MATH202X-F01-UX1

Fall 2019

Final Exam

Name: Answer Key

Instructions. You have 120 minutes. No calculators allowed. *Show all your work* in order to receive full credit.

1. A triangle has the following vertices

$$A = (2, 2, 1),\ B = (1, -1, 2),\ C = (0, 0, 0).$$

(a) Find the cosine of the angle at vertex B in the triangle. Is the angle acute, obtuse, or right?

Solution:

$\vec{BC} = \langle -3, 1, -5\rangle$ and $\vec{BA} = \langle 1, -2, -1\rangle - \langle 3, -1, 2\rangle = \langle -2, -1, -3\rangle$. Thus $\vec{BC} \cdot \vec{BA} = (-3)(-2) + (1)(-1) + (-5)(-3) = 20 > 0$, so the angle is acute.

(b) What is the area of the triangle ABC? (*Hint:* The area of a triangle is half of that of a parallelogram.)

Solution:

$\vec{BA} \times \vec{BC} = \langle 8, -1, -5\rangle$ so the area is $||\vec{BA} \times \vec{BC}|| = \sqrt{64 + 1 + 25} = \sqrt{90} = 3\sqrt{10}$.

(c) Give the equation of the plane containing ABC.

Solution:

(d) Find the intersection of the plane containing ABC and the line through $(6, 0, 3)$ with direction $\mathbf{v} = \langle 2, -1, 1\rangle$.

Solution:

Using the normal $\vec{BA} \times \vec{BC}$, and point C, gives $8x - 1y - 5(z + 3) = 0$, or $8x - y - 5z = 15$.

2. Use the chain rule to find $\dfrac{\partial f}{\partial s}$ where $f(x, y, z) = xy^2 + 2z$ and $(x, y, z) = (s - t, st, e^t)$. The final answer should be in terms of s and t only.

Solution:

$$\begin{aligned}\frac{\partial f}{\partial u} &= f_x x_u + f_y y_u + f_z z_u = y^2(1) + 2xy(v) + 2(0)\\ &= (uv)^2 + 2(u - v)(uv)v = u^2v^2 + 2(u - v)uv^2 = 3u^2v^2 - 2uv^3\\ &= uv^2(3u - 2v)\end{aligned}$$

3. Consider the parameterized surface:

$$\mathbf{r}(u, v) = \left\langle 2u, v, u^2 \right\rangle \qquad , \qquad 0 \le u \le 1\ ,\ 0 \le v \le u.$$

Find the surface area of $\mathbf{r}(u, v)$.

Solution:

4. Find and classify all the critical points of

$$f(x, y) = \frac{1}{2}x^2 - xy + \frac{1}{3}y^3.$$

Solution: The function f is continuously differentiable and so the only critical points correspond to $\nabla f = \vec{0}$:

$$\nabla f = \left\langle x-y, -x+y^2 \right\rangle = \vec{0} \quad \Rightarrow \quad \begin{cases} x = y \\ -y+y^2 = 0 \end{cases} \quad \Rightarrow \quad y = 0 \text{ or } 1$$

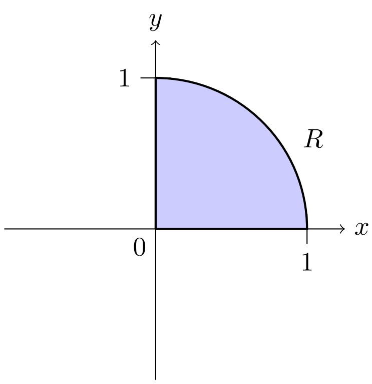
so the critical points are $(0,0)$ and $(1,1)$. Since $f_{xx} = 1$, $f_{yy} = 2y$ and $f_{xy} = -1$, we have:

$$d(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 2y - 1.$$

Thus,

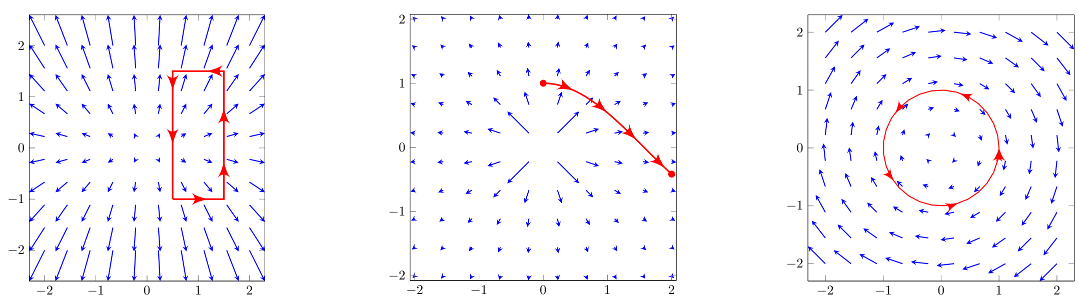
- $d(0,0) < 0$ so $(0,0)$ is a saddle point;
- $d(1,1) > 0$ and $f_{xx}(1,1) > 0$ so $(1,1)$ is a relative minimum.

5. Find the center of mass of the planar lamina with density $\rho(x,y) = 4y$ described by the region R illustrated below. You may use that the mass of the lamina is $m = \frac{4}{3}$ without computing it.



Solution:

6. Consider the following sketches of vector fields (rescaled).



(a) $\mathbf{F_1} = \langle x, 2y \rangle$ (b) $\mathbf{F_2} = \left\langle \dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2} \right\rangle$ (c) $\mathbf{F_3} = \langle y, -x \rangle$

Indicate whether work done by the vector field on a particle moving along the given curve will be positive, negative or zero. Briefly justify your choices.

(a)

(b)

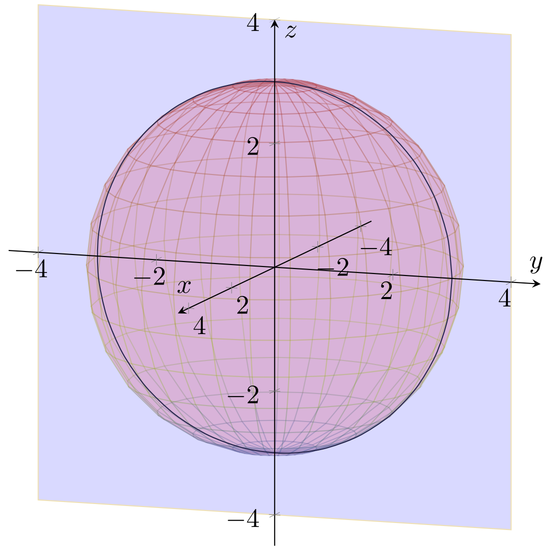
(c)

7. Consider the surfaces given by:

$$S_1 : \theta = \frac{\pi}{2} \quad \text{(cylindrical)} \qquad , \qquad S_2 : \rho = 3 \quad \text{(spherical)}$$

(a) Sketch and describe the surfaces and their intersection.

Solution: The surface S_1 is the yz-plane whereas S_2 is a sphere of radius 3 centered at the origin. Their intersection is a circle centered at the origin of radius 3.



(b) Parametrize the surface S_2 giving bounds on the parameters.

Solution: $S_2 : \quad \vec{r}(u, v) = \langle 3\cos u \sin v, 3\sin u \sin v, 3\cos v\rangle \quad , \quad 0 \le u \le 2\pi \ , \ 0 \le v \le \pi$

(c) Parametrize the intersection of S_1 and S_2 giving bounds.

Solution: $S_1 \cap S_2$: taking $u = \frac{\pi}{2}$: $\vec{r}(v) = \langle 0, 3\sin v, 3\cos v\rangle \quad , \quad 0 \le v \le 2\pi$.

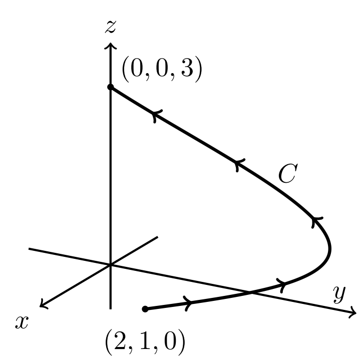
8. A force field is given by

$$\mathbf{F}(x, y, z) = \langle 2xy, x^2 + z\sin(yz), y\sin(yz) + 2z - 1\rangle.$$

(a) This field is conservative. Find *all* potential functions for it.

