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Nonlinear Optimization

MIT MANAGEMENT SLOAN SCHOOL

15.060: Data, Models, and Decisions

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Class 18 (Nov 24)

A Classification of Optimization Problems

		Decision Variables	
		All continuous	Some (or all) integer
Objective Function & Constraints	All linear	Linear Optimization	Integer Linear Optimization*
	Some (or all) non-linear	Nonlinear Optimization	Integer Nonlinear Optimization

* also known as Discrete Optimization

Nonlinear Optimization is Very Versatile

Examples of important applications of nonlinear optimization

- Portfolio optimization
- Revenue/Price/Promotions Optimization
- Salesforce Optimization
- Marketing Mix Optimization
- Predictive Analytics
  - When we build Linear Regression and Logistic Regression models, we are actually solving Nonlinear Optimization problems!
- Numerous applications in science and engineering …

Optimizing Campaign Expenditure

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This is a hypothetical example designed for <u>educational</u> purposes only

- You are in the year 20xx and the next U.S. Presidential Election will be held on “the Tuesday next after the first Monday in the month of November”

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- Three weeks before the election campaign, one of the candidates is asking for your advice on how to use analytics to help allocate the remaining campaign funds in the final weeks of the campaign

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- Three weeks before the election campaign, one of the candidates is asking for your advice on how to use analytics to help allocate the remaining campaign funds in the final weeks of the campaign
- In particular, the campaign has $20 million left to spend on State-level campaign efforts (advertising, get-out-the-vote efforts , etc.)

Optimizing Campaign Expenditure?

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- You are in the year 20xx and the next U.S. Presidential Election will be held on "the Tuesday next after the first Monday in the month of November"
- Three weeks before the election campaign, one of the candidates is asking for your advice on how to use analytics to help allocate the remaining campaign funds in the final weeks of the campaign
- In particular, the campaign has $20 million left to spend on State-level campaign efforts (advertising, get-out-the-vote efforts , etc.)
- They want to know how to allocate the remaining funds across the states

The State of the Campaign

- At this stage, the campaign is confident their candidate has **229** electoral votes almost guaranteed

- Their opponent has **242** electoral votes almost guaranteed

- There are only three swing states left:
  - FL (29 electoral votes)
  - OH (18 electoral votes)
  - PA (20 electoral votes)

Electoral Vote Math

Electoral votes secured by our candidate	229
Electoral votes secured by the opponent	242
Electoral votes up for grabs	67 (= 29 + 18 + 20)
Total votes in the Electoral College	538
Total votes needed by a candidate to win	270
To win the election, our candidate needs	41 votes!

How best to spend the $20m campaign funds that remain?

<u>Proposal</u>: Allocate the remaining funds *in proportion to the number of electoral votes* in each of the remaining contested states

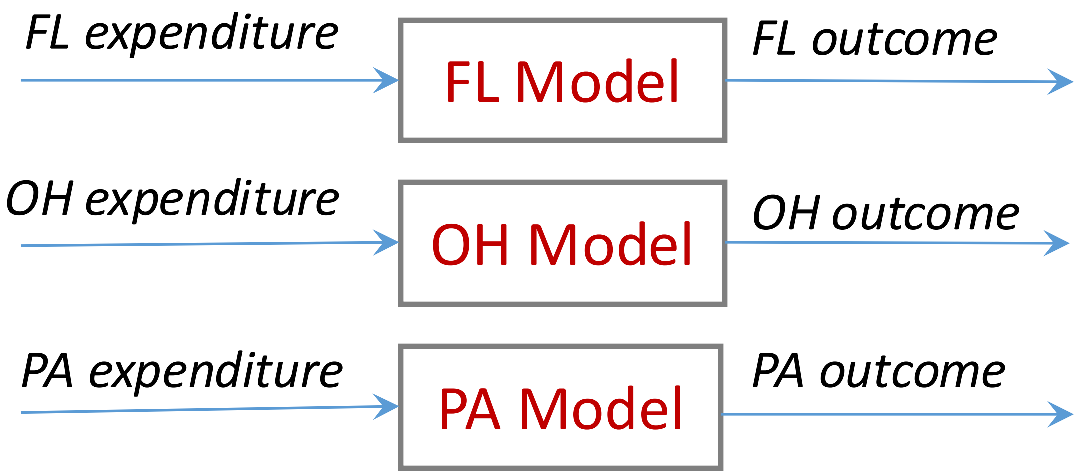
State	**FL**	**OH**	**PA**
Electoral votes	29	18	20
Expenditure [$ million]	20 * 29/67 = **8.66**	20 * 18/67 = **5.37**	20 * 20/67 = **5.97**

- **How good is this plan?**
- **Can we devise a better plan?**

Proposed Approach

<u>Step 1</u>: Predictive

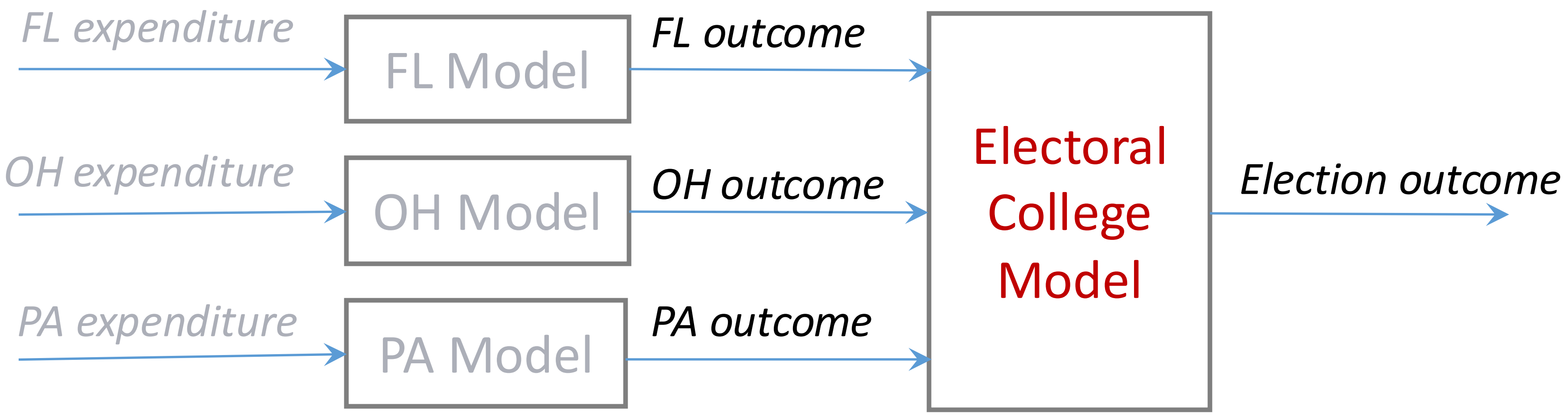
- Develop State-level models that relate expenditure in a state to the election outcome in that state.



Proposed Approach

<u>Step 1</u>: Predictive

- Develop State-level models that relate expenditure in a state to the election outcome in that state.
- Combine the State-level models with the logic of the Electoral College to predict the outcome of the election given an expenditure plan.



Proposed Approach

Step 1: Predictive

- Develop State-level models that relate proposed expenditure in a state to the election outcome in that state.
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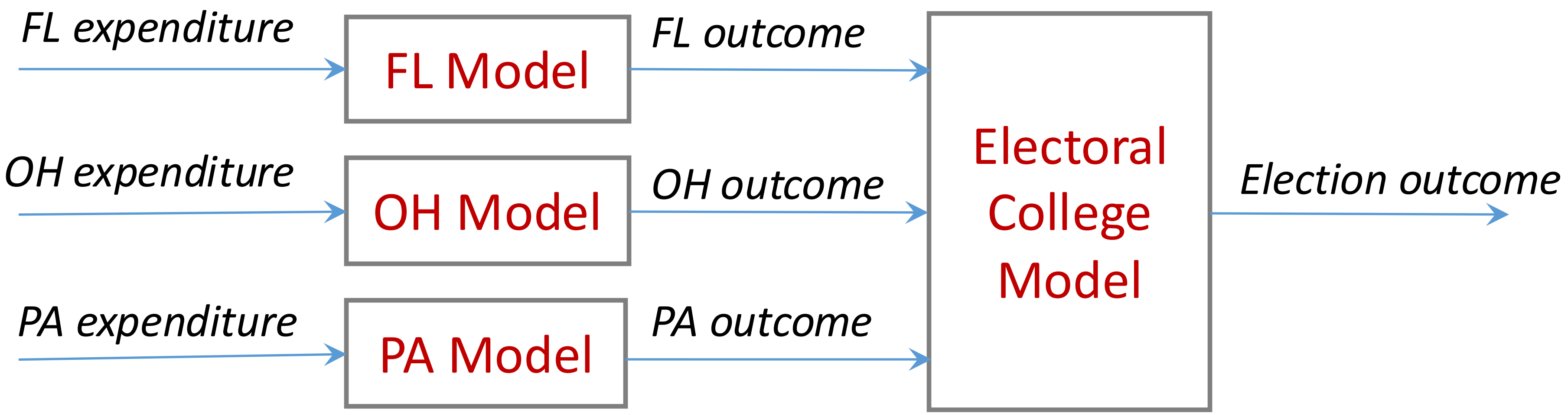

Step 2: Prescriptive

- Use the predictive models to formulate an expenditure optimization model

Building a Predictive Model

Building a predictive model

- We would like to build a predictive model that takes as input the expenditure and gives us the probability of a win.

- What kind of historical data would such a model need?

Building a predictive model

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FL expenditure → FL Model → *FL outcome*

- What kind of historical data would such a model need?

- Data that would be nice to have:
  - Expenditure of past campaigns in this state and whether the candidate won
  - Also: Features of the candidate, opponent, political/economic/social landscape (to build a more accurate model)

Expenditure	Features/ Independent variables	Win/Loss?
\$10m	…	Win
\$8m	…	Win
\$2m	…	Win
\$1m	…	Loss
\$0m	…	Loss

What can we do if there is limited data?

