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# ENME 570

## Assignment 2: Vortex Panel Method

UNIVERSITY OF CALGARY

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Due: December 1, 2024

# Introduction

The **vortex panel method** (VPM) is a numerical technique used to analyze potential flow around a 2D object, typically an airfoil. The method involves dividing the surface of the airfoil into a series of discrete panels, each carrying a vortex of constant strength. By enforcing the boundary condition that the flow must remain tangent to the surface of the airfoil at each panel, a system of equations is established to determine the vortex strength distribution along the surface. Additionally, the Kutta condition at the trailing edge is applied to ensure realistic flow behavior. Once the vortex strengths are calculated, they are used to determine the circulation and subsequently the coefficient of lift via the Kutta-Joukowski Theorem. This approach provides a fast, effective way to estimate lift and pressure characteristics of airfoils without solving the full Navier-Stokes equations, making it ideal for early-stage airfoil design and analysis. [1]

This assignment is to develop the code for a VPM and validate its robustness against literature sources for multiple airfoils. The airfoils chosen for this assignment are:

- NACA 0012 (thin symmetric)
- NACA 0025 (thick symmetric)
- NACA 2412 (thin cambered)

# Methodology

## Discretization

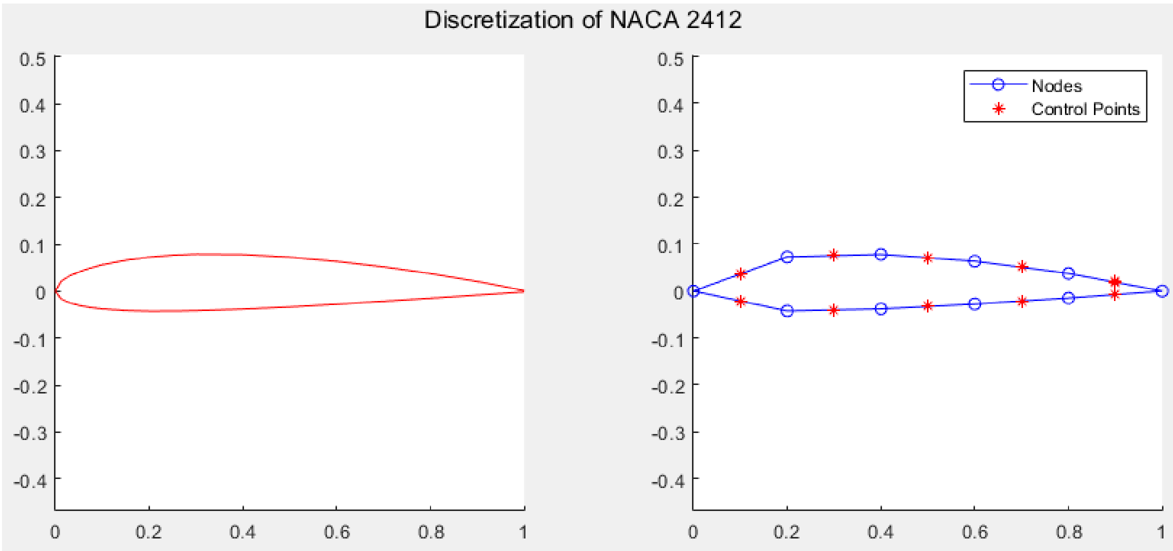



*Figure 1: Discretization of NACA 2412 with 10 panels*

Figure 1 shows plot of the an airfoils being analyzed in this assignment. The left plot is the raw data of the airfoil shape, taken from airfoiltools.com [2]. The right plot is the discretized airfoil visualized to validate the method of discretization. This figure is discretized into 10 panels for a clear visualization of the panel nodes and control points. The actual analysis was conducted after testing the sensitivity of the solution to various panel resolutions, which resulted in 100 panels being chosen.

The splines are also initially split into their respective top and bottom surfaces. This ensures there is a node at the exact leading and trailing edge of the airfoil. However, this requires that the number of panels remain an even number.

The methodology of discretization is linear along the x-axis. The raw (x,y) datapoints are first fitted to a spline, and then reverted back into (x,y) datapoints according to the specified amount of panels to be used. However, because the points are spaced along the x-axis, panels with large changes in the y direction will have less refinement. The effect of this is most evident in thicker airfoils such as the NACA 0025.

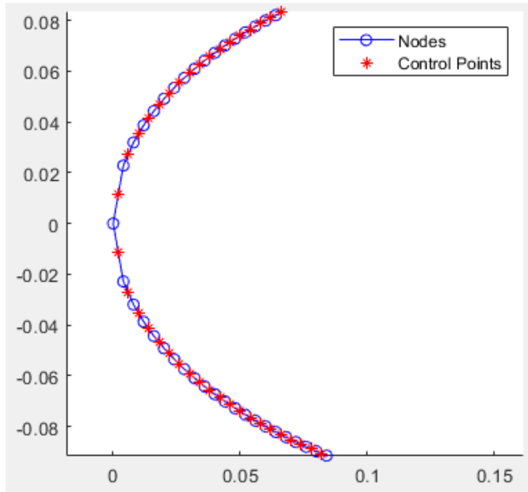



*Figure 2: Discretization of NACA 0025 with 500 panels*

This is a source of error that can be fixed through a different method of discretization. Future improvements should discretize the airfoil along the spline so that each panel's length is approximately equal.

The data structures containing the coordinates of the control points and panel nodes are in order from the top trailing edge surface, around the airfoil, and end at the bottom trailing edge surface. This means the trailing edge panels are located at the first and last indexes of every matrix. This will be useful for implementing the Kutta condition.

