CSC373H1 Summer 2014 Assignment 3

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Question #	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total	

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"We declare that we have not used any outside help in completing this assignment."

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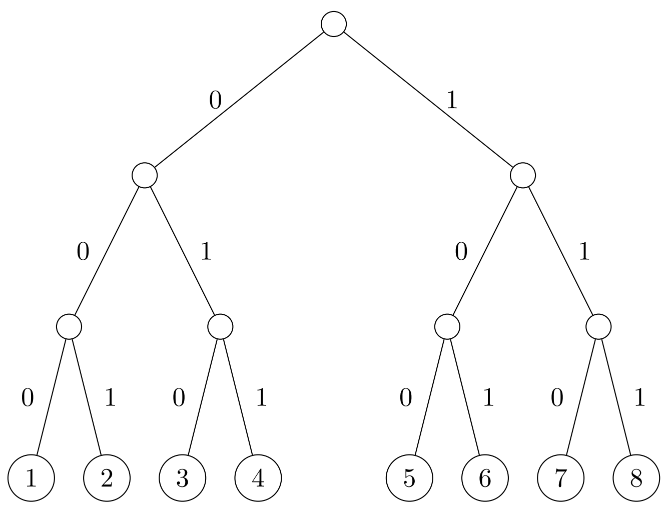
Q1. Suppose G(V, E) is representative of a network where V = {s, t, v_1, ..., v_n} and E is the set of edges that make up this network. Suppose we have sets S and T such that s ∈ S, t ∈ T, (S ∪ T) - {s, t} = V, and S ∩ T = ∅. Also, suppose that for all possible arrangement of vertices in S and T, c(S, T) = f(S, T), which means that G(V, E) has an exponential number of minimum cuts between s and t. Precisely, 2^n minimum cuts as each vertex is either in S or T and cannot be in both, thus it is much like a binary string of |V| elements, which can be expressed in 2^n possible ways.

To construct a graph that is representative of there being 2^n possible minimum cuts, a tree like structure seems appropriate since there are 2^n leaves in a full and complete tree for n levels in the tree. Taking inspiration from Huffman encoding, we wish to have a graph that for each leaf in a full tree will represent a unique binary string that represents whether a vertex is in S or T. Suppose we arrange the the string as follows, v_1, ..., v_n, where if v_i is in S then v_i = 0 and v_i = 1 if it is in T. The tree/graph can be constructed so that each left edge appends 0 to the vertex string, and WLOG each right edge appends 1. Finally, when reaching any leaf we have an n length binary string that represents a unique cut S, T. In total there are 2^n such possible cuts.

For example, suppose we had |V| = 3 and we are certain that there are an exponential number of minimum cuts. We construct a full and complete binary tree as follows to represent the minimum cuts:



If we were to trace each leaf from the node, in a similar fashion to Huffman encoding, we would have 1: 000, 2: 001, 3: 010, 4: 011, 5: 100, 6: 101, 7: 110, and 8: 111. So, it is clear that each binary string representing v_1, v_2, v_3 will represent all possible sets S, T such that in total the constructed graph can represent an exponential number of minimum cuts.

Q2. Claim:

Function f, such that f(S) is the number of edges (u, v) with $u \in S$, $v \in V\backslash S$, is submodular.

Proof:

Using the fact provided in the question, all that is necessary to show that f is submodular is to show that for any two subsets $A, B \subseteq V$, $f(A) + f(B) \geqslant f(A \cup B) + f(A \cap B)$.

We begin by establishing a fact. Suppose you have disjoint sets $X_1, X_2, ..., X_k \subseteq V$, $k > 1$.

<u>Case 1:</u> Assume that in each X_i the set of vertices can be divided into two sets, those that exist as startpoints of edges in E and those that exist as endpoints in E. However, $\forall\ u \in X_i$, $\nexists\ v \in X_i$ such that $(u, v) \in E$. It follows that $f(X_i) \geqslant 0$. Additionally, suppose that $\forall\ u \in X_1 \cup X_2 \cup ... \cup X_k$, $\exists\ v \in X_1 \cup X_2 \cup ... \cup X_k$, such that $(u, v) \in E$. It follows that $f(X_1 \cup X_2 \cup ... \cup X_k) = 0$, and so $f(X_1) + f(X_2) + ... + f(X_k) \geqslant f(X_1 \cup X_2 \cup ... \cup X_k)$.

<u>Case 2:</u> Assume that $\forall\ u \in X_i \nexists\ v \in X_j$, such that $(u, v) \in E$. Also, $i \neq j$ and $1 \geqslant i, j \geqslant k$. So, it would follow that $f(X_1) + f(X_2) + ... + f(X_k) = f(X_1 \cup X_2 \cup ... \cup X_k)$.

Both Case 1 and Case 2 represent the least ideal and most ideal cases, respectively. It follows that in all cases the function f on the union of any series of disjoint subsets of V will result in a value less than or equal to the summation of values of f operating on each subset seperately, thus, $f(X_1) + f(X_2) + ... + f(X_k) \geqslant f(X_1 \cup X_2 \cup ... \cup X_k)$.

Now, suppose we have some arbitrary subsets $A, B \subseteq V$. Indeed, it is possible to seperate both A and B as follows. Let A' represent all vertices from A that do not appear in B, and conversely let A" be those vertices that do appear in both A and B. Similarly, Let B' represent all vertices from B that do not appear in A, and let B" be those vertices that appear in A and B. Then, f(A) + f(B) = f(A') + f(A") + f(B') + f(B"). Clearly, A" = B" = $(A \cap B)$, then $f(A) + f(B) = (f(A') + f(A \cap B) + f(B')) + f(A \cap B)$. It follows by our construction of A' and B' that the sets A', $(A \cap B)$, and B' are disjoint. Then using the fact proven above, $(f(A') + f(A \cap B) + f(B')) \geqslant f(A' \cup (A \cap B) \cup B')$. However, $(A' \cup (A \cap B) \cup B') = (A \cup B)$. Finally,
$f(A) + f(B) = (f(A') + f(A \cap B) + f(B')) + f(A \cap B) \geqslant f(A \cup B) + f(A \cap B)$.

Indeed, $f(A) + f(B) \geqslant f(A \cup B) + f(A \cap B)$ implies that function f, such that f(S) is the number of edges (u, v) with $u \in S$, $v \in V\backslash S$, is submodular, by the equivalence of submodular functions.

Q3.

Q4.

Q5.

Q6.

Q7.

Q8.