#1 (10 pts)

$$\ddot{x} + (x^2 + \dot{x}^2 - \mu)\dot{x} + x = 0$$

a) $\begin{cases} x_1 = x \\ x_2 = \dot{x} \end{cases}$ $\quad \dot{x} = \begin{bmatrix} x_2 \\ -x_1 - (x_1^2 + x_2^2 - \mu)x_2 \end{bmatrix}$

Eq. point $(0,0)$ (only one)

$$A = \begin{bmatrix} 0 & 1 \\ -1 - 2x_1x_2 & -(x_1^2 + x_2^2 - \mu) - 2x_2 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}$$

b) Compare linear vs. nonlinear system

<u>linearization:</u> $\mu = 0$ $\quad \begin{matrix} \tau = 0 \\ \delta = 1 \end{matrix}$ $\Big\rangle$ center

$\mu < 0$ $\quad \begin{matrix} \tau = -\mu \\ \delta = 1 \end{matrix}$ $\Big\rangle$ stable focus $(0 < \mu < 2)$
stable node $(\mu \leq -2)$

$\mu > 0$ $\quad \begin{matrix} \tau = \mu \\ \delta = 1 \end{matrix}$ $\Big\rangle$ unstable focus $(0 < \mu < 2)$
unstable node $(\mu \geq 2)$

<u>Nonlinear:</u>

let $V = \frac{1}{2}(x_1^2 + x_2^2)$

$$\dot{V} = -x_2^2(x_1^2 + x_2^2 - \mu) = x_2^2(\mu - x_1^2 - x_2^2)$$

$\mu \leq 0$, $\dot{V} \leq 0 \longrightarrow$ origin is G.A.S.

$\mu > 0 \longrightarrow$ depends on if $\mu > (x_1^2 + x_2^2)$ or not. To see what is happening, let $r^2 = (x_1^2 + x_2^2)$

If $r > \sqrt{\mu} \longrightarrow \dot{V} < 0$ and $\dot{V} = 0$ when $r = \sqrt{\mu}$

If $r < \sqrt{\mu} \longrightarrow \dot{V} > 0 \Longrightarrow$ there is a limit cycle.

(10 pts)

#2 a) $x = \begin{bmatrix} \gamma \\ \dot{\gamma} \\ E_q \end{bmatrix}$ $\qquad$ $I\dot{x}_2 = P - b x_2 - c_1 x_3 \sin(x_1)$
$\tau \dot{x}_3 = -c_2 x_3 + c_1 \cos(x_1) + E_F$

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{1}{I}(P - b x_2 - c_1 x_3 \sin(x_1)) \\ \frac{1}{\tau}(-c_2 x_3 + c_1 \cos(x_1) + E_F) \end{bmatrix}$$

b) The system is continuously differentiable $\Rightarrow$ locally Lipschitz

The term $c_1 x_3 \sin(x_1)$ is not globally bounded $\Rightarrow$ not globally Lipschitz

#3 (15 pts)

a) True – this is the definition of being positive definite

b) False – it is negative semi-definite.

c) False – uniqueness is guaranteed when a system is locally Lipschitz which is a weaker condition than continuous differentiability.

d) True – it is strictly increasing & zero at the origin.

e) False – will be zero along the line $x_1 = -x_2$.

#4 (10 pts)

$$V = x_1^2 + \frac{x_2^2}{2+\sin(t)}$$

$$\underbrace{x_1^2 + \tfrac{1}{3}x_2^2}_{W_1(x)} \le V \le \underbrace{x_1^2 + x_2^2}_{W_2(x)}$$

$$\dot V = 2x_1\dot x_1 + \frac{2}{(2+\sin t)} x_2 \dot x_2 - \frac{x_2^2}{(2+\sin t)^2}\cos(t)$$

$$= 2x_1x_2 + \frac{2x_2}{(2+\sin t)}\left(-x_2 - (2+\sin t)x_1\right) - \frac{x_2^2 \cos t}{(2+\sin t)^2}$$

$$= \cancel{2x_1x_2} - \frac{2x_2^2}{2+\sin t} \cancel{-2x_1x_2} - \frac{x_2^2\cos t}{(2+\sin t)^2} \le -\frac{1}{2}x_2^2$$

conservative. true min is @ -0.637

$\rightarrow$ The system is stable. That is all that may be determined w/ out using an Invariance-like theorem (LaSalle's does not apply).

