WHAT HAS BEEN GOING ON IN THE WORLD OF CHAOS: FROM 1975 TO FUTURE

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Abstract. In this article, Li-Yorke chaos and its subsequent promotion in high dimensions are shown. We summarize many methods of judging chaos, including Li-Yorke theorem, and derive the ideas behind the concepts introduced. Moreover, we analyze their advantages, disadvantages and differences. Finally, we elaborate the nature of chaos and its future.
Key words. Li-Yorke chaos, Expanding fixed point, Snap-back repeller, Jacobian matrix

1. Introduction

Chaos, which is interpreted as "a state of total confusion with no order" in the Cambridge dictionary, reflects the irregularity and unpredictability of the natural world. Einstein said that as far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality, which means that we cannot use mathematics to completely predict the future of the world, but can mathematics explain why the world is irregular and unpredictable? In 1975, Li and Yorke published the article "Period Three Implies Chaos" in American Mathematics Monthly, introducing the term "chaos" into mathematics[10]. In the next few years, the study of Chaos was in the ascendant.

However, Li-Yorke theorem only applies to the one-dimensional space, and in the following years, it was generalized by many mathematicians. Marotto firstly gave a good definition of chaos in higher dimension and an easy way to test[11]. However, some error was found later and the condition of was then improved by Shi and Chen[13]. But the conditions given by Shi and Chen became stronger, so that there were fewer cases that can be tested. In 1981, Kloeden gave a better way to test chaos defined by Marotto[7]. However, his judgment criteria was still limited and could not be applied to all chaotic phenomena. So far, there are still no sufficient and necessary conditions to determine the chaos given by Marotto, and no one has given a unified definition of chaos.

This article is divided into two parts. The first is an in-depth analysis and explanation of the Li-Yorke theorem. We will make supplementary proofs for the wrong or unobvious parts of the theorem. The second part summarizes the follow-up promotion of Li-York's theorem. Here, three theorems are mainly mentioned. All remarks and some examples are given by the author.

2. An in-depth analysis of Li-Yorke chaos

Theorem 2.1 (Li-Yorke[10]). *Let J be an interval and let $F : J \to J$ be continuous. Assume there is a point $a \in J$ for which the points $b = F(a)$, $c = F^2(a)$ and $d = F^3(a)$, satisfy*

$$d \le a < b < c \text{ (or } d \ge a > b > c)$$

Then

- T_1: *for every $k = 1, 2, \cdots$, there is a periodic point in J having period k.*
- T_2: *there is an uncountable set $S \subset J$ (containing no periodic points), which satisfies the following conditions: For every $p, q \in S$ with $p \neq q$,*

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0 \tag{2.1}$$

and

$$\liminf_{n\to\infty} |F^n(p) - F^n(q)| = 0 \tag{2.2}$$

For every $p \in S$ and periodic point $q \in J$,

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0 \tag{2.3}$$

Before giving the proof, we first introduce three lemmas. Since the proof of the original text has many difficulties and incomprehensibility, we give supplementary explanations and proofs in addition to copying.

Lemma 2.2. [10] *Let $G : I \to R$ be continuous, where I is an interval. For any compact interval $I_1 \in G(I)$, there is a compact interval $Q \subset I$ such that $G(Q) = I_1$*

Proof. Suppose $I_1 = [G(p), G(q)]$, If $p < q$, denote by r the last point of $[p, q]$ where $G(r) = G(p)$, s the first point after r where $G(s) = G(q)$, then $G([r, s]) = I_1$. Similar reasoning applies when $p > q$. □

Remark 1. Firstly, continuous functions map connected sets to connected sets, and compact sets to compact sets. But the inverse of the two propositions are not valid. However, here we are not to select the entire preimage, but just prove the existence of the interval. Why can we get the first point and the last point here: because a single point is a closed set, and the original image under a continuous function is also closed, so the maximum and minimum value must be reached in the bounded interval. The last claim is given by intermediate value theorem.

Lemma 2.3. [10] *Let $F : J \to J$ be continuous and let $\{I_n\}_{n=0}^{\infty}$ be a sequence of compact intervals with $I_n \subset J$ and $I_{n+1} \subset F(I_n)$ for all n. Then there is a sequence of compact intervals Q_n, such that $Q_{n+1} \subset Q_n \subset l_0$ and $F^n(Q_n) = I_n$ for $n \ge 0$. For any $x \in Q = \cap Q$, we have $F^n(x) \in I_n$ for all n.*

Proof. Define $Q_0 = I_0$. Then $F^0(Q_0) = I_0$. If Q_{n-1} has been defined so that $F^{n-1}(Q_{n-1}) = I_{n-1}$, then $I_n \subset F(I_{n-1}) = F^n(Q_{n-1})$. By lemma 0 applied to $G = F^n$ on Q_{n-1} there is a compact interval $Q_n \subset Q_{n-1}$ such that $F^n(Q_n) = I_n$. This completes the induction. □

Remark 2. Just apply lemma 0 and by induction: $Q_0 = I_0$, $I_n \subset F(I_{n-1}) = F(F^{n-1}(Q_{n-1})) = F^n(Q_{n-1})$. The last claim is of great use in the proof of T2.

Lemma 2.4. *Let $G : J \to R$ be continuous. Let $I \subset J$ be a compact interval. Assume $I \subset (G(I))$. Then there is a point $p \in I$ such that $G(p) = p$*

Proof. Let $I = [\beta_0, \beta_1]$. Choose α_i such that $G(\alpha_i) = \beta_i$. It follows $\alpha_0 - G(\alpha_0) \geq 0$ and $\alpha_1 - G(\alpha_1) \leq 0$ and continuity implies $G(\beta) - \beta$ must be 0 for some β in I. □

Assume $d \leq a < b < c$ as in the theorem. The proof for the case $d \geq a > b > c$ is similar and so is omitted. Write $K = [a, b]$ and $L = [b, c]$.

