

Math 651 – Supplementary homework exercises  
Simplicial homology

1. The  $n$ -ball  $B^n = \{z \in \mathbb{R}^n \mid \|z\| \leq 1\}$  is triangularized as the  $n$ -simplex. Use this to compute all simplicial homology groups for  $B^2$ . In doing so, give complete arguments to determine all cycles and boundaries.
2. Consider the Figure-8 space  $X$  as a simplicial complex of 5 vertices and 6 edges forming 2 triangles with a vertex in common.
  - (a) Compute the simplicial homology groups of  $X$ , and give representatives of generators for each.
  - (b) If you “glue” on a 2-simplex to  $X$  to fill in one of the triangles, yielding a new space  $X'$ , some homology groups might change. Compute them.
  - (c) If a complex  $X$  is composed only of  $n$ -simplices for  $n < m$ , and one or more  $m$  simplices is glued on so their boundaries are already in  $X$ , which homology groups *might* change? Which *cannot* change? Explain.
3. Consider the “parachute”  $Z$  formed by identifying the 3 corners of a 2-simplex. As described, this is *not* a simplicial complex.
  - (a) Give a triangularization of it that is a simplicial complex (so each simplex is uniquely determined by distinct vertices).
  - (b) Use your triangularization to compute homology groups.
  - (c) Explain informally why this space has as a deformation retract the Figure-8. Use this to determine its fundamental group  $G = \pi_1(Z, z_0)$  and verify that  $H_1(Z) \cong G/[G, G]$ .
4. The Torus and Figure-8 have different fundamental groups  $\pi_1$ , but the same first homology group  $H_1$ . Nonetheless they can be distinguished by homology. How?