• NONLINEAR DYNAMICS •

a very introductory lecture (FR, 13:15 A-1, 204)

≡ theory and phenomenology of (dissipative) nonlinear dynamical systems

- dynamical system → anything (physical, chemical, biological) that evolves with time
  → we consider 'simple' systems = parametrized by only a few variables
- nonlinear (NL)
  → realistic, far from 'small' perturbations
  → literally: equations itself are NL
- dissipative → dissipating energy, loosing information during the evolution

LOTKA-VOLTERRA (PREDATOR-PREY MODEL)

Example of NL dynamical system:

$$\begin{cases} \dot{x} = x(-\alpha + \beta y) \leftarrow \text{PREDATOR} \\ \dot{y} = y(\delta - \gamma x) \leftarrow \text{PREY} \end{cases}$$

$[\, x, y > 0 \quad (\alpha, \beta, \gamma, \delta > 0)]$

- $\dot{x} = \frac{d}{dt} x$
- L would be $\dot{\mathbf{x}} = A\mathbf{x}$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\frac{dy}{dt} = \delta y$$

$$\int_{y_0}^{y} \frac{dy'}{y'} = \int_{t_0=0}^{t} \delta \, dt'$$

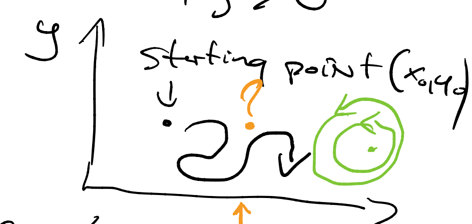
$$\ln y - \ln y_0 = \delta t$$

$$e^{\ln \frac{y}{y_0}} = e^{\delta t} \quad \text{exp}$$

$$y = y_0 e^{\delta t} \text{ growth}$$

- preys y have unlimited food supply
  ↳ reproduce exponentially with rate δ:
- rate of predation $= \gamma x y$ ← rate that they both meet
  - note minus sign
- predator population grows with $\beta x y$
  → same as rate of predation but with different constant
- predator population decay exponentially in the absence of prey with a rate $-\alpha$: $x = x_0 e^{-\alpha t}$

$x, y > 0$



- we could solve the problem numerically → lAbs
- use qualitative analytical method

* what dynamics?
* is it possible for P-P to coexist?

QUALITATIVE METHODS

- (finding) fixed points x^*
- determining their (type of) stability
  ↳ • linearization, Jordan forms
    • reconstruction of PHASE PORTRAIT
- checking the change (of behavior or existence) of fixed points when changing parameters
  ↳ bifurcations
  (rapid change in the nature of solutions/phase portraits)
- ...

PLAN for today

(many other methods)
- Lyapunov stability
- stability of orbits
- (deterministic) chaos

FIXED POINTS

- let's start with the simple possible NL differential eq: (ODE)

$$\dot{x} = f(x), \quad x \in \mathbb{R}^1 \quad \text{(1D system)}$$

FIXED POINT x^* ← if the system enters such point it remains there

→ condition for x^*: $f(x^*) = 0$ →
1. solve ⇒ x_i^*, $i = 1, 2, \dots$
2. $\forall_i \; f'(x)\big|_{x=x^*} \equiv \lambda_i$

- if $\lambda_i > 0$ then x_i^* is unstable
  $\lambda_i < 0 \Rightarrow x_i^*$ is stable

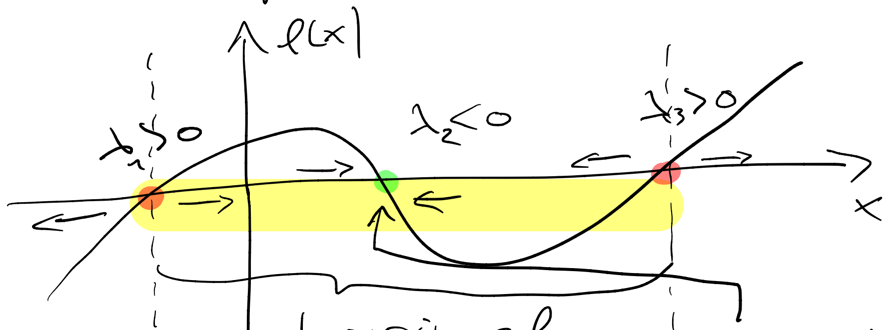
- interpretation — analytical

expand near to x^*: $f(x) \approx \underbrace{f(x^*)}_{=0} + \overbrace{(x - x^*)}^{\delta} \underbrace{f'(x^*)}_{=\lambda} = \delta\lambda$

$$\boxed{\begin{aligned} \delta &= x - x^* \\ \dot{\delta} &= \dot{x} - 0, \quad x^* = \text{const}(t) \end{aligned}}$$

