Section 7.1 - Integration by Parts

Example 1: Evaluate the following integrals.

(a) $\int x\cos(2x)dx$ $= uv - \int v\,du$

$u = x$ $\quad v = \frac{1}{2}\sin(2x)$
$du = dx$ $\quad dv = \cos(2x)dx$

$= \frac{1}{2}x\sin(2x) - \frac{1}{2}\int \sin(2x)\,dx$

$= \frac{1}{2}x\sin(2x) - \frac{1}{2}\left(-\frac{1}{2}\cos(2x)\right) + C$

$= \boxed{\frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C}$

(b) $\int \frac{\ln x}{x^3}dx$ $= \int x^{-3}\ln x\,dx$

$u = \ln x$ $\quad v = -\frac{1}{2}x^{-2}$
$du = \frac{1}{x}dx$ $\quad dv = x^{-3}dx$

$= -\frac{1}{2}x^{-2}\ln x + \frac{1}{2}\int x^{-2}\cdot\frac{1}{x}\,dx$

$= -\frac{\ln x}{2x^2} + \frac{1}{2}\int x^{-3}\,dx$

$= \frac{-\ln x}{2x^2} + \frac{1}{2}\frac{x^{-2}}{(-2)} + C$

$= \boxed{\frac{-\ln x}{2x^2} - \frac{1}{4x^2} + C}$

LONER $\longrightarrow$ (c) $\int \arctan(3x)dx$ $= x\arctan(3x) - \int \frac{3x}{1+9x^2}\,dx$

$u = \arctan(3x)$ $\quad v = x$
$du = \frac{3}{1+9x^2}dx$ $\quad dv = dx$

$= x\arctan(3x) - \int \frac{3x}{w}\frac{dw}{18x}$

substitute:
$w = 1+9x^2$
$dw = 18x\,dx$
$\frac{dw}{18x} = dx$

$= x\arctan(3x) - \frac{1}{6}\int \frac{1}{w}\,dw$

$= x\arctan(3x) - \frac{1}{6}\ln|w| + C$

$= \boxed{x\arctan(3x) - \frac{1}{6}\ln|1+9x^2| + C}$

Section 7.2 - Trigonometric Integrals

Example 2: Evaluate the following integrals.

(a) $\int_0^{\pi/2} \sin^5\theta\, d\theta = \int_0^{\pi/2} \sin^4\theta \sin\theta\, d\theta$

$u=\cos\theta$
$du=-\sin\theta d\theta$
$x=0,\ u=1$
$x=\pi/2,\ u=0$

$= \int_0^{\pi/2} (1-\cos^2\theta)^2 \sin\theta\, d\theta$

$= -\int_1^0 (1-u^2)^2 du$

$= \int_0^1 (1-2u^2+u^4)\, du$

$= \left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right)\Big|_0^1$

$= 1 - 2/3 + 1/5$

$= 15/15 - 10/15 + 3/15$

$= \boxed{8/15}$ ✓

(b) $\int \sin^4(5x)dx = \int (\sin^2(5x))^2 dx$

$= \int \left(\frac{1}{2}(1-\cos(10x))\right)^2 dx$

$= \frac{1}{4}\int \left(1 - 2\cos(10x) + \cos^2(10x)\right) dx$

$= \frac{1}{4}\int (1 - 2\cos(10x) + \frac{1}{2}(1+\cos 20x))\, dx$

$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos(10x) + \frac{1}{2}\cos(20x)\right) dx$

$= \boxed{\frac{1}{4}\left(\frac{3}{2}x - \frac{1}{5}\sin(10x) + \frac{1}{40}\sin(20x)\right) + C}$ ✓

Example 3: Evaluate the following integrals.

(a) $\int \tan^5\theta \sec^3\theta\, d\theta = \int \tan^4\theta \sec^2\theta \sec\theta\tan\theta\, d\theta$

$u=\sec\theta$
$du=\sec\theta\tan\theta d\theta$

$= \int (\sec^2\theta - 1)^2 \sec^2\theta \sec\theta\tan\theta\, d\theta$

$= \int (u^2-1)^2 u^2\, du$

$= \int (u^4 - 2u^2 + 1) u^2\, du$

$= \int (u^6 - 2u^4 + u^2)\, du$ ✓

$= \boxed{\frac{1}{7}\sec^7\theta - \frac{2}{5}\sec^5\theta + \frac{1}{3}\sec^3\theta + C}$