Proof of T1

Proof. Let k be a positive integer. For $k > 1$, let $\{I_n\}$ be the sequence of intervals $I_n = L$ for $n = 0, 1, \cdots, k-2$ and $I_{k-1} = K$, and define I_n to be periodic inductively, $I_{n+k} = I_n$ for $n = 0, 1, 2, \cdots$. If $k = 1$, let $I_n = L$ for all n.
Note that $Q_k \subset Q_0$ and $F^k(Q_k) = Q_0$, $G = F^k$ has a fixed point p_k in Q_k. p_k cannot have period less than k for F; otherwise $F^{k-1}(p_k) = b$, contrary to $F^{k+1}(p_k) \in L$. □

Remark 3. Why $F^{k-1}(p_k) = b$ if p_k has period less than k? Because we know that $Q_{t-1} \subset Q_t$ for all t, and $p_k \in Q_k$, thus it is in all Q_t for $t \leq k$, Because $F^{k-1}(Q_{k-1}) = I_{k-1} = K$, implying that $F^{k-1}(p_k) \in K$, and if p_k has period less than k, then there must be a $0 \leq t < k$, such that $F^{k-1}(p_k) = F^t(p_k)$, and because $F^t(Q_t) = I_t = L$, $F^{k-1}(p_k) = F^t(p_k) \in L$, implying that $F^{k-1}(p_k) \in L$, So, $F^{k-1}(p_k) = b$.

Proof of T2

Proof. Let $\mathcal{M}$ be the set of sequences $M = \{M_n\}_{n=1}^{\infty}$ of intervals with

$$M_n = K \text{ or } M_n \in L, \text{ and } F(M_n) \supset M_{n+1} \tag{2.4}$$

$$\text{if } M_n = K \text{ then } n \text{ is the square of an integer and } M_{n+1}, M_{n+2} \subset L, \tag{2.5}$$

where $K = [a, b]$ and $L = [b, c]$. Of course if n is the square of an integer, then $n+1$ and $n+2$ are not, so the last require is redundant. For $M \in M$, let $P(M, m)$ denote the number of i's in $\{1, \cdots n\}$ for which $M_i = K$. For each $r \in (3/4, 1)$, choose $M^r = \{M_n^r\}_{n=1}^{\infty}$ to be a sequence in M such that

$$\lim_{n\to\infty} P(M^r, n^2)/n = r \tag{2.6}$$

***Remark* 4.** Because only when n is the square of an integer can $M_n = K$, $P(M^r, n^2) \leq n$. But for any r, the sequence that satisfies the condition $\lim_{n\to\infty} P(M^r, n^2) = nr$ exists. For example, let all M_n that are not K be taken as L. In this case if M_t is K, then M_{t+1} must be L, but if M_t is L, either M_{t+1} is L or K does not violate the condition $F(M_n) \supset M_{n+1}$, so that we can take the set of $M_t = K$ appropriately sparse. At the same time, we should also note that for a given r, there may be many such sequences, but we only need to find one.

Let $\mathcal{M}_0 = \{M^r : r \in (3/4, 1)\} \subset M$. Then $\mathcal{M}_0$ is uncountable since $M^{r_1} \neq M^{r_2}$ for $r_1 \neq r_2$. For each $M^r \in \mathcal{M}_0$, by lemma 1, there exists a point x_r with $F^n(x_r) \in M_n^r$ for all n. Let $S = \{x_r : r \in (3/4, 1)\}$. Then S is also uncountable. For $x \in S$, let $P(x, n)$ denote the number of i's in $\{1, \cdots, n\}$ for which $F^i(x) \in K$. We can never have $F^k(x_r) = b$, because then x_r would eventually have period 3, contrary to 2.5. Consequently $P(x_r) = P(M^r, n)$ for all n, and so

$$\rho(x_r) = \lim_{n\to\infty} P(X_r, n^2) = r$$

for all r.

***Remark* 5.** The M_n^r here is equivalent to the I_n in the lemma 1, so such X_r can be taken. Why "then x_r would eventually have period 3"? If we have "$F^k(x_r) = b$", $F^{k+2}(x_r) = d$. However, $F^{k+2}(x_r) \in M_n^{k+2}$, which implies that $F^{k+2}(x_r) \in K \cup L$, we can have $d = a$, which means that x_r would eventually have period 3. Why "contrary to 2.5"? If $F^k(x_r) = b$, $F^{k+2}(x_r) = a \in M_{k+2} = K$, and $F^{k+5}(x_r) = a \in M_{k+5} = K$. After excluding the case of $F^k(x_r) = b$, we can make the last claim. Finally, we should note that there is an error in the original text, the final formula should be changed to $\rho(x_r) = \lim_{n\to\infty} P(X_r, n^2)/n = r$

We claim that
for $p, q \in S$, with $p \neq q$, there exists infinite many n's such that $F^n(p) \in K$ and $F^n(q) \in L$ or vise versa.
We may assume $\rho(p) > \rho(q)$. Then $P(p, n) - P(q, n) \to \infty$, and so there must be infinitely many n's such that $F^n(p) \in K$ and $F^n(q) \in L$.

Since $F^2(b) = d \leq a$ and F^2 is continuous, there exists $\delta > 0$ such that $F^2(x) < (b+d)/2$ for all $x \in [b-\delta, b] \subset K$, then 2.5 implies $F^(n+1)(p) \in L$ and $F^{(n+2)}(p) \in L$. Therefore $F^n(p) < b - \delta$. If $F^n(q) \in L$, then $F^n(q) \geq b$ so

$$|F^n(p) - F^n(q)| > \delta$$

For any $p, q \in S$, $p \neq q$, it follows

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| \leq \delta$$

The technique may be similar used to prove 2.3 is satisfied.

***Remark* 6.** We have already suppose that for a given r, we only take one sequence M_r and take only one x_r. So if $q \neq p$, Their corresponding values of r are different. Even if $F^n(p) = K$ and $F^n(q) = L$, they may gradually approach the point b. In order to show that they have enough distance, we must take a δ such that $F^n(p) < b - \delta$. The claim that the proof of B is similar: If q is a periodic point, $p \in S$, then $P(q, n) = r_1 n$ for some $r_1 \geq 0$, and $P(p, n) = r_2\sqrt{n}$ for some $r_2 \in (3/4, 1)$. The condition $|P(p, n) - P(q, n)| \to \infty$ still holds.

Since $F(b) = c$, $F(c) = d \leq a$, we may choose intervals $[b^n, c^n]$, $n = 0, 1, 2, \cdots$ such that

(a) $[b, c] = [b^0, c^0] \supset [b^1, c^1] \supset \cdots \supset [b^n, c^n] \supset \cdots$
(b) $F(x) \in (b^n, c^n)$ for all $x \in (b^{n+1}, c^{n+1})$
(c) $F(b^{n+1}) = c^n$, $F(c^{n+1}) = b^n$.

Let $A = \bigcap_{n=0}^{\infty}[b^n, c^n]$, $b^* = \inf A$ and $c^* = \sup A$, then $F(b^*) = c^*$ and $F(c^*) = b^*$, because of (c).