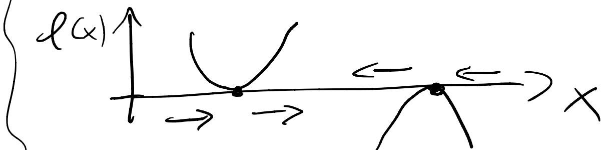
$$\dot{\delta} = \delta\lambda$$
$$\delta(t) = \delta_0 e^{\lambda t} \longrightarrow \begin{array}{ll} \lambda < 0 & \text{stable} \\ \lambda > 0 & \text{unstable} \end{array}$$

• interpretation – graphical



BTW: what if $\lambda=0$?

— "one side" stability



~~HARMONIC OSCILLATOR~~

$\ddot{x} = -x \quad \Updownarrow$ equiv

$$\begin{cases}\dot{x} = y\\ \dot{y} = -x\end{cases}$$

$$\dot{x}\ddot{x} = -x\dot{x}$$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2\right) = -\frac{d}{dt}\left(\frac{1}{2}x^2\right)$$

$$\frac{d}{dt}\Big(\underbrace{\frac{1}{2}\dot{x}^2+\frac{1}{2}x^2}_{const(t)}\Big) = 0$$

— integral of motion (E)

PHASE SPACE (PORTRAITS)

• point in PS explicitly determines the system state (or configuration)

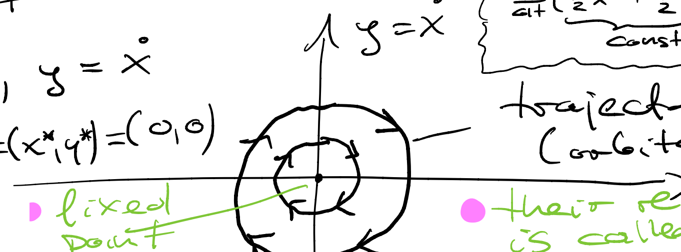
1st. example: harmonic oscillator:

$$\frac{1}{2}\dot{x}^2+\frac{1}{2}x^2 = const$$

$$\frac{1}{2}y^2+\frac{1}{2}x^2 = const,\quad y=\dot{x}$$

$$x^* = (x^*, y^*) = (0,0)$$

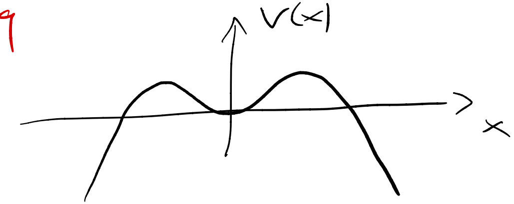
let's draw possible trajectories in (x,y) space = space of (positions, velocities) = PHASE SPACE



• their representation in PS is called PHASE PORTRAIT

2nd. example $\ddot{x} = -x + x^3$ → Potential energy $V(x) = \frac{1}{2}x^2 - \frac{1}{4}x^4$

