
CSC373H1 Summer 2014 Assignment 3

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Q1. Construction:

To construct a family of graphs where $G(V, E)$ has exponential number of minimum cuts between source and terminal, we begin with n being the number of vertices that appear between the source and the terminal, we will also assume that the capacity on each edge is 1. If $n = 0$ then there are no vertices between the source and the terminal and thus $V = \{s, t\}$ and $E = \emptyset$. Clearly, a single minimum-cut exists since we can only $s \in S$ and $t \in T$, no edges exist to place a capacity on, so indeed $2^{|V|-2} = 2^0 = 1$.

For arbitrary n , $n \geq 1$, and $n = |V| - 2$, then $G(V, E)$ would be constructed such that $V = \{s, t, v_1, v_2, \dots, v_n\}$ and $E = \{(s, v_1), (v_1, t), (s, v_2), (v_2, t), \dots, (s, v_n), (v_n, t)\}$ (i.e. the only edges connected to a vertex are those that have s as their startpoint, or t as their endpoint). This arrangement is chosen because every cut in $G(V, E)$ will be a minimum cut. By the arrangement of $G(V, E)$, and since all capacities are the same. Whether a vertex v is or is not in the set S will not affect the summation of outgoing edge capacities, either the capacity from (v, t) or from (s, v) , respectively, will be used in computing the minimum cut. Therefore we have a maximum number of minimum cuts in G .

Proof of exponential minimum cuts:

From the description above, in our construction of $G(V, E)$, for any $v_i \in V$ the vertex can be contained within or outside the set that describes a cut in G and the cut is still a minimum cut. If we think of each minimum-cut as being described by a binary string for $n = i$, $i \geq 1$, $(q_{v_1}, q_{v_2}, \dots, q_{v_n})$, such that:

$$q_{v_i} = \begin{cases} 1 & \text{if } v_i \in \text{minimum cut set} \\ 0 & \text{if } v_i \notin \text{minimum cut set} \end{cases}$$

Base: Let $n = 1$, and let the B_1 be the set of binary strings describing the minimum cuts in $G(V, E)$. Then B_1 contains two binary strings such that $B_1 = \{0, 1\}$. Indeed, $|B_1| = 2 = 2^n = 2^{|V|-2}$.

IH: Let $n = i$, $i > 1$, and let B_i be the set of binary strings describing the minimum cuts in $G(V, E)$, such that $V = \{s, t, v_1, v_2, \dots, v_i\}$ and $E = \{(s, v_1), (v_1, t), (s, v_2), (v_2, t), \dots, (s, v_i), (v_i, t)\}$. Suppose that $|B_i| = 2^i = 2^{|V|-2}$.

IS: Let $n = i + 1$, it follows from the construction description that $V' = \{s, t, v_1, v_2, \dots, v_i, v_{i+1}\}$ and $E' = \{(s, v_1), (v_1, t), (s, v_2), (v_2, t), \dots, (s, v_i), (v_i, t), (s, v_{i+1}), (v_{i+1}, t)\}$. Clearly, the addition of v_{i+1} extends $G(V, E)$ to $G(V', E')$. The minimum cuts in $G(V', E')$ are composed from the minimum cuts in $G(V, E)$ except that v_{i+1} is either inside or outside these previous cuts. Thus, to compute the set of minimum cuts as binary strings, B_{i+1} , simply take each binary string in B_i and append a 0 to be a minimum cut where v_{i+1} is outside the set, and then again take each binary string in B_i and append a 1 to be a minimum cut where v_{i+1} is inside the set. Thus, $|B_{i+1}| = 2 * (|B_i|) = 2 * 2^i = 2^{i+1}$.

By proof of induction this construction will have exponential minimum cuts between the source and the terminal. ■

Q2. Claim:

Function f , such that $f(S)$ is the number of edges (u, v) with $u \in S$, $v \in V \setminus S$, is submodular.

Proof:

Using the fact provided in the question, all that is necessary to show that f is submodular is to show that for any two subsets $A, B \subseteq V$, $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$.