# Mathematical Model

The vortex panel method is a numerical method based on potential flow theory to develop efficient analyses of aerodynamics without the need to solve the fluid domain. The mathematical model for this method begins with simulating the wrapping of a vortex sheet around the surface of the airfoil, with vortex strengths $\gamma(s)$ varying such that the body surface becomes a streamline of the flow and the Kutta condition is satisfied. Through the equations outlined in Anderson's *Fundamentals of Aerodynamics* [1], this can be discretized and computed as a system of linear equations to solve for these vortex strengths.

$$V_\infty \cos(\Phi_i - \alpha) - \sum_{j=1}^{n} \frac{\gamma_j}{2\pi} J_{i,j} = 0$$

This equation is a statement that satisfies the body surface becoming a streamline of the flow, in a form that is computable with discrete panels of constant vortex strength. Converting this equation into matrix form:

$$\frac{1}{2\pi} J_{nxn} \gamma_{nx1} = -V_\infty \cos(\Phi_{nx1} - \alpha)$$

This becomes a system of $n$ equations with $n$ unknowns, $\gamma_1, \gamma_2, \dots, \gamma_n$. Expanding each matrix shows the system to be solved.

$$\frac{1}{2\pi}\begin{bmatrix} J(1,1) & \cdots & J(1,n) \\ \vdots & \ddots & \vdots \\ J(n,1) & \cdots & J(n,n) \end{bmatrix}\begin{bmatrix} \gamma(1) \\ \vdots \\ \gamma(n) \end{bmatrix} = \begin{bmatrix} -V_\infty \cos(\Phi(1) - \alpha) \\ \vdots \\ -V_\infty \cos(\Phi(\mathrm{n}) - \alpha) \end{bmatrix}$$

To apply the Kutta condition at the trailing edge, an extra equation must be added to the system of equations, and subsequently one must be removed to keep the system of equations unique and solvable. A sensitivity analysis was conducted to determine which panel to omit. In this code, the upper leading edge panel was excluded to adopt the Kutta condition, which is located at the $i = \frac{n}{2} - 1$ index in the matrix. The condition states:

$$\gamma_1 + \gamma_n = 0$$

Therefore, this is implemented directly into the the J matrix.

$$\frac{1}{2\pi}\begin{bmatrix} J(1,1) & & \cdots & & J(1,n) \\ \vdots & & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & & \vdots \\ J(n,1) & & \cdots & & J(n,n) \end{bmatrix}\begin{bmatrix} \gamma(1) \\ \vdots \\ \gamma(n) \end{bmatrix} = \begin{bmatrix} -V_\infty \cos(\Phi(1) - \alpha) \\ \vdots \\ 0 \\ \vdots \\ -V_\infty \cos\,(\Phi(n) - \alpha) \end{bmatrix}$$

Now to define the terms of this equation.

- $\Phi$ : the angle of the panel with respect to the chord (x-axis)
- $\alpha$ : angle of attack
- $V_\infty$ : freestream velocity
- $\gamma$ : constant vortex strength of panel
- $J$ : influence coefficient matrix

From the University of Calgary's ENME 570: Aerodynamics lecture notes [3],

$$J(i,j) = \frac{C}{2}\ln\left(\frac{S_j^2 + 2AS_j^2 + B}{B}\right) + \frac{D - AC}{E}\left(arctan\left(\frac{S_j + A}{E}\right) - arctan\left(\frac{A}{E}\right)\right)$$

$$A = -(x_i - X_j)cos\Phi_j - (y_i - Y_j)sin\Phi_j$$

$$B = (x_i - X_j)^2 + (y_i - Y_j)^2$$

$$C = -\cos\,(\Phi_i - \Phi_j)$$

$$D = (x_i - X_j)cos\Phi_i - (y_i - Y_j)sin\Phi_i$$

$$E = \sqrt{B - A^2}$$

Where,

- S : length of panel
- (x,y) : coordinates of panel control point
- (X,Y) : coordinates of panel nodes

The diagonal terms of J were set to zero, as they represent the influence of a panel on itself. Special case handling was done for when E is imaginary where only the real component is taken.

Each of these coefficients is computed for each entry of J(i,j) for every panel. The solution to $\gamma$ is calculated in MATLAB and subsequently the circulation. Using the Kutta-Joukowski Theorem, the coefficient of lift can be determined from the circulation. [1]

$$\Gamma = \sum_{i=1}^{n} \gamma_i S_i$$

$$Cl = \frac{2\Gamma}{V_\infty(1)}$$

The Cl is determined for each airfoil at angles of attack ranging from 0 to 16 degrees. The lift curves are plotted in the following section.

# Results and Discussion

## Panel Omission Sensitivity Analysis

The sensitivity test was conducted on both the top and bottom surfaces of the NACA 2412 airfoil since its asymmetry will cause greater variation in its solution, making it more sensitive to these changes. The methodology of this analysis is to begin with six panels that represent a region of the airfoil surface. There will be a panel representing the leading edge, midsection, and trailing edge on both the upper and lower surfaces.

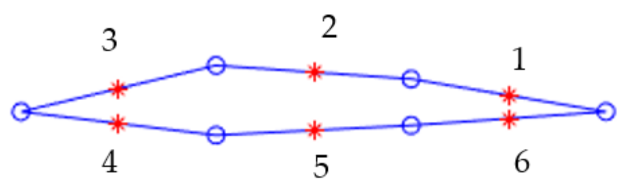



*Figure 3: Discretization of NACA 2412 into 6 panels*

The VPM code will be executed six times, each time replacing one of these six panels with the Kutta condition to determine the general region that affects the solution the least. These plots are shown next to literature.

