*Alchemist by Juame Plensa (location: MIT W20)*

# *Discrete Optimization*

MIT MANAGEMENT SLOAN SCHOOL

15.060: Data, Models, and Decisions

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Class 16 (Nov 17)

# A Classification of Optimization Problems

| | | Decision Variables | |
|---|---|---|---|
| | | All continuous | Some (or all) integer |
| Objective Function & Constraints | All linear | Linear Optimization | Integer Linear Optimization* |
| | Some (or all) non-linear | Nonlinear Optimization | Integer Nonlinear Optimization |

* also known as Discrete Optimization

# In some problem settings, fractional solutions are acceptable

In our linear optimization examples so far (School Bus, Optima Autos), when variables represent large quantities of something, **fractional solutions are ok**.

- 23.6 Type A buses => The optimal is between 23 and 24 buses.
- We can round to the nearest integer. Some constraints may be slightly broken (e.g., budget is 10.01 million), but usually not a big deal.

# We need Discrete Optimization when fractional solutions are not acceptable

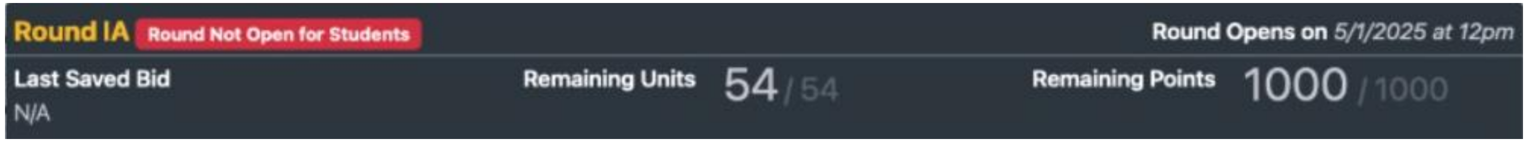
- Many business settings involve **binary decisions**: whether to do something or not (e.g., build a factory, enter a partnership, assign a shift).
- We model **binary decisions** with **binary decision variables,** since fractional solutions do not make sense e.g., *If 1 = do something and 0 = don't do something, what does 0.6 even mean?*
- Business problems also typically involve complex IF/THEN conditions and non-linearities, and optimization with binary variables is a powerful tool to handle these conditions
- In DMD, we will cover:
    - Supply chain design and operation (next class)
    - Retail pricing/promotions (the last class)

# Today's Class

- **Discrete optimization example:** Optimizing course selections at Sloan

- The power of binary variables to represent logical and non-linear relationships using linear constraints

- (If time permits) How discrete optimization problems are solved efficiently in practice




# Optimizing Course Selection

(Note: Not safe for use in the spring!)

# Course selection problem statement

- You are considering 10 courses named A, B, C, ..., J for Spring 2026. Each class gives you some **utility** towards your academic and professional goals.

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| Utility | 10 | 2 | 4 | 2 | 5 | 4 | 8 | 7 | 6 | 6 |

- You would like to choose which courses to take to **maximize total utility**.

- However, there are some constraints regarding:
  - Bidding
  - Credits
  - Schedule

# Course selection problem data

- **Required Bids**
  - You have 1000 points to allocate across the 10 courses.
  - We make a simplifying assumption that bidding = enrolling

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| Required Bid | 200 | 50 | 150 | 400 | 50 | 0 | 150 | 50 | 180 | 100 |

- **Credit Hours**
  - You need to take at least 36 and not more than 54 credit hours

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| Credit Hours | 12 | 9 | 9 | 12 | 6 | 6 | 9 | 6 | 9 | 6 |

# Course selection problem data

- **Schedule**
  - Some courses are whole-semester, others are only H3 or H4 half-semester courses.
  - You have a strong preference (ok, fine, let's make it a hard constraint 🙂) for taking no more than 3 courses on any given day

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| H3 | ✓ | ✓ | ✓ | ✓ | ✓ | | ✓ | | ✓ | |
| H4 | ✓ | ✓ | ✓ | ✓ | | ✓ | ✓ | ✓ | ✓ | ✓ |

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| Monday | ✓ | ✓ | | | ✓ | | ✓ | | | ✓ |
| Tuesday | | | ✓ | ✓ | | ✓ | | ✓ | ✓ | |
| Wednesday | ✓ | ✓ | | | ✓ | | ✓ | | | ✓ |
| Thursday | | | ✓ | ✓ | | ✓ | | ✓ | ✓ | |
| Friday | | | | | | | | | | |

# Course bidding problem data summary

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| Utility | 10 | 2 | 4 | 2 | 5 | 4 | 8 | 7 | 6 | 6 |
| Required Bid | 200 | 50 | 150 | 400 | 50 | 0 | 150 | 50 | 180 | 100 |
| Credit Hours | 12 | 9 | 9 | 12 | 6 | 6 | 9 | 6 | 9 | 6 |

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| H3 | ✓ | ✓ | ✓ | ✓ | ✓ | | ✓ | | ✓ | |
| H4 | ✓ | ✓ | ✓ | ✓ | | ✓ | ✓ | ✓ | ✓ | ✓ |
| Monday | ✓ | ✓ | | | ✓ | | ✓ | | | ✓ |
| Tuesday | | | ✓ | ✓ | | ✓ | | ✓ | ✓ | |
| Wednesday | ✓ | ✓ | | | ✓ | | ✓ | | | ✓ |
| Thursday | | | ✓ | ✓ | | ✓ | | ✓ | ✓ | |
| Friday | | | | | | | | | | |

**Which courses should you take to maximize utility while satisfying all the constraints?**

# Let’s write down the formulation

- Decision Variables
  - For each course (A, B, C, …, J), should you enroll or not enroll.
- Objectives
  - Maximize the utility of the courses that you do enroll in
- Constraints
  - Bids
  - Credits
  - Schedules

# Formulating the Course Selection Problem
## The Decision Variables

- Define *binary decision variables* for whether or not you bid/enroll in a course

$$y_A, y_B, \ldots, y_J$$

- For example:
  - $y_A = 1$ means you enroll in course A.
  - $y_A = 0$ means you do not enroll in course A.

- Note: In this simplified model, we assume that enrolling in a course is your core decision. You will simply bid the minimum number of points required to get a spot and assume you will get a spot for sure.

# Formulating the Course Selection Problem
## Objective Function

- **Variables:** whether or not you enroll in a course

$$y_A, y_B, \ldots, y_J \qquad = 0 \text{ or } 1$$

- **Objective:** Our goal is to maximize the utility of the courses we enroll in.

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| Utility | 10 | 2 | 4 | 2 | 5 | 4 | 8 | 7 | 6 | 6 |

$$\text{maximize} \qquad 10\, y_A + 2\, y_B + 4\, y_C + 2\, y_D + 5\, y_E + 4\, y_F + 8\, y_G + 7\, y_H + 6\, y_I + 6\, y_J$$

- Is the objective function linear?

# Formulating the Course Selection Problem
## Objective Function

- **Variables:** whether or not you bid/enroll in a course

$$y_A, y_B, \ldots, y_J \qquad = 0 \text{ or } 1$$

- **Objective:** Our goal is to maximize the utility of the courses we bid/enroll in.

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| Utility | 10 | 2 | 4 | 2 | 5 | 4 | 8 | 7 | 6 | 6 |

$$\text{maximize} \qquad 10\, y_A + 2\, y_B + 4\, y_C + 2\, y_D + 5\, y_E + 4\, y_F + 8\, y_G + 7\, y_H + 6\, y_I + 6\, y_J$$

- Is the objective function linear? Yes, it is a weighted sum of the binary variables.

Suppose you have a Solver that can optimize integer variables. How can you “force” it to solve for binary variables?

# Integer variables + Constraints = Binary variables

$y_A , y_B , \ldots , y_J$ integral

$y_A , y_B , \ldots , y_J \geq 0$ $\Longleftrightarrow$ $y_A , y_B , \ldots , y_J$ binary

$y_A , y_B , \ldots , y_J \leq 1$

# Formulating the Course Bidding Problem
## Constraints

| | | |
|---|---|---|
| Variables | (binary) | $y_A$ , $y_B$ , … , $y_J$ $\geq 0, \leq 1$ and <u>integral</u> |
| Bidding | (1000 points budget) | ??? |
| Credits | (54 max credits) | ??? |
| | (36 min credits) | ??? |
| Schedule | (MW H3 load) | ??? |
| | (MW H4 load) | ??? |
| | (TR H3 load) | ??? |
| | (TR H4 load) | ??? |

# In-class EXERCISE!

# Dos and Donts when writing constraints

- You can use $\leq$, $\geq$ and =

$$y_C + y_D + y_I \leq 3$$

$$y_C + y_D + y_I \geq 3$$

$$y_C + y_D + y_I = 3$$

- You cannot use <, > or $\neq$

$$y_C + y_D + y_I < 3$$

$$y_C + y_D + y_I > 3$$

$$y_C + y_D + y_I \neq 3$$

# Write down the constraints for the budget, credits and MW H3 load

Variables (binary) $y_A , y_B , \ldots , y_J \geq 0, \leq 1$ and <u>integral</u>

Bidding (1000 points budget)

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| Required Bid | 200 | 50 | 150 | 400 | 50 | 0 | 150 | 50 | 180 | 100 |

Credits (54 max credits)

(36 min credits)

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| Credit Hours | 12 | 9 | 9 | 12 | 6 | 6 | 9 | 6 | 9 | 6 |

Schedule (MW H3 load)

(MW H4 load)

(TR H3 load)

(TR H4 load)

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| H3 | ✓ | ✓ | ✓ | ✓ | ✓ | | ✓ | | ✓ | |
| H4 | ✓ | ✓ | ✓ | ✓ | | ✓ | ✓ | ✓ | ✓ | ✓ |
| Monday | ✓ | ✓ | | | ✓ | | ✓ | | | ✓ |
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| Thursday | | | ✓ | ✓ | | ✓ | | ✓ | ✓ | |
| Friday | | | | | | | | | | |

# Formulating the Course Selection Problem
## Constraints

Variables (binary) $y_A, y_B, \ldots, y_J \geq 0, \leq 1$ and <u>integral</u>

Bidding (1000 points budget) $200\, y_A + 50\, y_B + \ldots + 100\, y_J \leq 1000$

Credits (54 max credits) (36 min credits)

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| Credit Hours | 12 | 9 | 9 | 12 | 6 | 6 | 9 | 6 | 9 | 6 |

Schedule (MW H3 load) (MW H4 load) (TR H3 load) (TR H4 load)

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| H3 | ✓ | ✓ | ✓ | ✓ | ✓ | | ✓ | | ✓ | |
| H4 | ✓ | ✓ | ✓ | ✓ | | ✓ | ✓ | ✓ | ✓ | ✓ |
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| Thursday | | | ✓ | ✓ | | ✓ | | ✓ | ✓ | |
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# Formulating the Course Selection Problem
## Constraints

Variables (binary) $y_A , y_B , \ldots , y_J \geq 0, \leq 1$ and <u>integral</u>

Bidding (1000 points budget) $200\, y_A + 50\, y_B + \ldots + 100\, y_J \leq 1000$

Credits (54 max credits) $12\, y_A + 9\, y_B + \ldots + 6\, y_J \leq 54$

(36 min credits) $12\, y_A + 9\, y_B + \ldots + 6\, y_J \geq 36$

Schedule

(MW H3 load)

(MW H4 load)

(TR H3 load)

(TR H4 load)

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|---|---|---|---|---|---|---|---|---|---|---|
| H3 | ✓ | ✓ | ✓ | ✓ | ✓ | | ✓ | | ✓ | |
| H4 | ✓ | ✓ | ✓ | ✓ | | ✓ | ✓ | ✓ | ✓ | ✓ |
| Monday | ✓ | ✓ | | | ✓ | | ✓ | | | ✓ |
| Tuesday | | | ✓ | ✓ | | ✓ | | ✓ | ✓ | |
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# Formulating the Course Selection Problem

## Constraints

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| Schedule | (MW H3 load) | $y_A + y_B + y_E + y_G \leq 3$ |
| | (MW H4 load) | |
| | (TR H3 load) | |
| | (TR H4 load) | |

# Formulating the Course Selection Problem

## Constraints

| | | |
|---|---|---|
| Variables | (binary) | $y_A, y_B, \ldots, y_J \geq 0, \leq 1$ and <u>integral</u> |
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| | (MW H4 load) | $y_A + y_B + y_G + y_J \leq 3$ |
| | (TR H3 load) | $y_C + y_D + y_I \leq 3$ |
| | (TR H4 load) | $y_C + y_D + y_F + y_H + y_I \leq 3$ |

# The Complete Formulation

**Maximize:** $10\, y_A + 2\, y_B + 4\, y_C + 2\, y_D + 5\, y_E + 4\, y_F + 8\, y_G + 7\, y_H + 6\, y_I + 6\, y_J$

over variables: $y_A, y_B, \ldots, y_J$

**Subject To:**

| | |
|---|---|
| (binary) | $y_A, y_B, \ldots, y_J \geq 0, \leq 1$ and <u>integral</u> |
| (points budget) | $200\, y_A + 50\, y_B + \ldots + 100\, y_J \leq 1000$ |
| (54 max credits) | $12\, y_A + 9\, y_B + \ldots + 6\, y_J \leq 54$ |
| (36 min credits) | $12\, y_A + 9\, y_B + \ldots + 6\, y_J \geq 36$ |
| (MW H3 load) | $y_A + y_B + y_E + y_G \leq 3$ |
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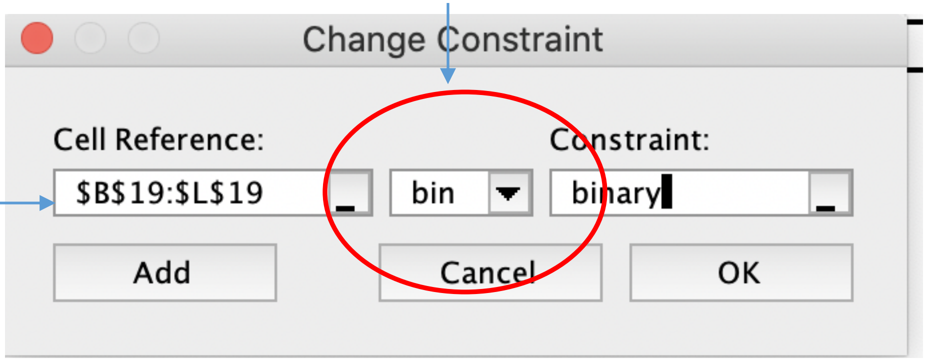
| (TR H3 load) | $y_C + y_D + y_I \leq 3$ |
| (TR H4 load) | $y_C + y_D + y_F + y_H + y_I \leq 3$ |

The only new piece here is we need to tell Solver that these decision variables are <u>binary</u>.

