Homework 1 - Monte Carlo Methods

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Problem 1

The standard Laplace distribution has density [f(x) = 0.5e^{-|x|}, x \in (-\infty, \infty)]. Please provide an algorithm that uses the inverse transformation method to generate a random sample from this distribution. Use the (U(0,1)) random number generator in

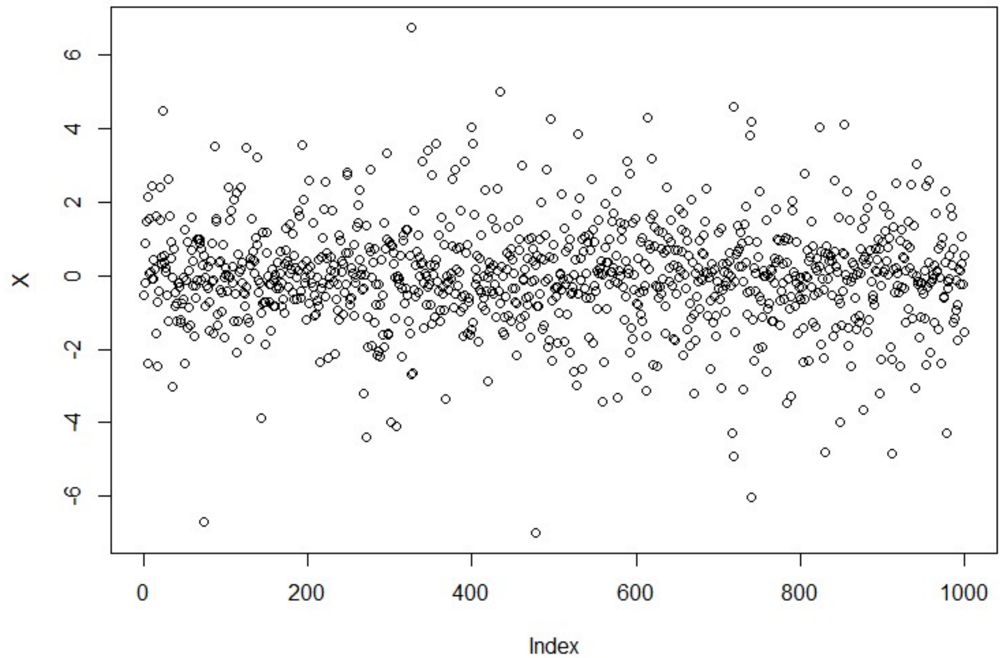
Answer: your answer starts here…

X = (F^{-1}(U))

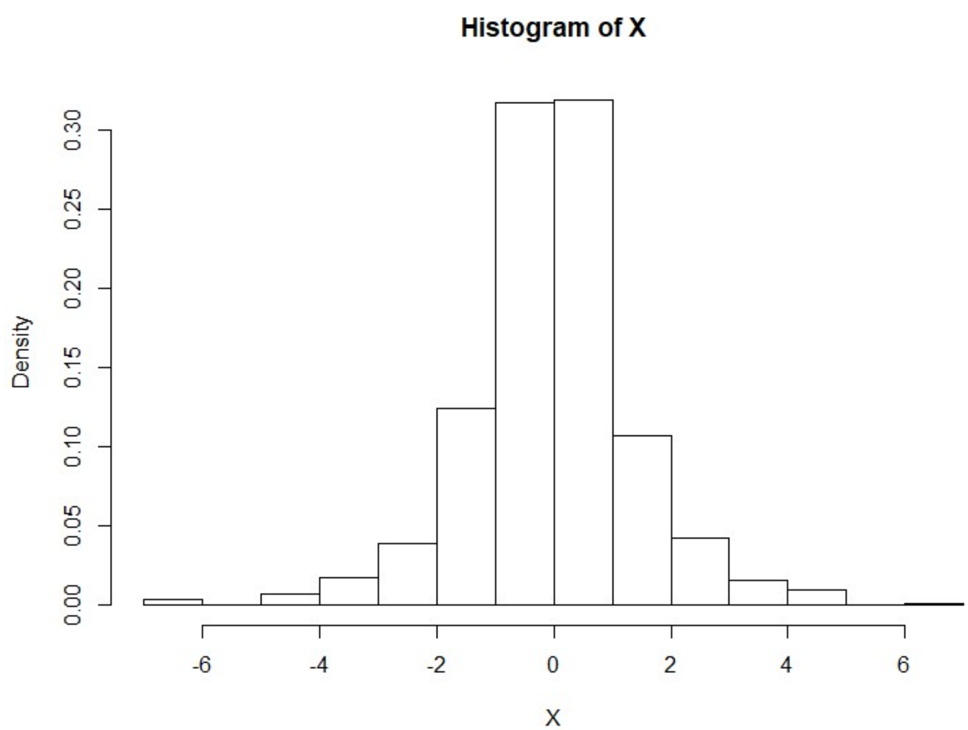
= (F(x))

