MATH253X-UX1 Midterm Exam 2 Name: Answer Key
Spring 2016

Instructions. You have 60 minutes. Closed book, closed notes, no calculator. *Show all your work* in order to receive full credit.

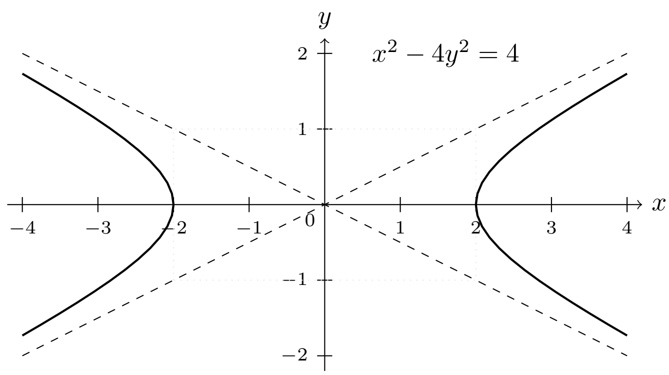
1. Show that $\lim_{(x,y)\to(2,-1)} \frac{xy+2}{x^2+4y}$ does not exist.

Solution: Setting $x = 2$ and letting $y \to -1$ to approach $(2, -1)$ along the line $(2, y)$, we see $\lim_{y\to -1} \frac{2y+2}{4+4y} = \frac{1}{2}$. Setting $y = -1$ and letting $x \to 2$ to approach $(2, -1)$ along the line $(x, -1)$, we see $\lim_{x\to 2} \frac{2-x}{x^2-4} = -\frac{1}{4}$. Since these limits are different, $\boxed{\text{the original multivariable limit does not exist.}}$

2. Consider the function $z = f(x, y) = x^2 - 4y^2$.

(a) Sketch the level curve $z = 4$.

Solution:



(b) Use Lagrange multipliers to find the absolute maximum z_{max} of f on the line $2x + y = 15$.

Solution: Define $g(x, y) = 2x + y$. Then we must solve (along with the constraint $g(x, y) = 15$):

$$\nabla f = \lambda \nabla g \iff \langle 2x, -8y \rangle = \lambda \langle 2, 1 \rangle \iff \begin{cases} 2x = 2\lambda \\ -8y = \lambda \end{cases} \iff \lambda = x = -8y$$

Substituting into the constraint, we have:

$$2(-8y) + y = 15 \iff y = -1$$

and so $x = -8(-1) = 8$. Thus,

$$z_{\text{max}} = f(8, -1) = 64 - 4 = \boxed{60}.$$

(c) What is the geometrical relationship between $2x + y = 15$ and the level curves $z = z_{\text{max}}$ at their intersection?

Solution: Note that the line $2x + y = 15$ can be parametrized by $\langle x, 15 - 2x \rangle$ so its direction is $\langle 1, -2 \rangle$ and $\nabla g \perp \langle 1, -2 \rangle$. But then ∇f is orthogonal to (the tangent of) the level curve at any point so also at $(8, -1)$ along $z = z_{\text{max}}$. Now since ∇f and ∇g are parallel, then so are the line and the tangent to the level curve. They also share the point $(8, -1)$ so $\boxed{\text{the line } 2x + y = 15 \text{ is the tangent to the level curve } z = z_{\text{max}} \text{ at } (8, -1)}$.

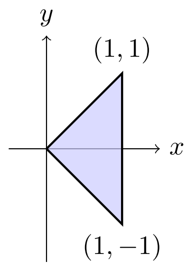
3. Consider the double integral:

$$I = \iint_R e^{x^2}\, dA$$

where R is the triangular region with vertices $(0,0)$, $(1,1)$, and $(1,-1)$.

(a) Write I as an iterated integral in two ways.

Solution: Let's sketch the region R:



We have the lines $y = x$, $y = -x$ and $x = 1$ so the double integral can be written as either of these forms in rectangular coordinates:

$$\boxed{I = \int_0^1 \int_{-x}^{x} e^{x^2}\, dy\, dx = \int_{-1}^{0} \int_{-y}^{1} e^{x^2}\, dx\, dy + \int_0^1 \int_y^1 e^{x^2}\, dx\, dy}$$

(b) Compute the integral using the form of your choice.