***Remark* 7.** b^{n+1} is the last point in $[b^n, c^n]$ satisfying $F(b^{n+1}) = c^n$, c^{n+1} is the first point satisfying $F(c^{n+1}) = b^n$, and $b^n < c^n$, simply by induction. And we should note that it is possible that $b^* = c^*$.

In order to prove 2.2 we must be more specific in our choice of the sequences M^r. In addition to our previous requirements on $M \in M$, we will assume that if $M_k = K$ for both $k = n^2$ and $(n+1)^2$ then $M_k = [b^{2n-(2j-1)}, b^*]$ for $k = n^2 + (2j - 1)$, $M_k = [c^*, c^{2n-2j}]$ for $k = n^2 + 2j$ where $j = 1, \cdots, n$. For the remaining k's which are not squares of integers, we assume $M_k = L$.

It is easy to check that these requirements are consistent with 2.4 and 2.5, and that we can still choose M^r so as to satisfy 2.6. From the fact that $\rho(x)$ may be thought of as the limit of the fraction of n's for which $F^{n^2}(x) \in K$, it follows that for any r^*, $r \in (3/4, 1)$ there exist infinitely many n such that $M_k^r = M_k^{r^*} = K$ for both $k = n^2$ and $(n + 1)^2$. To show 2.2,let $x_r \in S$ and $x_{r^*} \in S$. Since $b^n \to b^*$, $c^n \to c^*$ as $n \to \infty$, for any $\epsilon > O$ there exists N with $|b^n - b^*| < \epsilon/2$, $|c^n - c^*| < \epsilon/2$ for all $n > N$. Then, for any n with $n > N$ and $M_k^r = M_k^{r^*} = K$ for both $k = n^2$ and $(n + 1)^2$, we have

$$F^{n^2+1}(x_r) \in M_k^r = [b^{2n-1}, b^*]$$

with $k = n^2 + 1$ and $F^{n^2+1}(x_r)$ and $F^{n^2+1}(x_{r^*})$ both belong to $[b^{2n-1}, b^*]$. Therefore, $|F^{n^2+1}(x_r) - F^{n^2+1}(x_{r^*})| < \epsilon$. Since there are infinitely many n with this property, $\liminf_{n\to\infty} |F^n(x_r) - F^n(x_{r^*})| = 0$.

***Remark* 8.** Here, we see the ingenuity of the number 3/4: It guarantees that

$$\liminf_{n\to\infty} \frac{\{n : M_{n^2}^r = M_{n^2}^{r^*} = K\}}{n} > (\frac{3}{4})^2 > \frac{1}{2}.$$

□

3. Further development in general space

In $\mathbb{R}_n$, "period three implies chaos" may not be true.

Example 3.1. Consider the mapping f as below.

$$f\left(x_1, x_2\right) = \left(-\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2, -\frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2\right)$$

f is a map from the unit disc $D = \left\{\left(x_1, x_2\right) \in \mathbb{R}^2 - x_1^2 + x_2^2 \leq 1\right\}$ to itself. Then each point except the origin has period three, but f cannot be chaotic.

Definition 3.2 (Marotto[11])**.** We introduce the two concepts of Expanding fixed point and Snap-back repeller here.

(1) **Expanding fixed point**: Let F be differentiable in $B_r\left(x^*\right)$. The point $x^* \in \mathbb{R}^n$ is an expanding fixed point of fin $B_r\left(x^*\right)$ if $F\left(x^*\right) = x^*$ and all eigenvalues of $DF(x)$ exceed 1 in norm for all $x \in B_r\left(x^*\right)$.

(2) **Snap-back repeller**: Assume that x^* is an expanding fixed point of F in $B_r\left(x^*\right)$ for some $r > 0$. Then x^* is said to be a snap-back repeller of F if there exists an eventually fixed point $x_0 \in B_r\left(x^*\right)$ with $x_0 \neq x^*, F^M\left(x_0\right) = x^*$ and the determinant $\det DF^M\left(x_0\right) \neq 0$ for some positive integer M.

Note that only the condition of Expanding fixed point cannot lead to chaos, such as the following example.

Example 3.3. Consider the mapping f as below.

$$f(x) = (2 - \|x\|)x$$

f keeps the angle of x unchanged and squeezes to the boundary. The origin is the fixed point of f, and the norms of eigenvalues of f at the origin is 2, so the origin of f is Expanding fixed point, but f is not chaotic, Because except for the origin and the boundary point, which are fixed points, the other points have no period and gradually approach the boundary.

Theorem 3.4 (Marotto[11])**.** *If f possesses a snap-back repeller, then system $x_{k+1} = f(x_k)$ is chaotic in the following generalized sense of Li–Yorke:*

(1) *There is a positive integer N such that for each integer $p \geq N$, F has a point of period p;*

(2) *There is a "scrambled set" of F, i.e. an uncountable set S containing no periodic points of F, such that*
   (a) *$F(S) \subset S$,*
   (b) *for every $X, Y \in S$ with $X \neq Y$,*

$$\limsup_{k\to\infty} \|F^k(X) - F^k(Y)\| > 0$$

   (c) *for every $X \in S$ and any periodic point Y of F,*

$$\liminf_{k\to\infty} \|F^k(X) - F^k(Y)\| > 0$$

(3) *There is an uncountable subset* S_0 *of* S *such that for every* $X, Y \in S_0$*:*

$$\liminf_{k\to\infty} \|F^k(X) - F^k(Y)\| = 0$$

The existence of a snap-back repeller for the one-dimensional mapping f is equivalent to the existence of a point of period-3 for the map f^n for some positive integer n[11].

In order to show that the theorem is true, we first give two lemmas without proof.

Lemma 3.5. [11] *Let* Z *be a snap-back repeller of* F*. Then for some* $s > 0$ *there exists* $Y_0 \in B_s^0(Z)$ *and an integer* L *such that* $F^k(Y_0) \notin B_s(Z)$ *for* $1 \leqslant k < L$ *and* $F^L(Y_0) = Z$*. Also,* $\left|DF^L(Y_0)\right| \neq 0$ *and* Z *is expanding in* $B_s(Z)$

Lemma 3.6. [11] *Let* $\{C_k\}_{k=0}^{\infty}$ *be a sequence of compact sets in* $\mathbb{R}^n$ *and let* $H : \mathbb{R}^n \to \mathbb{R}^n$ *be continuous. If* $H[C_k] \supset C_{k+1}$ *for all* $k \geqslant 0$*, then there exists a non-empty compact set* $C \subset C_0$ *with* $H^k(X) \in C_k$ *for all* $X \in C$ *and* $k \geqslant 0$.

