
CSC373H1 Summer 2014 Assignment 4

Names: John Armstrong, Henry Ku

SNs\CDF username: 993114492\g2jarmst, 998551348\g2kuhenr

| Question # | Score |
|------------|-------|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| Total | |

Acknowledgements:

"We declare that we have not used any outside help in completing this assignment."

Name: John Armstrong, Henry Ku

Date: August 2, 2014

Q1. The Mute Prison

Claim: The mute prison problem is NP-complete.

Proof:

1. Show the mute prison problem is NP.
2. Show the mute prison problem is NP-hard.

1. Suppose we are given a certificate S and have access to value k and matrix T . We can verify that the certificate is satisfiable in the following way. Suppose each element in S represents an inmate. Verification would involve iterating on each inmate in the following way:

```

for inmate in S do
    j = 1;
    while j ≤ m do
        if T[inmate, j] then
            for (otherinmate ≠ inmate) in S do
                if T[otherinmate, j] then
                    S is not a subset of inmates who do don't speak the same language;
                end
            end
        end
        j++;
    end
end
end

```

Clearly, the verification that S is a subset where no two inmates speak the same language can run in polynomial time $O(mn^2)$. Once this verification is complete all that is left to do is to verify that $|S| \geq k$, which is $O(1)$. Therefore the mute prison problem is NP. ■

2. To show that the mute prison problem is NP-hard we must perform a reduction using an NP-complete problem. We will use a reduction on NP-complete 3-SAT in CNF, in order to show $3\text{-SAT} \leq_p \text{Mute Prison Problem}$.

Properties of Reduction

Suppose that ϕ is an instance of 3-SAT and C_1, C_2, \dots, C_m are the clauses of ϕ . By construction of 3-SAT in CNF we have $C_i = (z_{i1} \vee z_{i2} \vee z_{i3})$. In the reduction each C_i 's boolean value will represent a boolean value for each language, L_i , spoken by some inmate(s), precisely, $L_i = C_i = (z_{i1} \vee z_{i2} \vee z_{i3})$. Each boolean value for L_i has a specific mean:

$$L_i = \begin{cases} 1 & \text{if } L_i \text{ is spoken by at most 1 inmate} \\ 0 & \text{if } L_i \text{ is spoken by at least 1 inmate} \end{cases}$$

Producing L_1, L_2, \dots, L_m will take polynomial time since we iterate through each C_i and perform a boolean operation on each z_i in C_i which takes $O(m)$.

Finally, the mute prison problem requires a matrix T to produce the subset of inmates S . Let T be an $m \times m$ matrix, so that no inmates are left without a language. The rows in T will represent inmates and the columns will represent languages such that column i represents L_i . The algorithm that performs the reduction will

iterate through each L_i . If $L_i = 1$ then set $T[i, i] = 1$, else if $L_i = 0$ then $T[1, i] = T[2, i] = \dots = T[m, i] = 1$. Assigning all inmates to speak L_i , when $L_i = 0$, will guarantee that $|S| = 0$. Alternatively, $\forall i$, if $L_i = 1$ then $|S| = m$. So that if ϕ satisfies 3-SAT, then T will satisfy the mute prison problem if we set $k = m$. Again this process is polynomial as it iterates through m L_i 's and assigns at most m inmates the language L_i , so it will run $O(m^2)$.

ϕ of 3-SAT is satisfiable \rightarrow L and k of mute prison problem is satisfiable

Suppose ϕ of 3-SAT is satisfiable, then each clause C_1, C_2, \dots, C_m is satisfied. A set of L_1, \dots, L_m is produced such that $\forall L_i, L_i = 1$. Then we form matrix T of size $m \times m$, such that T resembles the identity matrix as each $T[i, i] = 1$. Also, $k = m$, so that when S is assembled all m inmates speak a different language, then $|S| \geq k$ is satisfied.

