PHYS 381 – Computational Physics I (Winter 2025)

Assignment #2: The Pendulum problem

Due date: February 17, 2025

Group members:

Member #1: Jared Crebo (30085839)

Authors' contributions:

This assignment was completed solo. All work presented in this document and the related code was completed by Jared Crebo.

Abstract (0.5 points):

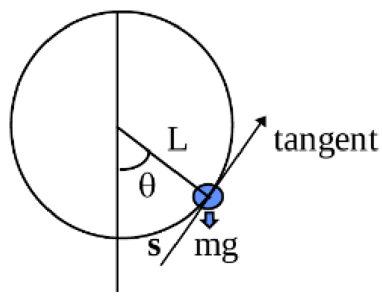
This assignment applies different numerical methods to simulate a swinging pendulum. The physics were simulated with linear and nonlinear equations, with the trapezoid and Runge Kutta methods, and with free oscillation, damped, and driven-damped oscillations. With each subsequent method, the physics being simulated will differ despite being simulations of the same physical phenomena. The purpose of the assignment is to know how the different assumptions in each method can affect your outcome and what numerical methods are advantageous or disadvantageous for a problem.

It was observed that the linear pendulum equations with the trapezoid method operated in a simple harmonic manner. When applying the nonlinear pendulum equations, the period of oscillation increased for the same initial conditions. Switching the numerical method from the trapezoid to the fourth-order Runge Kutta method, the period of oscillation increases again. Lastly, the effects of damping and driving forces were examined which show how the real physics of an oscillation would be modelled. With each iteration, the physics being simulated becomes more realistic at the cost of computation.

Introduction (0.5 points):

The simple pendulum is a system that is commonly analyzed in physics classes due its simplicity and its ability to demonstrate the dynamics of a mass-spring-damper system in a real context that students can understand. This assignment uses various numerical methods to approximate the dynamic motion of the pendulum with various initial conditions. The pendulum will be analyzed in both its linear and nonlinear forms. The effects of damping and driven oscillations will also be investigated. Two numerical methods will be used to analyze these physical phenomena: the trapezoidal rule and the Runge-Kutta method. Their advantages and disadvantages will be discussed and made applicable to the context of the physics being simulated.

Figure 1: Pendulum system



The first-order equations of motion for the pendulum system is derived in [1]:

$$\frac{d\theta}{dt} = \omega, \qquad \frac{d\omega}{dt} = f(\theta, \omega, t)$$

$$f(\theta, \omega, t) = -\frac{g}{L}\sin\theta - k\omega + Acos(\phi t)$$

$-\frac{g}{L}sin\theta$ represents the free oscillation of the pendulum in nonlinear form without damping effects or external driving forces acting on the pendulum. This form is nonlinear because of $sin\theta$ which is approximated as $sin\theta \approx \theta$ to linearize the equation. $-k\omega$ represents the real effects of damping on the pendulum such as air resistance and friction that would bring the oscillation to rest over time. $Acos(\phi t)$ is the external driving force with an amplitude, A, and a driving frequency of ϕ. The equation of motion in its linear form is:

$$\frac{d\theta}{dt} = \omega, \qquad \frac{d\omega}{dt} = f(\theta, \omega, t)$$

$$f(\theta, \omega, t) = -\frac{g}{L}\theta - k\omega + Acos(\phi t)$$

The differences between the linear and nonlinear forms will be investigated in this assignment.

Methods (1 point):

The equations of motion, whether linear or nonlinear, are a system of coupled ordinary differential equations. The first method to solve these equations is derived from the simple Euler method, where:

$$\theta_{n+1} = \theta_n + \omega_n \Delta t$$

$$\omega_{n+1} = \omega_n + f(\theta_n, \omega_n, t_n)\Delta t$$

The trapezoid rule improves this method by calculating the change in θ over one timestep using the average ω between the two times. This results in the method being implicit, rather than explicit, because ω_{n+1} is required to solve for itself (it appears on both sides of the equation). This intermediate ω_{n+1} is approximated using the Taylor series in the same method as the simple Euler:

$$\omega_{n+1}^0 = \omega_n + f(\theta_n, \omega_n, t_n)\Delta t$$

And then the subsequent equations for the trapezoidal method are:

$$\theta_{n+1} = \theta_n + \frac{\Delta t}{2}[\omega_n + \omega_{n+1}^0]$$

$$\omega_{n+1} = \omega_n + \frac{\Delta t}{2}[f(\theta_n, \omega_n, t_n) + f(\theta_{n+1}, \omega_{n+1}^0, t_{n+1})]$$

The second-order Runge-Kutta method is similar to the trapezoid method, but the Taylor series expansion is performed at the midpoint of the timestep which improves the order of accuracy from $O(\Delta t)$ to $O(\Delta t^2)$.

