta = y_0 \left(\frac{x + x'}{xx'}\right) = \frac{y_0}{x} + \frac{y_0}{x'}$$

Finally substituting the values, we get,

$$\tan\theta = \tan\alpha + \tan\beta$$

□

There are two important properties of projectile motion. One is more general, and that is that for classical mechanics, motion is time-reversible. That is, if we do the transformation $t \to -t$, the motion doesn't change. The idea is that the acceleration remains invariant, and the velocity reverses, so the trajectory is the same, for forces of the form $F(x)$. We will say more about this later, when newton's laws are introduced, but till then you can use it as a fact.

The second is that the time to attain some vertical displacement can be found in terms of the maximum height alone, in fact the motion is independent of range.

Consider the vertical position of the particle to be s at some time t. Then,

$$s = u_y t - \frac{1}{2} g t^2$$

Also, note that if the maximum height is h,

$$u_y = \pm\sqrt{2gh}$$

Where the positive value is during the ascent and the negative is during the descent. Then,

$$s = \pm\sqrt{2gh}t - \frac{1}{2} g t^2$$

Solving for t gives us,

$$t = \pm\sqrt{\frac{2h}{g}} \pm \sqrt{\frac{2(h-s)}{g}} \tag{3.12}$$

If the initial velocity is positive and $s > 0$,

$$t = \sqrt{\frac{2h}{g}} \pm \sqrt{\frac{2(h-s)}{g}} \tag{3.13}$$

the two values correspond to the fact that the particle will have the same vertical displacement at two instances of time.

If the initial velocity is positive but $s < 0$, i.e., the particle starts at some height but then transverses below it,

$$t = \sqrt{\frac{2h}{g}} + \sqrt{\frac{2(h-s)}{g}} \tag{3.14}$$

is the only reasonable solution.

If the initial velocity is negative, then clearly the particle will attain a lower position. The maximum height in this case is simply the initial height of the particle.

$$t = -\sqrt{\frac{2h}{g}} + \sqrt{\frac{2(h-s)}{g}} \tag{3.15}$$

Thus qualitatively, projectile motion is independent of its horizontal range. If some projectiles have the same maximum height, their time of flight *will be the same*.

Exercises

3.2.1. *Projectile with Vectors.* [2]

We can integrate the kinematic equations vectorially, to get

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{g}t,$$

and for the displacement,

$$\mathbf{s}(t) = \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2.$$

(a) Show that these two equations are, in fact true. Justify our integration of a vector when discussing velocity and acceleration. *Hint: How can we integrate a vector? Try to use its components.*

(b) Note that the vectors $\mathbf{v}_0 t$, $\mathbf{g}t^2/2$ are parallel to vector $\mathbf{v}_0$ and $\mathbf{g}$ respectively. Use this to find the range and time of flight. *Hint: You will find Figure 3.4 useful.*

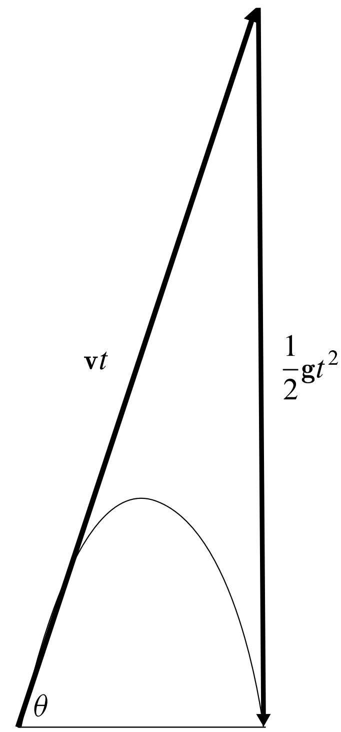


Figure 3.4: Projectile with Vectors

3.2.2. *Projectile Motion in tilted axes.* [2]

Consider the case of a projectile launched along tilted ground (or a wedge), as in, fig. 3.7. If we were to figure out d, we could use a number of ways. The equation of the projectile is,

$$y = x\tan(\theta+\alpha) - \frac{gx^2}{2v_0^2\cos^2(\theta+\alpha)}$$

and of the wedge is,

$$y = x\tan\theta$$

We could simply equate them to figure out the distance d, which is the range of the projectile. However, let us look at alternate way of doing this, using tilted axis. This is the first time we're using the *transformation* of our reference frame. There are many more to come after this.

