Andronov-Hopf Bifurcation Analysis of a Hassell-Type Population Matrix

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Abstract

We discuss the chaotic dynamics of a two-generation (dimensional) Hassel-Type population matrix where the fertility rates decay with population size. The simple map exhibits many extraordinary dynamical behavior. We find and analyze two different bifurcation types, adding to the current research in population dynamics concluding that even the most apparently simplified models widely accepted by population biologists as time-tested theories exhibit ergodic properties.

1 Preliminaries

First we establish our population model. At any time t we split the population X_t into $n+1$ distinct age classes.

$$X_t = \begin{pmatrix} x_{0,t} \\ x_{1,t} \\ \vdots \\ x_{n,t} \end{pmatrix}$$

We can regard the total population at a time t as $X = x_0 + x_1 + \cdots + x_n$. We call the the Leslie matrix

$$A = \begin{pmatrix} f_0 & f_1 & \cdots & & f_n \\ p_0 & 0 & \cdots & 0 & 0 \\ 0 & p_1 & & & \vdots \\ \vdots & & \ddots & \vdots & \\ 0 & & \cdots & p_{n-1} & 0 \end{pmatrix}$$

where p_i is interpreted as the probability of age class i surviving and f_i is the fertility of that class. The model then is $\mathbf{X_{t+1}} = A\mathbf{X_t}$. The analysis of this model is fairly well understood and the literature is vast.

We consider now the non-linear Leslie population model where the fertility rates decay with population size. The Leslie matrix variation is then written as

$$A = \begin{pmatrix} \frac{f_0}{(1+x_0+x_1+\cdots+x_n)^\beta} & \frac{f_1}{(1+x_0+x_1+\cdots+x_n)^\beta} & \cdots & & \frac{}{(1+x_0+} \\ p_0 & 0 & \cdots & 0 & \\ 0 & p_1 & & & \\ \vdots & & \ddots & \vdots & \\ 0 & & \cdots & p_{n-1} & \end{pmatrix}$$

The associated non-linear transformation is

$$T\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{f_0x_0+f_1x_1+\cdots+f_nx_n}{(1+x_0+x_1+\cdots+x_n)^\beta} \\ p_1x_1 \\ \vdots \\ p_{n-1}x_{n-1} \end{pmatrix}$$

2 Two Generation Model

For the rest of this discussion, we will analyze the properties of this non-linear Leslie matrix variation as applied to a population with only two age classes, x and y.

The associated transformation is then

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{f_1x+f_2y}{(1+x+y)^\beta} \\ p_1x \end{pmatrix}$$

We'll study the special case when $f_1 = f_2$ and $\beta > 1$. Solving the kernel of this transformation will yield the fixed points, and we'll analyze further the stability of these points for the remainder of this paper. Equivalently, the non-linear Leslie matrix can be written as

$$A = \begin{pmatrix} \frac{f_1}{(1+x+y)^\beta} & \frac{f_1}{(1+x+y)^\beta} \\ p_1 & 0 \end{pmatrix}$$

The equillibrium solution is then fround from the set of equations:

$$x = \frac{f(x+y)}{(1+x+y)^{\beta}}$$
$$y = px$$

Starting from the second equation,

$$x = \frac{y}{p}$$

And substitute this into the first equation:

$$\frac{y}{p} = \frac{f(\frac{y}{p}+y)}{(1+\frac{y}{p}+y)^{\beta}}$$

$$\frac{y}{p} = \frac{\frac{fy}{p}+fy}{(1+\frac{y}{p}+y)^{\beta}}$$

$$\frac{f+fp}{(1+\frac{y}{p}+y)^{\beta}} = 1$$

$$f+fp = (1+\frac{y}{p}+y)^{\beta}$$

$$\sqrt[\beta]{f+fp} - 1 = \frac{y}{p}+y$$

$$\sqrt[\beta]{f+fp} - 1 = (\frac{1}{p}+1)y$$

$$y^* = \frac{\sqrt[\beta]{f+fp}-1}{(\frac{1}{p}+1)}$$

And substitute this into the *substituted* equation to solve for x:

$$\frac{y}{p} = x$$

$$x^* = \frac{(\sqrt[\beta]{f+fp}-1)p}{(\frac{1}{p}+1)} = \frac{\sqrt[\beta]{f+fp}-1}{p+1}$$

