PHYS 381 – Computational Physics I (Winter 2025)

Assignment #3: The Pendulum problem

Due date: March 10, 2025

Group members:

Member #1: Jared Crebo (30085839)

Authors' contributions:

This assignment was completed solo. All work presented in this document and the related code was completed by Jared Crebo.

Abstract (0.5 points):

This assignment investigates the effects of air resistance on projectile motion using three models: linear air resistance, quadratic air resistance, and no air resistance. The trajectories are modelled using the Explicit Euler method, a numerical method of first order accuracy, and are plotted against each other for comparison. These air resistance models are plotted for a variety of initial conditions with varying initial velocities, masses, and launch angles to determine the effects of the assumptions made in each model. Results show that air resistance significantly impacts projectiles with low mass and high velocity. It also reveals the relationship between the launch angle and projectile height and distance travelled. These findings highlight the importance of accounting for air resistance in real-world projectile motion.

Introduction (0.5 points):

Projectile motion is one of the earliest concepts taught in the field of classical physics. Commonly, projectile motion is modeled using the basic kinematic equations under the assumption that air resistance is negligible. Perhaps this is true on the moon, but on earth there exists a force of air resistance that acts on the projectile.

$$\vec{F} = -f(v)\hat{u}$$

$\hat{u}$ is the unit vector in the direction of the projectile's velocity, v, and $f(v) = bv + cv^2$. This air resistance force depends on the velocity of the object as well as the shape of the object and the properties of the medium it is travelling through. The shape and air properties are captured in the coefficients $b = BD$ and $c = CD^2$, where $B = 1.6 * 10^{-4}\ Ns/m^2$, and $C = 0.25\ Ns^2/m^4$ for a spherical object of diameter D.

When modelling the trajectory of a 2D projectile, there are four initial conditions: x_o, y_o, v_{xo}, v_{yo} corresponding to the initial (x,y) coordinates and velocities in 2D Euclidean space. Two equations are required to govern the motion of the projectile, one that updates the x component and one that updates the y component.

$$x_{i+1} = x_i + v_{xi}\Delta t$$

$$y_{i+1} = y_i + v_{yi}\Delta t$$

The function $f(v)$ is used to calculate the v_{xi} and v_{yi} components at each timestep. However, this function is a nonlinear function and requires simplification to be able to iterate. Depending on the scale of the velocity, it can be assumed that either the linear term or the quadratic term dominates the function, and the other component can be dropped.

For small velocities, $f(v) \approx bv$, and therefore:

$$\frac{dv_x}{dt} = -\frac{b}{m}v_x$$

$$\frac{dv_y}{dt} = -g - \frac{b}{m}v_y$$

And for larger velocities, $f(v) \approx cv^2$, and therefore:

$$\frac{dv_x}{dt} = -\frac{c}{m}\sqrt{v_x^2 + v_y^2}v_x$$

$$\frac{dv_y}{dt} = -g - \frac{c}{m}\sqrt{v_x^2 + v_y^2}v_y$$

From question 1, $f(v) = \begin{cases} bv, & if\ Dv < 2 * 10^{-5} \\ bv + cv^2, & if\ 2 * 10^{-5} < Dv < 0.1 \\ cv^2, & if\ 0.1 < Dv \end{cases}$

Methods (1 point):

The numerical method used in this assignment is the Explicit Euler method. The Euler method is a basic method of solving ordinary differential equations of the form:

$$\frac{dy}{dt} = f(t, y)$$

Given an initial condition (t_o, y_o), the method approximates the solution using the formula:

$$y_{i+1} = y_i + f(t_i, y_i)\Delta t$$

Where Δt is an increment in time, y_i is the current value of the function, $f(t_i, y_i)$ is the slope dy/dt, and y_{i+1} is the next value that is calculated at time $t + \Delta t$. The explicit Euler method has an accuracy of $O(\Delta t)$, which is a low order of accuracy and can lead to large errors if Δt is not small enough or if the function contains a rapid change.

In the context of projectile motion, these are computed for $\frac{dvy}{dt} = f(v)$ and $\frac{dvx}{dt} = f(v)$ in 2D space.

Code workflow (1 point):

The code in question 4a begins by importing the relevant packages and defining the parameters of the projectile such as its mass and diameter.