Now we give the proof of theorem 3.4.

Proof. Proof of (1). Without loss of generality we can assume that:

$$X_0 \in B_r^0(Z) \quad \text{and} \quad F^k(X_0) \notin B_r(Z) \quad \text{for } 1 \leqslant k < M. \tag{3.1}$$

Otherwise, replacing X_0, r and M with the quantities Y_0, s and L respectively, provided by Lemma 3.5. The following analysis could then be carried out in terms of these new variables.

Now, since $F^M(X_0) = Z$ and $\left|DF^M(X_0)\right| \neq 0$, then for some $\epsilon > 0$ satisfying $0 < \epsilon < r$ there exists a continuous and $1-1$ function G defined on $B_\epsilon(Z)$ with $G(Z) = X_0$ and:

$$G^{-1}(X) = F^M(X) \quad \text{for all } X \in G[B_\epsilon(Z)] \tag{3.2}$$

For notational convenience let Q be the compact set defined by $Q = G[B_\epsilon(Z)]$. Because of 3.1, we can assume without loss of generality that $X_0 \in Q \subset B_r(Z)$ and:

$$F^m[Q] \subset \mathbb{R}^n - B_r(Z) \quad \text{for } 1 \leqslant m < M \tag{3.3}$$

If not, then we could choose a smaller ϵ such that this is true. Also, since Z is expanding in $B_r(Z)$ then F^{-1} exists in $B_r(Z)$, and thus $Q \subset B_r(Z)$ implies that:

$$F^{-m}[Q] \subset B_r(Z) \quad \text{for } m \geqslant 0 \tag{3.4}$$

In addition, for any $X \in Q, F^{-h}(X) \to Z$ as $k \to \infty$, so there exists an integer $J = J(X) \geqslant 0$ such that $F^{-J}(X) \in B_\epsilon^0(Z)$. By continuity, therefore, we have $\delta = \delta(X) > 0$ with $F^{-J}\left[B_\delta^0(X)\right] \subset B_\epsilon(Z)$. Consider the collection of open sets: $D = \left\{B_\delta^0(X) : \text{for all } X \in Q\right\}$. The set D constitutes an open cover of the compact set Q, and thus a finite sub-collection D_0 of D also covers Q, where:

$$D_0 = \left\{B_\delta^0\left(X_i\right) : i = 1, \ldots, L\right\}$$

Letting

$$T = \max\left\{J\left(X_i\right) : i = 1, \ldots, L\right\}$$

we have $F^{-T}(X) \in B_\epsilon(Z)$ for any $X \in Q$. Since $\epsilon < r, Z$ is also expanding in $B_\epsilon(Z)$, so:

$$F^{-k}[Q] \subset B_\epsilon(Z) \quad \text{for all } k \geqslant T \tag{3.5}$$

For each $k \geqslant T$ consider the function $F^{-k} \circ G$ defined for all $X \in B_\epsilon(Z)$. Since G is continuous (and $1-1$) in $B_\epsilon(Z)$ and F^{-k} is continuous (and 1-1) in $G\left[B_\epsilon(Z)\right]$, then $F^{-K} \circ G$ is continuous (and (1-1) in $B_\epsilon(Z)$. So, from 3.5 and the definition of Q, $F^{-k} \circ G\left[B_\epsilon(Z)\right] \subset B_\epsilon(Z)$, and, consequently, $F^{-k} \circ G$ must have a fixed point $Y_k \in B_\epsilon(Z)$ by the Brouwer fixed point theorem. That is, $F^{-k} \circ G\left(Y_k\right) = Y_k$ for all $k \geqslant T$. Note, therefore, that $F^k\left(Y_k\right) = \left(F^k \circ F^{-k} \circ G\right)\left(Y_k\right) = G\left(Y_k\right)$. From 3.2 $F^{M+k}\left(Y_k\right) = F^M \circ G\left(Y_k\right) = G^{-1} \circ G\left(Y_k\right) = Y_k$, and Y_k is thus a fixed point of F^{M+k}. We shall show that Y_k^r cannot have period less than $M+k$. From above $F^k\left(Y_k\right) = G\left(Y_k\right)$ and $Y_k \in B_\epsilon(Z)$. Hence

$$F^k\left(Y_k\right) \in Q = G\left[B_\epsilon(Z)\right] \quad \text{for all } k \geqslant T \tag{3.6}$$

Taking F^{-k} of the point $F^k\left(Y_k\right)$ in 3.6 yields: $Y_k \in F^{-k}[Q]$. Letting $m = -n+k$ in 3.4, we thus obtain: $F^n\left(Y_k\right) \in F^{n-k}[Q] \subset B_r(Z)$ for all n satisfying $0 \leqslant n \leqslant k$. Also, 3.3 and 3.6 imply that $F^{m+k}\left(Y_k\right) \notin B_r(Z)$ for all m satisfying $1 \leqslant m < M$. So:

$$\begin{aligned} &F^n\left(Y_k\right) \in B_r(Z) && \text{for } 0 \leqslant n \leqslant k \\ &F^n\left(Y_k\right) \notin B_r(Z) && \text{for } k+1 \leqslant n < M+k \end{aligned}$$

and

$$F^{M+k}\left(Y_k\right) = Y_k$$

It is clear, therefore, that Y_k cannot have period less than $M+k$. Hence, letting $N = M+T$ and $p = M+k$ for all $k \geqslant T$ proves (1).

Proof of (2). Let the integers M, T and N be as in the proof of (i) and let U and V be the two compact sets defined by:

$$U = F^{M-1}[Q] \quad \text{and} \qquad V' = B_\epsilon(Z)$$

Claim 1. $U \cap V = \varnothing$.

Proof. From 3.3 $U = F^{M-1}[Q] \subset \mathbb{R}^n - B_r(Z)$. Since $\epsilon < r$, then $U \subset \mathbb{R}^n - B_\epsilon(Z)$. So, $V = B_\epsilon(Z)$ implies that $U \cap V = \varnothing$. □

Claim 2. $V \subset F^{\mathrm{N}}[U]$.

