CLUSTER ALGEBRAS ARISING FROM SURFACES

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ABSTRACT. We review some basic concept and properties of cluster algebras arising from bordered surfaces with marked points. We analyze the relationship between the cluster algebras, upper subordinate algebras, and lower cluster algebras derived from the surfaces, and do some calculations for three cases of surfaces.

1. INTRODUCTION

Cluster algebras were introduced by Fomin and Zelevinsky as a tool for studying total positivity and dual canonical bases in Lie theory [2]. Later, the connection between the theory of cluster algebras and other fields in mathematics have been discovered. Cluster algebras now have wide applications in Teichmüller theory, tropical geometry, integrable systems, and Poisson geometry.

Cluster algebras are closely related to the triangulations of a bordered surface, which was defined and studied by M. Gekhtman, M. Shapiro, and A. Vainshtein and, in a more general setting, by V. Fock and A. Goncharov, making use of foundational work by R. Penner and V. Fock.

In the second section, we introduce some basic background from combinatorial topology of surfaces. In the third section, we give the definition of cluster algebras from quiver and illustrate the how the cluster algebras arise from a bordered surface with marked points. In the fourth section, we introduce the Laurent phenomenon, define the upper cluster algebras and lower cluster algebras, and analyze the properties and their relationships. In the last section, we calculate the cluster algebras from three classic type of surfaces: unpunctured d-gons, once punctured d-gons and once punctured closed surfaces.

2. TRIANGULATED SURFACES AND BORDERED SURFACES

This section introduce some basic background from combinatorial topology of surfaces.

2.1. **Bordered surface with marked points.** We first set up some notation and terminology to be used in this paper.

Definition 2.1 (Bordered surfaces with marked points)**.** Let $\mathbf{S}$ be a connected oriented 2-dimensional Riemann surface with boundary $\partial\mathbf{S}$. Fix a finite set $\mathbf{M} \subset \mathbf{S}$ of *marked points* in the closure of $\mathbf{S}$. Marked points in the interior of $\mathbf{S}$ are called *punctures*.

we always assume that $\mathbf{M}$ is nonempty, and there is at least one marked point on each connected component of the boundary of $\mathbf{S}$.

However, we do not consider the following cases:

(1) a sphere with one, two or three punctures;
(2) an unpunctured or once-punctured monogon;
(3) an unpunctured digon;
(4) an unpunctured triangle.

where an m-gon is a disk with m marked points on the boundary.

We define the following data for a bordered surface with marked points $(\mathbf{S}, \mathbf{M})$.

(1) g is the genus of the original Riemann surface;
(2) b is the number of boundary components;
(3) p is the number of punctures;
(4) c is the number of marked points on the boundary of S.

Up to homeomorphism, $(\mathbf{S}, \mathbf{M})$ can be determined by these four numbers.

2.2. **Arcs and triangulations.** We now present two important concept: Arcs and triangulations and we consider arcs up to isotopy.

Definition 2.2 (Isotopy)**.** A *homotopy* between two continuous maps $f, g : X \to Y$ is a continuous map $h : [0,1] \times X \to Y$ such that $h(0,x) = f(x)$ and $h(1,x) = g(x)$. An *isotopy* is a homotopy h such that for all $t \in [0,1]$ the map $h(t,-) : X \to h(t,X)$ is a homeomorphism.

Definition 2.3 (Arcs)**.** An arc γ in $(\mathbf{S}, \mathbf{M})$ is a curve in $\mathbf{S}$, considered up to isotopy, such that the following four conditions are satisfied.

(1) the endpoints of γ are both in $\mathbf{M}$;
(2) γ does not cross itself, except that its endpoints may coincide;
(3) except for the endpoints, γ is disjoint from $\mathbf{M}$ and from $\partial\mathbf{S}$;
(4) γ does not cut out an unpunctured monogon or an unpunctured digon.

We denote by $\mathbf{A}^\circ(\mathbf{S}, \mathbf{M})$ the all arcs in $(\mathbf{S}, \mathbf{M})$.

Remark 2.4. Curves which connected two marked points on the boundary of $\mathbf{S}$ and lie entirely on the boundary are called *boundary segment*. However, boundary segments are actually not arcs

Remark 2.5. A *loop* is an arc whose endpoints coincide.

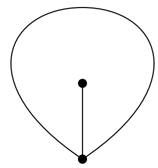

FIGURE 1. Self-fold ideal triangle

Notably, $\mathbf{A}^{\circ}(\mathbf{S}, \mathbf{M})$, The set of all arcs in $(\mathbf{S}, \mathbf{M})$ is typically infinite, and the cases when $\mathbf{A}^{\circ}(\mathbf{S}, \mathbf{M})$ is finite are given by the following proposition.

Proposition 2.6. [1] *The set* $\mathbf{A}^{\circ}(\mathbf{S}, \mathbf{M})$ *of all arcs in* $(\mathbf{S}, \mathbf{M})$ *is finite if and only if* $(\mathbf{S}, \mathbf{M})$ *is an unpunctured or once-punctured polygon.*

Definition 2.7 (Compatibility of arcs)**.** Two arcs are called *compatible* if they do not intersect in the interior of $\mathbf{S}$; To be specific, there are curves in their respective isotopy classes which do not intersect in the interior of $\mathbf{S}$.

Remark 2.8. Each arc is compatible with itself.

Given the concept of arcs, we describe the ideal triangulation of a surface with marked points.

Definition 2.9 (Ideal triangulations)**.** An *ideal triangulation* is a maximal collection of distinct pairwise compatible arcs (together with all boundary segments). The arcs of a triangulation cut the surface $\mathbf{S}$ into *ideal triangles.* The three sides of an ideal triangle may be not distinct, and triangles that have only two distinct sides are called *self-fold* triangles.

Remark 2.10. A self-fold triangle actually consists of a loop and an folded arc which is shown in Figure 1.

Example 2.11. Let $(\mathbf{S}, \mathbf{M})$ be a once punctured triangle, there are 10 ideal triangulations, which can be obtained from the four cases in Figure 2 by rotations.

