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Midterm Exam

Math 2210Q, Section 12

Instructions: You have 70 minutes to take this exam and 5 minutes to upload a clear and legible copy of your answers to HuskyCT. **On each question show all your work and explain your reasoning.** You may refer to your textbook during the exam, but no other outside assistance is permitted.

(1) Solve the following system of equations: $x + y + z = 9$, $2x + 3y = 8$, $x - y - z = -7$.

$$\begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & 3 & 0 & 8 \\ 1 & -1 & -1 & -7 \end{bmatrix}$$

$$\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6 \end{matrix} \qquad \begin{matrix} x = 1 \\ y = 2 \\ z = 6 \end{matrix}$$

(2) Find all solutions to the matrix equation $\begin{bmatrix} 2 & 8 & 0 \\ 0 & 0 & c \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$.

(3) For which value(s) of c does the above equation have a unique solution?

(4) Determine if the following vectors are linearly independent: $\begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}0\\2\\0\end{bmatrix}, \begin{bmatrix}-2\\-3\\-6\end{bmatrix}$.

$\det \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & -3 \\ 3 & 0 & -6 \end{bmatrix}$ col 2 expansion

$0 \cdot -1 \cdot \begin{vmatrix} 0 & -3 \\ 3 & -6 \end{vmatrix} + 0 \cdot -1 \cdot \begin{vmatrix} 1 & -2 \\ 0 & -3 \end{vmatrix} =$

$2 \begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix}$ $ad - bc = -6 - -6 = 0$

$\det = 0$

Vectors are dependent since determinate is 0

(5) Is $\begin{bmatrix}4\\5\\6\end{bmatrix}$ in Span $\left(\begin{bmatrix}0\\2\\3\end{bmatrix}, \begin{bmatrix}1\\-4\\-6\end{bmatrix}, \begin{bmatrix}2\\8\\12\end{bmatrix}\right)$?

$$\begin{bmatrix}0 & 1 & 2\\ 2 & -4 & 8\\ 3 & -6 & 12\end{bmatrix}\vec{x} = \begin{pmatrix}4\\5\\6\end{pmatrix}$$

augment

$$\left[\begin{array}{ccc|c}0 & 1 & 2 & 4\\ 2 & -4 & 8 & 5\\ 3 & -6 & 12 & 6\end{array}\right]$$

$$\begin{bmatrix}1 & -2 & 4 & \frac{5}{2}\\ 0 & 1 & 2 & 4\\ 3 & -6 & 12 & 6\end{bmatrix}$$

$$\begin{bmatrix}1 & -2 & 4 & \frac{5}{2}\\ 0 & 1 & 2 & 4\\ 0 & 0 & 0 & -\frac{3}{2}\end{bmatrix}$$

Impossible

no, $\begin{bmatrix}4\\5\\6\end{bmatrix}$ is not in span$\{\ldots\}$

(6) Let A be the matrix representing the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $\begin{bmatrix}1\\0\end{bmatrix} \mapsto \begin{bmatrix}7\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix} \mapsto \begin{bmatrix}0\\c\end{bmatrix}$. For what value(s) of c does A have an inverse?

all real numbers ~~ex~~ except

$\mathbb{R}$ $\quad$ 0

$\neq 0$

(7) Supposing c is such that A is invertible, what is the matrix A^{-1}?

$$\left[\begin{array}{cc|cc} 7 & 0 & 1 & 0 \\ 0 & c & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{7} & 0 \\ 0 & 1 & 0 & \frac{1}{c} \end{array}\right]$$

matrix A^{-1}

$$\begin{bmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{c} \end{bmatrix}$$

For the following statements let A be an $n \times n$ matrix. Determine whether each statement is **True** or **False**. Only answer **True** if the statement is always true.

(8) A is invertible if and only if the diagonal is non-zero.

True

(9) If the determinant is A is non-zero, then the linear transformation represented by A is one-to-one.

True

(10) The image of A forms a subset of $\mathbb{R}^n$, but not always a subspace.

False

(11) The kernel of A forms a subspace of the codomain.

False

(12) A maps all non-zero vectors to a non-zero vectors if and only if A is invertible.

True

(13) The columns of A are linearly dependent if and only if the determinant is non-zero.

False

(14) A is row equivalent to the identity matrix if and only if A is invertible.

True

(15) If A is invertible then, $\det(A^{-1}) = -\det(A)$.

False

(16) A may be onto, but not one-to-one.

True

(17) If $n = 2$ and $\det(A) = 3$, then A sends $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix}$.

