1 a) $PDF(x) = f(x) = \frac{d}{dx} F(x) = \frac{d}{dx}\left(\frac{x}{8}\right) = f(x) = \frac{1}{8}$

$0 \leq x \leq 8$

b) i. $P(x=2) = f(2) = \frac{1}{8}$

b) ii. $P(x<2)$ and $P(x \leq 2)$ ford continues random variable, $P(x<2) = P(x \leq 2)$

$P(x<2) = \frac{2}{8} = \frac{1}{4} = .25$

c) $P(x \leq m) = \frac{1}{2}$

$\int_0^m f(x)\,dx = \frac{1}{2}$ $\quad \int_0^m \left(\frac{1}{8}\right)dx = \frac{1}{8}[x]_0^m = \frac{1}{2}$ $\quad \frac{m}{8} = \frac{1}{2}$

$m = 4$ $\quad m = \frac{8}{2} = 4$

d) $P(x \geq 4) = 1 - P(x < 4)$

$1 - F(4)$

$1 - .5$

$\frac{1}{2}$

2. a) ~~note~~ Let $d_X(x)$ denote PDF of x:

$$d_X(x) = \frac{d}{dx} F_X(x) = -e^{-\left(\frac{x}{\beta}\right)^\alpha} \cdot \frac{-1}{\beta^\alpha} \alpha x^{\alpha-1}$$

$$d_X(x) = \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^\alpha} \qquad x \geq 0$$

b) Q_1 defined such that $P(X \leq Q_1) = .25$

$$1 - e^{-\left(\frac{Q_1}{\beta}\right)^\alpha} = .25 \rightarrow e^{-\left(\frac{Q_1}{\beta}\right)^\alpha} = .75 \rightarrow -\left(\frac{Q_1}{\beta}\right)^\alpha = \ln(.75)$$

$$Q_1 = -\beta\left(\ln(.75)\right)^{\frac{1}{\alpha}}$$

c) $P(x<2)$

$$\int_0^2 \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^\alpha} dx$$

by following above integrating we get

$$-e^{-\frac{x^\alpha}{\beta^\alpha}}\Big]_0^2$$

$$1 - e^{-\frac{2^\alpha}{\beta^\alpha}}$$

$P(x>4) = 1 - P(x<4)$

$$P(X>4) = \int_0^4 \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^\alpha} dx$$

$$= -e^{-\left(\frac{x^\alpha}{\beta}\right)}\Big]_0^4$$

$$= 1 - e^{-\frac{4^\alpha}{\beta^\alpha}}$$

$$1 - \left[1 - e^{-\frac{4^\alpha}{\beta^\alpha}}\right] = e^{-\frac{4^\alpha}{\beta^\alpha}}$$

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3. a) $y = x_1 + x_2 + 2x_3$

By addition property of normal distribution

$\sum a_i \mu_i \sim N(\sum a_i \mu_i, \sum a_i^2, \sigma^2$

$y = x_1 + x_2 + 2x_3 \sim N(\mu_1 + \mu_2 + 2\mu_3, \sigma_1^2 + \sigma_2^2 + 2^2\sigma_3^2)$

$y \sim N((5+5+2\cdot5), 4+4+2^2\cdot4)$

$y \sim N(20, 24)$

$P(y \geq 20) = P\left(\frac{x-\mu}{\sigma} \geq \frac{20-\mu}{\sigma}\right)$

$P(Z \geq \frac{20-20}{6})$ $P(Z \geq 0$

Since, Z is standard normal random variable

$P(Z \leq 0)$ $P(Z \geq 0)$

$P(Z \geq 0) = .5$ via Z-Table

$P(x_1 + x_2 + 2x_3 \geq 20) = .5$

3b. $P(x \geq 5)$

$\bar{x} = \frac{x_1 + x_2 + x_3}{3} \sim N\left(\frac{\mu_1 + \mu_2 + \mu_3}{3}, \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{3^2}\right)$

$\bar{x} = \bar{x} \sim N(5, 4/3)$ $P(\bar{x} \geq 5) = P\left(\frac{\bar{x}-\mu}{\sigma} \geq \frac{5-\mu}{\sigma}\right)$

