Harvard School of Engineering and Applied Sciences — CS 152: Programming Languages

Math Review
Section and Practice Problems

Feb 3, 2017

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1 Sets and Logic

(a) **Sets 1.** How do we write the set of the numbers 1 through 10?

It could be: $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Or it could be: $\{x \mid x \in \mathbb{Z} \land 1 \leq x \leq 10\}$.

(b) **Sets 2.** If A = {1, 3, 5}, B = {3, 5, 6} and C = {1, 3, 7} (i) Verify that $A \cup (B \cap C) = (A \cup B)(A \cup C)$.

(c) **Logic 1.** Determine whether the following statements are true or false:

$\forall x.\forall y.(y > x) \implies (x = 0)$.

$\exists x.\forall y.(y > x) \implies (x = 0)$.

2 Induction

An inductive proof has four components: the statement to prove, the base case, the assumption, and the inductive step. Let P(x) be a predicate and m, n nonnegative integers. If $P(m)$ is true and $P(n) \to P(n+1)$ for all $n \leq m$, then $P(n)$ is true for all $n \leq m$.

(a) **Induction 1.** Use mathematical induction to prove that $1 + 2 + 3 + ... + n = n(n+1)/2$ for all positive integers n.

With strong induction, however, we say: If $P(m)$ is true and $P(m) \to P(m+1)$; $P(n) \to P(n+1)$ for all $n \leq n$, then $P(n)$ is true for all $n \leq m$.

(b) **Strong induction 1.** Imagine you have an unlimited supply of 4-cent and 5-cent postage stamps. Show, using induction, that you can make any amount of postage 12 cents or more using just these stamps.

(c) **Strong induction 2.** Let a_n be the sequence defined by $a_1 = 1, a_2 = 8$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{N}$.

Question to consider: Why is strong induction even necessary? In what cases is regular induction not enough?

(d) **Structural induction.** Two examples from the lecture notes (reflexive closure of $\to$, termination), with some kind of focus on set theoretic interpretations.

Structural induction 1. When proving semantic equivalence of big- and small-step semantics in lecture 4, induction on the number of steps $\langle e, \sigma \rangle \to^* \langle n, \sigma' \rangle$ is actually structural induction over the derivation of $\to^*$.

Structural induction 2. We prove termination by induction over expressions.

3 Contradiction

(a) **Contradiction 1.** Prove by contradiction that $\sqrt[3]{4}$ is irrational.

(b) **Contradiction 2.** Prove by contradiction that if $17n + 2$ is odd then n is odd.