# Excel’s Solver makes it easy to create integer or binary variables

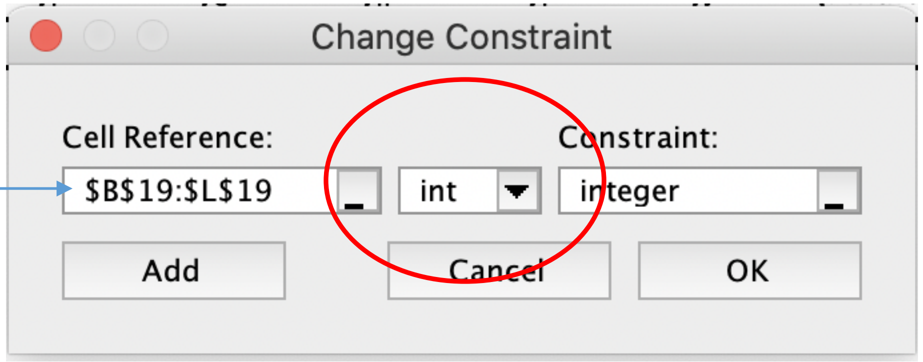
Choose the binary decision variables here




This is a shortcut for

- Integer
- >= 0
- <= 1

Choose the integer decision variables here



# Solving the basic formulation in Excel

**DECISIONS**

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| **Course** | **1.0** | **0.0** | **0.0** | **0.0** | **1.0** | **1.0** | **1.0** | **1.0** | **1.0** | **1.0** |

**OBJECTIVE**

| | |
|---|---|
| **Total utility** | **46** |

**CONSTRAINTS**

| | LHS | □ | RHS |
|---|---|---|---|
| **Points budget** | 730 | <= | 1,000 |
| **Course credit maximum** | 54 | <= | 54 |
| **Course credit minimum** | 54 | >= | 36 |
| **Mon, Wed H3 classes** | 3 | <= | 3 |
| **Tue, Thr H3 classes** | 1 | <= | 3 |
| **Mon, Wed H4 classes** | 3 | <= | 3 |
| **Tue, Thr H4 classes** | 3 | <= | 3 |
| **Binary constraints** | | | |

# Summary

| Model | Optimal Utility | Optimal Course Selection |
|---|---|---|
| Basic | 46 | **Bid on A, E, F, G, H, I, J** |

# Many realistic constraints are easily represented using binary variables

- A and B cannot be taken together (overlapping material)

- Must take at least one of B or C (graduation requirement)

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*Can you write down linear constraints to capture these relationships?*

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$$y_A + y_B \leq 1$$

- Must take at least one of B or C

# Many realistic constraints are easily represented using binary variables

- A and B cannot be taken together

$$y_A + y_B \leq 1$$

- Must take at least one of B or C

$$y_B + y_C \geq 1$$

# If-then constraints

Writing if-then constraints using math

# Let’s add another constraint

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| H3 | ✓ | ✓ | ✓ | ✓ | ✓ | | ✓ | | ✓ | |
| H4 | ✓ | ✓ | ✓ | ✓ | | ✓ | ✓ | ✓ | ✓ | ✓ |

- <u>If</u> you take H, <u>then</u> you must also take E (i.e., E is a prerequisite to H)

*Can you write down a linear constraint to capture this relationship?*

# Let's add another constraint

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| H3 | ✓ | ✓ | ✓ | ✓ | ✓ | | ✓ | | ✓ | |
| H4 | ✓ | ✓ | ✓ | ✓ | | ✓ | ✓ | ✓ | ✓ | ✓ |

- If you take H, then you must also take E (i.e., E is a prerequisite to H)

$$y_H \leq y_E$$

(If $y_H = 1$, then $y_E \geq 1$, so $y_E$ must be equal to 1. If $y_H = 0$ then $y_E$ can be either 1 or 0)

# Formulation with more constraints

**Maximize:** $10\ y_A + 2\ y_B + 4\ y_C + 2\ y_D + 5\ y_E + 4\ y_F + 8\ y_G + 7\ y_H + 6\ y_I + 6\ y_J$

over variables: $y_A, y_B, \ldots, y_J$

**Subject To:**

(binary) $y_A, y_B, \ldots, y_J \geq 0, \leq 1$ and <u>integral</u>

(points budget) $200\ y_A + 50\ y_B + \ldots + 100\ y_J \leq 1000$

(max credits) $12\ y_A + 9\ y_B + \ldots + 6\ y_J \leq 54$

(min credits) $12\ y_A + 9\ y_B + \ldots + 6\ y_J \geq 36$

(MW H3 load) $y_A + y_B + y_E + y_G \leq 3$

(MW H4 load) $y_A + y_B + y_G + y_J \leq 3$

(TR H3 load) $y_C + y_D + y_I \leq 3$

(TR H4 load) $y_C + y_D + y_F + y_H + y_I \leq 3$

(A B conflict) $y_A + y_B \leq 1$

(B or C required) $y_B + y_C \geq 1$

(E pre-req to H) $y_H \leq y_E$

Additional constraints

# Excel with additional constraints

| DECISIONS | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| Course | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 1.0 | 1.0 | 0.0 | 1.0 |

**OBJECTIVE**

| | |
|---|---|
| Total utility | 44 |

**CONSTRAINTS**

| | LHS | □ | RHS |
|---|---|---|---|
| Points budget | 700 | <= | 1,000 |
| Course credit maximum | 54 | <= | 54 |
| Course credit minimum | 54 | >= | 36 |
| Mon, Wed H3 classes | 3 | <= | 3 |
| Tue, Thr H3 classes | 1 | <= | 3 |
| Mon, Wed H4 classes | 3 | <= | 3 |
| Tue, Thr H4 classes | 3 | <= | 3 |
| Do not take A and B | 1.0 | <= | 1 |
| Take either B or C | 1.0 | >= | 1 |
| H requires E | 1.0 | <= | 1 |
| Binary constraints | | | |

# Summary

| **Model** | **Optimal Utility** | **Optimal Course Selection** |
|---|---|---|
| Basic | 46 | **Bid on A, E, F, G, H, I, J** |
| Additional Constraints | 44 | **Bid on A, C, E, F, G, H, J** |

# When we add a new constraint…

*Does the original optimal solution satisfy the new constraint?*

- **Yes** => The original solution is still optimal. We don’t need to resolve.

| Model | Optimal Utility | Optimal Course Selection |
|---|---|---|
| Basic | 46 | **Bid on A, E, F, G, H, I, J** |

**Additional Constraints**

☑ (A B conflict) $y_A + y_B \leq 1$

☑ (E pre-req to H) $y_H \leq y_E$

# When we add a new constraint…

*Does the original optimal solution satisfy the new constraint?*

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- **No** => We need to explicitly add the new constraint and re-solve.

| **Model** | **Optimal Utility** | **Optimal Course Selection** |
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**Additional Constraints**

- ✅ (A B conflict) $y_A + y_B \leq 1$
- ❌ (B or C required) $y_B + y_C \geq 1$
- ✅ (E pre-req to H) $y_H \leq y_E$

# When we add a new constraint…

*Does the original optimal solution satisfy the new constraint?*

- **Yes** => The original solution is still optimal. We don't need to resolve.
- **No** => We need to explicitly add the additional constraint and re-solve. The optimal objective value may stay the same or get worse (it cannot improve).

| Model | Optimal Utility | Optimal Course Selection |
|---|---|---|
| Basic | 46 | Bid on A, E, F, G, H, I, J |
| Additional Constraints | 44 | **Bid on A, C, E, F, G, H, J** |

**Additional Constraints**

- ✅ (A B conflict) $y_A + y_B \leq 1$
- ✅ (B or C required) $y_B + y_C \geq 1$
- ✅ (E pre-req to H) $y_H \leq y_E$

# More challenging: what if H has two pre-requisites?

- If you take H, you must also take **both E and F** (E and F are both pre-requisite to H)

???

- If you take H, you must also take either E or F (Either E or F can be a pre-requisite to H).

????

# More challenging: what if H has two pre-requisites?

- If you take H, you must also take **both E and F** (E and F are both pre-requisite to H)

$$y_H \leq y_E$$
$$y_H \leq y_F$$

- If you take H, you must also take either E or F (Either E or F can be a pre-requisite to H).

# More challenging: what if H has two pre-requisites?

- If you take H, you must also take **both E and F** (E and F are both pre-requisite to H)

$$y_H \leq y_E$$
$$y_H \leq y_F$$

- If you take H, you must also take either E or F (Either E or F can be a pre-requisite to H).

$$y_H \leq y_E + y_F$$

# Other examples of IF-THEN constraints

Let's say we have binary variables x, y and z. How would you encode each of the following if-then statements as a *linear* constraint?

- If x = 1 and y = 1, then z must be 1.

$$z \geq x + y - 1$$

- If either of x = 0 or y = 0, then z must be 0.

$$z \leq x$$

$$z \leq y$$

- If x and y are different, then z must be 1. Otherwise, no restrictions on z.

$$z >= x - y$$

$$z >= y - x$$

# Synergy

Applying IF-THEN constraints to model relationships between variables

# Course Bidding Formulation - "Synergy Effects"

- Recall the objective function:

**maximize** $10\, y_A + 2\, y_B + 4\, y_C + 2\, y_D + 5\, y_E + 4\, y_F + 8\, y_G + 7\, y_H + 6\, y_I + 6\, y_J$

- Suppose the utility of taking courses I and J *together* is 20 (greater than their sum of 12) because they allow you to complete the Analytics Certificate.
- How can you incorporate this into your model?

# Course Bidding Formulation - “Synergy Effects”

- Recall the objective function:

**maximize** $10\ y_A + 2\ y_B + 4\ y_C + 2\ y_D + 5\ y_E + 4\ y_F + 8\ y_G + 7\ y_H + 6\ y_I + 6\ y_J$

- Suppose the utility of taking courses I and J *together* is 20 (greater than their sum of 12) because they allow you to complete the Analytics Certification.
- How can you incorporate this into your model?

$8 = (20 – (6+6))$

**maximize** $10\ y_A + 2\ y_B + 4\ y_C + 2\ y_D + 5\ y_E + 4\ y_F + 8\ y_G + 7\ y_H + 6\ y_I + 6\ y_J + 8\ y_I\ y_J$

# Course Bidding Formulation - “Synergy Effects”

- Recall the objective function:

$$\textbf{maximize} \quad 10\, y_A + 2\, y_B + 4\, y_C + 2\, y_D + 5\, y_E + 4\, y_F + 8\, y_G + 7\, y_H + 6\, y_I + 6\, y_J$$

- Suppose the utility of taking courses I and J *together* is 20 (greater than their sum of 12) because they allow you to complete the Analytics Certification.
- How can you incorporate this into your model?

$8 = (20 – (6+6))$

$$\textbf{maximize} \quad 10\, y_A + 2\, y_B + 4\, y_C + 2\, y_D + 5\, y_E + 4\, y_F + 8\, y_G + 7\, y_H + 6\, y_I + 6\, y_J + 8\, y_I\, y_J$$

Will this work?

# Course Bidding Formulation - “Synergy Effects”

- Recall the objective function:

**maximize** $10\ y_A + 2\ y_B + 4\ y_C + 2\ y_D + 5\ y_E + 4\ y_F + 8\ y_G + 7\ y_H + 6\ y_I + 6\ y_J$

- Suppose the utility of taking courses I and J *together* is 20 (greater than their sum of 12) because they allow you to complete the Analytics Certification.
- How can you incorporate this into your model?