Consider the set of edges in $G(V, E)$, $E = \{e_1, e_2, \dots, e_n\}$. If we think of $f(S)$ as an iterative algorithm that goes through each $e_i \in E$ and checks whether the startpoint of $e_i \in S$ and whether the endpoint of $e_i \in V \setminus S$. Then we have $f_{e_i}(S)$:

$$f_{e_i}(S) = \begin{cases} 1 & \text{if startpoint of } e_i \text{ in } S \text{ and endpoint of } e_i \text{ in } V \setminus S \\ 0 & \text{otherwise} \end{cases}$$

And so, $f(S) = \sum_{i=1}^n f_{e_i}(S)$. Suppose (u, v) is an arbitrary edge in E , also suppose A, B are some arbitrary subsets of V . We wish to show in all cases, however vertices u and v appear or do not appear in A and/or B , that $f_{(u,v)}(A) + f_{(u,v)}(B) \geq f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$:

Case 1: $u \notin A$ and $u \notin B$: Clearly, $u \notin (A \cup B)$ and $u \notin (A \cap B)$, thus, $f_{(u,v)}(A) = f_{(u,v)}(B) = f_{(u,v)}(A \cup B) = f_{(u,v)}(A \cap B) = 0$. So, $f_{(u,v)}(A) + f_{(u,v)}(B) \geq f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

Case 2: $u \in A$ and $u \notin B$ (W.L.O.G. switching B for A in this case and all subcases the claim holds):

Case 2.1: $v \in A$: Clearly, $u, v \in (A \cup B)$ and $u \notin (A \cap B)$, it follows that $f_{(u,v)}(A) = f_{(u,v)}(B) = f_{(u,v)}(A \cup B) = f_{(u,v)}(A \cap B) = 0$. So, $f_{(u,v)}(A) + f_{(u,v)}(B) \geq f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

Case 2.2: $v \notin A$ and $v \in B$: So, $f_{(u,v)}(A) = 1$ and $f_{(u,v)}(B) = 0$, but since $u, v \in (A \cup B)$ and $u \notin (A \cap B)$, then $f_{(u,v)}(A \cup B) = f_{(u,v)}(A \cap B) = 0$. So, $f_{(u,v)}(A) + f_{(u,v)}(B) \geq f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

Case 2.3: $v \notin A$ and $v \notin B$: Again, $f_{(u,v)}(A) = 1$ and $f_{(u,v)}(B) = 0$. Also, $u \in (A \cup B)$ and $v \notin (A \cup B)$, then $f_{(u,v)}(A \cup B) = 1$. Again, $u \notin (A \cap B)$, then $f_{(u,v)}(A \cap B) = 0$. So, $f_{(u,v)}(A) + f_{(u,v)}(B) \geq f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

Case 3: $u \in A$ and $u \in B$ (W.L.O.G. switching B for A in this case and all subcases the claim holds):

Case 3.1: $v \in A$ and $v \in B$: Since $u, v \in A$ and B , then $f_{(u,v)}(A) = f_{(u,v)}(B) = 0$. Also, $u, v \in (A \cup B)$ and $(A \cap B)$, then $f_{(u,v)}(A \cup B) = f_{(u,v)}(A \cap B) = 0$. So, $f_{(u,v)}(A) + f_{(u,v)}(B) \geq f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

Case 3.2: $v \notin A$ and $v \in B$: It follows that $f_{(u,v)}(A) = 1$ and $f_{(u,v)}(B) = 0$. Since $u, v \in (A \cup B)$ then $f_{(u,v)}(A \cup B) = 0$, but $u \in (A \cap B)$ and $v \notin (A \cap B)$, thus $f_{(u,v)}(A \cap B) = 1$. Again, $f_{(u,v)}(A) + f_{(u,v)}(B) \geq f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

Case 3.3: $v \notin A$ and $v \notin B$: Clearly, $f_{(u,v)}(A) = 1$ and $f_{(u,v)}(B) = 1$. Also, $u \in (A \cup B)$ and $(A \cap B)$, but $v \notin (A \cup B)$ and $(A \cap B)$, then $f_{(u,v)}(A \cup B) = f_{(u,v)}(A \cap B) = 1$. Then, $f_{(u,v)}(A) + f_{(u,v)}(B) \geq f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

