Homework 2 on Newton's methods

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Due: 03/18/2020, Wednesday, by 1pm

Problem 1

Design an optimization algorithm to find the minimum of the continuously differentiable function $f(x) = -e^{-1}\sin(x)$ on the closed interval $[0, 1.5]$. Write out your algorithm and implement it into **R**.

Answer: your answer starts here...

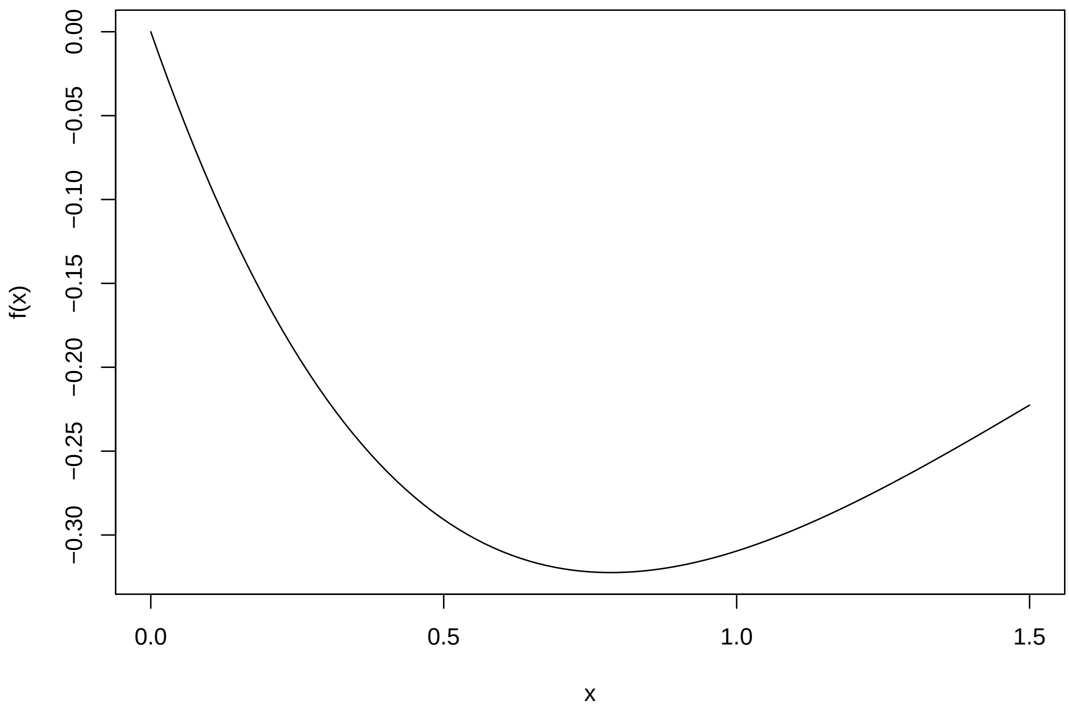
```
#R codes:

f = function(x){
  return(-sin(x)*exp(-x))
}


optimize(f, interval = c(0, 1.5))
```

```
## $minimum
## [1] 0.7854043
##
## $objective
## [1] -0.3223969
```

```
curve(f, from = 0, to = 1.5)
```




```r
w = .618
a =  0
b = 1.5
tol = 1e-10
i = 0  # iteration index
x1 = (1 - w) * (b - a) + a
x2 = x1 + w * (b - a) * (1 - w)
res = c(a, b, x1, x2)

findmin = function() {
  while (abs(b - a) > tol) {
    i <- i + 1
    if (f(b) > f(a)) {
      b <- x2
      x1 <- (1 - w) * (b - a) + a
      x2 <- x1 + w * (b - a) * (1 - w)
    }
    else {
      a <- x1
      x1 <- x2
      x2 <- x1 + w * (b - a) * (1 - w)
    }
    res <- rbind(res, c(i, a, b, x1, x2))
  }

    return(res)
```

```r
}

findmin()
```

```
##     [,1]      [,2]      [,3]      [,4]      [,5]
## res    0 1.5000000 0.5730000 0.9271140 0.0000000
##        1 0.5730000 1.5000000 0.9271140 1.1459565
##        2 0.5730000 1.1459565 0.7918694 0.9271306
##        3 0.5730000 0.9271306 0.7082779 0.7918796
##        4 0.7082779 0.9271306 0.7918796 0.8435455
##        5 0.7082779 0.8435455 0.7599501 0.7918836
##        6 0.7599501 0.8435455 0.7918836 0.8116184
##        7 0.7599501 0.8116184 0.7796874 0.7918851
##        8 0.7599501 0.7918851 0.7721493 0.7796884
##        9 0.7721493 0.7918851 0.7796884 0.7843475
##       10 0.7796884 0.7918851 0.7843475 0.7872269
##       11 0.7796884 0.7872269 0.7825681 0.7843477
##       12 0.7825681 0.7872269 0.7843477 0.7854475
##       13 0.7843477 0.7872269 0.7854475 