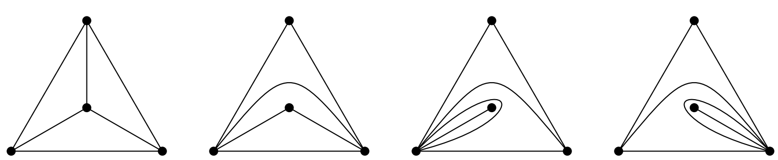

FIGURE 2. Ideal triangulations of a once-punctured triangle

Lemma 2.12. *Let $(\mathbf{S},\mathbf{M})$ be a bordered surface with marked points. The number of arcs in an ideal triangulation of $(\mathbf{S},\mathbf{M})$ is exactly*

$$n = 6g + 3b + 3p + c - 6,$$

where g is the genus of $\mathbf{S}$, b is the number of boundary components, p is the number of punctures and c is the number of marked points on the boundary of $\mathbf{S}$. The number n is called the rank of $(\mathbf{S},\mathbf{M})$.

Proof. Let $\chi(\mathbf{S})$ be the Euler characteristic of $\mathbf{S}$. We know that $\chi(\mathbf{S}) = 2-2g$ for a closed surface and $\chi(\mathbf{S}) = 2-2g-b$, if the boundary of $\mathbf{S}$ has b connected components.

Moreover, If $p = 0$, $\chi(\mathbf{S}) = f - e + v$, where $v = c + p$ is the number of vertices, $e = c + n$ is the number of edges, and f is the number of faces in any triangulation of $\mathbf{S}$ satisfying $3f = 2n + c$.

Now suppose that $p > 0$. After adding a puncture, we need to add 3 arcs to complete the triangulation, and thus adding a puncture increases n by 3 . □

Corollary 2.13. *For each positive n, there are only finitely many bordered surfaces with marked points whose rank is equal to n under homeomorphism.*

Example 2.14. Now we list all cases of the surfaces with marked points where n is small.

$n = 1$: unpunctured square (type A_1);
$n = 2$: unpunctured pentagon (type A_2);
once-punctured digon (type $A_1 \times A_1$);
annulus with one marked point on each boundary component (type $\widetilde{A}(1,1)$);
$n = 3$: unpunctured hexagon (type A_3);
once-punctured triangle (type A_3);
annulus with one marked point on one boundary component and two marked points on another (type $\widetilde{A}(2,1)$);
once-punctured torus.

$(\mathbf{S},\mathbf{M})$	g	b	p	c	n
$(n+3)$-gon	0	1	0	$n+3$	n
n-gon, 1 puncture	0	1	1	n	n
annulus, n_1+n_2 marked points	0	2	0	n_1+n_2	n_1+n_2
$(n-3)$-gon, 2 punctures	0	1	2	$n-3$	n
torus, 1 puncture	1	0	1	0	3

Lemma 2.15. *A bordered surface with marked points has a triangulation without self-folded triangles.*

Proof. Induction on n. Let $\mathbf{S}'$ be the close oriented surface obtained by gluing a disk to each boundary component of $\mathbf{S}$. If the genus of $\mathbf{S}'$ is positive, then there is a loop γ in $(\mathbf{S}, \mathbf{M})$ that goes around a handle in $\mathbf{S}'$. Cut $(\mathbf{S}, \mathbf{M})$ open along γ, and proceed by induction.

It remains to treat the case of a sphere with holes and punctures. In the absence of boundary, connect the punctures cyclically by nonintersecting arcs; the polygons on both sides of the resulting closed curve can be triangulated without self-folded triangles. If there is boundary, say with components labeled $1, \ldots, b$, then draw nonintersecting arcs connecting components 1 and 2, 2 and 3 , etc., and cut them open to reduce the claim to the case of one boundary component (a disk with punctures). There will be at least 3 marked points altogether, including at least one marked point on the boundary of the disk. Direct inspection shows that such a punctured disk can be triangulated without self-folded triangles. $\square$

Definition 2.16 (Arc complex)**.** The arc complex $\Delta^\circ(\mathbf{S}, \mathbf{M})$ is the clique complex for the compatibility relation. That is, $\Delta^\circ(\mathbf{S}, \mathbf{M})$ is the simplicial complex on the ground set $\mathbf{A}^\circ(\mathbf{S}, \mathbf{M})$ of all arcs in $(\mathbf{S}, \mathbf{M})$ whose simplices are collections of distinct mutually compatible arcs, and whose maximal simplices are the ideal triangulations.

Example 2.17. Figure 3 shows the arc complex of a once-punctured triangle.

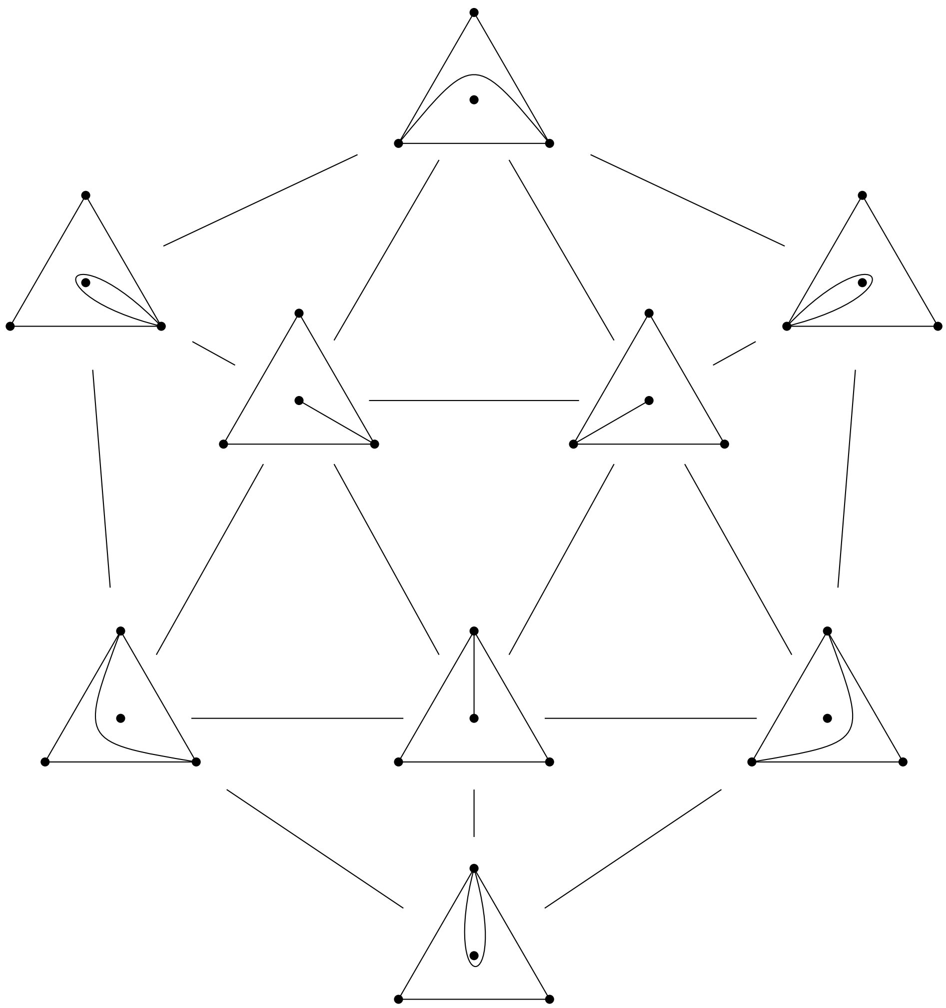

FIGURE 3. Graph $\Delta^\circ(\mathbf{S},\mathbf{M})$ for a once-punctured triangle

