How Does Excel Solve Integer Optimization?

- Unlike linear optimization, discrete optimization is typically quite challenging to solve.

- In general, to find the best solution, a solver may just need to enumerate over all possible solutions.
  - In our simple course-selection problem, there are $2^{10} = 1024$ possible course selections.
  - In real-world sized problems with millions of variables, there would be way too many solutions to enumerate.

- However, solvers are still routinely and efficient at finding **very good solutions, good enough for real-world use-cases.** *How do solvers do it?*

First, ignore the integrality constraints

Maximize:	$10\ y_A + 2\ y_B + 4\ y_C + 2\ y_D + 5\ y_E$ $+ 4\ y_F + 8\ y_G + 7\ y_H + 6\ y_I + 6\ y_J$
over variables:	$y_A, y_B, \ldots, y_J$
Subject To:	
(binary)	$y_A , y_B , \ldots , y_J \ \geq 0, \leq 1$ ~~and integral~~
(points budget)	$200\ y_A + 50\ y_B + \ldots + 100\ y_J \leq 1000$
(max credits)	$12\ y_A + 9\ y_B + \ldots + 6\ y_J \leq 54$
(min credits)	$12\ y_A + 9\ y_B + \ldots + 6\ y_J \geq 36$
(MW H3 load)	$y_A + y_B + y_E + y_G \leq 3$
(MW H4 load)	$y_A + y_B + y_G + y_J \leq 3$
(TR H3 load)	$y_C + y_D + y_I \leq 3$
(TR H4 load)	$y_C + y_D + y_F + y_H + y_I \leq 3$
(A B conflict)	$y_A + y_B \leq 1$
(B or C required)	$y_B + y_C \geq 1$
(E pre-req to H)	$y_H \leq y_E$

Solve in Excel and check if the solution is integral

Maximize: $10\, y_A + 2\, y_B + 4\, y_C + 2\, y_D + 5\, y_E + 4\, y_F + 8\, y_G + 7\, y_H + 6\, y_I + 6\, y_J$

over variables: $y_A, y_B, \ldots, y_J$

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(A B conflict)	$y_A + y_B \leq 1$
(B or C required)	$y_B + y_C \geq 1$
(E pre-req to H)	$y_H \leq y_E$

Solve in Excel / Solver

Optimal Solution

$y_A = 1.0$
$y_B = 0.0$
$y_C = 1.0$
$y_D = 0.0$
$y_E = 1.0$
$y_F = 0.0$
$y_G = 1.0$
$y_H = 1.0$
$y_I = 0.0$
$y_J = 1.0$

If the solution has all integers, then we are done! Why?

Take one of the fractional variables and …

Maximize: $10\ y_A + 2\ y_B + 4\ y_C + 2\ y_D + 5\ y_E + 4\ y_F + 8\ y_G + 7\ y_H + 6\ y_I + 6\ y_J$

over variables: $y_A, y_B, \ldots, y_J$

Subject To:

(binary)	$y_A, y_B, \ldots, y_J \geq 0, \leq 1$ ~~and integral~~
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(TR H4 load)	$y_C + y_D + y_F + y_H + y_I \leq 3$
(A B conflict)	$y_A + y_B \leq 1$
(B or C required)	$y_B + y_C \geq 1$
(E pre-req to H)	$y_H \leq y_E$

Solve in Excel / Solver

Optimal Solution

$y_A = 0.7$
$y_B = 0.0$
$y_C = 0.7$
$y_D = 0.0$
$y_E = 1.0$
$y_F = 0.3$
$y_G = 1.0$
$y_H = 1.0$
$y_I = 0.0$
$y_J = 1.0$

Branch into two optimization problems

Maximize: $10\ y_A + 2\ y_B + 4\ y_C + 2\ y_D + 5\ y_E + 4\ y_F + 8\ y_G + 7\ y_H + 6\ y_I + 6\ y_J$

over variables: $y_A, y_B, \ldots, y_J$

Subject To:

