1) a)

A and B are two events with $P(A) > 0$ and $P(B) > 0$ and

$A \cap B \neq 0$

i. $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1$

ii. $P(A \cap B \geq P(A) + P(B) - 1$

iii. $P(A \cup B) \geq P(A \cap B)$

From ii. and iii, we know

iv. $P(A \cup B) \geq P(A) + P(B) - 1$

and from i we know

V. $P(A \cup B) \leq P(A) + P(B)$

From iV and V we know

$P(A) + P(B) - 1 \leq P(A \cup B) \leq P(A) + P(B)$

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1) b) Given A, B are independant

$P(A \cap B) = P(A) \cdot P(B)$

$P(A) = .3$, $P(B) = .4$

$P(A \cup B) =$ unknown

$P(A \cap B) = P(A) \cdot P(B) = (.3) . (.4) = .12$

$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A) \cdot P(B)}{P(A)} = P(B) = .4$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = .3 + .4 - .12$

$P(A \cup B) = .58$

2)a) $P[(A\cup B)'] = P(A'\cap B')$

$= 1 - P(A\cup B)$

$= 1 - [P(A) + P(B) - P(A\cap B)]$

$= 1 - P(A) - P(B) + P(A\cap B)$

$= 1 - P(A) - P(B) + P(A)\cdot P(B)$

$= P(A') - P(B)[1 - P(A)]$

$= P(A') - P(B)\cdot P(A')$

$= P(A')[1 - P(B)]$

$= P(A')\cdot P(B')$

$\therefore P(A'\cap B') = P(A')\cdot P(B')$ therefore, the only A' and B' are independent i.e A and B are independent

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2)b) $P(A) = .8 \quad P(B) = .5$

i) $P(A'\cup B')$

$P(A') + P(B') - P(A'\cap B')$

$P(A') + P(B') - P(A')\cdot P(B')$

$P(A') = 1 - .8 = .2,\ P(B') = 1 - .5 = .5$

$P(A') + P(B') - P(A')\cdot P(B') = .2 + .5 - .2\cdot .5$

$P(A'\cup B') = .6 \qquad .7 - .1 = .6$

ii) $P(A'|B') = \dfrac{P(A'\cap B')}{P(B')} = \dfrac{P(A')\cdot \cancel{P(B')}}{\cancel{P(B')}} = \cancel{P(A')}$

$P(A') = .2$

$\therefore \boxed{P(A'|B') = .2}$

3.) a) .55

$S < \begin{matrix} .45 < \begin{matrix} .7 \\ .3 \end{matrix} \\ .55 < \begin{matrix} .2 \\ .8 \end{matrix} \end{matrix}$

b) $(.45 \cdot .7) + (.55 \cdot .2) = .425$

c) $P(R \mid SAS) = \frac{P(R \cap SAS)}{P(SAS)} = \frac{.315}{.425} = .7412$

4)

$P(\text{red} \mid B_1) = 10$ $P(\text{Green} \mid B_1) = 3$
$P(\text{Red} \mid B2) = 6$ $P(\text{Green} \mid B_2) = 4$

a) $P(\text{red from } B1) = \frac{10}{10+3} = \frac{10}{13} = \boxed{.769}$

b) $P(\text{Red from } B2) = \frac{10}{13} \cdot \frac{7}{11} + \frac{3}{13} \cdot \frac{6}{11} = \frac{88}{143} = .616$

5) a) Distrobution of x is a <u>binomial distrobution</u>
Binomial distrobution describes Prob. of # of success out of a Sample size n, with rate P,
# of S, where S is stop and shop shopper, success rate P = 50%, Sample size is n = 25

b) $n = 25$ $P = .5$ # of SOS shopper = X
$X \sim \text{Binomial}(n = 25, P = .5)$
$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$
we have to find the $P(X \geq 8) = 1 - P(X \leq 7)$
$P(X = x) = \binom{25}{x} .5^x (1 - .5)^{25-x} = \binom{25}{x} .5^{25}$
Hence, $P(X \leq 7) = \sum_{i=0}^{7} P(X = i) = \sum_{i=0}^{7} \binom{25}{i} (.5)^{25} = .5^{25} \sum_{i=0}^{7} \binom{25}{i}$
$.5^{25} (1 + 25 + 300 + 2300 + 12650 + 53130 + 177100 + 480700)$
$= .0216$ where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ Thus $P(X \geq 8) = (1 - .0216) = .9784$