(b) $\int \tan^2\theta \sec^4\theta\, d\theta = \int \tan^2\theta \sec^2\theta \sec^2\theta\, d\theta$

$u=\tan\theta$
$du=\sec^2\theta d\theta$

$= \int \tan^2\theta(\tan^2\theta+1)\sec^2\theta\, d\theta$

$= \int u^2(u^2+1)\, du$

$= \int (u^4 + u^2)\, du$

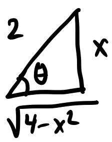
$= \boxed{\frac{1}{5}\tan^5\theta + \frac{1}{3}\tan^3\theta + C}$ ✓

Section 7.3 - Trigonometric Substitution

Example 4: Evaluate $\int \frac{x^2}{(4-x^2)^{3/2}}dx$ $= \int \frac{4\sin^2\theta \cdot 2\cos\theta}{\left(\sqrt{4-4\sin^2\theta}\right)^3}\,d\theta$

$x = 2\sin\theta$
$dx = 2\cos\theta\,d\theta$

$\frac{x}{2} = \sin\theta$



$$= 8\int \frac{\sin^2\theta\cos\theta}{\left(\sqrt{4(1-\sin^2\theta)}\right)^3}\,d\theta$$

$$= 8\int \frac{\sin^2\theta\cos\theta}{(2\cos\theta)^3}\,d\theta$$

$$= \frac{8}{8}\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta$$

$$= \int \tan^2\theta\,d\theta$$

$$= \int (\sec^2\theta - 1)\,d\theta$$

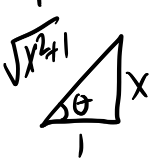
$$= \tan\theta - \theta + C$$

$$= \boxed{\frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C}$$

Example 5: Evaluate $\int \frac{1}{(x^2+1)^2}dx$ $= \int \frac{\sec^2\theta}{(\tan^2\theta+1)^2}\,d\theta$

$x = \tan\theta$
$dx = \sec^2\theta\,d\theta$

$\frac{x}{1} = \tan\theta$



$$= \int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$$

$$= \int \frac{1}{\sec^2\theta}\,d\theta$$

$$= \int \cos^2\theta\,d\theta$$

$$= \frac{1}{2}\int (1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\cdot 2\sin\theta\cos\theta\right) + C$$

$$= \frac{1}{2}\left(\tan^{-1}x + \frac{x}{\sqrt{x^2+1}}\cdot\frac{1}{\sqrt{x^2+1}}\right) + C$$

$$= \boxed{\frac{1}{2}\left(\tan^{-1}x + \frac{x}{x^2+1}\right)_3 + C}$$

Section 7.4 - Integration by Parts

Example 6: Give the partial fraction decomposition for the following.

(a) $\dfrac{x^2+4}{x(x^2-4)} = \dfrac{x^2+4}{x(x+2)(x-2)}$

factors

$= \boxed{\dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-2}}$

(b) $\dfrac{x^2+4}{x^2(x-4)}$

$= \boxed{\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-4}}$

(c) $\dfrac{x^2+1}{x(x^2+4)}$

does not factor

$= \boxed{\dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}}$

Example 7: Evaluate $\displaystyle\int \frac{3x^2-2}{x^2-2x-8}dx = \int \left(3 + \frac{6x+22}{x^2-2x-8}\right)dx$

① long divide:

$$\begin{array}{r} 3 \\ x^2-2x-8 \,\big)\, 3x^2+0x-2 \\ -(3x^2-6x-24) \\ \hline 6x+22 \end{array}$$

② Partial Fraction

$$\left[\frac{6x+22}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}\right](x-4)(x+2)$$

$6x+22 = A(x+2) + B(x-4)$

$x=-2 \Rightarrow -12+22 = -6B \Rightarrow 10 = -6B$
$B = -5/3$

$x=4 \Rightarrow 24+22 = 6A \Rightarrow 46 = 6A \Rightarrow A = 23/3$

$$= \int \left(3 + \frac{(23/3)}{x-4} + \frac{(-5/3)}{x+2}\right)dx$$

$$= \boxed{3x + \frac{23}{3}\ln|x-4| - \frac{5}{3}\ln|x+2| + C}$$

Example 8: Evaluate $\displaystyle\int \frac{5x^2+3x-2}{x^3+2x^2}dx = \int \left(\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\right)dx$