Solution:
$f(x, y, z) = x^2y - \cos(yz) + z^2 - z + C.$

(b) How much work is done by $\mathbf{F}$ as a particle moves along the path C:



Solution:

We need to compute $W = \int_C \mathbf{F} \cdot d\mathbf{r}$. Since $\mathbf{F} = \nabla f$ and C starts at $\mathbf{r}(0) = (2, 0, 0)$ and ends at $\mathbf{r}(2\pi) = (2, 0, 2\pi)$, by the FTLI,

$$W = f(2,0,2\pi) - f(2,0,0) = (-\cos(0) + (2\pi)^2 - 2\pi) - (-\cos(0)) \\ = 4\pi^2 - 2\pi.$$

9. Reverse the order of integration of the following.

$$\int_0^{16} \int_{x/4}^{\sqrt{x}} f(x,y)\, dy\, dx$$

Solution:

10. Suppose the temperature at a point (x, y) is given by

$$T = 100 - 0.5x^2 - y^2,$$

with x, y measured in meters, and T in degrees, and you are currently at the point $(3, 2)$. The positive x-axis points east, and the positive y-axis points north.

(a) If you walk northwest, will the temperature increase or decrease? At what rate?

Solution:

$$\nabla z|_{(60,40)} = \langle -.001x, -.002y \rangle|_{(60,40)} = \langle -.06, -.08 \rangle.$$

In the direction $\mathbf{u} = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle$, the directional derivative is

$$\langle -.06, -.08 \rangle \cdot \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle = \frac{.06 - .08}{\sqrt{2}} = -\frac{.02}{\sqrt{2}}$$

Thus I would descend at a rate of $\frac{.02}{\sqrt{2}}$ meters/meter.

(b) In which direction does the temperature increase most rapidly? At what rate?

Solution:

The slope is greatest in the gradient direction $\langle -.06, -.08 \rangle$. The rate of ascent in that direction is $||\langle -.06, -.08 \rangle|| = \sqrt{(-.06)^2 + (-.08)^2} = \sqrt{.01} = .1$ meters/meter.

11. Sketch the curve $\mathbf{r}(t) = \left\langle \cos t, \sin t, \frac{t}{2} \right\rangle$ for $0 \le t \le 6\pi$ and compute its length.

Solution:

12. Let the surface S be that part of the paraboloid $z = 4 - x^2 - y^2$ above the plane $z = 0$.

(a) Calculate the flux of $\mathbf{F} = \langle x, y, 0 \rangle$, through the surface S oriented using the upward pointing normals.

Solution:

$$\begin{aligned}
\mathbf{r}_u &= \langle \cos v, \sin v, -2u \rangle \\
\mathbf{r}_v &= \langle -u \sin v, u \cos v, 0 \rangle \\
\mathbf{r}_u \times \mathbf{r}_v &= \langle 2u^2 \cos v, 2u^2 \sin v, u \rangle \\
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^2 \langle u \cos v, u \sin v, 0 \rangle \cdot \langle 2u^2 \cos v, 2u^2 \sin v, u \rangle\, du\, dv \\
&= \int_0^{2\pi} \int_0^2 2u^3 \cos^2 v + 2u^3 \sin^2 v\, du\, dv \\
&= \int_0^{2\pi} \int_0^2 2u^3\, du\, dv = 16\pi
\end{aligned}$$

(b) Using the Divergence Theorem, calculate the flux of $\mathbf{F}$ through the closed surface given by S together with a disc in the plane $z = 0$. Recall that the Divergence Theorem states that, for appropriate S and Q,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_Q \operatorname{div} \mathbf{F}\, dV.$$

Solution:

$$\begin{aligned}\iint_{S'} \mathbf{F} \cdot d\mathbf{S} = \iiint_Q \nabla \cdot \mathbf{F} dV = \iiint_Q 2dV \\ = 2\int_0^{2\pi}\int_0^2\int_0^{4-r^2} r\,dz\,dr\,d\theta = 4\pi\int_0^2 (4-r^2)r\,dr \\ = 4\pi\left(2r^2 - \frac{r^4}{4}\right)\Big|_0^2 = 16\pi\end{aligned}$$

13. An object moves in space with velocity $\mathbf{v}(t) = \langle \sin t, 2\cos t, 1\rangle$. At $t = 0$, it has position $\langle 0, 0, -1\rangle$. Find the object's trajectory as a function of time.

