Lagrangian submanifolds of the standard  $\mathbf{C}^n$ Jarek Kędra University of Aberdeen joint with Jonny Evans (UCL)

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Essentially, a monotone lagrangian in  $\mathbf{C}^n$  cannot have too complicated topology. For example, it can't admit a metric of negative curvature, in dimension three it has to be a product  $\mathbf{S}^1 \times \Sigma$  and, in general, its simplicial volume has to be zero.

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- Examples of monotone embeddings.

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- For example, every real line on the plane  ${f C}$  is Lagrangian.

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- The standard sphere  $\mathbf{S}^2 \subset \mathbf{R}^3 \subset \mathbf{C}^2$  is not a Lagranian submanifold.

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### Theorem

Theorem [Evans-K.]

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### Corollary

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- $\pi_1(L)$  is not hyperbolic.
- L does not admit a Riemannian metric of negative curvature. Remark: Eliashberg and Viterbo obtained the last statement without the monotonicity assumption.

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- Every summand of *L* has vanishing simplicial volume.
- If dim L = 3 then  $L = \mathbf{S}^1 \times \Sigma$ . Remark: Fukaya obtained the same statement without the monotonicity assumption but assuming that L is prime.
- o If  $f: L \to \mathbb{C}^3$  is a Lagrangian immersion with k double points then resolving the double points can produce a monotone embedding only if  $L = \mathbb{S}^3$  and k = 1.

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$$\langle \mu_f, u(\partial \mathbf{D}^2) \rangle = 2.$$

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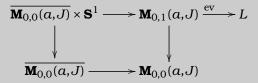
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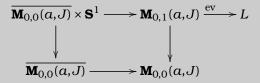
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The composition of the maps in the top row gives the required map of nonzero degree.

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Monotone Lagrangian immersions obey the h-principle.0 Theorem

If  $f: L \to \mathbf{C}^n$  is a K-monotone Lagrangian immersion and  $e: K \to \mathbf{C}^m$  be a K-monotone Lagrangian embedding. Then there is a monotone Lagrangian embedding

$$K \times L \rightarrow \mathbf{C}^{m+n}$$
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