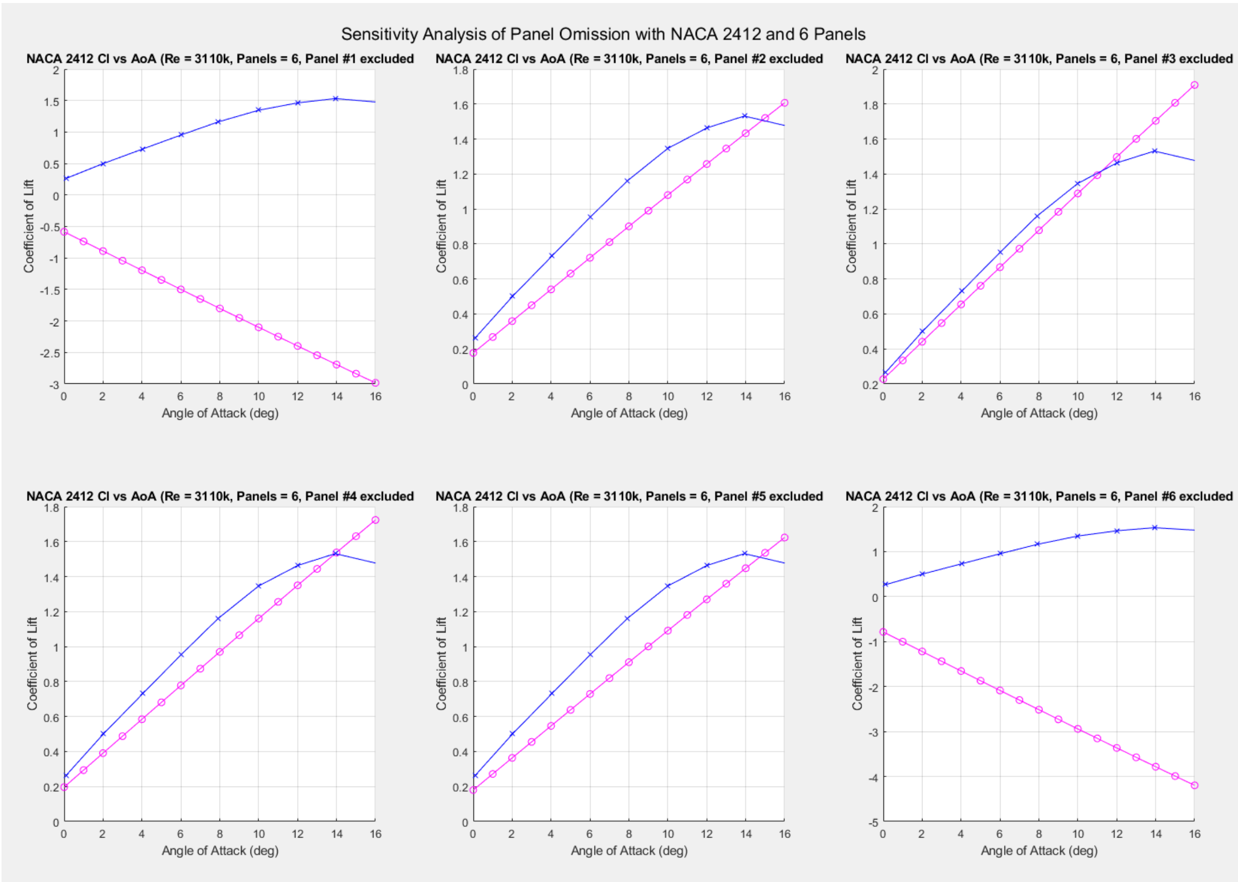



*Figure 4: Sensitivity Analysis of Panel Omission with Naca 2412 and 6 Panels*

Evidently, the 3rd panel gives the most appropriate solution for Cl which corresponds to the leading edge panel of the upper surface. The panel resolution is once again refined by a factor of six (36 panels), so each panel in the last test is split into six new panels and the process is repeated within the 3rd panel of the last test.

