Exam 2
Math 201 F01

Name: Key

04/16/12

1. (12 pts. – 6 pts. each) Using any means you like, give Maclaurin series for the following functions, writing out all terms to *at least degree 5*.

   (a) xe^{-x}

   $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$

   $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \cdots$

   $xe^{-x} = x - x^2 + \frac{x^3}{2} - \frac{x^4}{3!} + \frac{x^5}{4!} - \cdots$

   (b) $\dfrac{\cos x}{1+x^2}$

   $\boxed{\text{Solution 1}}$: $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots$

   $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$

   $\frac{\cos x}{1+x^2} = \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right)\left(1 - x^2 + x^4 - \cdots\right) = 1 + \left(-1-\frac{1}{2}\right)x^2 + \left(\frac{1}{4!} + \frac{1}{2} + 1\right)x^4 + \cdots$

   $= 1 - \frac{3}{2}x^2 + \frac{37}{24}x^4 + \cdots$

   $\boxed{\text{Solution 2}}$:

   $$\begin{array}{r|l} & 1 - \frac{3}{2}x^2 + \frac{37}{24}x^4 + \cdots \\ \hline 1+x^2 & 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots \\ & 1 + x^2 \\ \hline & -\frac{3}{2}x^2 + \frac{x^4}{4!} - \cdots \end{array}$$

   $$\begin{array}{l} -\frac{3}{2}x^2 - \frac{3}{2}x^4 \\ \hline \frac{37}{24}x^4 \end{array}$$

2. (12 pts. – 6 pts. each) Determine the following limits, showing your work.

   (a) $\displaystyle\lim_{x\to 0} \frac{\cos(7x) - 1}{x^2} \underset{\frac{0}{0},\ \text{L'Hopital}}{=} \lim_{x\to 0} \frac{-7\sin(7x)}{2x} \underset{\frac{0}{0},\ \text{L'Hopl}}{=} \lim_{x\to 0} \frac{-49\cos(7x)}{2} = -\frac{49}{2}$

   (b) $\displaystyle\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln x} = e^{\lim_{x\to\infty} \frac{\ln x}{x}} = e^0 = 1$

   $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x} \underset{\frac{\infty}{\infty},\ \text{L'Hopital}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$

3. (20 pts.) A function $f(x)$ is defined by the series

$$f(x) = \sum_{n=0}^{\infty} \frac{5^n}{(n+2)2^n}(x-3)^n.$$

(a) (5 pts.) Determine the radius of convergence of the series.

$$\left| \frac{\frac{5^{n+1}}{(n+3)2^{n+1}}(x-3)^{n+1}}{\frac{5^n}{(n+2)2^n}(x-3)^n} \right| = \frac{5}{2}\,\frac{n+2}{n+3}\,|x-3| \xrightarrow[n\to\infty]{\text{as}} \frac{5}{2}|x-3| < 1$$

$$|x-3| < \frac{2}{5} \qquad R = \frac{2}{5}$$

(b) (6 pts.) Determine the interval of convergence of the series.

$x = 3 + \frac{2}{5}$ series is $\sum \frac{5^n}{(n+2)2^n}\left(\frac{2}{5}\right)^n = \sum \frac{1}{n+2}$, harmonic diverges

$x = 3 - \frac{2}{5}$ series is $\sum \frac{5^n}{(n+2)2^n}\left(\frac{-2}{5}\right)^n = \sum \frac{(-1)^n}{n+2}$, alternating harmonic, converges

$$\left[3 - \frac{2}{5}, 3 + \frac{2}{5}\right) = \left[\frac{13}{5}, \frac{17}{5}\right)$$

(c) (5 pts.) Give a series for $\int f(x)\,dx$.

$$C + \sum_{n=0}^{\infty} \frac{5^n}{2^n(n+2)(n+1)}(x-3)^{n+1}$$

(d) (4 pts.) What is $f^{(9)}(3)$?

$$\frac{f^{(9)}(3)}{9!} = \frac{5^9}{(9+2)2^9} \quad \text{so} \quad f^{(9)}(3) = \frac{5^9}{11 \cdot 2^9} \cdot 9!$$

