This provides the context for the equations used in Question 2.

A.1 Pendulum

Consider the simple pendulum shown in Figure A.1, where l denotes the length of the rod and m the mass of the bob. Assume the rod is rigid and has zero mass. Let θ denote the angle subtended by the rod and the vertical axis through the pivot point. The pendulum swings in the vertical plane. The bob of the pendulum moves in a circle of radius l. To write the equation of motion of the pendulum, let us identify the forces acting on it. There is a downward gravitational force equal to mg, where g is the acceleration due to gravity. There is also a frictional force resisting the motion, which we assume to be proportional to the speed of the bob with a coefficient of friction k. Suppose also that a torque T is applied at the pivot point in the direction of θ. By taking moments about the pivot point, we obtain

$$ml^2 \frac{d^2\theta}{d\,t^2} + mgl\sin\theta + kl^2\frac{d\theta}{dt} = T$$

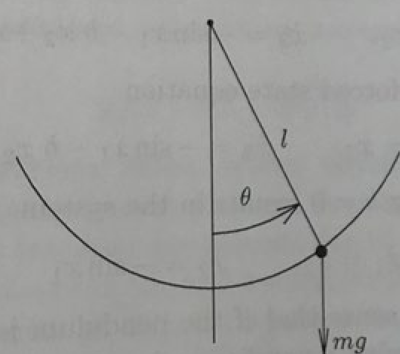

Figure A.1: Pendulum.

When $T = 0$, the equilibrium points are obtained by setting $d\theta/dt = d^2\theta/dt^2 = 0$, which shows that the pendulum has infinitely many equilibrium points at $(\theta, d\theta/dt) = (n\pi, 0)$, for $n = 0, \pm 1, \pm 2, \ldots$. From the physical description of the pendulum, it is clear that the pendulum has only two equilibrium positions corresponding to the equilibrium points $(0, 0)$ and $(\pi, 0)$. Other equilibrium points are repetitions of these two positions, which correspond to the number of full swings the pendulum would make before it rests at one of the two equilibrium positions. For example, if the pendulum makes m complete $360°$ revolutions before it rests at the downward vertical position, then, mathematically, we say that the pendulum approaches the equilibrium point $(2m\pi, 0)$. In our investigation of the pendulum, we will limit our attention to the two "nontrivial" equilibrium points at $(0, 0)$ and $(\pi, 0)$. Physically, we can see that these two equilibrium points are quite distinct from each other. While the pendulum can indeed rest at $(0, 0)$, it can hardly maintain rest at $(\pi, 0)$ because infinitesimally small disturbance from that equilibrium will take the pendulum away. The difference between the two equilibrium points is in their stability properties.

We normalize the pendulum equation so that all variables are dimensionless. By linearization at $(0, 0)$ it can be seen that when there is no friction ($k = 0$) and no applied torque ($T = 0$) the pendulum oscillates around the downward vertical position with frequency Ω where $\Omega^2 = g/l$. We define the dimensionless time $\tau = \Omega t$. Then

$$\frac{d\theta}{dt} = \Omega \frac{d\theta}{d\tau} \quad \text{and} \quad \frac{d^2\theta}{dt^2} = \Omega^2 \frac{d^2\theta}{d\tau^2}$$

Defining the dimensionless torque $u = T/(m_o g l)$ for some nominal mass m_o and denoting the first and second derivatives with respect to τ by $(\dot{\cdot})$ and $(\ddot{\cdot})$, respectively, we obtain the normalized equation

$$\ddot{\theta} + \sin\theta + b\,\dot{\theta} = c\,u \tag{A.1}$$

where $b = k/(m\Omega)$ and $c = m_o/m$. In our study of the pendulum equation we sometimes deal with the mass m as uncertain parameter, which results in uncertainty in the coefficient b and c of (A.1). To obtain a state model of the pendulum, we take the state variables as $x_1 = \theta$ and $x_2 = \dot{\theta}$, which yields

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - b\,x_2 + c\,u \tag{A.2}$$

When $u = 0$ we have the unforced state equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - b\,x_2 \tag{A.3}$$

Neglecting friction by setting $b = 0$ results in the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 \tag{A.4}$$

which is conservative in the sense that if the pendulum is given an initial push, it will keep oscillating forever with a nondissipative energy exchange between kinetic and potential energies. This, of course, is not realistic, but gives insight into the behavior of the pendulum.