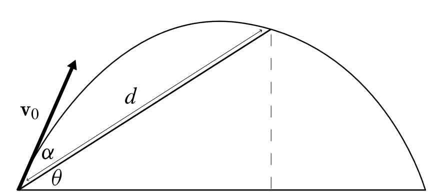


Figure 3.5: Projectile on tilted ground

(a) Rotate the axes by the angle of inclination of the wedge, θ, so that the x axis lies along the plane, and y axis perpendicular to it. Now, find the kinematic equations along the two axes.

(b) Solve these to find the time at which the projectile hits the wedge, and the distance it travels.

3.2.3. *Projectile with drag, Morin* [4]

A ball is thrown with speed v_0 at an angle θ. Let the drag acceleration from the air take the form $a = -\alpha\mathbf{v}$.

(a) Find $x(t)$ and $y(t)$.

(b) Assume that the drag coefficient takes the value that makes the magnitude of the initial drag acceleration equal to g. If your goal is to have x be as large as possible when y achieves its maximum value (you don't care what this maximum value actually is), show that θ should satisfy $\sin\theta = (\sqrt{5} - 1)/2$, which just happens to be the inverse of the golden ratio.

3.3 Polar Co-ordinates

The polar co-ordinate axes are the two-dimensional subset of the 3-d cylindrical co-ordinates. Any point in polar co-ordinates is depicted by a system of two unit vectors, $\mathbf{e}_r$ and $\mathbf{e}_\theta$. However, each of these are dependent on the position of the particle and may be written as $\mathbf{e}_r(\theta)$ and $\mathbf{e}_\theta(\theta)$.

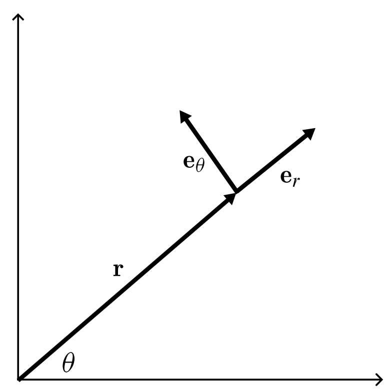


Figure 3.6: Polar-coordinates

Here, $\mathbf{e}_r$ points in the direction of increasing radius (along the radial vector), and $\mathbf{e}_\theta$ points in the direction of increasing θ (tangent to the radial vector). These two unit vectors are *orthogonal* at any point.

In Figure 3.6, if $\mathbf{e}_r$ makes an angle θ with the horizontal, then,

$$\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \tag{3.16a}$$

$$\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \tag{3.16b}$$

Which follows by adding projections of $\mathbf{i}$ and $\mathbf{j}$ along the radial and tangential directions.

Now let us formulate kinematics in polar co-ordinates. The position vector $\mathbf{r}$ can be written as

$$\mathbf{r} = r\mathbf{e}_r \tag{3.17}$$

Velocity follows normally,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt}$$

Or does it? What even is $d\mathbf{e}_r/dt$? The answer lies in the time-derivative of the unit vectors. Using the definitions in Equation (3.16), we get:

$$\frac{d\mathbf{e}_r}{dt} = \dot{\theta}\mathbf{e}_\theta \tag{3.18a}$$

$$\frac{d\mathbf{e}_\theta}{dt} = -\dot{\theta}\mathbf{e}_r \tag{3.18b}$$

Using these results, we get that

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \tag{3.19}$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta \tag{3.20}$$

The most interesting result is the various terms of acceleration in Equation (3.20). First, let us consider the terms along $\mathbf{e}_r$. The term $\ddot{r}$ is the *radial acceleration*. It is the change in the radial speed (and thus the radius). The second term $-r\dot{\theta}^2$ is the *centripetal acceleration*. This term accounts for change in direction of tangential velocity and is responsible for the particle's curved path.