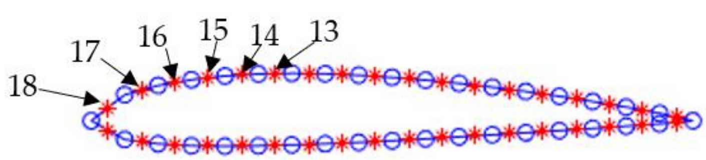



*Figure 5: Discretization of NACA 2412 into 36 panels*

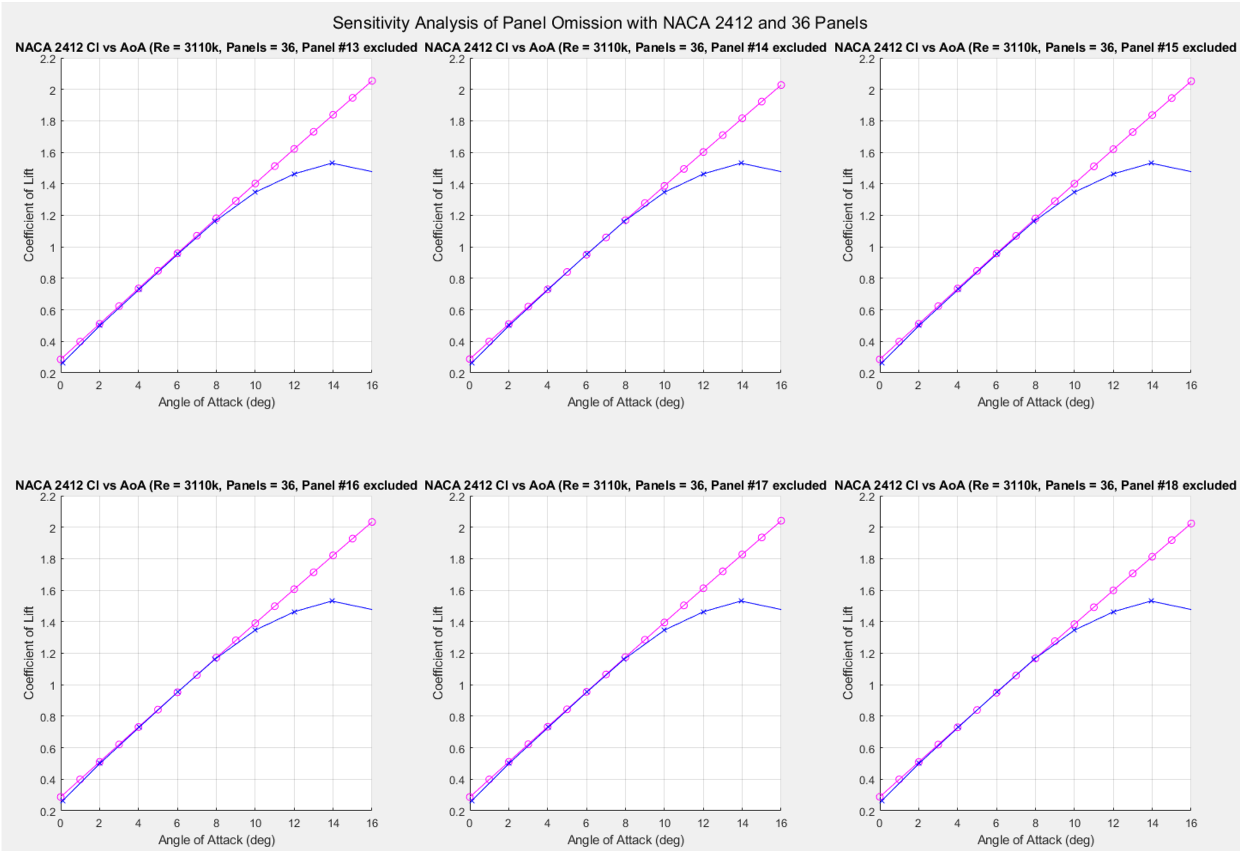



*Figure 6: Sensitivity Analysis of Panel Omission with NACA 2412 and 36 panels*

It was determined after only two iterations of this analysis to omit the upper leading edge surface to implement the Kutta condition. Between each plot in Figure 6 there is minimal difference in the effect of which panel is replaced and taking it to be the exact leading edge allows for easy indexing.

# Panel Resolution Sensitivity Analysis

The grid convergence study is a standardized method of determining whether the solution of a numerical method is independent of its discretization. In this case, it will determine whether the solution of the vortex panel method is independent of the panel resolution.

NACA 2412 was used as the test subject for the number of panels since it is the most sensitive and does not have its lift curve anchored at (0,0) like the symmetric airfoils. Both the slope of the curve and its $Cl_{\alpha=0}$ will be taken as the solution and a typical grid convergence study will be conducted. The results of this study are tabulated below.

*Table 1: Grid Convergence Study for Vortex Panel Method*

| No. Panels | 50 | 100 | 200 | $GCI_{50\text{-}100}$ | $GCI_{100\text{-}200}$ |
|---|---|---|---|---|---|
| Slope = $\Delta Cl$ / $\Delta\alpha$ | 0.109404 | 0.110564 | 0.109977 | 1.61% | 0.82% |
| $Cl_{\alpha=0}$ | 0.286361 | 0.311761 | 0.309584 | 1.15% | 0.10% |

With the GCI for all refinement levels less than 2%, the mid-level refinement of 100 panels can be taken as the appropriate panel resolution that produces an independent solution.