(binary) $y_A, y_B, \ldots, y_J \geq 0, \leq 1$ and <u>integral</u>

…

(E pre-req to H) $y_H \leq y_E$

Maximize: $10\ y_A + 2\ y_B + 4\ y_C + 2\ y_D + 5\ y_E + 4\ y_F + 8\ y_G + 7\ y_H + 6\ y_I + 6\ y_J$

over variables: $y_A, y_B, \ldots, y_J$

Subject To:

(binary) $y_A, y_B, \ldots, y_J \geq 0, \leq 1$ and <u>integral</u>

…

(E pre-req to H) $y_H \leq y_E$

(Fix $y_A = 1$) $y_A = 1$

Maximize: $10\ y_A + 2\ y_B + 4\ y_C + 2\ y_D + 5\ y_E + 4\ y_F + 8\ y_G + 7\ y_H + 6\ y_I + 6\ y_J$

over variables: $y_A, y_B, \ldots, y_J$

Subject To:

(binary) $y_A, y_B, \ldots, y_J \geq 0, \leq 1$ and <u>integral</u>

…

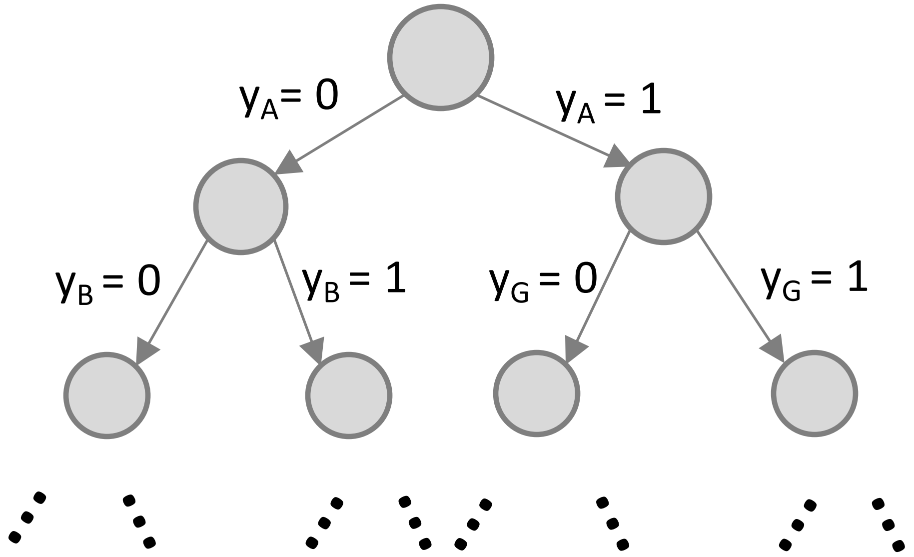
(E pre-req to H) $y_H \leq y_E$

(Fix $y_A = 0$) $y_A = 0$

We have turned 1 problem with 10 variables into 2 problems with 9 variables.

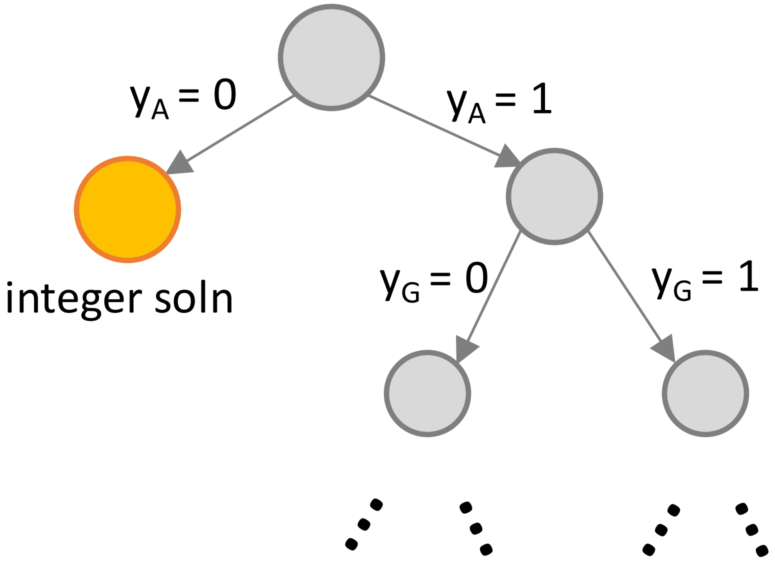
Take the *better* of the two optimal solutions.

