CSC373H1 Summer 2014 Assignment 3

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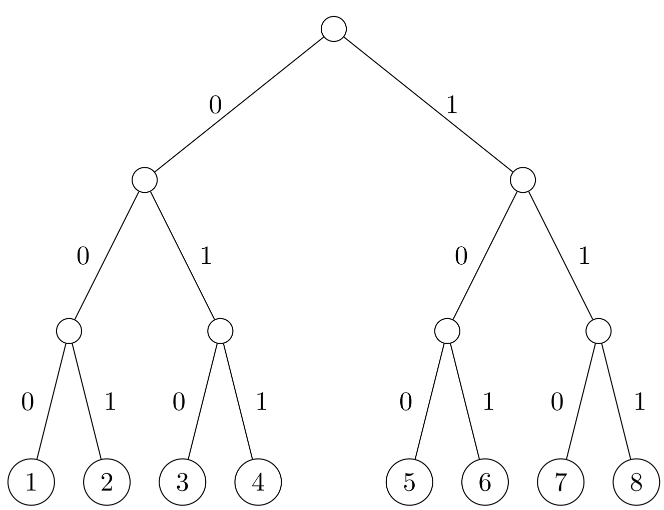
"We declare that we have not used any outside help in completing this assignment."

Name: John Armstrong, Henry Ku Date: July 17, 2014

Q1. Suppose G(V, E) is representative of a network where V = {s, t, v_1, ..., v_n} and E is the set of edges that make up this network. Suppose we have sets S and T such that s $\in$ S, t $\in$ T, (S $\cup$ T) - {s, t} = V, and S $\cap$ T = $\emptyset$. Also, suppose that for all possible arrangement of vertices in S and T, c(S, T) = f(S, T), which means that G(V, E) has an exponential number of minimum cuts between s and t. Precisely, 2^n minimum cuts as each vertex is either in S or T and cannot be in both, thus it is much like a binary string of |V| elements, which can be expressed in 2^n possible ways.

To construct a graph that is representative of there being 2^n possible minimum cuts, a tree like structure seems appropriate since there are 2^n leaves in a full and complete tree for n levels in the tree. Taking inspiration from Huffman encoding, we wish to have a graph that for each leaf in a full tree will represent a unique binary string that represents whether a vertex is in S or T. Suppose we arrange the the string as follows, v_1, ..., v_n, where if v_i is in S then v_i = 0 and v_i = 1 if it is in T. The tree/graph can be constructed so that each left edge appends 0 to the vertex string, and WLOG each right edge appends 1. Finally, when reaching any leaf we have an n length binary string that represents a unique cut S, T. In total there are 2^n such possible cuts.

For example, suppose we had |V| = 3 and we are certain that there are an exponential number of minimum cuts. We construct a full and complete binary tree as follows to represent the minimum cuts:



If we were to trace each leaf from the node, in a similar fashion to Huffman encoding, we would have 1: 000, 2: 001, 3: 010, 4: 011, 5: 100, 6: 101, 7: 110, and 8: 111. So, it is clear that each binary string representing v_1, v_2, v_3 will represent all possible sets S, T such that in total the constructed graph can represent an exponential number of minimum cuts.

Q2. Claim:

Function f, such that f(S) is the number of edges (u, v) with $u \in S$, $v \in V\backslash S$, is submodular.

Proof:

Using the fact provided in the question, all that is necessary to show that f is submodular is to show that for any two subsets $A, B \subseteq V$, $f(A) + f(B) \geqslant f(A \cup B) + f(A \cap B)$.

Consider the set of edges in G(V, E), $E = \{e_1, e_2, ..., e_n\}$. If we think of f(S) as an iterative algorithm that goes through each $e_i \in E$ and checks whether the startpoint of $e_i \in S$ and whether the endpoint of $e_i \in V\backslash S$. Then we have $f_{e_i}(S)$:

$$f_{e_i}(S) = \begin{cases} 1 & \text{if startpoint of } e_i \text{ in S and endpoint of } e_i \text{ in } V\backslash S \\ 0 & \text{otherwise} \end{cases}$$

And so, $f(S) = \sum_{i=1}^{n} f_{e_i}(S)$. Suppose (u, v) is an arbitrary edge in E, also suppose A, B are some arbitrary subsets of V. We wish to show in all cases, however vertices u and v appear or do not appear in A and/or B, that $f_{(u,v)}(A) + f_{(u,v)}(B) \geqslant f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$:

Case 1: $u \notin A$ and $u \notin B$: Clearly, $u \notin (A \cup B)$ and $u \notin (A \cap B)$, thus, $f_{(u,v)}(A) = f_{(u,v)}(B) = f_{(u,v)}(A \cup B) = f_{(u,v)}(A \cap B) = 0$. So, $f_{(u,v)}(A) + f_{(u,v)}(B) \geqslant f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

Case 2: $u \in A$ and $u \notin B$ (W.L.O.G. swtiching B for A in this case and all subcases the claim holds):

Case 2.1: $v \in A$: Clearly, $u, v \in (A \cup B)$ and $u \notin (A \cap B)$, it follows that $f_{(u,v)}(A) = f_{(u,v)}(B) = f_{(u,v)}(A \cup B) = f_{(u,v)}(A \cap B) = 0$. So, $f_{(u,v)}(A) + f_{(u,v)}(B) \geqslant f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

Case 2.2: $v \notin A$ and $v \in B$: So, $f_{(u,v)}(A) = 1$ and $f_{(u,v)}(B) = 0$, but since $u, v \in (A \cup B)$ and $u \notin (A \cap B)$, then $f_{(u,v)}(A \cup B) = f_{(u,v)}(A \cap B) = 0$. So, $f_{(u,v)}(A) + f_{(u,v)}(B) \geqslant f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

Case 2.3: $v \notin A$ and $v \notin B$: Again, $f_{(u,v)}(A) = 1$ and $f_{(u,v)}(B) = 0$. Also, $u \in (A \cup B)$ and $v \notin (A \cup B)$, then $f_{(u,v)}(A \cup B) = 1$. Again, $u \notin (A \cap B)$, then $f_{(u,v)}(A \cap B) = 0$. So, $f_{(u,v)}(A) + f_{(u,v)}(B) \geqslant f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

Case 3: $u \in A$ and $u \in B$ (W.L.O.G. swtiching B for A in this case and all subcases the claim holds):

Case 3.1: $v \in A$ and $v \in B$: Since u, $v \in A$ and B, then $f_{(u,v)}(A) = f_{(u,v)}(B) = 0$. Also, $u, v \in (A \cup B)$ and $(A \cap B)$, then $f_{(u,v)}(A \cup B) = f_{(u,v)}(A \cap B) = 0$. So, $f_{(u,v)}(A) + f_{(u,v)}(B) \geqslant f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

Case 3.2: $v \notin A$ and $v \in B$: It follows that $f_{(u,v)}(A) = 1$ and $f_{(u,v)}(B) = 0$. Since $u, v \in (A \cup B)$ then $f_{(u,v)}(A \cup B) = 0$, but $u \in (A \cap B)$ and $v \notin (A \cap B)$, thus $f_{(u,v)}(A \cap B) = 1$. Again, $f_{(u,v)}(A) + f_{(u,v)}(B) \geqslant f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

Case 3.3: $v \notin A$ and $v \notin B$: Clearly, $f_{(u,v)}(A) = 1$ and $f_{(u,v)}(B) = 1$. Also, $u \in (A \cup B)$ and $(A \cap B)$, but $v \notin (A \cup B)$ and $(A \cap B)$, then $f_{(u,v)}(A \cup B) = f_{(u,v)}(A \cap B) = 1$. Then, $f_{(u,v)}(A) + f_{(u,v)}(B) \geqslant f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

In all cases in which the vertices of some edge $(u, v) \in E$ might appear in $A, B \subseteq V$ it has been shown that $f_{(u,v)}(A) + f_{(u,v)}(B) \geqslant f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds. It must follow then that $\sum_{i=1}^{n} f_{e_i}(A) + \sum_{i=1}^{n} f_{e_i}(B) \geqslant \sum_{i=1}^{n} f_{e_i}(A \cup B) + \sum_{i=1}^{n} f_{e_i}(A \cap B) \rightarrow f(A) + f(B) \geqslant f(A \cup B) + f(A \cap B)$. Clearly, function f is submodular by the equivalence of submodular functions. ■

Q3.

Q4.

Q5.

Q6.

Q7.

Q8.