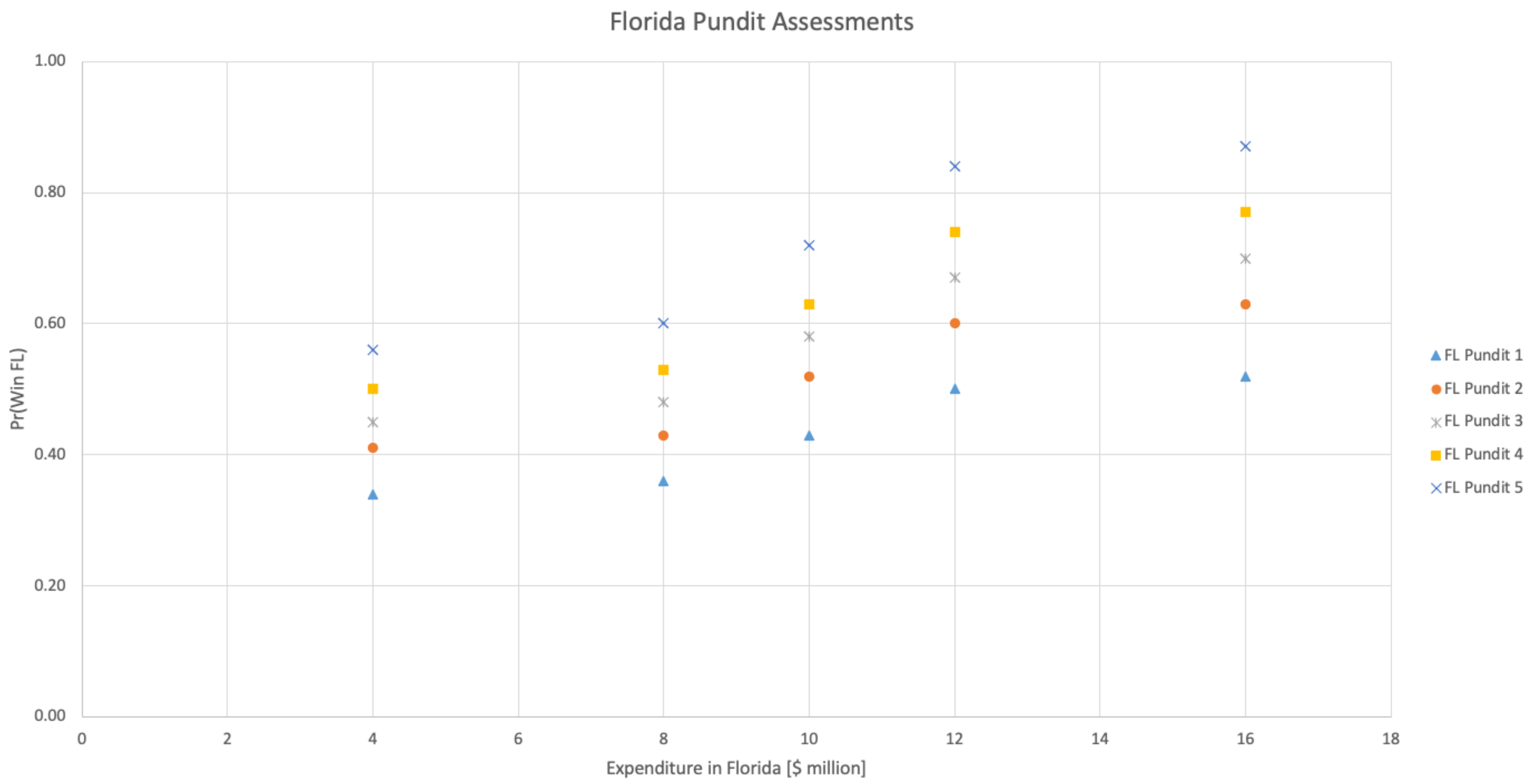
- The campaign has very limited reliable historical data on how campaign expenditure influences election results in swing states.

- Luckily, the campaign has **pundits (experts)** in each state, who know the state well, and who can provide (subjective) assessments of the chances of winning the state given certain levels of expenditure.

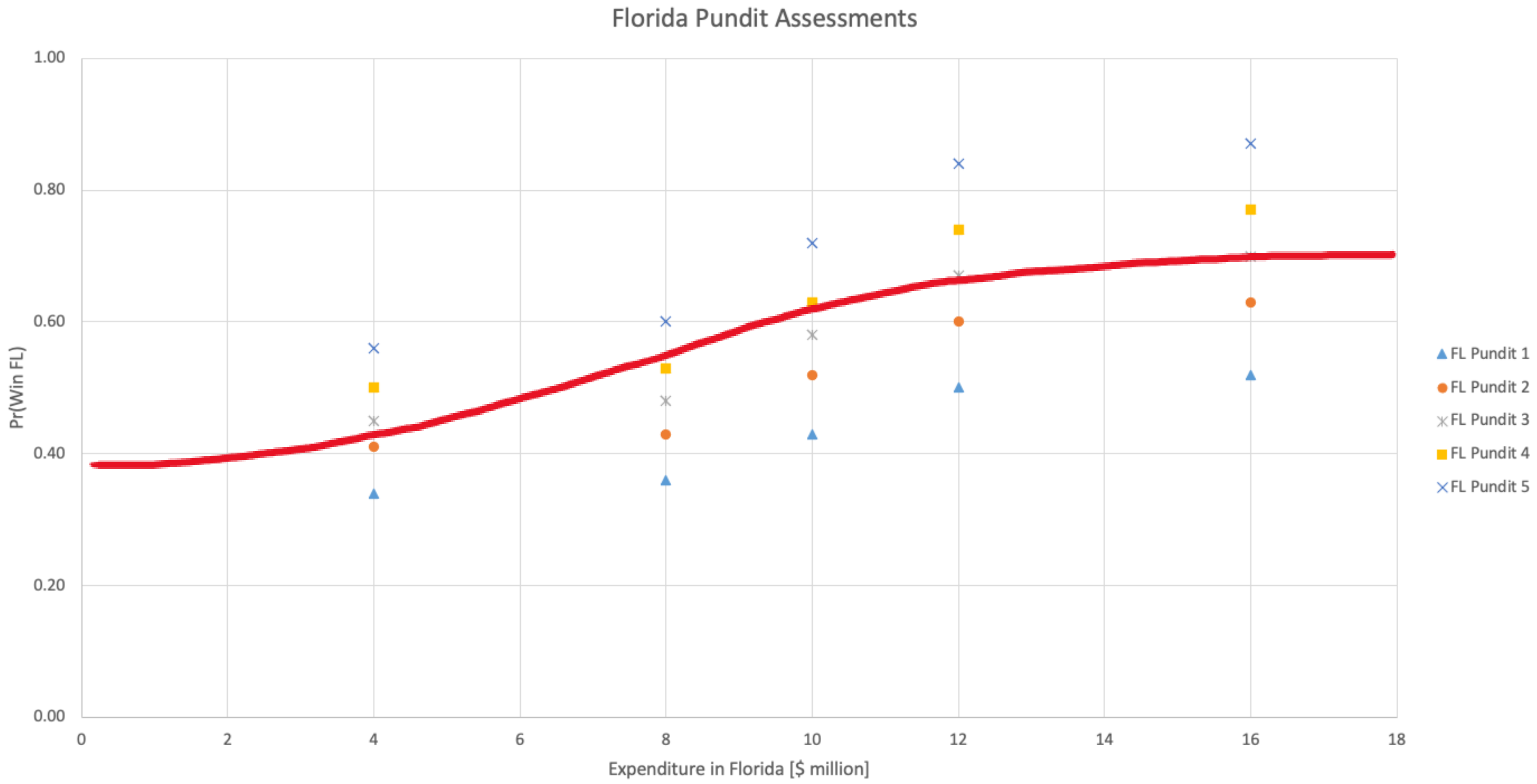
Pundit Assessments - Florida

Florida	Probability of Winning Florida for Expenditure Level				
	$4 million	$8 million	$10 million	$12 million	$16 million
FL Pundit 1	0.34	0.36	0.43	0.50	0.52
FL Pundit 2	0.41	0.43	0.52	0.60	0.63
FL Pundit 3	0.45	0.48	0.58	0.67	0.70
FL Pundit 4	0.50	0.53	0.63	0.74	0.77
FL Pundit 5	0.56	0.60	0.72	0.84	0.87

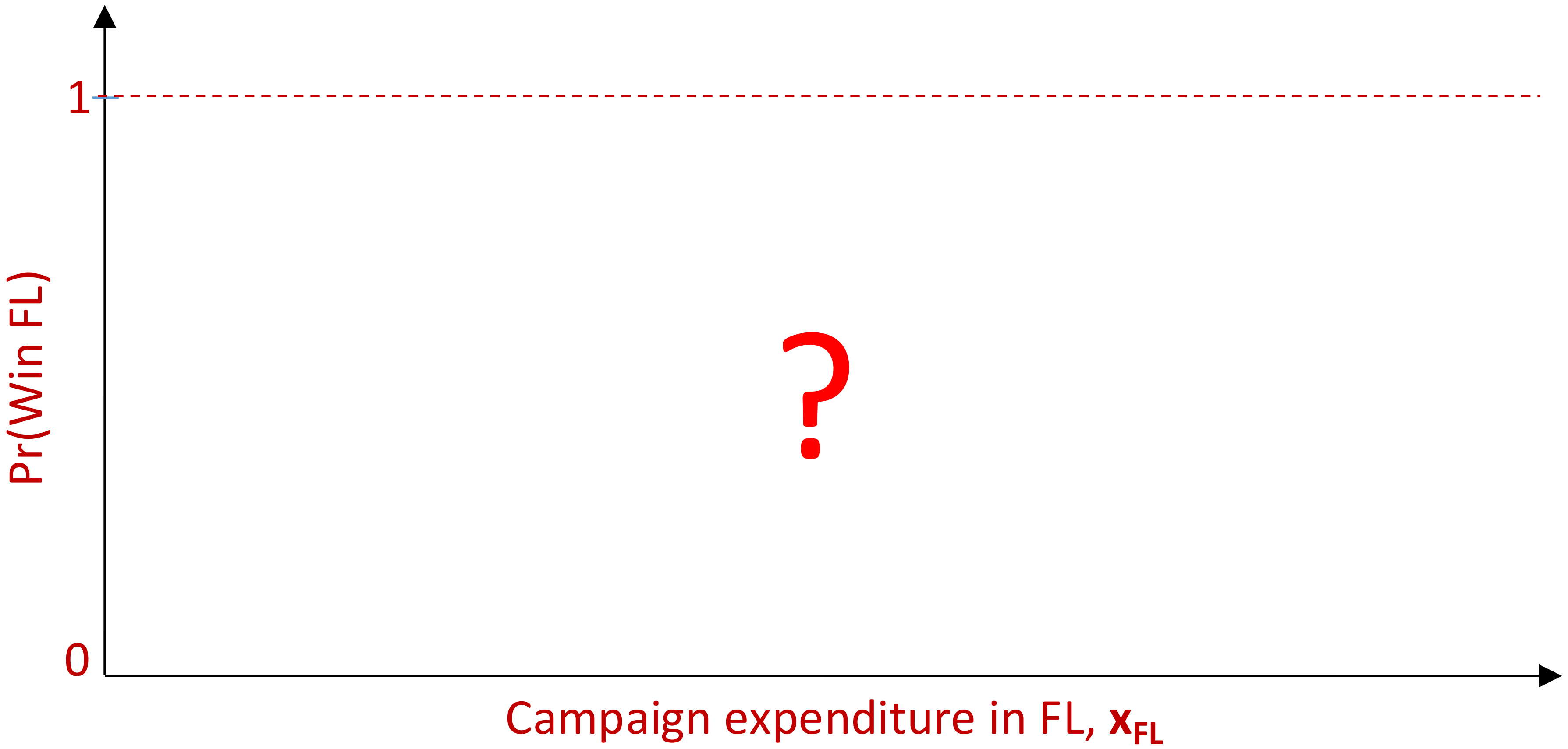
Let’s visualize the pundit assessments for Florida