# Symmetric Airfoil Analysis

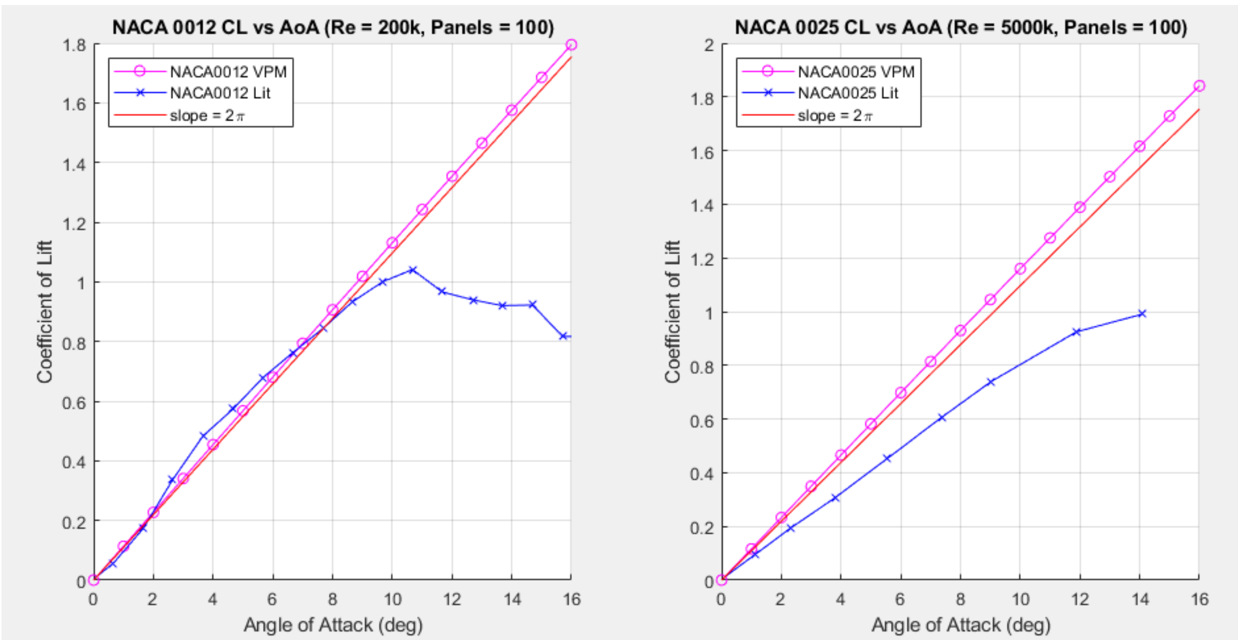



*Figure 7: Cl vs AoA of symmetric airfoils using VPM and literature data [4][5]*

One notable characteristic of the VPM is its linear lift curve. This limitation arises because the VPM does not simulate the fluid domain and therefore cannot account for the effects of viscosity or flow separation. As a result, the method provides an idealized solution that omits critical nonlinear behaviours such as stall that are observed in real-world aerodynamics [1]. In Figure 7, the literature data also exhibits linear behaviour until it reaches stall, driven by flow separation at that angle of attack at which point the VPM data deviates dramatically.

For a thin airfoil, it can be derived from classical thin airfoil theory that the slope of the lift curve should equal $2\pi$, and should intersect (0,0) where no AoA generates no lift [1]. This is plotted next to the VPM results for reference.

The VPM and theoretical lift slope of the NACA 0012 behave similarly to the literature values, while the lift curve of the NACA 0025 deviates from the literature data. The thickness of the airfoil should cause it to be lower than the $2\pi$ slope, yet it trends like a thin airfoil. Some potential sources of error for this will be discussed in a later section.

## Cambered Airfoil Analysis

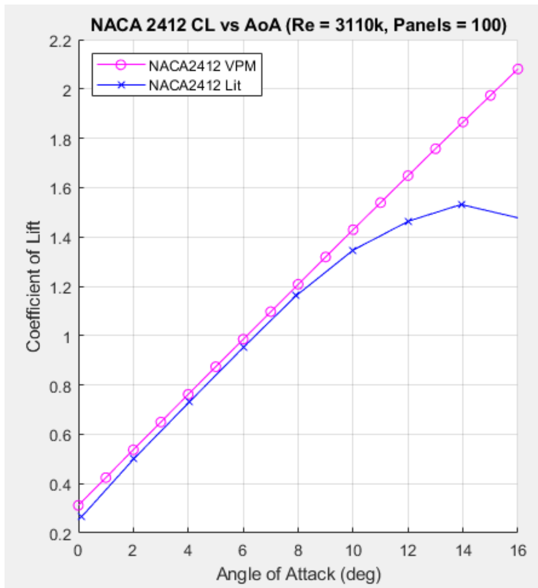



*Figure 8: Cl vs AoA of cambered airfoil using VPM and literature data [6]*

The characteristics mentioned regarding the VPM in the analysis of the symmetric airfoils hold true for this figure as well. The linear lift curve does not reflect the real-world behaviour of stall as shown in the literature data. The lift curve generated by the VPM contains a similar slope to the literature curve, stall notwithstanding. This demonstrates an accurate approximation of the linear region of the lift curve for both cambered and symmetric thin airfoils. The VPM curve for NACA 0025 does not demonstrate an accurate approximation, suggesting that perhaps it is the thickness of the airfoil that is the cause. Future iterations should test various thicknesses of airfoils to identify if this is the case.

# Sources of Error

The types of error in this assignment can be separated into two categories:

1) Systematic error: due to the method itself
2) Random error: due to variability in the implementation

**Systematic Sources of Error**

- Discretization Error
  - The spatial discretization of the airfoil shape will affect the solution without proper resolution of panels, especially around regions of high curvature where the vortex strength will very greatly between panels such as the leading edge.
  - The discretization of the vortex sheet along the airfoil makes the assumption that each panel contains a constant vortex strength. The vortex strength along the airfoil is continuous in potential flow theory.
  - Modelling any continuous curve with discrete panels requires a sensitivity analysis to the number of discrete panels used, otherwise it is unknown whether the solution is independent of the discretization.
- Potential Flow Theory
  - The mathematical model behind the vortex panel method is derived from potential flow theory. Potential flow theory assumes the flow is inviscid and incompressible. This prevents the vortex panel method from modelling nonlinear flow behaviour such as flow separation and boundary layer effects.