(f(x) = 0.5e^{-|x|},~x \in(-\infty, \infty))

```
set.seed(123)
U <- runif(1000)
X <- (U < 0.5) * log(2*U) + (U >= 0.5) * -log(2-2*U)

plot(X)
```




```
hist(X, prob = T)
```



#Problem 2

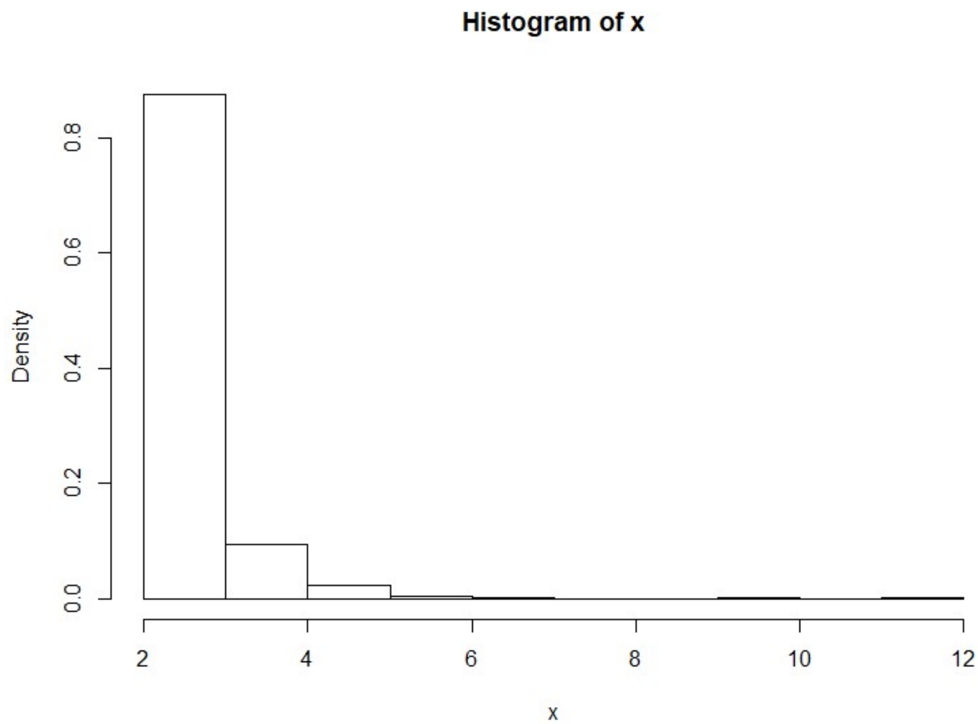
Use the inverse transformation method to derive an algorithm for generating a Pareto random number with (U\sim U(0,1)), where the Pareto random number has a probability density function [f(x; \alpha, \gamma)=\frac{\gamma\alpha^{\gamma}}{x^{\gamma+1}} I{x\ge \alpha}] with two parameters (\alpha >0) and (\gamma>0). Use visualization tools to validate your algorithm (i.e., illustrate whether the random numbers generated from your function truely follows the target distribution.)

[F(x) = 1 - \left(\frac{\alpha}{x}\right)^{\gamma},~~x \ge \alpha,~~\alpha > 0,~\gamma > 0]

[x = \frac{\alpha}{(1-u)^{1/\gamma}}]

Answer: your answer starts here…

```
set.seed(1001)
U <- runif(1000)
xdens = function(gamma = 5, alpha = 2, x = U) {
  alpha/((1-U)^(1/gamma))
}

x <- xdens(5, 2, U)

hist(x, prob = T)
```