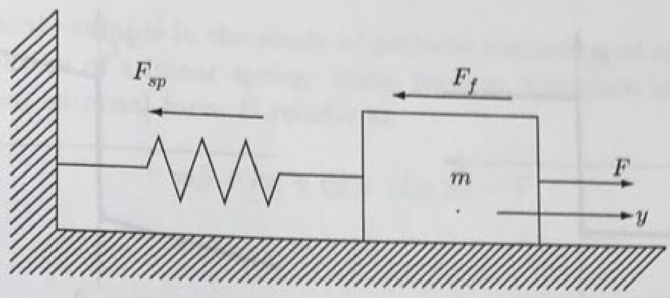

Figure A.2: Mass-spring mechanical system

A.2 Mass–Spring System

In the mass–spring mechanical system, shown in Figure A.2, we consider a mass m sliding on a horizontal surface and attached to a vertical surface through a spring. The mass is subjected to an external force F. We define y as the displacement from a reference position and write Newton's law of motion

$$m\ddot{y} + F_f + F_{sp} = F$$

where F_f is a resistive force due to friction and F_{sp} is the restoring force of the spring. We assume that F_{sp} is a function only of the displacement y and write it as $F_{sp} = g(y)$. We assume also that the reference position has been chosen such that $g(0) = 0$. The external force F is at our disposal. Depending upon F, F_f, and g, several interesting time-invariant and time-varying models arise.

For a relatively small displacement, the restoring force of the spring can be modeled as a linear function $g(y) = ky$, where k is the spring constant. For a large displacement, however, the restoring force may depend nonlinearly on y. For example, the function

$$g(y) = k(1 - a^2y^2)y, \quad |ay| < 1$$

models the so-called *softening spring*, where, beyond a certain displacement, a large displacement increment produces a small force increment. On the other hand, the function

$$g(y) = k(1 + a^2y^2)y$$

models the so-called *hardening spring*, where, beyond a certain displacement, a small displacement increment produces a large force increment.

The resistive force F_f may have components due to static, Coulomb, and viscous friction. When the mass is at rest, there is a static friction force F_s that acts parallel to the surface and is limited to $\pm\mu_s mg$, where $0 < \mu_s < 1$ is the static friction coefficient. This force takes whatever value, between its limits, to keep the mass at rest. For motion to begin, there must be a force acting on the mass to overcome the static friction. In the absence of an external force, $F = 0$, the static friction

