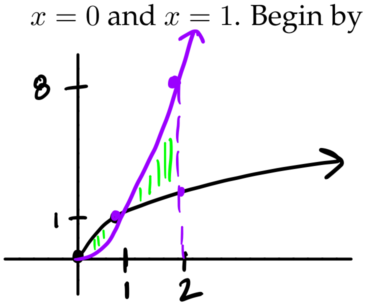
Chapter 6 and 8 Review

6.1 Areas Between Curves

Example 1: Set up, but do not solve, an integral (or integrals) that finds the area between $y = \sqrt{x}$, $y = x^3$, $x = 0$ and $x = 1$. Begin by graphing the region bounded by these curves.

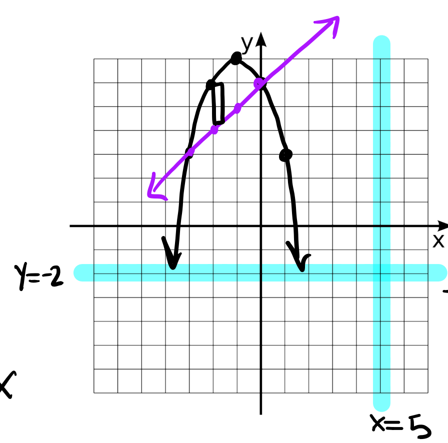


$$\boxed{A = \int_0^1 (\sqrt{x} - x^3)\,dx + \int_1^2 (x^3 - \sqrt{x})\,dx}$$

6.2 The Disc Method & 6.3 The Shell Method

- When we use the **disc method** of Section 6.2 we slice Perpendicular to the axis of rotation. Here, we need to find the area of a general slice, $A(x)$ or $A(y)$ and sum over all of our slices.
- When you use the **shell method** of Section 6.3 we slice parallel to the axis of rotation. Here you need to find the radius and the height of a slice.

Example 2: Sketch the region bounded by the curves $y = 6 - 2x - x^2$ and $y = x + 6$. Then, **SET UP BUT DO NOT SOLVE** an integral that would find the volume when this region is rotated about the following axes. Specify the method you use.



$y = 6 - 2x - x^2$

x	y
0	6
1	6-2-1=3
-1	6+2-1=7
-2	6+4-4=6
-3	6+6-9=3

(a) x-axis Slice // → $\boxed{\text{DISC}}$

$$\boxed{V = \pi \int_{-3}^{0} \left((6-2x-x^2)^2 - (x+6)^2\right) dx}$$

(b) y-axis Slice ⊥ to y-axis $\boxed{\text{Shell}}$

$$V = 2\pi \int_{-3}^{0} (-x)(6-2x-x^2-(x+6))\,dx$$

$$\boxed{V = 2\pi \int_{-3}^{0} (-x)(-3x-x^2)\,dx}$$

(c) $y = -2$ Slice // to axis $\boxed{\text{DISC}}$

$$V = \pi \int_{-3}^{0} (6-2x-x^2+2)^2 - (x+6+2)^2\,dx$$

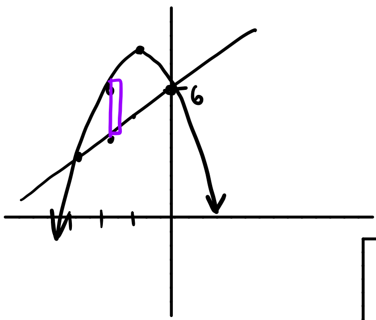
$$\boxed{V = \pi \int_{-3}^{0} (8-2x-x^2)^2 - (x+8)^2\,dx}$$

(d) $x = 5$ Slice ⊥ to axis $\boxed{\text{SHELL}}$

$$V = 2\pi \int_{-3}^{0} (5-x)(6-2x-x^2-(x+6))\,dx$$

$$\boxed{V = 2\pi \int_{-3}^{0} (5-x)(-3x-x^2)\,dx}$$

Example 3: Consider the region bounded by the curves $y = 6 - 2x - x^2$ and $y = x + 6$ from Example 2. Suppose that this region is the base of a solid and the cross sections of your solid are rectangles with height 3 times the base. Set up, but do not solve, an integral that gives the volume of this solid.