Then, our **equillibrium solution** is:

$$(x^*, y^*) = (\frac{\sqrt[\beta]{f+fp}-1}{p+1}, \frac{\sqrt[\beta]{f+fp}-1}{(\frac{1}{p}+1)})$$

With the necessary conditions that $p \neq 0$ and $p \neq 1$. These seem like fine conditions to impose on a populational model.

Now, we'll analyze the stability of the fixed point by linearizing the system. We'll calculate the Jacobian.

$$\mathcal{J}_A = \begin{pmatrix} \frac{\partial}{\partial x}[\frac{f(x+y)}{(1+x+y)^{\beta}}] & \frac{\partial}{\partial y}[\frac{f(x+y)}{(1+x+y)^{\beta}}] \\ \frac{\partial}{\partial x}[p_1 x] & \frac{\partial}{\partial y}[p_1 x] \end{pmatrix}$$

$$= \begin{pmatrix} \frac{f(-\beta x+x-\beta y+y+1)}{(1+x+y)^{\beta+1}} & \frac{f(-\beta x+x-\beta y+y+1)}{(1+x+y)^{\beta+1}} \\ p_1 & 0 \end{pmatrix}$$

It will be beneficial to rewrite the Jacobian as such:

$$\mathcal{J}_A = \begin{pmatrix} \frac{f(-\beta(x+y)+x+y+1)}{(1+x+y)^{\beta+1}} & \frac{f(-\beta(x+y)+x+y+1)}{(1+x+y)^{\beta+1}} \\ p_1 & 0 \end{pmatrix}$$

And call $1 + x^* + y^* \equiv Q$. We now just note that $Q = \sqrt[\beta]{fp+f}$. Investigate the matrix $\mathcal{J}_A(x^*, y^*)$:

$$\mathcal{J}_A(x^*, y^*) = \begin{pmatrix} \frac{f(-\beta(Q-1)+Q)}{(Q)^{\beta+1}} & \frac{f(-\beta(Q-1)+Q)}{(Q)^{\beta+1}} \\ p_1 & 0 \end{pmatrix}$$

The eigenvalues of this matrix determine the stability of the fixed point. The characteristic polynomial (leading to the eigenvalues) is

$$\mathcal{P}_{\mathcal{J}_A}(\lambda) = \lambda^2 - tr(\mathcal{J}_A(x^*, y^*))\lambda + det(\mathcal{J}_A(x^*, y^*))$$

where $tr(\star)$ and $det(\star)$ are the trace and determinant, respectively. We'll write $\mathcal{J}_A$ for $\mathcal{J}_A(x^*, y^*)$ when the context is clear.

$$tr(\mathcal{J}_A) = \frac{f(-\beta(Q-1)+Q)}{Q^{\beta+1}}$$

$$det(\mathcal{J}_A) = \frac{-p_1 f(-\beta(Q-1)+Q)}{Q^{\beta+1}}$$

with roots given by

$$\lambda_{1,2} = \frac{tr(\mathcal{J}_A) \pm \sqrt{tr(\mathcal{J}_A)^2 - 4det(\mathcal{J}_A)}}{2}$$

3 Stability from Characteristic Polynomial

Recall that for a dynamical system's fixed point to be asymptotically stable, we require that the eigenvalues modulus be strictly less than one, or $|\lambda| < 1$ for both eigenvalues when λ is the root of the characteristic equation. Here, we'll investigate what properties the coefficients of the polynomial will exhibit in order to maintain this property. We have the equation

$$\lambda^2 + a_1\lambda + a_2 = 0$$

We're interested in the conditions that guarantee the roots will be less than one. By the nature of being a root, we can rewrite this equation as

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$