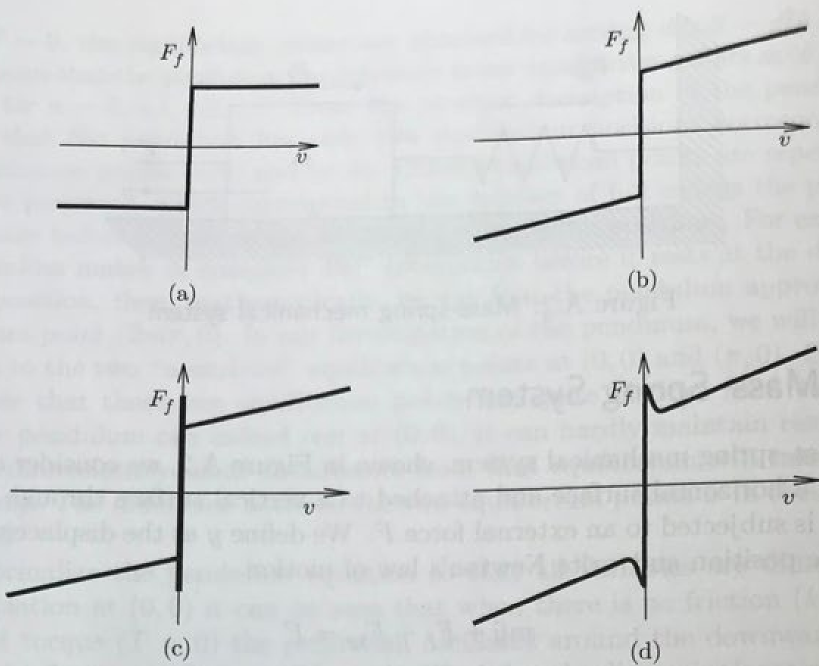

Figure A.3: Examples of friction models. (a) Coulomb friction; (b) Coulomb plus linear viscous friction; (c) static, Coulomb, and linear viscous friction; (d) static, Coulomb, and linear viscous friction—Stribeck effect.

force will balance the restoring force of the spring and maintain equilibrium for $|g(y)| \leq \mu_s mg$. Once motion has started, the resistive force F_f, which acts in the direction opposite to motion, is modeled as a function of the sliding velocity $v = \dot{y}$. The resistive force due to *Coulomb friction* F_c has a constant magnitude $\mu_k mg$, where μ_k is the kinetic friction coefficient, that is,

$$F_c = \begin{cases} -\mu_k mg, & \text{for} \quad v < 0 \\ \mu_k mg, & \text{for} \quad v > 0 \end{cases}$$

As the mass moves in a viscous medium, such as air or lubricant, there will be a frictional force due to viscosity. This force is usually modeled as a nonlinear function of the velocity; that is, $F_v = h(v)$, where $h(0) = 0$. For small velocity, we can assume that $F_v = cv$. Figure A.3 shows various examples of friction models. In Figure A.3(c), the static friction is higher than the level of Coulomb friction, while Figure A.3(d) shows a similar situation, but with the force decreasing continuously with increasing velocity, the so-called *Stribeck effect*.

The combination of a hardening spring, linear viscous friction, and a periodic external force $F = A\cos\omega t$ results in the Duffing's equation

$$m\ddot{y} + c\dot{y} + ky + ka^2y^3 = A\cos\omega t \tag{A.5}$$

which is a classical example in the study of periodic excitation of nonlinear systems.

The combination of a linear spring, static friction, Coulomb friction, linear viscous friction, and external force F results in

$$m\ddot{y} + ky + c\dot{y} + \eta(y, \dot{y}) = F$$

where

$$\eta(y, \dot{y}) = \begin{cases} \mu_k mg \ \text{sign}(\dot{y}), & \text{for} \quad |\dot{y}| > 0 \\ -ky, & \text{for} \quad \dot{y} = 0 \ \text{ and } \ |y| \le \mu_s mg/k \\ -\mu_s mg \ \text{sign}(y), & \text{for} \quad \dot{y} = 0 \ \text{ and } \ |y| > \mu_s mg/k \end{cases}$$

The value of $\eta(y, \dot{y})$ for $\dot{y} = 0$ and $|y| \le \mu_s mg/k$ is obtained from the equilibrium condition $\ddot{y} = \dot{y} = 0$. With $x_1 = y$, $x_2 = \dot{y}$, and $u = F$, the state model is

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = [-kx_1 - cx_2 - \eta(x_1, x_2) + u]/m \tag{A.6}$$

Let us note two features of this state model. First, with $u = 0$ it has an equilibrium set, rather than isolated equilibrium points. Second, the right-hand side is a discontinuous function of the state. The discontinuity is a consequence of the idealization we adopted in modeling friction. One would expect the transition from static to sliding friction to take place in a smooth way, not abruptly as our idealization suggests.[1] The discontinuous idealization, however, allows us to carry out *piecewise linear analysis* since in each of the regions $\{x_2 > 0\}$ and $\{x_2 < 0\}$, we can use the model

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = [-kx_1 - cx_2 - \mu_k mg \ \text{sign}(x_2) + u]/m$$

to predict the behavior of the system via linear analysis.