$A(x) = \text{base} * \text{height}$

$= \text{base} * 3\,\text{base}$

$= 3(6 - 2x - x^2 - (6 + x))^2$

$= 3(-3x - x^2)^2$

$$\boxed{V(x) = \int_{-3}^{0} 3(-3x - x^2)^2\,dx}$$

6.4 Work

Example 4: A force of 100 N is required to hold a spring stretched from its natural length of 10 cm to a length of 20 cm. How much work is done in stretching the spring from 20 cm to 30 cm? (0.1 m to 0.2 m)

convert to m → 20 cm is 10 cm (0.1 m) past rest

① get force equation

$f(x) = kx$

$100 = k(0.1)$

$100 = k(1/10)$

$k = 1000$

② get work → watch the bounds!

$W = \int_{0.1}^{0.2} 1000x\,dx \quad = \boxed{15\text{ J}}$

$= 500x^2 \Big|_{1/10}^{2/10}$

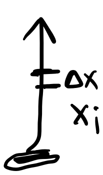
$= 500(4/100 - 1/100)$

$= 500(3/100)$

Example 5: A heavy rope, 50 feet long, weighs 100 lbs (or 2 a lb per foot) and is lying on the ground. How much work is done in pulling the rope so that it is hanging completely stretched out?

$W_i = x_i\text{ ft } 2\text{ lb/ft } \Delta x\text{ ft}$

$W = \int_0^{50} 2x\,dx$



$= x^2 \Big|_0^{50}$

$= \boxed{2500\text{ ft-lbs}}$

6.5 Average Value

Example 6: n a certain city, the temperature (in $^\circ$ F) t hours after 9 AM is modeled by the function $T(t) = 50 + 20\sin\left(\frac{\pi t}{12}\right)$.

$t=0$ to $t=12$

(a) Find the average temperature during the period from 9 AM to 9 PM. Give an exact answer and an answer rounded to the nearest thousandth.

$$T_{ave} = \frac{1}{12}\int_0^{12}\left(50 + 20\sin\left(\tfrac{\pi}{12}t\right)\right)dt$$

$$= \frac{1}{12}\left(50t - \frac{240}{\pi}\cos\left(\tfrac{\pi}{12}t\right)\right)\Big|_0^{12}$$

$$= \frac{1}{12}\left(\left(50 * 12 - \frac{240}{\pi}\cos\pi\right) - \left(0 - \frac{240}{\pi}\cos 0\right)\right)$$

$$= 50 + 20/\pi + 20/\pi = \boxed{50 + 40/\pi \ {}^\circ F}$$

(b) Explain why the Mean Value for integrals applies to the equation for $T(t)$ on any interval $[a, b]$ and find the time $t = c$, such that $T(c)$ equals the average value from part (a). Give an exact and approximate answer with proper units.

The MVT for integrals applies as $T(c)$ is continuous

Find where: $T_{ave} = T(c)$

$$50 + 40/\pi = 50 + 20\sin\left(\tfrac{\pi}{12}t\right)$$

$$40/\pi = 20\sin\left(\tfrac{\pi}{12}t\right)$$

$$2/\pi = \sin\left(\pi/12\, t\right)$$

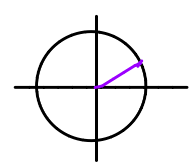
$$\sin^{-1}(2/\pi) = \pi/12\, t$$

$$\boxed{t = \frac{12}{\pi}\sin^{-1}(2/\pi) \text{ hrs}}$$

$\boxed{\text{after 9 AM}}$

8.1 Arc Length

Example 7: Find the arc length of the curve $y = 1 + \ln(\cos x)$, $0 \le x \le \pi/6$.



