**MATH 531** **Homework 2** **Spring 2024**

**(Due: 29th January, Monday)**

1. Let $\{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k\}$ be a set of $k$ orthonormal vectors in $\mathbb{R}^n$ that span a subspace $\mathcal{W}$ and let $Z$ be a matrix formed by taking these as column vectors.

$$Z = [\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k]$$

( Note in R this would be the `cbind` function to form the vectors into a matrix. ) Show that $Z^T Z = I$ where $I$ is the identity matrix. What is the dimension of this identity? (Subsequently when it is important we will notate the identity matrix as $I_m$ where $m$ refers to the dimension. )

2. Show that the trace of the matrix $ZZ^T$ is $k$. Under what circumstances will $ZZ^T = I$?

3. Define: The *projection* of $\mathbf{y} \in \mathbb{R}^n$ onto a subspace $\mathcal{W}$ as the vector $\mathbf{u}$ such that $\mathbf{y} = \mathbf{u} + \mathbf{v}$, $\mathbf{u} \in \mathcal{W}$ and $\mathbf{v} \in \mathcal{W}^\perp$.

   From lecture we know that if a projection exists it is unique – so we can talk about *the* projection.

   (This is not exactly how projection is defined in the course reference notes but is how it was presented in class lecture, followed by a proof of existence and uniqueness. )

   From Problem (1) define the $n \times n$ matrix $P = ZZ^T$. Show that for any $\mathbf{y}$, $\mathbf{u} = P\mathbf{y}$ is the projection.

   Show that $(I - P)\mathbf{y}$ is the projection onto $\mathcal{W}^\perp$.

4. Let $X$ be an $n \times k$ matrix where the columns are linearly independent. Show that $X^T X$ has full rank ($k$) and show that $P = X(X^T X)^{-1} X^T$ is a projection onto the subspace spanned by the columns of $X$.

5. Classical treatments of linear algebra tend to present the singular value decomposition (svd) at the end of a course – if at all. However, it is an extremely useful way to categorize all matrices and hence all linear maps between $\mathbb{R}^k$ and $\mathbb{R}^n$. Let $X$ be any $n \times k$ matrix, then

$$X = UDV^T$$

   with

   - $U$ an $n \times k$ matrix where the columns form an orthonormal basis ( $U^T U = I_k$)
   - $D$ is a $k \times k$ diagonal matrix with nonnegative elements (some of the $D$ can be zero and the convention is to sort these in descending order.)
   - $V$ is an $k \times k$ orthonormal matrix ($V^T V = I_k$)

   (The proof of this is long and involved and we will just take this result on trust!)

   If $A$ is a square matrix and invertible show that $A^{-1} = VD^{-1}U^T$.

   Show that $X^T X = VD^2 V^T$.

   Based on this previous result explain how to construct a G-inverse for $X^T X$, that is identify a matrix, A, such that $X^T X(A) X^T X = X^T X$.

6. When was the singular value decomposition invented? If possible identify its inventor(s)? Is it difficult to compute the SVD of a matrix in R?