Does the solution make intuitive sense?

- Expenditure in OH is zero since you can only increase the win probability slightly.

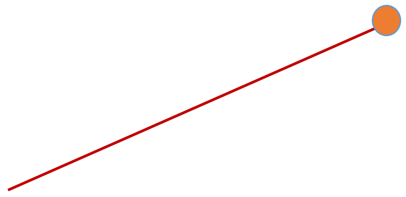
- For PA, we are still on the increasing part of the curve, whereas for FL we are almost at the part where the probability flattens. But it makes sense to go further to the right than PA, since FL is the state that we MUST win.

Nonlinear Optimization is much harder to solve than Linear Optimization due to Local vs Global optima

In linear Optimization, we don't have local optima

Objective Function

(linear e.g., maximize 3x + 5)



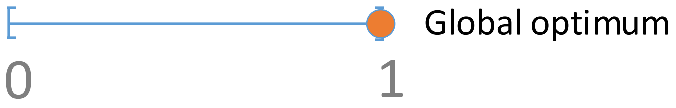
Constraints

(linear)

$x \geq 0$

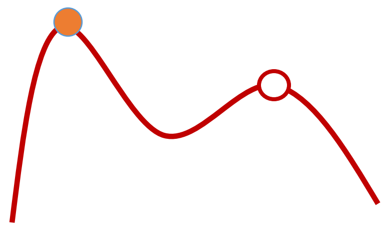
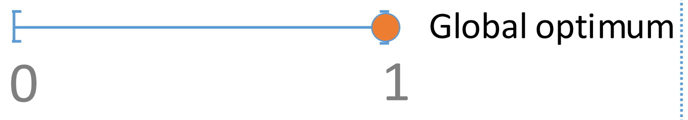
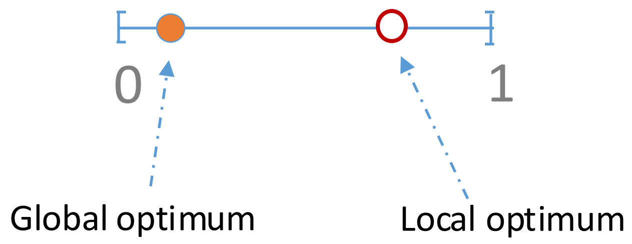
$x \leq 1$