$= P(Z \geq \frac{5-5}{4/3}) = P(Z \geq 0) = .5$

3c. $P(\bar{X} \geq C) = 0.2$

$= \bar{X} \sim N(5, 4/3)$

$P\left(\frac{\bar{X}-M}{6} \geq \frac{C-M}{6}\right) = .2$

$P\left(Z \geq \frac{C-M}{6}\right) = .2 \quad P\left(Z \leq -\frac{C-M}{6}\right) = .2$

$P\left(\frac{x-M}{6} \geq \frac{C-M}{6}\right) = .2$

$\frac{C-M}{6} = .84$

$C = M + (6)(.84)$

$5 + \frac{4}{3} \cdot 0.84 = 6.12$

From statistical / table

4. $X \sim$ Exponential $\lambda = \frac{1}{4}$

a) $f(x) = \lambda e^{-\lambda x}$

$f(x) = \begin{cases} \frac{1}{4} e^{-\frac{x}{4}} & x > 0 \\ 0 & x \leq 0 \end{cases}$

4b)i. Variance $= \frac{1}{\lambda^2} = \frac{1}{\left(\frac{1}{4}\right)^2} = 16$

4bii) $P(x \geq 8) = e^{-\lambda x} = e^{-\frac{1}{4} \times 8} = 0.1353$

4c) $P(3 \leq x \leq 7) = P(x \leq 7) - P(x \leq 3)$

$\left\{1 - e^{-\frac{1}{4} \times 7}\right\} - \left\{1 - e^{-\frac{1}{4} \times 3}\right\} = .2986$

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5. Let $\alpha \sim \exp(\lambda)$, then $F(\alpha) = P(\alpha \leq x) = 1 - e^{-\lambda \alpha}$

$P[X > x] = 1 - P[X \leq x] = e^{-\lambda \alpha}$

then. $\frac{P[\alpha > d+b]}{P[\alpha > d]} \Rightarrow \frac{e^{-\lambda(d+b)}}{e^{-\lambda a}} \Rightarrow e^{-\lambda b} \Rightarrow P[x > b)$

$\Rightarrow \frac{P(x > (d+b))}{P(\alpha > d)} = P(x > b)$

---

6. Given that $x_i | i = 1, 2, 3 \ldots n)$ are indept. random sample from $U(0, \theta)$ i.e

$x_i \overset{iid}{\sim} U(0, \theta)$

we know from even distrobution $E(\alpha) = \frac{\theta + 0}{2} = \frac{\theta}{2}$

$E(2x) = \theta$

a) $E(2X_1) = 2E(\alpha_1) = 2\frac{\theta}{2} = \theta$

$E[2\bar{x}] = 2E(\bar{x}) = 2E[\frac{\sum \alpha_i}{n}] = \frac{2\sum E(x_i)}{n}$

$\frac{2\sum\theta}{2n} = \frac{2n\theta}{2n} = E(2\bar{x}) = \theta$ both

$2x_1$ and $2\bar{x}$ are unbiased estimators for θ

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6b) since on uniform $(0,\theta)$; $V(X) = \frac{(\theta-0)^2}{12} = \frac{\theta^2}{12}$
and here $2x$ and $2\bar{x}$ are unbiased estimators for θ, so MSE = Variance

$$MSE(2x_1) = V(2x_1) = 2^2 V(x) = 4\frac{\theta^2}{12} = \frac{\theta^2}{3}$$

$$MSE(2\bar{x}) = V(2\bar{x}) = 2^2 V(\bar{x}) = 4V(\bar{x}) = 4V\left(\sum \frac{x_i}{n}\right)$$

$$\Rightarrow 4V\left(\sum \frac{x_i}{n}\right) = 4\frac{1}{n^2}\sum V(x_i) = 4\frac{1}{n^2}\sum \frac{\theta^2}{12}$$

$$\Rightarrow \frac{4\theta^2}{12n} = \frac{\theta}{3n}$$

so $mse(2\bar{x}) = MSE(2x_i)$

Hence $MSE(2\bar{x}) < MSE(2x_i)$

So $2\bar{x}$ is better than $2x_i$

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7) $x_1 + x_2 + \dots x_n \sim \text{Poisson}(\lambda)$

pmf $P(X=x) = \frac{e^{-\lambda}\lambda^{x_i}}{x_i}$

a) For Poisson(λ), $E(x) = \lambda$ method of moment estim equates the sample mean with population mean.