$$\boxed{y' = \frac{-\sin x}{\cos x}}$$

$$L = \int_a^b \sqrt{1 + (y')^2}\, dx$$

$$= \int_0^{\pi/6} \sqrt{1 + (-\tan x)^2}\, dx$$

$$= \int_0^{\pi/6} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_0^{\pi/6} \sqrt{\sec^2 x}\, dx$$

$$= \int_0^{\pi/6} \sec x\, dx$$

$$= \ln|\sec x + \tan x|\,\Big|_0^{\pi/6}$$

$$= \ln\left|\frac{1}{(\sqrt{3}/2)} + \frac{1/2}{\sqrt{3}/2}\right| - \ln|1 + 0|$$

$$= \ln\left|2/\sqrt{3} + 1/\sqrt{3}\right|$$

$$= \ln\left|3/\sqrt{3}\right|$$

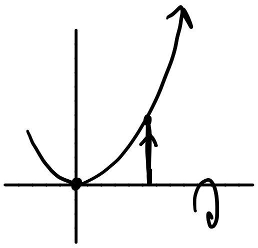
$$= \boxed{\ln\sqrt{3}}$$

8.2 Surface Area

Example 8: The curve $y = x^2$ is rotated about the following axes. Set up an integral that finds the area of the resulting surface for $0 \le x \le 2$. $y' = 2x$

(a) x-axis.

$$SA = 2\pi \int y\, ds$$

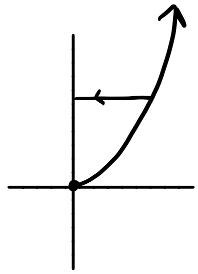


$$= 2\pi \int_0^2 x^2 \sqrt{1+(2x)^2}\, dx$$

$$= \boxed{2\pi \int_0^2 x^2 \sqrt{1+4x^2}\, dx}$$

(b) y-axis.

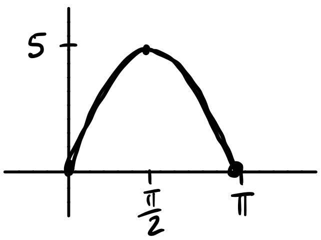
$$SA = 2\pi \int x\, ds$$



$$= \boxed{2\pi \int_0^2 x \sqrt{1+4x^2}\, dx}$$

8-3 Moments, Centers of Mass and Centroids

Example 9: Find the centroid of the region bounded by $y = 5\sin x$ and $y = 0$ for $0 \le x \le \pi$.



$$A = \int_0^{\pi} 5\sin x\, dx = -5\cos x \Big|_0^{\pi} = -5(\cos\pi - \cos 0) = -5(-1-1) = 10$$

$$\bar{x} = \frac{1}{A}\int_a^b x\,(f(x)-g(x))\,dx$$

$$= \frac{5}{10}\int_0^{\pi} x\sin x\, dx \quad \begin{bmatrix} u = x & v = -\cos x \\ du = dx & dv = \sin x\, dx \end{bmatrix}$$

$$= \frac{1}{2}\left(-x\cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x\, dx\right)$$

$$= \frac{1}{2}\left(-\pi\cos\pi + 0 + \sin x \Big|_0^{\pi}\right)$$

$$= \frac{1}{2}(\pi + \sin\pi - \sin 0)$$

$$= \boxed{\pi/2} \leftarrow$$ you can als say $\bar{x} = \pi/2$ by symmetry

$$\bar{y} = \frac{1}{2A}\int_a^b (f(x))^2 - (g(x))^2\, dx$$

$$= \frac{1}{20}\int_0^{\pi} 25\sin^2 x\, dx$$

$$= \frac{5}{4}\cdot\frac{1}{2}\int_0^{\pi}(1-\cos 2x)\,dx$$

$$= \frac{5}{8}\left(x - \frac{1}{2}\sin 2x\right)\Big|_0^{\pi}$$

$$= \frac{5}{8}\left(\pi - \tfrac{1}{2}\sin 2\pi - \left(0 - \tfrac{1}{2}\sin 0\right)\right)$$

$$= \boxed{\frac{5\pi}{8}}$$