# Midterm Exam: Nonlinear Systems
# Winter 2020

To get full credit you must *show all your work* and *clearly mark your answers*. If you can complete a problem without doing any derivations, then *explain why this is true*. The exam is due at the beginning of class on Tuesday 3/10/20.

The following apply:

- For this exam you may use our textbook, class notes, and files posted on LearningSuite (i.e. lecture notes, assignment solutions, etc.), Matlab, Python, and their respective help documentation.
- You are not allowed to search other textbooks, previous course exams, or the internet for answers.
- All work must be completely your own. You are not allowed to collaborate or ask questions of other individuals. Since other students may be taking the exam at different times, then please don't comment, even generally, on the exam.
- To get full credit you must **fully justify all your answers** and they must be **clear and easy to follow**. If you can complete a problem without doing any derivations, then **explain why this is true**.
- If a question is unclear or you don't think you have enough information to proceed, either ask for clarification or state your assumptions and work the problem from there.

There are seven questions to this exam which are spread out over three pages (including this cover sheet).

I certify that the solutions to this exam represent my own work and that I did not consult with any other individual or use reference material other than what was allowed as stipulated above.

Jason Elmiegan

Signature

1).

$$\ddot{x} + (x^2 + \dot{x}^2 - \mu)\dot{x} + x = 0$$

$$z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad \dot{z} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} z_2 \\ -z_1 - (z_1^2 + z_2^2 - \mu) z_2 \end{bmatrix}$$

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = -z_1 - (z_1^2 + z_2^2 - \mu) z_2$$

$$z^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ ✓

$$A = \frac{\partial f}{\partial z} = \begin{bmatrix} 0 & 1 \\ -1 - 2z_1 z_2 & -z_1^2 - 3z_2^2 + \mu \end{bmatrix}\Bigg|_{z^*} = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}$$ ✓

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda - \mu \end{vmatrix} = \lambda(\lambda - \mu) + 1$$
$$= \lambda^2 - \mu\lambda + 1$$
$$\lambda = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

* When $\mu < 0 \Rightarrow$ Asy. Stable / Exp. Stable $\begin{cases} |\mu| < 2 \Rightarrow \text{focus} \\ |\mu| > 2 \Rightarrow \text{node} \end{cases}$

$\mu > 0 \Rightarrow$ unstable

$\mu = 0 \Rightarrow$ undetermined (center for linear)

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$$V = \frac{1}{2}(z_1^2 + z_2^2)$$
$$\dot{V} = z_1 z_2 + z_2 (-z_1 - (z_1^2 + z_2^2 - \mu) z_2)$$
$$= z_1 z_2 - z_1 z_2 - (z_1^2 + z_2^2 - \mu) z_2^2$$
$$= -(z_1^2 + z_2^2 - \mu) z_2^2 < 0 \quad \text{when} \quad \mu < z_1^2 + z_2^2$$

stable when $\mu = z_1^2 + z_2^2$

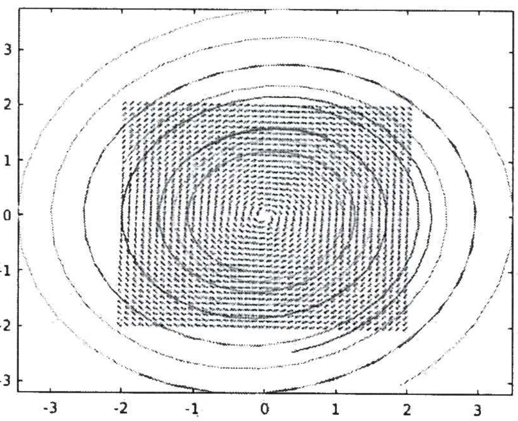
Asy Stable when $\mu < z_1^2 + z_2^2$

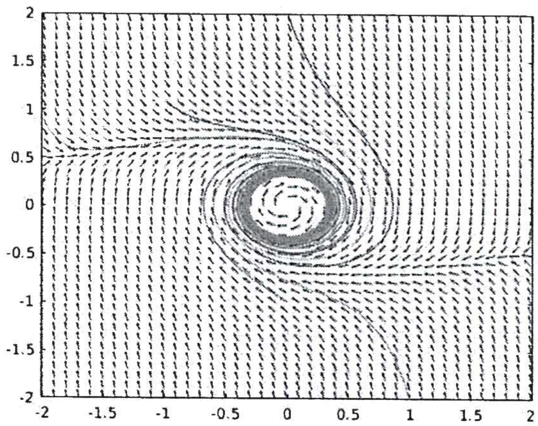
1)

