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Problem 3

Part a

The conjugate prior of an exponential distribution with rate, λ is a Gamma distribution with paramaters Gamma$(\lambda; \alpha, \beta)$

$$\frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} exp{-\lambda\beta}$$

The posterior is the likelihood times the conjugate prior.

$$L(\lambda)\Gamma(\lambda; \alpha, \beta)$$

$L(\lambda) = \prod \lambda exp(-\lambda x) = \lambda^n exp(-\lambda n\bar{x})$

Therefore,

$p(\lambda) = \lambda^n exp(-\lambda n\bar{x}) \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} exp(-\lambda\beta)$

Rearrange terms to get

$= \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{(\alpha+n)-1} exp(-\lambda(\beta + n\bar{x}))$

The prior predictive distribution is the integral of the posterior distribution with respect to the prior.

So, the prior predictive distribution $= \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{(\alpha+n)-1} exp(-\lambda(\beta + n\bar{x}))$

Removing the first term from the integral as it is not related to lambda,

$$\frac{\beta^\alpha}{\Gamma(\alpha)} \int_\lambda \lambda^{(\alpha+n)-1} exp(-\lambda(\beta + n\bar{x}))$$

By multiplying through by a factor of $\frac{(\beta+n\bar{x})^{\alpha+n}}{\Gamma(\alpha+n)}$,

the integral can be written as:

$$\int_\lambda \Gamma(\lambda; \alpha + n, \beta + n\bar{x}) d\lambda$$

This simplifies to 1 by the rules of exponential family distributions.

So,

p(λ) =

$$\frac{\beta^\alpha}{\Gamma(\alpha)}\frac{\Gamma(\alpha+n)}{(\beta+n\bar{x})^{\alpha+n}}$$

$$\frac{\beta^\alpha}{\Gamma(\alpha)}\frac{\Gamma(\alpha+n)}{(\beta+n\bar{x})^{\alpha+n}}$$

$$\left(\frac{\beta}{\beta+n\bar{x}}\right)^\alpha\left(\frac{1}{\beta+n\bar{x}}\right)^n\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

Part b

The posterior predictive distribution of X_2 given X_1 =

$$\int p(X_2|\lambda)p(\lambda|X_1)d\theta$$

where $p(\lambda|X_1)$ is the posterior distribution of X_1 and $p(X_2|\lambda)$ is the likelihood of X_2. So, this can be rewritten as,

$$\int \lambda^n exp(-\lambda n\bar{x})\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{(\alpha+n)-1}exp(-\lambda(\beta+n\bar{x}))d\lambda$$

After rearranging terms, this can be written as,

$$\frac{\beta^\alpha}{\Gamma(\alpha)}\int_\lambda \lambda^{(\alpha+2n)-1}exp(-\lambda(\beta+2n\bar{x}))d\lambda$$

The integral is a gamma distribution, $\Gamma(\lambda;\alpha+2n,\beta+2n\bar{x})$, so by multiplying through by a factor of $\frac{(\beta+2n\bar{x})^{\alpha+2n}}{\Gamma(\alpha+2n)}$

The posterior predictive distribution can be found to be,

$$\left(\frac{\beta}{\beta+2n\bar{x}}\right)^\alpha\left(\frac{1}{\beta+2n\bar{x}}\right)^{2n}\frac{\Gamma(\alpha+2n)}{\Gamma(\alpha)}$$

Part c

$\frac{X_1+X_2}{2}$ is the mean of X_1 and X_2

Since they are exchangeable exponential distributions, the mean is equal to $\frac{1}{\lambda}$, where the L(λ) is given above.

So, the likelihood of $\frac{X_1+X_2}{2}$ is equal to $\lambda^{-n}exp(\lambda n\bar{x})$.

Using the definition that the posterior predictive distribution = $\int$ likelihood * posterior distribution

$$p(\lambda) = \int_\lambda \lambda^{-n}exp(\lambda n\bar{x})\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{(\alpha+n)-1}exp(-\lambda(\beta+n\bar{x}))$$

This means that the posterior predictive distribution will be

$$\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}exp(-\lambda\beta)$$

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Which by observation is the gamma distribution, $\Gamma(\lambda; \alpha, \beta)$

This is the same as the conjugate prior.