**maximize** $10\ y_A + 2\ y_B + 4\ y_C + 2\ y_D + 5\ y_E + 4\ y_F + 8\ y_G + 7\ y_H + 6\ y_I + 6\ y_J + 8\ y_I\ y_J$

This is not linear ☹

# Making the objective linear – “synergy effects”

~~maximize~~ $\cancel{10\, y_A + 2\, y_B + 4\, y_C + 2\, y_D + 5\, y_E + 4\, y_F + 8\, y_G + 7\, y_H + 6\, y_I + 6\, y_J + 8\, y_I\, y_J}$

maximize $10\, y_A + 2\, y_B + 4\, y_C + 2\, y_D + 5\, y_E + 4\, y_F + 8\, y_G + 7\, y_H + 6\, y_I + 6\, y_J + 8\, z$

We’ll create a new variable z and use *linear constraints* to ensure that $z = y_I\, y_J$

**All possible values of $y_I$ and $y_J$**

| $y_I$ | $y_J$ | $y_I\, y_J$ | z |
|---|---|---|---|
| 0 | 0 | 0 | |
| 0 | 1 | 0 | |
| 1 | 0 | 0 | |
| 1 | 1 | 1 | |

# Making the objective linear – “synergy effects”

~~maximize~~ $\cancel{10\, y_A + 2\, y_B + 4\, y_C + 2\, y_D + 5\, y_E + 4\, y_F + 8\, y_G + 7\, y_H + 6\, y_I + 6\, y_J + 8\, y_I\, y_J}$

maximize $10\, y_A + 2\, y_B + 4\, y_C + 2\, y_D + 5\, y_E + 4\, y_F + 8\, y_G + 7\, y_H + 6\, y_I + 6\, y_J + 8\, z$

We’ll create a new variable z and use *linear constraints* to ensure that $z = y_I\, y_J$

**All possible values of $y_I$ and $y_J$**

| $y_I$ | $y_J$ | $y_I\, y_J$ | $z$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | |

When either $y_I = 0$ or $y_J = 0$, we can enforce that $z = 0$ with the constraints:

- $z \leq y_I$
- $z \leq y_J$

# Making the objective linear – “synergy effects”

~~maximize~~ $\cancel{10\, y_A + 2\, y_B + 4\, y_C + 2\, y_D + 5\, y_E + 4\, y_F + 8\, y_G + 7\, y_H + 6\, y_I + 6\, y_J} + \cancel{8\, y_I\, y_J}$

maximize $10\, y_A + 2\, y_B + 4\, y_C + 2\, y_D + 5\, y_E + 4\, y_F + 8\, y_G + 7\, y_H + 6\, y_I + 6\, y_J + 8\, z$

We’ll create a new variable z and use *linear constraints* to ensure that $z = y_I\, y_J$

**All possible values of $y_I$ and $y_J$**

| $y_I$ | $y_J$ | $y_I\, y_J$ | $z$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 |

When either $y_I = 0$ or $y_J = 0$, we can enforce that $z = 0$ with the constraints:

- $z \le y_I$
- $z \le y_J$

When $y_I = 1$ and $y_J = 1$, we want to enforce that $z = 1$ with the constraints:

- $z \ge y_I + y_J - 1$

# Model with Analytics Certificate Synergy

**Maximize:** $10\ y_A + 2\ y_B + 4\ y_C + 2\ y_D + 5\ y_E + 4\ y_F + 8\ y_G + 7\ y_H + 6\ y_I + 6\ y_J + 8\ z$

over variables: $y_A, y_B, \ldots, y_J$ and $z$

**Subject To:**

| | |
|---|---|
| (binary) | $y_A, y_B, \ldots, y_J, z \geq 0, \leq 1$ and integral |
| (points budget) | $200\ y_A + 50\ y_B + \ldots + 100\ y_J \leq 1000$ |
| (max credits) | $12\ y_A + 9\ y_B + \ldots + 6\ y_J \leq 54$ |
| (min credits) | $12\ y_A + 9\ y_B + \ldots + 6\ y_J \geq 36$ |
| (MW H3 load) | $y_A + y_B + y_E + y_G \leq 3$ |
| (MW H4 load) | $y_A + y_B + y_G + y_J \leq 3$ |
| (TR H3 load) | $y_C + y_D + y_I \leq 3$ |
| (TR H4 load) | $y_C + y_D + y_F + y_H + y_I \leq 3$ |
| (A B conflict) | $y_A + y_B \leq 1$ |
| (B or C required) | $y_B + y_C \geq 1$ |
| (E pre-req to H) | $y_H \leq y_E$ |

Analytics Certificate Synergy

- $z \leq y_I$
- $z \leq y_J$
- $z \geq y_I + y_J - 1$

# Synergy Excel Model

**DECISIONS**

| | A | B | C | D | E | F | G | H | I | J | Z |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Course | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 1.0 |

**OBJECTIVE**

| | |
|---|---|
| Total utility | 47 |

**CONSTRAINTS**

| | LHS | □ | RHS |
|---|---|---|---|
| Points budget | 830 | <= | 1,000 |
| Course credit maximum | 51 | <= | 54 |
| Course credit minimum | 51 | >= | 36 |
| Mon, Wed H3 classes | 3 | <= | 3 |
| Tue, Thr H3 classes | 2 | <= | 3 |
| Mon, Wed H4 classes | 3 | <= | 3 |
| Tue, Thr H4 classes | 2 | <= | 3 |
| Do not take A and B | 1.0 | <= | 1 |
| Take either B or C | 1.0 | >= | 1 |
| H requires E | 0.0 | <= | 1 |
| If I not chosen, Z is 0 | 1.0 | <= | 1 |
| If J not chosen, Z is 0 | 1.0 | <= | 1 |
| If either I or J is chosen, Z is 1 | 1.0 | >= | 1 |
| Binary constraints | | | |

Constraints that link z with $y_I$ and $y_J$

# Summary

| Model | Optimal Utility | Optimal Course Selection |
|---|---|---|
| Basic | 46 | **Bid on A, E, F, G, H, I, J** |
| Additional Constraints | 44 | **Bid on A, C, E, F, G, H, J** |
| Synergy Effect | 47 | **Bid on A, C, E, G, I, J** **(+ certificate)** |

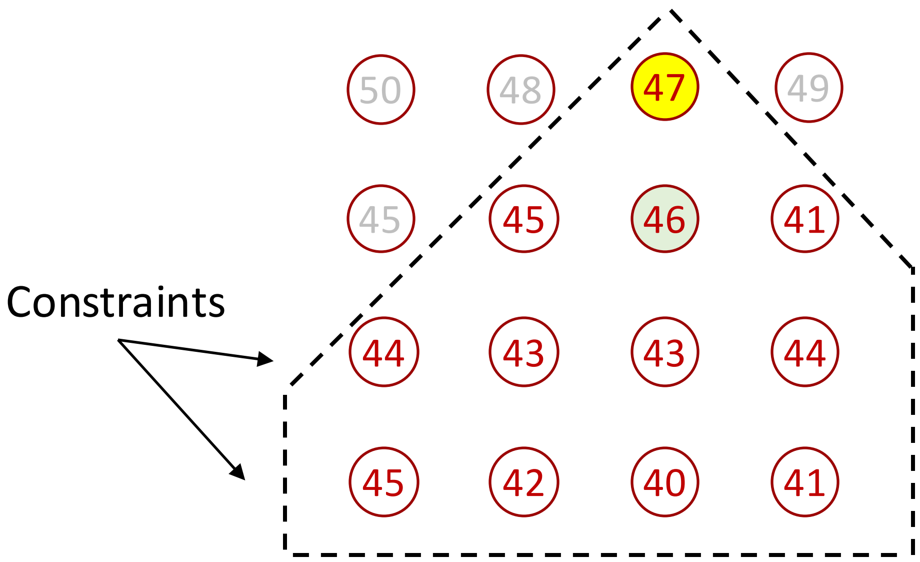
**Note**: Objective value increased because we *changed the objective function* with the new analytics certificate.

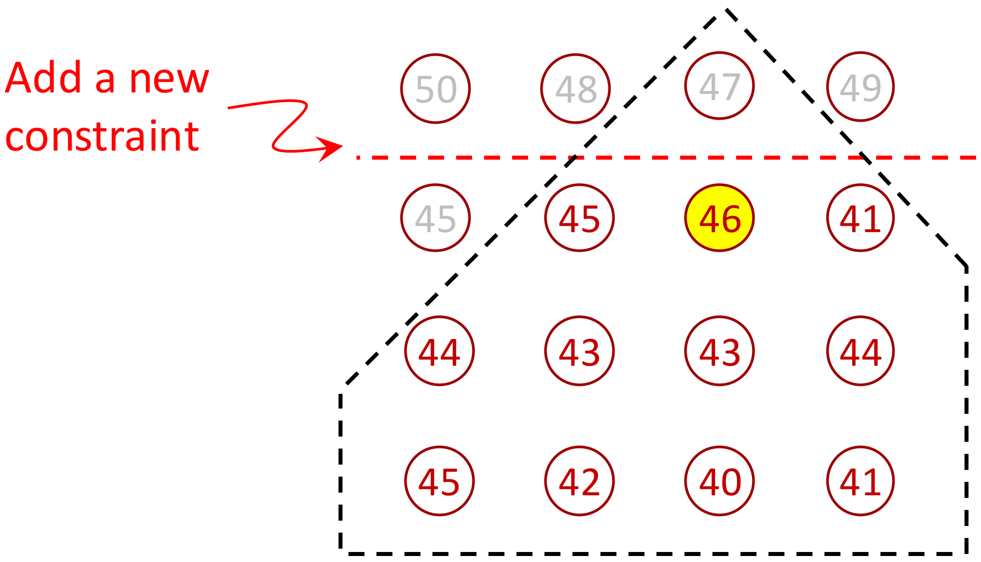
# Excluding AN optimal solution

# Excluding the optimal solution

- The model tells you to bid for / enroll in classes **A, C, E, G, I, J** for a total utility of 47.
- For some reason, you are not happy with this course selection. You can’t really express it as a constraint.
- You would like to exclude A, C, E, G, I and J, but find the best one from all the remaining ones
- **Approach**: Add a constraint which makes **A, C, E, G, I, J** infeasible, but does not exclude any other solutions.

# Excluding the optimal solution

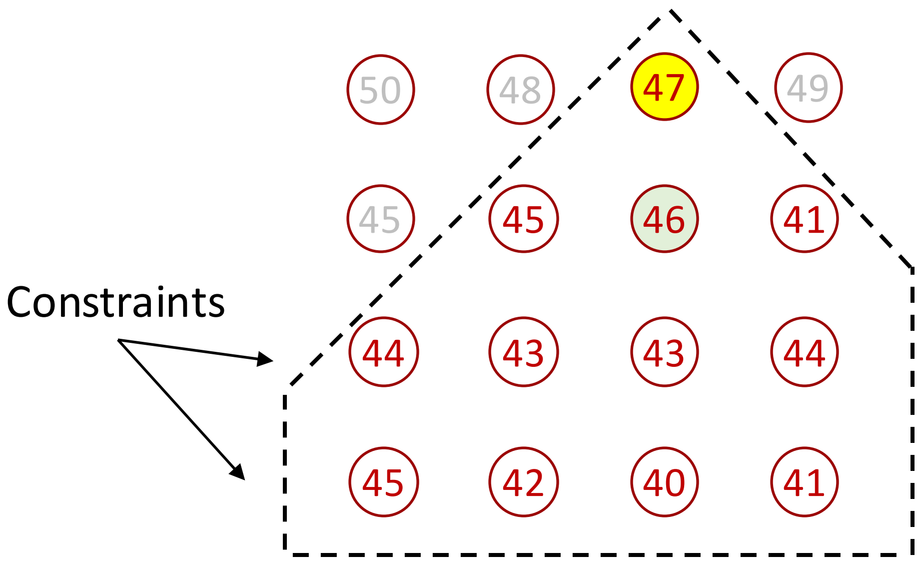


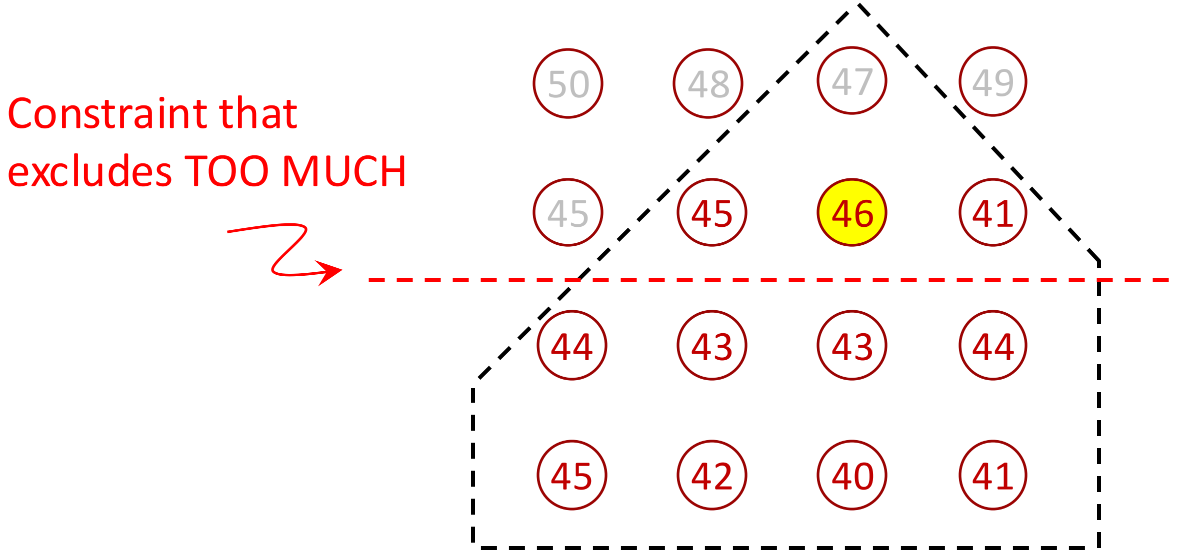

- Each circle represents a solution, while the dotted region are the constraints.
- The number inside represents the objective value. 47 is the best objective value, 46 is the second-best

- If we can add a constraint that **only cuts off this one solution**, then re-solving the optimization will give us the best of the rest

# Excluding the optimal solution




- Each circle represents a solution, while the dotted region are the constraints.
- The number inside represents the objective value. 47 is the best objective value, 46 is the second-best




- If this new constraint excludes more than just the solution we want to eliminate, then there’s a **chance that the second-optimal solution is also excluded**.