A.11 Inverted Pendulum on a Cart

Consider the inverted pendulum on a cart of Figure A.12 [82]. The pivot of the pendulum is mounted on a cart that can move in a horizontal direction. The cart is driven by a motor that exerts a horizontal force F. The figure shows also the forces acting on the pendulum, which are the force mg at the center of gravity, a horizontal reaction force H, and a vertical reaction force V at the pivot. Writing horizontal and vertical Newton's laws at the center of gravity of the pendulum yields

$$m\,\frac{d^2}{dt^2}(y + L\sin\theta) = H \quad \text{and} \quad m\,\frac{d^2}{dt^2}(L\cos\theta) = V - mg$$

Taking moments about the center of gravity yields the torque equation

$$J\ddot{\theta} = VL\sin\theta - HL\cos\theta$$

while a horizontal Newton's law for the cart yields

$$M\ddot{y} = F - H - k\dot{y}$$

Here m is the mass of the pendulum, M the mass of the cart, L the distance from the center of gravity to the pivot, J the moment of inertia of the pendulum with respect to the center of gravity, k a friction coefficient, y the displacement of the pivot, θ the angular rotation of the pendulum (measured clockwise), and g the acceleration due to gravity. We will derive two models of this system, a fourth-order model that describes the motion of the pendulum and cart when F is viewed as the control input, and a second-order model that describes the motion of the pendulum when the cart's acceleration is viewed as the input.

Carrying out the indicated differentiation to eliminate H and V, we obtain

$$H = m\frac{d^2}{dt^2}(y + L\sin\theta) = m\frac{d}{dt}(\dot{y} + L\dot{\theta}\cos\theta) = m(\ddot{y} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta)$$

$$V = m\frac{d^2}{dt^2}(L\cos\theta) + mg = m\frac{d}{dt}(-L\dot{\theta}\sin\theta) + mg = -mL\ddot{\theta}\sin\theta - mL\dot{\theta}^2\cos\theta + mg$$

Substituting H and V in the $\ddot{\theta}$- and $\ddot{y}$-equations yields

$$\begin{aligned} J\ddot{\theta} &= -mL^2\ddot{\theta}(\sin\theta)^2 - mL^2\dot{\theta}^2\sin\theta\cos\theta + mgL\sin\theta \\ &\quad -mL\ddot{y}\cos\theta - mL^2\ddot{\theta}(\cos\theta)^2 + mL^2\dot{\theta}^2\sin\theta\cos\theta \\ &= -mL^2\ddot{\theta} + mgL\sin\theta - mL\ddot{y}\cos\theta \end{aligned}$$

$$M\ddot{y} = F - m(\ddot{y} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta) - k\dot{y}$$

The foregoing two equations can be rewritten as

$$\begin{bmatrix} J + mL^2 & mL\cos\theta \\ mL\cos\theta & m + M \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} mgL\sin\theta \\ F + mL\dot{\theta}^2\sin\theta - k\dot{y} \end{bmatrix}$$

Hence,

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{y} \end{bmatrix} = \frac{1}{\Delta(\theta)} \begin{bmatrix} m + M & -mL\cos\theta \\ -mL\cos\theta & J + mL^2 \end{bmatrix} \begin{bmatrix} mgL\sin\theta \\ F + mL\dot{\theta}^2\sin\theta - k\dot{y} \end{bmatrix} \tag{A.40}$$

where

$$\Delta(\theta) = (J + mL^2)(m + M) - m^2L^2\cos^2\theta \geq (J + mL^2)M + mJ > 0$$

Using $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = y$, and $x_4 = \dot{y}$ as the state variables and $u = F$ as the control input, the state equation is given by

$$\dot{x}_1 = x_2 \tag{A.41}$$

$$\dot{x}_2 = \frac{1}{\Delta(x_1)}\left[(m + M)mgL\sin x_1 - mL\cos x_1(u + mLx_2^2\sin x_1 - kx_4)\right] \tag{A.42}$$

$$\dot{x}_3 = x_4 \tag{A.43}$$

$$\dot{x}_4 = \frac{1}{\Delta(x_1)}\left[-m^2L^2g\sin x_1\cos x_1 + (J + mL^2)(u + mLx_2^2\sin x_1 - kx_4)\right] \tag{A.44}$$