MATH 531 **Homework 3** **Spring 2024**

(Due: February 5, 2024)

1. Suppose $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{\beta}$ is an estimable function, i.e., $\boldsymbol{c}^{\mathrm{T}} \in \mathcal{R}(\boldsymbol{X})$, the row space of $\boldsymbol{X}$. Then show that

   (a) $\mathcal{R}(\boldsymbol{X}) = \mathcal{R}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}) = \mathcal{C}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X})$.

   (b) $\boldsymbol{c}^{\mathrm{T}} \in \mathcal{R}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X})$ if and only if $\boldsymbol{c}^{\mathrm{T}}G\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X} = \boldsymbol{c}^{\mathrm{T}}$, where G is any generalized inverse of $\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}$.

   Remark. Note that, to check whether $\boldsymbol{c}^{\mathrm{T}} \in \mathcal{R}(\boldsymbol{X})$ is often a tedious job in practice but from (a) and (b), we get an equivalent condition for estimability of a linear function, i.e., by checking whether $\boldsymbol{c}^{\mathrm{T}}G\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X} = \boldsymbol{c}^{\mathrm{T}}$.

2. Consider the linear model
$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon},$$
   with $\mathbb{E}(\boldsymbol{\varepsilon}) = 0$ and $\mathrm{Var}(\boldsymbol{\varepsilon}) = \sigma^2 I$. Suppose $\mathrm{Rank}(\boldsymbol{X}) \leq p$. Suppose $\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{\beta}$ and $\boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{\beta}$ are both estimable functions, then show that $\mathrm{Cov}[\boldsymbol{c}_1^{\mathrm{T}}\widehat{\boldsymbol{\beta}}, \boldsymbol{c}_2^{\mathrm{T}}\widehat{\boldsymbol{\beta}}] = \sigma^2\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{G}\boldsymbol{c}_2$, where $\boldsymbol{G}$ is any generalized inverse of $\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}$.

3. If $\mathrm{Rank}(\boldsymbol{X})$ has full rank, p, then we know $\widehat{\boldsymbol{\beta}} = (\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X})^{-1}\boldsymbol{X}^{\mathrm{T}}\boldsymbol{y}$ starting from the normal equations as described in the class and we also know $\mathbb{E}(\widehat{\boldsymbol{\beta}}) = \boldsymbol{\beta}$ is an unbiased estimator. Show:

   (a) $\mathrm{Cov}(\widehat{\boldsymbol{\beta}}) = \sigma^2(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X})^{-1}$.

   (b) $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{\beta}$ is estimable for any $\boldsymbol{c}^{\mathrm{T}} \in \mathbb{R}^p$.

   (c) $\mathrm{Var}(\boldsymbol{c}^{\mathrm{T}}\widehat{\boldsymbol{\beta}}) = \sigma^2\boldsymbol{c}^{\mathrm{T}}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X})^{-1}\boldsymbol{c}$, for any $\boldsymbol{c}^{\mathrm{T}} \in \mathbb{R}^p$.

   (d) $\mathrm{Cov}(\boldsymbol{c}_{\mathbf{1}}^{\mathrm{T}}\widehat{\boldsymbol{\beta}}, \boldsymbol{c}_{\mathbf{2}}^{\mathrm{T}}\widehat{\boldsymbol{\beta}}) = \sigma^2\boldsymbol{c}_{\mathbf{1}}^{\mathrm{T}}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X})^{-1}\boldsymbol{c}_{\mathbf{2}}$, for any $\boldsymbol{c}_{\mathbf{1}}^{\mathrm{T}}, \boldsymbol{c}_{\mathbf{2}}^{\mathrm{T}} \in \mathbb{R}^p$.