Definition 2.18. A flip is a transformation of an ideal triangulation T that removes an arc γ and replaces it with a (unique) different arc γ' that, together with the remaining arcs, forms a new ideal triangulation T'.

Lemma 2.19. *The arc complex is a pseudomanifold with boundary, i.e., each maximal simplex is of the same dimension and each simplex of codimension 1 is contained in at most two maximal simplices.*

The boundary of $\Delta^\circ(\mathbf{S},\mathbf{M})$ consists of collections of arcs which contain a loop bounding a punctured monogon but do not contain the arc enclosed.

Theorem 2.20. *The arc complex is contractible except when* $(\mathbf{S}, \mathbf{M})$ *is a polygon, i.e., a disk with no punctures.*

The dual graph of a pseudomanifold has maximal simplices as its vertices, with edges connecting maximal simplices sharing a codimension-1 face.

Definition 2.21. $\mathbf{E}^\circ(\mathbf{S}, \mathbf{M})$ is the dual graph of $\Delta^\circ(\mathbf{S}, \mathbf{M})$. The vertices of $\mathbf{E}^\circ(\mathbf{S}, \mathbf{M})$ are the ideal triangulations of $(\mathbf{S}, \mathbf{M})$, and the edges correspond to the flips.

Example 2.22. Graph $\mathbf{E}^\circ(\mathbf{S}, \mathbf{M})$ for a once-punctured triangle is shown in Figure 4

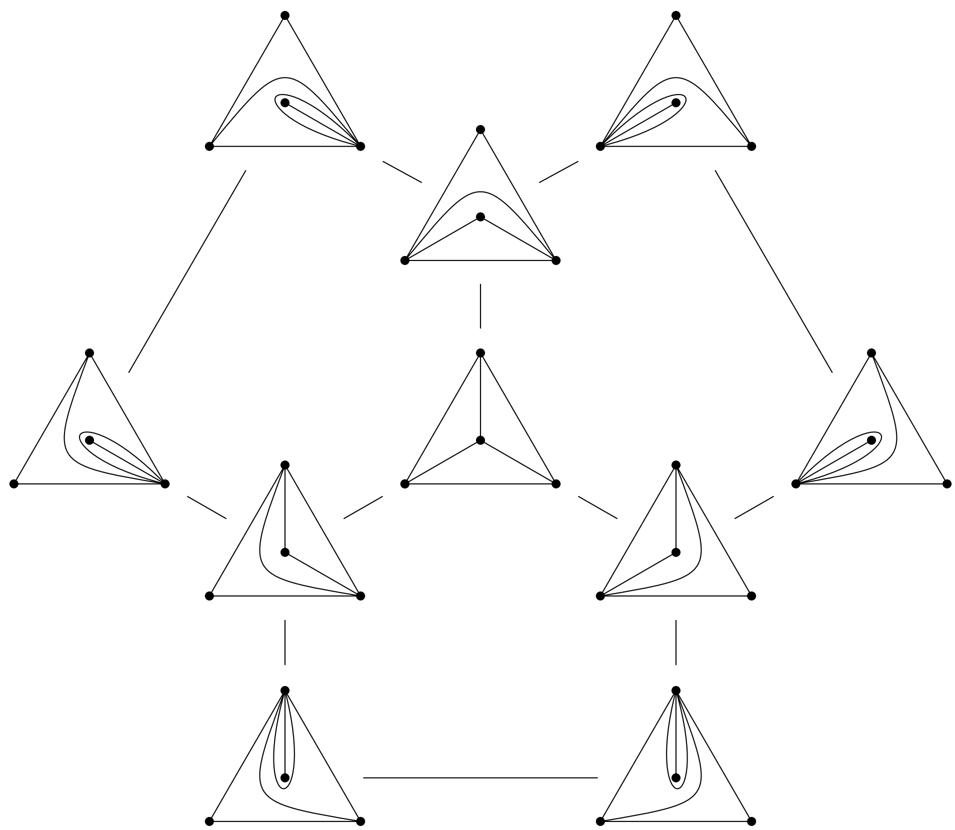

FIGURE 4. Graph $\mathbf{E}^\circ(\mathbf{S}, \mathbf{M})$ for a once-punctured triangle

3. Cluster Algebra

Definition 3.1 (Quiver)**.** A quiver Q is an oriented graph given by a set of vertices Q_0, a set of arrows Q_1, and two maps $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ taking an arrow to its source and target, respectively.

A quiver Q is finite if the sets Q_0 and Q_1 are finite. A loop of a quiver is an arrow α whose source and target coincide. A 2 -cycle of a quiver is a pair of distinct arrows β and γ such that $s(\beta) = t(\gamma)$ and $t(\beta) = s(\gamma)$.

Definition 3.2 (Quiver Mutation)**.** Let k be a vertex of Q. We define the mutated quiver $\mu_k(Q)$ as follows: it has the same set of vertices as Q, and its set of arrows is obtained by the following procedure:

(1) for each subquiver $i \to k \to j$, add a new arrow $i \to j$;
(2) reverse all allows with source or target k;
(3) remove the arrows in a maximal set of pairwise disjoint 2-cycles.