attracting repelling



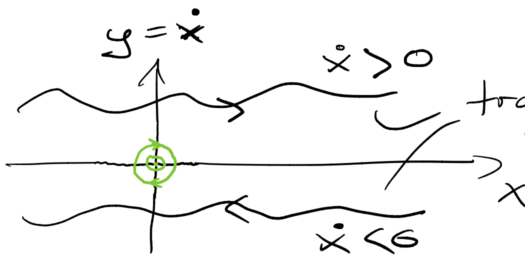
- integral of motion

$$\underbrace{\frac{1}{2}\dot{x}^2}_{K} + \frac{1}{2}x^2 - \frac{1}{4}x^4 = C$$

$$\dot{x} = \pm\sqrt{2C - x^2 + \frac{1}{2}x^4}$$

- fixed points

$(0,0)$, $(-1,0)$, $(1,0)$



trajektorie występują parami ale o przeciwnych kierunkach

- method of isoclines

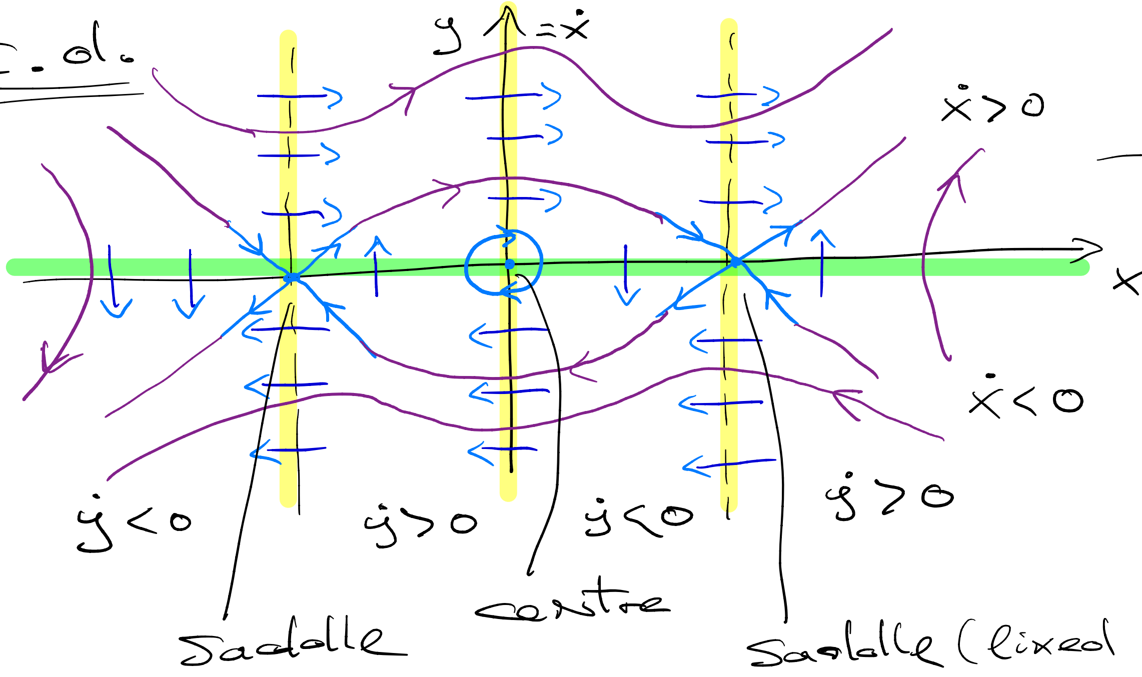
$$\ddot{x} = -x + x^3 = -x(1-x^2) = -x(1-x)(1+x) \iff \begin{cases} \dot{x} = y \\ \dot{y} = -x(1-x)(1+x) \end{cases}$$

two families of curves (lines):

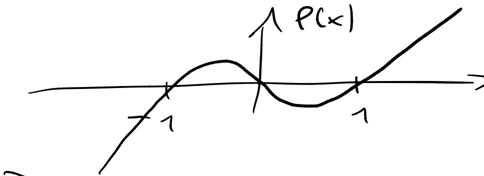
$y = 0 \longrightarrow \dot{x} = 0$ - velocity $= 0 \Rightarrow$ trajectories are vertical

$x = 0 \vee x = \pm 1 \longrightarrow \dot{y} = 0$ - velocity is constant $\Rightarrow$ trajectories are horizontal

isoclines (trajectory cross them with the same SLOPE) c.d.



$$-x(1-x)(1+x) = f(x)$$



LINEARIZATION

- stability in linear equations

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n, \quad \text{general solution } x = e^{At} x_0 =$$

$$= \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \right) x_0$$

A – $n \times n$ matrix, $x^* = 0$

— let's stick to simplest case: 2D, $n=2$,

$x \in \mathbb{R}^2$
A – 2×2 matrix
$x^* = (0,0)$

— to be able to calculate A^n
we have to diagonalize it = find eigenvalues

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

characteristic equation:

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - \underbrace{(a+d)}_{Tr\,A}\lambda + \underbrace{ad - bc}_{Det\,A} = 0$$

$$\lambda^2 - T\lambda + D = 0$$

$$\lambda_{1,2} = \frac{T}{2} \pm \frac{1}{2}\sqrt{T^2 - 4D}$$

We can always find eigenvalues but transformation P not always exists

— after diagonalization
A can have 3 possible forms (JORDAN FORMS)
– for which we exactly know how to calculate powers, A^n

1) $\lambda_1 \neq \lambda_2$

$$\Lambda \text{ (A after diagonalization)} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$\longleftrightarrow$ eigenvectors are linearly independent and transformation matrix $P^{-1} A P = \Lambda$ exists

diagonal matrix

— first Jordan form $P = (v_1, v_2)$

– for $\begin{pmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{pmatrix} = \Lambda$ we can easily get $e^{\Lambda t} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t}\end{pmatrix}$

2) $\lambda_1, \lambda_2 = \rho \pm i\omega$ (complex conjugate pair) $\xrightarrow[\text{nondiagonalizable}]{\text{can be}}$

* second Jordan form: $\Lambda = \begin{pmatrix}\rho & -\omega \\ \omega & \rho\end{pmatrix} = D + C$

& $[D, C] = 0$

commute!