Branching doubles the number of optimization problems we must solve.



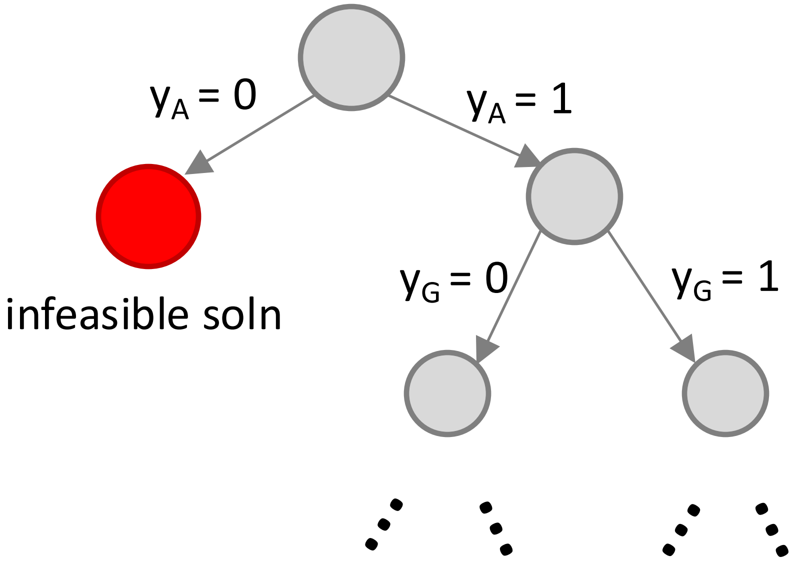
- We continue to branch, each time *doubling* the number of Linear Programs that we must solve. Yikes!

Ways that this enumeration is “smarter”



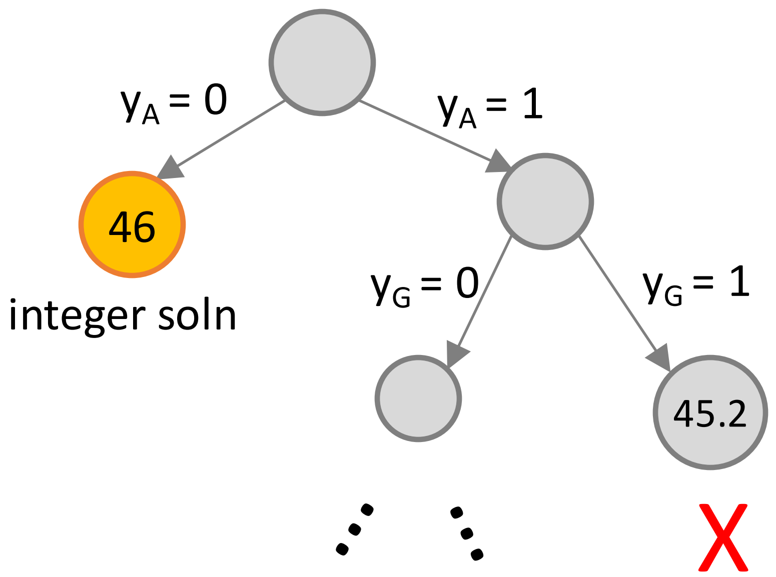
- Every time we hit an integer solution, we can stop branching.