Proof. From the definition of $U, F[U] = F \circ F^{M-1}[Q] = F^M[Q]$. But, from 3.2 and the definition of $Q, F^M[Q] = F^M \circ G\left[B_\epsilon(Z)\right] = B_\epsilon(Z)$. So, $F[U] = B_\epsilon(Z)$, and hence $F^N[U] = F^{N-1}[F[U]] = F^{N-1}\left[B_\epsilon(Z)\right]$. Now since Z is expanding in $B_\epsilon(Z), F^{N-1}\left[B_\epsilon(Z)\right] \supset B_\epsilon(Z)$, and therefore $F^N[U] \supset B_\epsilon(Z) = V$ $\square$

Claim 3. $U \subset F^N\left[L'\right]$ and $l \subset F^{\mathrm{N}}[\mathrm{V}]$

Proof. $F^{\mathrm{v}}[\mathrm{V}] = F^{\mathrm{N}}\left[B_\epsilon(Z)\right] \supset B_{\mathrm{f}}(Z)$ since Z is expanding in $B_\epsilon(Z)$. So, $F^N[V] \supset V$ Also, letting $k = T+1$ in 3.5 yields $F^{-T-1}[Q] \subset B_\epsilon(Z)$, and therefore $F^{N-T-1}[Q] \subset F^N\left[B_\epsilon(Z)\right] = F^N[V]$. But, $U = F^{M-1}[Q] = F^{N-T-1}[Q]$ and thus $L \subset F^{\mathrm{v}}[\mathrm{V}]$. $\square$

Now, let H be the function defined by: $H(X) = F^N(X)$ for all $X \in B_r(Z)$ We summerize the properties of L, V and H :

$$\begin{array}{l} \inf\{\|X-Y\| : X \in U \text{ and } Y \in I\} > 0 \\ V \subset H[U] \quad \text{ and } \quad U, V \subset H[V] \end{array} \tag{3.7}$$

The remainder of the proof of (2) is essentially identical to that of the corresponding result in [10].

Let A be the set of sequences $E = \{E_n\}_{n=1}^{\infty}$ where E_n equals either U or V, and if $E_n = U$ then $E_{n+1} = E_{n+2} = V$. Let $R(E, n)$ be the number of E_i's which equal U for $1 \leqslant i \leqslant n$. For each $w \in (0,1)$ choose $E^w = \{E_n^w\}_{n=1}^{\infty}$ to be a sequence in A satisfying:

$$\lim_{n \to \infty} \frac{R\left(E^u, n^2\right)}{n} = w$$

If B is defined by: $B = \left\{E^k : w \in (0,1)\right\} \subset A$, then B is uncountable. Also, from 3.7 $H\left[E_n^w\right] \supset E_{n+1}^w$, and, therefore, by Lemma 3.6 for each $E^w \in B$ there is a point $X_w \in U \cup V$ with $H^n\left(X_w\right) \in E_n^w$ for all $n \geqslant 1$. Letting $S_H = \{H^n\left(X_n\right) : n \geqslant 0 \text{ and } E^w \in B\}$, then $H\left[S_H\right] \subset S_H, S_H$ contains no periodic points of H, and there exists an infinite number of n's such that $H^n(X) \in U$ and $H^n(Y) \in V$ for any $X, Y \in S_H$ with $X \neq Y$. (See [10].)

Now combining this last statement with 3.7 implies that for any $X, Y \in S_H$ with $X \neq Y$:

$$L_1 = \limsup_{n \to \infty} \|H^n(X) - H^n(Y)\| > 0$$

Therefore, letting $S = \{F^n(X) : X \in S_H \text{ and } n \geqslant 0\}$ and recalling that $H(X) = F^N(X)$, we see that $F[S] \subset S, S$ contains no periodic points of F, and for any $X, Y \in S$ with $X \neq Y$:

$$\limsup_{n \to \infty} \|F^n(X) - F^n(Y)\| \geqslant L_1 > 0$$

We thus have (2)a and (2)b. In a similar manner (2)c can be proven.

Proof of (3). First note that since Z is expanding in $B_\epsilon(Z)$, if we define $D_n = H^{-n}\left[B_\epsilon(Z)\right]$ for all $n \geqslant 0$, then given $\delta > 0$ there exists $J = J(\delta)$

such that $\|X - Z\| < \delta$ for all $X \in D_n$ and $n > J$. The proof of (3) again parallels the proof of the corresponding result in [10].

For any sequence $E^w = \{E_n^w\}_{n=1}^{\infty} \in A$ we shall further restrict the E_n^w in the following manner: if $E_n^k = U$ then $n = m^2$ for some integer m. Also, if $E_n^w = U$ for both $n = m^2$ and $n = (m+1)^2$ then $E_n^w = D_{2m-k}$ for $n = m^2+k$ where $k = 1, \ldots, 2m$. For the remaining n's we shall assume $E_n^w = V$.

It can be easily checked that these sequences still satisfy $H\left[E_n^w\right] \supset E_{n+1}^w$, and thus by Lemma 3.6 there exists a point X_w with $H^n\left(X_u\right) \in E_n^w$ for all $n \geqslant 0$. Now, defining $S_0 = \left\{X_w : w \in \left(\frac{4}{5}, 1\right)\right\}$ then S_0 is uncountable, $S_0 \subset S_H \subset S$ and for any $s, t \in \left(\frac{4}{5}, 1\right)$ there exist infinitely many m's such that $H^n\left(X_s\right) \in E_n^s = D_{2m-1}$ and $H^n\left(X_t\right) \in E_n^t = D_{2m-1}$ where $n = m^2+1$. But from above, given $\delta > 0, \|X - Z\| < \delta/2$ for all $X \in D_{2m-1}$ and m sufficiently large. Thus, for all $\delta > 0$ there exists an integer m such that $\|H^n\left(X_s\right) - H^n\left(X_t\right)\| < \delta$ where $n = m^2 - 1$. Since δ is arbitrary we have:

$$L_2 = \liminf_{n\to\infty} \|H^n\left(X_s\right) - H^n\left(X_t\right)\| = 0$$

Therefore, for any $X, Y \in S_0$:

$$\liminf_{n\to\infty} \|F^n(X) - F^n(Y)\| \leqslant L_2 = 0$$

and (3) is proven.

$\square$

***Remark* 9.** Assuming f to be differentiable in $B_r\left(x^*\right)$, Marotto claimed that the logical relationship $A \Rightarrow B$ holds, where:

- A. All eigenvalues of the Jacobian $Df\left(x^*\right)$ of system at the fixed point $x^* = f\left(x^*\right)$ are greater than 1 in norm.
- B. There exist some $s > 1$ and $r > 0$ such that $\|f(x) - f(y)\| > s\|x - y\|$ for all $x, y \in B_r\left(x^*\right)$

Unfortunately, two counterexamples have been given in [1] and [9] to show that $A \Rightarrow B$ is not necessarily true, and there exists an error in the proof given by Marotto.

