
CSC373H1 Summer 2014 Assignment 4

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Acknowledgements:

"We declare that we have not used any outside help in completing this assignment."

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Date: August 8, 2014

Q1. The Mute Prison

Claim: The mute prison problem is NP-complete.

Proof:

1. Show the mute prison problem is NP.
2. Show the mute prison problem is NP-hard.

1. Suppose we are given a certificate S and have access to value k and matrix T . We can verify that the certificate is satisfiable in the following way. Suppose each element in S represents an inmate. Verification would involve iterating on each inmate in the following way:

```

for inmate in  $S$  do
     $j = 1$ ;
    while  $j \leq m$  do
        if  $T[inmate, j]$  then
            for ( $otherinmate \neq inmate$ ) in  $S$  do
                if  $T[otherinmate, j]$  then
                     $S$  is not a subset of inmates who do don't speak the same language;
                    return 0;
                end
            end
        end
         $j++$ ;
    end
end
return 1;

```

Clearly, the verification that S is a subset where no two inmates speak the same language can run in polynomial time $O(mn^2)$. Once this verification is complete all that is left to do is to verify that $|S| \geq k$, which is $O(1)$. Therefore the mute prison problem is NP. ■

2. To show that the mute prison problem is NP-hard we must perform a reduction using an NP-complete problem. We will use a reduction on NP-complete 3-SAT in CNF, in order to show $3\text{-SAT} \leq_p \text{Mute Prison Problem}$.

Properties of Reduction

Suppose that ϕ is an instance of 3-SAT and C_1, C_2, \dots, C_m are the clauses of ϕ . By construction of 3-SAT in CNF we have $C_i = (z_{i1} \vee z_{i2} \vee z_{i3})$. In the reduction each C_i 's boolean value will represent a boolean value for each language, L_i , spoken by some inmate(s), precisely, $L_i = C_i = (z_{i1} \vee z_{i2} \vee z_{i3})$. Each boolean value for L_i has a specific mean:

$$L_i = \begin{cases} 1 & \text{if } L_i \text{ is spoken by at most 1 inmate} \\ 0 & \text{if } L_i \text{ is spoken by at least 1 inmate} \end{cases}$$

Producing L_1, L_2, \dots, L_m will take polynomial time since we iterate through each C_i and perform a boolean or operation on each z_i in C_i which takes $O(m)$.

Finally, the mute prison problem requires a matrix T to produce the subset of inmates S . Let T be an $m \times m$

matrix, so that no inmates are left without a language. The rows in T will represent inmates and the columns will represent languages such that column i represents L_i . The algorithm that performs the reduction will iterate through each L_i . If $L_i = 1$ then set $T[i, i] = 1$, else if $L_i = 0$ then set $T[1, i] = T[2, i] = \dots = T[m, i] = 1$. Assigning all inmates to speak L_i , when $L_i = 0$, will guarantee that $|S| = \emptyset$. Alternatively, $\forall i$, if $L_i = 1$ then $|S| = m$. So that if ϕ in 3-SAT is satisfiable, then T will satisfy the mute prison problem if we set $k = m$. Again this process is polynomial as it iterates through m L_i 's and assigns at most m inmates the language L_i , so it will run $O(m^2)$.

ϕ of 3-SAT is satisfiable \rightarrow L and k of mute prison problem is satisfiable

Suppose ϕ of 3-SAT is satisfiable, then each clause C_1, C_2, \dots, C_m is satisfied. A set of L_1, \dots, L_m is produced such that $\forall L_i, L_i = 1$. Then we form matrix T of size $m \times m$, such that T resembles the identity matrix as each $T[i, i] = 1$. Also, $k = m$, so that when S is assembled all m inmates speak a different language, then $|S| \geq k$ is satisfied.

L and k of mute prison problem is satisfiable $\rightarrow \phi$ of 3-SAT is satisfiable

Suppose that T and k of the mute prison problem are satisfiable. Also, suppose $|S| = m = k$. Suppose T is an $m \times m$ matrix that resembles an identity matrix. We will attribute the m columns in T to variables L_1, \dots, L_m , such that, $1 \leq i \leq m$, and set $L_i = 1$ if the column has at most one entry equal to 1, and set $L_i = 0$ otherwise. Since T and k satisfy the problem then all $L_i = 1$. We then form m clauses of a 3-SAT CNF, call them C_1, \dots, C_m . Each C_i relates to L_i , so that the boolean value of $C_i = (z_{i1} \vee z_{i2} \vee z_{i3}) = 1$. Thus set any one of the z_{i1}, z_{i2} , or z_{i3} to 1 (or true) to set C_i to 1. If a column in T has more than one entry with 1 then clearly the mute prison problem would not be satisfied and some $C_i = 0$ (or false) and ϕ would not be satisfied. It follows that all C_i equal 1 since all L_i equal 1, thus $\phi = (C_1 \wedge C_2 \wedge \dots \wedge C_m)$ is satisfiable.