Linear

Non-Linear

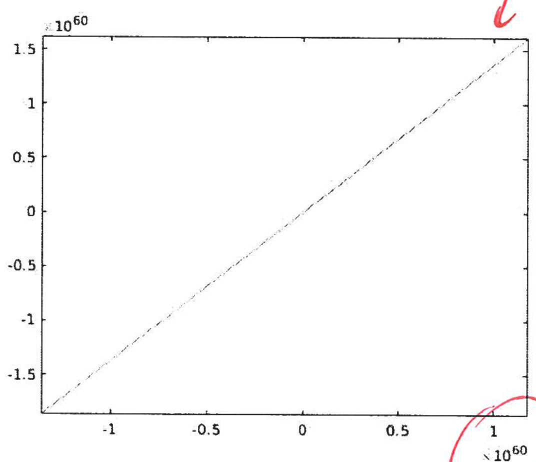
mu = 0.1

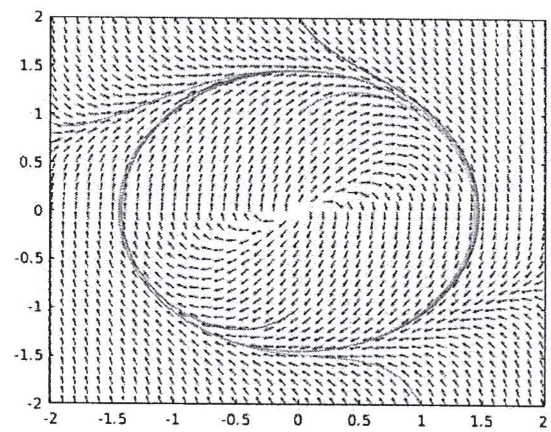




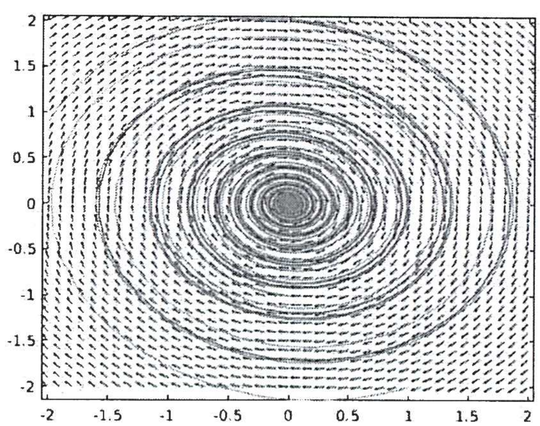
mu = 2.1

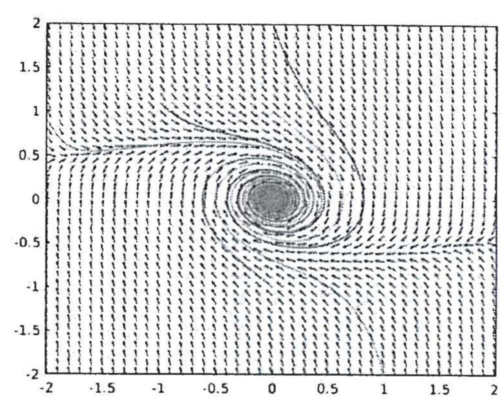
what is happening here?



