MATH 531 Homework 6
Fisher information
March 12, 2024

Each subsection counts for 10 points – 50 points total and the extra credit counts for 5.

1. Assume a linear model
$$\mathbf{y} = X\beta + \mathbf{e}$$
where $\mathbf{y}$ is an vector of length n and X is a full rank, $n \times p$ matrix. $\mathbf{e}_i$ are independent $N(0, \sigma^2\Omega)$. Also suppose that β_T is "true" value for β and $\hat{\beta}$ the Generalized Least Squares (GLS) /MLE estimate and $\hat{\sigma}$ the MLE for σ. Here Ω is a known correlation matrix.

   (a) Identify the formulas for the MLEs $\hat{\beta}$ and $\hat{\sigma}_{MLE}$.

   (b) Derive the Fisher information for this model when evaluated at the true values of β and σ^2. (You should reparametrize σ^2 as ω as in the lectures to make derivatives easier.)

   (c) Do the GLS estimates for β have a covariance matrix that achieves the Cramer-Rao lower bound (the inverse Fisher information matrix)?

   (d) The GLS estimate for σ^2 based on previous homework differs slightly from the MLE but it is unbiased.
$$\hat{\sigma}^2_{GLS} = \frac{1}{(n-p)}(\mathbf{y} - X\hat{\beta})^T\Omega^{-1}(\mathbf{y} - X\hat{\beta})$$
and has variance $\frac{2\sigma^4}{(n-p)}$
Show that that the variance of this estimate achieves the Cramer-Rao lower bound in the limit as $n \to \infty$.

2. Now supposed that Ω above depends on an additional statistical parameter, say $\Omega(\alpha)$. (This is common in models for spatial and time series data.) Derive the Fisher Information for this extended model. Explain why finding the GLS estimate in this case is problematic.