Maybe we can “fit” a curve to this data?

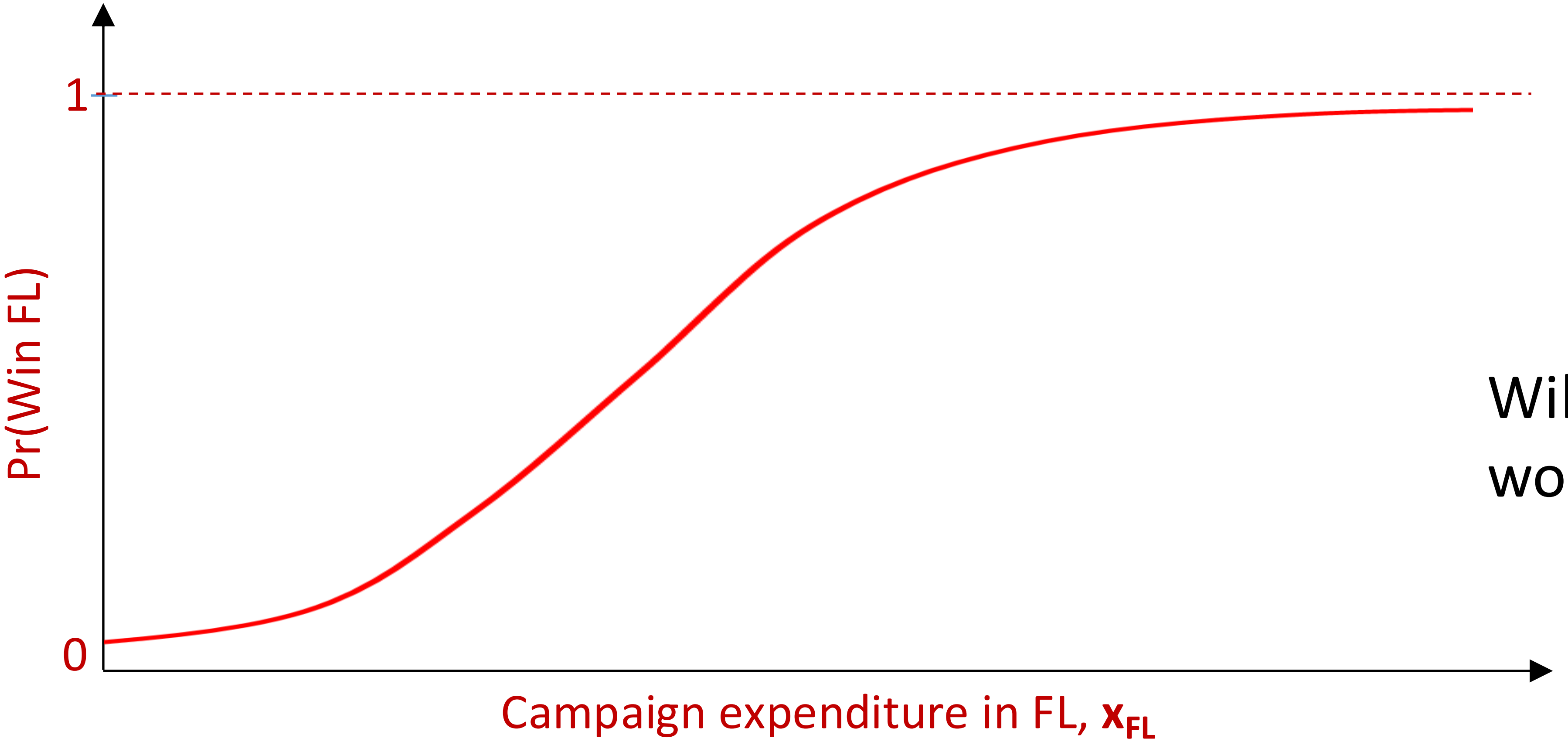


State Level Model



Propose a model
that relates campaign expenditure in FL
to the probability of winning FL

State Level Model

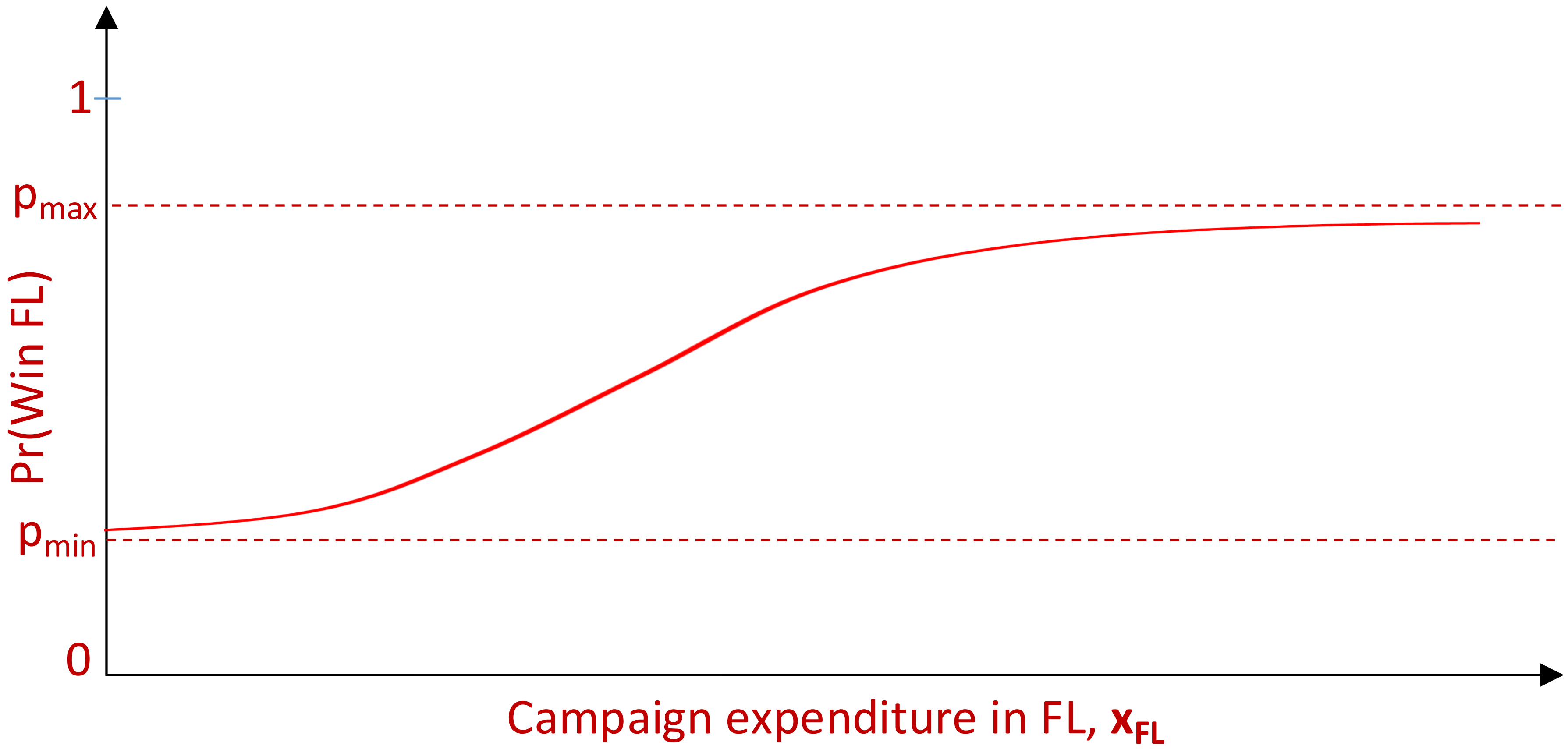


Will this work?

