**HITTING TIME AND THE KEMENY CONSTANT**

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Abstract. In this paper, we prove that the hitting time cannot be controlled by the Kemeny constant.

**Key words.** Markov chain, hitting time, Kemeny constant

First, Construct a complete graph $K_n$ with $n$ vertices $\{a_1, a_2, \cdots, a_n\}$, with each edge having the weight 1, and let $a_0$ be an added vertices such that it is only connected with $a_1$ and the edge connecting $a_0$ and $a_1$ has the weight $w$.

This weighted graph can naturally induce a Markov chain with each vertex as the state, where

$$p_{ij} = \frac{w(i,j)}{\sum_i w(i,j)}$$

We will prove that the hitting time from $a_i (i \geq 1)$ cannot be controlled by the Kemeny constant by the following two claim:

Denote $E_i(\tau_j) := E[\inf\{t \geq 0 : X(t) = a_j | X(0) = a_i\}]$, $E_i(\tau_j^+) := E[\inf\{t > 0 : X(t) = a_j | X(0) = a_i\}]$ First:

$$\lim_{w \to 0} E_1(\tau_0) = \infty$$

It is clear since $E_0(\tau_0^+) = E_1(\tau_0) + 1$ and $E_0(\tau_0^+) = \frac{1}{\pi(0)} \to 0 \ as \ w \to 0$.

However, the kemeny constant can be controlled by a constant when $w < 1$:

Denote

$$K = \sum_{i \neq 0} \pi(i) E_0(\tau_i) = \sum_{i \neq 0} \pi(i) E_1(\tau_i) + 1$$
$$= \sum_{i \neq 0,1} \pi(i) E_1(\tau_i) + 1 < E_1(\tau_2) + 1$$

The last inequality holds since the symmetry of the states $a_2, \cdots, a_n$ and $\sum_{i\neq 0,1} \pi(i) < 1$

However, $E_1(\tau_2)$ is an increasing function of $t$, and thus is controlled by the case when $w < 1$.

So in this example, the hitting time cannot be controlled by the Kemeny constant.

However, when we fix $t = 1$, we find that the hitting time can be controlled by the product of $n + 1$ and Kemeny constant by the following data, and further, we find that

$$\lim K(n+1) - n^2 = 2$$

and this formula is verified by progressive analysis in the paper submitted by Qi Zhou.

| n | K | K(n+1) | $t_{hit}$ |
|---|---|---|---|
| 1 | 0.5 | 1 | 1 |
| 2 | 1.5 | 4.5 | 5 |
| 3 | 2.5417 | 10.167 | 9 |
| 4 | 3.5 | 17.5 | 16 |
| 5 | 4.4455 | 26.673 | 25 |
| 10 | 9.2652 | 101.92 | 100 |
| 20 | 19.142 | 401.98 | 400 |
| 50 | 49.059 | 2502 | 2500 |
| 100 | 99.028 | 10002 | 10000 |
| 300 | 299.01 | 90002 | 90000 |

Conjecture: In an unweighted graph with $n$ vertices, the following relationship exists:

$$\max_{i,j} E_i(\tau_j^+) \leq nK$$

where $K$ is the kemeny constant of this graph. This result can be obtained through a lot of computer practice, but the author has not given proof here.

However, I think it can be divided into the following steps to prove the conclusion:

In a graph with $n$ vertices, fix one vertex $V_0$ and look into the following function $f(w(i,j) : 1 \leq i \neq j \leq n)$:

$$f = \frac{\max_{i,j} E_i(\tau_j^+)}{K}$$

this is a function of all weight of edges, which is 0 or 1, and to prove that this function monotonically increases as $w(0,i)$, and monotonously decreases as $w(i,j)(i,j \neq 0)$.

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