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\frac{1}{3}$ False

(18) Let $A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & -4 & 0 \\ 4 & 3 & 1 \end{bmatrix}$.

Use cofactor expansion along the 2^{nd} row of A to compute $\det(A)$. Check your work with the cofactor expansion along the 3^{rd} column of A.

$$2\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} - (-4)\begin{vmatrix} 5 & 0 \\ 4 & 1 \end{vmatrix} \cancel{+ 0\begin{vmatrix} 5 & 2 \\ 4 & 3 \end{vmatrix}}$$

$2 \cdot 2 = 4 \qquad 4 \cdot 5 = 20$

-24

$$\cancel{0|\ | - 0|\ |} + 1\begin{vmatrix} 5 & 2 \\ 2 & -4 \end{vmatrix}$$

$5 \cdot -4$

$-20 - 2 \cdot 2$

4

$= -24$

(19) A sphere in $\mathbb{R}^3$ has volume $\frac{4}{3}\pi r^3$. Compute the volume of the image of the sphere under the linear transformation corresponding to the matrix A from (18).

(20) Let $A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 4 & 0 \\ 2 & 7 & 2 \end{bmatrix}$. Compute A^{-1}.

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 & 1 & 0 \\ 2 & 7 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 2 & 7 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right] \cdot \frac{1}{2} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 4 & -3 & 0 \\ -1 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(21) Suppose I tell you that an $n \times n$ matrix B is invertible. Write down **at least three** non-trivial things that you now know about B and the linear transformation corresponding to B.

- it is a square matrix
- its det(B) is not 0
- Bx = 0 has 1 trivial Solution
- B's collums form a linearly indue Set

Let $A = \begin{bmatrix} -3 & 3 \\ 1 & 2 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 3 & 1 \end{bmatrix}$. For each of the following questions compute the given quantity or explain why it is not possible to compute.

(22) BA

Rows of B x Cols of A

$$\begin{bmatrix} 0 & 2 \\ 0 & 8 \end{bmatrix}$$

(23) AB Rows of A x Cols of B

$$\begin{bmatrix} 3 & 9 & 9 \\ 2 & 6 & 0 \\ -1 & -3 & -1 \end{bmatrix}$$

(24) $A^T B$

$$A \cancel{\begin{bmatrix} -3 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}}^T \to \begin{bmatrix} -3 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & -2 \\ 1 & 3 & 1 \end{bmatrix}$$

dimensions impossible

(25) $(BA)^{-1}$

$$\begin{bmatrix} 0 & 2 \\ 0 & 8 \end{bmatrix}^{-1}$$

not possible, is singular

(26) $B^T A^T$

Row $B^T \cdot$ Col A^T

$$\begin{bmatrix} 0 & 1 \\ 0 & 3 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 9 & 6 & 3 \\ 9 & 0 & 1 \end{bmatrix}$$

(27) Construct a 3×3 matrix that has image (column space) equal to Span $\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}7\\-1\\6\end{bmatrix}, \begin{bmatrix}2\\0\\2\end{bmatrix}\right)$.

Time

(28) Construct a 3×3 matrix that has kernel (null space) equal to Span $\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}7\\-1\\6\end{bmatrix}, \begin{bmatrix}2\\0\\2\end{bmatrix}\right)$.

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$

$x_1 - x_2 = 0$ $\quad A\begin{bmatrix}1\\-1\\0\end{bmatrix} = 0 \quad A\begin{bmatrix}2\\0\\2\end{bmatrix} = 0$

$7x_1 - x_2 = 6$

$2x_1 = 2$ $\quad A\begin{bmatrix}7\\-1\\6\end{bmatrix} = 0 \quad x_1\begin{bmatrix}1\\7\\2\end{bmatrix} + x_2\begin{bmatrix}-1\\-1\\0\end{bmatrix} = \begin{matrix}0\\6\\2\end{matrix}$

$x_1 = 1$

$x_1 = x_2$

$x_2 = 1$

$\begin{bmatrix} 1 & -1 & 0 \\ 7 & -1 & 6 \\ 2 & 0 & 2 \end{bmatrix}$

(29) Consider the set $H := \left\{ \begin{bmatrix} 0 \\ b \\ c \end{bmatrix} : b, c \in \mathbf{R} \right\}$.[1] Prove H is a subspace under the vector space operations inherited from $\mathbf{R}^3$.

Time

(30) Define new vector space operations so that H is still a vector space, but no longer a subspace of $\mathbf{R}^3$.

Time

[1]The symbol ':=' is read 'defined to be equal to.' It is apparently borrowed from computer science.

Bonus: Given an invertible linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$, describe the possible images of a parabola under A.

Time