3) $\lambda_1 = \lambda_2$ (degeneracy) $\xrightarrow[\text{diagon.?}]{\text{non-}}$ but third Jordan form exists: $\begin{pmatrix}\lambda & 1 \\ 0 & \lambda\end{pmatrix}$ → $\boxed{\Lambda^n = \begin{pmatrix}\lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n\end{pmatrix}}$

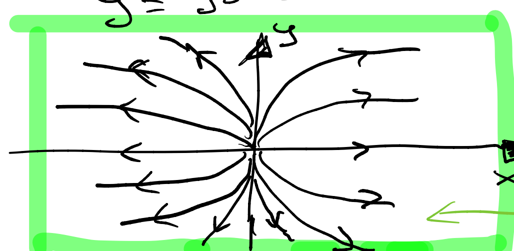
Therefore, for any cases we can solve the system and learn behaviour next to the fixed point!

Example (NODES)

$$\begin{pmatrix}\dot x \\ \dot y\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} \longrightarrow \begin{array}{l} x = x_0 e^{\lambda_1 t} \\ y = y_0 e^{\lambda_2 t} \end{array} \longrightarrow \begin{array}{l} \ln\frac{x}{x_0} = \lambda_1 t \\ \ln\frac{y}{y_0} = \lambda_2 t \end{array} \longrightarrow \frac{\lambda_2}{\lambda_1}\ln\frac{x}{x_0} = \ln\frac{y}{y_0}$$

$\lambda_1 > \lambda_2 > 0$

(JORDAN 1st form)



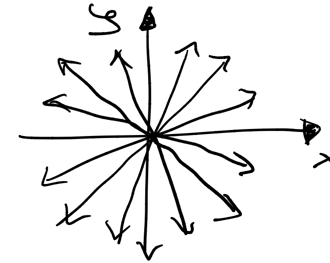
$$\left\{ \begin{array}{l} \frac{y}{y_0} = \left(\frac{x}{x_0}\right)^{\frac{\lambda_2}{\lambda_1}} \\ \frac{\lambda_2}{\lambda_1} < 1 \end{array} \right\}$$

UNSTABLE ($\lambda_1 < \lambda_2 < 0$ stable)

(STAR)

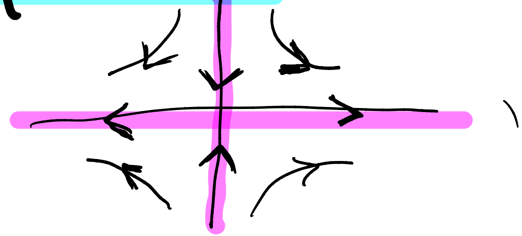
$\lambda_1 = \lambda_2 = \lambda \neq 0$ (3rd form)

$\frac{\lambda_1}{\lambda_2} = 1 \longrightarrow \frac{y}{y_0} = \frac{x}{x_0} \longrightarrow$ trajectories are just straight lines

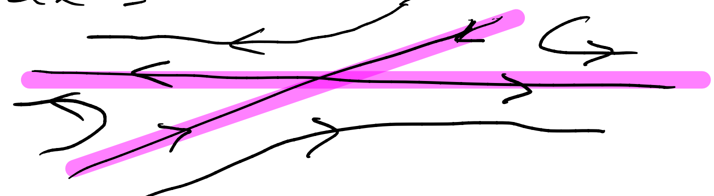


(SADDLE)

$\lambda_1 > 0 > \lambda_2$



note that in original basis (before translating to J. form) axes can be non orthogonal:

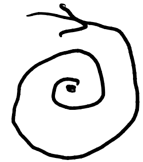


(SPIRAL)

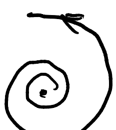
(2nd form)

$\rho < 0$
$\omega > 0$

$\rho < 0$
$\omega < 0$



$\rho > 0$
$\omega > 0$



$\rho > 0$
$\omega < 0$

(CENTER)

