Lagrangian submanifolds of the standard $\mathbf{C}^n$

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University of Aberdeen
joint with Jonny Evans (UCL)

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Essentially, a monotone lagrangian in $\mathbf{C}^n$ cannot have too complicated topology. For example, it can't admit a metric of negative curvature, in dimension three it has to be a product $\mathbf{S}^1 \times \Sigma$ and, in general, its simplicial volume has to be zero.

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- Examples of monotone embeddings.

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- The standard sphere $\mathbf{S}^2 \subset \mathbf{R}^3 \subset \mathbf{C}^2$ is not a Lagranian submanifold.

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Gromov proved that there are no exact Lagrangian embeddings into $\mathbf{C}^n$. In particular, there are no Lagrangian embeddings of simply connected manifolds.

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- If $\dim L = 3$ then $L = \mathbf{S}^1 \times \Sigma$. Remark: Fukaya obtained the same statement without the monotonicity assumption but assuming that $L$ is prime.
- If $f : L \to \mathbf{C}^3$ is a Lagrangian immersion with $k$ double points then resolving the double points can produce a monotone embedding only if $L = \mathbf{S}^3$ and $k = 1$.

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$$\begin{array}{ccccc} \overline{\mathbf{M}_{0,0}(a,J)} \times \mathbf{S}^1 & \longrightarrow & \mathbf{M}_{0,1}(a,J) & \xrightarrow{\text{ev}} & L \\ \downarrow & & \downarrow & & \\ \overline{\mathbf{M}_{0,0}(a,J)} & \longrightarrow & \mathbf{M}_{0,0}(a,J) & & \end{array}$$

The composition of the maps in the top row gives the required map of nonzero degree. $\square$

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Thank you for listening!