Feasible Region



Linear Optimization

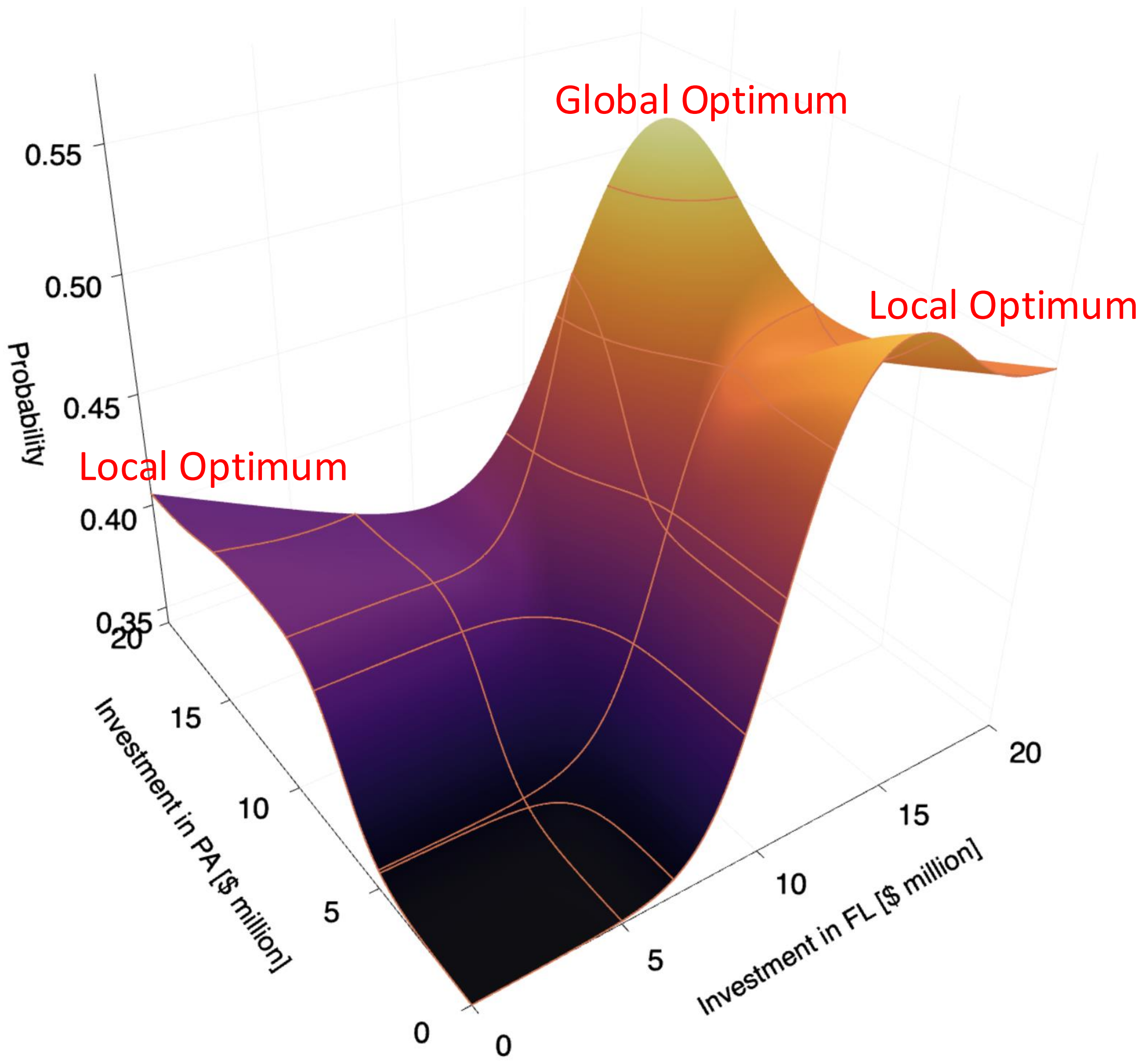
But in Nonlinear Optimization, we do

	Linear Optimization	Nonlinear Optimization
Objective Function	(linear e.g., maximize 3x + 5) 	(nonlinear) 
Constraints	(linear) $x \geq 0$ $x \leq 1$	(linear) $x \geq 0$ $x \leq 1$
Feasible Region		

Linear Optimization

Nonlinear Optimization

Local vs Global Optima



- Non-linear functions may contain *local optima*.
- A local optimum is the "top of a hill", a point that is the best among all the neighboring points.
- A global optimum is the "top of the highest hill"

Roughly speaking, solvers find a local optimum that is the "nearest" to the starting point.