Solution:
Since $\mathbf{r}''(t) = \langle \sin t, 2\cos t, 1\rangle$,

$$\mathbf{r}'(t) = \langle -\cos t + c, 2\sin t + d, t + e\rangle.$$

Since $\mathbf{r}'(0) = \langle 0, 0, -1\rangle$,

$$\mathbf{r}'(t) = \langle -\cos t + 1, 2\sin t, t - 1\rangle,$$

so

$$\mathbf{r}(t) = \langle -\sin t + t + f, -2\cos t + g, (1/2)t^2 - t + h\rangle.$$

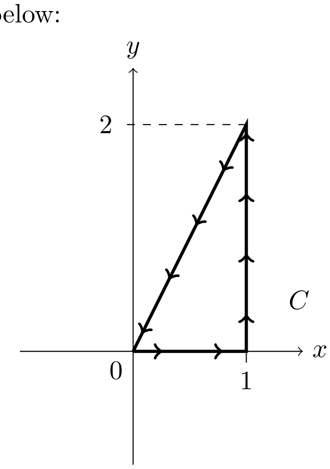
Since $\mathbf{r}(0) = \langle 0, 1, -4\rangle$,

$$\mathbf{r}(t) = \langle -\sin t + t, -2\cos t + 3, (1/2)t^2 - t - 4\rangle.$$

14. Throughout this problem, we consider the following line integral:

$$I = \oint_C (x-y)dx + xy^2dy$$

where C is the closed curve shown below:



(a) Use Green's theorem to evaluate the line integral I.

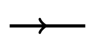
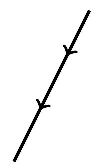
Solution: The curve C encloses a simply bounded region R such that:

$$I = \iint_R \left(xy^2\right)_x - (x-y)_y \, dA = \iint_R y^2 + 1 \, dA = \int_0^1 \int_0^{2x} y^2 + 1 \, dy \, dx$$
$$= \int_0^1 \left[\frac{y^3}{3} + y\right]_0^{2x} dx = \int_0^1 \frac{8x^3}{3} + 2x \, dx = \frac{2x^4}{3} + x^2 \Big|_0^1 = \frac{5}{3}.$$

(b) Use a parametrization to fully set up $\int_{C'} (x-y)\, dx + xy^2 \, dy$ where C' is the diagonal segment of C. Do NOT evaluate.

Solution: Split I into three parts:

$$I = I_1 + I_2 + I_3 \quad \text{where} \quad I_k = \int_{C_k} (x-y)dx + xy^2 dy \quad \text{and}$$

C_1 :  ; C_2 :  ; C_3 : 

Along C_1, $0 \le x \le 1$, $y = 0$, $dy = 0$ so

$$I_1 = \int_0^1 (x - 0)\, dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}.$$

Along C_2, $x = 1$, $dx = 0$, $0 \le y \le 2$ so

$$I_2 = \int_0^2 1 \cdot y^2 \, dy = \left[\frac{y^3}{3}\right]_0^2 = \frac{8}{3}.$$

Along C_3, x decreases from 1 to 0, $y = 2x$, $dy = 2dx$ so

$$I_3 = \int_1^0 (x - 2x)\, dx + x(2x)^2\,(2dx) = \int_1^0 \left(-x + 8x^3\right) dx = \frac{-x^2}{2} + 2x^4 \Big|_1^0 = \frac{1}{2} - 2 = -\frac{3}{2}.$$

Therefore,

$$I = \frac{1}{2} + \frac{8}{3} - \frac{3}{2} = \frac{8}{3} - 1 = \frac{5}{3}.$$