EECS 498 HW2

Prob. 1

Define a point f in n-dimension space: $f \coloneqq X \to y, where\ y\ is\ constant\ R\ and\ X\ is\ a\ constant\ R^n$

$Let\ x_1, x_2 \in X\ \ (x_1 = x_2)$

$f(x_1) = f(x_2) = y, \forall t \in [0,1],$

$\Rightarrow f(tx_1 + (1-t)x_2) = f(x_1) = f(x_2) = y, \qquad tf(x_1) + (1-t)f(x_2) = ty + (1-t)y = y$

$\Rightarrow f(tx_1 + (1-t)x_2) = y\ \le\ \ tf(x_1) + (1-t)f(x_2) = y$

$\therefore\ a\ single\ point\ is\ convex$

Prob. 2

$Let\ f_1(x) = |2-5x|, f_2(x) = 2x, f_3(x) = 8e^{(-4x)}, f_4(x) = -1$

$f_1 = |2-5x|$ is convex since it is a combination of hyperplane

$f_2 = 2x$ is convex since it is a hyperplane

$f_3 = 8e^{(-4x)}$ is convex since $f_3'' = 128e^{(-4x)} > 0$

$f_4 = -1$ is convex since it is a hyperplane

Since $f_1,\ f_2,\ f_3,\ f_4$ are convex and operation of sum of convex functions preserves convexity

$\Rightarrow f(x)$ is convex

Prob. 3

$$f'(x) = 0.5e^{(0.5x+2)} - 0.5e^{(-0.5x-0.5)} + 4 \Rightarrow f'(5) \approx 48.9837$$

By Symmetric Difference Quotient

$$h = 0.001, f\ \acute{}\ _{h=0.001}(5) - f'^{(5)} = -1.874 \times 10^{-6}$$

$$h = 10^{-4}, f\ \acute{}\ _{h=10^{-4}}(5) - f'(5) = -1.856 \times 10^{-8}$$

$$h = 10^{-5}, f\ \acute{}\ _{h=10^{-5}}(5) - f'(5) = 1.688 \times 10^{-9}$$

$$h = 10^{-6}, f\ \acute{}\ _{h=10^{-6}}(5) - f'^{(5)} = -5.658 \times 10^{-8}$$

Numerical results with $h = 10^{-3}, 10^{-4}, 10^{-5}$ are smaller than the analytical result, except the result with $h = 10^{-5}$ and it is also the closest result to the analytical one.

Prob. 4

$$\min_x \begin{bmatrix} -3 & 2 \end{bmatrix} x - 1$$

$$f \coloneqq \begin{bmatrix} 0 & -1 \\ 0 & -5 \\ -5 & 4 \end{bmatrix} x \le \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}$$

$$g \coloneqq \begin{bmatrix} 1 & 1 \end{bmatrix} x = 2.3$$

Prob. 5

There would be more than one subgradient when $3x^2 - 2, 2x - 1$ intersects

$\Rightarrow (3x^2 - 2) - (2x - 1) = 0 \Rightarrow x = 1/3 \ or \ 1$

$$\delta f(x) = \begin{cases} 6x, x > 1 \ or \ x < 1/3 \\ 2, \quad 1/3 < x < 1 \\ \quad [2,6], x = 1 \\ \quad 2, \quad x = 1/3 \end{cases}$$

Prob. 6

(a) Lagrange dual function

Lagrange function: $L(x, \lambda, \nu) = c^t x + \lambda^T (\mathrm{G}x - \mathrm{h}) + \nu^T (Ax - b)$

Dual function:

$$g(\lambda, \nu) = \inf_{x \in \mathrm{D}} c^t x + \lambda^T (\mathrm{G}x - \mathrm{h}) + \nu^T (Ax - b) = -h^T \lambda - b^T \nu + \inf_x (G^T \lambda + A^T \nu + c)^T x$$

$\lambda \ is$ a $vector \ of \ Lagrange \ mutipliers$

ν is a vector of Lagrange multipliers

(b) Dual problem

Maximize $-h^T \lambda - b^T \nu$

Subject to $G^T \lambda + A^T \nu + c = 0$

$\lambda \succeq 0$

(c) Optimal value of dual problem relation with primal optimal?

Since we assume primal is feasible and bounded, then the strong duality holds.

$\Rightarrow p^* = d^*$