L and k of mute prison problem is satisfiable $\rightarrow \phi$ of 3-SAT is satisfiable

Suppose that T and k of the mute prison problem are satisfiable. Also, suppose $|S|$ is at least $m=k$. Choose only the first m inmates from S , and extract only their rows from T to form a new matrix T' . It will follow that in T' there will be only m columns where there is at most one entry with the value 1. We will attribute these m columns with variables L_1, \dots, L_m , such that, $1 \leq i \leq m, L_i = 1$. We then form m clauses of a 3-SAT CNF, call them C_i, \dots, C_m . Each C_i relates to L_i , so that the boolean value of $C_i = (z_{i1} \vee z_{i2} \vee z_{i3}) = 1$. Thus set any one of the z_{i1}, z_{i2} , or z_{i3} to 1. It follows that all $C_i = 1$, thus $\phi = (C_1 \wedge C_2 \wedge \dots \wedge C_m)$ is satisfiable.

So, ϕ of 3-SAT is satisfiable \Leftrightarrow L and k of mute prison problem is satisfiable. Also, because the reduction was shown to be polynomial it is proven that the mute prison problem is NP-hard. ■

By the proofs 1. and 2. it follows that the mute prison problem is NP-complete. ■

Q2. The Nonsense Prerequisites

Claim: The nonsense prerequisites problem is NP-complete.

Proof:

1. Show the nonsense prerequisites problem is NP.
2. Show the nonsense prerequisites problem is NP-hard.

1. Suppose we know $G(V, E)$ and k and we are given E' as a certificate. We verify the certificate with the following algorithm:

```

 $E'' = E - E'$ ;
Produce function  $w$ , such that  $\forall (u, v) \in E'', w(u, v) = -1$ ;
Produce new  $G'(V, E'', w)$ ;
for  $v$  in  $V$  do
    Perform Bellman-Ford( $G', w, v$ );
    for each edge  $(u, v) \in G'.E''$  do
        if  $v.d > u.d + w(u, v)$  then
            | There is a cycle and the certificate is not satisfiable.
        end
    end
end

```

If there is a cycle in $G'(V, E'')$ then setting each edge in G' to a weight -1 will produce a negative edge cycle which, after relaxations, we can identify easily. Given that $G(V, E'')$ may or may not be connected, to locate a cycle in the graph we must perform the relaxation with Bellman-Ford $|V|$ times. Bellman-Ford runs at $O(VE)$, it is executed $|V|$ times in the verifier, thus we have $O(V^2E)$ for our algorithm. Since $|V| = n$, and $|E| = O(n^2)$, the verifier runs $O(n^4)$. So the verifier is polynomial and then the nonsense prerequisites problem is NP. ■

2. To show the nonsense prerequisites problem is NP-hard, as directed by the problem set, we will perform a reduction using NP-complete problem VECTOR COVER. So, we will show $VECTOR\ COVER \leq_p The\ Nonsense\ Prerequisites\ Problem$.

Properties of Reduction

Take the $G(V, E)$ and k given to the VECTOR COVER problem. k represents $|S| \leq k$, such that $S \subseteq V$ such that if $(u, v) \in E$, then $u \in S$ or $v \in S$. However, in the nonsense prerequisites, the k corresponds to edges that when removed from the graph will make it acyclic. It follows that the reduction must somehow convert the vertices in G to represent edges. This is done by splitting each vertex in two, so given $V = \{v_1, v_2, \dots, v_n\}$, produce $V' = \{v_{pre-1}, v_{post-1}, v_{pre-2}, v_{post-2}, \dots, v_{pre-n}, v_{post-n}\}$, and (v_{pre-i}, v_{post-i}) is a directed edge such that $(v_{pre-i}, v_{post-i}) \in E'$. Also, we must create a circumstance in the new graph where each undirected edge $(v_i, v_j) \in E$, becomes directed edges $(v_{post-i}, v_{pre-j}) \in E'$ and $(v_{post-j}, v_{pre-i}) \in E'$. This construction guarantees in $G'(V', E')$ that when we enter any v_{pre-i} we can walk a path $v_{pre-i} \rightarrow v_{post-i} \rightarrow v_{pre-j} \rightarrow v_{post-j} \rightarrow v_{pre-j}$, and indeed this is a cycle. Thus, we have a cycles, such that if $(v_i, v_j) \in E$, then the cycle is limited to the new vertices $\{v_{pre-i}, v_{post-i}, v_{pre-j}, v_{post-j}\}$. So the reduction is complete and can easily be performed in polynomial time. $O(n\alpha(m + n))$ to produce new directed edges from m existing edges, splitting vertices in V and creating new edges, and adding them to the new graph G' using make-set, union, and link.