Now consider the terms along $\mathbf{e}_\theta$. The term $r\ddot{\theta}$ is the *tangential acceleration* due to change in tangential speed:

$$a_t = r\ddot{\theta} = \dot{v}$$

The other term $2\dot{r}\dot{\theta}$ is the *coriolis acceleration*, which we'll discuss with rotating frames.

Example 3.5

The movement of a particle in a circle about some point is called circular motion. If we move to the origin, r does not change with time. Thus, $\dot{r} = \ddot{r} = 0$. This gives us $v = \dot{\theta}r$. $\dot{\theta}$ is often refered to as the *Angular Speed*, ω.

We also find that $a_r = -m\omega^2 r$ and $a_\theta = r\ddot{\theta}$. If $\ddot{\theta} = 0$, then the motion is called uniform circular motion, and the acceleration is perpendicular to the velocity.

3.4 Optimisation Problems

These are a class of math problems with almost no physics involved. The following problem, for instance, is a purely geometrical one. These problems rely on tricks, and can be fun puzzles.

A man who can run at a constant speed stands at a point with coordinates $(4, 5)$ and has to walk to the x-axis, then to the point $(3, 7)$. What path must the man walk along? **Question**

A large subset of these questions can be done purely by calculus, but that is not always the most elegant way to go about it. We'll cover questions of both kinds, sometimes you do need to do the effort and the elegant solution may not be so elegant.

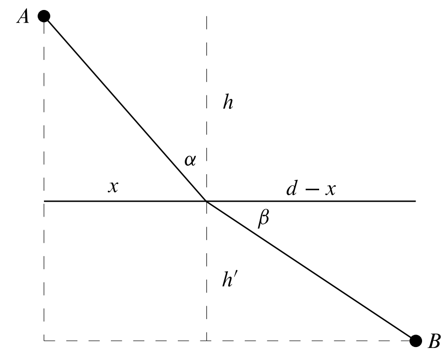


Figure 3.7: One of the possible paths to reach from A to B.

Example 3.6

A man can run at a speed u and swim at a speed v. In the following diagram, the man starts at point A and has to save his friend from drowning at point B. Assuming he likes his friend, what should he do?

Solution. The man should call a lifeguard. Now, let's figure out how the lifeguard should run from point A to B. The time taken to reach the man is:

$$t = \frac{\sqrt{x^2 + h^2}}{u} + \frac{\sqrt{(d - x)^2 + h'^2}}{v}$$

where d is the horizontal distance between A, B, and h, h' is vertical distance of A and B to the shore, and hence independent of the path taken.

The derivative of this with respect to the x has to be 0 for the best path of the lifeguard:

$$\frac{x}{u\sqrt{x^2 + h^2}} = \frac{(d - x)}{v\sqrt{(d - x)^2 + h'^2}}$$

But $\sin\alpha = x/\sqrt{x^2 + h^2}$ and $\sin\beta = (d - x)/\sqrt{(d - x)^2 + h'^2}$, so we have,

$$\boxed{\frac{u}{\sin\alpha} = \frac{v}{\sin\beta}} \tag{3.21}$$

□

This path follows the laws of refraction. In fact, this hints at a more general result for light, that being that it follows the path of least time. This is called Fermat's principle. Of course, we don't care about what light does at this point, but it is a very cool result. Note for one that we paid no attention to which reference frame we were talking about, and it doesn't matter (unless its rotation).