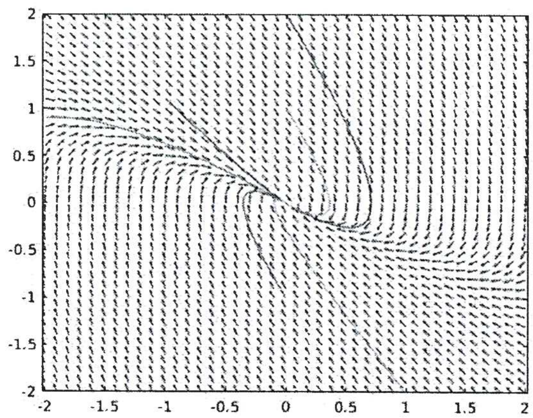



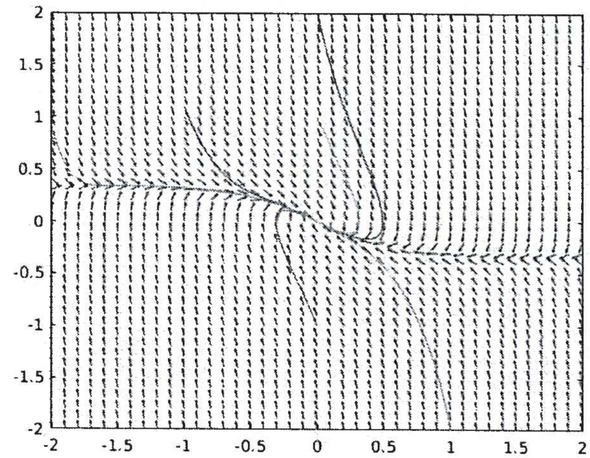
mu = -0.1



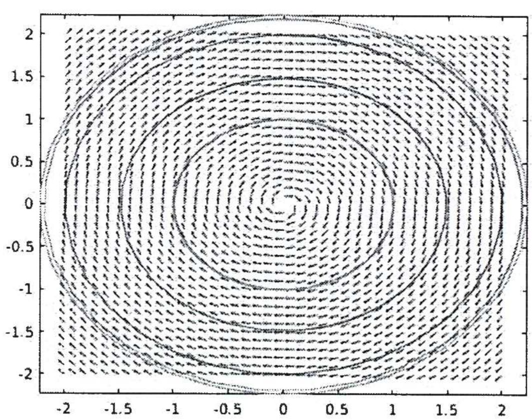


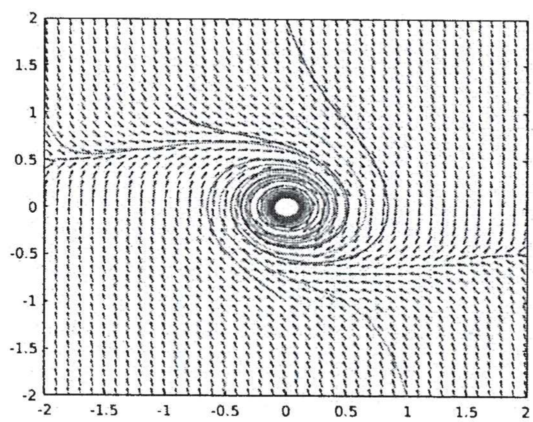
mu = -2.1




mu = 0




The differences between the linear system and the nonlinear system can be seen on the above plots with different values of mu for the linear and nonlinear systems. For the most part the linear system approximates the nonlinear system closely. Where the linear system breaks down is where mu > 0 and mu = 0. When mu > 0 the nonlinear system is very unstable but the nonlinear system appears to find a center to oscillate around. This therefore means that the linear system is accurate up to that specific center. When mu = 0, you cannot approximate the nonlinear system with the linear system. But if we to anyway we would see that the linear system shows a center while the nonlinear shows a stable focus.

→ no centers in nonlinear systems. This is actually a stable limit cycle that can be seen when doing Lyapunov analysis on the system.