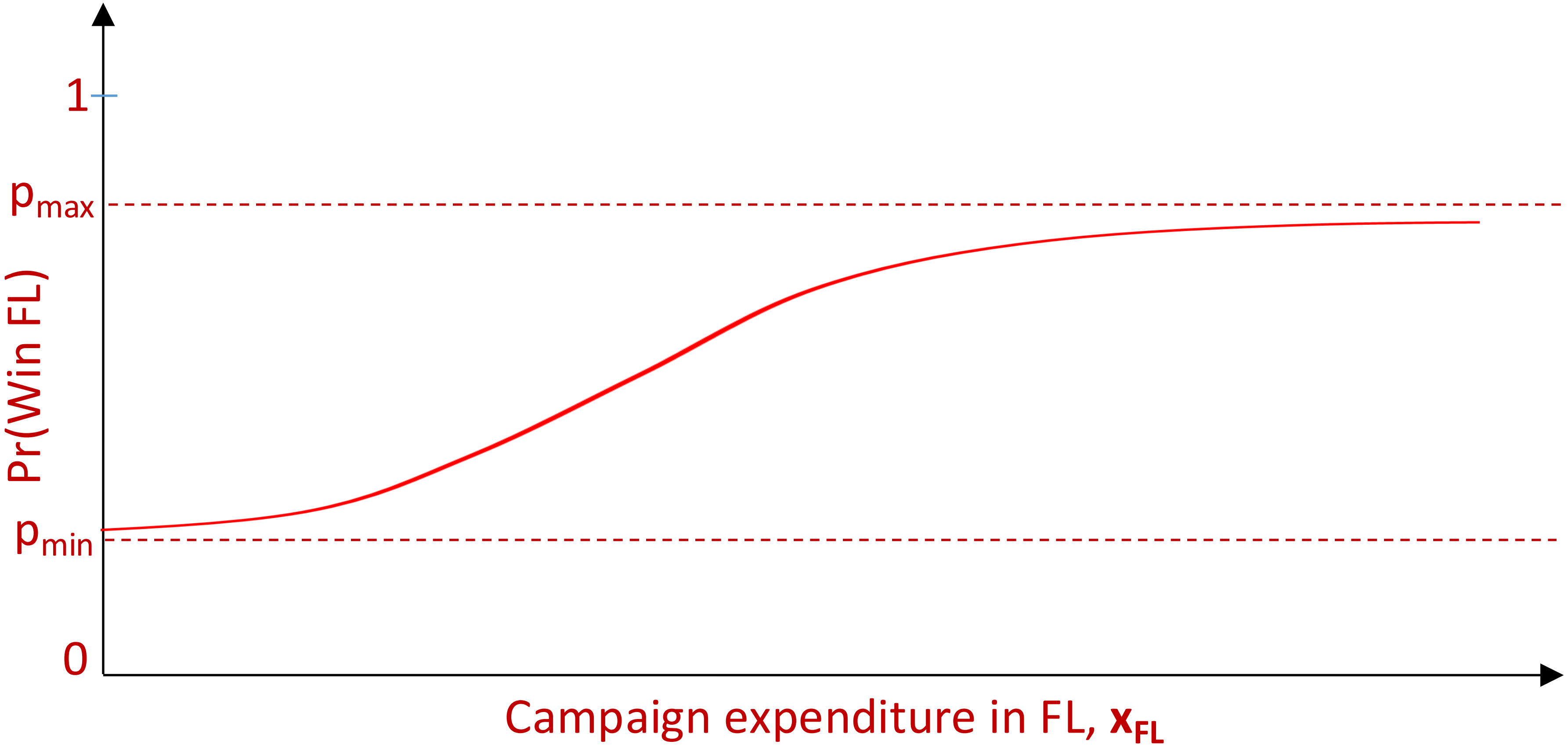
Recall:
Logistic curve

$$\boxed{Pr(Win\ FL) = \frac{1}{1 + e^{-(b_0 + b_1 x_{FL})}}}$$

A Better State Level Model?



A Better State Level Model



A better model:
Scaled logistic curve

$$\boxed{Pr(Win\ FL) = p_{min} + \frac{(p_{max} - p_{min})}{1 + e^{-(b_0 + b_1 x)}}}$$

A Better State Level Model

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How can we estimate the parameters p_{min}, p_{max}, b_0, and b_1 from the above data?

$$\boxed{Pr(Win\ FL) = p_{min} + \frac{(p_{max} - p_{min})}{1 + e^{-(b_0 + b_1 x)}}}$$

A Better State Level Model

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$$\boxed{Pr(Win\ FL) = p_{min} + \frac{(p_{max} - p_{min})}{1 + e^{-(b_0 + b_1 x)}}}$$

We can use <u>optimization to find the best-fit curve</u> for the pundit data (like how we found the best-fitting line in Linear Regression)

Pundit Estimation

For any choice of p_{max}, p_{min}, b_0 and b_1, we can calculate the sum of squared *errors* of that pundit model. We'd like to choose p_{max}, p_{min}, b_0 and b_1 to minimize this error.

$$\min_{p_{min},\, p_{max},\, b_0,\, b_1} \; Total\ Prediction\ Error$$

$$\Downarrow$$

$$\min_{p_{min},\, p_{max},\, b_0,\, b_1} \; Sum\ of\ (pundit\ - model)^2$$

$$\Downarrow$$

$$\min_{p_{min},\, p_{max},\, b_0,\, b_1} \sum_{i=1}^{n} \left(y_i - \left(p_{min} + \frac{(p_{max} - p_{min})}{1 + e^{-(b_0 + b_1 x_i)}} \right) \right)^2$$

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Aside: Regression is a non-linear optimization problem! Recall that we find the parameters that minimize the sum of squared errors.

Estimating parameters in Excel

CURVE FITTING USING NONLINEAR OPTIMIZATION

"DECISIONS"

p_min	0.45
p_max	0.70
b_0	-10.16
b_1	1.02

OBJECTIVE

Objective function	0.249	<-

$$p_{min} + \frac{(p_{max} - p_{min})}{1 + e^{-(b_0 + b_1 x)}}$$

CONSTRAINTS *None*

CALCULATIONS

Observation	Data		Model	Residual	Square of
i	x_i	y_i (= FL pundits)	Prediction		Residual
1	4.00	0.34	0.45	0.11	0.01
2	4.00	0.41	0.45	0.04	0.00
3	4.00	0.45	0.45	0.00	0.00
4	4.00	0.50	0.45	-0.05	0.00
5	4.00	0.56	0.45	-0.11	0.01
6	8.00	0.36	0.48	0.12	0.01
7	8.00	0.43	0.48	0.05	0.00
8	8.00	0.48	0.48	0.00	0.00
9	8.00	0.53	0.48	-0.05	0.00
10	8.00	0.60	0.48	-0.12	0.01

Parameter Estimation Results

- Suppose we do this for each state and obtain the following parameter estimates:

Parameter	State		
	FL	OH	PA
p_{min}	0.45	0.50	0.40
p_{max}	0.70	0.60	0.80
b_0	-10	-5	-7
b_1	1	1	1

Parameter Estimation Results

Parameter	State		
	FL	**OH**	**PA**
p_{min}	0.45	0.50	0.40
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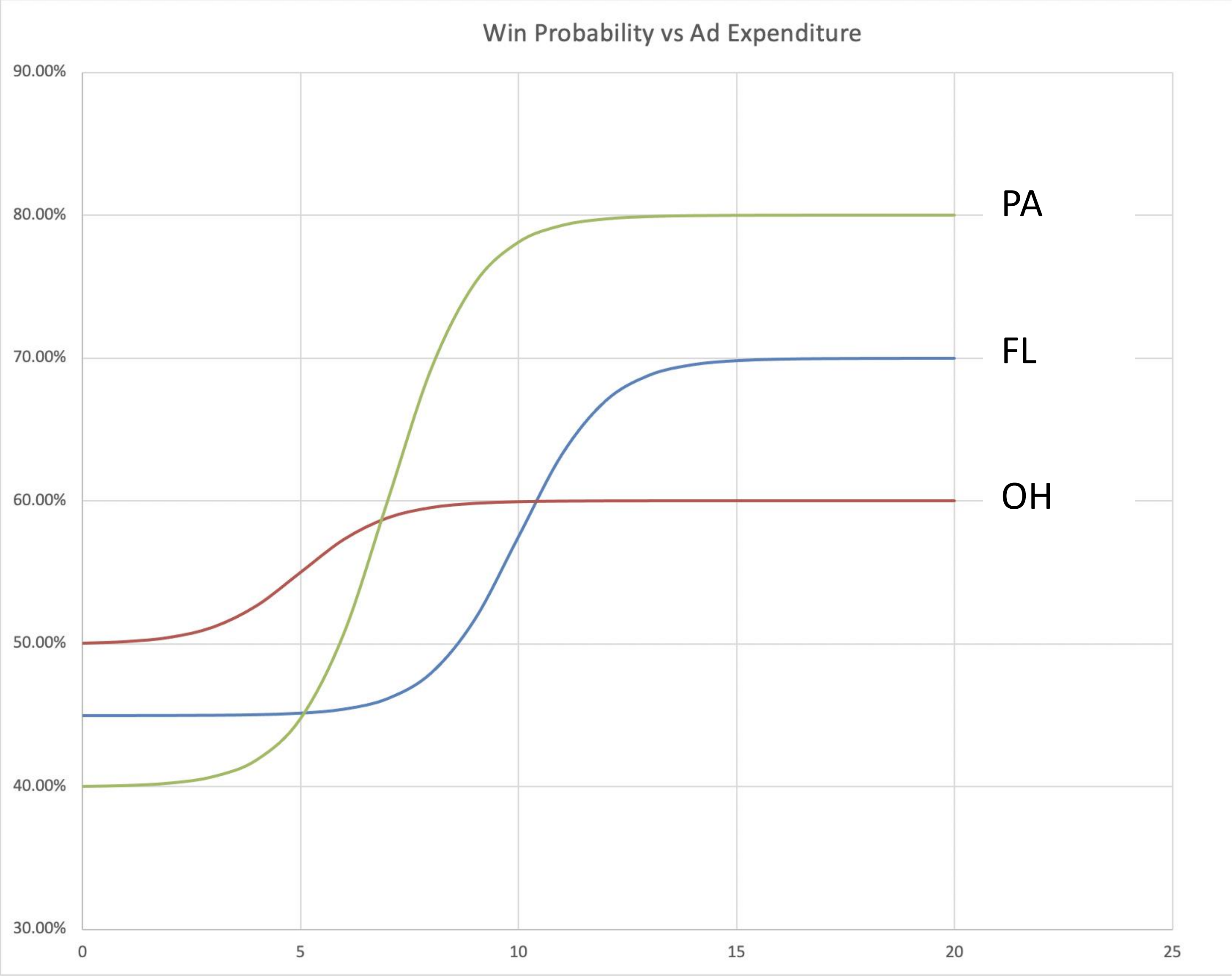
b_0	-10	-5	-7
b_1	1	1	1

Pr(Win FL) Pr(Win OH) Pr(Win PA)

$$\left(0.45 + \frac{(0.70 - 0.45)}{1 + e^{-(x_{FL} - 10)}}\right)\left(0.50 + \frac{(0.60 - 0.50)}{1 + e^{-(x_{OH} - 5)}}\right)\left(0.40 + \frac{(0.80 - 0.40)}{1 + e^{-(x_{PA} - 7)}}\right)$$

Predictive Models for FL, OH and PA



Formulating an Optimization Model

Let's Formulate the Optimization Problem

Decision variables

Objective function

Constraints

Decision Variables

Decision variables

x_{FL} : $millions to spend in Florida
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Objective function

Constraints

Objective Function

Decision variables

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Objective function

What's the Objective Function?