Proposition 3.3. *Mutation is an involution, that is, $\mu_k^2(Q) = Q$ for each vertex k.*

Definition 3.4. Let Q be a quiver (as in Definition 2.1.1) with m vertices, n of them mutable. Let us label the vertices of Q by the indices $1, \ldots, m$ so that the mutable vertices are labeled $1, \ldots, n$. The extended exchange matrix of Q is the $m \times n$ matrix $\tilde{B}(Q) = (b_{ij})$ defined by

$$b_{ij} = \begin{cases} \ell & \text{if there are } \ell \text{ arrows from vertex } i \text{ to vertex } j \text{ in } Q \\ -\ell & \text{if there are } \ell \text{ arrows from vertex } j \text{ to vertex } i \text{ in } Q \\ 0 & \text{otherwise.} \end{cases}$$

The exchange matrix $B(Q)$ is the $n \times n$ skew-symmetric submatrix of $\tilde{B}(Q)$ occupying the first n rows:

$$B(Q) = (b_{ij})_{i,j \in [1,n]}$$

To illustrate, consider the Markov quiver Q shown in Figure 2.10. Then

$$\tilde{B}(Q) = B(Q) = \pm \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

where the sign depends on the labeling of the vertices.

Remark 3.5. While the definition of $\tilde{B}(Q)$ depends on the choice of labeling of the vertices of Q by the integers $1, \ldots, m$, we often consider extended exchange matrices up to a simultaneous relabeling of rows and columns $1, 2, \ldots, n$, and a relabeling of the rows $n+1, n+2, \ldots, m$.

Lemma 3.6. *Let k be a mutable vertex of a quiver Q. The extended exchange matrix $\tilde{B}(\mu_k(Q)) = \left(b'_{ij}\right)$ of the mutated quiver $\mu_k(Q)$ is given* *by*

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i=k \text{ or } j=k; \\ b_{ij}+b_{ik}b_{kj} & \text{if } b_{ik}>0 \text{ and } b_{kj}>0; \\ b_{ij}-b_{ik}b_{kj} & \text{if } b_{ik}<0 \text{ and } b_{kj}<0; \\ b_{ij} & \text{otherwise.} \end{cases}$$

Definition 3.7. An $n \times n$ matrix $B = (b_{ij})$ with integer entries is called skew-symmetrizable if $d_i b_{ij} = -d_j b_{ji}$ for some positive integers $d_1, \ldots, d_n$. In other words, a matrix is skew-symmetrizable if it differs from a skewsymmetric matrix by a rescaling of its rows by positive scalars.

An $m \times n$ integer matrix, with $m \geq n$, whose top $n \times n$ submatrix is skew-symmetrizable is called an extended skew-symmetrizable matrix.

We are now ready to define the notion of matrix mutation.

Definition 3.8. Let $\tilde{B} = (b_{ij})$ be an $m \times n$ extended skew-symmetrizable integer matrix. For $k \in [1, n]$, the matrix mutation μ_k in direction k transforms $\tilde{B}$ into the $m \times n$ matrix $\mu_k(\tilde{B}) = \left(b'_{ij}\right)$ whose entries are given by lemma 3.6.

Proposition 3.9. *We can know that the matrix mutation has the following properties:*

1. *the mutated matrix $\mu_k(\tilde{B})$ is again extended skew-symmetrizable, with the same choice of $d_1, \ldots, d_n$;*
2. $\mu_k\left(\mu_k(\tilde{B})\right) = \tilde{B}$*;*
3. $\mu_k(-\tilde{B}) = -\mu_k(\tilde{B})$*;*
4. $\mu_k\left(B^T\right) = (\mu_k(B))^T$*, where B^T denotes the transpose of B;*
5. *if $b_{ij} = b_{ji} = 0$, then $\mu_i\left(\mu_j(\tilde{B})\right) = \mu_j\left(\mu_i(\tilde{B})\right)$. For $b \in \mathbb{R}$, let $\operatorname{sgn}(b)$ be 1,0 , or -1, depending on whether b is positive, zero, or negative.*

Definition 3.10. Let B be a skew-symmetrizable matrix. The skewsymmetric matrix $S(B) = (s_{ij})$ defined by

$$s_{ij} = \operatorname{sgn}(b_{ij}) \sqrt{|b_{ij} b_{ji}|}$$

is called the skew-symmetrization of B. Note that $S(B)$ has real (not necessarily integer) entries. Exercise 2.7.9 shows that skew-symmetrization commutes with mutation (extended verbatim to matrices with real entries).

Proposition 3.11. *For any skew-symmetrizable matrix B and any k, we have*

$$S(\mu_k(B)) = \mu_k(S(B))$$

3.1. **Clusters and seeds.**

Definition 3.12 (Labeled seeds)**.** Choose $m \geq n$ positive integers. Let $\mathcal{F}$ be an ambient field of rational functions in n independent variables over $\mathbb{Q}(x_{n+1}, \ldots, x_m)$. A labeled seed in $\mathcal{F}$ is a pair $(\mathbf{x}, Q)$, where $\mathbf{x} = (x_1, \ldots, x_m)$ forms a free generating set for $\mathcal{F}$, and Q is a quiver on vertices $1, 2, \ldots, n, n+1, \ldots, m$, whose vertices $1, 2, \ldots, n$ are called mutable, and whose vertices $n+1, \ldots, m$ are called frozen.

We refer to $\mathbf{x}$ as the (labeled) extended cluster of a labeled seed $(\mathbf{x}, Q)$. The variables $\{x_1, \ldots, x_n\}$ are called cluster variables, and the variables $c = \{x_{n+1}, \ldots, x_m\}$ are called frozen or coefficient variables.

Definition 3.13 (Seed mutations)**.** Let $(\mathbf{x}, Q)$ be a labeled seed in $\mathcal{F}$, and let $k \in \{1, \ldots, n\}$. The seed mutation μ_k in direction k transforms $(\mathbf{x}, Q)$ into the labeled seed $\mu_k(\mathbf{x}, Q) = (\mathbf{x}', \mu_k(Q))$, where the cluster $\mathbf{x}' = (x'_1, \ldots, x'_m)$ is defined as follows: $x'_j = x_j$ for $j \neq k$, whereas $x'_k \in \mathcal{F}$ is determined by the exchange relation

$$x'_k x_k = \prod_{\substack{\alpha \in Q_1 \\ s(\alpha)=k}} x_{t(\alpha)} + \prod_{\substack{\alpha \in Q_1 \\ t(\alpha)=k}} x_{s(\alpha)}.$$

Remark 3.14. Note that arrows between two frozen vertices of a quiver do not affect seed mutation (they do not affect the mutated quiver or the exchange relation). For that reason, one may omit arrows between two frozen vertices. Correspondingly, when one represents a quiver by a matrix, one often omits the data corresponding to such arrows. The resulting matrix B is hence an $m \times n$ matrix rather than an $m \times m$ one.