So, ϕ of 3-SAT is satisfiable \Leftrightarrow L and k of mute prison problem is satisfiable. Also, because the reduction was shown to be polynomial it is proven that the mute prison problem is NP-hard. ■

By the proofs 1. and 2. it follows that the mute prison problem is NP-complete. ■

Q2. The Nonsense Prerequisites

Claim: The nonsense prerequisites problem is NP-complete.

Proof:

1. Show the nonsense prerequisites problem is NP.
2. Show the nonsense prerequisites problem is NP-hard.

1. Suppose we know $G(V, E)$ and k and we are given E' as a certificate. We verify the certificate with the following algorithm:

```

 $E'' = E - E'$ ;
Produce function  $w$ , such that  $\forall (u, v) \in E'', w(u, v) = -1$ ;
Produce new  $G'(V, E'', w)$ ;
for  $v$  in  $V$  do
    Perform Bellman-Ford( $G', w, v$ );
    for each edge  $(u, v) \in G'.E''$  do
        if  $v.d > u.d + w(u, v)$  then
            There is a cycle and the certificate is not satisfiable;
            return 0;
        end
    end
end
return 1;

```

If there is a cycle in $G'(V, E'')$ then setting each edge in G' to a weight -1 will produce a negative edge cycle which, after relaxations, we can identify easily. Given that $G(V, E'')$ may or may not be connected, to locate a cycle in the graph we must perform the relaxation with Bellman-Ford $|V|$ times. Bellman-Ford runs at $O(VE)$, it is executed $|V|$ times in the verifier, thus we have $O(V^2E)$ for our algorithm. Since $|V| = n$, and $|E| = O(n^2)$, the verifier runs $O(n^4)$. So the verifier is polynomial and then the nonsense prerequisites problem is NP. ■

2. To show the nonsense prerequisites problem is NP-hard, as directed by the problem set, we will perform a reduction using NP-complete problem VECTOR COVER. So, we will show $\text{VECTOR COVER} \leq_p \text{The Nonsense Prerequisites Problem}$.

Properties of Reduction

Take the $G(V, E)$ and k given to the VECTOR COVER problem. k represents $|S| \leq k$, such that $S \subseteq V$ such that if $(u, v) \in E$, then $u \in S$ or $v \in S$. However, in the nonsense prerequisites, the k corresponds to edges that when removed from the graph will make it acyclic. It follows that the reduction must somehow convert the vertices in G to represent edges. This is done by splitting each vertex in two, so given $V = \{v_1, v_2, \dots, v_n\}$, produce $V' = \{v_{pre-1}, v_{post-1}, v_{pre-2}, v_{post-2}, \dots, v_{pre-n}, v_{post-n}\}$, and $\forall i, 1 \leq i \leq n$, (v_{pre-i}, v_{post-i}) is a directed edge such that $(v_{pre-i}, v_{post-i}) \in E'$. Also, we must create a circumstance in the new graph where each undirected edge $(v_i, v_j) \in E$, becomes directed edges $(v_{post-i}, v_{pre-j}) \in E'$ and $(v_{post-j}, v_{pre-i}) \in E'$. This construction guarantees in $G'(V', E')$ that when we enter any v_{pre-i} we can walk a path $v_{pre-i} \rightarrow v_{post-i} \rightarrow v_{pre-j} \rightarrow v_{post-j} \rightarrow v_{pre-j}$, and indeed this is a cycle. Thus, we have a cycles, such that if $(v_i, v_j) \in E$, then the cycle is limited to the new vertices $\{v_{pre-i}, v_{post-i}, v_{pre-j}, v_{post-j}\}$. So the reduction is complete and can easily be performed in polynomial time. $O(n\alpha(m + n))$ to produce new directed edges from m existing edges, split-

ting vertices in V and creating new edges, and adding them to the new graph G' using make-set, union, and link.