-2

2)

a) $I\ddot{\gamma} = p - b\dot{\gamma} - c_1 E_q \sin(\gamma)$

$\tau \dot{E}_q = -c_2 E_q + c_3 \cos(\gamma) + E_F$

$$x = \begin{bmatrix} \gamma \\ \dot{\gamma} \\ E_q \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad f(x) = \dot{x} = \begin{bmatrix} \dot{\gamma} \\ \ddot{\gamma} \\ \dot{E}_q \end{bmatrix} = \begin{bmatrix} x_2 \\ (p - b x_2 - c_1 x_3 \sin(x_1))\frac{1}{I} \\ (-c_2 x_3 + c_3 \cos(x_1) + E_F)\frac{1}{\tau} \end{bmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

b)

a)

$$f'(x) = \begin{bmatrix} 1 \\ (-b - c_1 \sin(x_1) - c_1 x_3 \cos(x_1))\frac{1}{I} \\ (-c_2 - c_3 \sin(x_1))\frac{1}{\tau} \end{bmatrix} \Rightarrow \text{Exists \& Continous}$$

$\therefore$ Continuously Differentiable

* Continously Differentiable $\Rightarrow$ Lipschitz $\Rightarrow$ continous

Lemma 3.2 (NS) because $f(x)$ is continoous

* Not Globally Lipschitz by Lemma 3.3 (NS)

because $f'(x)$ is not uniformly bounded

because of the $x_3 \cos(x_2)$ term.

✓

3)

a) True

b) False $\dot{V}(x) = -x_1^2 - 0x_2^2$ implies negative semi-deffinite

c) False. Thm 3.1 states that a function can be piecewise continous & satisfy the Lipschitz condition then there is a unique solution. This case violates this statement because a piecewise continous function is not continously differentiable.

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d) True $\frac{\partial}{\partial r}\left(\frac{r}{1+r}\right) = \frac{1}{1+r} - \frac{r}{(1+r)^2}$

$$= \frac{1+r-r}{(1+r)^2} = \frac{1}{(1+r)^2} > 0 \Rightarrow \text{strictly increasing.}$$

$$\alpha(0) = \frac{0}{1+0} = 0 \Rightarrow \text{class } \mathcal{K}$$

* See Example 4.16 (NS) 4th Bulet.

e) True X

/3

4)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 - (2+\sin(t))\,x_1$$

$$V = x_1^2 + \frac{x_2^2}{2+\sin(t)}$$

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t)$$

$$= \frac{-x_2^2\cos(t)}{(2+\sin(t))^2} + 2x_1x_2 + \frac{2x_2}{2+\sin(t)}\left(-x_2 - (2+\sin(t))\,x_1\right)$$

$$= \frac{-x_2^2\cos(t)}{(2+\sin(t))^2} - \frac{2x_2^2}{2+\sin(t)}$$

$$\le -0.5\,x_2^2 \checkmark$$

$$\le 0$$

$1 \le 2+\sin(t) \le 3$

$-1 \le \cos(t) \le 1$

if $t = \frac{3\pi}{4}$

$$\frac{-x_2^2\left(-\frac{\sqrt{2}}{2}\right)}{\left(2+\frac{\sqrt{2}}{2}\right)^2} - \frac{2x_2^2}{2+\frac{\sqrt{2}}{2}}$$

$$= \frac{-x_2^2(-2\sqrt{2})}{(4+\sqrt{2})^2} - \frac{4x_2^2}{4+\sqrt{2}}$$

$$= -x_2^2\left(\frac{-2\sqrt{2}}{(4+\sqrt{2})^2} + \frac{4}{4+\sqrt{2}}\right)$$

$$= -0.64230079197\,x_2^2$$

$$\le -0.5\,x_2^2$$

Also:

$$x_1^2 + \frac{x_2^2}{3} \le V = x_1^2 + \frac{x_2^2}{2+\sin(t)} \le x_1^2 + x_2^2$$