Theorem 3.7 (Shi and Chen[13]). *Let $f : R^n \to R^n$ be a map with a fixed point $z \in R^n$. Assume that*

(a) *f is continuously differentiable in a neighborhood of z and all the eigenvalues of $Df(z)$ have absolute values larger than 1, which implies that there exist a positive constant r and a norm $\|\cdot\|$ in R^n such that f is expanding in $B_r(z)$ in norm $\|\cdot\|$;*

(b) *z is a snap-back repeller of f with $f^m\left(x_0\right) = z$ for some $x_0 \in B_r^0(z), x_0 \neq z$, and some positive integer m. Furthermore, f is continuously differentiable in some neighborhoods of $x_0, x_1, \ldots, x_{m-1}$, respectively, and $\det Df\left(x_j\right) \neq 0$ for $0 \leq j \leq m-1$, where $x_j = f\left(x_{j-1}\right), 0 \leq j \leq m-1$*

Then, all the results of the Marotto theorem hold.

In the proof of this theorem, the author just perfected and supplemented the original step (A) to (B), and added continuously differentiable conditions. Most of the proof process is the same. In order to avoid redundancy, I won't repeat it.

Remark 10. There are some disadvantages of Marotto's result:

- The difference equations are required to be continuously differentiable rather than only continuous;
- It is applicable only to difference equation with repellers but not to those with saddle points.

It can also be seen that while perfecting the theorem, the conditions required for chaos are also strengthened, which leads to many deficiencies in the application. In order to improve this result, Kloeden has made the following efforts.

Definition 3.8 (l-ball)**.** An l -ball is defined as a closed ball of finite radius in $\mathbb{R}^l$ in terms of the Euclidean distance on $\mathbb{R}^l$. Such a ball of radius r centered on a point $z_0 \in \mathbb{R}^l$ is denoted by $B^l(z_0; r)$.

Definition 3.9 (expanding)**.** A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is called expanding on a set $A \subset \mathbb{R}^n$ if there exists a constant $\lambda > 1$ such that

$$\lambda \|x - y\| \leq \|f(x) - f(y)\|$$

for all $x, y \in A$. Note that such a mapping is one-to-one on A.

Similarly, without proof, the following two results are given.

Lemma 3.10. [7] *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping, which is one-to-one on a compact subset $K \subset \mathbb{R}^n$. Then there exists a continuous mapping $g : f(K) \to K$ such that $g(f(x)) = x$ for all $x \in K$*

Lemma 3.11. [7] *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping and let $\{K_i\}_{i=0}^{\infty}$ be a sequence of compact sets in $\mathbb{R}^n$ such that $K_{i+1} \subseteq f(K_i)$ for $i = 0, 1, 2, \ldots$ Then there exists a nonempty compact set $K \subseteq K_0$ such that $f^i(x_0) \in K_i$ for all $x_0 \in K$ and all $i \geq 0$.*

Theorem 3.12 (Kloeden[7])**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping and suppose that there exist nonempty compact sets A and B, and integers $1 \leq l \leq n$ and $n_1, n_2 \geq 1$ such that:*

(a) *A is homeomorphic to an l-ball;*
(b) *$A \subseteq f(A)$*
(c) *f is expanding on A;*
(d) *$B \subseteq A$*
(e) *$f^{n_1}(B) \cap A = \emptyset$;*
(f) *$A \subseteq f^{n_1+n_2}(B)$;*
(g) *$f^{n_1+n_2}$ is one-to-one on B.*

Then, all the results of the Marotto theorem hold.

Proof. The proof is similar to that used by Marotto, except Lemma 3.10 is used instead of the inverse mapping theorem and the Brouwer fixed point theorem is used on homeomorphisms of l -balls rather than n -balls. The Brouwer fixed point theorem says that a continuous mapping from an n -dimensional closed ball into itself must have fixed point. From the continuity of f and assumption (f) there exists a nonempty, compact subset $C \subseteq B$ such that $A = f^{n_1+n_2}(C)$.

By (g)$f^{n_1+n_2}$ is one-to-one on C, and by Lemma 3.10 there exists a continuous function $g : A \to C$ such that $g\left(f^{n_1+n_2}(x)\right) = x$ for all $x \in C$. Note that $f^{n_1}(C) \cap A = \emptyset$ by (e).

Now f is one-to-one on A by (c), so by Lemma 3.10 f has a continuous inverse $f_A^{-1} : f(A) \to A$. By (b) $C \subset A \subseteq f(A)$, so $f_A^{-k}(C) \subset A$ holds for all $k \geq 0$ For each $k \geq 0$ the mapping $f_A^{-k} \circ g : A \to A$ is a continuous mapping from a homeomorphism of an l -ball into itself, so by the Brouwer fixed point theorem there exists a point $y_k \in A$ such that $f_A^{-k}\left(g\left(y_k\right)\right) = y_k$.

In fact $y_k \in f^{-k}(C)$ and so $f^{n_1+k}\left(y_k\right) = f^{n_1+k}\left(f_A^{-k}\left(g\left(y_k\right)\right)\right) = f^{n_1}\left(g \leq ft(y_k)\right)$ $\in f^{n_1}(C)$ as $g\left(y_k\right) \in C$. Hence $f^{n_1+n_k}\left(y_k\right) \notin A$ as $f^{n_1}(C) \cap A = \varnothing$.

Also $f^{n_1+n_2+k}\left(y_k\right) = f^{n_1+n_2}\left(g\left(y_k\right)\right) = y_k$ Now for $k \geq n_1 + n_2$ the point y_k is a periodic point of period $p = n_1+n_2+k$. To see this note that p cannot be less than or equal to k because $f^j\left(y_k\right) \in f_A^{-k+j}(C) \subset A$ for $1 \leq j \leq k$ and then the whole cycle would belong to A in contradiction to the fact that $f^{n_1+k}\left(y_k\right) \notin A$. Also p cannot lie between k and n_1+n_2+k when $k \geq n_1+n_2$ because $f^{n_1+n_2+k}\left(y_k\right) = y_k$ and so p would have to divide n_1+n_2+k exactly, which is impossible when $k \geq n_1 + n_2$. Hence, the difference equation (8) has a periodic point of period p for each $p \geq N = 2\left(n_1 + n_2\right)$.