Constraints

Objective Function

<u>Decision variables</u>

x_{FL} : $millions to spend in Florida
x_{OH} : $millions to spend in Ohio
x_{PA} : $millions to spend in Pennsylvania

<u>Objective function</u>

Maximize Pr(Winning the Election)

<u>Constraints</u>

Given our models, what's the probability of winning?

x_{FL} *FL expenditure* → FL Model → *Win FL prob* → Pr (Win FL)

$$0.45 + \frac{(0.70 - 0.45)}{1 + e^{-(x_{FL} - 10)}}$$

x_{OH} *OH expenditure* → OH Model → *Win OH prob* → Pr (Win OH)

$$0.50 + \frac{(0.60 - 0.50)}{1 + e^{-(x_{OH} - 5)}}$$

x_{PA} *PA expenditure* → PA Model → *Win PA prob* → Pr (Win PA)

$$0.40 + \frac{(0.80 - 0.40)}{1 + e^{-(x_{PA} - 7)}}$$

Probability of Election Win?

Paths to Victory

Path	Electoral Votes
Win FL and OH and PA	29 + 18 + 20 = 67
Win FL and OH, lose PA	29 + 18 = 47
Win FL and PA, lose OH	29 + 20 = 49

[Win FL] *<u>and</u>* *[don't lose both PA and OH]*

$$\Pr(\text{Winning the Election}) = \Pr(\text{Win FL})\ [\ 1 - \Pr(\text{Lose OH}) \times \Pr(\text{Lose PA})\ ]$$

Objective Function in Terms of Expenditure Allocations

Pr(Winning the Election) = Pr(Win FL) [1 - Pr(Lose OH) x Pr(Lose PA)]

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Objective Function in Terms of Expenditure Allocations

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Objective Function in Terms of Expenditure Allocations

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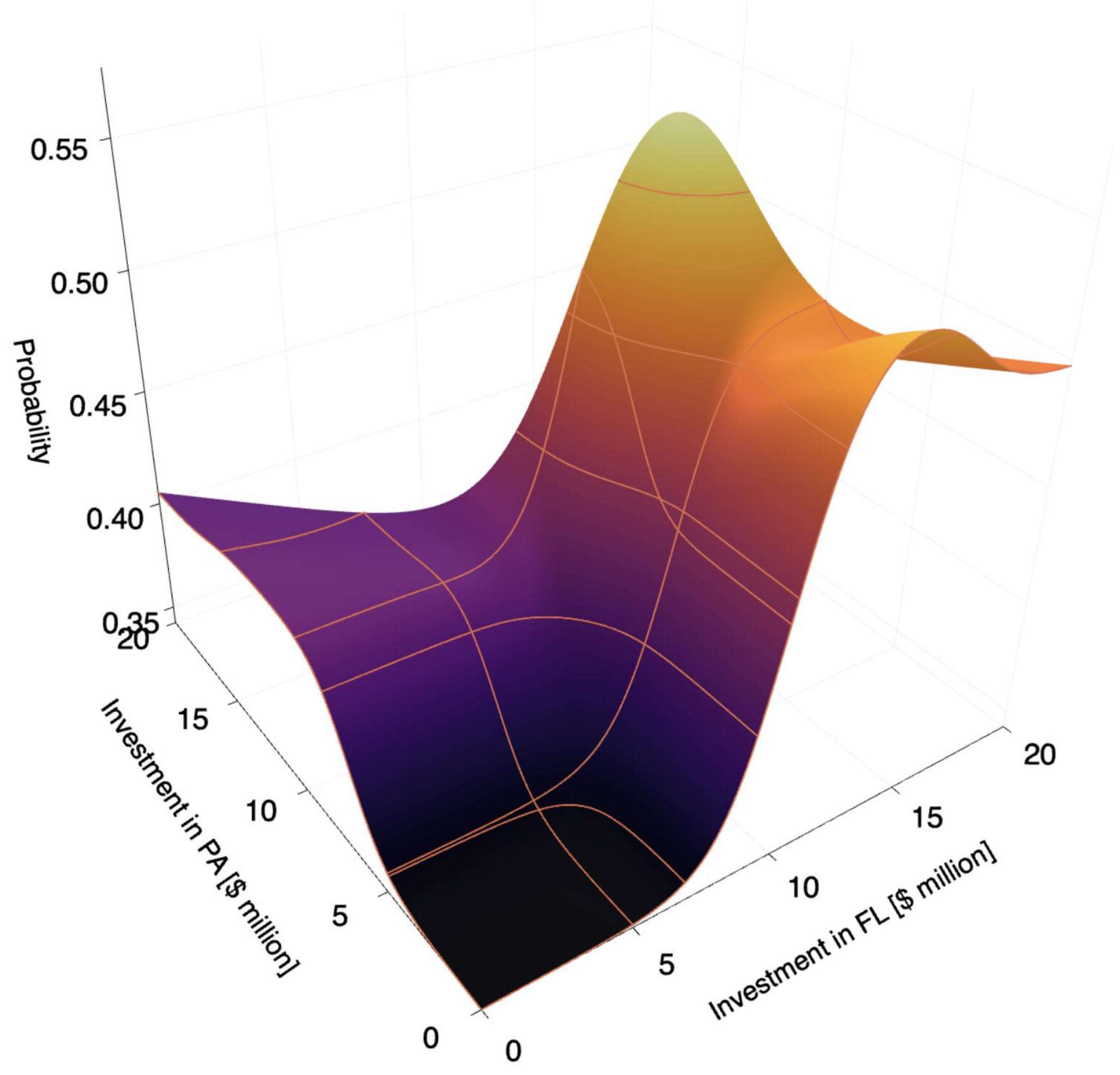
Let’s visualize the Objective Function!



We set OH expenditure to be
= 20 – FL expenditure
– PA expenditure

We can use this objective function to predict the probability of a win for any expenditure plan

Plan	Expenditure [$ million]			Predicted Outcome
	FL	OH	PA	
No expenditure	0	0	0	31.53%

$$\left(0.45+\frac{(0.70-0.45)}{1+e^{-(0-10)}}\right)\left[1-\left(1-0.50-\frac{(0.60-0.50)}{1+\mathrm{e}^{-(0-5)}}\right)\left(1\ -\ 0.40-\frac{(0.80-0.40)}{1+\mathrm{e}^{-(0-7)}}\right)\right]$$

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Current Plan	8.66	5.37	5.97	39.23%

$$\left(0.45 + \frac{(0.70 - 0.45)}{1 + e^{-(8.66 - 10)}}\right)\left[1 - \left(1 - 0.50 - \frac{(0.60 - 0.50)}{1 + e^{-(5.37 - 5)}}\right)\left(1 - 0.40 - \frac{(0.80 - 0.40)}{1 + e^{-(5.97 - 7)}}\right)\right]$$

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But we want to find the best plan, so let's return to the optimization formulation

Objective Function

Decision variables

x_{FL} : $millions to spend in Florida
x_{OH} : $millions to spend in Ohio
x_{PA} : $millions to spend in Pennsylvania

Objective function

Maximize

$$\left(0.45 + \frac{(0.70 - 0.45)}{1 + e^{-(x_{FL} - 10)}}\right)\left[1 - \left(1 - 0.50 - \frac{(0.60 - 0.50)}{1 + e^{-(x_{OH} - 5)}}\right)\left(1 - 0.40 - \frac{(0.80 - 0.40)}{1 + e^{-(x_{PA} - 7)}}\right)\right]$$

Constraints

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Decision variables

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Objective function

Maximize

$$\left(0.45+\frac{(0.70-0.45)}{1+e^{-(x_{FL}-10)}}\right)\left[1-\left(1-0.50-\frac{(0.60-0.50)}{1+\mathrm{e}^{-(x_{OH}-5)}}\right)\left(1-0.40-\frac{(0.80-0.40)}{1+\mathrm{e}^{-(x_{PA}-7)}}\right)\right]$$

Constraints

What constraints do we need?

The complete formulation

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Maximize

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Constraints

subject to $x_{FL} + x_{OH} + x_{PA} \leq 20$

$x_{FL}, x_{OH}, x_{PA} \geq 0$

This is a nonlinear optimization model

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Objective function

Non-linear objective function

Maximize

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Constraints

subject to $x_{FL} + x_{OH} + x_{PA} \leq 20$

$x_{FL}, x_{OH}, x_{PA} \geq 0$

Linear constraints

Solving the NLO Formulation

Switch to Excel

Recommendations

Plan	Expenditure [$ million]			Predicted Outcome
	FL	OH	PA	
No expenditure	0	0	0	31.53%
Current Plan	8.66	5.37	5.97	39.23%
Solver Solution (starting from no expenditure)	0	7.91	12.09	41.31%