In all cases in which the vertices of some edge $(u, v) \in E$ might appear in $A, B \subseteq V$ it has been shown that $f_{(u,v)}(A) + f_{(u,v)}(B) \geq f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds. It must follow then that $\sum_{i=1}^n f_{e_i}(A) + \sum_{i=1}^n f_{e_i}(B) \geq \sum_{i=1}^n f_{e_i}(A \cup B) + \sum_{i=1}^n f_{e_i}(A \cap B) \rightarrow f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$. Clearly, function f is submodular by the equivalence of submodular functions. ■

Q3. Consider the restructuring of LP formulation of maximum flow into standard form:

maximize: $\vec{c} \cdot \vec{x}$

subject to: $A\vec{x} \leq \vec{b}$, $\vec{x} \geq 0$.

Suppose this LP operates on $G(V, E)$ such that paths $P = \{p_1, \dots, p_n\}$ and $E = \{e_1, \dots, e_m\}$.

Let $\vec{c} = [1, \dots, 1]$, such that $|\vec{c}| = n$.

Let $\vec{x} = [x_1, \dots, x_n]$, such that $x_i = f_{p_i}$, maximum flow along path p_i .

Let A be an $m \times n$ matrix and let A be defined as follows:

$$A[i, j] = \begin{cases} 1 & \text{edge } e_i \text{ appears on path } p_j \\ 0 & \text{edge } e_i \text{ does not appear on path } p_j \end{cases}$$

Let $\vec{b} = [c(e_1), \dots, c(e_m)]$, such that $|\vec{b}| = m$.

Using the restructured LP, we construct the dual as follows:

minimize: $\vec{b} \cdot \vec{y}$

subject to: $A^t \vec{y} \geq \vec{c}$, $\vec{y} \geq 0$.

Let $\vec{y} = [y_1, \dots, y_m]$, y_i is a value associated with e_i .

Again, we define \vec{c} and \vec{b} as the same in the original construction.

Now, A^t is an $n \times m$ matrix, and so A^t is defined as follows:

$$A^t[i, j] = \begin{cases} 1 & \text{in path } p_i \text{ the edge } e_j \text{ appears} \\ 0 & \text{in path } p_i \text{ the edge } e_j \text{ does not appear} \end{cases}$$

The conclusion for the dual LP is that the objective function and constraints represent a construction of a max-flow min-cut, such that when the dual LP is complete its solution will equal the max-flow calculated by the original LP. It can now be stated:

$$y_i = \begin{cases} 1 & \text{edge } e_i \text{ connects sets } S \text{ and } T^* \\ 0 & \text{edge } e_i \text{ does not connect sets } S \text{ and } T^* \end{cases}$$

*clearly S and T represent the set of vertices separated by the minimum cut of $G(V, E)$.

Then the objective function represents the summation of the capacity of those edges where $y_i = 1$, and the summation will be the maximum flow in the network $G(V, E)$. The n path constraints, $A^t \vec{y} \geq \vec{c}$, represent the fact that on each path at least one edge connects S to T . Thus, for path p_i , $\sum_{j=1}^m a_{ij} y_j \geq 1$.

Q4. Polytope P is represented by $Ax \leq b$. Let $x \in \mathbb{R}^n$, so x is a point within P if $Ax \leq b$ holds. More specifically, let A be an $m \times n$ matrix, and b a series of m constants, such that $a_{i1}x_1 + \dots + a_{in}x_n \leq b_i$ represents a closed half-space for each facet of the polytope, so that when all constraints are combined they represent all internal and boundary points of P .*

The LP must be designed so that we find a centre point y of a ball in P , such that all points on the surface of the ball that intersect in the plane where P exists are also in P . Suppose some point x exists at the border of P , then for some i , specifically the i that represents the facet on which x exists, $a_{i1}x_1 + \dots + a_{in}x_n = b_i$. So, if the ball is the largest possible ball to fit in P the surface points must be as close to the boundaries as possible.