$\therefore$ $x=0$ is uniformly stable by Thm 4.8 (NS)

where

$$w_1 = x_1^2 + \frac{x_2^2}{3}$$
$$w_2 = x_1^2 + x_2^2 \checkmark$$

5)

$$I_1 \dot{w}_1 = (I_2 - I_3)\, w_2 w_3 + M_1 + d_1$$
$$I_2 \dot{w}_2 = (I_3 - I_1)\, w_1 w_3 + M_2 + d_2$$
$$I_3 \dot{w}_3 = (I_1 - I_2)\, w_1 w_2 + M_3 + d_3$$

a)

$$\dot{z} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \qquad M = -K \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} - \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$x = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{I_1}\left[(I_2 - I_3)\, x_2 x_3 - K x_1 - x_4 + d_1\right] \\ \frac{1}{I_2}\left[(I_3 - I_1)\, x_1 x_3 - K x_2 - x_5 + d_2\right] \\ \frac{1}{I_3}\left[(I_1 - I_2)\, x_1 x_2 - K x_3 - x_6 + d_3\right] \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$V = \frac{1}{2}\left(I_1 x_1^2 + I_2 x_2^2 + I_3 x_3^2 + (x_4 - d_1)^2 + (x_5 - d_2)^2 + (x_6 - d_3)^2\right)$$

$$= (I_2 - I_3)\, x_1 x_2 x_3 - K x_1^2 - x_1 x_4 + d_1 x_1$$
$$+ (I_3 - I_1)\, x_1 x_2 x_3 - K x_2^2 - x_2 x_5 + d_2 x_2$$
$$+ (I_1 - I_2)\, x_1 x_2 x_3 - K x_3^2 - x_3 x_6 + d_3 x_3$$
$$+ x_4 x_1 + x_5 x_2 + x_6 x_3 - d_1 x_1 - d_2 x_2 - d_3 x_3$$

$$= -K x_1^2 - K x_2^2 - K x_3^2 \le 0 \Rightarrow \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \text{ converge to zero Asymtotically}$$

only negative semi-definite. Why can you guarantee?

6)

$$\dot{x}_1 = x_2 + x_1 x_3$$
$$\dot{x}_2 = -x_1 - x_2 + x_2 x_3$$
$$\dot{x}_3 = -x_1^2 - x_2^2$$

Case #1

~~$x_1 > 0$, $x_2 > 0$, $x_3 > 0$ $\Rightarrow$ $\dot{x}_3 < 0$~~

$$V = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$

$$\dot{V} = x_1(x_2 + x_1 x_3) + x_2(-x_1 - x_2 + x_2 x_3) + x_3(-x_1^2 - x_2^2)$$
$$= \cancel{x_1 x_2} + \cancel{x_1^2 x_3} - \cancel{x_1 x_2} - x_2^2 + \cancel{x_2^2 x_3} - \cancel{x_1^2 x_3} - \cancel{x_2^2 x_3}$$
$$= -x_2^2 < 0$$

La Salle's

$D = \mathbb{R}^2$

$\Omega = \{(x_1, x_2, x_3) \in D \mid V(x) \le c,\ c > 0\}$

$E =$ all $x_1, x_3$ w/in $\Omega$

$$x_2 \equiv 0 \Rightarrow \left.\begin{array}{l} x_1 x_3 = 0 \\ -x_1^2 = 0 \end{array}\right\} \Rightarrow x_1 = 0$$

