Midterm Exam: Nonlinear Systems
Winter 2020

To get full credit you must *show all your work* and *clearly mark your answers*. If you can complete a problem without doing any derivations, then *explain why this is true*. The exam is due at the beginning of class on Tuesday 3/10/20.

The following apply:

- For this exam you may use our textbook, class notes, and files posted on LearningSuite (i.e. lecture notes, assignment solutions, etc.), Matlab, Python, and their respective help documentation.
- You are not allowed to search other textbooks, previous course exams, or the internet for answers.
- All work must be completely your own. You are not allowed to collaborate or ask questions of other individuals. Since other students may be taking the exam at different times, then please don't comment, even generally, on the exam.
- To get full credit you must **fully justify all your answers** and they must be **clear and easy to follow**. If you can complete a problem without doing any derivations, then **explain why this is true**.
- If a question is unclear or you don't think you have enough information to proceed, either ask for clarification or state your assumptions and work the problem from there.

There are seven questions to this exam which are spread out over three pages (including this cover sheet).

I certify that the solutions to this exam represent my own work and that I did not consult with any other individual or use reference material other than what was allowed as stipulated above.

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Signature

1. Consider the following nonlinear oscillator, where $x \in R$ and $\mu \in R$:

$$\ddot{x} + (x^2 + \dot{x}^2 - \mu)\dot{x} + x = 0.$$

   a) Rewrite the system in state-space form, find its equilibrium point(s), and linearize about each one.
   b) Compare the solutions and phase portraits of the nonlinear system and its linearization(s) for $\mu < 0, \mu = 0, and\ \mu > 0$. Comment on any differences between the two.

2. A generator can be represented by the equations

$$I\ddot{\gamma} = \rho - b\dot{\gamma} - c_1 E_q \sin(\gamma), \qquad \tau \dot{E}_q = -c_2 E_q + c_3 \cos(\gamma) + E_F$$

Where γ is an angle, E_q is a voltage, ρ is a mechanical input power, E_F is field voltage (input), b is a damping coefficient, I is an inertial coefficient, τ is a time constant, and $c_1, c_2, and\ c_3$ are positive constants. For the problems below, assume that ρ and E_F are constants.

   a) Use $\gamma, \dot{\gamma}, and\ E_q$ as the state variables to put the system into state space form.
   b) Determine if your system is (a) continuously differentiable; (b) locally Lipschitz; (c) continuous; (d) globally Lipschitz. If the condition only holds for a specific domain, then specify that domain.

3. **True or False:** Let $x = [x_1, x_2]^T \in R^2$. If false, then explain your reasoning.
   a. $V(x) = x^T P x$, where $P = P^T$, is positive definite if and only if P is positive definite.
   b. If $\dot{V}(\boldsymbol{x}) = -x_1^2$ along solutions of $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, then $\dot{V}$ is negative definite.
   c. If a system $\dot{\boldsymbol{x}} = f(x)$ is **not** continuously differentiable, then we can't guarantee that there will be a unique solution, $x(t)$.
   d. The function $f(r) = \frac{r}{1+r}$ is a class-$\mathcal{K}$ function.
   e. $V(x) = \frac{1}{2}(x_1 + x_2)^2$ is radially unbounded.

4. Consider the system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 - (2 + \sin(t))x_1$$

Show that $V = x_1^2 + \frac{x_2^2}{2+\sin(t)}$ is a valid Lyapunov candidate function and indicate what is the strongest statement (if any) that you can say about the system concerning stability? Fully justify your response.

5. Consider a rigid spacecraft that evolves according to the Euler equations,

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 + M_1 + d_1$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_1\omega_3 + M_2 + d_2$$
$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 + M_3 + d_3,$$

where $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]^T \in R^3$ is the angular velocity with respect to an inertial frame from a body-fixed frame. And $I_1 > I_2 > I_3 > 0$ are the principle moments of inertial and $\boldsymbol{M} = [M_1, M_2, M_3]^T$ is the external torque applied by means of a rocket thruster. And the spacecraft is subjected to a constant external disturbance $\boldsymbol{d} = [d_1, d_2, d_3]^T$.

a) Implement the dynamic state feedback control law given by:

$$\boldsymbol{M} = -K\boldsymbol{\omega} - \boldsymbol{z}$$
$$\dot{\boldsymbol{z}} = \boldsymbol{\omega}.$$

Where $K > 0$. What is the new state space form?

b) Use Lyapunov's direct method to prove that this controller yields a stable closed-loop system and ensures asymptotic convergence of $\boldsymbol{\omega}(t)$ to zero for any $\boldsymbol{d}$ and any initial conditions $\boldsymbol{\omega}(0)$.

6. Which, if any, of the state variables for the following system are guaranteed to converge to zero:

$$\dot{x}_1 = x_2 + x_1x_3$$
$$\dot{x}_2 = -x_1 - x_2 + x_2x_3$$
$$\dot{x}_3 = -x_1^2 - x_2^2$$

7. Given the following system:

$$\dot{x}_1 = -x_2^3$$
$$\dot{x}_2 = x_1 - x_2$$

What is the strongest stability condition you can say about it? Is it stable, asymptotically stable, exponentially stable, globally asymptotically stable, globally exponentially stable?