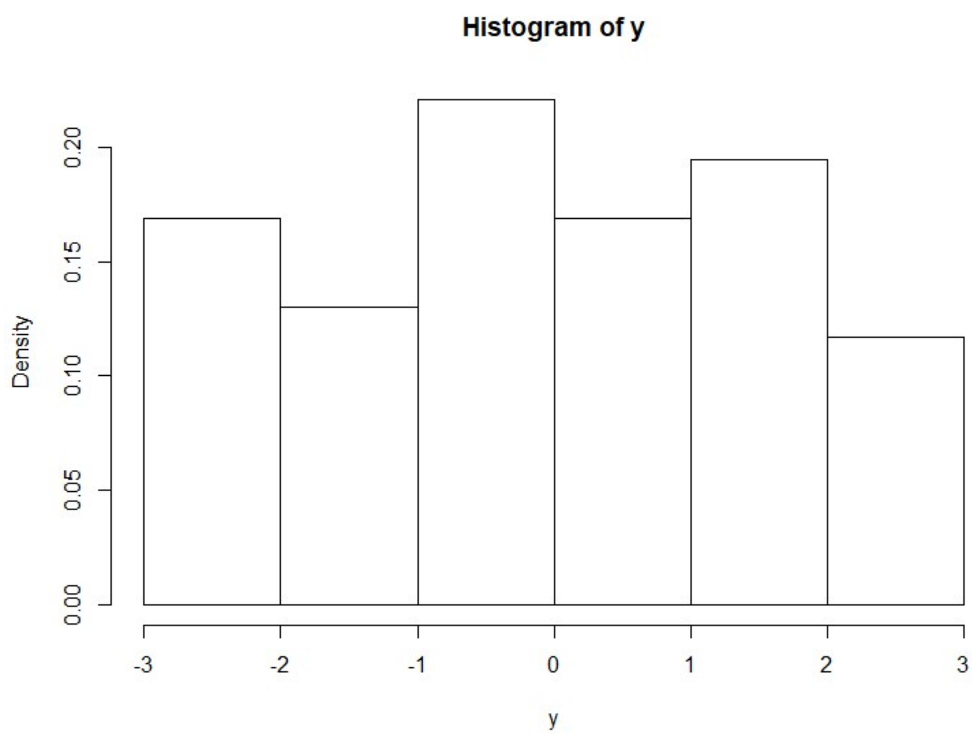
#Problem 3

Construct an algorithm for using the acceptance/rejection method to generate 100 pseudorandom variable from the pdf [f(x) = \frac{2}{\pi \beta^2} \sqrt{\beta^2-x^2}, ,, -\beta \le x \le \beta.] The simplest choice for (g(x)) is the (U(-\beta, \beta)) distribution but other choices are possible as well. Use visualization tools to validate your algorithm (i.e., illustrate whether the random numbers generated from your function truely follows the target distribution.)

Answer: your answer starts here…

```
set.seed(1001)

accrej <- function(fdens, gdens, beta, M = (4/pi), x){
  x = runif(100, min = -beta, max = beta)
  return(x[runif(length(x)) <= fdens(x, beta) / (M * gdens(x, beta))])
}
```

```r
xdens = function(x, beta){
  return((2/(pi*beta^2))*sqrt(beta^2 - x^2) * (abs(x) <= beta))
}

unifdens = function(x, beta){
  return((1/(2*beta))*(abs(x) <= beta))
}


y = accrej(xdens, unifdens, 3, 4/pi)

hist(y, prob = T)
```



#Problem 4

Develop two Monte Carlo methods for the estimation of (\theta=\int_0^1 e^{x^2} dx) and implement in

Answer: your answer starts here…

```
n = 10000
u = runif(n)
y = sum(exp(u^2))/n
x = median(exp(u^2))

xy = tibble(
  median = x,
  mean = y
) %>%
  knitr::kable()
```

#Problem 5

Show that in estimating (\theta=E\sqrt{1-U^2}) it is better to use (U^2) rather than (U) as the control variate, where (U\sim U(0,1)). To do this, use simulation to approximate the necessary covariances. In addition, implement your algorithms in

Using U^2 the variance is lower than using U as the control variate

The variance is lower for u^2 (-4.43) than for u (-4.09) as the control variate as shown below.

```
gfun<-function(x){
  sqrt(1 - x^2)
}
mfun<-function(x) {
  x^2
}
mfun2<-function(x) {
  x
}
set.seed(123)
uran<-runif(10000)
```

```
ga<-gfun(uran)
ma<-mfun(uran)
ma2<-mfun2(uran)


theta1<-mean(ga)
theta2<-mean(ma2)
hha<- pi/4 + (ga-ma)
hha2<- pi/4 + (ga-ma2)
theta1a<-mean(hha)
theta2a<-mean(hha2)

c(var(ga), var(hha))
```

```
## [1] 0.04848323 0.26344282
```

```
c(var(ga), var(hha2))
```

```
## [1] 0.04848323 0.24693579
```

```
(var(ga)-var(hha))/var(ga)
```

```
## [1] -4.43369
```

```
(var(ga)-var(hha2))/var(ga)
```

```
## [1] -4.093221
```

#Problem 6 Obtain a Monte Carlo estimate of [\int_1^\infty \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx] by importance sampling and evaluate its variance. Write a

Use a normal distribution to implement importance sampling on the function above.

I generated a random sample of a uniform distribution from 0 to 1.

```
ncandidates <- 100000;
M <- exp(-1)
u = runif(ncandidates)
x <- rnorm(ncandidates)
Mfun <- function(x){
  x^2*exp(-x^2/2)/sqrt(2*pi)
}
pfun <- function(x){
  dnorm(x)
}
accrej <- function(Mfun, pfun, M, x){
  ncandidates = length(x)
  u = rexp(ncandidates)
  accepted <- NULL      # Initialize the vector of accepted values
  for(i in 1:ncandidates) {
    if(u[i] <= Mfun(x[i])/(M*pfun(x[i])))
      accepted <- c(accepted, x[i])  # Accept x[i]
  }
  return(accepted)
}

y = accrej(Mfun, pfun, 1/exp(1), x)

hist(u)
hist(y, prob = T)

sum(y*(y>1))/length(y*(y>1))
```