Ways that this enumeration is “smarter”



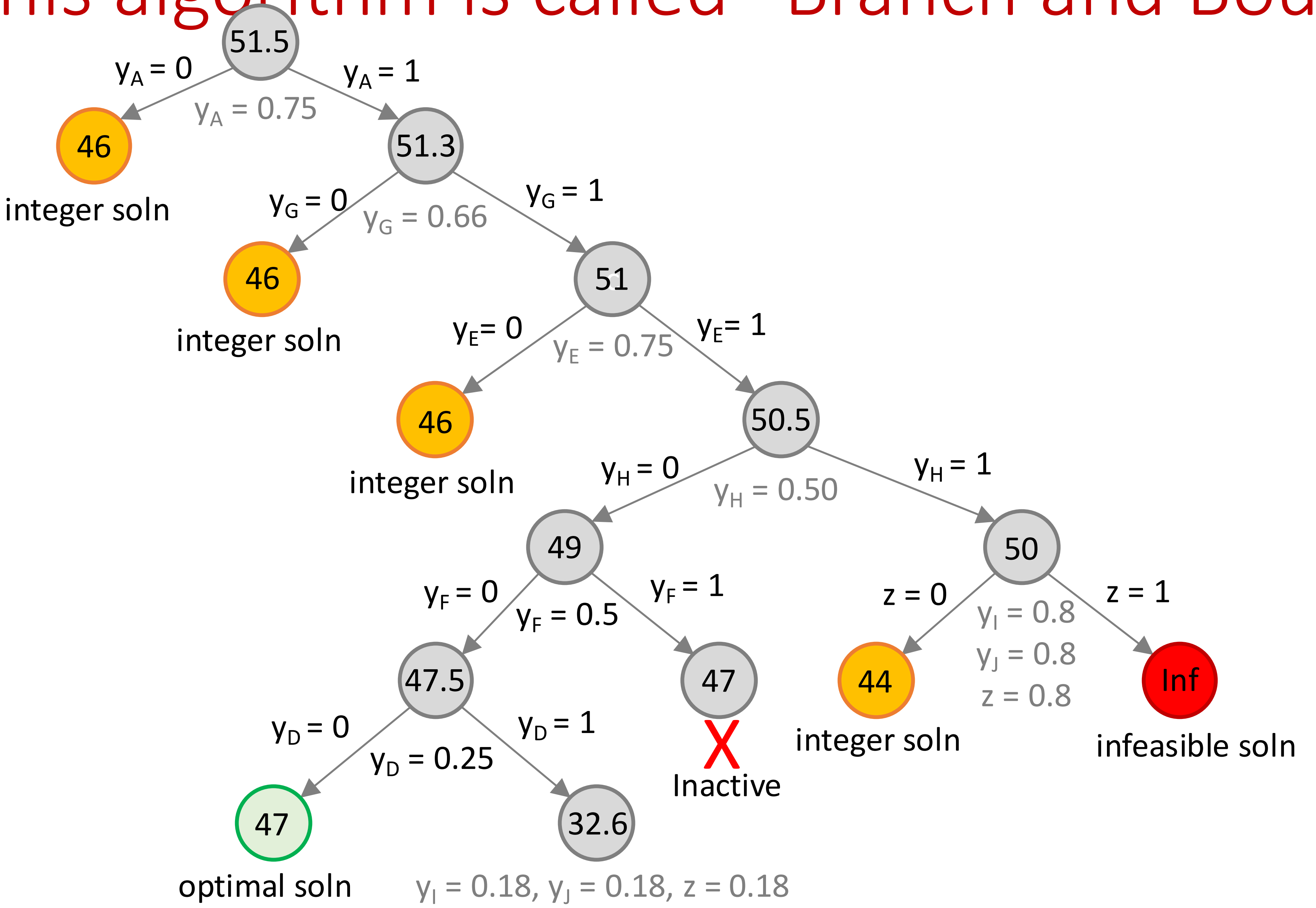
- Every time we hit an integer solution, we can stop branching.
- If the linear optimization relaxation is *infeasible*, then we can also stop branching.