The first function, *position_air_resistance_quad*, passes the mass, C coefficient, diameter, and initial conditions. This function assumes a quadratic dominant air resistance and uses the equations 6 and 7 from [1] to simulate the projectile's motion. It makes use of Python lists rather than NumPy arrays because lists are dynamic and thus allow for changing size. This is required since the length of the simulation depends on how long the projectile takes to hit the ground, which will vary depending on the initial conditions. The code uses a while loop with the breaking condition that $y > 0$. For every iteration, *dvy* and *dvx* are computed using equations 6 and 7 and are added to the previous value of *vy* and *vx*. From there, *y* and *x* are updated with:

$$x_{i+1} = x_i + v_{xi}\Delta t$$

$$y_{i+1} = y_i + v_{yi}\Delta t$$

This iterates until the breaking condition is met and the function returns the three lists containing the time, *x*, and *y* coordinates at each timestep.

The following function, *position_air_resistance_lin* is almost identical to *position_air_resistance_quad*, except it passes the B coefficient instead of the C coefficient since this function assumes a linear dominant air resistance. It therefore uses the equations 4 and 5 from [1] to compute *dvy* and *dvx* at each timestep. From this point in the code, the rest is identical to the first function and the *y* and *x* coordinates are updated and returned.

The third function, *position_vacuum*, is more simplistic and uses the basic kinematic equations to solve for *x* and *y*.

$$x_{i+1} = x_i + v_{xo}t$$

$$y_{i+1} = y_i + v_{yo}t - 0.5gt^2$$

The only parameters passed into the function are the mass and the initial conditions, since this function assumes no air resistance. Again, it makes use of the while loop and iterates under the condition that the projectile is above the ground.

The final section of this code is a set of plots comparing the trajectories of all 3 functions, given the same parameters. Each figure varies its initial velocity, mass, and launching angles. The results of these figures will be discussed in the following section.

Results and analysis (1 point):

Figure 1: Projectile Motion, Lower Mass, Lower Velocity

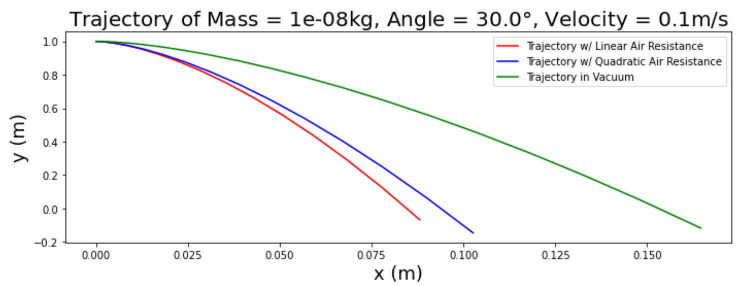


Figure 2: Projectile Motion, Higher Mass, Lower Velocity

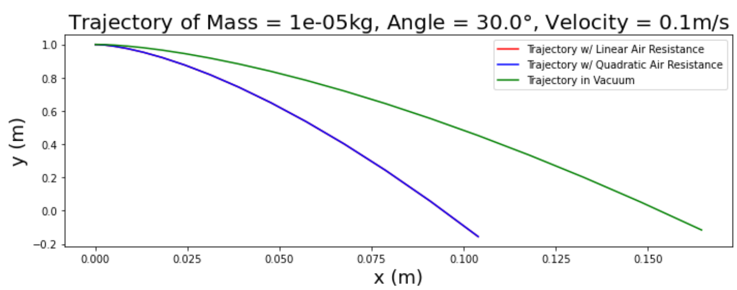


Figure 1 and 2 show the trajectories of six projectiles launched with the same initial conditions, except for their mass and assumptions. Each plot in the figure represents the trajectory of the projectile using the linear air resistance model, the quadratic air resistance model, and the projectile motion in a vacuum. Between the two graphs, the mass is increased by a factor of 10^3. At very low masses, the force of air resistance differs between the linear and quadratic motions. At higher masses, the difference between the linear and quadratic motions is negligible. This is likely because projectiles with higher mass have more inertia and are therefore less affected by the air resistance.

