Section 11.1 - Sequences

The Big Question: How do you tell whether a *sequence* converges or diverges??

Find $\lim_{n\to\infty} a_n$. If this limit is a number the sequence converges to that number. If $\lim_{n\to\infty} a_n = \pm\infty$ or DNE the seq. diverges.

Example 1: Determine whether the sequence is convergent or divergent. If it is convergent find its limit.

(a) $a_n = \dfrac{2n^2+1}{3n^2+2}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(2n^2+1)\,1/n^2}{(3n^2+2)\,1/n^2}$$

$$= \lim_{n\to\infty} \frac{2+1/n^2}{3+2/n^2}$$

$$= 2/3$$

$\boxed{\text{So } a_n \text{ converges to } 2/3}$

(b) $a_n = \dfrac{n}{n^2+1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n \cdot 1/n}{(n^2+1)\,1/n}$$

$$= \lim_{n\to\infty} \frac{1}{n+1/n}$$

$$= \boxed{0}$$

$\boxed{a_n \text{ converges to } 0}$

Example 2: Determine whether the sequence is convergent or divergent. If it is convergent find its limit.

(a) $a_n = \dfrac{n}{\ln n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\ln n}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{1}{1/n}$$

$$= \lim_{n\to\infty} n$$

$$= \infty$$

$\boxed{\text{Thus } a_n \text{ diverges}}$

(b) $a_n = \dfrac{n^3}{n^2+1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(n^3)\,1/n^2}{(n^2+1)\,1/n^2}$$

$$= \lim_{n\to\infty} \frac{n}{1+1/n^2}$$

$$= \infty$$

$\boxed{\text{Thus } a_n \text{ diverges}}$

Section 11.2 - Series

The Big Question: How do you tell if a *series* diverges?

Given $\sum a_n$ if $\lim_{n\to\infty} a_n \neq 0$ the series diverges

The Next Big Question: Suppose you have a series $\sum a_n$. What if I tell you that $a_n \to 0$ but $s_n \to 5$, where s_n is the n-th partial sum. What can you say about the convergence/ divergence of $\sum a_n$?

You can say $\sum a_n = 5$. Remember, $s_n = a_1 + a_2 + \cdots + a_n$, if $\lim_{n\to\infty} s_n = 5$ this means you are adding all a_n's, so you found the sum.

Question: In this section you learned how to find the **exact** sum for two different types of series. What are these types and how do you find the sum?

Geo: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ if $|r|<1$

diverges if $|r|>1$

Telescoping: find S_n and then take $\lim_{n\to\infty} S_n$

Example 3: Find the sum of the series.

(a) $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{3n}}$ $= \sum_{n=0}^{\infty} \frac{(-3)^{n+1-1}}{2^{3(n+1)}}$

$= \sum_{n=0}^{\infty} \frac{(-3)^n}{2^{3n} \cdot 2^3}$

$= \sum_{n=0}^{\infty} \frac{1}{8}\left(\frac{-3}{8}\right)^n$

$= \frac{1/8}{1+3/8}$

$= \frac{1}{8+3}$

$= \boxed{\frac{1}{11}}$

(b) $\sum_{n=1}^{\infty} (e^{1/n} - e^{1/(n+2)})$ telescoping

$S_n = (e - e^{1/3}) + (e^{1/2} - e^{1/4}) + (e^{1/3} - e^{1/5}) + \cdots$

$\cdot + (e^{1/n-2} - e^{1/n}) + (e^{1/n-1} - e^{1/n+1}) + (e^{1/n} - e^{1/n+2})$

$S_n = e + e^{1/2} - e^{1/n+1} - e^{1/n+2}$

$S = \lim_{n\to\infty} S_n$

$= \lim_{n\to\infty} (e + \sqrt{e} - e^{1/n+1} - e^{1/n+2})$

$= e + \sqrt{e} - e^0 - e^0$

$= \boxed{e + \sqrt{e} - 2}$

Example 4: Determine whether $\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n+1}\right)$ is convergent or divergent.

As $\lim_{n\to\infty} \ln\left(\frac{n}{3n+1}\right) = \ln\left(\lim_{n\to\infty} \frac{n}{3n+1}\right)$

$= \ln\left(\lim_{n\to\infty} \frac{1}{3+1/n}\right)$

$= \ln(1/3) \neq 0$

Since $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges by the test for divergence.

Section 11.3 - The Integral Test and p-Series

Example 5: Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ is convergent or divergent.

Let $f(x) = \frac{1}{x\sqrt{\ln x}}$, note $f(x)$ is cts, pos, + dec on $[2,\infty)$.

Now $\int_2^\infty \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x\sqrt{\ln x}}\,dx$ $\left[\begin{array}{l} u=\ln x \\ du = 1/x\,dx \end{array}\right]$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} u^{-1/2}\,dx$$

$$= \lim_{b\to\infty} 2u^{1/2} \Big|_{\ln 2}^{\ln b}$$

$$= \lim_{b\to\infty} 2\sqrt{\ln b} - 2\sqrt{\ln 2} = \infty$$

As the integral diverges, so does the series.

Example 6: Estimate $\sum_{n=1}^{\infty} \frac{1}{n^6}$ using s_5. What is the error in this estimate?

