Section 1.1: Portfolios

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Portfolio vs. Single Project (from Misbehaving)

In a meeting with 23 executives plus the CEO of a major company economist Richard Thaler poses the following question:

Suppose you were offered an investment opportunity for your division (each executive headed a separate/independent division) that will yield one of two payoffs. After the investment is made, there is a 50% chance it will make a profit of $2 million, and a 50% chance it will lose $1 million. Thaler then asked by a show of hands who of the executives would take on this project. Of the twenty-three executives, only three said they would do it.

Anything wrong with that?

Portfolio vs. Single Project (from Misbehaving)

Then Thaler asked the CEO a question. If these projects were independent, that is, the success of one was unrelated to the success of another, how many of the projects would he want to undertake? His answer: all of them! By taking on twenty three projects, the firm expects to make \$11.5 million (since each of them is worth an expected half million), and a bit of mathematics reveals that the chance of losing any money overall is less than 10%.

Portfolio vs. Single Project (from Misbehaving)

Companies, CEO's, managers have to be careful in setting incentives that avoid what psychologist and behavior economists call "narrow framing"... otherwise, what can be perceived to be bad for one manager may be very good for the entire company!

Sums of Random Variables

Suppose you play a game n times and the winning from the ith play is represented by the random variable X_i, $i = 1, 2, \ldots, n$.

We assume the each play of the game is independent of the others and it is the same game each time.

So, the X_i are IID (independent and identically distributed) .

What are the mean and variance of the *total* winnings?

$$T = X_1 + X_2 + X_3 + \ldots + X_n.$$

Let $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$.

$$T = X_1 + X_2 + X_3 + \ldots + X_n.$$

It turns out...

The mean of the sum:

$$\begin{aligned} E(T) &= E(X_1) + E(X_2) + \ldots + E(X_n) \\ &= \mu + \mu + \ldots + \mu \\ &= n\,\mu \end{aligned}$$

The variance of the sum:

$$\begin{aligned} Var(T) &= Var(X_1) + Var(X_2) + \ldots + Var(X_n) \\ &= \sigma^2 + \sigma^2 + \ldots + \sigma^2 \\ &= n\,\sigma^2 \end{aligned}$$

Portfolio vs. Single Project (from Misbehaving)

for an individual executive:

$X_i, i = 1, 2, \ldots, 23.$

x	$P(X = x)$
-1	.5
2	.5

$E(X_i) = \mu$: $.5 * (-1) + .5 * 2 = 0.5$

$Var(X_i) = \sigma^2$: $.5 * (-1 - .5)^2 + .5 * (2 - .5)^2 = 2.25$

σ: 1.5

μ/σ: $.5/1.5 = 0.3333333$ (why am I calculating this??)

Portfolio vs. Single Project (from Misbehaving)

for CEO:

$$T = X_1 + X_2 + X_3 + \ldots + X_n.$$

$E(T)$: 23*.5 = 11.5

$Var(T)$: 23*2.25 = 51.75

$sd(T)$: sqrt(51.75) = 7.193747

$E(T)/sd(T)$: 11.5/7.193747 = 1.598611

For the CEO, the mean is much bigger relative to the standard deviation than it is for the individual managers !!!

The Normal Distribution – Approximating repeated trials (Very Useful Idea!)

If we are taking on all 23 projects, $E(Profits) = 0.5 \times 23 = 11.5$ and the $Var(Profits) = 2.25 \times 23 = 7.19^2 = 51.75$...

We can now approximate the distribution of Profits via a $N(11.5, 7.19^2)$... therefore the $Pr(Profits > 0) \approx 0.94$ (why?)

In summary, in many situations, including large sums, if you can figure out the mean and variance of the random variable of interest, you can use a normal distribution to approximate the calculation of probabilities.

Option Pricing Aside

`https://youtu.be/A5w-dEgIU1M?si=7oup5vRGTE1oUzdx`

Let's run a simulation to price an option...

Covariance

- A measure of *dependence* between two random variables...
- It tells us how two unknown quantities tend to move together

The Covariance is defined as (for discrete X and Y):

$$Cov(X, Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} Pr(x_i, y_j) \times [x_i - E(X)] \times [y_j - E(Y)]$$

- What are the units of $Cov(X, Y)$?