$G(V, E)$, k of VECTOR COVER is satisfiable \rightarrow

$G^*(V^*, E^*)$, k of the nonsense prerequisite is satisfiable

Suppose using undirected $G(V, E)$ and k , VECTOR COVER is satisfied. Suppose also that we have access to $S = \{s_1, \dots, s_q\}$, which is the vertex cover of G and $|S| \leq k$. We perform the reduction and have $G^*(V^*, E^*)$. It follows in G^* any cycles is limited to $\{v_{pre-i}, v_{post-i}, v_{pre-j}, v_{post-j}\}$. To break a cycle in G^* we could remove any edge from the cycle, but to do this efficiently we need to remove edges that break many cycles at once. This is precisely $E' = \{(s_{pre-1}, s_{post-1}), \dots, (s_{pre-q}, s_{post-q})\}$, because in G^* the edges in E' that correspond to vertices in S , are precisely the set of edges that appear in all cycles. Thus $|E'| = |S| \leq k$, and so $G^*(V^*, E^*)$ and k of the nonsense prerequisite is satisfiable.

$G(V, E)$, k of the nonsense prerequisite is satisfiable \rightarrow

$G^*(V^*, E^*)$, k of VECTOR COVER is satisfiable

Suppose $G(V, E)$, k when used in the nonsense prerequisite problem is satisfiable. Now, to establish a contradiction, suppose the original graph, $G^*(V^*, E^*)$ and k in VERTEX COVER were not satisfiable. This would mean that the set of vertex cover $S \subseteq V$, $|S| > k$. But since $G(V, E)$ and k were satisfiable then $|E'| \leq k$. But by the construction of the reduction this is impossible. Since every $(v_{pre-i}, v_{post-i}) \in E'$ corresponds to a vertex $v_i \in S$, this will mean that there is some $v_i \in S$ that is not represented in E' , since $|E'| < |S|$. This means that there is some cycle left over in G when $E - E'$ is performed. So then a contradiction is reached based on our original assumption, and so $G^*(V^*, E^*)$, k of VECTOR COVER must be satisfiable.

By proving both directions, it follows that $G(V, E)$, k of VECTOR COVER is satisfiable \leftrightarrow

$G^*(V^*, E^*)$, k of the nonsense prerequisite is satisfiable. ■

Additionally, since the reduction can be performed in polynomial time, then the nonsense prerequisite problem is NP-hard. ■

Q3. T-rex Christmas

1. $\{0, 1\}$ -integer programming formulation:

Minimize:

P

Subject To:

a)

$$\begin{aligned} x_q^1 &\in \{0, 1\} & \forall q \text{ s.t. } q \text{ represents a path of transferring a present, } i \rightarrow j, \text{ such that } L[i, j] = 1 \\ x_q^0 &\in \{0, 1\} \end{aligned}$$

b)

$$x_q^1 + x_q^0 \geq 1$$

c)

$$\sum_{\forall q} x_q^{f(q, m)} \leq P \quad \forall m \in \{0, 1, \dots, n-1\}$$

Explanation of Constraints:

a) x_q^1 and x_q^0 represent the direction (x_q^1 clockwise and x_q^0 counter-clockwise) a gift q will be passed from i to j , such that $i, j \in \{0, 1, \dots, n-1\}$. If $x_q^1 = 1$ then the gift will be sent clockwise, if $x_q^1 = 0$ it will not be sent clockwise (same for x_q^0 but counter-clockwise).

b) This constraint limits x_q^1 and x_q^0 so both cannot be 0, and it is tightly bound when only one of the variables equals 1.

c) Assume some linear function f , and also assume m represents a pass $(m, m+1[n])$. $f(q, m) = 1$ if on the clockwise direction of passing a gift from i to j a pass in $(m, m+1[n])$ occurs. Conversely, $f(q, m) = 0$ if on the counter-clockwise direction of passing a gift from i to j a pass in $(m, m+1[n])$ occurs. We can assume linearity of f because the computation of the intersection of gift direction and passes will have been computed previously and values will be contained in some 2-D matrix which f has access to at will. Finally, each summation for each m will represent, when the IP completes, each P_m . Clearly if the pass $(m, m+1[n])$ is used during the passing of gift q , then $x_q^{f(q, m)} = 1$, then it is added to P_m . Additionally, the constraint is limited to P , as a tight bound, P is some value that will equal the maximum of all P_m 's.

