Math 651 – Supplementary homework exercises
Simplicial homology

1. The n-ball $B^n = \{z \in \mathbb{R}^n \mid ||z|| \leq 1\}$ is triangularized as the n-simplex. Use this to compute all simplicial homology groups for B^2. In doing so, give complete arguments to determine all cycles and boundaries.

2. Consider the Figure-8 space X as a simplicial complex of 5 vertices and 6 edges forming 2 triangles with a vertex in common.

   (a) Compute the simplicial homology groups of X, and give representatives of generators for each.

   (b) If you"glue" on a 2-simplex to X to fill in one of the triangles, yielding a new space X', some homology groups might change. Compute them.

   (c) If a complex X is composed only of n-simplices for $n < m$, and one or more $m + 1$ simplices is glued on so their boundaries are already in X, which homology groups *might* change? Which *cannot* change? Explain.

3. Consider the "parachute" Z formed by identifying the 3 corners of a 2-simplex. As described, this is *not* a simplicial complex.

   (a) Give a triangularization of it that is a simplicial complex (so each simplex is uniquely determined by distinct vertices).

   (b) Use your triangularization to compute homology groups.

   (c) Explain informally why this space has as a deformation retract the Figure-8. Use this to determine its fundamental group $G = \pi_1(Z, z_0)$ and verify that $H_1(Z) \equiv G/[G, G]$.

4. The Torus and Figure-8 have different fundamental groups π_1, but the same first homology group H_1. Nonetheless they can be distinguished by homology. How?