As you may notice, the principle doesn't really depend on what the media or distances are, in fact this is a general result.

A beautiful application of this principle is the Brachistochrone problem, which is given at the end of this section.

Example 3.7

Consider a projectile thrown with a velocity v_0 making an angle α with horizontal, on wedge of inclination θ, which slopes downwards. Find out the angle α such that the projectile has maximum range.

Solution. The locus of the projectile is,

$$y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$$

and of the wedge is,

$$y = -x\tan\theta$$

At the range, they will be equal so,

$$x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha} = -x\tan\theta \implies \tan\alpha + \tan\theta = \frac{gx}{2v_0^2\cos^2\alpha}$$

which gives us the desired range,

$$x = \frac{2v_0^2(\tan\alpha + \tan\theta)\cos^2\alpha}{g}$$

So the condition for maximum range is simply,

$$\frac{dx}{d\alpha} = 0.$$

This also kind of illustrates a nice idea, since we need to get the derivative 0, we can safely discard any multiplicative or additive constants right away which will either have a zero differentiation, or can be divided by since the other side of the equation is simply 0.

We get that,

$$\begin{aligned}
\frac{d}{d\alpha}\left\{(\tan\alpha + \tan\theta)\cos^2\alpha\right\} &= 0\\
\frac{d}{d\alpha}\left\{\frac{\sin 2\alpha}{2} + \tan\theta\cos^2\alpha\right\} &= 0\\
\cos 2\alpha - \tan\theta\sin 2\alpha &= 0\\
\tan 2\alpha &= \cot\theta\\
\tan 2\alpha &= \tan\left(\frac{\pi}{2} - \theta\right)\\
\implies \alpha &= \frac{\pi}{4} - \frac{\theta}{2}
\end{aligned}$$

□

An easier way to find x is to use Exercise 3.2.2., and then we can continue in the same manner.

We'll see this idea in the problems, but when the accelerations or velocities are complicated, but it is often helpful to consider the *velocity space* or the position space, which is just the vector space of all velocities, instead of our usual position space.

Example 3.8

A bug wishes to jump over a cylindrical log of radius R lying on the ground, so that it just grazes the top of the log horizontally as it passes by. What is the minimum launch speed v required to do this?

Solution. First of all, since the the bug grazes the top of the log horziontally, v_y at the top is just 0. Since $v^2 = v_0^2 - 2gh$, this implies that $v_{0,y} = \sqrt{2gh}$. Now we just need to find v_x.

The idea here is of radius of curvature. If v_x is too low, the trajectory of the bug will just pass through the log. If it's too high, our velocity is not minimal. At the most optimal trajectory, the *curvature* of the trajectory is the same as that of the log, so the bug is doing circular motion at the top of the log with radius r. Using polar-cordinates, and $\dot{r} = 0$, we find that $v_x = gR$, since $\mathbf{g}$ points radially at top.

Thus, $v_{\text{min}} = \sqrt{v_y^2 + v_x^2} = \sqrt{5gR}$. □

Exercises

3.4.4. *Projectile from a height* [3]

Consider a projectile thrown from height h with velocity v. What is its maxium horizontal range? There's two ways we will tackle this problem:

(a) Write down the equation of trajectory, but shifted so that $y \to y - h$. At $x = R$, the range, we should have $y = 0$ (or don't change y, but note that projectile motion is time reversible, so $y = h$ at $x = R$). Then take the derivative of it with respect to θ, while noting that for optimal trajectory $dR/d\theta = 0$. Find θ and then R.

(b) Again write down the same shifted equation of trajectory. Now, write down every trignometric function in terms of $\tan\theta$. You should get a quadratic. For such a trajectory to exist, $\tan\theta$ must have real roots. Use this to get R.

3.4.5. *River and Drift* [2]

A swimmer can swim with velocity u in still water. If the river has a velocity v, what is the minimum distance through which she can cross the river? We'll do this in two ways as well.