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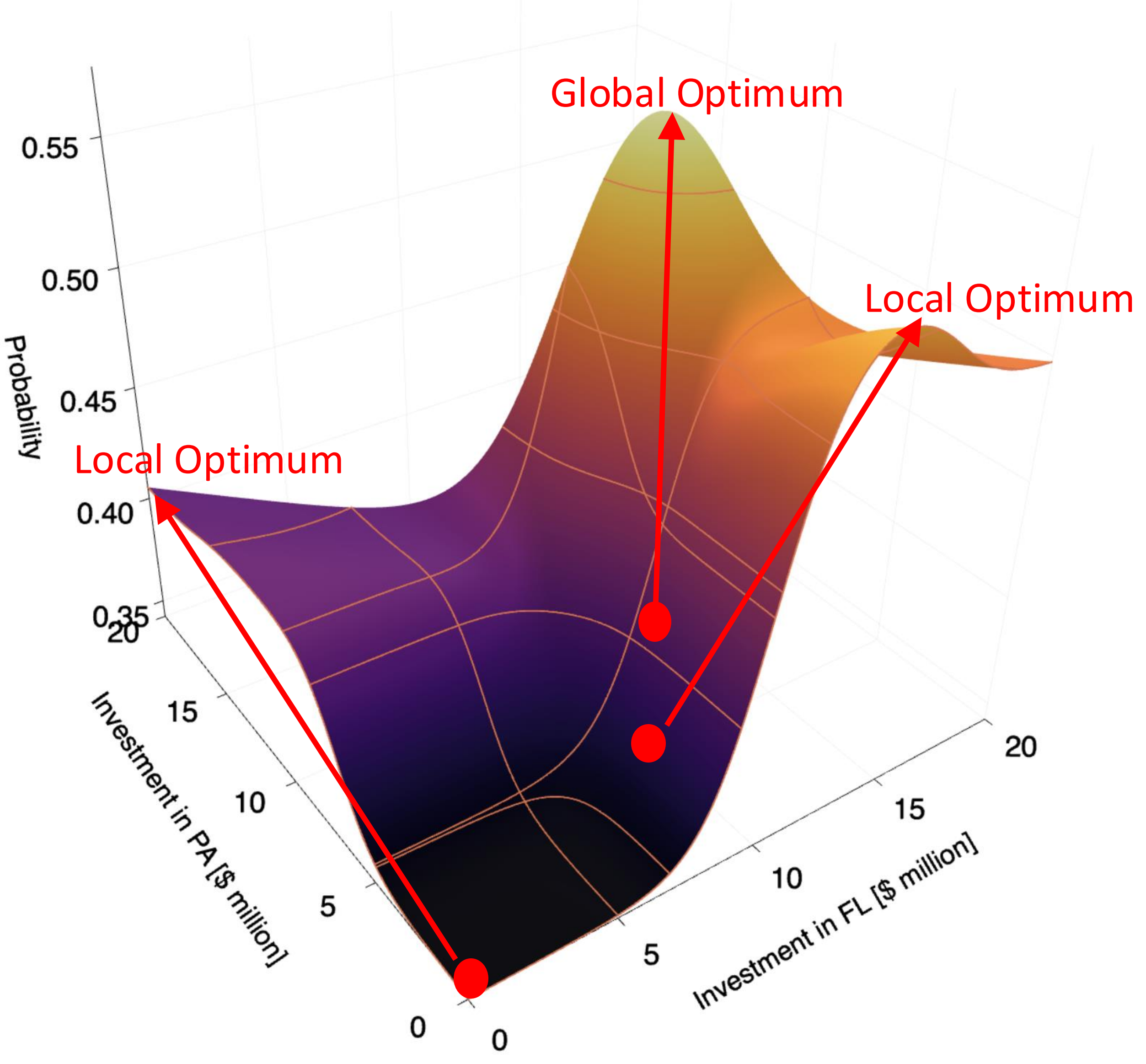


- Solvers work by starting at a point, then going "uphill"
- Solvers stop when they reach a local optimum
- In practice, we try many starting points and select the "best of the local optima"

Sensitivity Report in Nonlinear Optimization

Variable Cells

Cell	Name	Final Value	Reduced Gradient
C14	Advertising money invested in state FL	11.87172055	0
D14	Advertising money invested in state OH	0	-0.02446252
E14	Advertising money invested in state PA	8.12827984	0

