Nonlinear Systems: Homework Assignment #2

Due by midnight on Thursday Jan 30th.

Question 1: Problem 1.9 in NC (Note: NC refers to Hassan Khalil's Nonlinear Control textbook)

Figure 1.5 shows the hydraulic system where liquid is stored in an open tank. The cross-sectional area of the tank, $A(h)$, is a function of h, the height of the liquid level above the bottom of the tank. The liquid volume v is given by $v = \int_0^h A(\lambda)d\lambda$. For a liquid of density ρ, the absolute pressure p is given by $p = \rho g h + p_a$, where p_a is the atmospheric pressure (assumed constant) and g is the acceleration due to gravity. The tank receives liquid at a flow rate w_i and loses liquid through a valve that obeys the flow-pressure relationship $w_0 = k\sqrt{p - p_a}$. The rate of change of v satisfies $\dot{v} = w_i - w_0$. Take w_i to be the control input and h to be the output.

a) Using h as the state variable, determine the state model.
b) Using $p - p_a$ as the state variable, determine the state model.
c) Find a constant input that maintains a constant output at $h = r$.

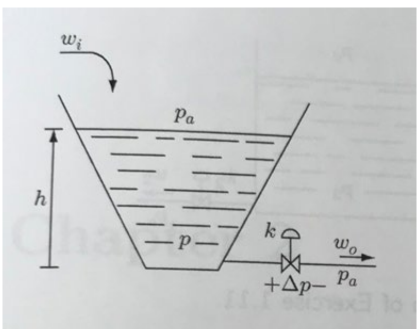

Figure 1.5: Exercise 1.9.

Question 2: Problem 1.12 parts a, b, and K in NC

For each of the following systems, investigate local and global Lipschitz properties. Assume that input variables are continuous functions of time.

a) The pendulum equation (A.2)

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - b\,x_2 + c\,u \tag{A.2}$$

b) The mass-spring systems (A.6)

$$\eta(y, \dot{y}) = \begin{cases} \mu_k mg \operatorname{sign}(\dot{y}), & \text{for } |\dot{y}| > 0 \\ -ky, & \text{for } \dot{y} = 0 \text{ and } |y| \le \mu_s mg/k \\ -\mu_s mg \operatorname{sign}(y), & \text{for } \dot{y} = 0 \text{ and } |y| > \mu_s mg/k \end{cases}$$

The value of $\eta(y, \dot{y})$ for $\dot{y} = 0$ and $|y| \le \mu_s mg/k$ is obtained from the equilibrium condition $\ddot{y} = \dot{y} = 0$. With $x_1 = y$, $x_2 = \dot{y}$, and $u = F$, the state model is

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = [-kx_1 - cx_2 - \eta(x_1, x_2) + u]/m \tag{A.6}$$

k) The inverted pendulum on a cart (A.41)-(A.44)

$$\Delta(\theta) = (J + mL^2)(m + M) - m^2L^2\cos^2\theta \ge (J + mL^2)M + mJ > 0$$

Using $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = y$, and $x_4 = \dot{y}$ as the state variables and $u = F$ as the control input, the state equation is given by

$$\begin{aligned}
\dot{x}_1 &= x_2 && \text{(A.41)}\\
\dot{x}_2 &= \frac{1}{\Delta(x_1)}\left[(m + M)mgL\sin x_1 - mL\cos x_1(u + mLx_2^2\sin x_1 - kx_4)\right] && \text{(A.42)}\\
\dot{x}_3 &= x_4 && \text{(A.43)}\\
\dot{x}_4 &= \frac{1}{\Delta(x_1)}\left[-m^2L^2g\sin x_1\cos x_1 + (J + mL^2)(u + mLx_2^2\sin x_1 - kx_4)\right] && \text{(A.44)}
\end{aligned}$$

Context for these equations can be found in the attached document, as well as in the Nonlinear Systems book (Chapter 1.1).

Question 3: Problem 1.14 from NC:

Find a diffeomorphism $z = T(x)$ that transforms the system

$$\dot{x}_1 = \sin(x_2)\,, \dot{x}_2 = -x_1^2 + u, y = x_1$$

Into

$$\dot{z}_1 = z_2, \dot{z}_2 = a(z) + b(z)u, y = z_1$$

and give the definitions of a and b.

Question 4: Problem 2.1 part (1) from NC:

For the system:

$$\dot{x}_1 = -x_1^3 + x_2, \dot{x}_2 = x_1 - x_2^3$$

(a) Find all equilibrium points and determine their types.
(b) Construct and discuss the phase portrait.

Question 5: Problem 2.8 from NC:

Consider the system:

$$\dot{x}_1 = x_2, \dot{x}_2 = -0.5x_1 + 1.5x_2 + 0.5u$$

Construct and discuss the phase portrait for $u = 0$, the feedback control $u = 0.9x_1 - 3.2x_2$, and the constrained feedback control $u = sat(0.9x_1 - 3.2x_2)$.

Question 6: Problem 2.10 from NC:

The elementary processing units in the central nervous system are the neurons. The FitzHugh-Nagumo model [49] is a dimensionless model that attempts to capture the dynamics of a single neuron. It is given by

$$\dot{u} = u - \frac{1}{3}u^3 - \omega + I, \dot{\omega} = \epsilon(b_0 + b_1 u - \omega)$$

Where u, ω, and $I \geq 0$ are the membrane voltage, recovery variable, and applied current, respectively. The constants ϵ, b_0, and b_1 are positive.

a) Find all the equilibrium points and determine their types when $b_1 > 1$.
b) Repeat part a) when $b_1 < 1$.
c) Let $\epsilon = 0.1, b_0 = 2$, and $b_1 = 1.5$. For each of the values $I = 0$ and $I = 2$, construct the phase portrait and discuss the qualitative behavior of the system.
d) Repeat c) with $b_1 = 0.5$.

Question 7:

Using ode45 in Matlab (or equivalent), simulate the Lorenz system

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz$$

$$\dot{z} = xy - bz$$

with $\sigma = 10, b = \frac{8}{3}, and\ r = 28$. Use plot3 to plot solutions for the initial conditions (0,2,0), (0,-2,0), and (0,2.01,0). Briefly describe the behavior of the solutions and their dependence on the initial conditions. Please include your Matlab figures and source code.

Question 8:

The behavior of 2D linearized systems can be immediately determined using a trace τ, determinant δ graph. Use the figure below to label where in the graph you will find stable nodes, unstable nodes, saddle points, stable foci, unstable foci, and centers.