Nonlinear Systems: Homework Assignment #3

Due by midnight on Thursday Feb 13th.

Question 1:

For a scalar nonautonomous differential equation in the form $\dot{x} = -a(t)x$, define sufficient conditions on $a(t)$, so that the equilibrium of the scalar dynamics is (1) stable and (2) asymptotically stable.

Question 2:

Consider the Euler equations,

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 + \mu_1$$

$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_1\omega_3 + \mu_2$$

$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 + \mu_3$$

where $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]^T \in R^3$ is the angular velocity with respect to an inertial frame from a body-fixed frame. And $I_1 > I_2 > I_3 > 0$ are the principle moments of inertial. Let $\boldsymbol{h}_G = [I_1\omega_1, I_2\omega_2, I_3\omega_3]^T$ represent the angular momentum of the body about its center of mass, G. $\boldsymbol{\mu} = [\mu_1, \mu_2, \mu_3]^T$ are the torques that represent the control of the system. When $\boldsymbol{\mu} = \mathbf{0}$ these are the Euler's equations for a free rigid body.

a) Assume $\boldsymbol{\mu} = \mathbf{0}.$ Show that $\|\boldsymbol{h}_G\|$ is conserved, i.e. $\frac{d}{dt}\|\boldsymbol{h}_G\| = 0.$
b) Find the equilibrium points of the system when $\|\boldsymbol{h}_G\| = \lambda \neq 0$, still assume $\boldsymbol{\mu} = \mathbf{0}$.
c) Linearize the system about each equilibrium point. What do the non-zero eigenvalues of the Jacobian matrix suggest about the local behavior of the solutions around each equilibrium point?
d) Use MATLAB to plot the phase portrait of the system on the surface of the angular momentum sphere. Let $I_1 = 3, I_2 = 2, I_3 = 1, and\ \lambda = 1.$ Include MATLAB figure and code. (Hint: use ode45 to compute $\omega(t)$ for a variety of initial conditions that satisfy $\|\boldsymbol{h}_G\| = \lambda$; for each solution use plot3 to plot $\boldsymbol{h}_G(t)$.
e) Briefly describe what happens to a free rigid body spun around an axis very nearly aligned with the $\boldsymbol{b}_1$axis (first body axis). The $\boldsymbol{b}_2$ axis? The $\boldsymbol{b}_3$ axis?

Question 3: (NS: 3.20)

Show that if $f: \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz on $W \subset \mathbb{R}^n$ then $f(x)$ is uniformly continuous on W.

Question 4: (NS: 4.3(3))

Consider the system:

$$\dot{x}_1 = x_2(1 - x_1^2)$$
$$\dot{x}_2 = -(x_1 + x_2)(1 - x_1^2)$$

Use a quadratic Lyapunov function candidate to show the origin is asymptotic stability.

Question 5: (NS: 4.14)

Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -g(x_1)(x_1 + x_2)$$

Where g is locally Lipschitz and $g(y) \geq 1$ for all $y \in \mathbb{R}$. Verify that $V(x) = \int_0^{x_1} y g(y) dy + x_1 x_2 + x_2^2$ is positive definite for all $x \in \mathbb{R}^2$ and radially unbounded, and use it to show that the equilibrium point $x = 0$ is globally asymptotically stable.

Question 6: (NS: 4.16)

Show that the origin of

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1^3 - x_2^3$$

Is globally asymptotically stable.

Question 7: (NS: 4.21)

A gradient system is a dynamical system of the form $\dot{x} = -\nabla V(x)$, where $\nabla V(x) = \left[\frac{\partial V}{\partial x}\right]^T$ and $V: D \subset \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable.

a) Show that $\dot{V}(x) \leq 0$ for all $x \in D$, and $\dot{V}(x) = 0$ if and only if x is an equilibrium point.
b) Take $D = \mathbb{R}^n$. Suppose the set $\Omega_c = \{x \in \mathbb{R}^n | V(x) \leq c\}$ is compact for every $c \in \mathbb{R}$. Show that every solution of the system is defined for all $t \geq 0$.
c) Continuing with part (b), suppose $\nabla V(x) \neq 0$, except for a finite number of points $p_1, \dots, p_r$. Show that for every solution $x(t)$, $\lim_{t\to\infty} x(t)$ exists and equals one of the points $p_1, \dots, p_r$.

Question 8: (NS: 4.27)

Consider the system

$$\dot{x}_1 = -x_2 x_3 + 1, \quad \dot{x}_2 = x_1 x_3 - x_2, \quad \dot{x}_3 = x_3^2(1 - x_3)$$

a) Show that the system has a unique equilibrium point.

b) Using linearization, show that the equilibrium point asymptotically stable. Is it globally asymptotically stable?