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Discrete Optimization

MIT MANAGEMENT SLOAN SCHOOL

15.060: Data, Models, and Decisions

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Class 16 (Nov 17)

A Classification of Optimization Problems

		Decision Variables	
		All continuous	Some (or all) integer
Objective Function & Constraints	All linear	*Linear Optimization* ✓	*Integer Linear Optimization**
	Some (or all) non-linear	*Nonlinear Optimization*	*Integer Nonlinear Optimization*

* also known as Discrete Optimization

In some problem settings, fractional solutions are acceptable

In our linear optimization examples so far (School Bus, Optima Autos), when variables represent large quantities of something, **fractional solutions are ok**.

- 23.6 Type A buses => The optimal is between 23 and 24 buses.
- We can round to the nearest integer. Some constraints may be slightly broken (e.g., budget is 10.01 million), but usually not a big deal.

We need Discrete Optimization when fractional solutions are not acceptable

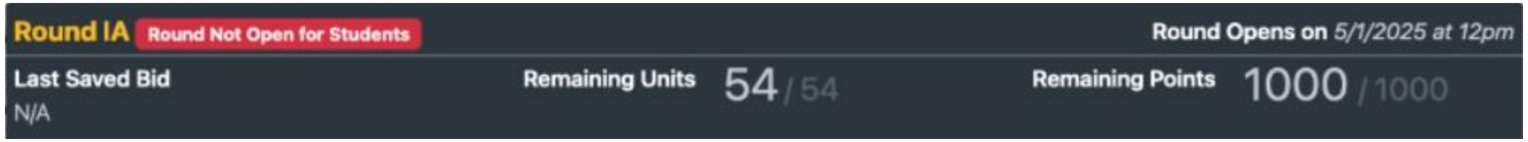
- Many business settings involve **binary decisions**: whether to do something or not (e.g., build a factory, enter a partnership, assign a shift).
- We model **binary decisions** with **binary decision variables,** since fractional solutions do not make sense e.g., *If 1 = do something and 0 = don't do something, what does 0.6 even mean?*
- Business problems also typically involve complex IF/THEN conditions and non-linearities, and optimization with binary variables is a powerful tool to handle these conditions
- In DMD, we will cover:
  - Supply chain design and operation (next class)
  - Retail pricing/promotions (the last class)

Today’s Class

- **Discrete optimization example:** Optimizing course selections at Sloan

- The power of binary variables to represent logical and non-linear relationships using linear constraints

- (If time permits) How discrete optimization problems are solved efficiently in practice



Optimizing Course Selection

(Note: Not safe for use in the spring!)

Course selection problem statement

- You are considering 10 courses named A, B, C, …, J for Spring 2026. Each class gives you some **utility** towards your academic and professional goals.

	A	B	C	D	E	F	G	H	I	J
Utility	10	2	4	2	5	4	8	7	6	6

- You would like to choose which courses to take to **maximize total utility**.

- However, there are some constraints regarding:
  - Bidding
  - Credits
  - Schedule

Course selection problem data

- **Required Bids**
  - You have 1000 points to allocate across the 10 courses.
  - We make a simplifying assumption that bidding = enrolling

	A	B	C	D	E	F	G	H	I	J
Required Bid	200	50	150	400	50	0	150	50	180	100

- **Credit Hours**
  - You need to take at least 36 and not more than 54 credit hours

	A	B	C	D	E	F	G	H	I	J
Credit Hours	12	9	9	12	6	6	9	6	9	6

Course selection problem data

- **Schedule**
  - Some courses are whole-semester, others are only H3 or H4 half-semester courses.
  - You have a strong preference (ok, fine, let's make it a hard constraint 🙂) for taking no more than 3 courses on any given day

	A	B	C	D	E	F	G	H	I	J
H3	✓	✓	✓	✓	✓		✓		✓	
H4	✓	✓	✓	✓		✓	✓	✓	✓	✓

	A	B	C	D	E	F	G	H	I	J
Monday	✓	✓			✓		✓			✓
Tuesday			✓	✓		✓		✓	✓	
Wednesday	✓	✓			✓		✓			✓
Thursday			✓	✓		✓		✓	✓	
Friday										

Course bidding problem data summary

	A	B	C	D	E	F	G	H	I	J
Utility	10	2	4	2	5	4	8	7	6	6
Required Bid	200	50	150	400	50	0	150	50	180	100
Credit Hours	12	9	9	12	6	6	9	6	9	6

	A	B	C	D	E	F	G	H	I	J
H3	√	√	√	√	√		√		√	
H4	√	√	√	√		√	√	√	√	√
Monday	√	√			√		√			√
Tuesday			√	√		√		√	√	
Wednesday	√	√			√		√			√
Thursday			√	√		√		√	√	
Friday										

Which courses should you take to maximize utility while satisfying all the constraints?