# Excluding a Solution

- The (previous) optimal solution was **A, C, E, G, I, J** :

$$y_A^* = y_C^* = y_E^* = y_G^* = y_I^* = y_J^* = 1 \text{ and } y_B^* = y_D^* = y_F^* = y_H^* = 0$$

# Excluding a Solution

- The optimal solution was **A, C, E, G, I, J** :

$$y_A^* = y_C^* = y_E^* = y_G^* = y_I^* = y_J^* = 1 \text{ and } y_B^* = y_D^* = y_F^* = y_H^* = 0$$

- We want to exclude just this one solution but none other. We will add a constraint that will make just this solution infeasible.

# Excluding Solutions

- The optimal solution was **A, C, E, G, I, J** :

$$y_A^* = y_C^* = y_E^* = y_G^* = y_I^* = y_J^* = 1 \text{ and } y_B^* = y_D^* = y_F^* = y_H^* = 0$$

- Consider the following expression:

$$y_A + y_C + y_E + y_G + y_I + y_J + (1 - y_B) + (1 - y_D) + (1 - y_F) + (1 - y_H)$$

For the solution **A, C, E, G, I, J :**
Value of this expression = 6

**+**

For the solution **A, C, E, G, I, J :**
Value of this expression = 4

= 10 (total)

# Excluding Solutions

- The optimal solution was **A, C, E, G, I, J** :

$$y_A^* = y_C^* = y_E^* = y_G^* = y_I^* = y_J^* = 1 \text{ and } y_B^* = y_D^* = y_F^* = y_H^* = 0$$

- Consider the following expression:

$$\underbrace{y_A + y_C + y_E + y_G + y_I + y_J}_{} + \underbrace{(1 - y_B) + (1 - y_D) + (1 - y_F) + (1 - y_H)}_{}$$

For the solution **A, C, E, G, I, J :**
Value of this expression = 6 **+** For the solution **A, C, E, G, I, J :**
Value of this expression = 4 = 10 (total)

For <u>any other</u> solution:
Value of this expression **+** For <u>any other</u> solution:
Value of this expression = ? (total)

# Excluding Solutions

- The optimal solution was **A, C, E, G, I, J** :

$$y_A^* = y_C^* = y_E^* = y_G^* = y_I^* = y_J^* = 1 \text{ and } y_B^* = y_D^* = y_F^* = y_H^* = 0$$

- Consider the following expression:

$$y_A + y_C + y_E + y_G + y_I + y_J + (1 - y_B) + (1 - y_D) + (1 - y_F) + (1 - y_H)$$

For the solution **A, C, E, G, I, J :**
Value of this expression = 6 **+** For the solution **A, C, E, G, I, J :**
Value of this expression = 4 = 10 (total)

For <u>any other</u> solution:
Value of this expression **+** For <u>any other</u> solution:
Value of this expression = 9 or less (total)

# Excluding Solutions

- The optimal solution was **A, C, E, G, I, J** :

$$y_A^* = y_C^* = y_E^* = y_G^* = y_I^* = y_J^* = 1 \text{ and } y_B^* = y_D^* = y_F^* = y_H^* = 0$$

- Consider the following expression:

$$y_A + y_C + y_E + y_G + y_I + y_J + (1 - y_B) + (1 - y_D) + (1 - y_F) + (1 - y_H)$$

  - ➢ This expression is equal to 10 for the solution **A, C, E, G, I, J**
  - ➢ *<u>For every other solution, the expression is guaranteed to be less than or equal to 9</u>*

# Excluding Solutions

- The optimal solution was **A, C, E, G, I, J** :

$$y_A^* = y_C^* = y_E^* = y_G^* = y_I^* = y_J^* = 1 \text{ and } y_B^* = y_D^* = y_F^* = y_H^* = 0$$

- Consider the following expression:

$$y_A + y_C + y_E + y_G + y_I + y_J + (1 - y_B) + (1 - y_D) + (1 - y_F) + (1 - y_H)$$

  - This expression is equal to 10 for the solution **A, C, E, G, I, J**
  - *For every other solution, the expression is guaranteed to be less than or equal to 9*

So, we add the constraint:

$$y_A + y_C + y_E + y_G + y_I + y_J + (1 - y_B) + (1 - y_D) + (1 - y_F) + (1 - y_H) \leq 9$$

This constraint eliminates **A, C, E, G, I, J** but no other solutions!

# Excluding Solutions

- The optimal solution was **A, C, E, G, I, J** :

$$y_A^* = y_C^* = y_E^* = y_G^* = y_I^* = y_J^* = 1 \text{ and } y_B^* = y_D^* = y_F^* = y_H^* = 0$$

- Consider the following expression:

$$y_A + y_C + y_E + y_G + y_I + y_J + (1 - y_B) + (1 - y_D) + (1 - y_F) + (1 - y_H)$$

  - This expression is equal to 10 for the solution **A, C, E, G, I, J**
  - *For every other solution, the expression is guaranteed to be less than or equal to 9*

So, we add the constraint:

$$y_A + y_C + y_E + y_G + y_I + y_J + (1 - y_B) + (1 - y_D) + (1 - y_F) + (1 - y_H) \leq 9$$

This constraint eliminates **A, C, E, G, I, J** but no other solutions!

IN GENERAL, THE RHS WILL BE THE NUMBER OF VARIABLES - 1

# Formulation with All Constraints, Synergy Effect,and Excluding Previous Optimal

maximize $10\,y_A + 2\,y_B + 4\,y_C + 2\,y_D + 5\,y_E + 4\,y_F + 8\,y_G + 7\,y_H + 6\,y_I + 6\,y_J + 8\,z$

subject to

(binary) $z, y_A, y_B, \ldots, y_J$

(points budget) $200\,y_A + 50\,y_B + \ldots + 100\,y_J \leq 1000$

(max credits) $12\,y_A + 9\,y_B + \ldots + 6\,y_J \leq 54$

(min credits) $12\,y_A + 9\,y_B + \ldots + 6\,y_J \geq 36$

(MW, H3 load) $y_A + y_B + y_E + y_G \leq 3$

(MW H4 load) $y_A + y_B + y_G + 6\,y_J \leq 3$

(TR H3 load) $y_C + y_D + y_I \leq 3$

(TR H4 load) $y_C + y_D + y_F + y_H \leq 3$

(A B conflict) $y_A + y_B \leq 1$

(B or C required) $y_B + y_C \geq 1$

(E pre-req to H) $y_H \leq y_E$

(Analytics Certificate Synergy)

- $z \leq y_I$
- $z \leq y_J$
- $z \geq y_I + y_J - 1$

(Exclude Previous Optimal Solution)

$$y_A + y_C + y_E + y_G + y_I + y_J + (1 - y_B) + (1 - y_D) + (1 - y_F) + (1 - y_H) \leq 9$$

# Updated Excel Solution

**Bid on B, C, E, G, H, I, J**

| DECISIONS | A | B | C | D | E | F | G | H | I | J | Z |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Course | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |

**OBJECTIVE**

| | |
|---|---|
| Total utility | 46 |

**CONSTRAINTS**

| | LHS | | RHS |
|---|---|---|---|
| Points budget | 730 | <= | 1,000 |
| Course credit maximum | 54 | <= | 54 |
| Course credit minimum | 54 | >= | 36 |
| Mon, Wed H3 classes | 3 | <= | 3 |
| Tue, Thr H3 classes | 2 | <= | 3 |
| Mon, Wed H4 classes | 3 | <= | 3 |
| Tue, Thr H4 classes | 3 | <= | 3 |
| Do not take A and B | 1.0 | <= | 1 |
| Take either B or C | 2.0 | >= | 1 |
| H requires E | 1.0 | <= | 1 |
| If I not chosen, Z is 0 | 1.0 | <= | 1 |
| If J not chosen, Z is 0 | 1.0 | <= | 1 |
| If either I or J is chosen, Z is 1 | 1.0 | >= | 1 |
| Exclude previous optimal solution | 3.0 | <= | 5 |
| Binary constraints | | | |

# Summary

| Model | Optimal Utility | Optimal Course Selection |
|---|---|---|
| Basic | 46 | **Bid on A, E, F, G, H, I, J** |
| Additional Constraints | 44 | **Bid on A, C, E, F, G, H, J** |
| Synergy Effect | 47 | **Bid on A, C, E, G, I, J (+ certificate)** |
| Exclude previous optimal solution | 46 | **Bid on B, C, E, G, H, I, J (+ certificate)** |

# Excluding Solutions

This approach can be used to exclude any number of previously-found solutions, by adding a constraint for each solution you want to exclude

# Excluding Two Solutions

- We want to exclude **A, C, E, G, I, J** and **B, C, E, G, H, I, J**

- To eliminate **A, C, E, G, I, J** we already have this in our model:

$$y_A + y_C + y_E + y_G + y_I + y_J + (1 - y_B) + (1 - y_D) + (1 - y_F) + (1 - y_H) \leq 9$$

- What constraint should we add to eliminate **B, C, E, G, H, I, J** ?

# Excluding Two Solutions

- We want to exclude **A, C, E, G, I, J** and **B, C, E, G, H, I, J**

- To eliminate **A, C, E, G, I, J** we already have this in our model:

$$y_A + y_C + y_E + y_G + y_I + y_J + (1 - y_B) + (1 - y_D) + (1 - y_F) + (1 - y_H) \leq 9$$

- What constraint should we add to eliminate **B, C, E, G, H, I, J** ?

$$y_B + y_C + y_E + y_G + yH + y_I + y_J + (1 - y_A) + (1 - y_D) + (1 - y_F) \leq 9$$

# Updated Excel Solution

**Bid on A, C, F, G, I, J**

| DECISIONS | A | B | C | D | E | F | G | H | I | J | Z |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Course | 1.0 | 0.0 | 1.0 | 0.0 | 0.0 | 1.0 | 1.0 | 0.0 | 1.0 | 1.0 | 1.0 |

OBJECTIVE

| | |
|---|---|
| Total utility | 46 |

CONSTRAINTS

| | LHS | | RHS |
|---|---|---|---|
| Points budget | 780 | <= | 1,000 |
| Course credit maximum | 51 | <= | 54 |
| Course credit minimum | 51 | >= | 36 |
| Mon, Wed H3 classes | 2 | <= | 3 |
| Tue, Thr H3 classes | 2 | <= | 3 |
| Mon, Wed H4 classes | 3 | <= | 3 |
| Tue, Thr H4 classes | 3 | <= | 3 |
| Do not take A and B | 1.0 | <= | 1 |
| Take either B or C | 1.0 | >= | 1 |
| H requires E | 0.0 | <= | 0 |
| If I not chosen, Z is 0 | 1.0 | <= | 1 |
| If J not chosen, Z is 0 | 1.0 | <= | 1 |
| If either I or J is chosen, Z is 1 | 1.0 | >= | 1 |
| Exclude previous optimal solution | 4.0 | <= | 5 |
| Exclude 2nd optimal solution | 2.0 | <= | 6 |
| Binary constraints | | | |

# Summary

| Model | Optimal Utility | Optimal Course Selection |
|---|---|---|
| Basic | 46 | **Bid on A, E, F, G, H, I, J** |
| Additional Constraints | 44 | **Bid on A, C, E, F, G, H, J** |
| Synergy Effect | 47 | **Bid on A, C, E, G, I, J** **(+ certificate)** |
| Exclude previous optimal solution | 46 | **Bid on B, C, E, G, H, I, J.** **(+ certificate)** |
| Excluding two solutions | 46 | **Bid on A, C, F, G, I, J** **(+ certificate)** |

Note that the optimal value didn't decrease when we excluded the second optimal solution

# Recap: The Power of Binary Variables

- Binary variables:
  - Are very useful to model business decisions that involve doing something or not doing something (as opposed to how much of something to do) …
  - … but make solving an optimization model more challenging

- Binary variables allow us to model IF-THEN relationships and non-linearities s using “tricks” so that the formulation remains linear.

- A vast array of real-world decision problems can be modeled with binary variables

# A Sample of Optimization Applications I have Worked On

- Airline/Airport Operations
  - Fleet Assignment, Crew Assignment, Gate Assignment, Staff Scheduling
- “Last Mile” Delivery and Routing
- Retail Pricing and Promotions
- Supply Chain Inventory Allocation

# What’s Next

- *Wednesday*: Discrete Optimization – Part 2 (Application to Supply Chain Optimization)

- *Thursday*: Deliverable #7 due

- *Friday*: Recitation on Discrete Optimization

- 1-on-1 Meetings
  - *Please book via https://calendly.com/ramamit*
  - *I have added more Calendly slots. If the Calendly times don’t work, please email my assistant Laura (brentrup@mit.edu) to find a time.*

# APPENDIX
# Optional How Integer Optimization Problems Are Solved in Practice

# How Does Excel Solve Integer Optimization?

- Unlike linear optimization, discrete optimization is typically quite challenging to solve.

- In general, to find the best solution, a solver may just need to enumerate over all possible solutions.
    - In our simple course-selection problem, there are $2^{10} = 1024$ possible course selections.
    - In real-world sized problems with millions of variables, there would be way too many solutions to enumerate.