0.7861272
##       14 0.7843477 0.7861272 0.7850275 0.7854476
##       15 0.7850275 0.7861272 0.7854476 0.7857072
##       16 0.7850275 0.7857072 0.7852872 0.7854476
##       17 0.7852872 0.7857072 0.7854476 0.7855468
##       18 0.7852872 0.7855468 0.7853863 0.7854476
##       19 0.7852872 0.7854476 0.7853485 0.7853863
##       20 0.7853485 0.7854476 0.7853863 0.7854097
##       21 0.7853863 0.7854476 0.7854097 0.7854242
##       22 0.7853863 0.7854242 0.7854008 0.7854097
##       23 0.7853863 0.7854097 0.7853953 0.7854008
##       24 0.7853953 0.7854097 0.7854008 0.7854042
##       25 0.7853953 0.7854042 0.7853987 0.7854008
##       26 0.7853953 0.7854008 0.7853974 0.7853987
##       27 0.7853974 0.7854008 0.7853987 0.7853995
##       28 0.7853974 0.7853995 0.7853982 0.7853987
##       29 0.7853974 0.7853987 0.7853979 0.7853982
##       30 0.7853979 0.7853987 0.7853982 0.7853984
##       31 0.7853979 0.7853984 0.7853981 0.7853982
##       32 0.7853981 0.7853984 0.7853982 0.7853983
##       33 0.7853981 0.7853983 0.7853982 0.7853982
##       34 0.7853981 0.7853982 0.7853981 0.7853982
##       35 0.7853981 0.7853982 0.7853982 0.7853982
##       36 0.7853982 0.7853982 0.7853982 0.7853982
##       37 0.7853982 0.7853982 0.7853982 0.7853982
##       38 0.7853982 0.7853982 0.7853982 0.7853982
##       39 0.7853982 0.7853982 0.7853982 0.7853982
##       40 0.7853982 0.7853982 0.7853982 0.7853982
##       41 0.7853982 0.7853982 0.7853982 0.7853982
##       42 0.7853982 0.7853982 0.7853982 0.7853982
##       43 0.7853982 0.7853982 0.7853982 0.7853982
##       44 0.7853982 0.7853982 0.7853982 0.7853982
##       45 0.7853982 0.7853982 0.7853982 0.7853982
##       46 0.7853982 0.7853982 0.7853982 0.7853982
##       47 0.7853982 0.7853982 0.7853982 0.7853982
```

```
## 48 0.7853982 0.7853982 0.7853982 0.7853982
## 49 0.7853982 0.7853982 0.7853982 0.7853982
```

```
min = last(findmin())
```