Solution: Note that we need to use the $dy\, dx$ order because e^{x^2} can not be integrated directly wrt x using elementary functions:

$$\begin{aligned}
I &= \int_0^1 \int_{-x}^{x} e^{x^2}\, dy\, dx = \int_0^1 \left[y e^{x^2} \right]_{y=-x}^{y=x} dx \\
&= \int_0^1 2x e^{x^2}\, dx = \left| \begin{matrix} u = x^2 & du = 2xdx \\ x = 0 & u = 0 \\ x = 1 & u = 1 \end{matrix} \right| \\
&= \int_0^1 e^u\, du = \left[e^u \right]_0^1 = \boxed{e - 1}
\end{aligned}$$

4. Find an equation of the tangent plane to the surface

$$x^2 y - z^2 + \ln(x+y) = 1$$

at the point $(x_0, y_0, z_0) = (-1, 2, 1)$.

Solution: For $F(x,y,z) = x^2 y - z^2 + \ln(x+y) = 1$, we find

$$\nabla F(x,y,z) = \left\langle 2xy + \frac{1}{x+y}, x^2 + \frac{1}{x+y}, -2z \right\rangle,$$

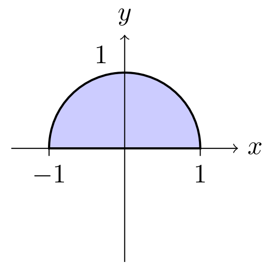
so $\nabla F(-1,2,1) = \langle -3, 2, -2 \rangle$. The tangent plane is thus given by

$$-3(x+1) + 2(y-2) - 2(z-1) = 0,$$

or

$$\boxed{-3x + 2y - 2z = 5.}$$

5. Compute the mass m of the planar lamina with density $\rho(x,y)=x^2y$ shown below.



Solution: Let's use polar coordinates: $\rho(r\cos\theta, r\sin\theta)=r^3\cos^2\theta\sin\theta$ and R will have constant bounds in (r,θ), that is $0\le r\le 1$ and $0\le\theta\le\pi$. Hence

$$m=\int_0^1\int_0^\pi r^3\cos^2\theta\sin\theta\, r\,d\theta\,dr=\int_0^1 r^4\left[-\frac{\cos^3\theta}{3}\right]_0^\pi dr=\int_0^1 r^4\left(\frac{1}{3}+\frac{1}{3}\right)dr=\frac{2}{3}\left[\frac{r^5}{5}\right]_0^1=\boxed{\frac{2}{5}}.$$

6. Find and classify all critical points of

$$f(x,y)=x^2y-2x+4y^2.$$

Solution: The gradient is

$$\nabla f=\langle f_x,f_y\rangle=\left\langle 2xy-2, x^2+8y\right\rangle$$

is defined everywhere and when setting it to the zero vector, we get $f_x=0$ if $xy=1$; that means $x,y\neq 0$ and $y=\frac{1}{x}$ so $f_y=0$ becomes

$$x^2+\frac{8}{x}=0 \quad\Rightarrow\quad x^3+8=0 \quad\Rightarrow\quad x=-2$$

This in turns means $y=-\frac{1}{2}$. So we have one critical point $\boxed{\left(-2,-\frac{1}{2}\right)}$. To classify it, we use the Second Partials Test:

$$f_{xx}=2y \quad,\quad f_{yy}=8 \quad,\quad f_{xy}=2x \quad\Rightarrow\quad d(x,y)=16y-4x^2$$

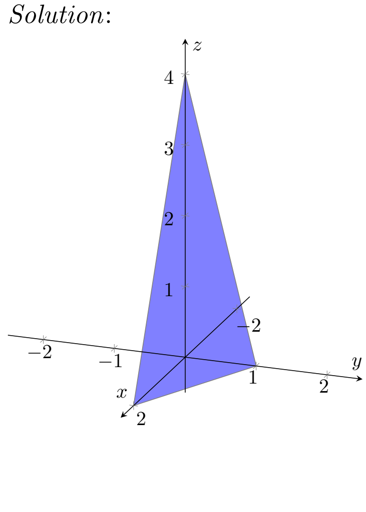
Now,

$$d\left(-2,-\frac{1}{2}\right)=-8-16<0 \text{ so } \boxed{\text{saddle point at } \left(-2,-\frac{1}{2},3\right)}.$$

7. Fully SET UP bounds and integrands but DO NOT EVALUATE the following double integrals.

(a) the volume below the plane $2x+4y+z=4$ in the first octant:

Solution:



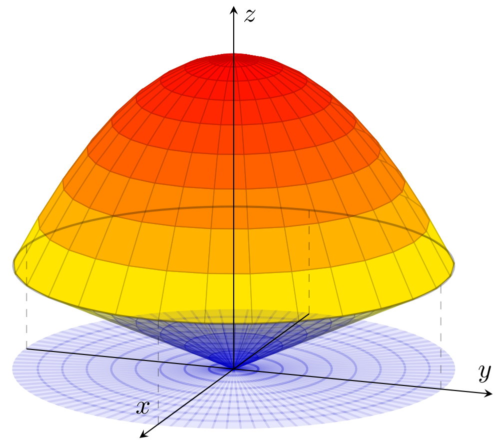
Solve for $z=4-2x-4y$ for the integrand, then set $z=0$ to get a boundary line in the xy-plane, the others being $x=0$, $y=0$. Finally for the order $dy\,dx$ set $y=0$ in the line to get the upper constant bound in x. Thus we have:

$$\boxed{V=\int_0^2\int_0^{1-\frac{x}{2}} 4-2x-4y\,dy\,dx}$$

(b) the volume of the solid bounded by the cone $z = \sqrt{x^2+y^2}$ and the inverted paraboloid $z = 6-x^2-y^2$ using polar coordinates.

Solution: The cone is below the paraboloid and for the base, we have a disk where the radius can be found using the intersection of the surfaces, i.e. set $\sqrt{x^2+y^2} = 6-x^2+y^2$ or in polar $r = 6-r^2$ for $r = \sqrt{x^2+y^2} \geq 0$. So $r^2+r-6=0$ which has for solutions $r=-3,2$ and we keep $r=2$. And so the volume is:

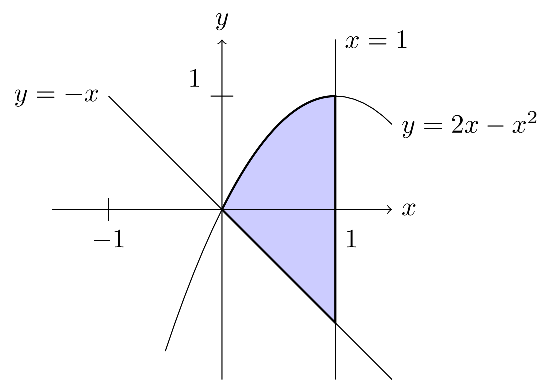
$$\boxed{V = \int_0^2 \int_0^{2\pi} (6-r^2-r)\, r\, d\theta\, dr}$$



(c) the surface area of $z = 4-x^2-y$ above the region R bounded by the graphs of $y=-x$, $y=2x-x^2$, $x=0$ and $x=1$ as sketched below:

Solution: The gradient is $\nabla z = \langle z_x, z_y\rangle = \langle -2x, -1\rangle$ so noting that R is vertically simple, we have that the surface area of our surface above R is:

$$\boxed{SA = \int_0^1 \int_{-x}^{2x-x^2} \sqrt{4x^2+2}\, dy\, dx}$$



8. Let

$$f(x,y) = \frac{x}{x-y}.$$

(a) Compute the maximum rate of change of f at the point $(1,2)$ and specify a unit vector in the direction where this maximum change occurs.

Solution: The gradient is

$$\nabla f(x,y) = \left\langle \frac{1(x-y)-x(1)}{(x-y)^2}, \frac{x}{(x-y)^2} \right\rangle = \left\langle \frac{-y}{(x-y)^2}, \frac{x}{(x-y)^2} \right\rangle.$$

So the maximum rate of change of f at $(1,2)$ is:

$$\|\nabla f(1,2)\| = \|\langle -2,1\rangle\| = \boxed{\sqrt{5}},$$

and a unit direction of greatest increase is

$$\boxed{\frac{\nabla f(1,2)}{\|\nabla f(1,2)\|} = \left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle.}$$

(b) Find the directional derivative of f at $(1,2)$ in the direction of $\mathbf{v} = 2\mathbf{i}+3\mathbf{j}$.

Solution: The direction we consider is $\mathbf{u} = \dfrac{\mathbf{v}}{||\mathbf{v}||} = \langle 2/\sqrt{13}, 3/\sqrt{13}\rangle$. Then

$$D_{\mathbf{u}}f(1,2) = \nabla f(1,2) \cdot \mathbf{u} = \langle -2, 1\rangle \cdot \langle 2/\sqrt{13}, 3/\sqrt{13}\rangle = \boxed{-1/\sqrt{13}.}$$

(c) Use the differential df to find an approximation of $f(1.1, 1.95)$.

Solution:

$$\begin{aligned} f(1.1, 1.95) &\approx f(1,2) + df \\ &= f(1,2) + f_x(1,2)(1.1-1) + f_y(1,2)(1.95-2) \\ &= -1 - 2(0.1) + 1(-0.05) = \boxed{-1.25} \end{aligned}$$