Ways that this enumeration is “smarter”



- Every time we hit an integer solution, we can stop branching.
- If the linear optimization relaxation is *infeasible*, then we can also stop branching.
- If the linear optimization relaxation has an objective value worse than an integer solution, then we can also stop branching.

This algorithm is called “Branch and Bound”.

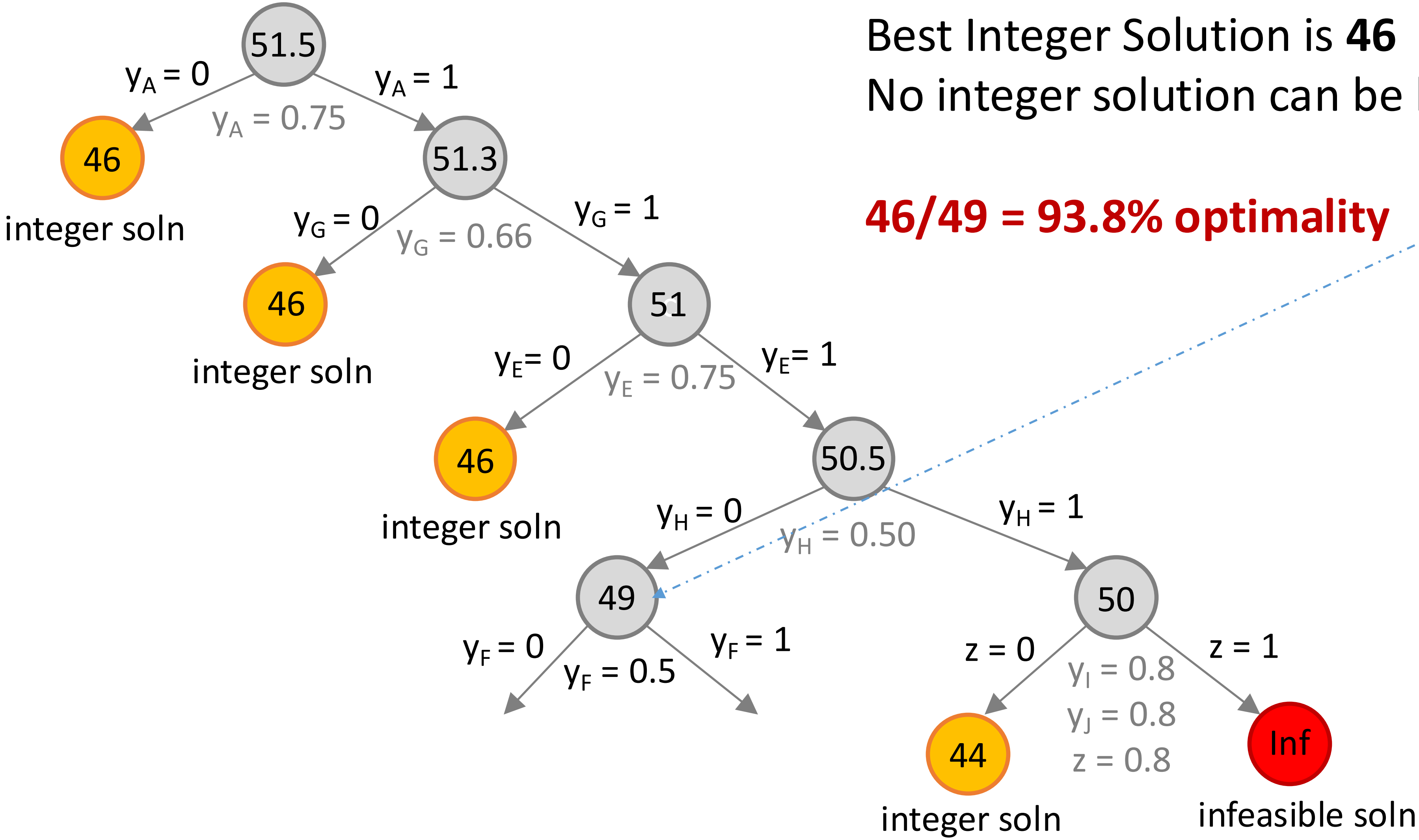


Branch and Bound

- Branch and bound can solve a binary optimization problems with N variables by **solving way fewer than 2^N** linear optimization problems.