$$\theta_{n+1/2} = \theta_n + \omega_n \frac{\Delta t}{2}$$

$$\omega_{n+1/2} = \omega_n + f(\theta_n, \omega_n, t_n) \frac{\Delta t}{2}$$

$$\theta_{n+1} = \theta_n + \omega_{n+1/2} \frac{\Delta t}{2}$$

$$\omega_{n+1} = \omega_n + f\left(\theta_{n+\frac{1}{2}}, \omega_{n+\frac{1}{2}}, t_{n+\frac{1}{2}}\right) \Delta t$$

In this assignment, the fourth-order Runge-Kutta method is used, which takes the Taylor series expansion at four points rather than one midpoint between the timestep. To organize the math, the segments are broken in to individual equations corresponding to each Taylor series expansion of its respective point.

$$k_{1,\theta} = \mathrm{\omega_n \Delta t}$$

$$k_{1,\omega} = f(\theta_n, \omega_n, t_n)\mathrm{\Delta t}$$

$$k_{2,\theta} = \left(\omega_n + \frac{k_{1,\omega}}{2}\right) \Delta t$$

$$k_{2,\omega} = f\left(\theta_n + \frac{k_{1,\theta}}{2}, \omega_n + \frac{k_{1,\omega}}{2}, t_n + \frac{\Delta t}{2}\right) \Delta t$$

$$k_{3,\theta} = \left(\omega_n + \frac{k_{2,\omega}}{2}\right) \Delta t$$

$$k_{3,\omega} = f\left(\theta + \frac{k_{2,\theta}}{2}, \omega + \frac{k_{2,\omega}}{2}, t_n + \frac{\Delta t}{2}\right) \Delta t$$

$$k_{4,\theta} = \left(\omega_n + \frac{k_{3,\omega}}{2}\right) \Delta t$$

$$k_{4,\omega} = f\left(\theta + \frac{k_{3,\theta}}{2}, \omega + \frac{k_{3,\omega}}{2}, t_n + \frac{\Delta t}{2}\right) \Delta t$$

$$\theta_{n+1} = \theta_n + (k_{1,\theta} + 2k_{2,\theta} + 2k_{3,\theta} + k_{4,\theta})/6$$

$$\omega_{n+1} = \omega_n + (k_{1,\omega} + 2k_{2,\omega} + 2k_{3,\omega} + k_{4,\omega})/6$$

Each constant *k* is the slope at a point, and the final iteration is a weighted average of these slopes with more weight given to the two midpoints k_2 and k_3.

Code workflow (1 point):

The code begins with a declaration of the parameters that will remain constant throughout the code. These are the number of transient timesteps, the acceleration of gravity, the length of the pendulum, the damping coefficient, and the driving force frequency of the oscillation.

A function, *f_nonlin*, is defined that takes in the variables of the pendulum function. These are the angle, the angular velocity, the current timestep, and the driving force amplitude of the oscillation. The function returns the second derivative of θ in its nonlinear form as mentioned in the Introduction.

The main function, *plot_nonlinear_pendulum*, allows for repeated test cases with varying initial conditions to examine the robustness of the code. The two initial conditions, θ_o and ω_o, and the external force amplitude, A, are passed through the parameters of the function. The angles and angular velocities at each timestep are stored in an array, with the initial conditions being placed at the 0^{th} index. The fourth-order Runge-Kutta method is used as per the Methodology above. At the end of the method, there is case handling for when $\theta > 2\pi$ or $\theta < -2\pi$ so that the final plot stays on the same axes $[-\pi, \pi]$. The function iterates over the fourth-order Runge-Kutta method and the case handling for every timestep except the last, since there is no next timestep after the last.

The subsequent results are plotted twice; once as a zoomed in section of the steady-state oscillations driven by A and ϕ, and again as a display of the entire motion of the pendulum from initial state until the last timestep.

Results and analysis (1 point):