Recommendations

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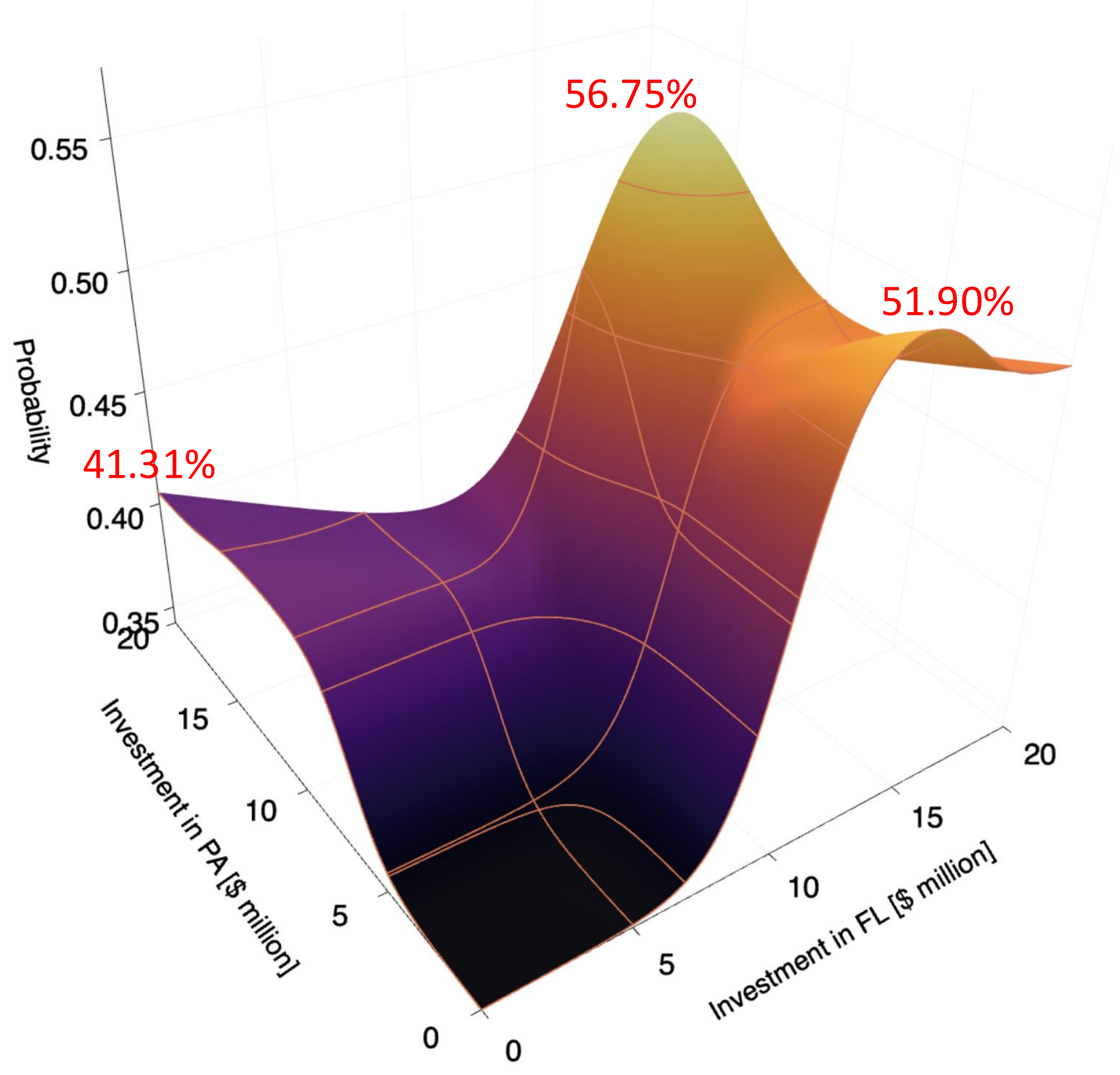
Solver Solution (starting from “Current Plan”)	11.87	0	8.13	56.75%

How the different solutions look visually



What is the best solution?

Recommendations

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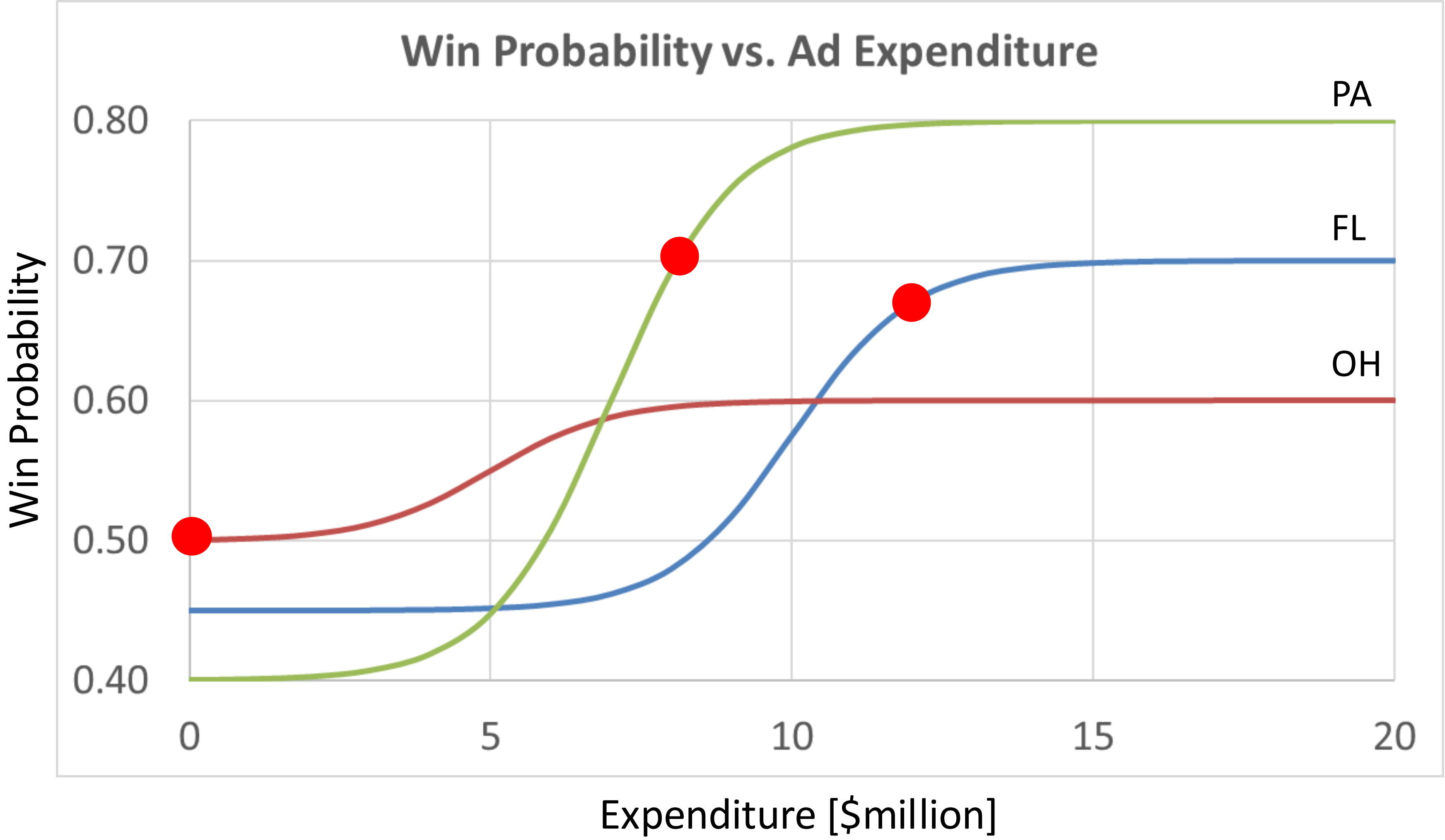
This is in fact the best solution!!

Optimal Expenditures for FL, OH and PA



Does the solution make intuitive sense?

- Expenditure in OH is zero since you can only increase the win probability slightly.

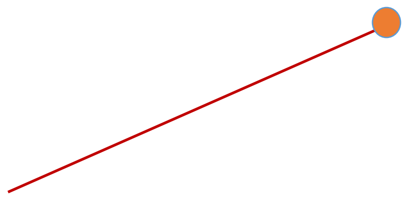
- For PA, we are still on the increasing part of the curve, whereas for FL we are almost at the part where the probability flattens. But it makes sense to go further to the right than PA, since FL is the state that we MUST win.

Nonlinear Optimization is much harder to solve than Linear Optimization due to Local vs Global optima

In linear Optimization, we don't have local optima

Objective Function

(linear e.g., maximize 3x + 5)



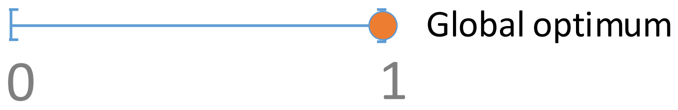
Constraints

(linear)

$x \geq 0$

$x \leq 1$

Feasible Region

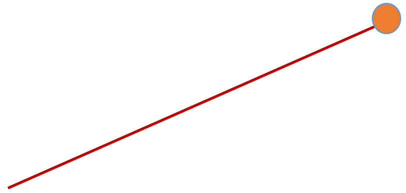


Linear Optimization

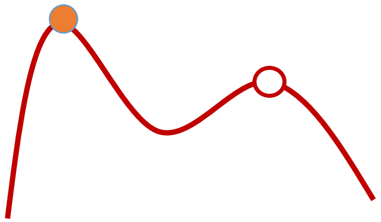
But in Nonlinear Optimization, we do

Objective Function

(linear e.g., maximize 3x + 5)



(nonlinear)



Constraints

(linear)

$x \geq 0$

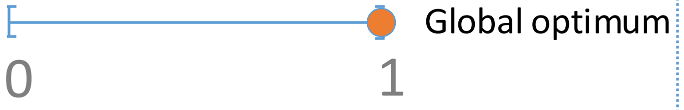
$x \leq 1$

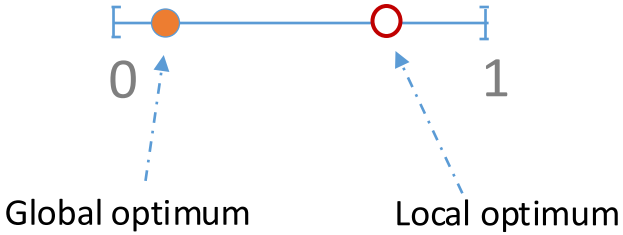
(linear)