- However, solvers are still routinely and efficient at finding **very good solutions, good enough for real-world use-cases.** *How do solvers do it?*

# First, ignore the integrality constraints

**Maximize:** $10\ y_A + 2\ y_B + 4\ y_C + 2\ y_D + 5\ y_E + 4\ y_F + 8\ y_G + 7\ y_H + 6\ y_I + 6\ y_J$

over variables: $y_A, y_B, \dots, y_J$

**Subject To:**

| | |
|---|---|
| (binary) | $y_A, y_B, \dots, y_J \geq 0, \leq 1$ ~~and integral~~ |
| (points budget) | $200\ y_A + 50\ y_B + \dots + 100\ y_J \leq 1000$ |
| (max credits) | $12\ y_A + 9\ y_B + \dots + 6\ y_J \leq 54$ |
| (min credits) | $12\ y_A + 9\ y_B + \dots + 6\ y_J \geq 36$ |
| (MW H3 load) | $y_A + y_B + y_E + y_G \leq 3$ |
| (MW H4 load) | $y_A + y_B + y_G + y_J \leq 3$ |
| (TR H3 load) | $y_C + y_D + y_I \leq 3$ |
| (TR H4 load) | $y_C + y_D + y_F + y_H + y_I \leq 3$ |
| (A B conflict) | $y_A + y_B \leq 1$ |
| (B or C required) | $y_B + y_C \geq 1$ |
| (E pre-req to H) | $y_H \leq y_E$ |

# Solve in Excel and check if the solution is integral

**Maximize:** $10\,y_A + 2\,y_B + 4\,y_C + 2\,y_D + 5\,y_E + 4\,y_F + 8\,y_G + 7\,y_H + 6\,y_I + 6\,y_J$

over variables: $y_A, y_B, \dots, y_J$

**Subject To:**

| | |
|---|---|
| (binary) | $y_A, y_B, \dots, y_J \geq 0, \leq 1$ ~~and integral~~ |
| (points budget) | $200\,y_A + 50\,y_B + \dots + 100\,y_J \leq 1000$ |
| (max credits) | $12\,y_A + 9\,y_B + \dots + 6\,y_J \leq 54$ |
| (min credits) | $12\,y_A + 9\,y_B + \dots + 6\,y_J \geq 36$ |
| (MW H3 load) | $y_A + y_B + y_E + y_G \leq 3$ |
| (MW H4 load) | $y_A + y_B + y_G + y_J \leq 3$ |
| (TR H3 load) | $y_C + y_D + y_I \leq 3$ |
| (TR H4 load) | $y_C + y_D + y_F + y_H + y_I \leq 3$ |
| (A B conflict) | $y_A + y_B \leq 1$ |
| (B or C required) | $y_B + y_C \geq 1$ |
| (E pre-req to H) | $y_H \leq y_E$ |

Solve in Excel / Solver $\Rightarrow$

Optimal Solution

$y_A = 1.0$
$y_B = 0.0$
$y_C = 1.0$
$y_D = 0.0$
$y_E = 1.0$
$y_F = 0.0$
$y_G = 1.0$
$y_H = 1.0$
$y_I = 0.0$
$y_J = 1.0$

If the solution has all integers, then we are done! Why?

# Take one of the fractional variables and …

| | |
|---|---|
| **Maximize:** | $10\ y_A + 2\ y_B + 4\ y_C + 2\ y_D + 5\ y_E$ $+ 4\ y_F + 8\ y_G + 7\ y_H + 6\ y_I + 6\ y_J$ |
| over variables: | $y_A, y_B, \ldots, y_J$ |
| **Subject To:** | |
| (binary) | $y_A, y_B, \ldots, y_J \geq 0, \leq 1$ ~~and integral~~ |
| (points budget) | $200\ y_A + 50\ y_B + \ldots + 100\ y_J \leq 1000$ |
| (max credits) | $12\ y_A + 9\ y_B + \ldots + 6\ y_J \leq 54$ |
| (min credits) | $12\ y_A + 9\ y_B + \ldots + 6\ y_J \geq 36$ |
| (MW H3 load) | $y_A + y_B + y_E + y_G \leq 3$ |
| (MW H4 load) | $y_A + y_B + y_G + y_J \leq 3$ |
| (TR H3 load) | $y_C + y_D + y_I \leq 3$ |
| (TR H4 load) | $y_C + y_D + y_F + y_H + y_I \leq 3$ |
| (A B conflict) | $y_A + y_B \leq 1$ |
| (B or C required) | $y_B + y_C \geq 1$ |
| (E pre-req to H) | $y_H \leq y_E$ |

Solve in Excel / Solver

Optimal Solution

$y_A = 0.7$
$y_B = 0.0$
$y_C = 0.7$
$y_D = 0.0$
$y_E = 1.0$
$y_F = 0.3$
$y_G = 1.0$
$y_H = 1.0$
$y_I = 0.0$
$y_J = 1.0$

# Branch into two optimization problems

**Maximize:** $10\ y_A + 2\ y_B + 4\ y_C + 2\ y_D + 5\ y_E + 4\ y_F + 8\ y_G + 7\ y_H + 6\ y_I + 6\ y_J$

over variables: $y_A, y_B, \ldots, y_J$

**Subject To:**

(binary) $y_A, y_B, \ldots, y_J \geq 0, \leq 1$ and <u>integral</u>

…

(E pre-req to H) $y_H \leq y_E$

---

**Maximize:** $10\ y_A + 2\ y_B + 4\ y_C + 2\ y_D + 5\ y_E + 4\ y_F + 8\ y_G + 7\ y_H + 6\ y_I + 6\ y_J$

over variables: $y_A, y_B, \ldots, y_J$

**Subject To:**

(binary) $y_A, y_B, \ldots, y_J \geq 0, \leq 1$ and <u>integral</u>

…

(E pre-req to H) $y_H \leq y_E$

(Fix $y_A = 1$) $y_A = 1$

---

**Maximize:** $10\ y_A + 2\ y_B + 4\ y_C + 2\ y_D + 5\ y_E + 4\ y_F + 8\ y_G + 7\ y_H + 6\ y_I + 6\ y_J$

over variables: $y_A, y_B, \ldots, y_J$

**Subject To:**

(binary) $y_A, y_B, \ldots, y_J \geq 0, \leq 1$ and <u>integral</u>

…

(E pre-req to H) $y_H \leq y_E$

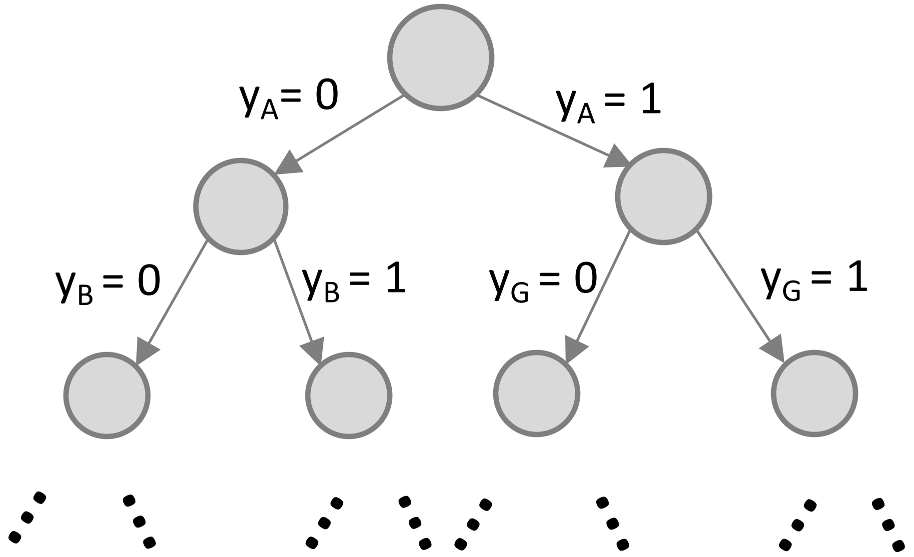
(Fix $y_A = 0$) $y_A = 0$

---

We have turned 1 problem with 10 variables into 2 problems with 9 variables.

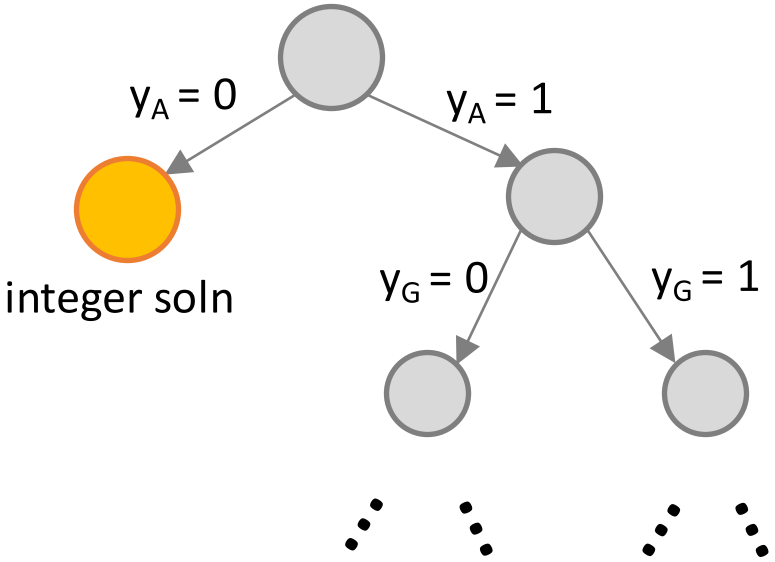
Take the *better* of the two optimal solutions.

# Branching doubles the number of optimization problems we must solve.



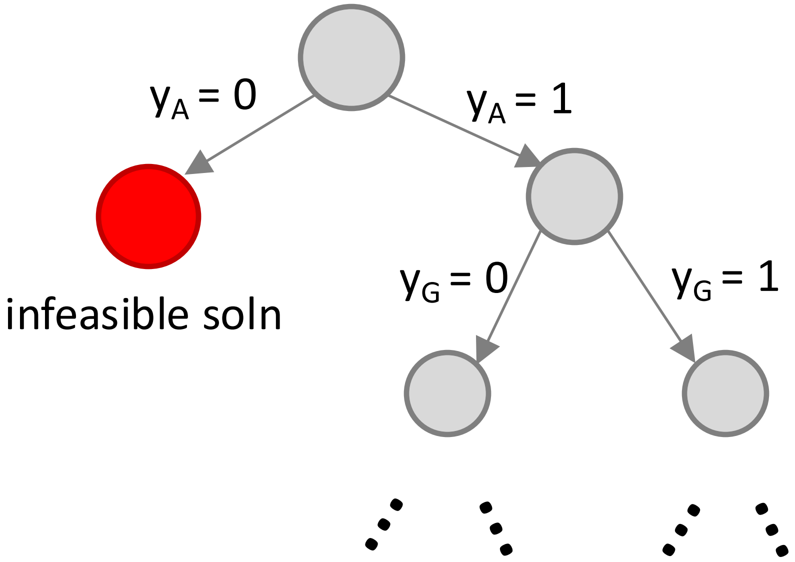

- We continue to branch, each time *doubling* the number of Linear Programs that we must solve. Yikes!

# Ways that this enumeration is “smarter”



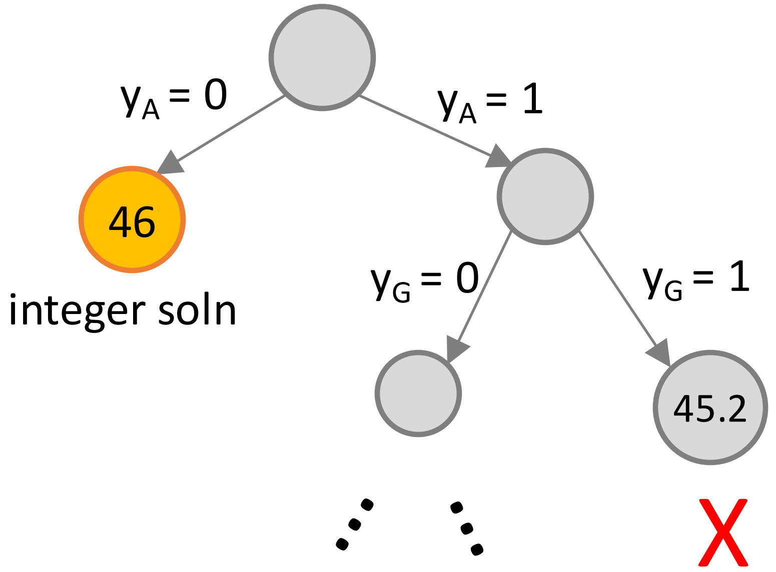

- Every time we hit an integer solution, we can stop branching.