Figure 2: Motion of Linear Pendulum Swing, Trapezoid Method

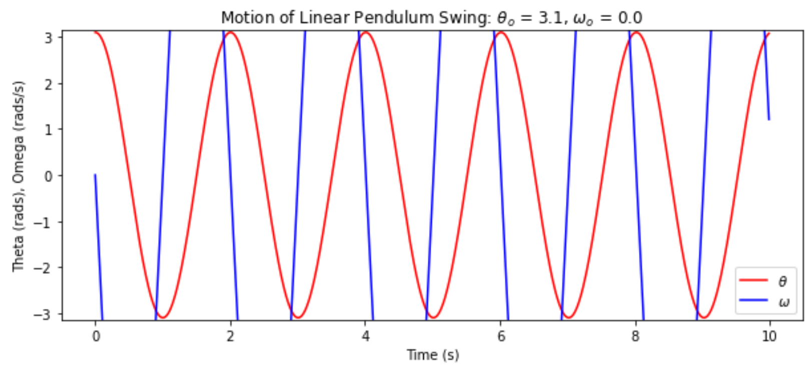
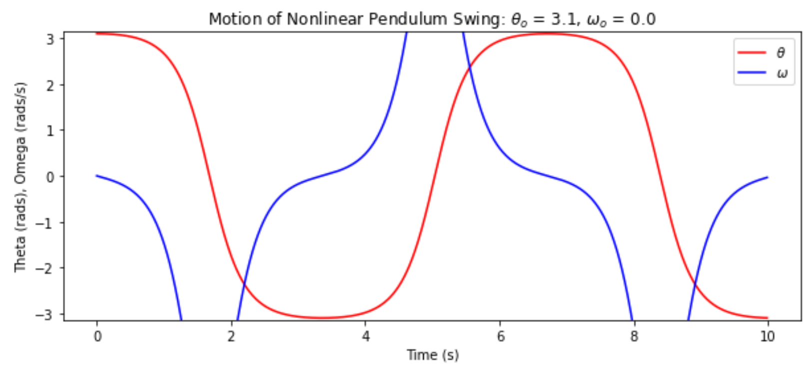


Figure 3: Motion of Nonlinear Pendulum Swing, Trapezoid Method



Figures 2 and 3 show the trapezoid method being used to approximate the motion of the linear and nonlinear pendulum equations with the same initial conditions. The linear equations behave as a perfect harmonic oscillator. The assumption being made to linearize the equations is the small angle approximation, which is only valid if the angle between timesteps is very small. The nonlinear equations are theoretically more accurate to the physical counterpart. The plot shows the nonlinear pendulum holds its position near the top of its oscillation before swinging back.

Figure 4: Motion of Nonlinear Pendulum Swing, Runge Kutta Method

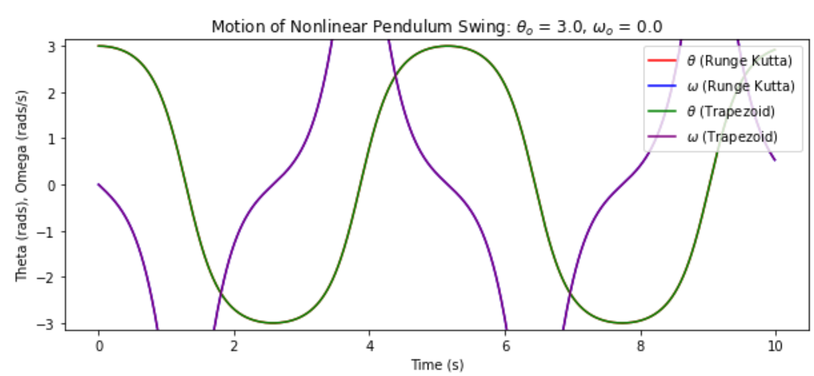


Figure 4 compares the difference between the trapezoid and Runge Kutta methods for the nonlinear pendulum equations. The oscillations are exactly identical because they are both using the nonlinear form of the equations and the timestep is small enough to retain accuracy across both methods. From how these methods are formulated, it can be said that the Runge Kutta method is more accurate since it is of fourth-order accuracy, whereas the trapezoid method is only second order accuracy. However, this discrepancy is not noticeable due to the small timestep used, therefore the trapezoid rule is preferred due to its simplicity.

Figure 5: Motion of Damped Nonlinear Pendulum Swing, Runge Kutta Method

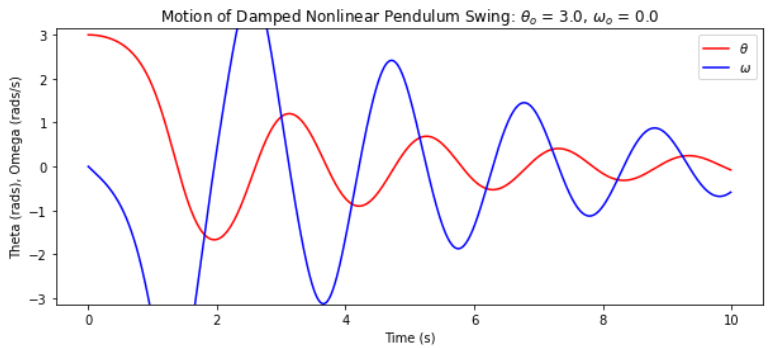


Figure 5 includes the element of damping on the oscillation, using a damping coefficient of $k = 0.5$. The damping coefficient simulates a force that acts in the opposite direction to the angular velocity. Without damping, the system will oscillate steadily forever, which does not represent the real physics of friction and air resistance that would slow the pendulum. The damping will cause the free oscillation of the pendulum to come to a rest, unless it is driven by an external force.