$\therefore \frac{1}{n}\sum_{i=1}^{n} x_i = \lambda$ $\quad$ $\left(MOM(\lambda) = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}\right)$

7b) likelihood function of λ

$$L(\lambda) = \prod_{i=1}^{n} f(x_i = f(x_i) \cdot f(x_2) \cdots\cdots f(x_n)$$

$$= \frac{e^{-\lambda}\lambda^{x_i}}{x_1!} \cdot \frac{e^{-\lambda}\lambda^{2}}{x_2!} \cdots\cdots \cdot \frac{e^{-\lambda}\lambda^{x_n}}{x_n!}$$

$$L(\lambda) = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{x_1! \cdot x_2! \cdots \cdot x_n!}$$

7c) Likelyhood function of $L(\lambda)$

$$\text{Log } L(\lambda) = \text{Log}\left(\frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{x_1! \cdot x_2! \cdots x_n!}\right)$$

$$= -n\lambda + \sum_{i=1}^{n} x_i \log(\lambda) - \text{Log}(x_1! \cdot x_2! \cdots x_n!)$$

$$= \frac{\partial}{\partial\lambda} \log(L(\lambda)) = 0$$

$$-n + \frac{\sum_{i=1}^{n} x_i}{\lambda} = 0 \Rightarrow \frac{\sum_{i=1}^{n} x_i}{n} = \lambda$$

$$\therefore \hat{\lambda}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$$

Using invariance property of MLE of λ is $\bar{x}$ then MLE of $g(\lambda)$ is $g(\bar{x})$

$$\Rightarrow \text{MLE}(\lambda^2) = \bar{x}^2$$

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8. a) MOM estimator is obtained by equating sample moments and population moments

$$\frac{1}{n}\sum x_i = E(x) = \frac{\theta}{2} \qquad f(x)\begin{cases} \frac{1}{\theta} & 0<x<\theta \\ 0, & \text{ow} \end{cases}$$

$$\bar{x} = \frac{\theta}{2}$$

$$\tilde{\theta} = 2\bar{x}$$

$\bar{x}$ is the sample moment

$$E(x) = \int_0^\theta x f(x)\,dx$$

$$= \int_0^\theta x/\theta \,dx$$

$$= \frac{1}{\theta}\cdot\frac{1}{2}(x^2)\Big|_0^\theta$$

$$= \frac{1}{2\theta}\theta^2 = \theta/2$$

MOM estimator of

$$(\theta, \tilde{\theta}) = 2\bar{x}$$

8b) def $T = \max(x_1, x_2, \dots x_n) = X_{(n)}$

PDF of T, $F_T(t) = \frac{nt^{n-1}}{\theta^n}$, $0 \le t \le \theta$

we need to find $E(t)$

$$= \int_t^{\infty} t f(t)\,dt = \int_0^\theta t\frac{nt^{n-1}}{\theta^n}\,dt = \int_0^\theta \frac{n}{\theta^n} t^n\,dt = \frac{n}{\theta^n}\int_0^\theta t^n\,dt$$

$$\Rightarrow \frac{n}{\theta^n}\frac{(t^{n+1})}{n+1}\Big|_0^\theta = \frac{n\theta}{n+1} = E(t) = \frac{n}{n+1}\theta$$

T is not an unbiased estimator of θ

8c) $E(T) = \frac{n}{n+1}\theta$, Let $V = \frac{n+1}{n} T$

Then $E(V) = E\left(\frac{n+1}{n} T\right) = \frac{n+1}{n} E(T) = \frac{n+1}{n} \frac{n}{n+1}\theta$

$\Rightarrow EV = \theta$ and $E\left(\frac{n+1}{n} T\right) = \theta$ Therefore,

$\frac{n+1}{n} T$ is an unbiased estimator of θ.