The minimum according to the golden ratio search method is 0.785. This closely approximates the result of optimize(), 0.785.

Problem 2

The Poisson distribution is often used to model "count" data — e.g., the number of events in a given time period.
The Poisson regression model states that

$$Y_i \sim \text{Poisson}(\lambda_i),$$

where

$$\log \lambda_i = \alpha + \beta x_i$$

for some explanatory variable x_i. The question is how to estimate α and β given a set of independent data $(x_1, Y_1), (x_2, Y_2), \ldots, (x_n, Y_n)$.

1. Modify the Newton-Raphson function from the class notes to include a step-halving step.
2. Further modify this function to ensure that the direction of the step is an ascent direction. (If it is not, the program should take appropriate action.)
3. Write code to apply the resulting modified Newton-Raphson function to compute maximum likelihood estimates for α and β in the Poisson regression setting.

The Poisson distribution is given by

$$P(Y = y) = \frac{\lambda^y e^{-\lambda}}{y!}$$

for $\lambda > 0$.

Answer: your answer starts here…

```
#R codes:

poissonstuff <- function(dat, betavec) {
  alpha = betavec[1]
  beta = betavec[2]
  log_lambda <- alpha + beta * dat$x
  lambda <- exp(log_lambda)
  loglik <- sum(dat$y * log_lambda - lambda - log(factorial(dat$y)))
  grad <- c(sum(dat$y * dat$x - dat$x * lambda),
            sum(dat$y - lambda))
  Hess <- matrix((-1) * c((-1) * sum(dat$x^2 * lambda),
                   rep((-1) * sum(dat$x * lambda), 2),
                   (-1) * sum(lambda)), ncol = 2)
  return(list(loglik = loglik, grad = grad, Hess = Hess))
}
```

```r
set.seed(22)
n <- 5000
truebeta <- c(1, .03)
x <- rnorm(n)
lambda <- exp(truebeta[1] + truebeta[2] * x)
y = rpois(n, lambda)
dat = list(x=x, y=y)
```

```r
NewtonRaphson <- function(dat, func, start, tol = 1e-5, maxiter = 200) {
  i <- 0
  cur <- start
  stuff <- func(dat, cur)
  l = 1
  prevloglik <- -Inf      # To make sure it iterates
  res = c(0, stuff$loglik, cur)
  while (i < maxiter && abs(stuff$loglik - prevloglik) > tol) {
    i = i + 1
    if (t(stuff$grad) %*% stuff$Hess %*% stuff$grad > 0) {
      Hess = stuff$Hess - 3*diag(max(stuff$Hess), nrow(stuff$Hess))
    }
    else {
      Hess = stuff$Hess
    }
    prev <- cur
    grad <- stuff$grad
    prevloglik <- stuff$loglik
    cur = prev - l * solve(Hess) %*% grad
    stuff = func(dat, cur)
    res <- rbind(res, c(i, stuff$loglik, cur)) # Add current values to results matrix
    while (stuff$loglik < prevloglik) {
      i = i + 1
      if (t(stuff$grad) %*% stuff$Hess %*% stuff$grad > 0) {
        Hess = stuff$Hess - 3*diag(max(stuff$Hess), nrow(stuff$Hess))
      }
      else {
        Hess = stuff$Hess
      }
      l = 0.5*l
      cur <- prev - l * solve(Hess) %*% grad
      stuff = func(dat, cur)
      res = rbind(res, c(i, stuff$loglik, cur))
    }
  }
  return(res)
}
```

```r
NewtonRaphson <- function(dat, func, start, tol = 1e-10, maxiter = 200) {
  i <- 0
  cur <- start
  stuff <- func(dat, cur)
  loglik <- stuff$loglik
  res <- c(0, stuff$loglik, cur)
  l = 1
```

```r
  prevloglik <- -Inf      # To make sure it iterates
  while (i < maxiter && abs(stuff$loglik - prevloglik) > tol) {
    i <- i + 1
    prevloglik <- stuff$loglik
    Hess <- stuff$Hess