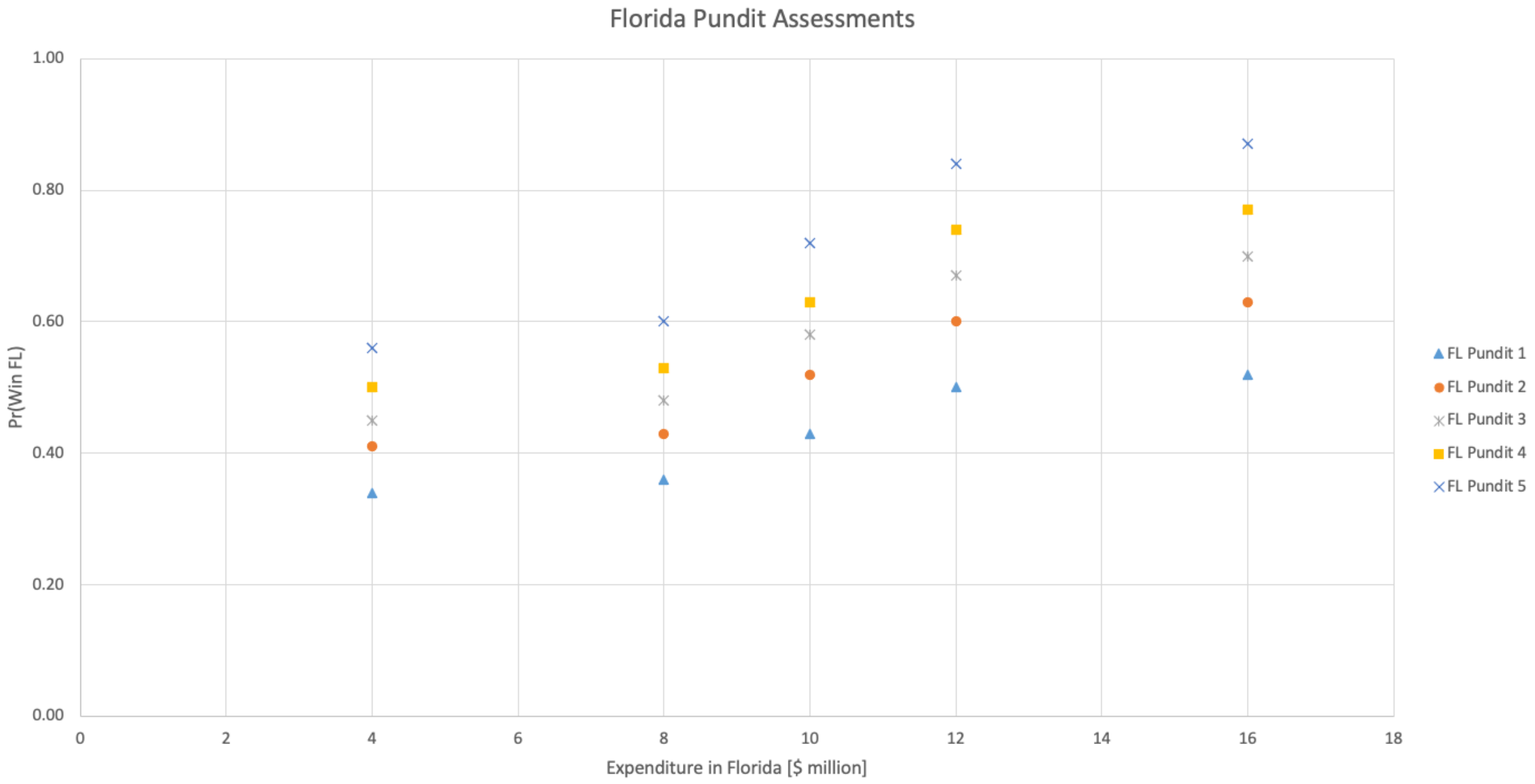
Constraints

Cell	Name	Final Value	Lagrange Multiplier
C21	Sum of expenditure [$millions] LHS	20.00000039	0.024594501

The Lagrange Multiplier indicates that a $1 million increase in total budget would yield a 2.46% increase in winning probability. Lagrange multipliers are similar to shadow prices <u>but are valid only at a point (as opposed to a range)</u>.

Summary

1) We started from **Data** (in this case, pundit assessments).



Summary

1) We started from **Data** (in this case, pundit assessments).
2) We used this data to estimate a **Predictive Model** (NLO model #1) to get S-shaped response functions for each state that allow us predict the probability of winning a state in terms of the expenditure in that state.

Pr(Win FL) Pr(Win OH) Pr(Win PA)

$$\left(0.45+\frac{(0.70-0.45)}{1+e^{-(x_{FL}-10)}}\right)\left(0.50+\frac{(0.60-0.50)}{1+e^{-(x_{OH}-5)}}\right)\left(0.40+\frac{(0.80-0.40)}{1+e^{-(x_{PA}-7)}}\right)$$

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3) We combined these regressions models to predict the overall probability of winning the election as a function of the expenditures in the three swing states

$$Pr(Win\ Election) =$$

$$\left(0.45 + \frac{(0.70 - 0.45)}{1 + e^{-(x_{FL} - 10)}}\right)\left[1 - \left(1 - 0.50 - \frac{(0.60 - 0.50)}{1 + \mathrm{e}^{-(x_{OH} - 5)}}\right)\left(1 - 0.40 - \frac{(0.80 - 0.40)}{1 + \mathrm{e}^{-(x_{PA} - 7)}}\right)\right]$$

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4) We formulated a **Nonlinear Optimization Model** (NLO model #2) to optimize the expenditure allocation across the three states with the objective of maximizing the probability of winning the election.