---

9) a) $X_i \sim n(\mu, \sigma^2)$

$E(X_i) = \mu \quad V(X_i) = \sigma^2$

① $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) \Rightarrow \frac{1}{n} \times \sum_{i=1}^{n} \mu \Rightarrow \frac{1}{\cancel{n}} \times \cancel{n}\mu$

$E(\bar{X}) = \mu$ and $var(\bar{X}) = V\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$

$\frac{1}{n^2}\left(\sum_{i=1}^{n} V(X_i)\right) = \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 \quad V(\bar{X}) = \frac{\sigma^2}{n}$

Hence $\bar{X}$ will be normally distributed

$\bar{X} \sim N(\mu, \sigma^2/n)$

9)

iii) Let $T_1 = x_1$, $T_2 = \frac{x_1 + 2x_2}{3}$ & $T_3 = \bar{x}$

① $E(T_1) = E(x_1) = \mu$ — x_1 is unbiased estimator of μ

and $var(T_1) = V(x_1)$

$V(T_1) = \sigma^2$

$T_2 = \frac{x_1 + 2x^2}{3}$ take expectation of both sides

$E(T_2) = E\left(\frac{x_1 + 2x_2}{3}\right) = \frac{1}{3}[E(x_1) + 2E(x_2)]$

$\frac{1}{3}[\mu + 2\mu) = \frac{1}{3}\, 3\mu$

$E(T_2) = \mu \rightarrow T_2$ is also unbiased estimator of μ

$var(T_2) = V\left(\frac{x_1 + 2x_2}{3}\right) \Rightarrow \frac{1}{9}[V(x_1) + 4V(x_2)]$

$\frac{1}{9}[\sigma^2 + 4\sigma^2) = \frac{5\sigma^2}{9}$

$T_3 = \bar{x}$

$E(T_3) = E(\bar{x}) = \mu$ $\quad \bar{x} \sim n(\mu, \sigma^2/n)$

$V(T_3) = \sigma^2/n$ $\quad T_3$ is unbiased estimator of μ

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4 c) T_1, T_2, T_3 are all unbiased estimators of μ, here

$$\text{Bias}(T_1) = E(T_1) - \mu = 0$$
$$\text{Bias}(T_2) = E(T2) - \mu = 0$$
$$\text{Bias}(T_3) = E(T_3) - \mu = 0$$

To find MSE of t_1, T_2, T_3,

$$MSE(t) = \text{var}\, t + \text{bias}(t)^2 = \sigma^2 + 0^2 = \sigma^2$$
$$MSE(T_2) = V(T_2) + \text{bias}(T_2)^2 = \frac{5\sigma^2}{9}$$
$$MSE(T3) = V(T_3) = \frac{\sigma^2}{n}$$

The estimator will be best when its MSE is minimum since n is less

Hence MSE (T_3) will be minimum than T_2 & T_1

T_3 is the best estimator among all

$T_3 = \bar{x}$

10. Given $x_1, x_2, \ldots x_n \sim N(m, \sigma^2)$
by definition of MLE, we can get
The MLE of parameter N and σ^2 as
$\hat{N} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\hat{\sigma} = s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2}$

Invariance property of MLE

If T is MLE for a parameter θ then
there is a function $g(\theta)$ of param. then the
MLE of $g(\theta)$ will be $g(T)$, hence, $\hat{g}(\theta) = g(T)$, this
is called invariance property of MLE

a) here, $g(m, \sigma) = m + \sigma$
then MLE is, $\hat{g}(m, \hat{\sigma}) = \hat{M} + \hat{\sigma} = \bar{x} + s$ ~~fem~~

$g(m, \sigma) = \frac{m}{\sigma}$
MLE is $\hat{g}(\hat{u}, \hat{\sigma}) = \hat{u} / \hat{\sigma} = \frac{\bar{x}}{s}$

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10.b) $P(\bar{x} > 5)$

$$= P\left(\frac{\bar{x}-m}{\sigma/\sqrt{n}} > \frac{5-m}{\sigma/\sqrt{n}}\right)$$

$$= P\left(z > \frac{5-m}{\sigma/\sqrt{n}}\right) \quad [z \sim n(0,1)]$$

$$1 - P\left(z > \frac{5-m}{\sigma/\sqrt{n}}\right) = 1 - \Phi\left(\sqrt{n}\,\frac{5-m}{\sigma}\right)$$

$$\Phi\left(-\sqrt{n}\,\frac{5-m}{\sigma}\right) = \Phi\left(\sqrt{n}\,\frac{m-5}{\sigma}\right)$$

So, $g(m,\sigma) = P(\bar{x} > 5) = \Phi\left(\sqrt{n}\,\frac{m-5}{\sigma}\right)$

MLE of $g(m,\sigma)$ is

$$\hat{g}(\hat{m}, \hat{\sigma}) = \Phi\left(\sqrt{n}\left(\frac{\bar{x}-5}{\hat{\sigma}}\right)\right)$$

This is required MLE