    prev <- cur
    grad <- stuff$grad
    cur <- prev - l * solve(Hess) %*% grad
    stuff <- func(dat, cur)
    if (t(grad) %*% Hess %*% grad > 0) {
      Hess = Hess - 3*diag(max(Hess), nrow(Hess))
    }
    else {
      Hess = Hess
    }
    cur = prev - l * t(Hess) %*% grad
    stuff = func(dat, cur)
    while (stuff$loglik < prevloglik) {
    l = 0.5*l
    i = i + 1
    prevloglik = stuff$loglik
    Hess = stuff$Hess
    prev = cur
    grad = stuff$grad
    cur <- prev - l * t(Hess) %*% grad
    stuff <- func(dat, cur)        # log-lik, gradient, Hessian
    }
    res <- rbind(res, c(i, stuff$loglik, cur)) # Add current values to results matrix
  }
  return(res)
}
```

```r
NewtonRaphson(list(x=x, y=y), poissonstuff, start = c(1, -2))
```

```
##     [,1]       [,2]     [,3]      [,4]
## res    0  -96057.68 1.000000 -2.000000
##        1 -144188.99 1.221782 -2.095149
##        2 -108443.00 1.067971 -2.028832
##        3 -104615.89 1.047792 -2.020431
##        4 -100423.33 1.024963 -2.010683
##        5  -98325.47 1.013123 -2.005622
##        6  -97213.96 1.006734 -2.002887
##        7  -96642.04 1.003414 -2.001464
##        8  -96351.50 1.001720 -2.000738
##        9  -96205.01 1.000863 -2.000370
##       10  -96131.46 1.000432 -2.000185
##       11  -96094.60 1.000216 -2.000093
##       12  -96076.15 1.000108 -2.000046
##       13  -96066.92 1.000054 -2.000023
##       14  -96062.30 1.000027 -2.000012
##       15  -96059.99 1.000014 -2.000006
##       16  -96058.84 1.000007 -2.000003
##       17  -96058.26 1.000003 -2.000001
```

```
##       18  -96057.97 1.000002 -2.000001
##       19  -96057.83 1.000001 -2.000000
##       20  -96057.76 1.000000 -2.000000
##       21  -96057.72 1.000000 -2.000000
##       22  -96057.70 1.000000 -2.000000
##       23  -96057.69 1.000000 -2.000000
##       24  -96057.69 1.000000 -2.000000
##       25  -96057.69 1.000000 -2.000000
##       26  -96057.68 1.000000 -2.000000
##       27  -96057.68 1.000000 -2.000000
##       28  -96057.68 1.000000 -2.000000
##       29  -96057.68 1.000000 -2.000000
##       30  -96057.68 1.000000 -2.000000
##       31  -96057.68 1.000000 -2.000000
##       32  -96057.68 1.000000 -2.000000
##       33  -96057.68 1.000000 -2.000000
##       34  -96057.68 1.000000 -2.000000
##       35  -96057.68 1.000000 -2.000000
##       36  -96057.68 1.000000 -2.000000
##       37  -96057.68 1.000000 -2.000000
##       38  -96057.68 1.000000 -2.000000
##       39  -96057.68 1.000000 -2.000000
##       40  -96057.68 1.000000 -2.000000
##       41  -96057.68 1.000000 -2.000000
##       42  -96057.68 1.000000 -2.000000
##       43  -96057.68 1.000000 -2.000000
##       44  -96057.68 1.000000 -2.000000
##       45  -96057.68 1.000000 -2.000000
##       46  -96057.68 1.000000 -2.000000
##       47  -96057.68 1.000000 -2.000000
##       48  -96057.68 1.000000 -2.000000
##       49  -96057.68 1.000000 -2.000000
##       50  -96057.68 1.000000 -2.000000
##       51  -96057.68 1.000000 -2.000000
##       52  -96057.68 1.000000 -2.000000
```

```r
poissonstuff(list(x=x, y=y), c(1, -2))
```

```
## $loglik
## [1] -96057.68
##
## $grad
## [1] 197341.3 -85893.6
##
## $Hess
##           [,1]      [,2]
## [1,]  487160.0 -197009.5
## [2,] -197009.5   99545.6
```

