INF5620 Project
Extended Navier-Stokes solver for Platelet Aggregation

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Abstract

The aim of the project is to solve an extended set of Navier-Stokes equations by the finite element method and *fenics*. The model I'm using is a simplification of model found in Aaron L. Fogelson article: **Continuum models of platelet aggregation: Formulation and mechanical properties**. I will focus mainly on equation 1 - 3 in this article. My solver is an extantion of *fenics* Navier-Stoke demo.

The Model

I focus on equation 1 - 3 from the article, witch are:

$$\rho\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\Delta\mathbf{u} + \mathbf{f}^g \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{2}$$

$$\frac{\partial\phi}{\partial t} + \mathbf{u}\cdot\nabla\phi = C\Delta\phi - R\left(c\right)\phi \tag{3}$$

The first two equation are Navier-Stokes equations for incompressible fluid. $\mathbf{u}\left(t,\mathbf{x}\right)$ is the fluid velocity field, $p\left(t,\mathbf{x}\right)$ is the pressure, ρ is the fluid's mass density(I'll sett this equal to 1, for simplicity.) and μ is the fluids viscosity(assumed to be constant). $\epsilon^{-3}\phi$ is the concentration of non-activated platelet(ϵ^{-3} is a scaling constant). C is a diffusion constant[1]. The term $R(c)\,\phi$ is the rate in which the non-activated platelets are converted to active platelets. This depends on the ADP concentration c, which vary. Since I'm not using the equation for c, I will have to manufacture a suitable function $c(\mathbf{x}, t)$[2].

Solving Navier-Stokes equation

Now I'm in detail going to talk about one way of solving the Navier-Stokes equations(1 and 2) with *fenics* and the finite element method. I will use Chorin's projection method. The idea is first to compute a tentative velocity($\mathbf{u}^\star$) by ignoring the pressure in equation 1 and then project the velocity onto the space of divergence free vector fields. Equation 1 then becomes:

$$\frac{\partial}{\partial t}\mathbf{u}^\star + \mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u}^\star = \mathbf{f}^g$$

Let $V = H_0^1(\Omega)$ be a Sobolev space and Ω the domain. I want to write the equation above as a weak formulation. I multiply with the function $\mathbf{v}(\mathbf{x})$ and integrate the space.

$$\int_\Omega \frac{\partial \mathbf{u}^\star}{\partial t}\cdot\mathbf{v} + (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{v} - \mu\Delta\mathbf{u}^\star\cdot\mathbf{v}d\mathbf{x} = \int_\Omega \mathbf{f}^g\cdot\mathbf{v}d\mathbf{x} \quad \forall\mathbf{v}\in V$$

Now if I integrate by parts the lest term in the left hand side(or use Green's formula), I obtain:

$$\int_\Omega \frac{\partial \mathbf{u}^\star}{\partial t}\cdot\mathbf{v} + (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{v} + \mu\nabla\mathbf{u}^\star\cdot\nabla\mathbf{v}d\mathbf{x} = \int_\Omega \mathbf{f}^g\cdot\mathbf{v}d\mathbf{x} \quad \forall\mathbf{v}\in V$$

Finely I discretesize the time(n) and space(h), $V_h \subset V$. Then $\mathbf{u}$ becomes $\mathbf{u}_h^n$. I define $\langle\cdot,\cdot\rangle$ as the $L^2(\Omega)$ norm. The above equation can be written as:

$$\langle D_t^n\mathbf{u}_h^\star, \mathbf{v}\rangle + \left\langle \mathbf{u}_h^{n-1}\cdot\nabla\mathbf{u}_h^{n-1}, \mathbf{v}\right\rangle + \langle\mu\nabla\mathbf{u}_h^\star, \nabla\mathbf{v}\rangle = \langle\mathbf{f}^{g,n}, \mathbf{v}\rangle \quad \forall\mathbf{v}\in V_h \tag{4}$$

The projection give us these two equation:

$$\frac{\mathbf{u}_h^n - \mathbf{u}_h^\star}{k_n} + \nabla p_h^n = 0 \tag{5}$$

[1]Fogelson article uses D_n and ϕ_n for C and ϕ.