Figure 3: Projectile Motion, Lower Mass, Higher Velocity

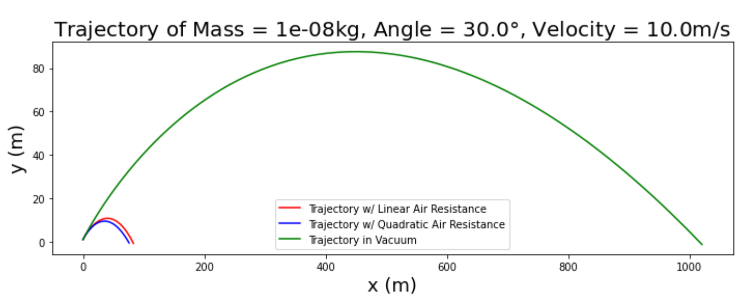
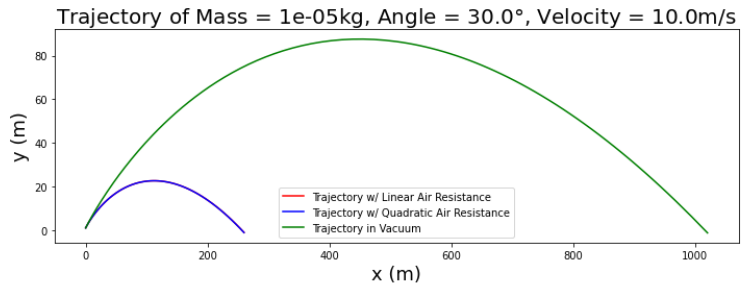
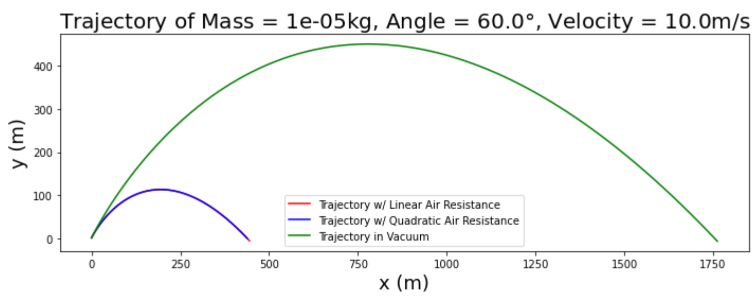


Figure 4: Projectile Motion, Higher Mass, Higher Velocity



Figures 3 and 4 prove this phenomenon is true for varied velocities as well. As shown in the figures, the linear and quadratic air resistance models are nearly identical for a projectile with a higher mass, regardless of its velocity. As the velocity increases, the discrepancy between the projectile in a vacuum and the projectiles affected by air resistance grows compared to Figures 1 and 2. This is because the force of air resistance is a function of velocity, so an increase in the initial velocity causes greater effects of air resistance.

Figure 5: Projectile Motion, Varied Launching Angle



Lastly, Figure 5 demonstrates that the total distance and height that the projectile travels are highly dependent on the launch angle. Comparing Figure 5 to Figure 4, a launch angle of 60° instead of 30° causes the air-resistance-projectile distance travelled to increase by ~60% and the vacuum-projectile distance travelled to increase by ~75%. This shows the relationship between the trajectory distance and the launch angle differs when accounting for air resistance.

Conclusions (0.5 points):

In this assignment, the effects of air resistance on projectile motion were analyzed through numerical methods. The Explicit Euler method was used to compute projectile trajectories under the assumptions of linear air resistance, quadratic air resistance, and no air resistance and their resulting behaviours were compared. The results demonstrate a significant impact of air resistance on projectile motion for objects with low mass or high velocity.

For projectiles with lower mass, the differences between linear and quadratic air resistance models were significant because the force of air resistance has a greater relative effect. When the mass was increased, the difference between both models became negligible because of the projectile's increased inertia.

As the initial velocity of the projectile increased, the force of air resistance was more significant as well. The evidence shows a major discrepancy between the trajectories of projectiles with high velocity in a vacuum and with air resistance, since air resistance is a function of velocity.

Lastly, the launch angle plays a critical role in determining the range and height of the projectile, regardless of the air resistance assumptions. However, the relationship between range, height, and the initial launch angle does differ between air resistance and vacuum models.

It is also important that these results are taken within the context of the numerical method being used, the Explicit Euler method, which is of first order accuracy and therefore is not very good at modelling curves with rapid changes in slope. This could affect the outcomes presented and can be improved upon by using higher order numerical methods.

References:

[1] PHYS 381 Assignment 3: Projectile motion under air resistance, *PHYS 381: Computational Physics I,* Department of Physics and Astronomy, University of Calgary, Winter 2025

Other (0.5 points):

This page is to be filled by the instructor or TAs ONLY.

Remaining 0.5 points are granted for following the template and overall quality of the report.

Was the assignment submitted before the due date (mark one option)?   YES   NO