Wronskians and Linear Independence: A Theorem Misunderstood by Many

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1 Introduction

Every mathematician—student, amateur, or professional—experiences a common wild emotional roller coaster occupational hazard. Discovering a proof after working on it for hours or days[1] leads to a sense of joy and achievement that can best be described as the mathematical version of a runner's high from completing a marathon.

But unlike completing a marathon or climbing a mountain, that joy is often followed by a sense of doubt. "Did I actually accomplish that?" "Did I make a mistake?" "Is there a trivial counterexample that I didn't see?" "Will the teacher or editor[2] be mean or angry if they find an error?"

While teachers may not advertise it, this happens to everyone and no student should feel embarrassed by these feelings. Indeed, many established mathematicians can believe for years that a theorem is true and proved. Yet, someone then comes along and changes everything. This is the story of just such an instance. Our protagonist noted that a theorem that had been "proved" true for decades actually had a simple counterexample! When he published his counterexample, he included the following:

In nearly all papers, one finds the proposition: If the determinant formed with n functions of the same variable, and their derivatives of orders $1, \ldots, (n-1)$ is identically zero, there is between these functions a homogenous linear relationship with constant coefficients.

This wording is too general.

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[1]or weeks, or months, or years, ...

[2]or advisor, or audience member, or referee, ...

2 Linear Independence of Vectors

In linear algebra, we learn that a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ are called *linearly independent* if the only solution to the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$ is when all the constants $= 0$. In other words, there is no non-trivial way to combine the given vectors to give the zero vector.

It turns out there is a very strong test to determine if $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subset \mathbf{R}^n$ are independent or not:

Vector Independence Theorem. *Form the $n \times n$ matrix where the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ form the columns of the matrix. The n vectors are linearly independent if and only if this matrix has nonzero determinant.*

What follows will be easier if we spend a few minutes reviewing the logic of the above theorem. If we define

$$p = \text{the determinant is nonzero},$$

$$q = \text{the vectors are independent},$$

and use the notation '$\sim p$' to mean *the negation of the statement p*, then the 'if and only if' nature of the Vector Independence Theorem places four true statements at our disposal.

Vector Independence Theorem. *The following statements are all true.*

(a) $p \implies q$ — *If the determinant is nonzero, then the vectors are independent. (True by the "if" part of the theorem.)*

(b) $\sim q \implies \sim p$ — *If the vectors are dependent, then the determinant is zero. (Called the "contrapositive" of the "if" part of the theorem, it is equivalent to the original implication.)*

(c) $q \implies p$ — *If the vectors are independent, then the determinant is nonzero. (Called the "converse" of $p \implies q$, its truth is never implied by $p \implies q$, but in this case is true by the "only if" portion of the theorem.)*

(d) $\sim p \implies \sim q$ — *If the determinant is zero, then the vectors are dependent. (Called the "inverse" of $p \implies q$, its truth is never implied by $p \implies q$, but in this case is true by the "only if" part of the theorem.)*

$\boxed{\text{Task 1}}$ Are the vectors $\vec{v}_1 = (1, 2, 3)$, $\vec{v}_2 = (4, 5, 6)$ and $\vec{v}_3 = (7, 8, 9)$ linearly independent or dependent?

$\det\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} = 0$ therefore Dependent

3 Linear Independence of Functions

It turns out there is an analogous story for the independence of *functions*.

A set of n differentiable functions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ on an interval I are called *linearly independent* if the only solution to $c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x) = 0$ for all $x \in I$ is when all the constants $= 0$. In other words, on the interval I, there is no non-trivial way to combine the given functions to give the zero function.

It would be useful if there were a test similar to the Vector Independence Theorem that would enable us to determine the independence of a set of functions. Given we are still talking about *linear* independence, we would hope that this might still involve the determinant of some matrix. The question is: "What matrix?"

The answer is a special matrix called the *Wronskian*[3] of the n functions. This is an $n \times n$ matrix where the n functions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ form the first row, their first derivatives $\{f_1'(x), f_2'(x), \ldots, f_n'(x)\}$ form the second row, their second derivatives $\{f_1''(x), f_2''(x), \ldots, f_n''(x)\}$ form the third row, etc.[4] The Scottish mathematician Thomas Muir (1844–1934) first named these matrices in his 1882 *Treatise on the theory of determinants* [Muir, 1882], in which he stated:

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194. If there be n functions of one and the same variable x, the determinant[5] which has in every case the element in its r^{th} row and s^{th} column the $(r-1)^{th}$ differential coefficient of the s^{th} function may be called the *WRONSKIAN* of the functions with respect to x. Thus, the Wronskian of y_1, y_2, y_3 with respect to x is

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \frac{dy_3}{dx} \\ \frac{d^2y_1}{dx^2} & \frac{d^2y_2}{dx^2} & \frac{d^2y_3}{dx^2} \end{vmatrix} \quad \text{or,} \quad \left| y_1 \quad \frac{dy_2}{dx} \quad \frac{d^2y_3}{dx^2} \right|$$

It may be shortly denoted by

$$W_x(y_1, y_2, y_3) \quad \text{or} \quad |y_{1(0)}, y_{2(1)}, y_{3(2)}|,$$

the enclosed suffixes referring to differentiations.