(a) Let the velocity of the swimmer make an angle θ with the vertical. The distance perpendicular to stream does not change with θ, so we can ignore it. Calculate and minimise then, the distance parallel to the flow of the river.

(b) Draw the *velocity* space of the resultant velocity of u and v. That is, by varying θ, consider how the resultant changes geometrically. Do this for both $v > u$ and $u \geqslant v$.

3.4.6. *Brachistochrone* [4]

Suppose we drop a ball from a point A to point B, vertically h and horizontally d apart along a curve. Find the curve which minimises the time of descent. You might find $v^2 = v_0^2 + 2g\Delta h$ helpful.

3.4.7. *Jumping over Roofs, Kalda* [3]

Two fences of heights h_1 and h_2 are erected on a horizontal plain, so that the tops of the fences are separated by a distance d. Show that the minimum speed needed to throw a projectile over both fences is $\sqrt{g(h_1 + h_2 + d)}$.

3.4.8. *Optimal Projectile* [2]

Find the minimum speed required to the reach the point (X, Y), by a projectile thrown from $(0, 0)$.

3.4.9. *Ping Pong 1, OPhO* [3]

A legal serve in ping-pong requires that the ball bounces on one side of the table and that the ball goes over the net. A certain world-class Olympic ping-pong player does the serve at the level of the table at a distance d from the net of height h.

The Olympic player can give the ball such a spin that the translational speed of the ball is conserved after a bounce but the direction of velocity can be controlled freely. What is the minimal serving speed v?

3.4.10. *Perpendicular Velocities, Morin* [3]

In the maximum-distance case of Example 3.7 show that the initial and final velocities are perpendicular to each other. *Hint: Use the fact that projectile motion is time reversible.*

3.4.11. *Parabolic Envelope, Knzhou* [5]

The boundary of the set of all points a projectile with fixed velocity can reach by varying θ is a parabola, with the focus at launch point. We can use this to geometrically solve many problems.

(a) Show that the trajectory that touches any point on the parabola must be tangent to it.

(b) Show that the velocity at the point of tangency must be perpendicular to the initial one, if we reach the point on the parabola with minimum velocity. Use this to solve the previous problem.

(c) Find the optimal angle in Example 3.7 and solve Exercise 3.4.7. using the envelope and (a), (b).

3.4.12. *Parabolic Envelope* [3]

Justify the claim that the boundary of boundary of the set of all points a projectile with fixed velocity can reach by varying θ is a parabola using the the second method of Exercise 3.4.4..

3.4.13. *Projectiles with Vector, V2.* [3]

When a projectile thrown along a wedge hits it, the sum of the scaled velocity and acceleration vector must equal the vector along the slope with the magnitude equal to the distance the projectile travels along the wedge.

Using this, the vector equations for $\mathbf{v}$, $\mathbf{r}$ and part (b) of the previous problem, show that $\mathbf{v}_0$ lies along the line joining focus and projection of point at which it hits the wedge on the directrix, and find the angle using some angle chase.

Do not use the reflexive property of parabola, in fact, this gives you a proof for it.

3.4.14. *Tired Flappy Bird, OPhO* [4]

A flappy bird can jump multiple times in the air. Each time it jumps mid-air, it can suddenly change its speed and direction. For every jump, the bird can decide when to jump and in which direction. Between jumps, the bird falls freely under gravity, which pulls it down at the acceleration g.

Say, our tired flappy bird starts off the cliff of height H with the jumping velocity $V[1] = V_0$. Subsequent jumps in mid-air have decreasing velocities, i.e. the n-th jump has speed $V[n] = V_0/n$, $(n > 1)$. This majestic Vietnamese animal wants to travel as far as possible horizontally before it lands on the ground. Find the maximum horizontal distance the bird can travel.