Ford vs. Tesla

- Assume a very simple joint distribution of monthly returns for Ford (F) and Tesla (T):

	t=-7%	t=0%	t=7%	Pr(F=f)
f=-4%	0.06	0.07	0.02	**0.15**
f=0%	0.03	0.62	0.02	**0.67**
f=4%	0.00	0.11	0.07	**0.18**
Pr(T=t)	**0.09**	**0.80**	**0.11**	**1**

Let's summarize this table with some numbers...

Ford vs. Tesla

	t=-7%	t=0%	t=7%	Pr(F=f)
f=-4%	0.06	0.07	0.02	**0.15**
f=0%	0.03	0.62	0.02	**0.67**
f=4%	0.00	0.11	0.07	**0.18**
Pr(T=t)	**0.09**	**0.80**	**0.11**	**1**

- $E(F) = 0.12$, $E(T) = 0.14$
- $Var(F) = 5.25$, $sd(F) = 2.29$, $Var(T) = 9.76$, $sd(T) = 3.12$
- What is the better stock?

Ford vs. Tesla

	t=-7%	t=0%	t=7%	Pr(F=f)
f=-4%	0.06	0.07	0.02	**0.15**
f=0%	0.03	0.62	0.02	**0.67**
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Pr(T=t)	**0.09**	**0.80**	**0.11**	**1**

$$
\begin{aligned}
Cov(F,T) = & (-7-0.14)(-4-0.12)0.06 + (-7-0.14)(0-0.12)0.03+ \\
& (-7-0.14)(4-0.12)0.00+(0-0.14)(-4-0.12)0.07+ \\
& (0-0.14)(0-0.12)0.62 + (0-0.14)(4-0.12)0.11+ \\
& (7-0.14)(-4-0.12)0.02 + (7-0.14)(0-0.12)0.02+ \\
& (7-0.14)(4-0.12)0.07 = 3.063
\end{aligned}
$$

Okay, the covariance in positive... makes sense, but can we get a more intuitive number?

Correlation

$$Corr(X,Y) = \frac{Cov(X,Y)}{sd(X)sd(Y)}$$

- What are the units of $Corr(X,Y)$? It doesn't depend on the units of X or Y!
- $-1 \leq Corr(X,Y) \leq 1$

In our Ford vs. Tesla example:

$$Corr(F,T) = \frac{3.063}{2.29 \times 3.12} = 0.428 \text{ (not too strong!)}$$

Linear Combination of Random Variables (aka Portfolios)

Is it better to hold Ford or Tesla? How about half and half?

To answer this question we need to understand the behavior of the weighted sum (linear combinations) of two random variables...

Let X and Y be two random variables:

- $E(aX + bY) = aE(X) + bE(Y)$
- $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2ab \times Cov(X, Y)$

Linear Combination of Random Variables

Applying this to a portfolio with 50% on Ford and 50% on Tesla...

- $E(0.5F + 0.5T) = 0.5E(F) + 0.5E(T) = 0.5 \times 0.12 + 0.5 \times 0.14 = 0.13$
- $Var(0.5F + 0.5T) = (0.5)^2 Var(F) + (0.5)^2 Var(T) + 2(0.5)(0.5) \times Cov(F, T) = (0.5)^2(5.25) + (0.5)^2(9.76) + 2(0.5)(0.5) \times 3.063 = 5.28$
- $sd(0.5F + 0.5T) = 2.297$

so, what is better? Holding Ford, Tesla or the 50-50 portfolio??

Linear Combination of Random Variables

More generally...

- $E(w_1 X_1 + w_2 X_2 + ... w_p X_p) = w_1 E(X_1) + w_2 E(X_2) + ... + w_p E(X_p) = \sum_{i=1}^{p} w_i E(X_i)$
- $Var(w_1 X_1 + w_2 X_2 + ... w_p X_p) = w_1^2 Var(X_1) + w_2^2 Var(X_2) + ... + w_p^2 Var(X_p) + 2 w_1 w_2 \times Cov(X_1, X_2) + 2 w_1 w_3 Cov(X_1, X_3) + ... = \sum_{i=1}^{p} w_i^2 Var(X_i) + \sum_{i=1}^{p} \sum_{j \neq i} w_i w_j Cov(X_i, X_j)$