# Ways that this enumeration is “smarter”



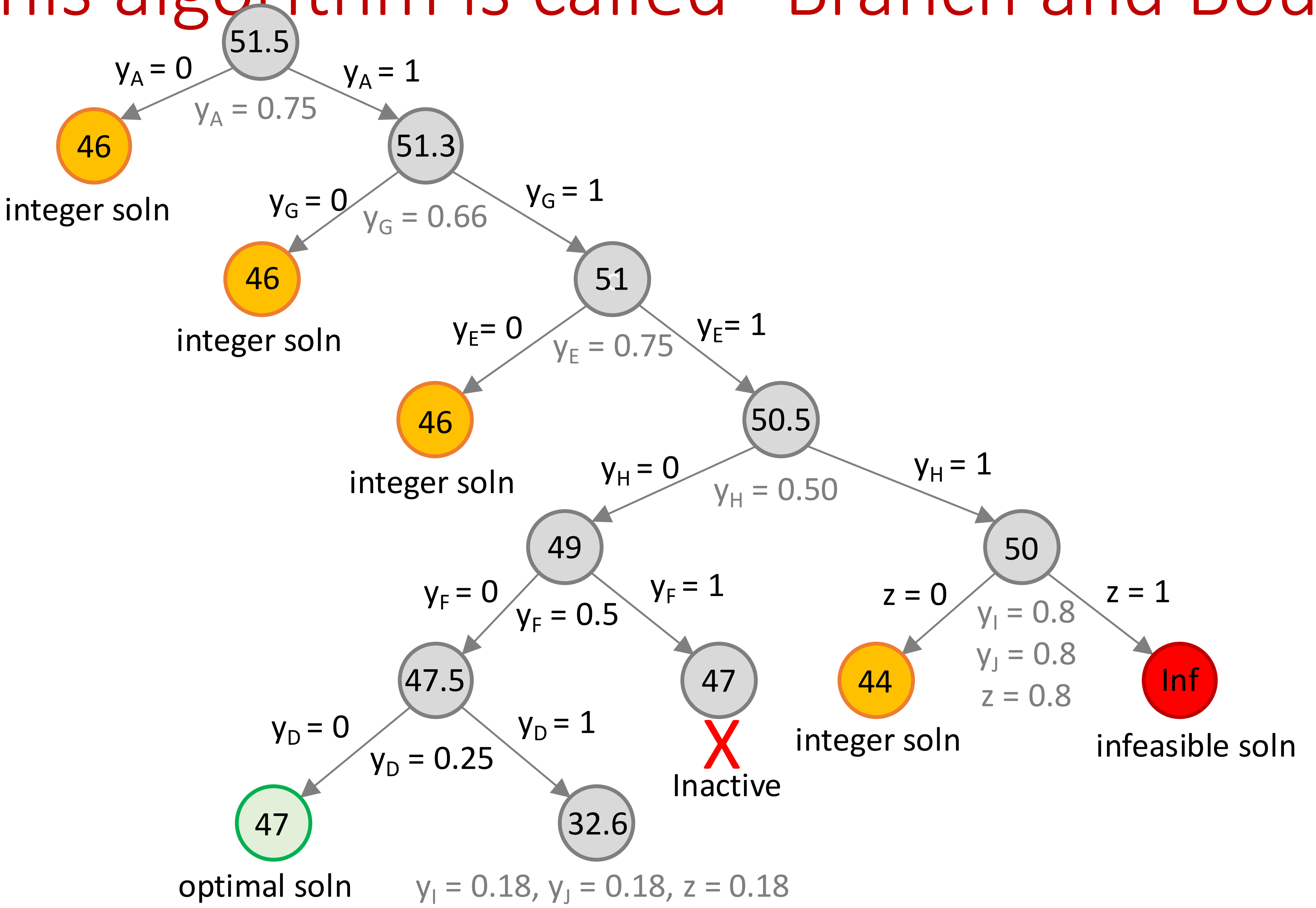

- Every time we hit an integer solution, we can stop branching.
- If the linear optimization relaxation is *infeasible*, then we can also stop branching.

# Ways that this enumeration is "smarter"




- Every time we hit an integer solution, we can stop branching.
- If the linear optimization relaxation is *infeasible*, then we can also stop branching.
- If the linear optimization relaxation has an objective value worse than an integer solution, then we can also stop branching.

# This algorithm is called “Branch and Bound”.

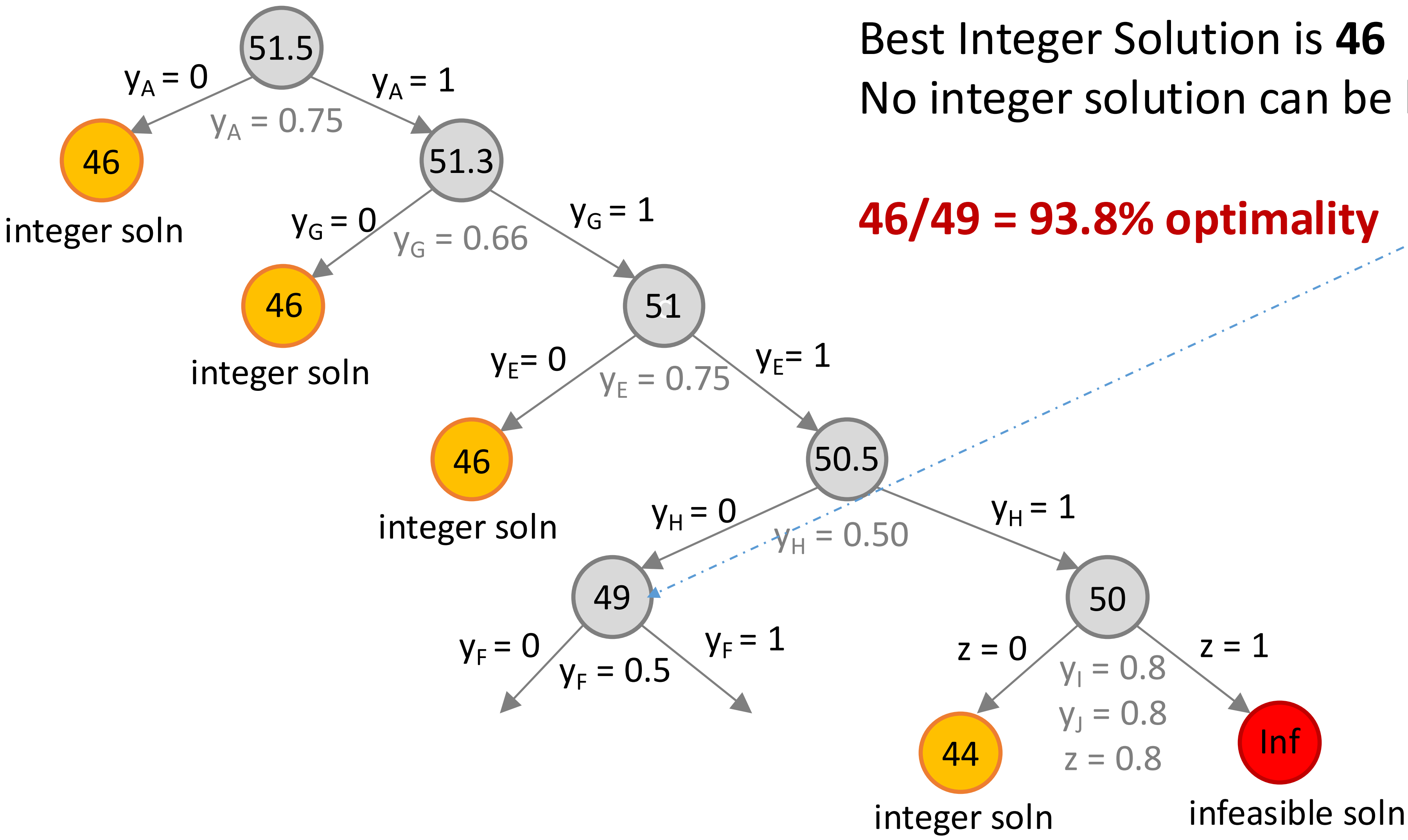


# Branch and Bound

- Branch and bound can solve a binary optimization problems with N variables by **solving way fewer than $2^N$** linear optimization problems.

- Practitioners typically set a time limit (e.g., 1 hour).
    - Branch-and-bound will find the **best solution** it can within that time limit.
    - Branch-and-bound will give tell you **how far you** are to the optimal solution.

# Time's up!




Best Integer Solution is **46**

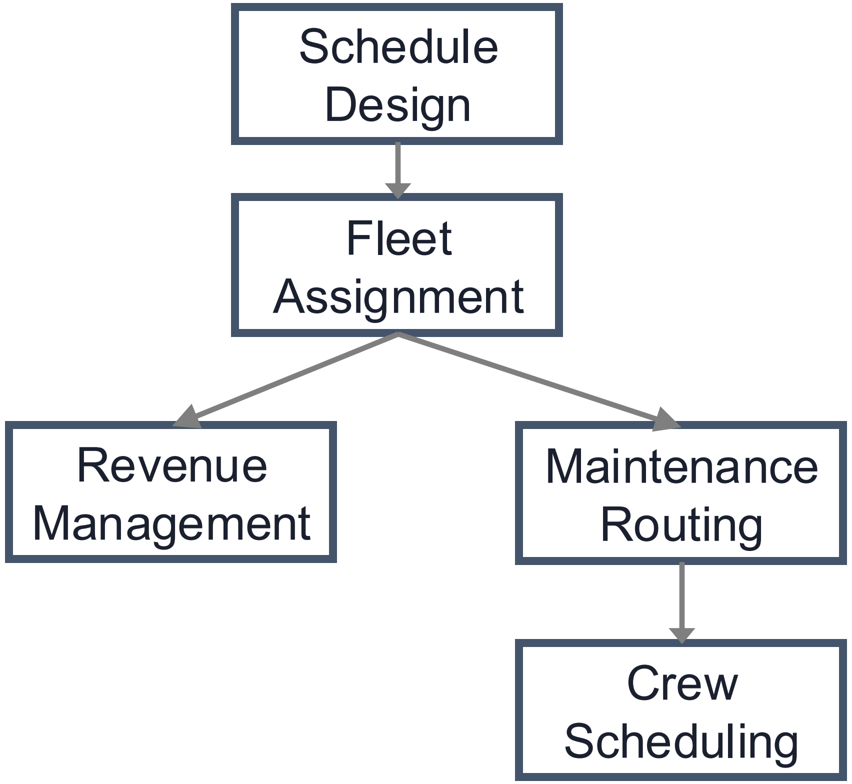
No integer solution can be better than **49**

**46/49 = 93.8% optimality**

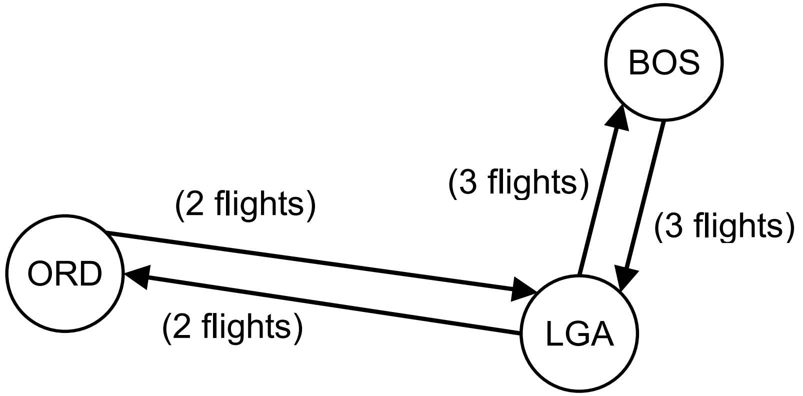
# APPENDIX
# <u>Optional</u>
# Real-world application of 0-1 integer optimization: AIRLINE FLEET ASSIGNMENT

# Applications of Optimization in Airline Operations

- Airlines use optimization models to plan many aspects of their operations such as assigning aircraft and crew to flights.
- Airlines also use optimization (especially linear programming) to manage their revenue.
- The scale of operations is huge. A major US airline would typically fly 1,500-2,500 domestic flights daily and offer over 100,000 different itinerary/fare combinations.



# Airline Fleet Assignment – Simplified Example (1/2)




| Flight # | From | To | Dept Time (EST) | Arr Time (EST) | Fare [$] | Demand [passengers] |
|---|---|---|---|---|---|---|
| 1 | LGA | BOS | 1000 | 1100 | 150 | 250 |
| 2 | LGA | BOS | 1100 | 1200 | 150 | 250 |
| 3 | LGA | BOS | 1800 | 1900 | 150 | 100 |
| 4 | BOS | LGA | 0700 | 0800 | 150 | 150 |
| 5 | BOS | LGA | 1030 | 1130 | 150 | 300 |
| 6 | BOS | LGA | 1800 | 1900 | 150 | 150 |
| 7 | LGA | ORD | 1100 | 1400 | 400 | 150 |
| 8 | LGA | ORD | 1500 | 1800 | 400 | 200 |
| 9 | ORD | LGA | 0700 | 1000 | 400 | 200 |
| 10 | ORD | LGA | 0830 | 1130 | 400 | 150 |

# Airline Fleet Assignment – Simplified Example (2/2)

| Fleet type | Number of aircraft owned | Capacity [seats] | Per flight operating cost [$000] | |
|---|---|---|---|---|
| | | | LGA - BOS | LGA – ORD |
| A | 1 | 120 | 10 | 15 |
| B | 2 | 150 | 12 | 17 |
| C | 2 | 250 | 15 | 20 |

# The Airline Fleet Assignment Problem

Given:

- Flight schedule: (set of daily flight legs)
- Estimated passenger demand
- Aircraft fleet characteristics
- Revenue and operating cost data

Find:

- A feasible fleet assignment (an allocation of fleet types to flight legs) that maximizes:

  Profit contribution = Revenue – Operating Costs

# Formulation – Decision Variables

Decision variables:

For each flight leg – fleet type combination, we need a binary variable that indicates whether that fleet type is assigned to that flight leg.