3.5 Frames and Symmetry

Often in a mechanics problem, switching frames proves to be very beneficial. It is especially useful if we have two moving objects, and we move to the frame of one, or move to a frame that offers high symmetry. Suppose we move to a frame with position vector $\mathbf{R}$ with respect to our original frame. Call the original frame S and the current one S'. If the position vector of an object in S os $\mathbf{r}$ and in S' is $\mathbf{r}'$, then,

$$\mathbf{r} = \mathbf{r}' + \mathbf{R} \tag{3.22}$$

The derivative of this gives

$$\mathbf{v} = \mathbf{V} + \mathbf{v}' \tag{3.23}$$

where $\dot{\mathbf{R}} = \mathbf{V}$ is the velocity of S' with respect to S.

We can go further, describing how accelerations are related, but that's not quite useful for kinematics, and is a whole chapter in itself. So we'll leave that for now. Also note that S' is not rotating.

Exercises

3.5.15. *Symmetrical Circles, Kalda* [2]

Two intersecting circles of radius r have centers a distance a apart. If one circle moves towards the other with speed v, what is the speed of one of the points of intersection?

3.5.16. *Rotating Circles, Kalda* [3]

Two circles of radius r intersect at the point O. One of the circles rotates about the point O with constant angular speed ω. The other point of intersection O' is originally a distance d from O. Find the speed of O' as a function of time.

A point rotating with angular speed ω about O just means in the frame in which O is the origin, and in polar co-ordinates, $\dot{\theta} = \omega$.

3.6 Invariants

Invariants are universal across physics. These give us *conservation laws*, many of which we'll encounter in the later chapters.

An *Invariant* is a quantity that remains constant throughout time. Identifying an invariant in a problem is very helpful, and can help us solve some seemingly impossible problems.

Example 3.9

A cat moves with speed v always directed towards a rat moving rectilinearly with speed v. They are initially at a distance ℓ from each other, and their initial velocities are perpendicular. After a long time they move along the same line, seperated by a distance d. Find d

Solution. Solving this problem using the techniques we have already discussed is incredibly hard. Whenever you do no know what to do, writing all the equations governing the system isn't a bad idea.

As I mentioned in the previous section, it might be good to move into the frame of one of them. Let's move into the frame of the fox. Let's orient our axis so that the seperation initially was along the y axis, and thus the cat moves with $v\mathbf{i}$.

Consider the system after time t, as in Figure 3.8. We don't really say anything nice in cartesian co-ordinates alone. But note that the velocity of the rat (aside from the horizontal one) because of switching the frames must point radially. So let's try out polar-coordinates.

The velocity along the x axis is $v - v\cos\theta$. The velocity along the radial direction is $v\cos\theta - v$. Thus,

$$\frac{dx}{dt} + \frac{dr}{dt} = v - v\cos\theta + v\cos\theta - v = 0 \implies \frac{d(x+r)}{dt} = 0.$$

So $x + r$ is a constant, independent of time. At $t = 0$, r is just ℓ and $x = 0$. After a long time, $x = r$. So, $d = r = \ell/2$. □

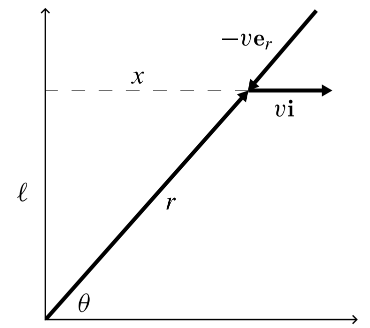


Figure 3.8: Velocities and distances in the frame of cat

It seems almost impossible to solve the previous problem without invariants, and in fact trying to solve without invariants is very difficult. You can look at this paper for an extensive discussion on the problem.

We'll only do one problem for this here, since most would require some knowledge of dynamics. But the idea is very important, and I urge you to keep it in mind.

Exercises

3.6.17. *Chasing Problem* [2]

Suppose the cat in the previous example has velocity $u > v$. How long does it take to catch the rabbit?