problem 3

Consider the ABO blood type data, where you have $N_{\text{obs}} = (N_A, N_B, N_O, N_{AB}) = (26, 27, 42, 7)$.

- design an EM algorithm to estimate the allele frequencies, P_A, P_B and P_O; and
- Implement your algorithms in R, and present your results..

Answer: your answer starts here...

```
blood_dat = tibble(
  obs = c(26, 27, 42, 7),
  type = c("A", "B", "O", "AB")
)

pars = tibble(
  p_o = .33,
  p_a = .33,
  p_b = .33
)

fpars = function(p_a = p_a, p_b = p_b, p_o = p_o){
  pars = tibble(
   p_o,
   p_a,
   p_b
  )
}
```

```
n_aa = function(df = blood_dat, pars){
  df %>%
  filter(type == "A") %>%
  pull(obs) %>%
  prod(., pars$p_a^2, (pars$p_a^2 + 2 * pars$p_a * pars$p_o)^(-1))
}

n_ao = function(df = blood_dat, pars){
  df %>%
  filter(type == "A") %>%
  pull(obs) %>%
  prod(., 2, pars$p_a, pars$p_o, (pars$p_a^2 + 2 * pars$p_a * pars$p_o)^(-1))
}

n_bb = function(df = blood_dat, pars){
  df %>%
  filter(type == "B") %>%
  pull(obs) %>%
  prod(., pars$p_b^2, (pars$p_a^2 + 2 * pars$p_b * pars$p_o)^(-1))
}

n_bo = function(df = blood_dat, pars){
  df %>%
  filter(type == "B") %>%
  pull(obs) %>%
  prod(., 2, pars$p_b, pars$p_o, (pars$p_a^2 + 2 * pars$p_b * pars$p_o)^(-1))
```

```
}

n_ab = blood_dat %>%
  filter(type == "AB") %>%
  pull(obs)

n_oo = blood_dat %>%
  filter(type == "O") %>%
  pull(obs)
```

```
fnobs = function(df = blood_dat, pars = pars) {
  nobs = tibble(
    N_AA = n_aa(blood_dat, pars),
    N_AO = n_ao(blood_dat, pars),
    N_BB = n_bb(blood_dat, pars),
    N_BO = n_bo(blood_dat, pars),
    N_AB = n_ab,
    N_OO = n_oo
  )
  return(nobs)
}

nobs = fnobs(blood_dat, pars)
```

```
nobs = tibble(
  N_AA = n_aa(blood_dat, pars),
  N_AO = n_ao(blood_dat, pars),
  N_BB = n_bb(blood_dat, pars),
  N_BO = n_bo(blood_dat, pars),
  N_AB = n_ab,
  N_OO = n_oo
)
```

```
lik = function(nobs, pars) {
    loglik = nobs$N_AA * log(pars$p_a^2) + nobs$N_AO * log(2*pars$p_a*pars$p_o) + nobs$N_BB * log(pars$p
    return(loglik)
}
```

```
delta = function(df = blood_dat, nobs = nobs, pars = c(p_a = .33, p_b = .33, p_o = .33), tol = 1e-10, ma
  i = 0
  p_o = pars$p_o
  p_b = pars$p_b
  p_a = pars$p_a
  prevpars = fpars(p_a, p_b, p_o)
  prevloglik = -Inf
  loglik = lik(nobs, pars)
  prevnobs = fnobs(blood_dat, pars)
  res = c(0, loglik, prevpars)
  while (i < maxiter && abs(loglik - prevloglik) > tol) {
    i = i + 1
    prevloglik = loglik
    prevnobs = nobs
```

```r
    lambda = (2) * sum(nobs)
    p_a = (2 * prevnobs$N_AA + prevnobs$N_AO + prevnobs$N_AB)/(lambda)
    p_b = (2 * prevnobs$N_BB + prevnobs$N_BO + prevnobs$N_AB)/(lambda)
    p_o = (2 * prevnobs$N_OO + prevnobs$N_AO + prevnobs$N_BO)/(lambda)
    pars = fpars(p_a, p_b, p_o)
    nobs = fnobs(blood_dat, pars)
    loglik = lik(nobs, pars)
    res = rbind(res, c(i, loglik, pars))
  }
  return(res)
}
```

```r
delta(blood_dat, nobs, pars)
```

```
##                    p_o       p_a       p_b
## res 0  -196.8239 0.33      0.33      0.33
##     1  -151.4549 0.5849673 0.2042484 0.2107843
##     2  -148.1427 0.6319013 0.1802591 0.1878395
##     3  -147.8275 0.6379783 0.1771892 0.1848325
##     4  -147.7893 0.6387278 0.1768208 0.1844514
##     5  -147.7842 0.638821  0.1767777 0.1844013
##     6  -147.7835 0.638833  0.1767729 0.1843941
##     7  -147.7834 0.6388347 0.1767725 0.1843929
##     8  -147.7833 0.6388349 0.1767724 0.1843926
##     9  -147.7833 0.638835  0.1767724 0.1843926
##     10 -147.7833 0.638835  0.1767724 0.1843926
##     11 -147.7833 0.638835  0.1767724 0.1843926
##     12 -147.7833 0.638835  0.1767725 0.1843926
##     13 -147.7833 0.638835  0.1767725 0.1843926
##     14 -147.7833 0.638835  0.1767725 0.1843926
##     15 -147.7833 0.638835  0.1767725 0.1843926
##     16 -147.7833 0.638835  0.1767725 0.1843926
##     17 -147.7833 0.638835  0.1767725 0.1843926
##     18 -147.7833 0.638835  0.1767725 0.1843926
```