

When we add a new constraint...

Does the original optimal solution satisfy the new constraint?

- **Yes** => The original solution is still optimal. We don't need to resolve.
- **No** => We need to explicitly add the additional constraint and re-solve. The optimal objective value may stay the same or get worse (it cannot improve).

| Model | Optimal Utility | Optimal Course Selection |
|------------------------|-----------------|-----------------------------------|
| Basic | 46 | Bid on A, E, F, G, H, I, J |
| Additional Constraints | 44 | Bid on A, C, E, F, G, H, J |

Additional Constraints

- (A B conflict) $y_A + y_B \leq 1$
- (B or C required) $y_B + y_C \geq 1$
- (E pre-req to H) $y_H \leq y_E$

More challenging: what if H has two pre-requisites?

- If you take H, you must also take **both E and F** (E and F are both pre-requisite to H)

???

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$$y_H \leq y_F$$

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$$y_H \leq y_E + y_F$$

Other examples of IF-THEN constraints

Let's say we have binary variables x , y and z . How would you encode each of the following if-then statements as a *linear* constraint?

- If $x = 1$ and $y = 1$, then z must be 1.

$$z \geq x + y - 1$$

- If either of $x = 0$ or $y = 0$, then z must be 0.

$$z \leq x$$

$$z \leq y$$

- If x and y are different, then z must be 1. Otherwise, no restrictions on z .

$$z \geq x - y$$

$$z \geq y - x$$

Synergy

Applying IF-THEN constraints to model relationships between variables

Course Bidding Formulation - “Synergy Effects”

- Recall the objective function:

$$\text{maximize } 10 y_A + 2 y_B + 4 y_C + 2 y_D + 5 y_E + 4 y_F + 8 y_G + 7 y_H + 6 y_I + 6 y_J$$

- Suppose the utility of taking courses I and J *together* is 20 (greater than their sum of 12) because they allow you to complete the Analytics Certificate.
- How can you incorporate this into your model?

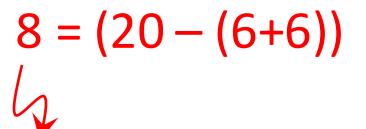
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$$8 = (20 - (6+6))$$


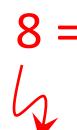
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$$8 = (20 - (6+6))$$


Will this work?

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This is not linear ☹

Making the objective linear – “synergy effects”

maximize ~~$10 y_A + 2 y_B + 4 y_C + 2 y_D + 5 y_E + 4 y_F + 8 y_G + 7 y_H + 6 y_I + 6 y_J + 8 y_I y_J$~~

maximize $10 y_A + 2 y_B + 4 y_C + 2 y_D + 5 y_E + 4 y_F + 8 y_G + 7 y_H + 6 y_I + 6 y_J + 8 z$

We'll create a new variable z and use *linear constraints* to ensure that $z = y_I y_J$

All possible values of y_I and y_J

| y_I | y_J | $y_I y_J$ | z |
|-------|-------|-----------|-----|
| 0 | 0 | 0 | |
| 0 | 1 | 0 | |
| 1 | 0 | 0 | |
| 1 | 1 | 1 | |

Making the objective linear – “synergy effects”

maximize ~~$10 y_A + 2 y_B + 4 y_C + 2 y_D + 5 y_E + 4 y_F + 8 y_G + 7 y_H + 6 y_I + 6 y_J + 8 y_I y_J$~~

maximize $10 y_A + 2 y_B + 4 y_C + 2 y_D + 5 y_E + 4 y_F + 8 y_G + 7 y_H + 6 y_I + 6 y_J + 8 z$

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All possible values of y_I and y_J

| y_I | y_J | $y_I y_J$ | z |
|-------|-------|-----------|-----|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | |

When either $y_I = 0$ or $y_J = 0$, we can enforce that $z = 0$ with the constraints:

- $z \leq y_I$
- $z \leq y_J$

Making the objective linear – “synergy effects”

maximize ~~$10 y_A + 2 y_B + 4 y_C + 2 y_D + 5 y_E + 4 y_F + 8 y_G + 7 y_H + 6 y_I + 6 y_J + 8 y_I y_J$~~

maximize $10 y_A + 2 y_B + 4 y_C + 2 y_D + 5 y_E + 4 y_F + 8 y_G + 7 y_H + 6 y_I + 6 y_J + 8 z$

We'll create a new variable z and use *linear constraints* to ensure that $z = y_I y_J$

All possible values of y_I and y_J

| y_I | y_J | $y_I y_J$ | z |
|-------|-------|-----------|-----|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 |

When either $y_I = 0$ or $y_J = 0$, we can enforce that $z = 0$ with the constraints:

- $z \leq y_I$
- $z \leq y_J$

When $y_I = 1$ and $y_J = 1$, we want to enforce that $z = 1$ with the constraints:

- $z \geq y_I + y_J - 1$

Model with Analytics Certificate Synergy

Maximize: $10 y_A + 2 y_B + 4 y_C + 2 y_D + 5 y_E + 4 y_F + 8 y_G + 7 y_H + 6 y_I + 6 y_J + 8 z$
over variables: y_A, y_B, \dots, y_J and z

Subject To:

(binary) $y_A, y_B, \dots, y_J, z \geq 0, \leq 1$ and integral

(points budget) $200 y_A + 50 y_B + \dots + 100 y_J \leq 1000$

(max credits) $12 y_A + 9 y_B + \dots + 6 y_J \leq 54$

(min credits) $12 y_A + 9 y_B + \dots + 6 y_J \geq 36$

(MW H3 load) $y_A + y_B + y_E + y_G \leq 3$

(MW H4 load) $y_A + y_B + y_G + y_J \leq 3$

(TR H3 load) $y_C + y_D + y_I \leq 3$

(TR H4 load) $y_C + y_D + y_F + y_H + y_I \leq 3$

(A B conflict) $y_A + y_B \leq 1$

(B or C required) $y_B + y_C \geq 1$

(E pre-req to H) $y_H \leq y_E$

Analytics Certificate Synergy

- $z \leq y_I$
- $z \leq y_J$
- $z \geq y_I + y_J - 1$

Synergy Excel Model

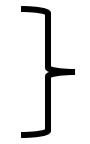
| DECISIONS | A | B | C | D | E | F | G | H | I | J | Z |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Course | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.0 | 1.0 |

OBJECTIVE

| | |
|---------------|----|
| Total utility | 47 |
|---------------|----|

CONSTRAINTS

| | LHS | <input type="checkbox"/> | RHS |
|------------------------------------|-----|--------------------------|-------|
| Points budget | 830 | \leq | 1,000 |
| Course credit maximum | 51 | \leq | 54 |
| Course credit minimum | 51 | \geq | 36 |
| Mon, Wed H3 classes | 3 | \leq | 3 |
| Tue, Thr H3 classes | 2 | \leq | 3 |
| Mon, Wed H4 classes | 3 | \leq | 3 |
| Tue, Thr H4 classes | 2 | \leq | 3 |
| Do not take A and B | 1.0 | \leq | 1 |
| Take either B or C | 1.0 | \geq | 1 |
| H requires E | 0.0 | \leq | 1 |
| If I not chosen, Z is 0 | 1.0 | \leq | 1 |
| If J not chosen, Z is 0 | 1.0 | \leq | 1 |
| If either I or J is chosen, Z is 1 | 1.0 | \geq | 1 |
| Binary constraints | | | |



Constraints that link z with y_I and y_J

Summary



| Model | Optimal Utility | Optimal Course Selection |
|------------------------|-----------------|--|
| Basic | 46 | Bid on A, E, F, G, H, I, J |
| Additional Constraints | 44 | Bid on A, C, E, F, G, H, J |
| Synergy Effect | 47 | Bid on A, C, E, G, I, J (+ certificate) |