2. LP relaxation:

Minimize:

P

Subject To:

a)

$$\begin{aligned} 0 &\leq x_q^1 \leq 1 & \forall q \text{ s.t. } q \text{ represents a path of transferring a present, } i \rightarrow j, \text{ such that } L[i, j] = 1 \\ 0 &\leq x_q^0 \leq 1 \end{aligned}$$

b)

$$x_q^1 + x_q^0 \geq 1$$

c)

$$\sum_{\forall q} x_q^{f(q, m)} \leq P \quad \forall m \in \{0, 1, \dots, n-1\}$$

Rounding Scheme:

Let all $x_1^{1*}, x_1^{0*}, x_2^{1*}, x_2^{0*}, \dots, x_s^{1*}, x_s^{0*}$ (s being the number of gifts to be exchanged) and P be the solution returned by the relaxed LP. Allow us to round all x_q^{1*} 's and x_q^{0*} 's to produce $y_1^1, y_1^0, y_2^1, y_2^0, \dots, y_s^1, y_s^0$, and P' . So, let this represent the solution to the integer programming such that:

$$y_q^1 \text{ or } y_q^0 = \begin{cases} 1 & \text{if corresponding } x_q^{1*} \text{ or } x_q^{0*} \geq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$P' = \max_{0 \leq r \leq n-1} (P'_r = \sum_{\forall q} y_q^{f(q, r)})$$

Proof that Rounding Scheme works:

All constraints in the interger program will be satisfied. For example a) is satisfied because y_q^1 or $y_q^0 \in \{0, 1\}$. Constraint b) is satisfied even in the unfortunate case where x_q^{1*} and x_q^{0*} equals $1/2$ or greater, then y_q^1 and y_q^0 will equal 1. Additionally both y_q^1 and y_q^0 cannot both equal 0 by the constraint b). Finally, in constraint c), P is a bound specified by the algorithm, and indeed with the specified values for y_q^1 and y_q^0 the constraint will be satisfied by replacing P with P' . Therefore we conclude that the result of our rounding scheme will be feasible in the integer program.

Proof of 2-approximation:

In both the relaxed linear program and rounding scheme function f will return the same results, such that the result of the relaxed linear program, since P is minimized, P will equal some P_m , for passes at $(m, m+1[n])$. Indeed suppose $P = P_m = \sum_{\forall q} x_q^{f(q, m)*}$. It follows, from our rounding scheme, that the rounding solution must be $P' = P'_r = \sum_{\forall q} y_q^{f(q, r)}$ representing the number of passes at some $(r, r+1[n])$. Then the proof is such that:

$$x_q^{1*} \geq \frac{1}{2} \rightarrow y_q^1 = 1$$

$$x_q^{0*} \geq \frac{1}{2} \rightarrow y_q^0 = 1$$

then,

$$2x_q^{1*} \geq 1 \rightarrow 2x_q^{1*} \geq y_q^1$$

$$2x_q^{0*} \geq 1 \rightarrow 2x_q^{0*} \geq y_q^0$$

finally,

$$P'_r = \sum_{\forall q} y_q^{f(q, r)} \leq 2P_r = 2 \sum_{\forall q} x_q^{f(q, r)}$$

$$P'_r = \sum_{\forall q} y_q^{f(q, r)} \leq 2P_m = 2 \sum_{\forall q} x_q^{f(q, m)} \quad \# \text{ Since } P_r \leq P_m$$

$$P'_r = \sum_{\forall q} y_q^{f(q, r)} \leq 2P \leq 2 \text{LPOPT} \leq 2 \text{IPOPT} \quad \# \text{ Since we established that } P = P_m$$

Therefore, the rounding scheme attains a 2-approximation. ■

Q4. Vertex Cover

1. $\{0, 1\}$ -integer programming formulation:

Minimize:

$$\sum_{u \in V} C_v(u)x_u + \sum_{(u,v) \in E} C_e(u,v)x_{(u,v)}$$

Subject To:

a)

$$\begin{aligned} x_u &\in \{0, 1\} & \forall u \in V \\ x_{(u,v)} &\in \{0, 1\} & \forall (u, v) \in E \end{aligned}$$

b)

$$x_u + x_v + x_{(u,v)} \geq 1 \quad \forall (u, v) \in E$$

Explanation of Constraints:

a)

$x_u \in \{0, 1\}$ represents whether or not vertex u is in our vertex cover S .