#5 (10 pts)

Letting $x = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}$

$$\dot{x} = \begin{bmatrix} \frac{1}{I_1}(I_2 - I_3)\omega_2\omega_3 - \frac{1}{I_1}(k\omega_1 - z_1 + d_1) \\ \frac{1}{I_2}(I_3 - I_1)\omega_1\omega_3 - \frac{1}{I_2}(k\omega_2 - z_2 + d_2) \\ \frac{1}{I_3}(I_1 - I_2)\omega_1\omega_2 - \frac{1}{I_3}(k\omega_3 - z_3 + d_3) \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

b)

Eq. point when $\omega = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ & $z = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

→ Use a change of variables to move eq. point to the origin.

Let $y = z - d$

Then $\begin{bmatrix} \dot{\omega}_1 = \frac{1}{I_1}(I_2 - I_3)\omega_2\omega_3 - \frac{1}{I_1}(k\omega_1 + y_1 \cancel{+ d_1}) + \frac{\cancel{d_1}}{I_1} \\ \dot{\omega}_2 = \quad \cdots \\ \dot{\omega}_3 = \quad \cdots \\ \dot{y}_1 = \omega_1 \\ \dot{y}_2 = \omega_2 \\ \dot{y}_3 = \omega_3 \end{bmatrix}$

Let $V = \underbrace{\frac{1}{2}(I\omega)\cdot\omega}_{\text{rotational K.E.}} + \frac{1}{2}\|y\|^2$

$= \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 + \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + \frac{1}{2}y_3^2$

$$\dot{V} = \dot{\omega}_1(I_1\omega_1) + \dot{\omega}_2(I_2\omega_2) + \dot{\omega}_3(I_3\omega_3) + y_1\dot{y}_1 + y_2\dot{y}_2 + y_3\dot{y}_3$$

$$= (I_2 - I_3)\omega_1\omega_2\omega_3 - (k\omega_1^2 + \omega_1 y_1 + \cancel{\omega_1 d_1}) + \cancel{\omega_1 d_1}$$

$$+ (I_3 - I_1)\omega_1\omega_2\omega_3 - (k\omega_2^2 + \omega_2 y_2)$$

$$+ (I_1 - I_2)\omega_1\omega_2\omega_3 - (k\omega_3^2 + \omega_3 y_3) + y_1\omega_1 + y_2\omega_2 + y_3\omega_3$$

$$= (-k\omega_1^2 - k\omega_2^2 - k\omega_3^2) \leq 0$$

→ Using the Invariance condition:

$\omega \equiv 0 \Rightarrow \dot{\omega} \equiv 0$ only when $z \equiv d$ & $\dot{z} \equiv 0$

$\Rightarrow$ G.A.S.

#6 (10 pts)

Eq. points: $(0, 0, x_3 = \text{anything})$

$$V = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_3^2$$

$$\dot{V} = x_1(x_2 + x_1x_3) + x_2(-x_1 - x_2 + x_2x_3) + x_3(-x_1^2 - x_2^2)$$

$$= -x_2^2$$

Using LaSalle's: $\quad x_2 \equiv 0 \quad \dot{x}_2 \equiv 0 \Rightarrow x_1 \equiv 0$

- x_1 must equal zero, but there is no requirement on x_3
- largest invariant set: $(x_2 = x_1 = 0,\ x_3 = \text{anything})$

\# 7 (10 pts)

Eq. point $(0,0)$ (only one)

Try linearization:

$$A = \begin{pmatrix} 0 & -3x_2^2 \\ 1 & -1 \end{pmatrix}_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \qquad \text{e-values: } \lambda = 0, -1$$

$\longrightarrow$ linearization is inconclusive.

Try direct method:

$$V = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2$$

$$\dot{V} = -x_1x_3^3 + x_2(x_1 - x_2) = -x_1x_2^3 + x_1x_2 - x_2^2$$

$\longrightarrow$ doesn't work, but looks close — try higher power on x_2

Let $V = \tfrac{1}{2}x_1^2 + \tfrac{1}{4}x_2^4$

$$\dot{V} = -x_1x_3^3 + x_2^3(x_1 - x_2) = \cancel{-x_1x_2^3} + \cancel{x_1x_2^3} - x_2^4$$

Using LaSalle's: $\quad x_2 \equiv 0,\ \dot{x}_2 \equiv 0 \Rightarrow x_1 = 0$

$\Rightarrow$ System is G.A.S. @ the origin