THE DISTANCE AND RESISTANCE DISTANCE ON FINITE POINTS

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Abstract. In this paper, we mainly focus on an interesting question: Is every distance the resistance distance. The answer is No, for the square root of the resistance distance is a Euclidean distance. However it is not a sufficient condition, even if restricted to the graph of finite points. We give an algorithm of constructing the electrical network for a given resistance distance between finite points. By the way, we can also use this algorithm to determine whether a given distance between finite point is a resistance distance and give a counterexample.
Key words. resistance distance, Euclidean distance, Laplacian matrix, adjacency matrix, Moore–Penrose inverse

1. Introduction

The concept 'Resistance distance' was first proposed by D.J.Klein, and M. Randić in 1993 [DM], which has many interesting results and applications in physics and computer science. It is easy to check that the resistance distance on graph is actually a distance, but is every distance the resistance distance?

In 2002, D. Babić and several other mathematicians gave a simple algorithm for resistance distance [DD], which provided convenience for the development of this paper. In this paper, we first discuss the result in [DD]. Then, we give a result that may seem disappointing: Not every distance are resistance distance, even it is over the finite points and is the square of a Euclidean distance.

In the following, G is a weighted, undirected and connected graph with no loops. Denote by $A(G)$, $D(G)$, $L(G)$ (or simply A, D, L in the absence of ambiguity) the adjacency matrix, the degree matrix, the Laplacian matrix of a given graph G, J the matrix containing all 1s, M^+ the Moore–Penrose inverse of M.

2. Resistance distance and its calculation

In graph theory, the resistance distance between two vertices i and j of a weighted, undirected and connected graph G with no loops, denoted as $r(i, j)$, is defined as the total energy consumed by the unit current from i to j, where the weight of each edge represents the conductance of this edge. It is easy to verify that resistance distance, which means that $r(i, j)$ satisfies the properties of nonnegativity, symmetry and triangle inequality.

We can use a simple algorithm to calculate the resistance distance between each two points[DD].

Theorem 2.1. *Given a graph $G = (V, E)$ with edge weights w_{ij} for edge $(i, j) \in E$, $|V| = n$. L is the Laplacian matrix of the G, denote $\Gamma(G) = (L(G) + \frac{1}{n}J)^{+}$. Then the resistance between two points i and j is*

$$r(i, j) = \Gamma(i, i) + \Gamma(j, j) - 2\Gamma(i, i)$$

Since the Laplacian L is symmetric and positive semi-definite so is $L+\frac{1}{n}J$, thus its pseudo-inverse Γ is also symmetric and positive semi-definite. Thus, there is a K such that $Gamma = KK^{\intercal}$ and we can write:

$$r(i, j) = \Gamma(i, i)+\Gamma(j, j)-2\Gamma(i, i) = K_iK_i^{\intercal}+K_jK_j^{\intercal}-K_iK_j^{\intercal}-K_jK_i^{\intercal} = |K_i-K_j|^2$$

showing that the square root of the resistance distance corresponds to the Euclidean distance in the space spanned by K.

3. The distance and resistance distance on finite point

Lemma 3.1. *Let A be a symmetric matrix and $B = A^{+}$. Then B is symmetric.*

Proof. Because B is the Moore–Penrose inverse of A, it satisfies:

- $ABA = A$;
- $BAB = B$;
- AB and BA are symmetric.

Because $A^{\intercal} = A$, taking the transposes of these equations implies that:

- $AB^{\intercal}A = A$;
- $B^{\intercal}AB^{\intercal} = B^{\intercal}$;
- $B^{\intercal}A$ and $AB^{\intercal}$ are symmetric.

Thus $B^{\intercal}$ is also the Moore–Penrose inverse of A. By uniqueness, it follows that B is symmetric. □

And it is easy to see the following proposition:

Proposition 3.2. *Given a graph $G = (V, E)$, $\Gamma(G)(i, j)$ is a symmetric matrix.*

Proposition 3.3. *Given a graph $G = (V, E)$, the sum of each row of $\Gamma(G)(i, j)$ is* 1.

The fist proposition is easily induced by Lemma 3.1, and the second proposition can be seen from the spectral decomposition of $L + \frac{1}{n}J$ and $(L + \frac{1}{n}J)^{+}$ that they both have an eigenvalue 1 and corresponding eigenvector $(1, \cdots, 1)^{\intercal}$.

Lemma 3.4. *If B is the Moore–Penrose inverse of A, then A is the Moore–Penrose inverse of B.*

It follows from the definition of Moore–Penrose inverse.

Now, we look at the algorithm of resistance distance, and together with the proposition of $\Gamma(G)$,we can get:

$$\Gamma(i,j) = \Gamma(j,i), \quad \forall i \neq j$$
$$\Gamma(i,i) + \Gamma(j,j) - 2\Gamma(i,j) = r(i,j) \quad \forall i \neq j$$
$$\sum_{i=1}^{n} \Gamma(i,j) = 1, \quad \forall j$$

The last two condition actually contain $\frac{n(n+1)}{2}$ linear equations for $\frac{n(n+1)}{2}$ variables, which implies that it must have a solution.

So, given a resistance distance, from the discussion above, we can use the method below to construct an electrical network conversely:

By simple calculation, we can get the result:

$$\Gamma(i,i) = \frac{n-1}{n^2}\sum_{j=1}^{n} r(i,j) - \frac{1}{2n^2}\sum_{j,k\neq i} r(j,k) + \frac{1}{n}$$

$$\Gamma(i,j) = -\frac{n^2-2n+2}{2n^2} r(i,j) + \frac{n-2}{n^2}(\sum_{k\neq i,j} r(i,k) + r(k,j)) - \frac{1}{2n^2}\sum_{k,l\neq i,j} r(k,l) + \frac{1}{n}$$

After getting $\Gamma(G)$, we can simply get

$$L = \Gamma^{+} - \frac{1}{n}J,$$

which is the Laplacian matrix of the graph, the adjacent matrix is obtain from $-L$ letting the diagonal be zero. So, we can see that the a_{ij} the conductance of the edge (i,j).

The symmetry of the adjacency matrix is naturally derived from the symmetry of the distance. However, this algorithm cannot guarantee that a_{ij} is nonnegative. In fact, we have the following counterexample:

Example 3.5. Assume there are four points A, B, C, D, and the distance between them are $d(A,B) = d(A,c) = d(B,C) = d(B,D) = d(C,D) = 1$, $d(A,D) = d$. It is a resistance distance if and only of $0 < d \leq 1.5$, when $1.5 < d \leq 2$, it cannot be a resistance distance.

We see the above counterexample does not contradict to the property of resistance distance that it is the square root of a Euclidean distance. Thus, this condition is not the sufficient condition for a distance to be a resistance distance.

References

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