For example:

$x_{1,A}$ = 1 if fleet type A is assigned to flight leg 1

= 0 otherwise

| Flight # | Fleet Type | | |
|---|---|---|---|
| | A | B | C |
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5 | | | |
| 6 | | | |
| 7 | | | |
| 8 | | | |
| 9 | | | |
| 10 | | | |

# Formulation – Objective Function

- What is the (estimated) profit contribution of flight 1 when assigned fleet type A?
  $\approx 150 \cdot \min(250,120) - 10{,}000 = 8{,}000$

- What is the (estimated) profit contribution of flight 2 when assigned fleet type B?
  $\approx 150 \cdot \min(250,150) - 12{,}000 = 10{,}500$

<u>Objective Function</u>:

maximize $8{,}000\, x_{1,A} + 10{,}500\, x_{1,B} + 22{,}500\, x_{1,C}$
$+ \ldots$

| Flight # | Fleet Type | | |
|---|---|---|---|
| | A | B | C |
| 1 | 8,000 | 10,500 | 22,500 |
| 2 | 8,000 | 10,500 | 22,500 |
| 3 | 5,000 | 3,000 | 0 |
| 4 | 8,000 | 10,500 | 7,500 |
| 5 | 8,000 | 10,500 | 22,500 |
| 6 | 33,000 | 10,500 | 7,500 |
| 7 | 33,000 | 43,000 | 40,000 |
| 8 | 33,000 | 43,000 | 60,000 |
| 9 | 33,000 | 43,000 | 60,000 |
| 10 | 33,000 | 43,000 | 10,000 |

# Formulation Constraints – (1/3)

Constraints:

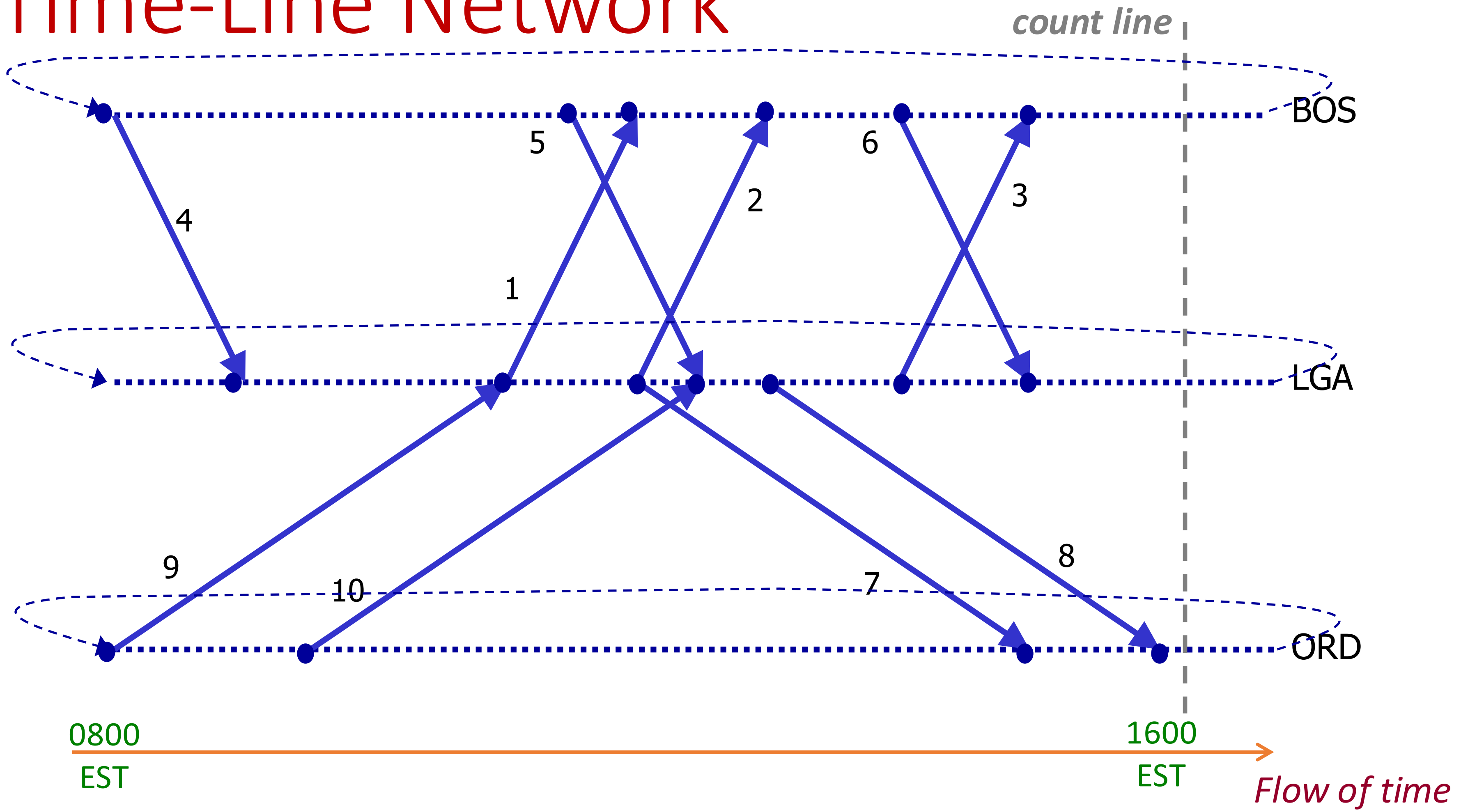
- Binary
  *Each assignment decision variable must be either 0 or 1*

  $x_{1,A}$ , $x_{1,B}$ , $x_{1,C}$ , … , $x_{10,A}$ , $x_{10,B}$ , $x_{10,C}$ are binary variables

- *Make sure each flight is assigned an airplane.*

  For flight 1: $x_{1,A} + x_{1,B} + x_{1,C} = 1$ and so on for each flight

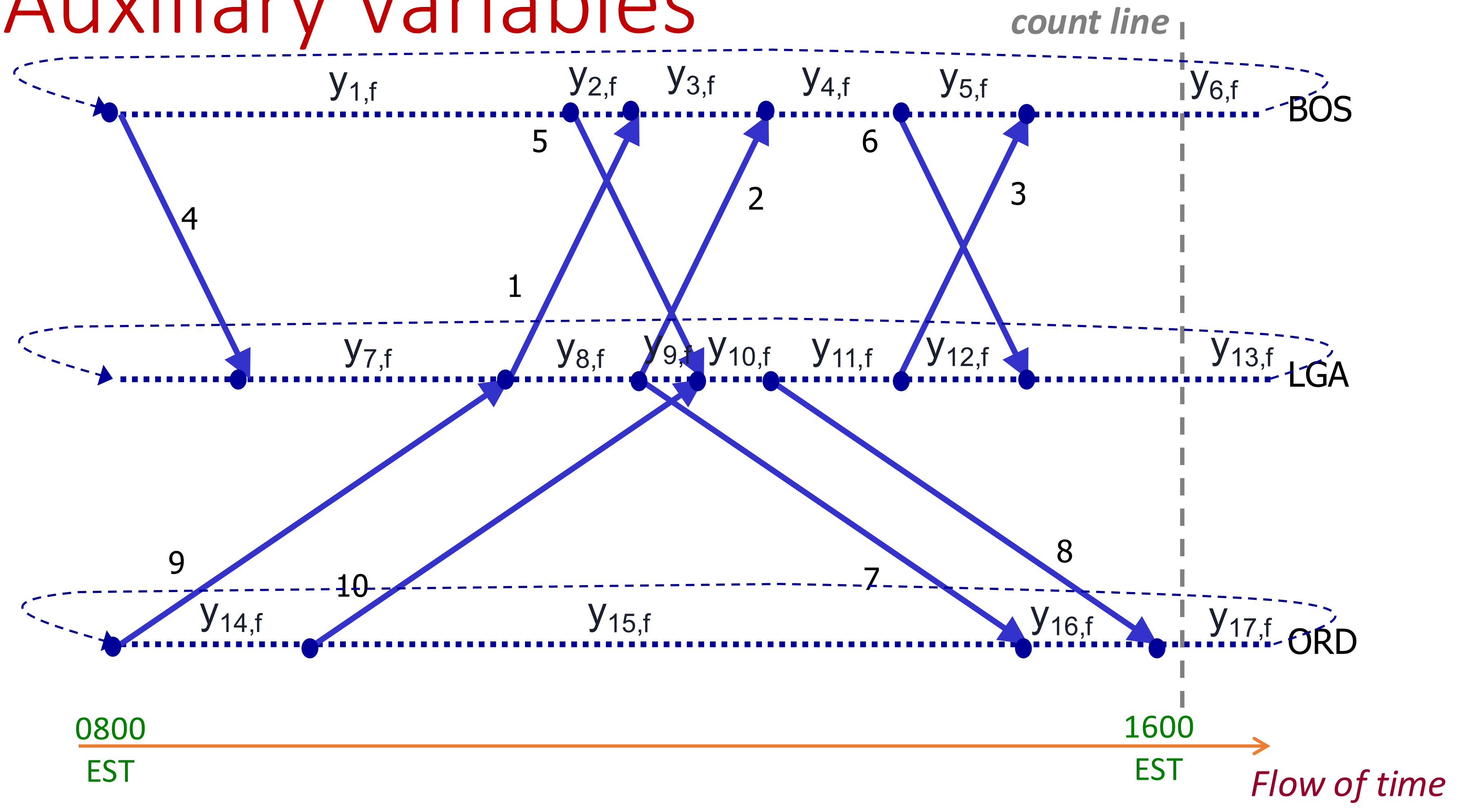
# Time-Line Network




Each ● is a node representing an airport at a specific time of day.


<u>Reference</u>: "The Fleet Assignment Problem: Solving a Large-Scale Integer Program" (1995) by C. A. Hane, C. Barnhart, E. L. Johnson, R. E. Marsten, G. L. Nemhauser, G. Sigismondi, *Mathematical Programming*, **70**, 211-232.

# Auxiliary Variables




Each ● is a node representing an airport at a specific time of day.

# Formulation Constraints – (2/3)

<u>Constraints</u> (cont.):

- Balance
  *make sure the number of inbound airplanes to a node equals the number of outbound planes for each fleet type.*

  For the first node at BOS & fleet type A: $y_{6,A} - x_{4,A} - y_{1,A} = 0$

  For the first node at BOS & fleet type B: $y_{6,B} - x_{4,B} - y_{1,B} = 0$

  and so on for each node and fleet type...

- Non-negativity (on the auxiliary y variables)
  *make sure the number of aircraft on the ground of each fleet type is not negative at any time.*

# Formulation Constraints – (3/3)

Constraints (cont.):

- Count
  *the number of airplanes on the ground and in the air at the "countline" of any given fleet type must not exceed fleet availability.*

$$y_{6,A} + y_{13,A} + y_{17,A} \leq 1$$

$$y_{6,B} + y_{13,B} + y_{17,B} \leq 2$$

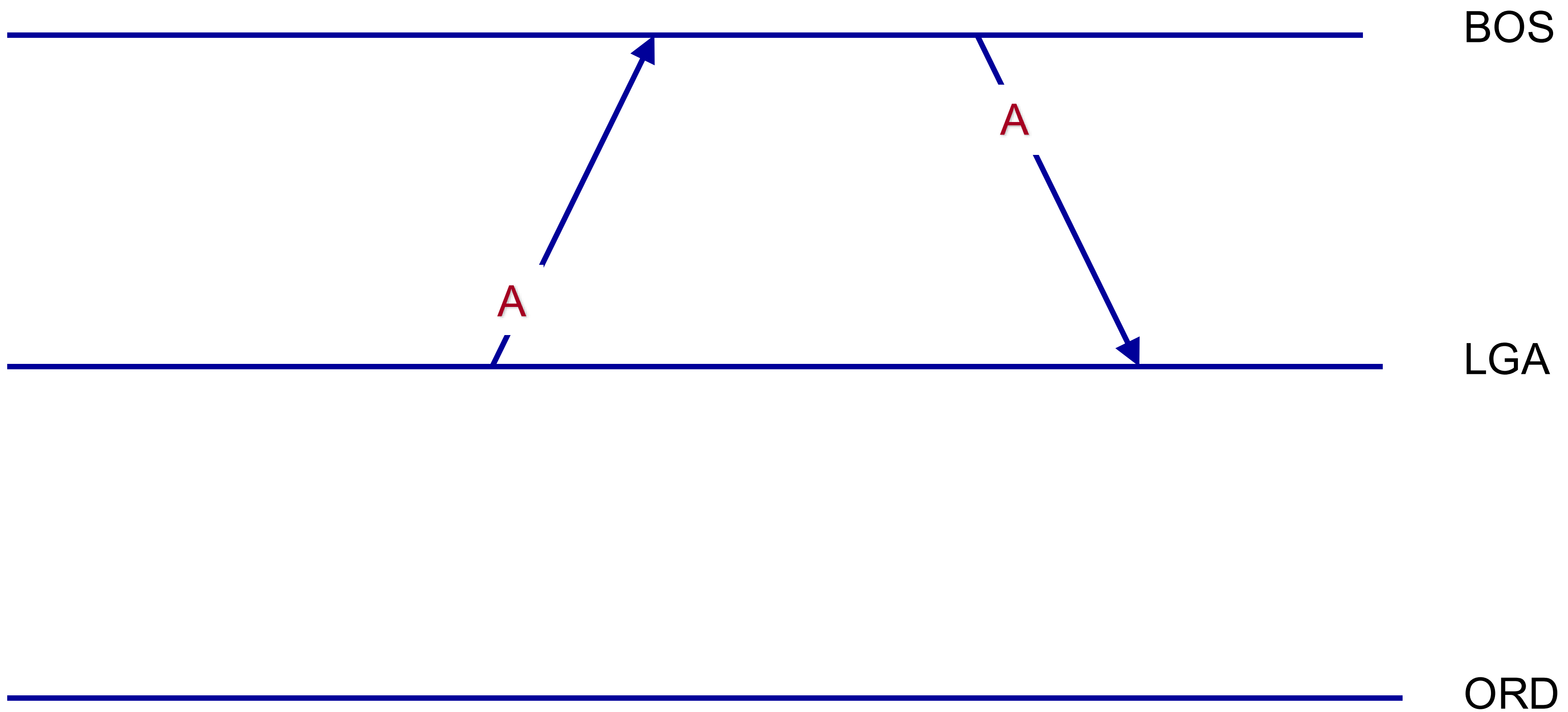
$$y_{6,C} + y_{13,C} + y_{17,C} \leq 2$$

# Formulation Constraints

Constraints:

- Binary
  *Each assignment decision variable must be either 0 or 1*
- Cover
  *make sure each flight is assigned an airplane.*
- Balance
  *make sure the number of inbound airplanes to a node equals the number of outbound planes for each fleet type.*
- Non-negativity (on the auxiliary y variables)
  *make sure the number of aircraft on the ground of each fleet type is not negative at any time.*
- Count
  *the number of airplanes on the ground and in the air at the "countline" of any given fleet type must not exceed fleet availability.*