If $x_u = 0$ then it is not in our vertex cover, and if $x_u = 1$, then it is in our vertex cover.

The second constraint $x_{(u,v)} \in \{0, 1\}$ represents whether or not the edge (u, v) is covered by the vertices in our vertex cover S . If $x_{(u,v)} = 0$ then either vertex u or v are in the cover S , if $x_{(u,v)} = 1$ then neither u nor v are in S .

b)

$x_u + x_v + x_{(u,v)} \geq 1$ shows that if either u , or v (or both) are in the cover, then we have at least one, x_u or x_v , equal to one. If u and v are not in the cover then x_u and x_v equals zero. So, in this case, $x_{(u,v)}$ must equal 1, and this indicates that the vertices in the edge (u, v) are not in the cover.

2. LP relaxation:

Minimize:

$$\sum_{u \in V} C_v(u)x_u + \sum_{(u,v) \in E} C_e(u,v)x_{(u,v)}$$

Subject To:

a)

$$\begin{aligned} 0 &\leq x_u \leq 1 & \forall u \in V \\ 0 &\leq x_{(u,v)} \leq 1 & \forall (u, v) \in E \end{aligned}$$

b)

$$x_u + x_v + x_{(u,v)} \geq 1 \quad \forall (u, v) \in E$$

Rounding Scheme:

$\forall u \in V$, and $\forall (u, v) \in E$ let all x_u^* and $x_{(u,v)}^*$ be the solution to the relaxed LP. Allow us to round all x_u^* 's and $x_{(u,v)}^*$'s to the interger programming solution, such that all y_u 's and $y_{(u,v)}$'s will be a feasible IP solution:

$$\begin{aligned} y_u &= \begin{cases} 1 & \text{if corresponding } x_u^* \geq \frac{1}{3} \\ 0 & \text{otherwise} \end{cases} \\ y_{(u,v)} &= \begin{cases} 1 & \text{if corresponding } x_{(u,v)}^* \geq \frac{1}{3} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Proof that Rounding Scheme works:

All constraints in the interger program will be satisfied.

Constraint a is satisfied since $y_u \in \{0, 1\}$. Second half of constraint a is satisfied as well since $y_{u,v} \in \{0, 1\}$

Constraint b is satisfied: if $x_u^* \geq 1/3$, then y_u will be 1 and this already satisfies constraint b.

If $x_u^* < 1/3$, then $x_{(u,v)}^*$ will be $\geq 1/3$ and $y_{(u,v)} = 1$ and thus constraint b still holds.

Therefore we conclude that the result of our rounding scheme will be feasible in the integer program.

Proof of 3-approximation:

HENRY TO DO (add intro text if needed)

$$x_u^* \geq \frac{1}{3} \rightarrow y_u = 1$$

$$x_{(u,v)}^* \geq \frac{1}{3} \rightarrow y_{(u,v)} = 1$$

then,

$$3x_u^* \geq 1 \rightarrow 3x_u^* \geq y_u$$

$$3x_{(u,v)}^* \geq 1 \rightarrow 3x_{(u,v)}^* \geq y_{(u,v)}$$

finally,

$$\sum_{u \in V} C_v(u)y_u + \sum_{(u,v) \in E} C_e(u,v)y_{(u,v)} \leq \sum_{u \in V} C_v(u)(3x_u^*) + \sum_{(u,v) \in E} C_e(u,v)(3x_{(u,v)}^*)$$

$$\sum_{u \in V} C_v(u)y_u + \sum_{(u,v) \in E} C_e(u,v)y_{(u,v)} \leq 3(\sum_{u \in V} C_v(u)x_u^*) + 3(\sum_{(u,v) \in E} C_e(u,v)x_{(u,v)}^*)$$

$$\sum_{u \in V} C_v(u)y_u + \sum_{(u,v) \in E} C_e(u,v)y_{(u,v)} \leq 3(\sum_{u \in V} C_v(u)x_u^* + \sum_{(u,v) \in E} C_e(u,v)x_{(u,v)}^*)$$

$$\sum_{u \in V} C_v(u)y_u + \sum_{(u,v) \in E} C_e(u,v)y_{(u,v)} \leq 3^* \text{LPOPT} \leq 3^* \text{IPOPT}$$

Therefore, the rounding scheme attains a 3-approximation. ■