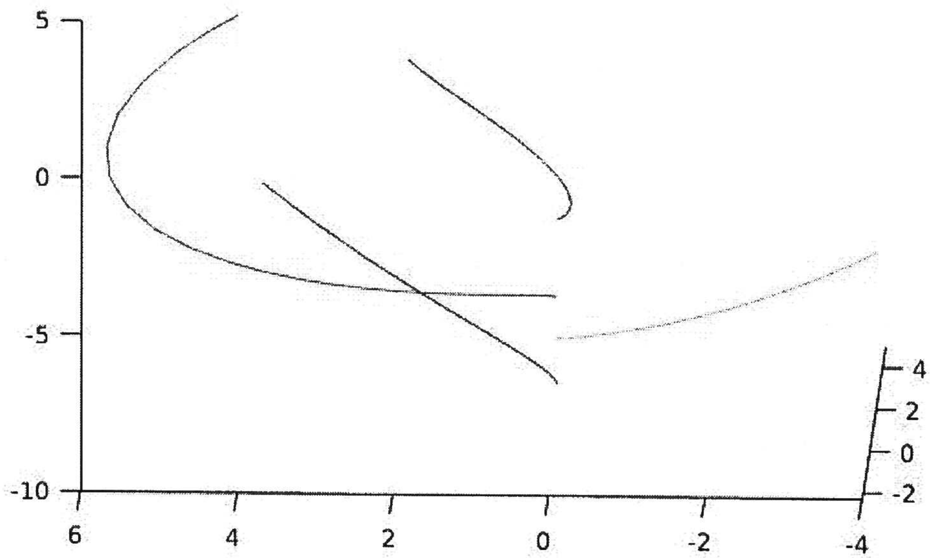
$$M = (0, 0, \mathbb{R})$$

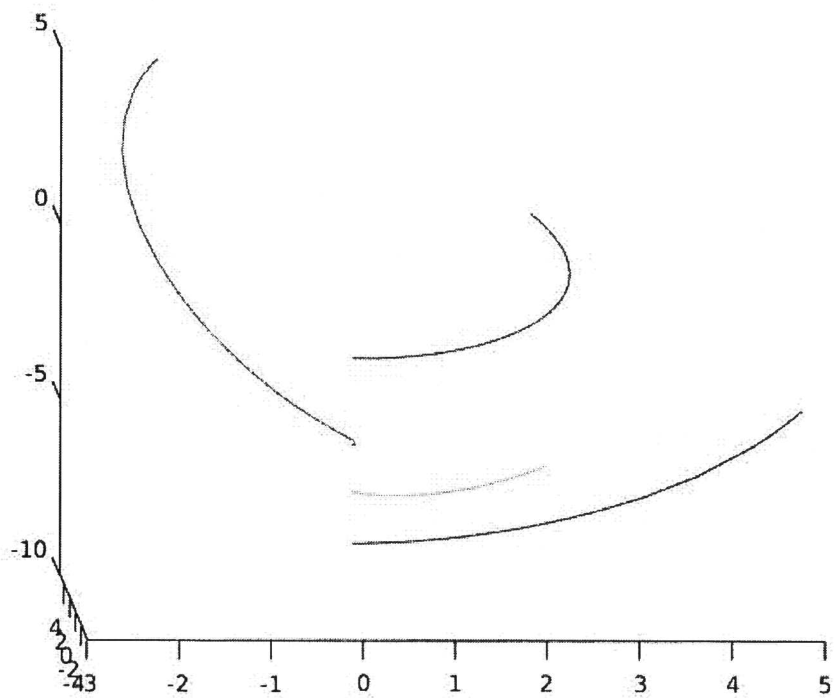
$\Rightarrow$ $x_1, x_2$ are guaranteed to converge to zero.

* This candidate function suggests $x_1$ & $x_2$ are guaranteed to converge to zero.
* Also see plots.

✓

6)
These plots show some sample trajectories converging to x1 = 0 and x2 = 0 but to different values of x3. This is consistent with results form a quadratic Lyapunov function.



7)

$$\dot{x}_1 = -x_2^3$$
$$\dot{x}_2 = x_1 - x_2$$

$$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$V = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4$$

$$\dot{V} = x_1(-x_2^3) + x_2^3(x_1 - x_2)$$

$$= -x_1 x_2^3 + x_1 x_2^3 - x_2^4$$

$$= -x_2^4 < 0$$

<u>La Salle's</u>

$$D = \mathbb{R}^2$$

$$\Omega = \{(x_1, x_2) \in D \mid V(x) \leq c,\ c > 0\}$$

$E$ = all $x_1$ w/in $\Omega$

$x_2 \equiv 0 \implies x_1 = 0$

$M = (0,0) \Rightarrow$ Globally Asymptotically Stable

<u>Thm 4.10</u>

$$V = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4 \leq$$

$\Rightarrow$ Not possible

✓

10/15