Example:

Ford, Tesla, GM.

$$P = w_1 F + w_2 T + w_3 G$$

$$E(P) = w_1 E(F) + w_2 E(T) + w_3 E(G)$$

$$\begin{aligned} Var(P) = {} & w_1^2 Var(F) + w_2^2 Var(T) + w_3^2 Var(G) \\ & + 2w_1 w_2 Cov(F, T) + 2w_1 w_3 Cov(F, G) + 2w_2 w_3 Cov(T, G). \end{aligned}$$

With lots of assets this gets complicated!! There many possible pairs of assets and corresponding covariance pairs representing the high dimensional dependence of the many input assets.

In practice you have to estimate all the covariances from data... it gets harder but the concept is the same.

Portfolios, another example...

- As before, let's assume that the annual returns on the SP500 are normally distributed with mean 6% and standard deviation of 15%, i.e., $SP500 \sim N(6, 15^2)$
- Let's also assume that annual returns on bonds are normally distributed with mean 2% and standard deviation 5%, i.e., $Bonds \sim N(2, 5^2)$
- What is the best investment?
- What else do I need to know if I want to consider a portfolio of SP500 and bonds?

Portfolios, another example...

- Additionally, let's assume the correlation between the returns on SP500 and the returns on bonds is -0.2.
- How does this information impact our evaluation of the best available investment?

Recall that for two random variables X and Y:

- $E(aX + bY) = aE(X) + bE(Y)$
- $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2ab \times Cov(X, Y)$

- One more very useful property... sum of normal random variables is a new normal random variable!

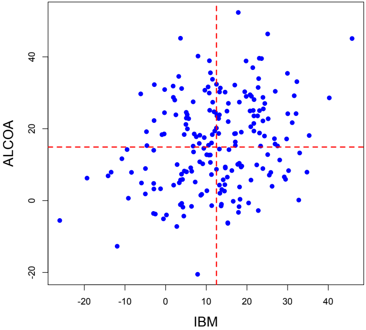
Portfolios once again...

- What is the behavior of the returns of a portfolio with 70% in the SP500 and 30% in Bonds?
- $E(0.7SP500 + 0.3Bonds) = 0.7E(SP500) + 0.3E(Bonds) = 0.7 \times 6 + 0.3 \times 2 = 4.8$
- $Var(0.7SP500 + 0.3Bonds) = (0.7)^2 Var(SP500) + (0.3)^2 Var(Bonds) + 2(0.7)(0.3) \times Corr(SP500, Bonds) \times sd(SP500) \times sd(Bonds) = (0.7)^2(15^2) + (0.3)^2(5^2) + 2(0.7)(0.3) \times -0.2 \times 15 \times 5 = 106.2$
- $Portfolio \sim N(4.8, 10.3^2)$
- Homework: good or bad? What now? Is there a better combination?

Building Portfolios (From Data)

- Let's assume we are considering 2 investment opportunities
  1. IBM stocks
  2. ALCOA stocks
- How should we start thinking about this problem?

Building Portfolios (From Data)



IBM	ALCOA
$\bar{X}_{IBM} = 12.5$	$\bar{X}_{Alcoa} = 14.9$
$s_{IBM} = 10.5$	$s_{Alcoa} = 14.0$

$$corr(IBM, ALCOA) = 0.33$$

Building Portfolios (From Data)

- How about combining these options? Is that a good idea? Is it good to have all your eggs in the same basket? Why?
- What if I place half of my money in ALCOA and the other half IBM...

In order to answer this question we need to understand how IBM and Alcoa *move together!*

Back to Building Portfolios

- So, by using what we learned about the means, variances and covariance, we get to:

$$\begin{aligned} E(P) &= 0.5\bar{X}_{IBM} + 0.5\bar{X}_{Alcoa} \\ Var(P) &= 0.5^2 * s^2_{IBM} + 0.5^2 * s^2_{Alcoa} + 2 * 0.5 * 0.5 * Cov(IBM, Alcoa) \end{aligned}$$

- $E(P)$ and $Var(P)$ refer to the estimated mean and variance of our portfolio

Building Portfolios (From Data)

Here are the results for different combinations of Alcoa and IBM...