- Practitioners typically set a time limit (e.g., 1 hour).
  - Branch-and-bound will find the **best solution** it can within that time limit.
  - Branch-and-bound will give tell you **how far you** are to the optimal solution.

Time's up!



Best Integer Solution is **46**

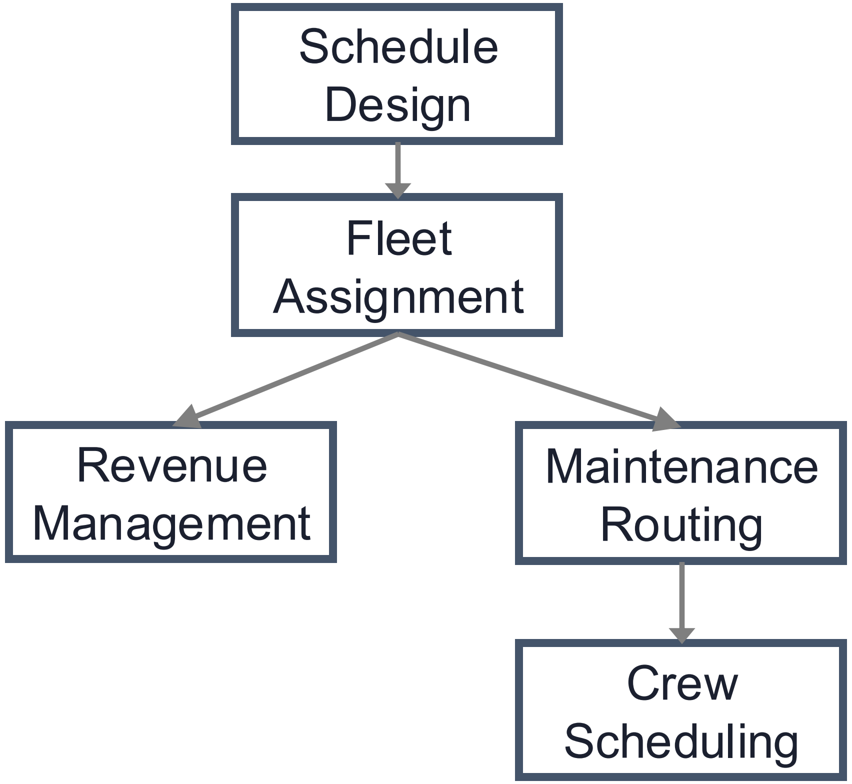
No integer solution can be better than **49**

46/49 = 93.8% optimality

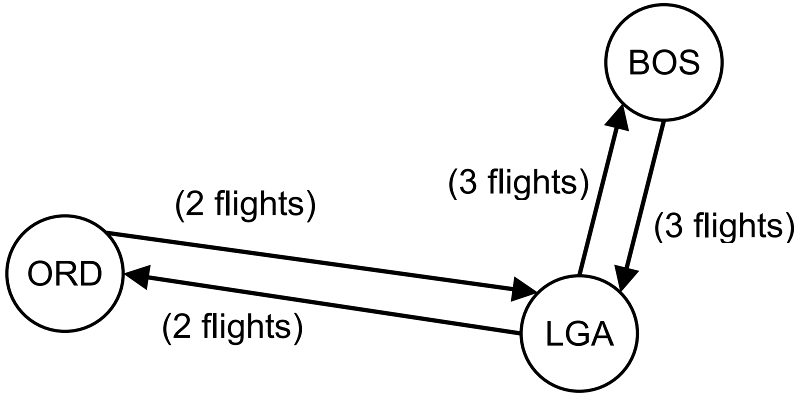
APPENDIX
<u>Optional</u>
Real-world application of 0-1 integer optimization: AIRLINE FLEET ASSIGNMENT

