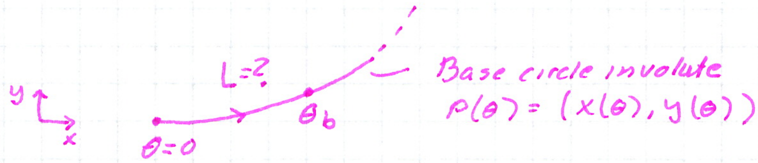
Analytical solution to the length of circle involute



$$\begin{cases} x(\theta) = r(\cos\theta + \theta\sin\theta) \\ y(\theta) = r(\sin\theta - \theta\cos\theta) \end{cases} \quad ; \quad \begin{aligned} x'(\theta) &= r\theta\cos\theta \\ y'(\theta) &= r\theta\sin\theta \end{aligned}$$

Line integral:

$$L = \int_{\theta=0}^{\theta_b} ds = \int_{\theta=0}^{\theta_b} \sqrt{(x'(\theta))^2 + (y'(\theta))^2}\, d\theta = \int_{\theta=0}^{\theta_b} \sqrt{(r\theta\cos\theta)^2 + (r\theta\sin\theta)^2}\, d\theta =$$

$$\int_{\theta=0}^{\theta_b} \sqrt{r^2\theta^2 \underbrace{(\sin^2\theta + \cos^2\theta)}_{=1}}\, d\theta = r\int_{\theta=0}^{\theta_b} \theta\, d\theta = r\left[\frac{1}{2}\theta^2\right]_{\theta=0}^{\theta_b} + C =$$

$$\frac{r\theta_b^2}{2} + C \quad , \quad C = L(0) = 0$$

$$\boxed{L(\theta) = \frac{r\theta^2}{2}}$$