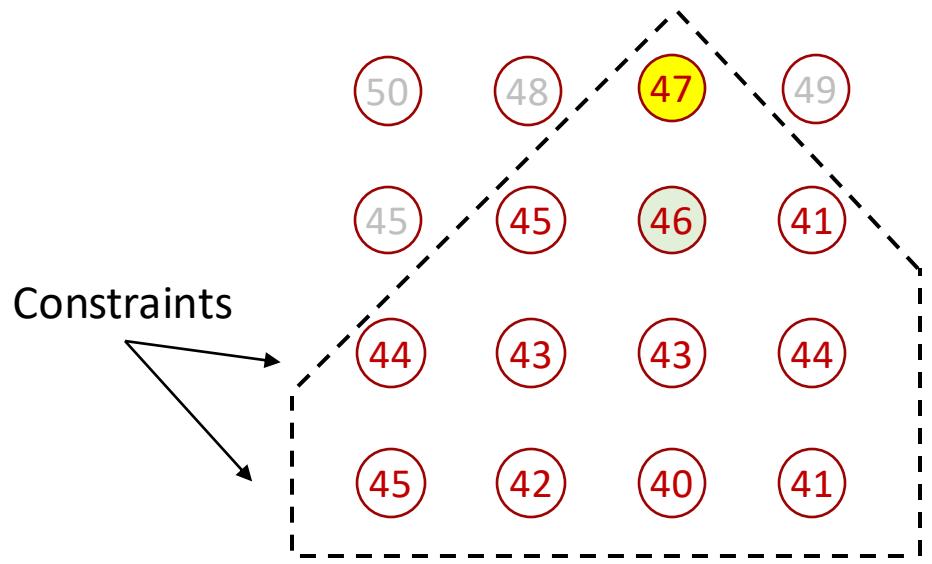
Note: Objective value increased because we
changed the objective function with the new
analytics certificate.

Excluding AN optimal solution

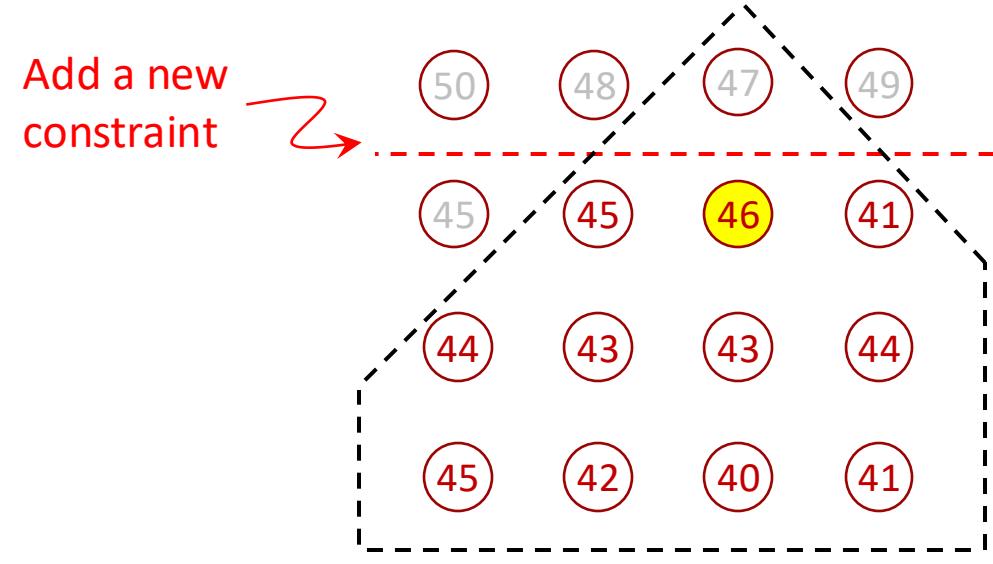
Excluding the optimal solution

- The model tells you to bid for / enroll in classes **A, C, E, G, I, J** for a total utility of 47.
- For some reason, you are not happy with this course selection. You can't really express it as a constraint.
- You would like to exclude A, C, E, G, I and J, but find the best one from all the remaining ones
- **Approach:** Add a constraint which makes **A, C, E, G, I, J** infeasible, but does not exclude any other solutions.