5)c) $P(x<5) = P(x \le 4)$

$P(x \le 4) = \sum_{i=0}^{4} P(X=i) = \sum_{i=0}^{4} \binom{25}{i}(.5)^{25} = .5^{25}(1+25+300+2300+12650)$

$= .0005$ hence $P(\cancel{x<5}) = P(x \le 4) = .0005$

The prob of number of stop and shop shoppers a sample is less than 5 is $.0005$

d) $P(2 \le x \le 9) = \sum_{i=2}^{9} P(x=i) = \sum_{i=2}^{9} \binom{25}{i} .5^{25} = \cancel{\ldots}$

$= .5^{25}(300+2300+12650+53130+177100+480700+1081575+2042975)$

$= .1148$

so the probability of # of SOS shoppers in a sample is between 2 and 9 inclusive is $.1148$

6) a) $E(2x_1) = 2E(x_1) = 2\left(\frac{\theta}{2}\right) = \theta$

$E[2\bar{x}] = 2E[\bar{x}] = 2E\left(\frac{\sum x_i}{n}\right) = \frac{2\sum E(x_i)}{n}$

$\frac{2\sum\theta}{2n} = \frac{2n\theta}{2n} = E(2\bar{x}) = \theta$ both

$2x_1$ and $2\bar{x}$ are unbiased estimators for θ

b) since uniform $(0,\theta)$; $V(x) = (\theta-0)^2 - \frac{\theta^2}{2}$ and ~~...~~ $2x$ and $2\bar{x}$ are ~~...~~ for θ, so MSE = Variance

$MSE(2x_1) = V(2x_i) = 2^2 V(x_1) = 4\left(\frac{\theta^2}{12}\right) = \frac{\theta^2}{3}$

$MSE(2\bar{x}) = V(2\bar{x}) = 2^2 V(\bar{x}) = 4V(\bar{x}) = V\left(\sum \frac{x_i}{n}\right)$

$\Rightarrow 4V\left(\frac{\sum x_i}{n}\right) = 4\frac{1}{n^2}\sum V(x_i) = 4\left(\frac{1}{n^2}\sum \frac{\theta^2}{12}\right)$

$\Rightarrow \frac{4\theta^2}{12n} = \frac{\theta}{3n}$ so $mse(2\bar{x}) = mse(2x_i)$

Hence $MSE(2\bar{x}) < MSE(2x_i)$ so $2\bar{x}$ is better than $2x_i$

7) a) $x_1 + x_2 + \dots x_n \sim \text{Poisson}(\lambda)$

pmf $P(X=x) = \frac{e^{-\lambda}\lambda^{x_i}}{x_i}$

For poisson(λ), $E(x)=\lambda$ method of moment then equals the sample mean with population mean.

$\therefore \frac{1}{n}\sum_{i=1}^{n} x_i = \lambda$ $\quad MOM(\lambda) = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$

b) likelihood function of λ

$L(\lambda) = \prod_{i=1}^{n} f(x_i = f(x_i)\cdot f(x_2)\cdots f(x_n) = \frac{e^{\lambda}\lambda^{x_i}}{x_1!}\cdot\frac{e^{-\lambda}\lambda^{2}}{x_2!}\cdots\frac{e^{\lambda}\lambda^{x_n}}{x_n!}$

