
CSC373H1 Summer 2014 Assignment 3

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Q1. Construction:

To construct a family of graphs where $G(V, E)$ has exponential number of minimum cuts between source and terminal, we begin with n being the number of vertices that appear between the source and the terminal, we will also assume that the capacity on each edge is 1. If $n = 0$ then there are no vertices between the source and the terminal and thus $V = \{s, t\}$ and $E = \emptyset$. Clearly, a single minimum-cut exists since we can only $s \in S$ and $t \in T$, no edges exist to place a capacity on, so indeed $2^{|V|-2} = 2^0 = 1$.

For arbitrary n , $n \geq 1$, and $n = |V| - 2$, then $G(V, E)$ would be constructed such that $V = \{s, t, v_1, v_2, \dots, v_n\}$ and $E = \{(s, v_1), (v_1, t), (s, v_2), (v_2, t), \dots, (s, v_n), (v_n, t)\}$ (i.e. the only edges connected to a vertex are those that have s as their startpoint, or t as their endpoint). This arrangement is chosen because every cut in $G(V, E)$ will be a minimum cut. By the arrangement of $G(V, E)$, and since all capacities are the same. Whether a vertex v is or is not in the set S will not affect the summation of outgoing edge capacities, either the capacity from (v, t) or from (s, v) , respectively, will be used in computing the minimum cut. Therefore we have a maximum number of minimum cuts in G .

Proof of exponential minimum cuts:

From the description above, in our construction of $G(V, E)$, for any $v_i \in V$ the vertex can be contained within or outside the set that describes a cut in G and the cut is still a minimum cut. If we think of each minimum-cut as being described by a binary string for $n = i$, $i \geq 1$, $(q_{v_1}, q_{v_2}, \dots, q_{v_n})$, such that:

$$q_{v_i} = \begin{cases} 1 & \text{if } v_i \in \text{minimum cut set} \\ 0 & \text{if } v_i \notin \text{minimum cut set} \end{cases}$$

Base: Let $n = 1$, and let the B_1 be the set of binary strings describing the minimum cuts in $G(V, E)$. Then B_1 contains two binary strings such that $B_1 = \{0, 1\}$. Indeed, $|B_1| = 2 = 2^n = 2^{|V|-2}$.

IH: Let $n = i$, $i > 1$, and let B_i be the set of binary strings describing the minimum cuts in $G(V, E)$, such that $V = \{s, t, v_1, v_2, \dots, v_i\}$ and $E = \{(s, v_1), (v_1, t), (s, v_2), (v_2, t), \dots, (s, v_i), (v_i, t)\}$. Suppose that $|B_i| = 2^i = 2^{|V|-2}$.

IS: Let $n = i + 1$, it follows from the construction description that $V' = \{s, t, v_1, v_2, \dots, v_i, v_{i+1}\}$ and $E' = \{(s, v_1), (v_1, t), (s, v_2), (v_2, t), \dots, (s, v_i), (v_i, t), (s, v_{i+1}), (v_{i+1}, t)\}$. Clearly, the addition of v_{i+1} extends $G(V, E)$ to $G(V', E')$. The minimum cuts in $G(V', E')$ are composed from the minimum cuts in $G(V, E)$ except that v_{i+1} is either inside or outside these previous cuts. Thus, to compute the set of minimum cuts as binary strings, B_{i+1} , simply take each binary string in B_i and append a 0 to be a minimum cut where v_{i+1} is outside the set, and then again take each binary string in B_i and append a 1 to be a minimum cut where v_{i+1} is inside the set. Thus, $|B_{i+1}| = 2 * (|B_i|) = 2 * 2^i = 2^{i+1}$.

By proof of induction this construction will have exponential minimum cuts between the source and the terminal. ■

Q2. Claim:

Function f , such that $f(S)$ is the number of edges (u, v) with $u \in S$, $v \in V \setminus S$, is submodular.

Proof:

Using the fact provided in the question, all that is necessary to show that f is submodular is to show that for any two subsets $A, B \subseteq V$, $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$.