2nd form

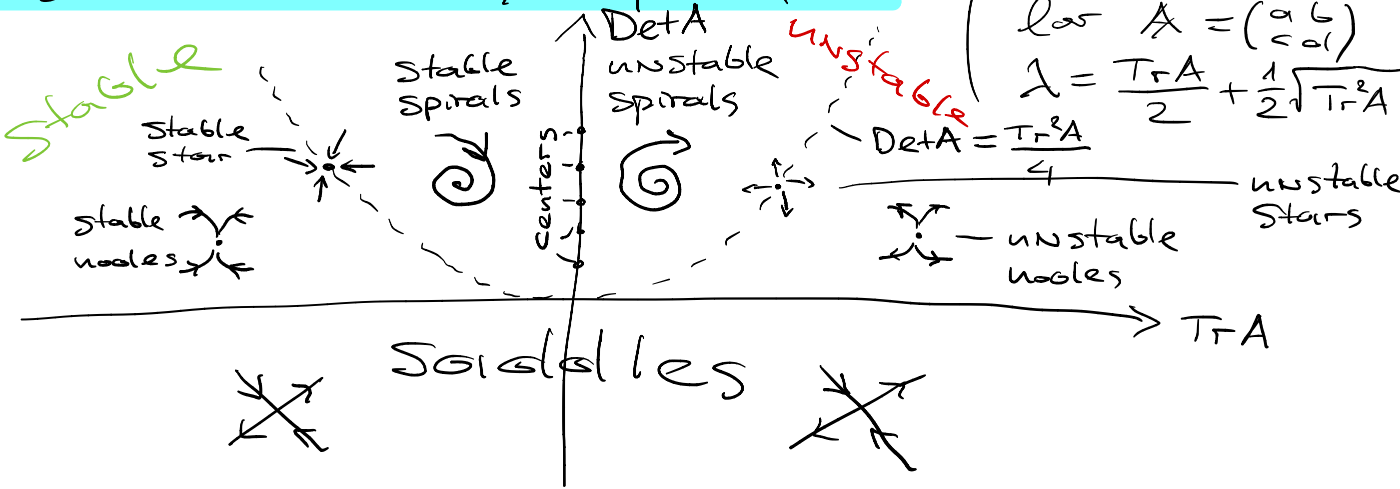
$\rho = 0$

orbits

Let's collect everything together

(characteristic equation for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\lambda = \frac{\mathrm{Tr}A}{2} \pm \frac{1}{2}\sqrt{\mathrm{Tr}^2 A - 4\,\mathrm{Det}A}$$
)



LINEARIZATION C.D.

— in 'most' cases NL equations can be well approximated by linear terms close to fixed points

$$\begin{cases} \dot{x} = f_x(x,y) \\ \dot{y} = f_y(x,y) \end{cases} \xrightarrow[\text{at } x^*]{\text{linearization}} \dot{\mathbf{x}} = A\,\mathbf{x}, \quad A = \left.\begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{pmatrix}\right|_{\mathbf{x} = \mathbf{x}^*}$$

at fixed point

— in 'some' cases including NL terms can change the behavior:

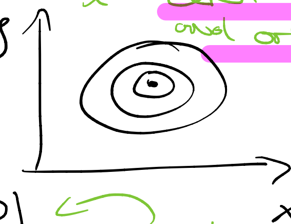
- Nodes ⟶ spirals
- Stars ⟶ spirals [• but spirals always remain spirals]
- centers ⟶ spirals / isolated orbits

Let's get back to Lotka-Volterra system

FINAL EXAMPLE

$x, y > 0$ — fundamental question: can both species coexist? → we are looking for center and orbits

$$\begin{cases} \dot{x} = x(-\alpha + \beta y) & \leftarrow \text{PREDATOR} \\ \dot{y} = y(\delta - \gamma x) & \leftarrow \text{PREY} \end{cases}$$



- fixed points: $x_0^* = (0,0)$, $x_1^* = \left(\frac{\delta}{\gamma}, \frac{\alpha}{\beta}\right)$