Example 3.15. Let Q be the quiver on two vertices 1 and 2 with a single arrow from 1 to 2 . Let $((x_1, x_2), Q)$ be an initial seed. Then if we perform seed mutations in directions $1, 2, 1, 2$, and 1 , we get the sequence of labeled seeds shown in Figure 2. Note that up to relabeling of the vertices of the quiver, the initial seed and final seed coincide.

Definition 3.16. Let $\mathbb{T}_n$ denote the n-regular tree whose edges are labeled by the numbers $1, \ldots, n$, so that the n edges incident to each vertex receive different labels. See Figure 3.1.

Definition 3.17 (Patterns)**.** Consider the n-regular tree $\mathbb{T}_n$ whose edges are labeled by the numbers $1, \ldots, n$, so that the n edges emanating from each vertex receive different labels. A cluster pattern is an assignment of a labeled seed $\Sigma_t = (\mathbf{x}_t, Q_t)$ to every vertex $t \in \mathbb{T}_n$, such that the seeds assigned to the endpoints of any edge are obtained

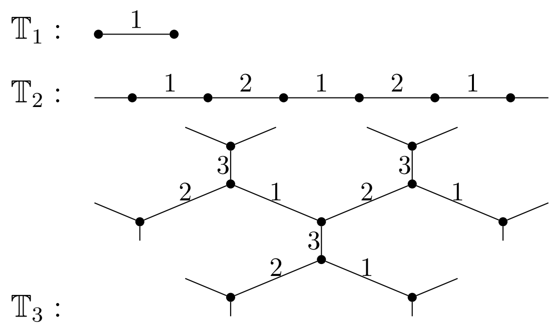

FIGURE 5. The n-regular trees $\mathbb{T}_n$ for $n = 1, 2, 3$.

from each other by the seed mutation in direction k. The components of $\mathbf{x}_t$ are written as $\mathbf{x}_t = (x_{1;t}, \dots, x_{n;t})$. Clearly, a cluster pattern is uniquely determined by an arbitrary seed.

Definition 3.18 (Cluster algebra)**.** Let $(\mathbf{x}_t, Q_t)$ be a seed pattern as above, and let

$$\mathcal{X} = \bigcup_{t \in \mathbb{T}_n} \mathbf{x}(t)$$

be the set of all cluster variables appearing in its seeds. We let the ground ring be $R = \mathbb{Z}\left[x_{n+1}^{\pm 1}, \dots, x_m^{\pm 1}\right]$, the ring of Laurent polynomials in the frozen variables. The cluster algebra $\mathcal{A}$ (of geometric type, over R) associated with the given seed pattern is the R-subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables: $\mathcal{A} = R[\mathcal{X}]$. To be more precise, a cluster algebra is the R-subalgebra $\mathcal{A}$ as above together with a fixed seed pattern in it.

Definition 3.19. Let T be an ideal triangulation, we define its signed adjacency matrix $B = B(T)$. The rows and columns of $B(T)$ are naturally labeled by the arcs in T, so that the rows and columns of $B(T)$ are numbered from 1 to n as customary. Let $\pi_T(i)$ denote the arc defined as follows: if there is a self-folded ideal triangle in T folded along i, then $\pi_T(i)$ is its remaining side (the enclosing loop); if there is no such triangle, set $\pi_T(i) = i$. For each ideal triangle Δ in T which is not self-folded, define the $n \times n$ integer matrix $B^\Delta = \left(b_{ij}^\Delta\right)$ by setting

$$b_{ij}^\Delta = \begin{cases} 1 & \text{if } \Delta \text{ has sides labeled } \pi_T(i) \text{ and } \pi_T(j), \\ & \text{with } \pi_T(j) \text{ following } \pi_T(i) \text{ in the clockwise order;} \\ -1 & \text{if the same holds, with the counter-clockwise order;} \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $B = B(T) = (b_{ij})$ is then defined by

$$B = \sum_{\Delta} B^{\Delta},$$

the sum over all ideal triangles Δ in T which are not self-folded. The $n \times n$ matrix B is skew-symmetric, and all its entries b_{ij} are equal to $0, 1, -1, 2$, or -2.

Also, we can extend the adjacency matrix to an $m \times n$ matrix $\widetilde{B}$ by adding frozen variables to the last $m - n$ rows.

Definition 3.20. Let $\mathcal{S}$ be a mutation equivalence class of seeds in $\mathcal{F}$. Let $\mathcal{X}$ denote the set of all cluster variables appearing in the seeds of $\mathcal{S}$. For all seeds $\Sigma = (\mathbf{x}, Q(B)) \in \mathcal{S}$. Cluster algebra $\mathcal{A}$ is the R-subalgebra of $\mathcal{F}$ generated by $\mathcal{X}$.

4. Upper and Lower cluster algebra

Theorem 4.1 (Laurent phenomenon)**.** [2] *Let $\mathcal{A}$ be a cluster algebra, and $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$ be a cluster in $\mathcal{A}$. As a subring of $\mathcal{F}$,*

$$\mathcal{A} \subset R\left[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}\right]$$

Definition 4.2. For a seed Σ, we denote by $\mathcal{U}(\Sigma)$ the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ given by

$$\mathcal{U}(\Sigma) = \mathbb{ZP}\left[\mathbf{x}^{\pm 1}\right] \cap \mathbb{ZP}\left[\mathbf{x}_1^{\pm 1}\right] \cap \cdots \cap \mathbb{ZP}\left[\mathbf{x}_n^{\pm 1}\right].$$

Definition 4.3. Let $\Sigma_\circ$ be a totally mutable seed. The upper cluster algebra $\overline{\mathcal{A}} = \overline{\mathcal{A}}(\Sigma_\circ)$ defined by $\Sigma_\circ$ is the intersection of the subalgebras $\mathcal{U}(\Sigma) \subset \mathcal{F}$ for all seeds $\Sigma \sim \Sigma_\circ$. In other words, $\overline{\mathcal{A}}$ consists of the elements of $\mathcal{F}$ which are Laurent polynomials over $\mathbb{ZP}$ in the cluster variables from any given seed that is mutation equivalent to $\Sigma_\circ$.

If all seeds mutation equivalent to a totally mutable seed $\Sigma_\circ$ are coprime, then the upper bound $\mathcal{U}(\Sigma)$ is independent of the choice of $\Sigma \sim \Sigma_\circ$, and so is equal to the upper cluster algebra $\overline{\mathcal{A}}(\Sigma_\circ)$.