$s_5 = 1/1^6 + 1/2^6 + 1/3^6 + 1/4^6 + 1/5^6 \approx \boxed{1.017305}$

$R_5 \le \int_5^\infty 1/x^6\,dx = \lim_{b\to\infty} \int_5^b x^{-6}\,dx = \lim_{b\to\infty} -\frac{1}{5}x^{-5}\Big|_5^b$

$$= \lim_{b\to\infty}\left(-\frac{1}{5b^5} + \frac{1}{5\cdot 5^5}\right) = \boxed{\frac{1}{5^6}}$$

Section 11.4 - The Comparison Tests

Example 7: Determine whether $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+1}}$ the series is convergent of divergent.

LCT w/ $a_n = \frac{n}{\sqrt{n^4+1}}$, $b_n = \frac{n}{\sqrt{n^4}} = \frac{n}{n^2} = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^4+1}} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{n^2}{\sqrt{n^4+1}} \cdot \frac{1/n^2}{1/n^2}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n^4}} = 1$$

Since $\sum_{n=1}^{\infty} b_n$ diverges $(p=1)$, $\sum_{n=1}^{\infty} a_n$ diverges by LCT.

Section 11.5 - Alternating Series

Example 8: Give an example of a series that is:

(a) conditionally convergent.

$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n}}$ conv. but $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$

diverges

(b) absolutely convergent.

$\sum_{n=1}^{\infty} 1/n^2$

Example 9: Determine whether the series is convergent.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(3n)}{1+(1.2)^n}$ $\boxed{\text{abs conv + DCT}}$

$$\left|\frac{\cos(3n)}{1+(1.2)^n}\right| < \frac{1}{1+1.2^n} \leq \frac{1}{(1.2)^n} = \left(\frac{5}{6}\right)^n$$

Thus $\sum_{n=1}^{\infty} \left|\frac{\cos(3n)}{1+(1.2)^n}\right|$ converges by DCT w/ $\sum_{n=1}^{\infty} (5/6)^n$ (geo $r = 5/6$).

Thus the given series is absolutely convergent and therefore convergent.

b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$ AST, $b_n = \frac{n}{n^2+2}$

① $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+2}$

$= \lim_{n\to\infty} \frac{1}{n+2/n} = 0$

② Show b_n is dec. w/ deriv.

$f(x) = \frac{x}{x^2+2}$

$f'(x) = \frac{(x^2+2) - x \cdot 2x}{(x^2+2)^2}$

$= \frac{x^2+2 - 2x^2}{(x^2+2)^2}$

$= \frac{2-x^2}{(x^2+2)^2} < 0$ if $x > \sqrt{2}$

Thus b_n is decreasing.

The given series conv. by AST.

Example 10: Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$ correct to within 0.0001.

By AST rem thm. $|\text{error}| < b_{n+1} < 0.0001$

So $\frac{1}{(n+1)^5} < 0.0001$

$\frac{1}{(n+1)^5} < \frac{1}{10{,}000}$

$10{,}000 < (n+1)^5$

$\sqrt[5]{10{,}000} < n+1$

$\sqrt[5]{10{,}000} - 1 < n$

$6.3 - 1 < n$

$5.3 < n$

So $\boxed{n \geq 6}$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} \approx \sum_{n=1}^{6} \frac{(-1)^{n+1}}{n^5} = \frac{1}{1} - \frac{1}{2^5} + \frac{1}{3^5} - \frac{1}{4^5} + \frac{1}{5^5} - \frac{1}{6^5}$$

$\approx \boxed{0.9720801}$

Section 11.6 - The Ratio and Root Tests

Example 11: Determine whether the series is convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2 9^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-5)^{2(n+1)}}{(n+1)^2 9^{n+1}} \cdot \frac{n^2\, 9^n}{(-5)^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-5)^{2n+2}\, n^2\, 9^n}{(n^2+2n+1)\, 9^{n+1}(-5)^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-5)^2\, n^2}{9(n^2+2n+1)}\right|$$

$$= \lim_{n\to\infty} \frac{25\, n^2}{9n^2+18n+9}$$

$$= \frac{25}{9} > 1$$

Thus the series diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{5^n n!}$

$$\lim_{n\to\infty}\left|\frac{1\cdot 3\cdot 5\cdots\quad (2(n+1)-1)}{5^{n+1}\,(n+1)!}\ \frac{5^n\cdot n!}{1\cdot 3\cdot 5\cdots\ (2n-1)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1\cdot 3\cdot 5\cdots\ (2n-1)(2n+1)}{5^{n+1}(n+1)\,n!}\ \frac{5^n\ n!}{1\cdot 3\cdot 5\cdots\ (2n-1)}\right|$$

$$= \lim_{n\to\infty} \frac{(2n+1)}{5(n+1)}$$

$$= \lim_{n\to\infty} \frac{2n+1}{5n+5}$$

$$= 2/5 < 1$$

Thus the series converges absolutely by the ratio test.

Example 12: For what values of x does the series $\displaystyle\sum_{n=1}^{\infty} (\ln x)^n$ converge?

Root test: series converges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$

Note $\lim_{n\to\infty} \sqrt[n]{|\ln x|^n} = \lim_{n\to\infty} |\ln x| = |\ln x|$

Need $|\ln x| < 1$ to converge.

or $-1 < \ln x < 1$

or $e^{-1} < x < e$

or $\boxed{1/e < x < e}$

Suggested Problems: # 1-31 odds, 34, 35, 41, 43, 43, 47-51 odds, 55, 57