Chapter 6

6. a) If x denotes the state at the root, and y the state at the other internal node, then of the 16 terms there is 1 ($x = \mathtt{G}, y = \mathtt{G}$) that is $(.25)(.9)^4$, 3 ($x = \mathtt{A}, \mathtt{C}, \mathtt{T}, y = \mathtt{G}$) that are $(.24)(.9)^2(.1/3)^2$, 6 ($x = \mathtt{G}, y = \mathtt{A}, \mathtt{C}, \mathtt{T}$ and $x = y = \mathtt{A}, \mathtt{C}, \mathtt{T}$) that are $(.24)(.9)(.1/3)^3$, and 6 (all others) that are $(.24)(.1/3)^4$. The sum is 0.16475185185.

   b) Of the 16 terms there are 2 ($x = \mathtt{G}, \mathtt{T}, y = \mathtt{G}$) that are $(.25)(.9)^3(.1/3)$, 3 ($x = y = \mathtt{T}$ and $x = \mathtt{A}, \mathtt{C}, y = \mathtt{G}$) that is $(.25)(.9)^2(.1/3)^2$, 4 ($x = \mathtt{T}, y = \mathtt{A}, y = \mathtt{C}$ and $x = y = \mathtt{C}, \mathtt{G}$) that are $(.24)(.9)(.1/3)^3$, and 7 (all others) that are $(.24)(.1/3)^4$. The sum is 0.012858950617284.

   c) Of the 16 terms there is 1 ($x = \mathtt{G}, y = \mathtt{G}$) that is $(.25)(.9)^3(.1/3)$, 2 ($x = \mathtt{G}, \mathtt{T}, y = \mathtt{T}$) that is $(.25)(.9)^2(.1/3)^2$, 4 ($x = \mathtt{A}, \mathtt{T}, y = \mathtt{C}$ and $x = \mathtt{C}, \mathtt{T}, y = \mathtt{A}$) that are $(.25)(.1/3)^4$, and 9 (all others) that are $(.24)(.9)^1(.1/3)^3$. The sum is 0.006601234567901.

10. Since the rooted tree has $n-1$ internal vertices, and $2n-2$ edges, there are 4^{n-1} terms, each of which is a product of $2n-1$ parameters.

17. To show $e^{Qt}\mathbf{u} = \mathbf{u}$, note that $Q\mathbf{u} = \mathbf{0}$, so

$$\begin{aligned} e^{Qt}\mathbf{u} &= (I + Qt + \frac{1}{2}Q^2t^2 + \frac{1}{3!}Q^3t^3 + \frac{1}{4!}Q^4t^4 + \dots)\mathbf{u} \\ &= I\mathbf{u} + tQ\mathbf{u} + \frac{1}{2}t^2Q^2\mathbf{u} + \frac{1}{3!}t^3Q^3\mathbf{u} + \frac{1}{4!}t^4Q^4\mathbf{u} + \dots \\ &= \mathbf{u} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \dots \\ &= \mathbf{u} \end{aligned}$$

27. Let $\mathbf{u}$ denote a column vector of 1s. Then using equation (6.12) we have

$$\mathbf{p}M = \mathbf{u}^T \operatorname{diag}(\mathbf{p})M = \mathbf{u}^T M^T \operatorname{diag}(\mathbf{p}) = (M\mathbf{u})^T \operatorname{diag}(\mathbf{p}) = \mathbf{u}^T \operatorname{diag}(\mathbf{p}) = \mathbf{p}.$$

30. b) If the entries of Q are q_{ij}, then

$$-\operatorname{Tr}(\operatorname{diag}(\mathbf{p})Q) = -\sum_{i=1}^{4} p_i q_{ii}.$$

   Since $-q_{ii}$ is the rate at which state i changes into a different state, is just a weighted average of the rates of change of the various states, weighted by the frequency at which those states occur. Alternately, since the p_i are probabilities, it is the expected value of the rate of change of a randomly chosen state.

   If this value is c, replacing Q by $\frac{1}{c}Q$ rescales so it will be 1.