Proposition 4.4. *Let $\Sigma = (\mathbf{x}, B)$ be a seed of geometric type. If the matrix B has full rank, then all seeds mutation equivalent to Σ are coprime.*

Definition 4.5. The lower bound $\mathcal{L}(\Sigma)$ associated with a seed Σ is defined by

$$\mathcal{L}(\Sigma) = \mathbb{ZP}\left[x_1, x_1', \ldots, x_n, x_n'\right]$$

Thus, $\mathcal{L}(\Sigma)$ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by the union of $n+1$ clusters $\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_n$.

Definition 4.6. The cluster algebra $\mathcal{A} = \mathcal{A}(\Sigma_\circ)$ associated with a totally mutable seed $\Sigma_\circ$ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by the union of all lower bounds $\mathcal{L}(\Sigma)$ for $\Sigma \sim \Sigma_0$.

The names 'lower bound' and 'upper bound' are justified by the obvious inclusions

$$L_{\mathbf{x}} \subseteq \mathcal{A} \subseteq \mathcal{U} \subseteq \mathcal{U}_{\mathbf{x}}$$

Theorem 4.7. *If $\mathcal{A}$ is totally coprime, then $\mathcal{U} = \mathcal{U}_{\mathrm{x}}$ for any seed $(\mathbf{x}, \mathbf{B})$.*

Mutating a seed can make coprime seeds non-coprime (and vice versa), so verifying a cluster algebra is totally coprime may be hard in general. A stronger condition is that the exchange matrix B has full rank; this is preserved by mutation, so it implies the cluster algebra $\mathcal{A}(\mathrm{B})$ is totally coprime.

Theorem 4.8. *If the exchange matrix B of a seed of $\mathcal{A}$ is full rank, then $\mathcal{A}$ is totally coprime. Of course, there are many totally coprime cluster algebras which are not full rank.*

Theorem 4.9. *If a seed Σ is coprime and acyclic, then $\mathcal{L}(\Sigma) = \mathcal{U}(\Sigma)$.*

Corollary 4.10. *If a cluster algebra possesses a coprime and acyclic seed, then it coincides with the corresponding upper cluster algebra.*

Theorem 4.11. *The cluster algebra $\mathcal{A}(\Sigma)$ associated with a totally mutable seed Σ is equal to the lower bound $\mathcal{L}(\Sigma)$ if and only if Σ is acyclic.*

Corollary 4.12. *Let $\mathcal{A} = \mathcal{A}(\Sigma)$ be the cluster algebra associated with a totally mutable acyclic seed Σ. Then:*

- *$\mathcal{A}$ is generated by $x_1, x'_1, \ldots, x_n, x'_n$.*
- *The standard monomials in $x_1, x'_1, \ldots, x_n, x'_n$ form a $\mathbb{ZP}$-basis of $\mathcal{A}$.*
- *The polynomials $x_j x'_j - P_j(\mathbf{x})$, for $j \in [1, n]$, generate the i deal $\mathcal{I}$ of relations among the generators $x_1, \ldots, x_n, x'_1, \ldots, x'_n$.*
- *These polynomials form a Gröbner basis for $\mathcal{I}$ with respect to any term order in which $x'_1, \ldots, x'_n$ are much more expensive than $x_1, \ldots, x_n$.*

5. Cluster algebra associated to certain surfaces

5.1. **d-gon with no punture.** If $(\mathbf{S}, \mathbf{M})$ is a d-gon with no puncture. We know that $m = 2d - 3$, $n = d - 3$. We give a triangulation of that surface as Figure 6, and the adjacency quivers is shown in Figure 7.

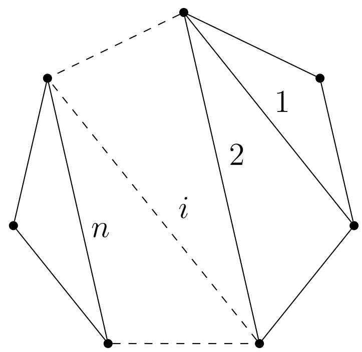

FIGURE 6. The triangulation of unpunctured d-gon

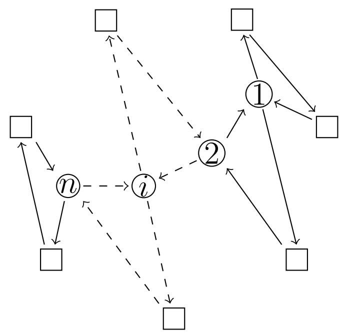

FIGURE 7. Adjacency quivers of triangulation of unpunctured d-gon

The signed adjacency matrix B is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

It is easy to know that $\mathcal{A}(\Sigma)$ is coprime and acyclic. Therefore

$$L_{\mathbf{x}} = \mathcal{A} = \mathcal{U} = \mathcal{U}_{\mathbf{x}}.$$

We can assume that the frozen variables are all 1. $\mathcal{A}$ is generated by $x_1, x_2, \cdots, x_n$ and $x'_1, x'_2, \cdots, x'_n$, where

$$x'_1 = \frac{x_2 + 1}{x_1}$$

$$x'_i = \frac{x_{i-1}x_{i+1} + 1}{x_i} \qquad \text{for } 1 < i < n$$

and

$$x'_n = \frac{x_{n-1}+1}{x_n}$$

Moreover, $A = R[x_1, x_2, \ldots, x_n, x'_1, x'_2, \ldots, x'_n,]$

5.2. **d-gon with one punture.** If $(\mathbf{S}, \mathbf{M})$ is a d-gon with one puncture. We know that $m = 2d$, $n = d$. We give a triangulation of that surface as Figure 8, and the adjacency quivers is shown in Figure 9.

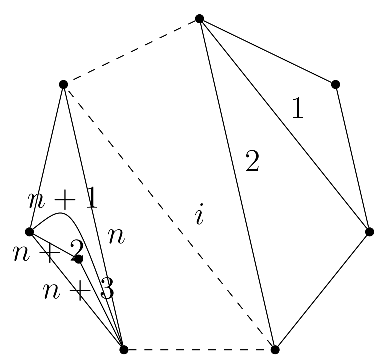

FIGURE 8. The triangulation of once punctured d-gon

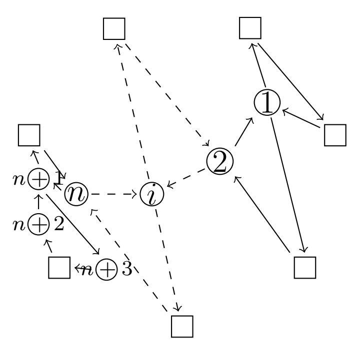

FIGURE 9. Adjacency quivers of triangulation of once punctured d-gon

We can also know that the signed adjacency matrix B is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \end{bmatrix}$$