Consider the set of edges in $G(V, E)$, $E = \{e_1, e_2, \dots, e_n\}$. If we think of $f(S)$ as an iterative algorithm that goes through each $e_i \in E$ and checks whether the startpoint of $e_i \in S$ and whether the endpoint of $e_i \in V \setminus S$. Then we have $f_{e_i}(S)$:

$$f_{e_i}(S) = \begin{cases} 1 & \text{if startpoint of } e_i \text{ in } S \text{ and endpoint of } e_i \text{ in } V \setminus S \\ 0 & \text{otherwise} \end{cases}$$

And so, $f(S) = \sum_{i=1}^n f_{e_i}(S)$. Suppose (u, v) is an arbitrary edge in E , also suppose A, B are some arbitrary subsets of V . We wish to show in all cases, however vertices u and v appear or do not appear in A and/or B , that $f_{(u,v)}(A) + f_{(u,v)}(B) \geq f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$:

Case 1: $u \notin A$ and $u \notin B$: Clearly, $u \notin (A \cup B)$ and $u \notin (A \cap B)$, thus, $f_{(u,v)}(A) = f_{(u,v)}(B) = f_{(u,v)}(A \cup B) = f_{(u,v)}(A \cap B) = 0$. So, $f_{(u,v)}(A) + f_{(u,v)}(B) \geq f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

Case 2: $u \in A$ and $u \notin B$ (W.L.O.G. switching B for A in this case and all subcases the claim holds):

Case 2.1: $v \in A$: Clearly, $u, v \in (A \cup B)$ and $u \notin (A \cap B)$, it follows that $f_{(u,v)}(A) = f_{(u,v)}(B) = f_{(u,v)}(A \cup B) = f_{(u,v)}(A \cap B) = 0$. So, $f_{(u,v)}(A) + f_{(u,v)}(B) \geq f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

Case 2.2: $v \notin A$ and $v \in B$: So, $f_{(u,v)}(A) = 1$ and $f_{(u,v)}(B) = 0$, but since $u, v \in (A \cup B)$ and $u \notin (A \cap B)$, then $f_{(u,v)}(A \cup B) = f_{(u,v)}(A \cap B) = 0$. So, $f_{(u,v)}(A) + f_{(u,v)}(B) \geq f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

Case 2.3: $v \notin A$ and $v \notin B$: Again, $f_{(u,v)}(A) = 1$ and $f_{(u,v)}(B) = 0$. Also, $u \in (A \cup B)$ and $v \notin (A \cup B)$, then $f_{(u,v)}(A \cup B) = 1$. Again, $u \notin (A \cap B)$, then $f_{(u,v)}(A \cap B) = 0$. So, $f_{(u,v)}(A) + f_{(u,v)}(B) \geq f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

Case 3: $u \in A$ and $u \in B$ (W.L.O.G. switching B for A in this case and all subcases the claim holds):

Case 3.1: $v \in A$ and $v \in B$: Since $u, v \in A$ and B , then $f_{(u,v)}(A) = f_{(u,v)}(B) = 0$. Also, $u, v \in (A \cup B)$ and $(A \cap B)$, then $f_{(u,v)}(A \cup B) = f_{(u,v)}(A \cap B) = 0$. So, $f_{(u,v)}(A) + f_{(u,v)}(B) \geq f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

Case 3.2: $v \notin A$ and $v \in B$: It follows that $f_{(u,v)}(A) = 1$ and $f_{(u,v)}(B) = 0$. Since $u, v \in (A \cup B)$ then $f_{(u,v)}(A \cup B) = 0$, but $u \in (A \cap B)$ and $v \notin (A \cap B)$, thus $f_{(u,v)}(A \cap B) = 1$. Again, $f_{(u,v)}(A) + f_{(u,v)}(B) \geq f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

Case 3.3: $v \notin A$ and $v \notin B$: Clearly, $f_{(u,v)}(A) = 1$ and $f_{(u,v)}(B) = 1$. Also, $u \in (A \cup B)$ and $(A \cap B)$, but $v \notin (A \cup B)$ and $(A \cap B)$, then $f_{(u,v)}(A \cup B) = f_{(u,v)}(A \cap B) = 1$. Then, $f_{(u,v)}(A) + f_{(u,v)}(B) \geq f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds.

In all cases in which the vertices of some edge $(u, v) \in E$ might appear in $A, B \subseteq V$ it has been shown that $f_{(u,v)}(A) + f_{(u,v)}(B) \geq f_{(u,v)}(A \cup B) + f_{(u,v)}(A \cap B)$ holds. It must follow then that $\sum_{i=1}^n f_{e_i}(A) + \sum_{i=1}^n f_{e_i}(B) \geq \sum_{i=1}^n f_{e_i}(A \cup B) + \sum_{i=1}^n f_{e_i}(A \cap B) \rightarrow f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$. Clearly, function f is submodular by the equivalence of submodular functions. ■

Q3.

Q4.

Q5.

Q6.

Q7.

Q8.