$\dfrac{5x^2+3x-2}{x^2(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+2}$

$5x^2+3x-2 = Ax(x+2) + B(x+2) + Cx^2$

$x=0 \Rightarrow -2 = 2B \Rightarrow \boxed{B=-1}$

$x=-2 \Rightarrow 20-6-2 = 4C \Rightarrow 12 = 4C$

$\boxed{C=3}$

$x=1 \Rightarrow 5+3-2 = 3A + 3B + C$

$6 = 3A - 3 + 3$

$6 = 3A$

$\boxed{A=2}$

$$= \int \left(\frac{2}{x} - x^{-2} + \frac{3}{x+2}\right)dx$$

$$= 2\ln|x| - \frac{x^{-1}}{(-1)} + 3\ln|x+2| + C$$

$$= \boxed{2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C}$$

Section 7.7 - Approximate Integration

Example 11: Approximate $\int_0^4 \sqrt{x}\cos x dx$ using $n = 4$ and the: $\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$

(a) Trapezoid Rule

$$T_4 = \frac{\Delta x}{2}\left(f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)\right)$$
$$= \tfrac{1}{2}\left(0 + 2\cos 1 + 2\sqrt{2}\cos 2 + 2\sqrt{3}\cos 3 + 2\cos(4)\right)$$
$$\approx \boxed{-2.417}$$

(b) Midpoint Rule

$$M_4 = \Delta x\left(f(0.5) + f(1.5) + f(2.5) + f(3.5)\right)$$
$$= 1\left(\sqrt{0.5}\cos(0.5) + \sqrt{1.5}\cos(1.5) + \sqrt{2.5}\cos(2.5) + \sqrt{3.5}\cos(3.5)\right)$$
$$\approx \boxed{-2.311}$$

(c) Simpson's Rule

$$S_4 = \frac{\Delta x}{3}\left(f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)\right)$$
$$= \tfrac{1}{3}\left(0 + 4\cos(1) + 2\sqrt{2}\cos(2) + 4\sqrt{3}\cos(3) + 2\cos(4)\right)$$
$$\approx \boxed{-2.394}$$

Section 7.8 - Improper Integrals

Example 12: Evaluate the integral or show that it is divergent.

(a) $\int_1^\infty x^3 e^{-x^4} dx = \lim_{b\to\infty} \int_1^b x^3 e^{-x^4} dx$

$u = -x^4$
$du = -4x^3 dx$
$\frac{du}{-4x^3} = dx$
$x=1,\ u=-1$
$x=b,\ u=-b^4$

$$= \lim_{b\to\infty} \int_{-1}^{-b^4} x^3 e^u \left(\frac{du}{-4x^3}\right)$$
$$= \lim_{b\to\infty} \int_{-1}^{-b^4} \left(-\tfrac{1}{4}e^u\right) du$$
$$= \lim_{b\to\infty} -\tfrac{1}{4}e^u \Big|_{-1}^{-b^4}$$
$$= \lim_{b\to\infty} \left(-\tfrac{1}{4}e^{-b^4} + \tfrac{1}{4}e^{-1}\right)$$
$$= \lim_{b\to\infty} \left(-\frac{1}{4e^{b^4}} + \frac{1}{4e}\right)$$
$$= \boxed{\frac{1}{4e}}$$

The integral converges to $1/4e$

(b) $\int_0^1 \frac{x-1}{\sqrt{x}} = \lim_{b\to 0^+} \int_b^1 \left(\frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right) dx$

$$= \lim_{b\to 0^+} \int_b^1 \left(x^{1/2} - x^{-1/2}\right) dx$$
$$= \lim_{b\to 0^+} \left(\tfrac{2}{3}x^{3/2} - 2x^{1/2}\right)\Big|_b^1$$
$$= \lim_{b\to 0^+} \left(\left(\tfrac{2}{3} - 2\right) - \left(\tfrac{2}{3}b^{3/2} - 2\sqrt{b}\right)\right)$$
$$= \tfrac{2}{3} - 2$$
$$= \tfrac{2}{3} - \tfrac{6}{3}$$
$$= \boxed{-4/3}$$

The integral converges to $-4/3$

Suggested Problems: Pg 537 # 5-15 odds, 23, 33, 41-49 odds, 63, 65