$x \geq 0$

$x \leq 1$

Feasible Region

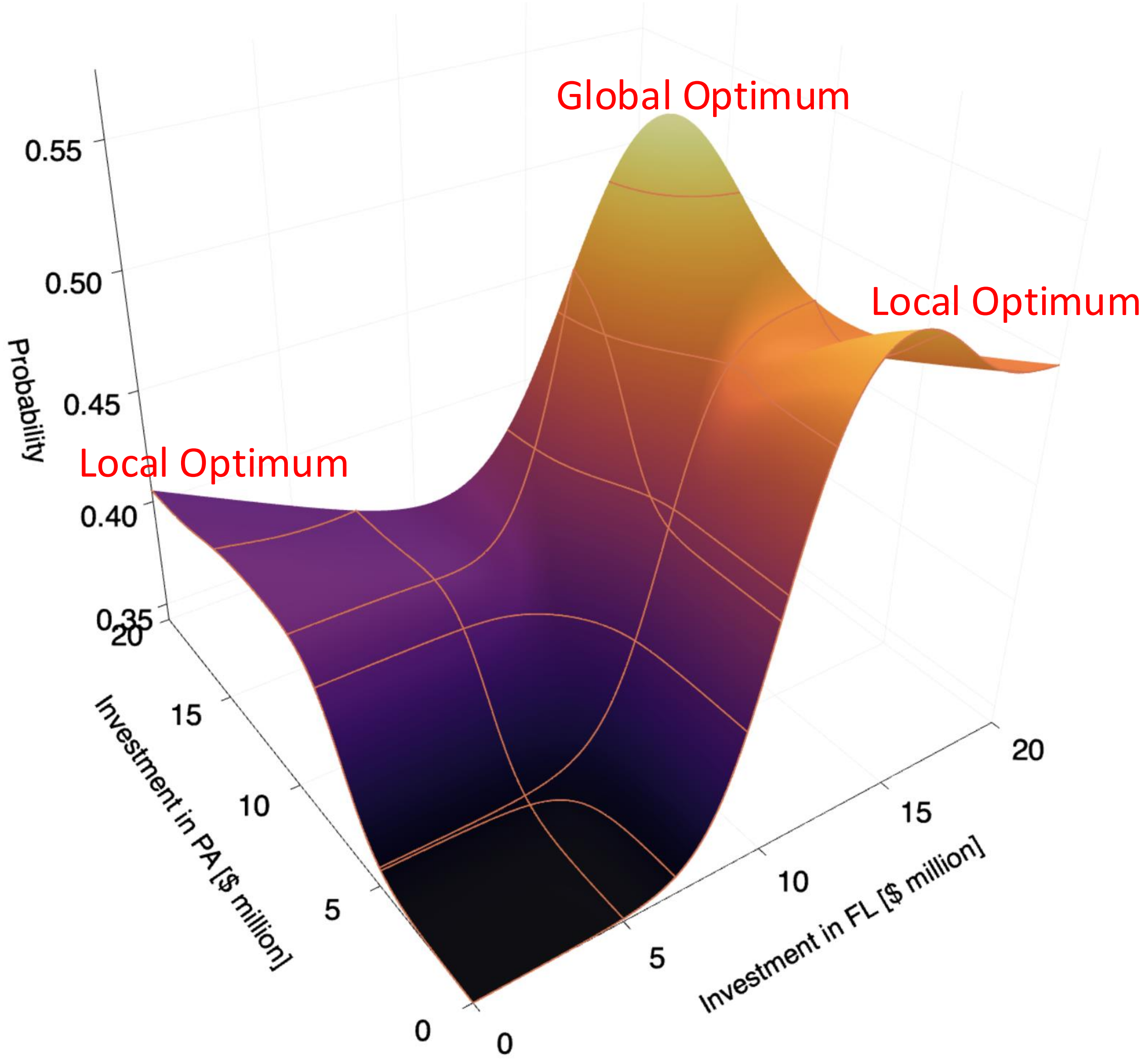




Linear Optimization

Nonlinear Optimization

Local vs Global Optima



- Non-linear functions may contain *local optima*.
- A local optimum is the “top of a hill”, a point that is the best among all the neighboring points.
- A global optimum is the “top of the highest hill”

Roughly speaking, solvers find a local optimum that is the “nearest” to the starting point.

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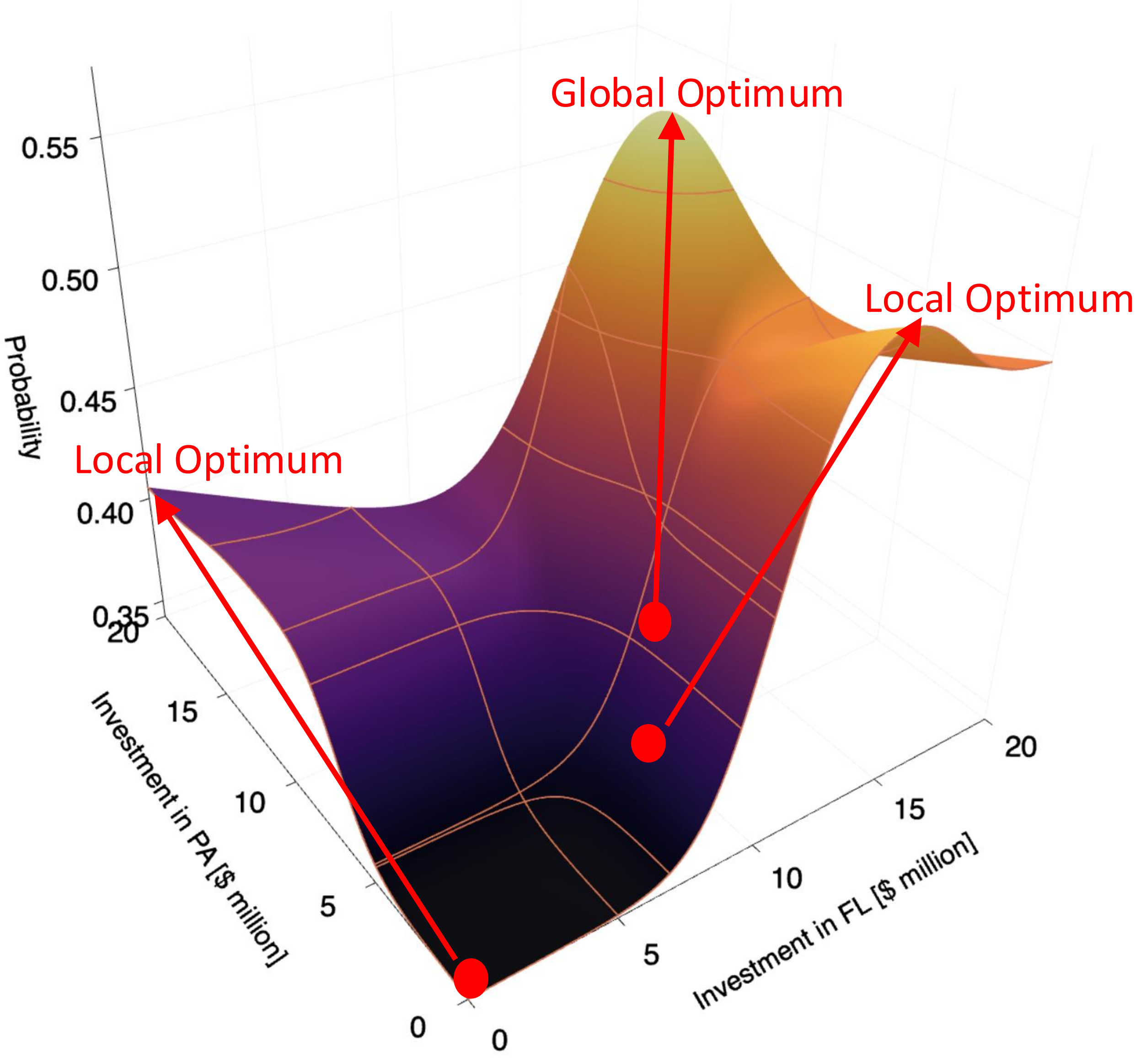


- Solvers work by starting at a point, then going "uphill"
- Solvers stop when they reach a local optimum
- In practice, we try many starting points and select the "best of the local optima"

Sensitivity Report in Nonlinear Optimization

Variable Cells

Cell	Name	Final Value	Reduced Gradient
C14	Advertising money invested in state FL	11.87172055	0
D14	Advertising money invested in state OH	0	-0.02446252
E14	Advertising money invested in state PA	8.12827984	0

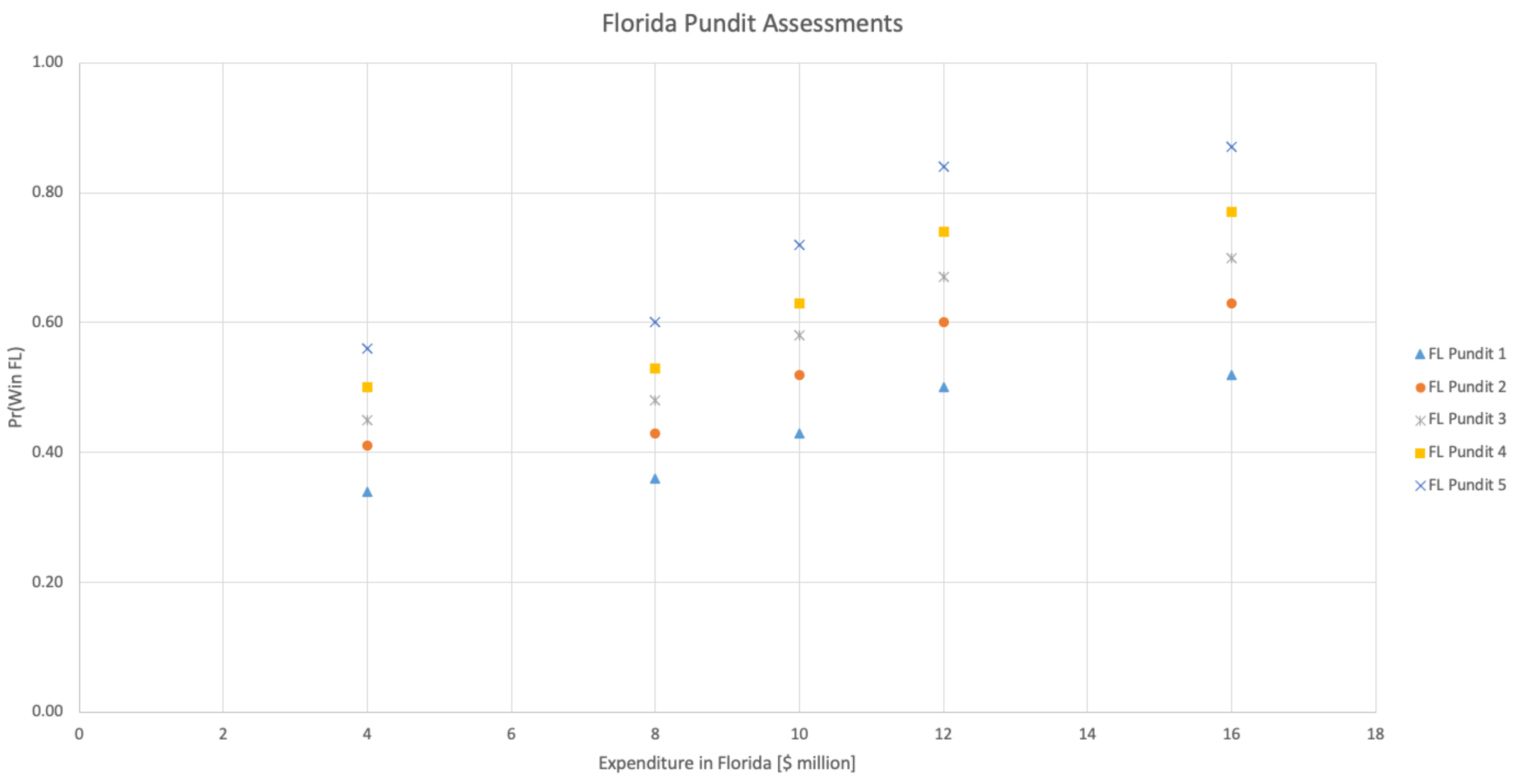
Constraints

Cell	Name	Final Value	Lagrange Multiplier
C21	Sum of expenditure [$millions] LHS	20.00000039	0.024594501

The Lagrange Multiplier indicates that a $1 million increase in total budget would yield a 2.46% increase in winning probability. Lagrange multipliers are similar to shadow prices <u>but are valid only at a point (as opposed to a range)</u>.