Decision variables

x_{FL} : $millions to spend in Florida
x_{OH} : $millions to spend in Ohio
x_{PA} : $millions to spend in Pennsylvania

Objective function

Non-linear objective function

Maximize $Pr(Win\ Election) =$

$$\left(0.45 + \frac{(0.70 - 0.45)}{1 + e^{-(x_{FL}-10)}}\right)\left[1 - \left(1 - 0.50 - \frac{(0.60 - 0.50)}{1 + e^{-(x_{OH}-5)}}\right)\left(1 - 0.40 - \frac{(0.80 - 0.40)}{1 + e^{-(x_{PA}-7)}}\right)\right]$$

Constraints

subject to $x_{FL} + x_{OH} + x_{PA} \leq 20$
$x_{FL}, x_{OH}, x_{PA} \geq 0$

Linear constraints

Summary

1) We started from **Data** (in this case, pundit assessments).
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3) We combined these regressions models to predict the overall probability of winning the election as a function of the expenditures in the three swing states
4) We formulated a **Nonlinear Optimization Model** (NLO model #2) to optimize the expenditure allocation across the three states with the objective of maximizing the probability of winning the election.
5) We solved the optimization model to generate optimal **Decisions**:
   Allocate $11.87m to FL and $8.13 to PA

In-Class Exercise (time permitting)

Optimizing Retail Shelf Space Allocation

See Canvas for problem and solution

Remember Two Years From Now …

- Non-Linear Optimization is very versatile and can capture the intricacies of real-world problems

- <u>However</u>,
    - Unlike Linear Optimization, the solution you get may not be the global optimum
    - Your starting point may have a big impact on where you end up. So try multiple runs of the solver from different starting points
    - Consider using solvers that are more advanced than Excel
    - Consult with an optimization expert to see if your NLO problem can "tricked" into becoming a Linear or Discrete Optimization problem
    - Since shadow price information is very limited, sensitivity analysis is best done by re-running the model with modified parameters

What's Next

- *Wednesday*: No class!

- *Friday*: No recitation!

- No deliverable due this week!

- 1-on-1 Meetings
  - *Please book via https://calendly.com/ramamit*
  - *I have added more Calendly slots. If the Calendly times don't work, please email my assistant Laura (brentrup@mit.edu) to find a time.*

Happy Thanksgiving!!

Another exercise! (if time permits)

- Imagine that now you have 5 states that you are deciding how to allocate your $20m expenditure across. However, you'd like to only spend in 3 out of these 5. You want Solver to help you find which 3 to invest in.

- How would you modify the following formulation? What new (binary) variables would you add?

Maximize $f(x_1, x_2, x_3, x_4, x_5)$

subject to:

$x_1 + x_2 + x_3 + x_4 + x_5 \leq 20$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

f is just some non-linear function of $x_1, \ldots, x_5$

Exercise Solution

- Imagine that now you have 5 states that you are deciding how to allocate your \$20m expenditure across. However, you'd like to only spend in 3 out of these 5. You want Solver to help you find which 3 to invest in.

- How would you modify the following formulation? What new (binary) variables would you add? y_i = binary variable for whether we choose to invest *anything* in the i-th state.

Maximize $f(x_1, x_2, x_3, x_4, x_5)$

subject to:

$x_1 + x_2 + x_3 + x_4 + x_5 \leq 20$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

$y_1 + y_2 + y_3 + y_4 + y_5 = 3$

y_1, y_2, y_3, y_4, y_5 are binary

$$\left.\begin{aligned} x_1 &\leq 20\, y_1 \\ x_2 &\leq 20\, y_2 \\ x_3 &\leq 20\, y_3 \\ x_4 &\leq 20\, y_4 \\ x_5 &\leq 20\, y_5 \end{aligned}\right\}$$

Linking constraints.

If Solver chooses $y_i = 0$, then x_i must be 0.
If Solver chooses $y_i = 1$, then x_i can be up to 20m.

Appendix (*OPTIONAL*)

Optimization in Practice: Salesforce Optimization