4. (10 pts.) Suppose the 6th degree Taylor polynomial at $c = 0$ is used to approximate $\sin(0.5)$. Give a numerical bound on how large the error could be. (You may leave your answer in a form where only a calculator is needed to evaluate it.)

$$\text{error} = |R_6(x)| = \left| \frac{f^{(7)}(z)}{7!}(.5-0)^7 \right| = \frac{|\cos z|}{7!}(.5)^7 \le \frac{1}{7!}(.5)^7$$

for some z between 0 & .5

$$f^{(7)}(x) = -\cos x$$

5. (24 pts. – 6 pts. each) Determine whether the following series converge or diverge. For each, *state what test you use*, and write enough to indicate that you checked all conditions necessary to apply the test.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$

alternating series test

$\frac{1}{\ln n} \to 0$ as $n \to \infty$

$\frac{1}{\ln(n)} > \frac{1}{\ln(n+1)}$

Converges

(b) $\sum_{n=1}^{\infty} \frac{2n^3+7n+1}{n^6-5n}$

limit comparison to $\sum \frac{1}{n^3}$ which converges (p-series) $p=3>1$

$$\frac{\frac{2n^3+7n+1}{n^6-5n}}{\frac{1}{n^3}} = \frac{2n^6+7n^4+n^3}{n^6-5n} \xrightarrow[n\to\infty]{\text{as}} 2 \neq \begin{smallmatrix}\infty\\0\end{smallmatrix}$$

So series converges

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{n^3+2}{5n^3+n^2}$

n^{th} term test

$\frac{n^3+2}{5n^3+n^2} \xrightarrow[\text{as } n\to\infty]{} \frac{1}{5}$, so $a_n \not\to 0$

diverges

(d) $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$

Direct comparison to $\sum \frac{1}{\sqrt{n}}$, which diverges (p-series $p=\frac{1}{2}<1$)

$\frac{\ln n}{\sqrt{n}} > \frac{1}{\sqrt{n}}$

So series diverges

Note: This can also be done using the integral test, but that requires integration by parts.

6. (12 pts. – 6 pts. each) Determine whether the following improper integrals converge or diverge. If they converge, compute their values.

(a) $\int_2^\infty \frac{1}{x \ln x}\,dx$ $= \lim_{b\to\infty} \int_2^b \frac{1}{x\ln x}\,dx$ But $\int_2^b \frac{1}{x\ln x}\,dx = \int_{\ln 2}^{\ln b} \frac{1}{u}\,du = \ln(u)\Big|_{\ln 2}^{\ln b}$ $u = \ln x$ $du = \frac{1}{x}dx$

So $\int = \lim_{b\to\infty} \ln(\ln b) - \ln(\ln 2) = \infty$

diverges

(b) $\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$ $= \lim_{a\to 0^+} \int_a^8 x^{-1/3}\,dx = \lim_{a\to 0^+} \frac{3}{2}x^{2/3}\Big|_a^8 = \lim_{a\to 0^+}\left(\frac{3}{2}8^{2/3} - \frac{3}{2}a^{2/3}\right)$

$= \frac{3}{2}8^{2/3} = 6$

7. (10 pts. – 5 pts. each) An instructor allows his students to earn extra credit by solving as many optional problems as they like. The first problem earns 10 pts., but then each problem is worth 90% of the previous one (*i.e.*, the second problem is worth only 9 pts., the third 8.1 pts., and so on).

(a) What is the maximum number of extra points a student can earn (by completing infinitely many problems)? Simplify your answer fully.

$10 + 10(.9) + 10(.9)^2 + 10(.9)^3 + \cdots = 10\frac{1}{1-.9} = \frac{10}{.1} = 100$

(b) How many points would be earned if 92 problems are completed? (You do not need to simplify your answer beyond expressing it *without* using '...' , 'Σ', or any equivalent notation.)

$10 + 10(.9) + \cdots + 10(.9)^{91} = 10\left(\frac{1-.9^{92}}{1-.9}\right) = \frac{10}{.1}(1-.9^{92})$

$= 100(1-.9^{92})$