Let’s write down the formulation

- Decision Variables
  - For each course (A, B, C, …, J), should you enroll or not enroll.
- Objectives
  - Maximize the utility of the courses that you do enroll in
- Constraints
  - Bids
  - Credits
  - Schedules

Formulating the Course Selection Problem

The Decision Variables

- Define *binary decision variables* for whether or not you bid/enroll in a course

$$y_A, y_B, \dots, y_J$$

- For example:
  - y_A = 1 means you enroll in course A.
  - y_A = 0 means you do not enroll in course A.

- <u>Note</u>: In this simplified model, we assume that enrolling in a course is your core decision. You will simply bid the minimum number of points required to get a spot and assume you will get a spot for sure.

Formulating the Course Selection Problem

Objective Function

- **Variables:** whether or not you enroll in a course

$$y_A, y_B, \ldots, y_J \qquad = 0 \text{ or } 1$$

- **Objective:** Our goal is to maximize the utility of the courses we enroll in.

	A	B	C	D	E	F	G	H	I	J
Utility	10	2	4	2	5	4	8	7	6	6

$$\text{maximize} \qquad 10\, y_A + 2\, y_B + 4\, y_C + 2\, y_D + 5\, y_E + 4\, y_F + 8\, y_G + 7\, y_H + 6\, y_I + 6\, y_J$$

- Is the objective function linear?

Formulating the Course Selection Problem
Objective Function

- **Variables:** whether or not you bid/enroll in a course

$$y_A, y_B, \ldots, y_J \qquad = 0 \text{ or } 1$$

- **Objective:** Our goal is to maximize the utility of the courses we bid/enroll in.

	A	B	C	D	E	F	G	H	I	J
Utility	10	2	4	2	5	4	8	7	6	6

$$\text{maximize} \qquad 10\, y_A + 2\, y_B + 4\, y_C + 2\, y_D + 5\, y_E + 4\, y_F + 8\, y_G + 7\, y_H + 6\, y_I + 6\, y_J$$

- Is the objective function linear? Yes, it is a weighted sum of the binary variables.

Suppose you have a Solver that can optimize integer variables. How can you “force” it to solve for binary variables?

Integer variables + Constraints = Binary variables

$y_A , y_B , \dots , y_J$ <u>integral</u>

$y_A , y_B , \dots , y_J \geq 0$ $\quad\Longleftrightarrow\quad$ $y_A , y_B , \dots , y_J$ <u>binary</u>

$y_A , y_B , \dots , y_J \leq 1$

Formulating the Course Bidding Problem
Constraints

Variables	(binary)	y_A , y_B , … , y_J $\geq 0, \leq 1$ and <u>integral</u>
Bidding	(1000 points budget)	???
Credits	(54 max credits)	???
	(36 min credits)	???
Schedule	(MW H3 load)	???
	(MW H4 load)	???
	(TR H3 load)	???
	(TR H4 load)	???

In-class EXERCISE!

Dos and Donts when writing constraints

- You can use $\leq$, $\geq$ and =

$$y_C + y_D + y_I \leq 3$$

$$y_C + y_D + y_I \geq 3$$

$$y_C + y_D + y_I = 3$$

- You cannot use <, > or $\neq$

$$y_C + y_D + y_I < 3$$

$$y_C + y_D + y_I > 3$$

$$y_C + y_D + y_I \neq 3$$

Write down the constraints for the budget, credits and MW H3 load

Variables (binary) $y_A, y_B, \ldots, y_J \geq 0, \leq 1$ and <u>integral</u>

Bidding (1000 points budget)

	A	B	C	D	E	F	G	H	I	J
Required Bid	200	50	150	400	50	0	150	50	180	100

Credits (54 max credits)

(36 min credits)

	A	B	C	D	E	F	G	H	I	J
Credit Hours	12	9	9	12	6	6	9	6	9	6

Schedule (MW H3 load)

(MW H4 load)

(TR H3 load)

(TR H4 load)

	A	B	C	D	E	F	G	H	I	J
H3	✓	✓	✓	✓	✓		✓		✓	
H4	✓	✓	✓	✓		✓	✓	✓	✓	✓
Monday	✓	✓			✓		✓			✓
Tuesday			✓	✓		✓		✓	✓	
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