# Example Solution – Type A Aircraft

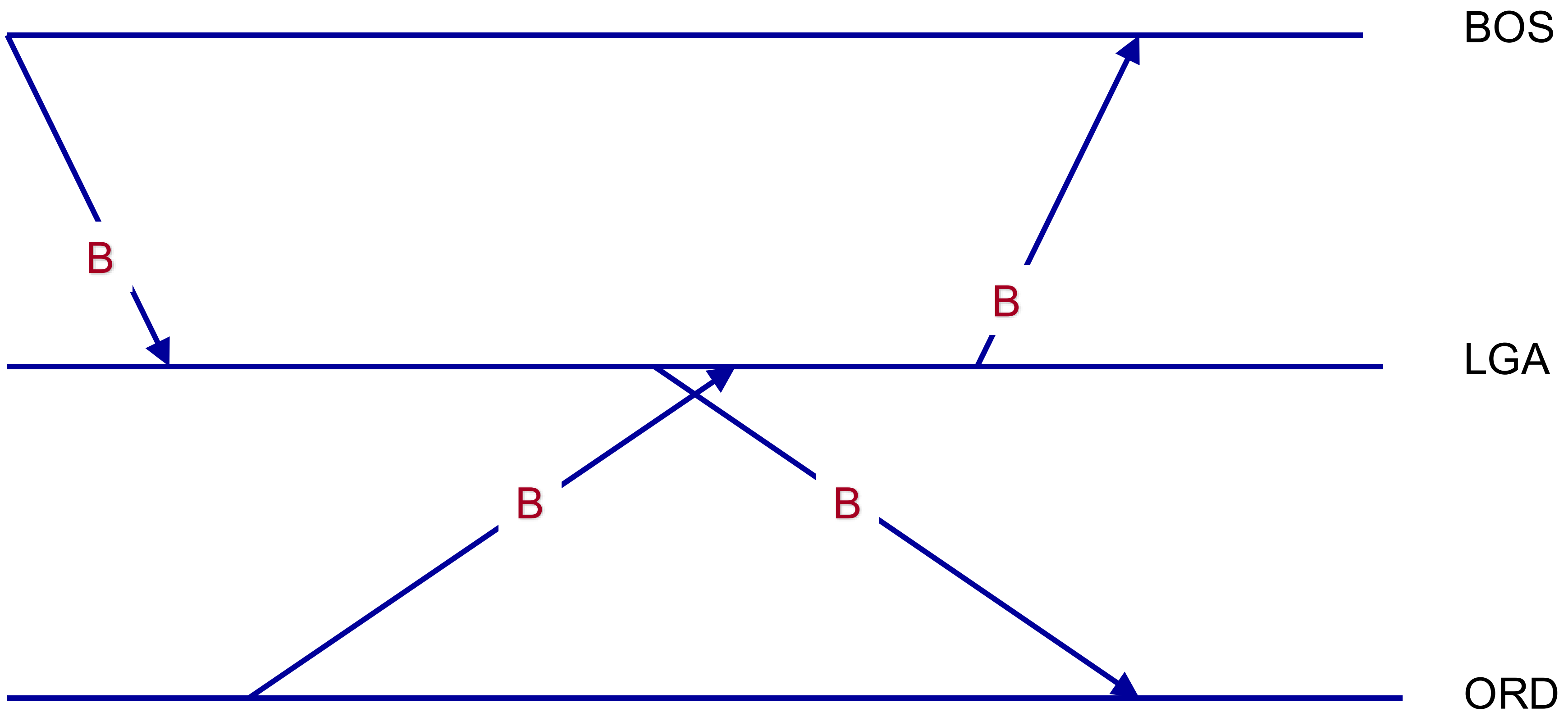



Revenue = $428,500

Cost = $148,000

Profit = $280,500

# Example Solution - Type B Aircraft

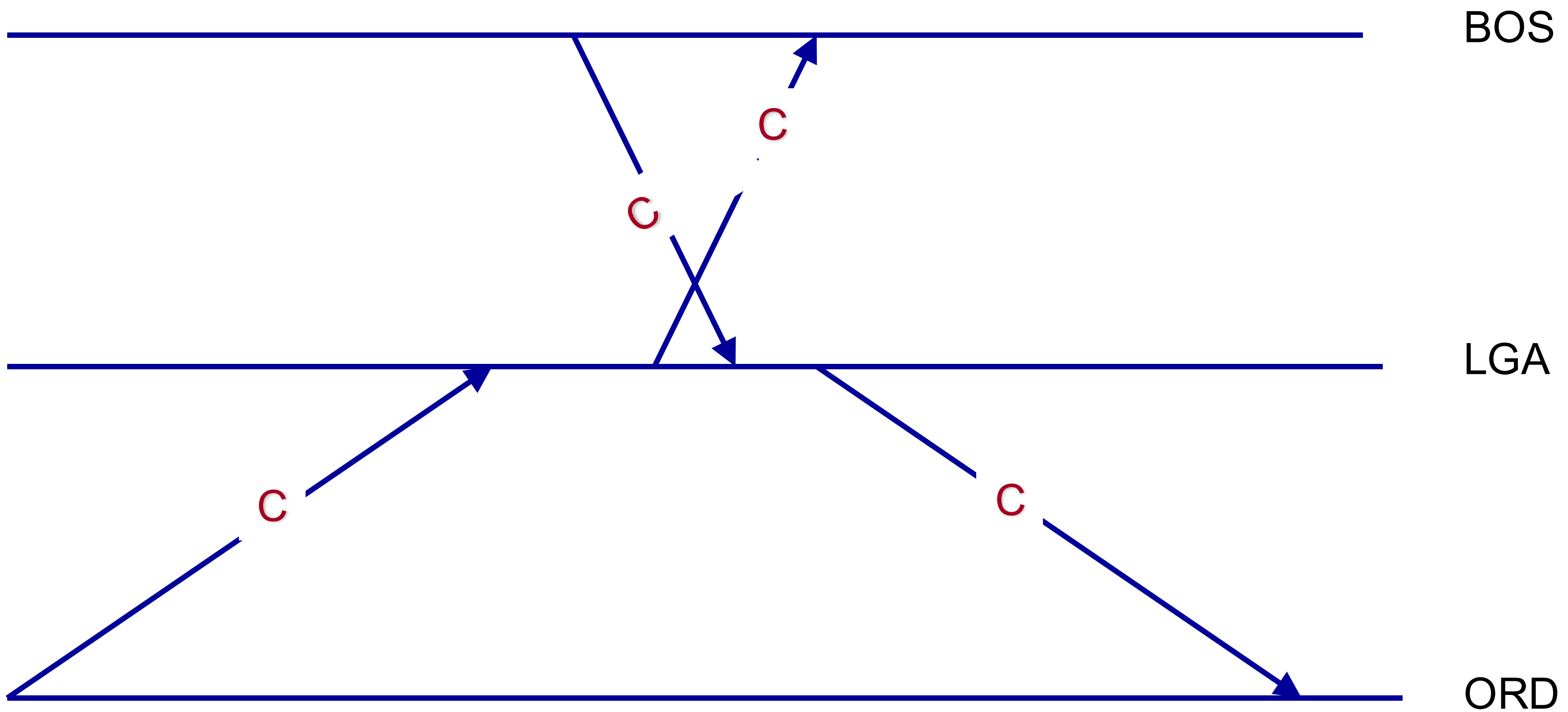



Revenue = $428,500

Cost = $148,000

Profit = $280,500

# Example Solution - Type C Aircraft

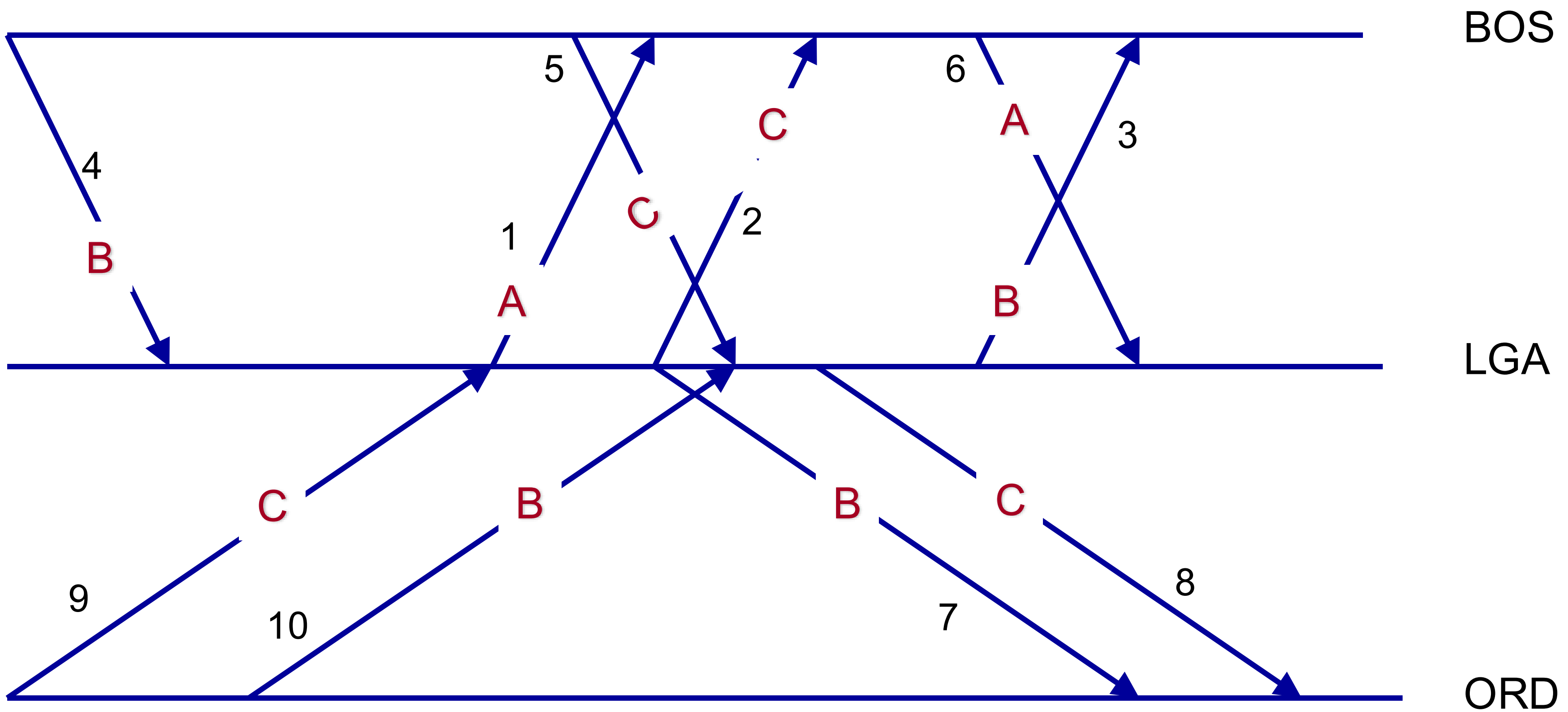



Revenue = $428,500

Cost = $148,000

Profit = $280,500

# Example Solution – All Aircraft Types




Revenue = $428,500

Cost = $148,000

Profit = $280,500

# Computational Experience with Actual Problems

- The "F" problem consists of 2,044 flights serving 76,741 itineraries covering 112 airports using a fleet of 9 aircraft types.
- The "A" problem consists of 1,888 flights serving 75,484 itineraries covering 106 airports using a fleet of 9 aircraft types.

**Table 5.** Comparison of models.

| Run | FAM | SFAM0 | SFAM4 | IFAM (1 hr.) | IFAM (5 hrs.) |
|---|---|---|---|---|---|
| F[0.8] | | | | | |
| Profit change | 0 | 43.234 | 45.834 | 45.834 | 45.834 |
| Solution time | 989 | 1,371 | 1,077 | 1,570 | 1,574 |
| F[1.0] | | | | | |
| Profit change | 0 | 14.55 | 32.922 | 29.333 | 32.993 |
| Solution time | 762 | 925 | 704 | 3,813 | 14,109 |
| F[1.2] | | | | | |
| Profit change | 0 | 32.596 | 41.421 | 28.86 | 28.86 |
| Solution time | 1,331 | 1,004 | 3,838 | 3,821 | 18,237 |
| F[1.5] | | | | | |
| Profit change | 0 | 71.477 | 73.585 | no | 8.979 |
| Solution time | 1,829 | 1,361 | 3,914 | solution | 18,267 |
| A[0.8] | | | | | |
| Profit change | 0 | 4.922 | 5.032 | (2.350) | 2.814 |
| Solution time | 517 | 884 | 1,005 | 3,762 | 18,171 |
| A[1.0] | | | | | |
| Profit change | 0 | 10.764 | 11.335 | (18.266) | (4.945) |
| Solution time | 776 | 930 | 871 | 3,806 | 18,203 |
| A[1.2] | | | | | |
| Profit change | 0 | 17.207 | 25.999 | no | (41.606) |
| Solution time | 877 | 1,007 | 1,676 | solution | 18,189 |
| A[1.5] | | | | | |
| Profit change | 0 | (11.729) | 1.157 | no | (63.785) |
| Solution time | 534 | 1,449 | 2,314 | solution | 18,307 |

*Notes.* Profit changes are reported in [$million/365 days] relative to FAM. Negative numbers are reported in parentheses. Solution times are reported in CPU seconds.

Reference: "Airline Fleet Assignment with Enhanced Revenue Modeling" (2009) by C. Barnhart, A. Farahat, M. Lohatepanont, *Operations Research*, **57**(1), 231-244.