ESCAPE PROBABILITY AND CATALAN NUMBER

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Abstract. In this paper, we state the relationship between the escape probability and the effective resistance in an electrical network, and use this property to calculate the escape probability of the binary tree in different ways, which induces a formula related to Catalan number. Generalizing this result, we get the generating function of Catalan number.
Key words. Markov chain, resistance, binary tree, Catalan number, generating function

1. Introduction

Resistance is an important concept in a circuit. The calculation of resistance plays an important role in various fields such as physics, chemistry, computer science and engineering. In 1993, the concept 'Resistance distance' was first proposed by D.J.Klein, and M.Randić [DM]. Since then, the calculation of resistance has been linked with the many mathematical concepts such as Laplacian matrix, spanning tree, and Markov chain, and has produced many interesting results.

In combinatorial mathematics, the Catalan number, which is named after the Belgian mathematician Eugéne Charles Catalan, occurs in various counting problems.

In this article, we first state an important conclusion in graph theory, which links the escape probability and the effective resistance. Next, we choose the simplest infinite network, binary tree, to calculate its escape probability using two different method, which induce an equation about the Catalan number. Finally, we generalized this calculation and get the generating function of Catalan number.

In this paper, denote by w_{ij} the weight of edge (i, j), $r(i, j)$ the effective resistance between two vertices i and j, C_i the i-th Catalan number: $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $\cdots$

2. The escape probability and the resistance

First, we present a theorem in graph theory, which can be seen in [V].

Theorem 2.1. *On a graph $G = (V, E)$ with edge weights w_{ij} for edge $(i, j) \in E$. Define an electric network on this graph by setting each edge $(i, j) \in E$ to be a resistor with resistance $r = \frac{1}{w_{ij}}$. Define a random walk on the graph by setting the transition probabilities to be $p_{ij} = \frac{w_{ij}}{s_i}$, where*

$s_i = \sum_{j\sim i} w_{ij}$. *For any pair of* $a, b \in V$*, we define* $p_{esc}(a \to b)$ *as the probability that, starting in* a *the random walk reaches* b *before returning to* a*. We call this the escape probability. Then for any pair of* $a, b \in V$*,*

$$p_{esc}(a \to b) = \frac{1}{s_a \cdot r(a,b)}.$$

Now, we give a binary tree, which is pictured below. The weight of each

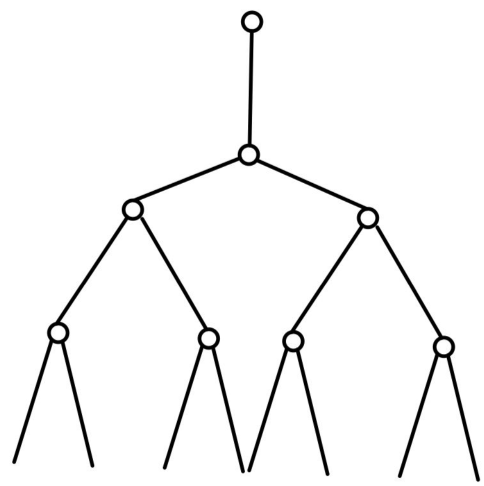

FIGURE 1. The binary tree

edge in the graph is 1, and by gluing the points of equal potential, the effective resistance from root to infinity can be easily calculated as

$$r(0,\infty) = \sum_{n=0}^{\infty}(\frac{1}{2})^n = 2$$

So, the probability of starting from the root and returning within a finite time is

$$P = 1 - \frac{1}{2} = \frac{1}{2}$$

3. THE MARKOV CHAIN ON THE BINARY TREE

Next, we use the traditional probability calculation method to calculate the probability of returning from the root.

We call the distance from each fixed point to the root its level, define by $\{a_k\}$ the state of k-th step as follows: If the k-th step is downwards, that is to the higher level, then $a_k = 1$, otherwise, $a_k = -1$. Returning to the starting point at exactly $2n$ steps means the following condition:

$$
\begin{aligned}
& a_i = 1 \text{ or } -1, \quad \forall 1 \leq i \leq 2n \\
& a_1 = 1, a_n = -1 \\
& \sum_{i=1}^{2n} a_i = 0 \\
& \sum_{i=1}^{k} a_i > 0, \quad \forall 1 \leq k < 2n
\end{aligned}
$$

It is easy to know that it is actually a Markov chain in one dimension: at each level, the probability of downwards in the next step is $\frac{2}{3}$, and the probability of upwards in the next step is $\frac{1}{3}$. For the case of returning to the root in $2n$ steps, the number of methods selected by a_n is the Catalan number C_{n-1}. Denote by t the number of steps required to return to the root for the first time. So we can calculate the probability of returning at exactly $2n$ steps is

$$P(t = 2n) = \frac{1}{3} C_{n-1} (\frac{2}{3})^{n-1} (\frac{1}{3})^{n-1}$$

Then the probability of returning to the root after finite steps is

$$\sum_{n=1}^{\infty} P(t = 2n) = \sum_{n=1}^{\infty} \frac{1}{3} C_{n-1} (\frac{2}{3})^{n-1} (\frac{1}{3})^{n-1}$$

We know it equals $\frac{1}{2}$. And we can get the formula:

$$\sum_{n=0}^{\infty} C_n (\frac{2}{9})^n = \frac{3}{2}$$

4. The generation function of the Catalan number

The calculation above example can be simplified as an infinitely long one-dimensional wire on the positive half x-axis, and the resistance between n and $n+1$ is $(\frac{1}{2})^{n-1}$. However, We can change the resistance between every two consecutive integer points to derive the generating function of Catalan number.

We assume that there is a wire on the positive half x-axis, and the resistance between two consecutive integer points n and $n+1$ is p^n, it is easy to calculate the resistance from the root to infinite is

$$r(0, \infty) = \sum_{0}^{\infty} p^n = \frac{1}{1-p}$$

.

The probability of walking to the left and the probability of walking to the right at each point are respectively $\frac{p}{1+p}$ and $\frac{1}{1+p}$. Same to the last section, we can get the formula

$$\frac{p}{1+p}\sum_{n=0}^{\infty}C_n(\frac{p}{1+p})^n(\frac{1}{1+p})^n = 1-\frac{1}{\frac{1}{1-p}} = p \tag{4.1}$$

Because p can be randomly chosen between 0 and 1, denote $x = \frac{p}{1+p}\frac{1}{1+p}$, and thus $p = \frac{1-2x-\sqrt{1-4x}}{2x}$. It is easy to know that $0 < x \leq \frac{1}{4}$, and the equation 4.1 can be rewritten as

$$\sum_{n=0}^{\infty}C_n x^n = \frac{1-\sqrt{1-4x}}{2x} \tag{4.2}$$

which gives the generating function of C_n.

References

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