Excluding the optimal solution

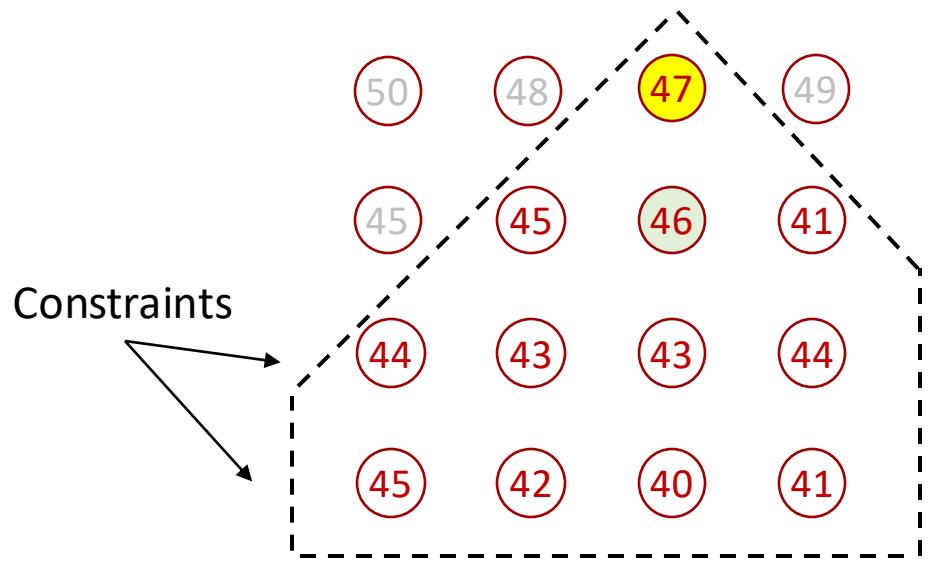


- Each circle represents a solution, while the dotted region are the constraints.
- The number inside represents the objective value. 47 is the best objective value, 46 is the second-best

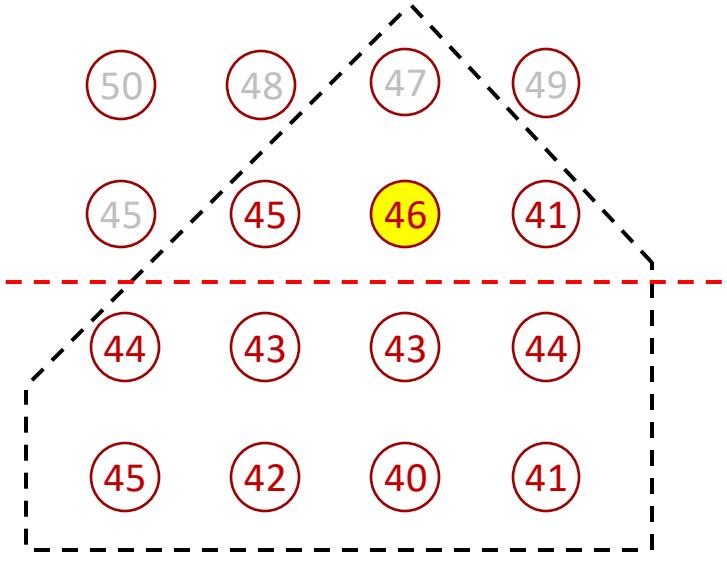


- If we can add a constraint that **only cuts off this one solution**, then re-solving the optimization will give us the best of the rest

Excluding the optimal solution



Constraint that
excludes TOO MUCH



- Each circle represents a solution, while the dotted region are the constraints.
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- If this new constraint excludes more than just the solution we want to eliminate, then there's a **chance that the second-optimal solution is also excluded**.