Task 2

(a) Compute $\det(W(x^2, x^3))$. $\det\begin{bmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{bmatrix} = x^4$

(b) Compute $\det(W(\cos x, \sin x, x))$. $\det\begin{bmatrix} \cos(x) & \sin(x) & x \\ -\sin(x) & \cos(x) & 1 \\ -\cos(x) & -\sin(x) & 0 \end{bmatrix} = x$

These matrices were named after the supremely determined yet frequently incorrect Polish thinker Józef Maria Hoene-Wroński[6] (1776–1853). In his *Réfutation de la Théorie des Fonctions analytiques*

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[3] The title of this project was almost "Wronskian: A Great Name for a Metal Band." I stand by that claim.

[4] Of course, the first $n-1$ derivatives of the functions must all be defined to form this matrix.

[5] It is not standard whether "Wronskian" refers to the matrix or the determinant of said matrix. It will hopefully be clear from context which is intended. In this project, we generally use W to refer to the Wronskian as a matrix, and $\det(W)$ to denote its determinant.

[6] Wroński may simply have been ahead of his time with a coarse personality that prevented acceptance of his ideas. His entry in the *Dictionary of Scientific Biography* notes that Wroński's interactions "indicates a marked psychopathic tendency, grandiose exaggeration of the importance of his own research, violent reaction to the slightest criticism, and repeated recourse to nonscientific media as allies against a supposed conspiracy. His aberrant personality, as well as the thesis of his esoteric philosophy (based on a revelation received on 15 August 1803 or, according to his other writings, 1804), tempt one to dismiss his work as the product of a gigantic fallacy engendered by a troubled and deceived mind. Later investigation of his writings, however, leads to a different conclusion. Hidden among the multitude of irrelevancies are important concepts that show him to have been a highly gifted mathematician whose contribution, unfortunately, was overshadowed by the imperative of his all-embracing absolute philosophy [Dobrzycki, 2008]."

de Lagrange (Refutation of Lagrange's Theory of analytic Functions),[7] Wroński developed certain "combinatorial sums" which ended up being the determinants of Wronskians [Hoene-Wroński, 1812, p.14], thereby justifying Muir's naming of the matrix.

The analogous theorem to the Vector Independence Theorem is:

Function Independence Theorem. *If* $\det(W(f_1(x), f_2(x), \ldots, f_n(x))) \neq 0$ *for some* $x_0 \in I$, *then* $f_1(x), f_2(x), \ldots, f_n(x)$ *are linearly independent on the interval* I.

As above, a quick analysis of the logic of the Function Independence Theorem will be helpful. Consider the following statements:

$p =$ the determinant of the Wronskian is nonzero for some $x_0 \in I$

$q =$ the functions are independent on I

Task 3

(a) With the above definitions of p and q, what is the statement of the Function Independence Theorem, both symbolically and in words? if and only if P then q $P \Rightarrow q$

(b) With the above definitions of p and q, what is the contrapositive of the Function Independence Theorem, both symbolically and in words? (Be careful to correctly negate the "for some" statement in p.) $\sim q \Rightarrow \sim p$ if the functions are independent, then the det is non-0 for some

(c) What two statements (both symbolically and in words) are NOT necessarily true because of the Function Independence Theorem? if q then P, if $\sim p$ then $\sim q$

(d) What statement (both symbolically and in words) would it make sense to call the "Converse Function Independence Theorem"? if q, then sometimes P

We want to concentrate on the statements coming from Task 3 above: the converse and inverse of the Function Independence Theorem, which remember are NOT implied true by theorem itself. In other words, the converse and inverse might still be true—they just aren't automatically true from the implication $p \implies q$.

Indeed this is exactly what happened. The Converse Function Independence Theorem appeared in many textbooks *with proof* in the mid- to late-1800's. These included Charles Hermite's 1873 *Cours d'analyse de lécole polytechnique* [Hermite, 1873], Hermann Laurent's 1885 *Traité d'analyse* [Laurent, 1885], Camille Jordan's 1887 *Cours d'analyse de l'école polytechnique, Tome Troisième* [Jordan, 1887], and more.

The problem is ... the Converse Function Independence Theorem is not true.

[7] According to one of his biographers, Wroński submitted this paper to the French Academy of Sciences, but later withdrew it after the committee assigned to review it issued a negative report; the members of that committee included Lagrange himself [Pragacz, 2007, p.9].