Applications of Optimization in Airline Operations

- Airlines use optimization models to plan many aspects of their operations such as assigning aircraft and crew to flights.
- Airlines also use optimization (especially linear programming) to manage their revenue.
- The scale of operations is huge. A major US airline would typically fly 1,500-2,500 domestic flights daily and offer over 100,000 different itinerary/fare combinations.



Airline Fleet Assignment – Simplified Example (1/2)



Flight #	From	To	Dept Time (EST)	Arr Time (EST)	Fare [$]	Demand [passengers]
1	LGA	BOS	1000	1100	150	250
2	LGA	BOS	1100	1200	150	250
3	LGA	BOS	1800	1900	150	100
4	BOS	LGA	0700	0800	150	150
5	BOS	LGA	1030	1130	150	300
6	BOS	LGA	1800	1900	150	150
7	LGA	ORD	1100	1400	400	150
8	LGA	ORD	1500	1800	400	200
9	ORD	LGA	0700	1000	400	200
10	ORD	LGA	0830	1130	400	150

Airline Fleet Assignment – Simplified Example (2/2)

Fleet type	Number of aircraft owned	Capacity [seats]	Per flight operating cost [$000]	
			LGA - BOS	LGA – ORD
A	1	120	10	15
B	2	150	12	17
C	2	250	15	20

The Airline Fleet Assignment Problem

<u>Given</u>:

- Flight schedule: (set of daily flight legs)
- Estimated passenger demand
- Aircraft fleet characteristics
- Revenue and operating cost data

<u>Find</u>:

- A feasible fleet assignment (an allocation of fleet types to flight legs) that maximizes:

  Profit contribution = Revenue – Operating Costs

Formulation – Decision Variables

Decision variables:

For each flight leg – fleet type combination, we need a binary variable that indicates whether that fleet type is assigned to that flight leg.

For example:

$x_{1,A}$ = 1 if fleet type A is assigned to flight leg 1

= 0 otherwise

Flight #	Fleet Type		
	A	B	C
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			

Formulation – Objective Function

- What is the (estimated) profit contribution of flight 1 when assigned fleet type A?
  $\approx 150 \cdot \min(250,120) - 10{,}000 = 8{,}000$

- What is the (estimated) profit contribution of flight 2 when assigned fleet type B?
  $\approx 150 \cdot \min(250,150) - 12{,}000 = 10{,}500$

<u>Objective Function</u>:

maximize $8{,}000\, x_{1,A} + 10{,}500\, x_{1,B} + 22{,}500\, x_{1,C}$
$+ \ldots$

Flight #	Fleet Type		
	A	B	C
1	8,000	10,500	22,500
2	8,000	10,500	22,500
3	5,000	3,000	0
4	8,000	10,500	7,500
5	8,000	10,500	22,500
6	33,000	10,500	7,500
7	33,000	43,000	40,000
8	33,000	43,000	60,000
9	33,000	43,000	60,000
10	33,000	43,000	10,000

Formulation Constraints – (1/3)

Constraints:

- Binary
  Each assignment decision variable must be either 0 or 1

  $x_{1,A}$, $x_{1,B}$, $x_{1,C}$, … , $x_{10,A}$, $x_{10,B}$, $x_{10,C}$ are binary variables

- *Make sure each flight is assigned an airplane.*

  For flight 1: $x_{1,A} + x_{1,B} + x_{1,C} = 1$ and so on for each flight