Write $D = f^{n_1}(C)$ and $h = f^N$. Then $A \cap D = \emptyset$ and

$$h(D) = f^N(D) = f^{2n_1+n_2}\left(f^{n_2}(D)\right) = f^{2n_1+n_2}(A) \supseteq A \tag{3.8}$$

in view of (b) and the definition of C. Also

$$h(A) = f^N(A) \supseteq A \tag{3.9}$$

by (b) and

$$h(A) = f^N(A) \supseteq f^{2(n_1+n_2)}\left(f_A^{-n_1-2n_2}(C)\right) = f^{n_1}(C) = D \tag{3.10}$$

as $f_A^{-n_1-2n_2}(C) \subset A$. Moreover as A and D are nonempty, disjoint compact sets it follows that

$$\inf\{\|x-y\|; x \in A, y \in D\} > 0 \tag{3.11}$$

The existence of a scrambled set S then follows exactly as in Marotto's proof [11] or in Li and Yorke [10]. It will be briefly outlined here for completeness.

Let $\mathcal{E}$ be the set of sequences $\xi = \{E_k\}_{k=1}^{\infty}$ where E_k is either A or D, and $E_{k+1} = E_{k+2} = A$ if $E_k = D$. Let $r(\xi, k)$ be the number of sets E_j equal to D for $1 \leq j \leq k$ and for each $\eta \in (0,1)$ choose $\xi^\eta = \left\{E_k^\eta\right\}_{k=1}^\infty$ to be a sequence in $\mathcal{E}$ satisfying

$$\lim_{k\to\infty} \frac{r\left(\xi^\eta, k^2\right)}{k} = \eta$$

Let $\mathcal{F} = \{\xi^\eta; \eta \in (0,1)\} \subset \mathcal{E}$. Then $\mathcal{F}$ is uncountable. Also from equations 3.8-3.10, $h\left(E_k^\eta\right) \supseteq E_{k+1}^\eta$ and so by Lemma 3.11 for each $\xi^\eta \in \mathcal{F}$ there is a point $x_\eta \in A \cup D$ with $h^k\left(x_\eta\right) \in E_k^\eta$ for all $k \geq 1$. Let $S_h = \left\{h^k\left(x_\eta\right); k \geq 0 \text{ and } \xi^\eta \in \mathcal{F}\right\}$. Then $h\left(S_h\right) \subset S_h$ S_h contains no periodic points of h, and there exists an infinite number of k's such that $h^k(x) \in A$ and $h^k(y) \in D$ for any $x, y \in S_h$ with $x \neq y$. Hence, from equation 3.11 for any $x, y \in S_h$ with $x \neq y$

$$L_1 = \limsup_{k\to\infty} \left\|h^k(x) - h^k(y)\right\| > 0$$

Thus letting $S = \left\{f^k(x); x \in S_h \text{ and } k \geq 0\right\}$ it follows that $f(S) \subset S, S$ contains no periodic points of f and for any $x, y \in S$ with $x \neq y$

$$\limsup \left\|f^k(x) - f^k(y)\right\| \geq L_1 > 0$$

This proves that the set S has properties (2)a and (2)b of a scrambled set. The remaining property (2)c can be proven similarly. See Li and Yorke for further details [10].

It remains now to establish the existence of an uncountable subset S_0 of the scrambled set S with the properties listed in part (3) of the definition of chaotic behavior. In contrast with Marotto's proof this is the first place where assumption (c) that f is expanding on A is required. Until now all that has been required is that f is one-to-one on A. From this, (b) and Lemma 3.10 follows the existence of a continuous inverse $f_A^{-1} : A \to A$. Hence by the Brouwer fixed point theorem there exists a point $a \in A$ such that $f_A^{-1}(a) = a$, or equivalently $f(a) = a$ Now because f is expanding on A it follows that f_A^{-1} is contracting A, i.e.

$$\left\|f_A^{-1}(x) - f_A^{-1}(y)\right\| \leq \lambda^{-1}\|x - y\|$$

for all $x, y \in A$ where $\lambda > 1$ is the coefficient of expansion of f on A. Hence for any $k \geq 1$ and all $x, y \in A$

$$\left\|f_A^{-k}(x) - f_A^{-k}(y)\right\| \leq \lambda^{-k}\|x - y\|$$

and in particular for any $x \in C \subset A$ and for $y = a$

$$\left\|f_A^{-k}(x) - a\right\| \leq \lambda^{-k}\|x - a\| \tag{3.12}$$

so $f_A^{-k}(x) \to a$ as $k \to \infty$ for all $x \in C$. Consequently, for any $\varepsilon > 0$ there exists an integer $j = j(x, \varepsilon)$ such that $f_A^{-j}(x) \in A \cap B^n(a; \varepsilon)$. Then by continuity there exists a $\delta = \delta(x, \varepsilon) > 0$ such that $f_A^{-1}\left(A \cap \operatorname{int} B^n(x; \delta)\right) \subset A \cap B^n(a; \varepsilon)$. Now the collection $s = \operatorname{int}\ B^n(x; \delta); x \in C\}$ constitutes an open cover of the compact set C, so there exists a finite sub-collection