$G(V, E)$, k of VECTOR COVER is satisfiable \rightarrow

$G^*(V^*, E^*)$, k of the nonsense prerequisite is satisfiable

Suppose using undirected $G(V, E)$ and k , VECTOR COVER is satisfied. Suppose also that we have access to $S = \{s_1, \dots, s_q\}$, which is the vertex cover of G and $|S| \leq k$. We perform the reduction and have $G^*(V^*, E^*)$. It follows in G^* any cycles is limited to $\{v_{pre-i}, v_{post-i}, v_{pre-j}, v_{post-j}\}$. To break a cycle in G^* we could remove any edge from the cycle, but to do this efficiently we need to remove an edges that break many cycles at once. This is precisely $E' = \{(s_{pre-1}, s_{post-1}), \dots, (s_{pre-q}, s_{post-q})\}$, because in G^* the edges in E' that correspond to vertices in S , are precisely the set of edges that appear in all cycles. Thus $|E'| = |S| \leq k$, and so $G^*(V^*, E^*)$ and k of the nonsense prerequisite is satisfiable.

$G(V, E)$, k of the nonsense prerequisite is satisfiable \rightarrow

$G^*(V^*, E^*)$, k of VECTOR COVER is satisfiable

Suppose $G(V, E)$, k when used in the nonsense prerequisite problem is satisfiable. Now, conversively suppose the original graph, $G^*(V^*, E^*)$, before reduction, and k in VERTEX COVER were not satisfiable. This would mean that the set of vertex cover $S \subseteq V$, $|S| \geq k$. But since $G(V, E)$ and k were satisfiable then $|E'| \leq k$. But by the construction of the reduction this is impossible. Since every $(v_{pre-i}, v_{post-i}) \in E'$ corresponds to a vertex $v_i \in S$, this will mean that there is some $v_i \in S$ that is not represented in E' , since $|E'| < |S|$. This means that there is some cycle left over in G when $E - E'$ is performed. So then a contradiction is reached, and so $G^*(V^*, E^*)$, k of VECTOR COVER must be satisfiable.

By proving both directions, it follows that $G(V, E)$, k of VECTOR COVER is satisfiable \leftrightarrow

$G^*(V^*, E^*)$, k of the nonsense prerequisite is satisfiable. ■

Additionally, since the reduction can be performed in polynomial time, then the nonsense prerequisite problem is NP-hard. ■

Q3. T-rex Christmas

1. $\{0, 1\}$ -integer programming formulation:

Minimize:

P

Subject To:

a)

$$\begin{aligned} x_q^1 &\in \{0, 1\} & \forall q \text{ s.t. } q \text{ represents a path } i \rightarrow j, \text{ such that } L[i, j] = 1 \\ x_q^0 &\in \{0, 1\} \end{aligned}$$

b)

$$x_q^1 + x_q^0 \geq 1$$

c)

$$\sum_{\forall q} x_q^{f(q, m)} \leq P \quad \forall m \in \{0, 1, \dots, n-1\}$$

Explanation of Constraints:

a) x_q^1 and x_q^0 represent the direction (x_q^1 clockwise and x_q^0 counter-clockwise) a gift q will be passed from i to j , such that $i, j \in \{0, 1, \dots, n-1\}$. If $x_q^1 = 1$ then the gift will be sent clockwise, if $x_q^1 = 0$ it will not be sent clockwise (same for x_q^0 but counter-clockwise).

b) This constraint limits x_q^1 and x_q^0 so both cannot be 0, and it tightly bound when only one of the variables equals 1.

