MATH 531 Homework 1 Spring 2025

1. Let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ be a set of k orthogonal vectors in $\mathbb{R}^n$. That is $\mathbf{x}_i^T \mathbf{x}_j = 0$ for all $i \neq j$. Show that this set is linearly independent when $k \leq n$. Also explain why this problem does not make sense when $k > n$!

2. Suppose $\mathcal{V}_n \subset \mathbb{R}^n$ is a vector space. Prove the following results.

   (a) If $\boldsymbol{x} \in \mathcal{V}_n$ and $\boldsymbol{x} \perp \mathcal{V}_n$ then $\boldsymbol{x} = \mathbf{0}$.

   (b) $\mathcal{V}_n^{\perp} = \{\boldsymbol{x} : \boldsymbol{x} \perp \mathcal{V}_n\}$ is a vector space.

3. Let $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_k\}$ be a basis of a vector space $\mathcal{W}$. Then show that $\boldsymbol{y} \in \mathcal{W}^{\perp}$ if and only if (aka iff) $\boldsymbol{y} \perp \boldsymbol{x}_i,\ i = 1, 2, \dots, k$.

4. Recall the Cauchy-Schwartz inequality for two vectors , $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$

   $$|\mathbf{x}^T \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$$

   use this inequality to show that the sample correlation coefficient between two data vectors is always in the range $[-1, 1]$

5. Recall the projection of a vector $\mathbf{y}$ onto $\mathbf{x}$ is given by $\hat{\mathbf{y}} = \beta \mathbf{x}$ with $\beta = (\mathbf{x}^T \mathbf{y})/(\mathbf{x}^T \underline{)}$ Using the fact that $\|\mathbf{y} - \hat{\mathbf{y}}\| \geq 0$ for all $\mathbf{x}$ and $\mathbf{y}$ prove the Cauchy-Schwartz inequality

6. Extra Credit Who were Cauchy and Schwarz in the equality given in the previous problem? If you could go to dinner with either one who would you choose?