**Random Sources of Error**

- Implementation Error (Human Error)
  - Many bugs were previously found in the implementation of the mathematical model of the vortex panel method, and it is possible that not all were accounted for. Iterations of testing with different airfoils should be conducted to reduce this error.
  - The method of discretization along the x-axis means it requires a very fine resolution to properly capture the curvature of the leading edge.
  - The discretization method forces the trailing edge panels to meet at (1,0) when the method of discretization does not close the loop of the airfoil spline. This may alter the panel angles of the trailing edge slightly from the contour of the actual airfoil as a means to complete the discretization. This error is minimized through proper panel resolution.
- Computational Error
  - Round-off or floating-point errors can occur due to finite precision in numerical computations which can lead to small variations in matrix calculations.

- Data Sources
  - The datapoints for the shape of the airfoil were taken from airfoiltools.com [2], an online airfoil database. If the airfoil shape taken is not geometrically identical to the airfoil used in the experiments from literature, this could cause a major discrepancy in the solution.
  - The NACA 0012 [4] and NACA 0025 [5] literature data were taken from a finite wing in a wind tunnel. The effects of downwash would lower the Cl in the literature data which would not be reflected in the vortex panel method.

# Conclusion

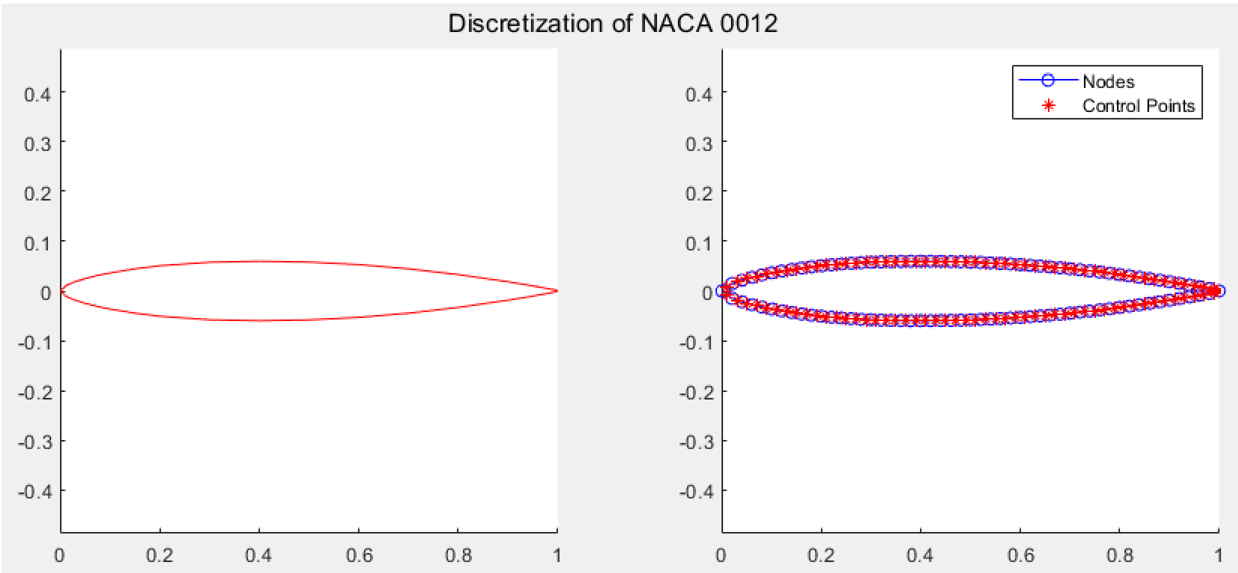
The vortex panel method was implemented in MATLAB to analyze the aerodynamic performance of symmetric and cambered airfoils. The methodology involved discretizing the airfoil geometry into panels, solving a system of equations for vortex strengths based on potential flow theory, and enforcing the Kutta condition to ensure physically realistic flow behavior at the trailing edge. The results demonstrated that the VPM accurately reproduced the theoretical linear lift curve for thin airfoils, consistent with classical thin airfoil theory. However, deviations from real-world behavior were observed due to the VPM's inability to account for nonlinear aerodynamic phenomena such as flow separation and stall.