Summary

1) We started from **Data** (in this case, pundit assessments).



Summary

1) We started from **Data** (in this case, pundit assessments).
2) We used this data to estimate a **Predictive Model** (NLO model #1) to get S-shaped response functions for each state that allow us predict the probability of winning a state in terms of the expenditure in that state.

Pr(Win FL) Pr(Win OH) Pr(Win PA)

$$\left(0.45 + \frac{(0.70 - 0.45)}{1 + e^{-(x_{FL}-10)}}\right)\left(0.50 + \frac{(0.60 - 0.50)}{1 + e^{-(x_{OH}-5)}}\right)\left(0.40 + \frac{(0.80 - 0.40)}{1 + e^{-(x_{PA}-7)}}\right)$$

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3) We combined these regressions models to predict the overall probability of winning the election as a function of the expenditures in the three swing states

$$Pr(Win\ Election) =$$

$$\left(0.45 + \frac{(0.70 - 0.45)}{1 + e^{-(x_{FL} - 10)}}\right)\left[1 - \left(1 - 0.50 - \frac{(0.60 - 0.50)}{1 + \mathrm{e}^{-(x_{OH} - 5)}}\right)\left(1 - 0.40 - \frac{(0.80 - 0.40)}{1 + \mathrm{e}^{-(x_{PA} - 7)}}\right)\right]$$

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4) We formulated a **Nonlinear Optimization Model** (NLO model #2) to optimize the expenditure allocation across the three states with the objective of maximizing the probability of winning the election.

Decision variables

x_{FL} : $millions to spend in Florida
x_{OH} : $millions to spend in Ohio
x_{PA} : $millions to spend in Pennsylvania

Objective function

Non-linear objective function

Maximize $Pr(Win\ Election) =$

$$\left(0.45+\frac{(0.70-0.45)}{1+e^{-(x_{FL}-10)}}\right)\left[1-\left(1-0.50-\frac{(0.60-0.50)}{1+e^{-(x_{OH}-5)}}\right)\left(1-0.40-\frac{(0.80-0.40)}{1+e^{-(x_{PA}-7)}}\right)\right]$$

Constraints

subject to $x_{FL}+x_{OH}+x_{PA} \le 20$ Linear constraints
$x_{FL}, x_{OH}, x_{PA} \ge 0$

Summary

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4) We formulated a **Nonlinear Optimization Model** (NLO model #2) to optimize the expenditure allocation across the three states with the objective of maximizing the probability of winning the election.
5) We solved the optimization model to generate optimal **Decisions**:
   Allocate $11.87m to FL and $8.13 to PA

In-Class Exercise (time permitting)

Optimizing Retail Shelf Space Allocation

See Canvas for problem and solution

Remember Two Years From Now …

- Non-Linear Optimization is very versatile and can capture the intricacies of real-world problems

- However,
  - Unlike Linear Optimization, the solution you get may not be the global optimum
  - Your starting point may have a big impact on where you end up. So try multiple runs of the solver from different starting points
  - Consider using solvers that are more advanced than Excel
  - Consult with an optimization expert to see if your NLO problem can "tricked" into becoming a Linear or Discrete Optimization problem
  - Since shadow price information is very limited, sensitivity analysis is best done by re-running the model with modified parameters

What's Next

- *Wednesday*: No class!

- *Friday*: No recitation!

- No deliverable due this week!

- 1-on-1 Meetings
  - *Please book via https://calendly.com/ramamit*
  - *I have added more Calendly slots. If the Calendly times don't work, please email my assistant Laura (brentrup@mit.edu) to find a time.*

Happy Thanksgiving!!

Another exercise! (if time permits)

- Imagine that now you have 5 states that you are deciding how to allocate your \$20m expenditure across. However, you'd like to only spend in 3 out of these 5. You want Solver to help you find which 3 to invest in.

- How would you modify the following formulation? What new (binary) variables would you add?

Maximize $f(x_1, x_2, x_3, x_4, x_5)$

subject to:

$x_1 + x_2 + x_3 + x_4 + x_5 \leq 20$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

f is just some non-linear function of $x_1, \ldots, x_5$

Exercise Solution

- Imagine that now you have 5 states that you are deciding how to allocate your \$20m expenditure across. However, you'd like to only spend in 3 out of these 5. You want Solver to help you find which 3 to invest in.

- How would you modify the following formulation? What new (binary) variables would you add? y_i = binary variable for whether we choose to invest *anything* in the i-th state.

Maximize $f(x_1, x_2, x_3, x_4, x_5)$

subject to:

$x_1 + x_2 + x_3 + x_4 + x_5 \leq 20$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

$y_1 + y_2 + y_3 + y_4 + y_5 = 3$

y_1, y_2, y_3, y_4, y_5 are binary

$$\left.\begin{aligned} x_1 &\leq 20\, y_1 \\ x_2 &\leq 20\, y_2 \\ x_3 &\leq 20\, y_3 \\ x_4 &\leq 20\, y_4 \\ x_5 &\leq 20\, y_5 \end{aligned}\right\}$$

Linking constraints.

If Solver chooses $y_i = 0$, then x_i must be 0.
If Solver chooses $y_i = 1$, then x_i can be up to 20m.

Appendix (*OPTIONAL*)

Optimization in Practice: Salesforce Optimization

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Similar methodology to the Campaign Expenditure example

- NLO models have been used to decide how large sales force (detailing resources) should be, and how it should be deployed (across different drugs/regions/physician specialties).

Optimization in Practice: Salesforce Optimization

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Optimization in Practice: Salesforce Optimization

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- "The response functions were estimated by a team of knowledgeable managers and salespeople using a modified Delphi technique".

Optimization in Practice: Salesforce Optimization

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- The predictive models consisted of S-shaped response curves relating incremental sales to incremental changes in sales force size.
- "The response functions were estimated by a team of knowledgeable managers and salespeople using a modified Delphi technique".

Impact

"… 8% annual sales increase. The model had important impacts on the strategic direction of the firm, helping change its focus to product markets with better future potential."

Reference: Lodish et al. (1988) "Sales Force Sizing and Deployment Using a Decision Calculus Model at Syntex Laboratories", *Interfaces*.

Appendix (*OPTIONAL*)

Portfolio Optimization is a Nonlinear Optimization Problem

Recall the Portfolio Diversification Problem

	AMZN	HD	WMT
Expected return	3.23%	2.04%	1.08%
Risk (std. dev. of returns)	8.40%	5.26%	5.38%

correlation matrix

	AMZN	HD	WMT
AMZN			
HD	0.42		
WMT	0.11	0.19	

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correlation matrix

	AMZN	HD	WMT
AMZN			
HD	0.42		
WMT	0.11	0.19	

If we invest fractions ***a***, ***b***, and ***c*** in AMZN, HD, WMT, respectively, then:

Portfolio Expected Return [in %] $= 3.23\,\boldsymbol{a} + 2.04\,\boldsymbol{b} + 1.08\,\boldsymbol{c}$

Portfolio Risk [in %]

$$= \sqrt{\begin{array}{c} \boldsymbol{a}^2\,(8.40)^2 + \boldsymbol{b}^2\,(5.26)^2 + \boldsymbol{c}^2\,(5.38)^2 \\ +2\boldsymbol{ab}\,(8.40)(5.26)(0.42) + 2\boldsymbol{ac}(8.40)(5.38)(0.11) + 2\boldsymbol{bc}(5.26)(5.38)(0.19) \end{array}}$$

Recall the Portfolio Diversification Problem (Lecture 2)

	AMZN	HD	WMT
Expected return	3.23%	2.04%	1.08%
Risk (std. dev. of returns)	8.40%	5.26%	5.38%

correlation matrix

	AMZN	HD	WMT
AMZN			
HD	0.42		
WMT	0.11	0.19	

If we invest fractions ***a***, ***b***, and ***c*** in AMZN, HD, WMT, respectively, then:

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Nonlinear!

Portfolio Risk [in %]

$$= \sqrt{\begin{array}{c} \boldsymbol{a}^2\,(8.40)^2 + \boldsymbol{b}^2\,(5.26)^2 + \boldsymbol{c}^2\,(5.38)^2 \\ +2\boldsymbol{ab}\,(8.40)(5.26)(0.42) + 2\boldsymbol{ac}(8.40)(5.38)(0.11) + 2\boldsymbol{bc}(5.26)(5.38)(0.19) \end{array}}$$

Formulation As a Nonlinear Optimization Problem

	Model 1	Model 2
Decision variables	**a**, **b**, **c**	**a**, **b**, **c**
Objective function	maximize Portfolio Expected Return	minimize Portfolio Risk
Constraints	**a** + **b** + **c** = 1	**a** + **b** + **c** = 1
	Portfolio Risk ≤ user-specified threshold	Portfolio Expected Return ≥ user-specified threshold
	a, **b**, **c** ≥ 0	**a**, **b**, **c** ≥ 0

Nonlinear!