$L(\lambda) = \frac{e^{-n\lambda}\lambda^{i=1}}{x_1!\cdot x_2!\dots x_n!}$

c) likelihood function of $L(\lambda)$

$\log L(\lambda) = \log\left(\frac{e^{n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{x_1!\, x_2!\dots x_n!}\right)$

$= -n\lambda + \sum_{i=1}^{n} x_i \log(\lambda) - \log(x_1!\cdot x_2!\dots x_n!)$

$= \frac{2}{2\lambda}\log(4\lambda) = 0$

$\frac{-n + \sum_{i=1}^{n} x_i}{\lambda} = 0 \Rightarrow \frac{\sum_{i=1}^{n} x_i}{n} = \lambda \quad \therefore MLE = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$

using variance property of MLE of λ is $Q\alpha$ then MLE of $g(\lambda)$ is $g(\bar{x}) \Rightarrow$ $MLE(x^2) = \bar{x}^2$

8 a) MOM estimator is obtained by equating sample moments and population moments

$\frac{1}{n}\sum x_i = 1 E(x) = \frac{\theta}{2}$ $\quad f(x)\begin{cases}\frac{1}{\theta}, & 0<x<\theta \\ 0, & m\end{cases}$

$\bar{x} = \frac{\theta}{2}$ $\quad \bar{x}$ is the sample ~~moment~~

$\tilde{\theta} = 2\bar{x}$

$E(x) = \int_0^{\theta} x f(x)\,dx = \int_0^{\theta}\frac{x}{\theta}\,dx = \frac{1}{\theta}\cdot\frac{1}{2}(x)^2\Big|_0^{\theta} = \frac{1}{2\cancel{\theta}}\theta^2 = \frac{\theta}{2}$

MOM estimate of

$(\theta, \tilde{\theta}) = 2\bar{x}$

8 b) def $T = \max(x_1, x_2, \dots x_n) = X_{(n)}$

PDF of T, $F_T(t) = \frac{n t^{n-1}}{\theta^n}$, $0 \le t \le \theta$

we need to find $E(t)$

$= \int_t^\theta t f(t)\,dt = \int_0^\theta t \frac{n t^{n-1}}{\theta^n}\,dt = \int_0^\theta \frac{n}{\theta^n} t^n\,dt = \frac{n}{\theta^n}\int_0^\theta t^n\,dt$

$\Rightarrow \frac{n}{\theta^n} \frac{(t^{n+1})}{n+1}\Big|_0^\theta = \frac{n\theta}{n+1} \therefore E(t) = \frac{n}{n+1}\theta$

T is not an unbiased estimator of θ

8c) $E(t) = \frac{n}{n+1}\theta$, Let $v = \frac{n+1}{n} t$

Then $E(v) = E(\frac{n+1}{n} T) = \frac{n+1}{n} E(t) = \frac{n+1}{n} \frac{n}{n+1}\theta$

$\Rightarrow Ev = \theta$ and $E(\frac{n+1}{n} T) = \theta$ Therefore,

$\frac{n+1}{n} T$ is an unbiased estimator of θ

9a) $x_i \sim n(\mu, \sigma^2)$

$E(x_i = \mu$ $v(x_i) = \sigma^2$

① $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$E(\bar{x}) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) \Rightarrow \frac{1}{n} \times \sum_{i=1}^{n} \mu \Rightarrow \cancel{\frac{1}{n}} \times \cancel{n}\, \mu$

$E(\bar{x}) = \mu$ and $var(\bar{x}) = V(\frac{1}{n}\sum_{i=1}^{n} x_i)$

$\frac{1}{n^2} \sum_{i=1}^{n} v(x_i) = \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2$ $V(\bar{x}) = \frac{\sigma^2}{n}$