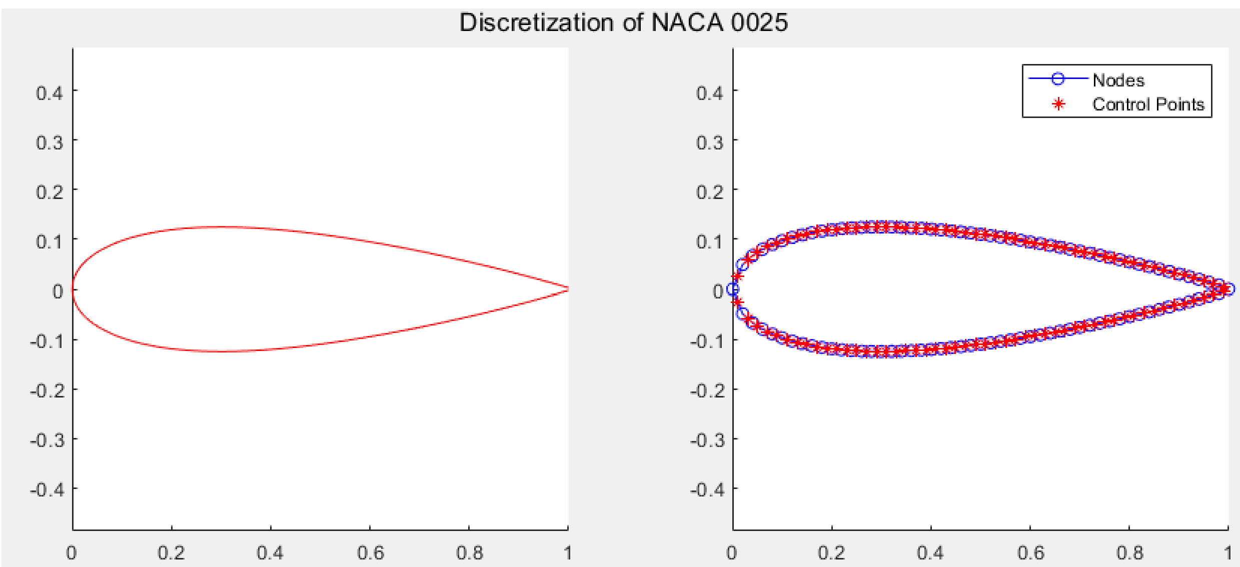
# References

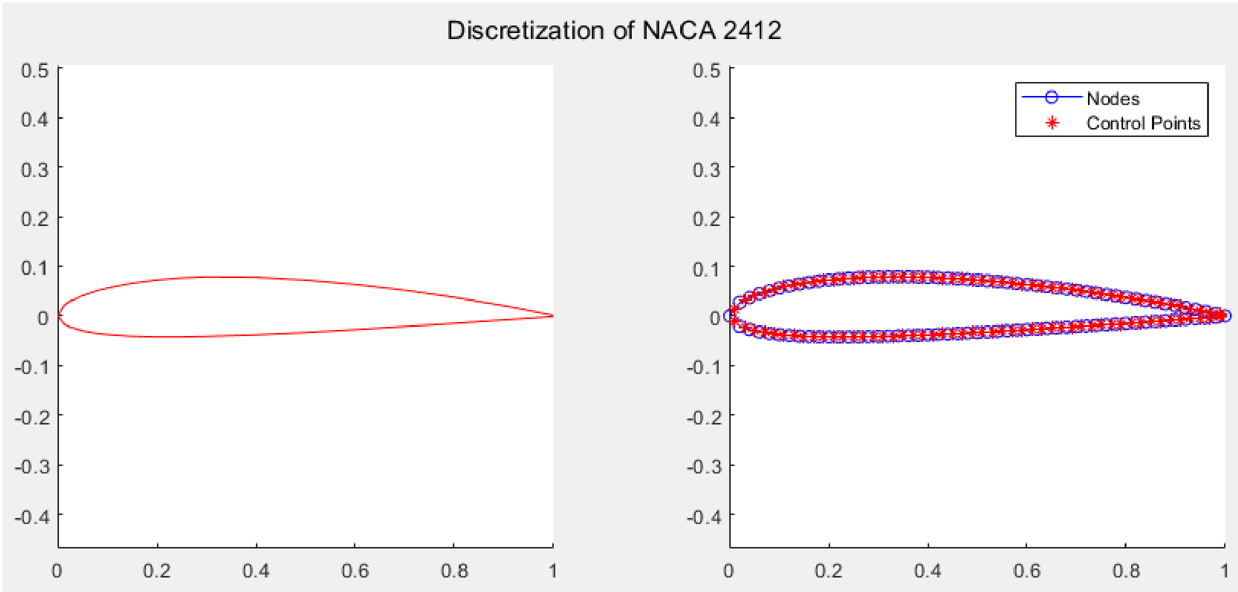

1. Anderson, John D. *Fundamentals of Aerodynamics*. McGraw-Hill, 2024.
2. “Airfoil Tools.” *Airfoil Tools*, www.airfoiltools.com/. Accessed 28 Nov. 2024.
3. VortexPanelExample. https://d2l.ucalgary.ca/d2l/le/content/618677/viewContent/6726388/View. Accessed 28 Nov. 2024.
4. De Paula, Adson A. *The airfoil thickness effects on wavy leading edge phenomena at low Reynolds number regime*. University of Sao Paulo Polytechnic School. 2016.
5. Bullivant, Kenneth W. *Tests of the NACA 0025 and 0035 Airfoils in the Full-Scale Wind Tunnel.* National Advisory Committee for Aeronautics. 1941.
6. Velkova, Cvetelina., et al. *The Impact of Different Turbulence Models at ANSYS FLUENT Over the Aerodynamic Characteristics of Ultra-Light Wing Airfoil NACA 2412*. Naval Academy Nikola Vaptsarov. Department of Technical Mechanics. 2017.

