LECTURE: CHAPTER 11 REVIEW (PART 2)

Section 11.8 - Power Series

Example 1: Find the radius of convergence and the interval of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n(x+2)^n}{n4^n}$.

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x+2)^{n+1}}{(n+1)\;4^{n+1}}\;\frac{n\,4^n}{(-1)^n(x+2)^n}\right| = \lim_{n\to\infty}\frac{|x+2|}{4}\frac{n}{(n+1)}$$

$$= \frac{|x+2|}{4}$$

converges if $\frac{|x+2|}{4} < 1 \Rightarrow |x+2| < 4$ so $\boxed{R=4}$

or $-4 < x+2 < 4 \Rightarrow -6 < x < 2$

at $x=-6$, $\sum_{n=1}^{\infty}\frac{(-1)^n(-6+2)^n}{n\,4^n} = \sum_{n=1}^{\infty}\frac{(-1)^n(-4)^n}{n\,4^n} = \sum_{n=0}^{\infty}\frac{1}{n}$ div., harmonic

at $x=2$, $\sum_{n=1}^{\infty}\frac{(-1)^n\,4^n}{n\,4^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ conv. by AST

$\boxed{\text{so IOC is } (-6,2]}$

Example 2: Find the interval of convergence of the following series.

(a) $\sum_{n=0}^{\infty} n!(x-2)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,(x-2)^{n+1}}{n!\,(x-2)^n}\right|$$

$$= \lim_{n\to\infty}(n+1)|x-2|$$

$$= \boxed{\infty}$$

Thus $R=0$ and the series only converges at its center $x=2$, so IOC is $\boxed{\{2\}}$

(b) $\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(x-2)^{n+1}}{(n+1)!}\;\frac{n!}{3^n(x-2)^n}\right|$$

$$= \lim_{n\to\infty}\frac{3\,|x-2|}{n+1}$$

$= 0 < 1$ for all x

Thus $\boxed{R=\infty}$ and $\boxed{\text{IOC is } \mathbb{R}}$ or $\boxed{(-\infty,\infty)}$

Section 11.9 - Representations of Functions by Power Series

Example 3: Find the Maclaurin series for f and its radius of convergence. Then find $\frac{d}{dx}$ for part (a) and $\int f(x)dx$ for (b) and the radii of convergence.

(a) $f(x) = \dfrac{x^2}{1+x^5}$

$$= \frac{x^2}{1-(-x^5)}$$

$$= \sum_{n=0}^{\infty} x^2(-x^5)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^2 x^{5n}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n x^{5n+2}}$$

$$\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n x^{5n+2} = \boxed{\sum_{n=0}^{\infty}(-1)^n(5n+2)x^{5n+1}}$$

converges if $|-x^5| < 1 \Rightarrow |x|^5 < 1$

$\Rightarrow |x| < 1$

$\boxed{R=1}$

(b) $f(x) = \dfrac{4}{x^3+8}$

$$= \frac{4}{8+x^3}$$

$$= \frac{4/8}{1-(-x^3/8)}$$

$$= \sum_{n=0}^{\infty} \frac{4}{8}\left(-\frac{x^3}{8}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\frac{(-1)^n x^{3n}}{2^{3n}}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{2^{3n+1}}}$$

$$\int \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{2^{3n+1}}\,dx = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{2^{3n+1}(3n+1)} + C}$$

Example 4: Find the power series for $f(x) = \tan^{-1} x$ using an integral or a derivative.

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

so $\tan^{-1}x = \displaystyle\int \frac{1}{1+x^2}\,dx$

$$= \int \frac{1}{1-(-x^2)}\,dx$$

$$= \int \sum_{n=0}^{\infty}(-x^2)^n\,dx$$

$$= \int \sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} + C$$

$\tan^{-1}(0) = C \Rightarrow C = 0$

So $\boxed{\tan^{-1}x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}}$

converges if $|-x^2| < 1$

$|x| < 1$

So $\boxed{R=1}$

Section 11.10 - Taylor and Maclaurin Series

Write the Maclaurin series an the interval of convergence for each of the following functions.

- $1/(1-x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \qquad R = 1$
- $e^x = \sum_{n=0}^{\infty} x^n/n! = 1 + x + x^2/2! + x^3/3! + \dots \qquad R = \infty$
- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - x^3/3! + x^5/5! - x^7/7! + \dots \qquad R = \infty$
- $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - x^2/2! + x^4/4! - x^6/6! + \dots \qquad R = \infty$
- $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)} = x - x^3/3 + x^5/5 - x^7/7 + \dots \qquad R = \infty$

Example 5: Find the Taylor series of $f(x) = e^{2x}$ at $a = 6$.

$f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

$f''(x) = 2^2 e^{2x}$

$f'''(x) = 2^3 e^{2x}$

$$f(x) = f(6) + f'(6)(x-6) + \frac{f''(6)}{2!}(x-6)^2 + \frac{f'''(6)}{3!}(x-6)^3 + \dots$$

$$= e^{12} + 2e^{12}(x-6) + \frac{2^2 e^{12}(x-6)^2}{2!} + \frac{2^3 e^{12}(x-6)^3}{3!} + \dots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{e^{12}\, 2^n (x-6)^n}{n!}}$$

Example 6: Find the Maclaurin series for f and its radius of convergence.

(a) $f(x) = \tan^{-1}(x^3)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{2n+1}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{2n+1}}$$

converges if $|x^3| < 1$

or $|x| < 1$

so $\boxed{R = 1}$

(b) $f(x) = xe^{3x}$

$$= x \cdot \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$

$$= x \sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{3^n x^{n+1}}{n!}}$$

$|3x| < \infty \Rightarrow |x| < \infty$

so $\boxed{R = \infty}$

Example 7: find the Maclaurin series for f and its radius of convergence.

(a) $f(x) = \sin\left(\frac{x^4}{2}\right)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x^4}{2}\right)^{2n+1}}{(2n+1)!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+4}}{2^{2n+1}(2n+1)!}}$$

$$\boxed{R = \infty}$$

(b) $f(x) = 10^x$

$$= (e^{\ln 10})^x$$

$$= e^{\ln 10 x}$$

$$= \sum_{n=0}^{\infty} \frac{(\ln 10\ x)^n}{n!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(\ln 10)^n x^n}{n!}}$$

$$\boxed{R = \infty}$$

Example 8: Evaluate $\int \frac{e^x}{x} dx$ as an infinite series.

$$\int \frac{e^x}{x} dx = \int \frac{1}{x} \sum_{n=0}^{\infty} \frac{x^n}{n!} dx$$

$$= \int \sum_{n=0}^{\infty} \frac{x^{n-1}}{n!} dx$$

$$= \int \left(x^{-1} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}\right) dx$$

$$= \boxed{\ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!} + C}$$

Example 9: Find the sum of the following series.

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{3^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi^{1/2})^{2n}}{3^{2n}(2n)!}$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\sqrt{\pi}/3\right)^{2n} / (2n)!$$

$$= \boxed{\cos\left(\sqrt{\pi}/3\right)}$$

(b) $1 - e + \frac{e^2}{2!} - \frac{e^3}{3!} - \cdots$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n e^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-e)^n}{n!}$$

$$= \boxed{e^{-e}}$$

$$= \boxed{1/e^e}$$