Similarly, we know that

$$L_{\mathbf{x}} = \mathcal{A} = \mathcal{U} = \mathcal{U}_{\mathbf{x}}.$$

We can assume that the frozen variables are all 1. $\mathcal{A}$ is generated by $x_1, x_2, \cdots, x_n$ and $x'_1, x'_2, \cdots, x'_n$, where

$$x'_1 = \frac{x_2 + 1}{x_1}$$

$$x'_i = \frac{x_{i-1}x_{i+1} + 1}{x_i} \qquad \text{for } 1 < i < n$$

and

$$x'_n = \frac{x_{n-1} + 1}{x_n}$$

Moreover, $A = R[x_1, x_2, \ldots, x_n, x'_1, x'_2, \ldots, x'_n,]$

We can assume that the frozen variables are all 1. $\mathcal{A}$ is generated by x_1, x_2, $\cdots$, x_n, $x_{n+1}, x_{n+2}, x_{n+3}$ and x'_1, $x'_2, \cdots, x'_n, x'_{n+1}, x'_{n+2}, x'_{n+3}$, where

$$x'_1 = \frac{x_2 + 1}{x_1}$$

$$x'_i = \frac{x_{i-1}x_{i+1} + 1}{x_i} \qquad \text{for } 1 < i < n+1$$

$$x'_{n+1} = \frac{x_n x_{n+2} + x_{n+3}}{x_{n+1}}$$

$$x'_{n+2} = \frac{x_{n+1} + 1}{x_{n+2}}$$

$$x'_{n+3} = \frac{x_{n+1} + 1}{x_{n+3}}$$

$A = R[x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, x_{n+3}, x'_1, x'_2, \ldots, x'_n, x'_{n+1}, x'_{n+2}, x'_{n+3}]$

5.3. **Closed surface with one puncture.** Now we analyze $\mathcal{A}(\Sigma)$ from $(\mathbf{S},\mathbf{M})$, once punctured closed surface.

When $g=1$: The Triangulation and adjacency quivers of $(\mathbf{S},\mathbf{M})$ is given in Figure 10

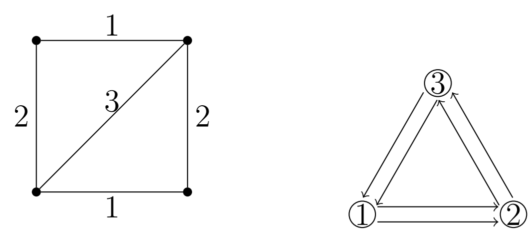

FIGURE 10. The triangulation of once puncture closed surface when $g=1$ and Markov cluster

The signed adjacency matrix of $(\mathbf{S},\mathbf{M})$ is

$$\begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

from which we can see that $\mathcal{A}(\Sigma)$ is coprime but not acyclic. And Markov cluster algebra as Figure 10 has infinitely many cluster variables. So it is not finite mutation type but not finite type.

Theorem 5.1. [4] *A skew-symmetrizable cluster algebra of rank 3 is finitely generated if and only if it has an acyclic seed.*

Corollary 5.2. *The cluster algebra arising from the once punctured closed surface whose $g=1$ is not finitely generated.*

Now we are interested in the upper cluster algebra from $(\mathbf{S},\mathbf{M})$. The following lemma gives us a good tool to analyze the upper cluster algebra.

Lemma 5.3. *If $\mathcal{A}$ is a cluster algebra with deep ideal $\mathbb{D}$, and $\mathcal{S}$ is a Noetherian ring such that $\mathcal{A}\subseteq\mathcal{S}\subseteq\mathcal{U}$, then the following are equivalent.*

(1) $\mathcal{S}=\mathcal{U}$.
(2) $\mathcal{S}$ *is normal and* $\operatorname{codim}(\mathcal{S}\mathbb{D})\geq 2$.
(3) $\mathcal{S}$ *is* $S2$ *and* $\operatorname{codim}(\mathcal{S}\mathcal{D})\geq 2$.
(4) $\operatorname{Ext}^1_{\mathcal{S}}(\mathcal{S}/\mathcal{S}\mathbb{D},\mathcal{S})=0$.
(5) $\mathcal{S}f=(\mathcal{S}f:(\mathcal{S}\mathcal{D})^{\infty})$, *where* $f:=x_1x_2\dots x_m$ *for some cluster* $\mathbf{x}=\{x_1,\dots,x_n\}$.

If $\mathcal{S}f\neq(\mathcal{S}f:(\mathcal{S}\mathcal{D})^{\infty})$, then $(\mathcal{S}f:(\mathcal{S}\mathcal{D})^{\infty})\,f^{-1}$ contains elements of $\mathcal{U}$ not in $\mathcal{S}$.

Theorem 5.4. *The upper cluster algebra from* $(\mathbf{S}, \mathbf{M})$ *is generated by*

$$x_1, x_2, x_3, \frac{x_1^2 + x_2^2 + x_3^2}{x_1 x_2 x_3}.$$

Proof. Let

$$\mathcal{S} = \mathbb{Z}\left[x_1, x_2, x_3, \frac{x_1^2 + x_2^2 + x_3^2}{x_1 x_2 x_3}\right]$$

be the subring of $\mathbb{Z}\left[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}\right]$.

The following identities imply that $L_{\mathbf{x}} \subset \mathcal{S}$.

$$x_1' = \frac{x_2^2 + x_3^2}{x_1} = x_2 x_3 \frac{x_1^2 + x_2^2 + x_3^2}{x_1 x_2 x_3} - x_1^2$$

$$x_2' = \frac{x_1^2 + x_3^2}{x_2} = x_1 x_3 \frac{x_1^2 + x_2^2 + x_3^2}{x_1 x_2 x_3} - x_2^2$$

$$x_3' = \frac{x_1^2 + x_2^2}{x_3} = x_1 x_2 \frac{x_1^2 + x_2^2 + x_3^2}{x_1 x_2 x_3} - x_3^2$$

□

The following identities imply that $\mathcal{S} \subset \mathcal{U}_{\mathbf{x}}$.

Since $\mathcal{S}$ is a complete intersection, and so it Cohen-Macaulay, and in particular it is $S2$.