[2]To understand the equations and symbols better, I advise you to take a look at the first few first pages of Fogelson article

$$\nabla \cdot \mathbf{u}_h^n = 0 \tag{6}$$

where k_h is the size of the local time step. I now multiply 5 with ∇q where $q \in Q_h \subset L^2(\Omega)$ and integrate.

$$\frac{1}{k_n} \int_\Omega \mathbf{u}_h^n \cdot \nabla q - \mathbf{u}_h^\star \cdot \nabla q + k_n \nabla p_h^n \cdot \nabla q dx = 0 \quad \forall q \in Q_h$$

I integrate by parts the first two terms. The first term becomes zero from equation 6 and $\mathbf{u} \mid_{d\Omega} = 0$.

$$\int_\Omega (\nabla \cdot \mathbf{u}_h^\star) \, q/k_n + (\nabla p_h^n \cdot \nabla q) \, dx = 0 \quad \forall q \in Q_h$$

This can be written as:

$$\langle \nabla p_h^n, \nabla q \rangle = - \langle \nabla \cdot \mathbf{u}_h^\star, q \rangle / k_n \quad \forall q \in Q_h \tag{7}$$

Finally I multiply equation 5 with $\mathbf{v} \in V_h$ and integrate. I obtain:

$$\langle \mathbf{u}_h^n, v \rangle = \langle \mathbf{u}_h^\star, v \rangle - k_n \langle \nabla p_h^n, v \rangle \quad \forall v \in V_h \tag{8}$$

The three equation 4, 7 and 8 is the Chorin's projection method scheme for solving Navier-Stokes equation. First solve for $\mathbf{u}_h^\star$ in equation 4, the solve the pressure p_h^n in equation 7 and finally find the velocity $\mathbf{u}_h^n$ in equation 8. Repeat this for each time-step.

Stability

When running the demo program I note that the speed is less then $1mm/s$. My model is suppose to model blood flow throw arteries. The velocities throw arteries is about $1000mm/s$, hence I need to turn up the pressure to acquire this velocity.
After increasing the velocity in the demo program, it became unstable. The velocity started fluxing and became non-physically high. It turned out to be advection-diffusion that cause the unstable behaver: $\mathbf{u} \cdot \nabla \mathbf{u} \gg \mu \Delta \mathbf{u}$. Similar problem will probably happen for the extension: $\mathbf{u} \cdot \nabla \phi \gg \mu \Delta \phi$.

The extension

As mention earlier I have extended an already existing program. Making it also solve equation 3. In order to do this I have to rewrite 3 to variation form. I multiply the test function $\psi \in \Psi \subset H_0^1(\Omega)$ and integrate.

$$\int_\Omega \left(D_t\phi\right)\psi + \mathbf{u}\cdot\left(\nabla\phi\right)\psi - C\left(\Delta\phi\right)\psi dx = -\int_\Omega R(\mathbf{x},t)\phi\psi dx \quad \forall\psi\in\Psi$$

after integrating by part and using backward euler as D_t, I obtain:

$$\int_\Omega \left(\frac{\phi^n-\phi^{n-1}}{k_n}\right)\psi+\mathbf{u}\cdot\left(\nabla\phi^n\right)\psi+C\left(\nabla\phi^n\cdot\nabla\psi\right)+R\left(\mathbf{x},t^n\right)\phi^n\psi dx = 0 \quad \forall\psi\in\Psi$$

I end up with the scheme:

$$\left\langle\phi^n-\phi^{n-1},\psi\right\rangle\frac{1}{k_n}+\left\langle\psi\nabla\phi^n,\mathbf{u}^n\right\rangle+C\left\langle\nabla\phi^n,\nabla\psi\right\rangle+\left\langle R\phi^n,\psi\right\rangle=0 \quad \forall\psi\in\Psi_h \tag{9}$$