Hence $\bar{X}$ will be normally distributed $\bar{x} \sim N(\mu, \sigma^2/n)$

b) Let $T_1 = x_1$, $T_2 = \frac{x_1 + 2x_2}{3}$ & $T_3 = \bar{x}$

① $E(T_1) = E(x_1) = \mu$ x is unbiased estimator of μ

and $var(T_1) = V(x_1)$ $V(T_1) = \sigma^2$

$T_2 = \frac{x_1 + 2x_2}{3}$ take expectation of both sides $E(T_2) = E(\frac{x_1 + 2x_2}{3}) = \frac{1}{3}[E(x_1) + 2E(x_2)]$

$\frac{1}{3}[u + 2u] = \frac{1}{\cancel{3}} \times \cancel{3}\, u$

$E(T_2) = u \rightarrow T_2$ is also unbiased estimator of $var(t_2) = V(\frac{x_1 + 2x_2}{3}) \Rightarrow \frac{1}{4}[v(x_1) + 4v(x_2)]$

$\frac{1}{4}(\sigma^2 + 4\sigma^2) = \frac{5\sigma^2}{9}$ $T_3 = \bar{x}$

9 b) cont. $T_3 = \bar{x}$

$E(T_3) = E(\bar{x}) = \mu$ $\bar{x} \sim n(\mu, \sigma^2/n)$

$V(T_3) = \sigma^2/n$ T_3 is an unbiased estimator of μ

9) c) T_1, T_2, T_3 are all unbiased estimators of μ, here

$$\boxed{\begin{aligned} Bias(T_1) &= E(T_1) - \mu = 0 \\ Bias(T_2) &= E(T_2) - \mu = 0 \\ Bias(T_3) &= E(T_3) - \mu = 0 \end{aligned}}$$

To find MSE of T_1, T_2, T_3

$$\boxed{\begin{aligned} MSE(T_1) &= Var\,T + bias(T)^2 = \sigma^2 + 0^2 = \sigma^2 \\ MSE(T_2) &= V(T_2) + bias(T_2)^2 = \frac{5\sigma^2}{4} \\ MSE(T_3) &= V(T_3) = \frac{\sigma^2}{n} \end{aligned}}$$

The estimator will be best that MSE is minimum where n is less

Hence MSE (T_3) will be min than T_2 & T_1

T_3 is the best estimator among all $T_3 = \bar{x}$

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10 a) here $g(\mu, \sigma) = \mu + \sigma$

then MLE is, $\hat{g}(\hat{\mu}, \hat{\sigma}) = \hat{\mu} + \hat{\sigma} = \bar{x} + \sigma + \mu$

$g(\mu, \sigma) = \frac{\mu}{\sigma}$

MLE is $\hat{g}(\mu, \hat{\sigma}) = \hat{\mu} / \hat{\sigma} = \frac{\bar{x}}{\sigma}$

If T is MLE for a parameter θ then there is a function $g(\theta)$ of param then the MLE of $g(\theta)$ will be $g(T)$ hence $\hat{g}(\theta) = g(T)$ this is the invariance property of MLE

10 b) $P(\bar{x}>5)$

$$= P\left(\frac{\bar{x}-m}{\sigma\sqrt{n}} > \frac{5-m}{\sigma\sqrt{n}}\right)$$

$$= P\left(z > \frac{5-m}{\sigma\sqrt{n}}\right) \quad [z \sim n(0,1)]$$

$$= 1 - P\left(z > \frac{5-m}{\sigma\sqrt{n}}\right) = 1 - \Phi\left(\sqrt{n}\,\frac{5-m}{\sigma}\right)$$

$$\Phi\left(-\sqrt{n}\,\frac{5-m}{\sigma}\right) = \Phi\left(\sqrt{n}\,\frac{m-5}{\sigma}\right)$$

So, $g(m,\sigma) = P(\bar{x}>5) = \Phi\left(\sqrt{n}\,\frac{m-5}{\sigma}\right)$

MLE of $g(m,\sigma)$ is $\hat{g}(\hat{m},\hat{\sigma}) = \Phi\left(\sqrt{n}\left(\frac{\bar{x}-5}{\hat{\sigma}}\right)\right)$

This is requires MLE