Suppose that $\vec{a}_i = [a_{i1}, \dots, a_{in}]$ is a unit vector for some i , such that $\|\vec{a}_i\| = 1$. Indeed, \vec{a}_i is the gradient unit vector to the facet represented by $a_{i1}x_1 + \dots + a_{in}x_n = b_i$, it is also perpendicular to the facet. Also, as a gradient vector, \vec{a}_i points in the direction outside the closed half-space defined by $a_{i1}x_1 + \dots + a_{in}x_n \leq b_i$. To discover if the ball at center y with radius $r \in \mathbb{R}$ can fit in P we check for each facet i if the surface point $z_i = (\vec{a}_i * r + y)$, perpendicular to the facet i , is within P . To do this we use the constraint $a_{i1}z_{i1} + \dots + a_{in}z_{in} \leq b_i$. To explain z_i further, $\vec{a}_i * r$ is an r length vector that when added to y , projects a point to the surface or boundary of the ball and is perpendicular to facet i . In order to find the largest such ball, the LP must be designed to calculate both the ball's maximum radius r and its centre point y . This is done by establishing z_i 's such that for all i , given constraints $a_{i1}z_{i1} + \dots + a_{in}z_{in} \leq b_i$, we want $a_{i1}z_{i1} + \dots + a_{in}z_{in}$ to be as close as possible to b_i for some maximum r and centre point y .

Thus, the LP is:

maximize:

r

subject to:

$$\vec{a}_i \cdot (\vec{a}_i * r + y) \leq b_i, \forall i, 1 \leq i \leq m$$

$$r \geq 0.$$

*explanation of polytope taken from wikipedia, http://en.wikipedia.org/wiki/Convex_polytope.

Q5. Claim:

$\exists s, t \geq 0, sA_1 + tA_2 = 0 \rightarrow sb_1 + tb_2 < 0$.

Proof:

Indeed, if $x \in P_1$ then $A_1x \leq b_1$, or if $x \in P_1$ then $A_2x \leq b_2$. Assume that P_1 and P_2 do not intersect, then $\nexists x$ such that $A_1x \leq b_1$ and $A_2x \leq b_2$. Then choose a point $y \notin P_1$ and $y \notin P_2$, such that $A_1y > b_1$ and $A_2y > b_2$, so that we might relate A_1 and A_2 to b_1 and b_2 .

Suppose we have found $s, t \geq 0$, such that $sA_1 + tA_2 = 0$. If we apply s and t to the respective inequalities the relation still holds because all values in s and t are greater than or equal to zero. Thus, $s(A_1y) > sb_1$ and $t(A_2y) > tb_2$. Combining them we have:

$$s(A_1y) + t(A_2y) > sb_1 + tb_2 \rightarrow$$

$$(sA_1)y + (tA_2)y > sb_1 + tb_2 \rightarrow$$

$$(sA_1 + tA_2)y > sb_1 + tb_2 \rightarrow$$

$$(0)y > sb_1 + tb_2 \rightarrow$$

$$0 > sb_1 + tb_2$$

Then it follows, $\exists s, t \geq 0, sA_1 + tA_2 = 0 \rightarrow sb_1 + tb_2 < 0$. ■

Indeed, to show that it is necessary for P_1 and P_2 to be non-intersecting in order for the statement to hold, suppose there was a point $x \in P_1$ and $x \in P_2$. We would have:

$$s(A_1x) + t(A_2x) \leq sb_1 + tb_2 \rightarrow$$

$$(sA_1)x + (tA_2)x \leq sb_1 + tb_2 \rightarrow$$

$$(sA_1 + tA_2)x \leq sb_1 + tb_2 \rightarrow$$

$$0 \leq sb_1 + tb_2$$

It would follow with both $sb_1 + tb_1 \geq 0$ and $sb_1 + tb_2 < 0$ being possible, that the claim would be impossible to prove.

Q6.

Q7.

Q8.