$s_0 = \{\operatorname{int} B^n(x_i; \delta_i); i = 1, 2, \ldots, L\}$ which also covers C. Let $T = T(\varepsilon) = \max\{j(x_i; \varepsilon); i = 1, 2, \ldots, L\}$. Then $f_A^{-T}(x) \in B^n(a; \varepsilon) \cap A$ for all $x \in C$ and so by equation 3.12 $f_A^{-k}(C) \subset B^n(a; \varepsilon) \cap A$ for all $k \geq T(\varepsilon)$. Let $H_k = h_A^{-k}(C)$ for all $k \geq 0$ where h_A^{-1} is a continuous inverse of $h = f^N$ on A. Then for any $\varepsilon > 0$ there exists a $J = J(\varepsilon)$ such that $\|x - a\| < \varepsilon/2$ for all $x \in H_k$ and all $k > J$. The remainder of the proof parallels that in Marotto [11] and in Li and Yorke [10] The sequences $\xi^n = \{E_k^n\}_{k=1}^{\infty} \in \mathcal{E}$ will be further restricted as follows: if $E_k^n = D$ then $k = m^2$ for some integer m and if $E_k^\eta = D$ for both $k = m^2$ and $k = (m+1)^2$ then $E_k^\eta = H_{2m-j}$ for $k = m^2 + j$ for $j = 1, 2, \ldots, 2m$. Finally for the remaining k's, $E_k^\eta = A$. Now these sequences still satisfy $h\left(E_k^\eta\right) \supset E_{k+1}^\eta$, so by Lemma 3.11 there exists a point x_η with $h^k(x_\eta) \in E_k^\eta$ for all $k \geq 0$. Let $S_0 = \{x_\eta : \eta \in ((4/5), 1)\}$. Then S_0 is uncountable, $S_0 \subset S_h \subset S$ and for any $s, t \in ((4/5), 1)$ there exist infinitely many m's such that $h^k(x_s) \in E_k^s = H_{2m-1}$ and $h^k(x_t) \in E_k^t = H_{2ml-1}$ where $k = m^2 + 1$. But from above, given any $\varepsilon > 0, \|x - a\| < \varepsilon/2$ for all $x \in H_{2m-1}$ provided m is sufficiently large. Hence, for any $\varepsilon > 0$ there exists an integer m such that $\left\|h^k(x_s) - h^k(x_t)\right\| < \varepsilon$ where $k = m^2 + 1$. As $\varepsilon > 0$ is arbitrary it follows that

$$L_2 = \liminf_{l \to \infty} \left\|h^k(x_s) - h^k(x_t)\right\| = 0$$

Thus for any $x, y \in S_0$

$$\liminf_{k \to \infty} \left\|h^k(x_s) - h^k(x_t)\right\| \leq L_2 = 0.$$

This completes the proof of Theorem 4. □

Remark 11. Here we explain the functions of the above two definitions l-ball and expansion: the introduction of l-ball is to deal with the influence of the saddle point, and the introduction of the above concept can be used to judge the chaos brought by the saddle point.

The condition of using expanding instead of the Jacobian matrix here is mainly to weaken the condition that the function is continuously differentiable. In this work, the concept of expanding can be introduced into a low-dimensional subspace.

Theorem 3.13 (Kloeden[7]). *Let $f : X \to X$ be a continuous mapping of a Banach space X into itself and suppose that there exist non-empty compact subsets A and B of X, and integers $n_1, n_2 \geq 1$ such that*

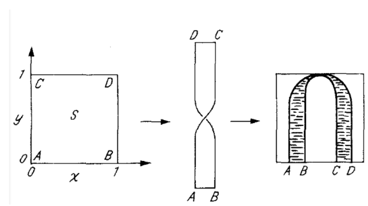
(a) *A is homeomorphic to a convex subset of X;*
(b) *$A \subseteq f(A)$;*
(c) *f is expanding on A, that is, there exists a constant $\lambda > 1$ such that $\lambda\|x - y\| \leq \|f(x) - f(y)\|$ for all $x, y \in A$;*
(d) *$B \subseteq A$*
(e) *$f^{n_1}(B) \cap A = \emptyset$;*
(f) *$A \subseteq f^{n_1+n_2}(B)$;*

(g) $f^{n_1+n_2}$ *is one-to-one on* B.

Then, all the results of the Marotto theorem hold.

Example 3.14 (Twisted horseshoe on I^2[2])**.** Consider the difference equation on the unit square I^2 in $\mathbb{R}^2$, which is defined in terms of the continuous mapping $f = (f_1, f_2)$ where

$$f_1(x, y) = \begin{cases} 2x & \text{for } 0 \le x \le \frac{1}{2}, \\ 2 - 2x & \text{for } \frac{1}{2} < x \le 1, \end{cases} \qquad f_2(x, y) = \frac{x}{2} + \frac{y}{10} + \frac{1}{4}$$



However, the above theorem is not applicable to diffeomorphisms such as the Hénon mapping[3] and the Smale horseshoe mapping[12].

Example 3.15 (Hénon map[3])**.**

$$\begin{cases} x_{n+1} = 1 - ax_n^2 + y_n \\ y_{n+1} = bx_n \end{cases}$$

4. The nature of chaos and the future of chaos

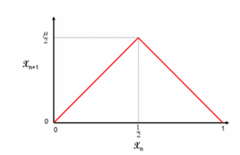
What caused the chaos? In longtime exploration, mathematicians have respectively given various conditions for judging chaos, but I believe that all chaotic conditions contain two unavoidable phenomena: **Expending** and **Folding over on itself**.

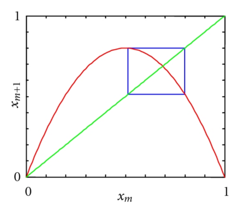
- **Expanding**: That is, the unstable fixed point. From the entire proof of Li-yoke theorem, lemma explains the existence of the fixed point, and since the image of f is included in the original image, we are bound to know it has properties $|f(x) - f(y)| \ge |x - y|$. Later, Marotto generalizes these concepts.
- **Folding over itself**: On the basis of a fixed point, if f^n is an expanded structure for all n, chaos will not occur as example 3.3. In Li-Yorke theorem, period three show the structure of "folding over it self". Since f maps K to L, there is bound to be a return structure. And Snap-back repeller was later used to explain the structure more clearly.

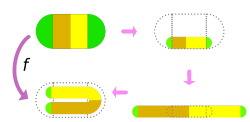
We use a few concrete examples to illustrate the existence of these two structures.

- **Tent map**: At its fixed point, the absolute value of the eigenvalue of the Jacobian matrix is the absolute value of the slope of the straight line segment, so it is easy to get that it is expanding. At the same time, because it maps half the domain to the entire domain , There must be a structure of ”folding over itself”.

- **Logistic function**: This is the same as the tent map, except that the Jacobian matrix at the fixed point is the slope of the tangent vector.
- **Smale Horseshoe**: This is one of the most famous examples of chaos in two-dimensional space. It can be seen that it has a fixed point and is expending. At the same time, there is also a folding, which means that there is a returning structure.







Finally, it is a pity that although we have conducted an in-depth analysis of chaos here, but the necessary and sufficient conditions to verify the chaos defined by Li-Yorke or Marotto have not yet been found. Moreover, a unified, well accepted, easy-to-test, and rigorous mathematical definition of chaos is still in the process of being revealed, and this work is far from complete.

The future of chaos is not only the perfection of its definitions and theorems, but also the application in various fields. So far, Some areas benefiting from chaos theory today are geology, mathematics, biology, computer science, economics[8], engineering[4], finance[5], physic[6] and so on.

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