Figure 6: Phase Plot of Driven-Damped Nonlinear Pendulum, Runge Kutta Method, A=1.07

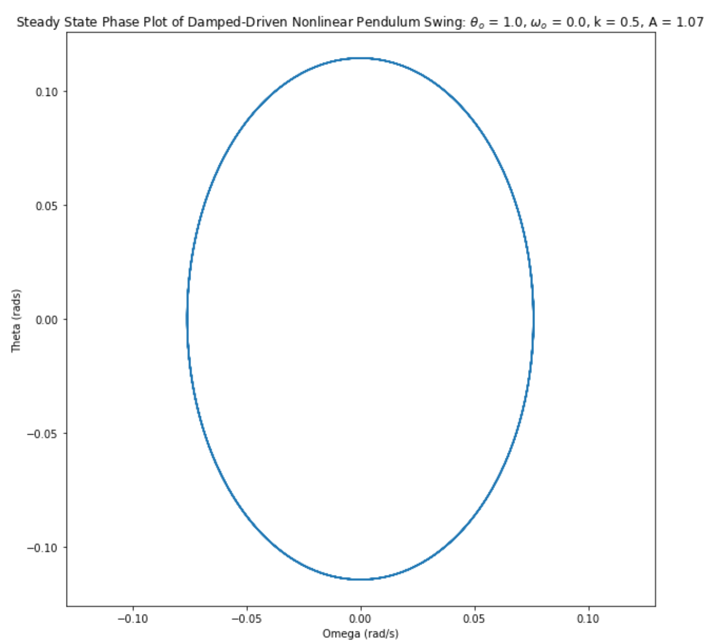
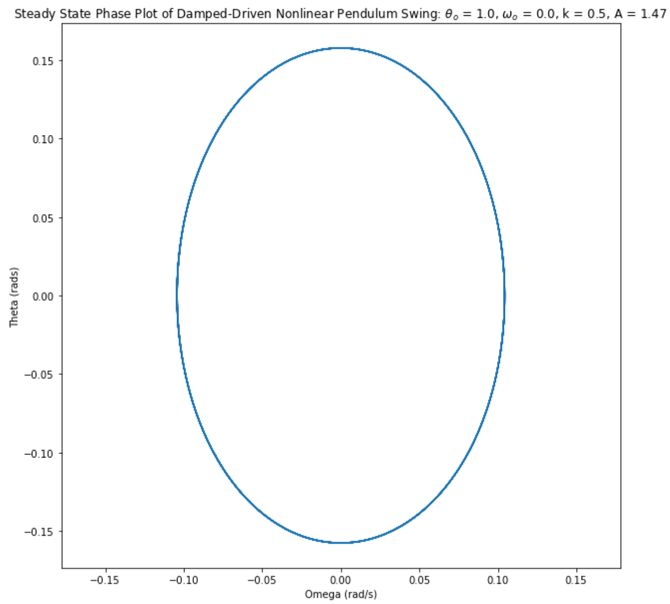


Figure 7: Phase Plot of Driven Damped Nonlinear Pendulum, Runge Kutta Method, A=1.47



Figures 7 and 8 use the same damping coefficient as in Figure 6, $k = 0.5$, except they introduce an external oscillatory force that continues driving the pendulum. The difference between Figures 7 and 8 is the amplitude of the external force. This will directly correlate to different amplitudes in the steady state θ and ω values in each figure. Figure 8 (A=1.47) shows a slightly higher amplitude than Figure 7 (A=1.07). And since their respective driving frequencies are equivalent, the steady state periods of oscillation are also equal.

Conclusions (0.5 points):

Overall, this assignment applied the theory of numerical methods learned in lectures to simulate a swinging pendulum. The physics were simulated with linear and nonlinear equations, with the trapezoid and Runge Kutta methods, and with free oscillation, damped, and driven-damped oscillations. With each subsequent method, the physics being simulated will differ despite being simulations of the same physical phenomena. The purpose being to know how different assumptions can affect your outcome and what numerical methods are advantageous or disadvantageous for a problem.

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References:

[1] PHYS 381 Assignment 1: Finding the Minima of Functions, *PHYS 381: Computational Physics I,* Department of Physics and Astronomy, University of Calgary, Winter 2025

Other (0.5 points):

This page is to be filled by the instructor or TAs ONLY.

Remaining 0.5 points are granted for following the template and overall quality of the report.

Was the assignment submitted before the due date (mark one option)? YES NO