c) Assume some linear function f , and also assume m represents a pass $(m, m+1[n])$. $f(q, m) = 1$ if on the clockwise direction of passing a gift from i to j a pass in $(m, m+1[n])$ occurs. Conversely, $f(q, m) = 0$ if on the counter-clockwise direction of passing a gift from i to j a pass in $(m, m+1[n])$ occurs. We can assume linearity of f as the computation of the intersection of gift direction and passes will have been computed previously and values will be contained in some 2-D matrix which f access at will. Finally, each summation for each m will represent, when the IP completes, each P_m . Clearly if the pass $(m, m+1[n])$ is used then then $x_q^{f(q, m)} = 1$, which will be added to P_m . Additionally, the constraint is limited to P , as a tight bound, P is some value that will equal the maximum of all P_m 's.

2. LP relaxation:

Minimize:

P

Subject To:

a)

$$\begin{aligned} 0 &\leq x_q^1 \leq 1 & \forall q \text{ s.t. } q \text{ represents a path } i \rightarrow j, \text{ such that } L[i, j] = 1 \\ 0 &\leq x_q^0 \leq 1 \end{aligned}$$

b)

$$x_q^1 + x_q^0 \geq 1$$

c)

$$\sum_{\forall q} x_q^{f(q, m)} \leq P \quad \forall m \in \{0, 1, \dots, n-1\}$$

Rounding Scheme:

Let all $x_0^{1*}, x_0^{0*}, x_1^{1*}, x_1^{0*}, \dots, x_s^{1*}, x_s^{0*}$ be the solution returned by the relaxed LP. Allow us to round all x_q^{1*} 's and x_q^{0*} 's to produce $y_0^1, y_0^0, y_1^1, y_1^0, \dots, y_s^1, y_s^0$, and let this represent the solution to the integer programming such that:

$$y_q^1 \text{ or } y_q^0 = \begin{cases} 1 & \text{if corresponding } x_q^{1*} \text{ or } x_q^{0*} \geq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Proof that Rounding Scheme works:

All constraints in the interger program will be satisfied. For example a) is satisfied because y_q^1 or $y_q^0 \in \{0, 1\}$. Constraint b) is satisfied even in the unfortunate case where x_q^{1*} and x_q^{0*} equals $1/2$ or greater, then y_q^1 and y_q^0 will equal 1. Additionally both y_q^1 and y_q^0 cannot both equal 0 by the constraint b). Finally, in constraint c), P is a bound specified by the algorithm, and indeed with the specified values for y_q^1 and y_q^0 the constraint will be satisfied. Therefore we conclude that the result of our rounding scheme will be feasible in the integer program.

Proof of 2-approximation:

In both the relaxed linear program and rounding scheme function f will return the same results, such that the result of the relaxed linear program, since P is minimized, P will equal some P_m . Indeed suppose $P_m = \sum_{\forall q} x_q^{f(q, m)*}$. It follows, from our rounding scheme, that the rounding solution would be $P'_m = \sum_{\forall q} y_q^{f(q, m)}$.

Then the proof is such that:

$$x_q^{1*} \geq \frac{1}{2} \rightarrow y_q^1 = 1$$

$$x_q^{0*} \geq \frac{1}{2} \rightarrow y_q^0 = 1$$

then,

$$2x_q^{1*} \geq 1 \rightarrow 2x_q^{1*} \geq y_q^1$$

$$2x_q^{0*} \geq 1 \rightarrow 2x_q^{0*} \geq y_q^0$$

finally,

$$P'_m = \sum_{\forall q} y_q^{f(q, m)} \leq 2P_m = 2 \sum_{\forall q} x_q^{f(q, m)} \quad \# \text{ Since we established that } P = P_m$$

$$P'_m = \sum_{\forall q} y_q^{f(q, m)} \leq 2P \leq 2 \text{LPOPT} \leq 2 \text{IPOPT}$$

Therefore we have that the rounding scheme attains a 2-approximation. ■

Q4. Vertex Cover