Let P be a prime ideal in $\mathcal{S}$ containing

$$\mathbb{D}_{\mathbf{x}} =< x_1 x_2 x_3, x_1' x_2 x_3, x_1 x_2' x_3, x_1 x_2 x_3' >$$

Since $x_1 x_2 x_3 \subset P$, at least one of $\{x_1, x_2, x_3\} \in P$ by primality. If any two x_i, x_j are, then

$$x_k^2 = x_i x_j x_k M - x_i^2 - x_j^2 \in P \Rightarrow x_k \in P$$

If only one $x_i \in P$, then $x_i' x_j x_k \in P$ implies that $x_i' \in P$ and $\frac{x_1^2+x_2^2+x_3^2}{x_1 x_2 x_3} \in P$. Additionally, $x_j^2 + x_k^2 = x_i x_j x_k M - x_i^2 \in P$. Therefore, P contains at least one of the four prime ideals

$$< x_1, x_2, x_3 >,$$

$$< x_1, x_2^2 + x_3^2, \frac{x_1^2 + x_2^2 + x_3^2}{x_1 x_2 x_3} >,$$

$$< x_2, x_1^2 + x_3^2, \frac{x_1^2 + x_2^2 + x_3^2}{x_1 x_2 x_3} >,$$

$$< x_3, x_1^2 + x_2^2, \frac{x_1^2 + x_2^2 + x_3^2}{x_1 x_2 x_3} >$$

Since $\{x_1, x_2\}$, $\left\{x_1, \frac{x_1^2+x_2^2+x_3^2}{x_1x_2x_3}\right\}$, $\left\{x_2, \frac{x_1^2+x_2^2+x_3^2}{x_1x_2x_3}\right\}$, and $\left\{x_3, \frac{x_1^2+x_2^2+x_3^2}{x_1x_2x_3}\right\}$ are each regular sequences in S, it follows that $\operatorname{codim}(\mathbb{D}_{\mathbf{x}}) \geq 2$ and $\mathcal{S} = \mathcal{U}$.

When $g = 2$ or $g = 3$: triangulation as figure 12 and cluster as figure 11. It is easy to see that they are both coprime but not acyclic.

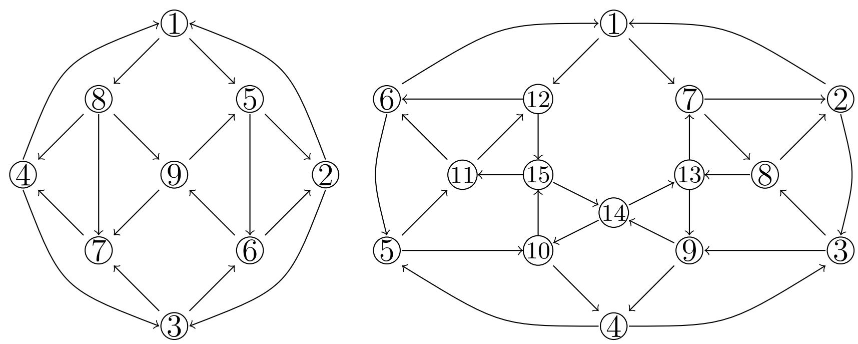

FIGURE 11. Adjacency quivers of triangulation of once punctured closed surfaces when $g = 2$ and $g = 3$.

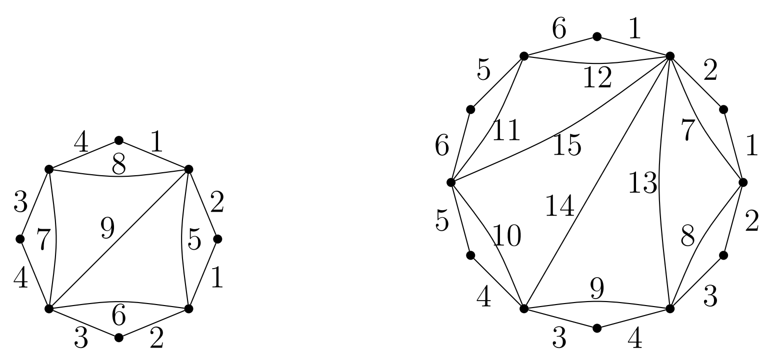

FIGURE 12. The triangulation of once puncture closed surfaces when $g = 2$ and $g = 3$.

Here we give an algorithm of computing upper cluster algebra, which has not been practiced by the author yet.

Input: A list $x_1, x_2, \cdots, x_n, f_1, f_2, \cdots, f_m$, with each f_i a Laurent polynomial in the $x_1, x_2, \cdots, x_n$.

(1) Compute the ideal of relations in $\widetilde{S} = \mathbb{Z}[x_1, x_2, ..., x_n, y_1, y_2, ..., y_m]$. This can be done by first computing the ideal of 'naive relations' $\widetilde{I}$ generated by elements of the form $x_1^{d_1}x_2^{d_2}\cdots x_n^{d_n}y_i - N_i$,

where $f_i = \frac{N_i}{x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}}$ is the Laurent polynomial representation of f_i... then saturating $\widetilde{I}$ with respect to the ideal generated by the single element $x_1, x_2 \cdots x_n$. The resulting saturated ideal I is the kernel of the map from $\widetilde{S}$ to $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, ..., x_n^{\pm 1}]$, so the quotient $S := \widetilde{S}/I$ is isomorphic to the ring generated by $x_1, x_2, ..., x_n, f_1, f_2, ..., f_m$.

(2) Write each of the mutations x_i' in terms of your generators. Depending on your generating set, this might be easy or hard.

(3) Compute the deep ideal D. This is generated by the products $x_1 x_2 \cdots x_n, x_1' x_2 \cdots x_n, x_1 x_2' \cdots x_n$, all the way to $x_1 x_2 \cdots x_n'$.

(4) Compute the saturation of the ideal in S generated by $x_1 x_2 \cdots x_n$ by the ideal D. If the result is just the ideal generated by $x_1 x_2 \cdots x_n$, then S is equal to the upper bound algebra. If not, then S does not equal the upper bound algebra.

Conjecture1: $\mathcal{A}(\Sigma)$ is not finitely finitely generated for a once punctured surface $(\mathbf{S}, \mathbf{M})$.

Conjecture2: For a once punctured surface $(\mathbf{S}, \mathbf{M})$, we can add the element

$$\angle(\mathbf{x}, T) = \sum_{\text{triangles } \{i,j,k\}} \left(\frac{x_k}{x_i x_j} + \frac{x_i}{x_j x_k} + \frac{x_j}{x_k x_i} \right) = \sum \frac{x_i^2 + x_j^2 + x_k^2}{x_i x_j x_k}$$

to make $\mathcal{A}(\Sigma)$ become its upper cluster algebra.

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