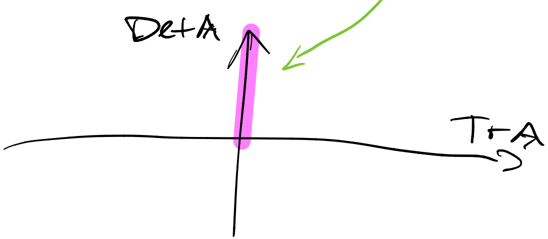
- linearization

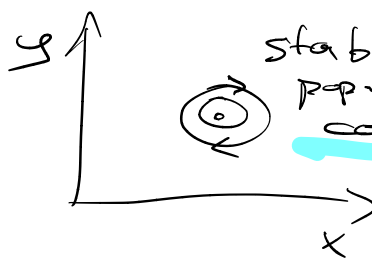
— Jacobian matrix
$$A = \begin{pmatrix} -\alpha + \beta y & \beta x \\ -\gamma y & \delta - \gamma x \end{pmatrix}_{x = x^*} \xrightarrow{x_0} \begin{pmatrix} -\alpha & 0 \\ 0 & \delta \end{pmatrix} = A_0 \qquad \xrightarrow{x_1} A_1$$

Not interested because we want finite populations

$$A_1 = \begin{pmatrix} 0 & \beta\delta/\gamma \\ -\gamma\alpha/\beta & 0 \end{pmatrix} \longrightarrow \begin{cases} \operatorname{Tr} A_1 = 0 \\ \operatorname{Det} A_1 = \alpha\delta > 0 \end{cases} \Rightarrow \text{center!}$$

$(\alpha, \beta, \gamma, \delta > 0)$

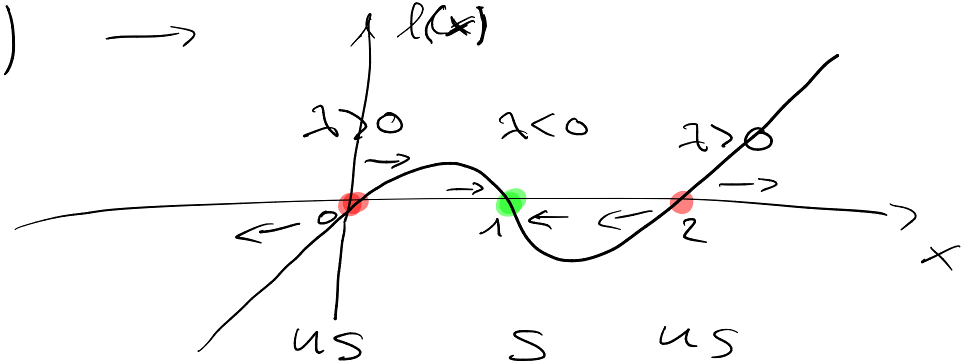




stable populations can coexist!

LABORATORY

$$\dot{x} = f(x) = x(x-1)(x-2) \longrightarrow$$



1) $x^* = 0, 1, 2$

3) det = D, tr = T

$$\begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix} \qquad \begin{pmatrix} -3 & -2 \\ -1 & -3 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$D = -5$	$D = 5$	$D = 7$	$D = 4$
$T = 0$	$T = 2$	$T = -6$	$T = 4$
Saddle	$\frac{T^2}{4} = 1 < 5 = D$	$\frac{T^2}{4} = \frac{36}{4} = 9 > 7 = D$	$\frac{T^2}{4} = \frac{16}{4} = 4 = D$
	unstable Spiral	stable node	unstable star

4)

JACOBIAN MATRIX

$$\begin{cases} \dot{x} = x(3-x-2y) = f_x \\ \dot{y} = y(2-x-y) = f_y \end{cases}$$

$$\begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{pmatrix} = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{pmatrix}$$

fixed points : $(0,0)$; $(0,2)$; $(3,0)$; $(1,1)$

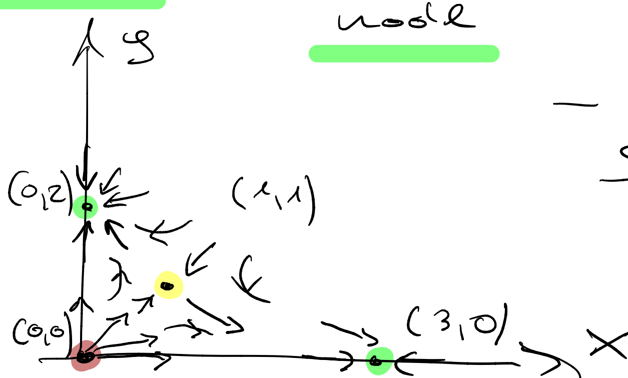
$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}; \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}; \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}; \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

$D = 6$, $T = 5$, unstable node

$D = 2$, $T = -3$, stable node

$D = 3$, $T = -4$, stable node

$D = -1$, Saddle



– both species cannot coexist.