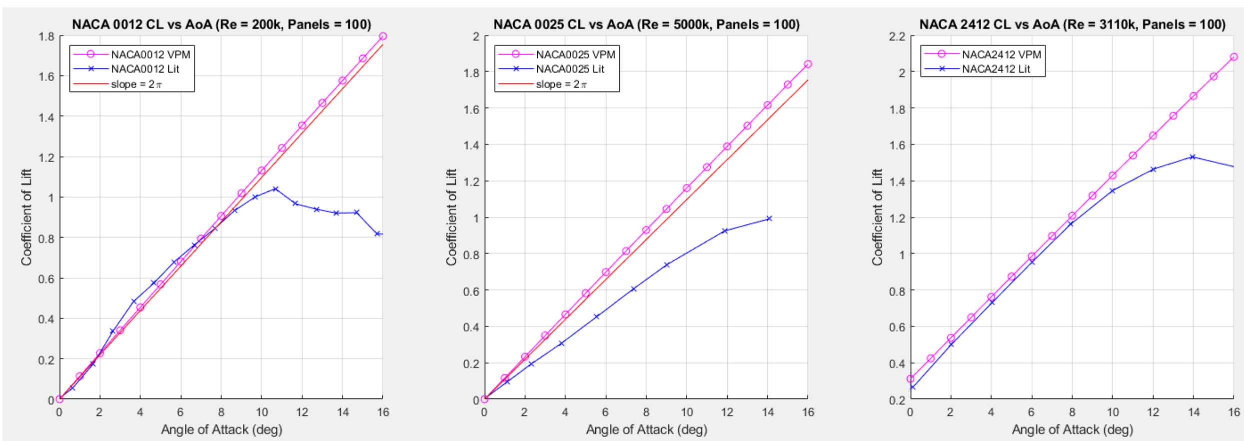
# Appendix

## MATLAB Figure Outputs (100 panels)



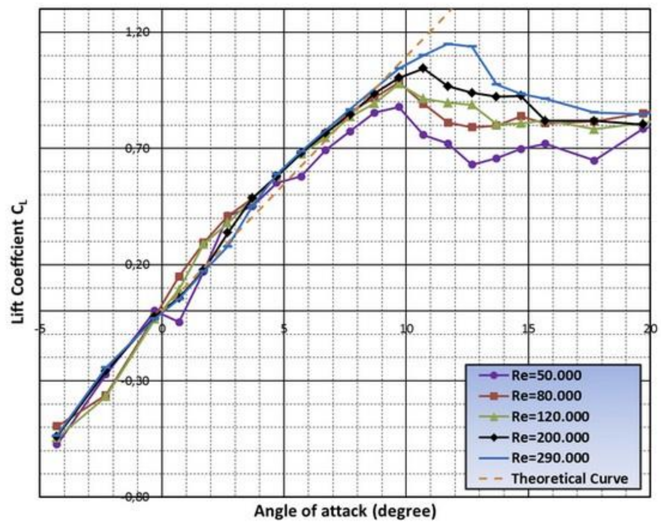




Discretization of NACA 2412
Nodes
Control Points


NACA 0012 CL vs AoA (Re = 200k, Panels = 100)
NACA0012 VPM
NACA0012 Lit
slope = 2π
Coefficient of Lift
Angle of Attack (deg)
NACA 0025 CL vs AoA (Re = 5000k, Panels = 100)
NACA0025 VPM
NACA0025 Lit
slope = 2π
Coefficient of Lift
Angle of Attack (deg)
NACA 2412 CL vs AoA (Re = 3110k, Panels = 100)
NACA2412 VPM
NACA2412 Lit
Coefficient of Lift
Angle of Attack (deg)

# Literature Figures

NACA 0012 Cl vs AoA [4]




NACA 0025 Cl vs AoA [5]

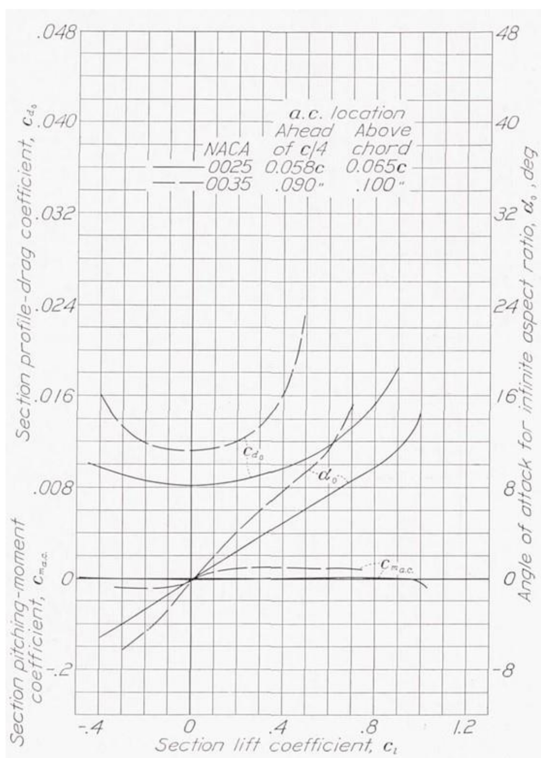



FIGURE 7.—Section characteristics of the NACA 0025 and 0035 airfoils at a Reynolds number of 3,200,000.

NACA 2412 Cl vs AoA [6]

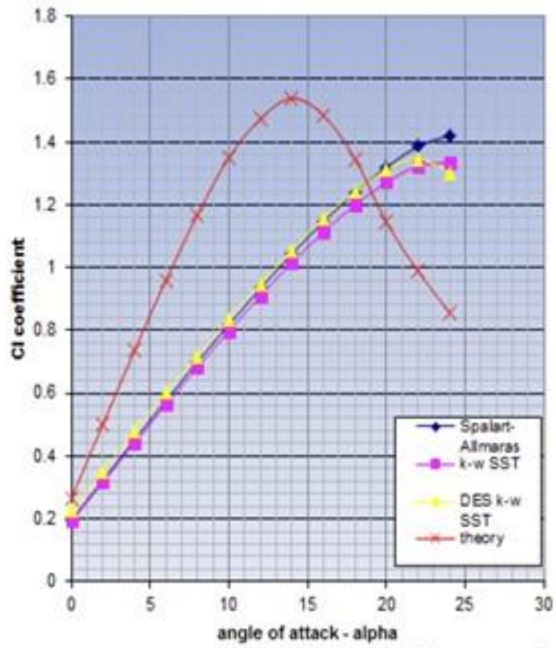



**Fig. 7 Comparison between lift coefficient $c_L$ of wing airfoil NACA 2412 with Spalart-Allmaras, k-w SST, DES k-w SST turbulence models and experiment (theory)**