LEVY

JAROD

HW 1

Exercise 1 :

1] The set $\{ x \in \mathbb{R}^N \mid \alpha_i \leq x_i \leq \beta_i,\ i = 1, \dots, N \}$ is the intersection of a finite number of halfspaces. So it's a polyhedon and so a convex set.

2] Let's take $A = (x_1, x_2)$ and $B = (y_1, y_2)$ in the hyperbolic set such as : $x_1 x_2 \geq 1$ and $y_1 y_2 \geq 1$

Let's show that $C = \lambda A + (1-\lambda) B$, $\lambda \in [0,1]$ also in S, the hyperbolic set.

$$C = (\lambda x_1 + (1-\lambda) y_1 , \lambda x_2 + (1-\lambda) y_2)$$

Let's take $C_1 = (\lambda x_1 + (1-\lambda) y_1)(\lambda x_2 + (1-\lambda) y_2)$

$$= \lambda^2 x_1 x_2 + \lambda(1-\lambda) x_1 y_2 + \lambda(1-\lambda) y_1 x_2 + (1-\lambda)^2 y_1 y_2$$

As $x_1 x_2 \geq 1$ and $y_1 y_2 \geq 1$ and λ, $1-\lambda$ both non negative, $\lambda \in (0,1]$

$$C_1 \geq \lambda^2 + 2\lambda(1-\lambda) + (1-\lambda)^2 = \lambda + 1 - \lambda = 1$$

So $C_1 \geq 1$ which means C is also in S. Thus S is convex since any line segment between 2 points in S lies in S.

So $S = \{ x \in \mathbb{R}^2_+ \mid x_1 x_2 \geq 1 \}$ is convex

3] Using a fixed y, we can use the same type of argument than for 1], $\{ x \mid \|x - x_0\|_2 < \|x - y\|_2$ for all $y \in S \}$ becomes an intersection of halfspaces (for each value of y).

So this set is convex.

4] This set is not convex and we will show it using a counterexample. Let's take:

$S = \{ (x,0) \mid x \leq -1 \text{ OR } x \geq 1 \}$

$T = \{ (x,0) \mid -1 < x < 1 \}$ in $\mathbb{R}^2$

We clearly see that this space is not convex, there is a gap in between and so $\{ x \mid \text{dist}(x,S) \leq \text{dist}(x,T) \}$ is <u>not a convex set</u>

5] $\{ x \mid x + S_2 \subseteq S_1 \}$

If $y \in S_2$: $\{ x \mid x + y \in S_1, \ y \in S_2 \}$

So : $\{ x \mid x \in S_1 - y, \ y \in S_2 \}$

As S_1 is convex, $S_1 - y$ is also convex as an intersection of convex set. So $\{ x \mid x + S_2 \subseteq S_1 \}$ is a <u>convex set</u>.

<u>Exercise 2</u> :

1] $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$

By computing the Hessian : $\nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we see that $\lambda_i = \pm 1$ with $i = 1,2$

So f is <u>not convex</u> and <u>not concave</u>

In 1]2] we prove that $\{ (x_1, x_2) \in \mathbb{R}^2_{++} \mid x_1 x_2 \geq 1 \}$ is convex, we can extend that to all $\alpha \in \mathbb{R}$ to have :

$$\{ (x_1, x_2) \in \mathbb{R}^2_{++} \mid x_1 x_2 \geq \alpha \} \text{ convex.}$$

So f is <u>quasiconcave</u> (and <u>not quasiconvex</u>)

2] $f(x_1, x_2) = \frac{1}{x_1 x_2}$ on $\mathbb{R}^2_{++}$

Again, we compute the Hessian Matrix : $\begin{bmatrix} 2/x_1^3 x_2 & -1/x_1^2 x_2^2 \\ -1/x_1^2 x_2^2 & 2/x_1 x_2^3 \end{bmatrix} = \nabla^2 f(x)$

Both 2nd order partial derivatives are positive as the domain is $\mathbb{R}^2_{++}$, the Hessian is positive semidefinite

And so $\underline{f \text{ is convex}}$ (and $\underline{\text{quasiconvex}}$ and $\underline{\text{not concave or quasiconcave}}$)

3] $f(x_1, x_2) = \frac{x_1}{x_2}$ on $\mathbb{R}^2_{++}$

$$\nabla^2 f(x) = \begin{bmatrix} 0 & -1/x_2^2 \\ -1/x_2^2 & \frac{2x_1}{x_2^3} \end{bmatrix}$$

This hessian is not positive or negative semidefinite. So f is $\underline{\text{not convex}}$ and $\underline{\text{not concave}}$

The set $\{(x_1, x_2) \in \mathbb{R}^2_{++} \mid \frac{x_1}{x_2} \leq \alpha\}$ defines halfspaces, that is to say it's linear! So f is $\underline{\text{quasilinear}}$.

4] $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ where $0 \leq \alpha \leq 1$, on $\mathbb{R}^2_{++}$

After computations, we found.

$$\frac{\partial^2 f}{\partial x_1^2} = \alpha(\alpha - 1) x_1^{\alpha-2} x_2^{1-\alpha}$$

$$\frac{\partial^2 f}{\partial x_2^2} = -\alpha(1-\alpha) x_1^\alpha x_2^{-\alpha-1}$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial^2 f}{\partial x_1 \partial x_2} = -\alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha \cdot 1}$$

As $x_1, x_2 > 0$ and $0 \leq \alpha \leq 1$,

$$\nabla^2 f(x) = -\alpha(1-\alpha) x_1^\alpha x_2^{1-\alpha} \begin{bmatrix} 1/x_1^2 & -1/x_1 x_2 \\ -1/x_1 x_2 & 1/x_2^2 \end{bmatrix} \preceq 0$$

And so f is $\underline{\text{concave}}$ (and $\underline{\text{quasiconcave}}$)

Exercise 3 :

2] $f(X,y) = y^T X^{-1} y$ on $S^N_{++} \times \mathbb{R}^N$

We can define: $g(\lambda) = \lambda^T X \lambda$ where $\lambda \in \mathbb{R}^N$. So now:

$$f(X,y) = \sup_{\lambda \in \mathbb{R}^N} g(\lambda)$$

If $g(\lambda)$ convex, f will also be convex by construction (supremum of convex function)

$$H = \nabla^2 g(\lambda) = 2X \succeq 0 \quad \text{because } X \in S^N_{++}$$

And so f is convex

1] Let's use the restriction to a line property:

$$g(t) = f(X + tV), \quad X \succ 0, \quad V \in S^N$$

$$g(t) = \mathrm{tr}((X + tV)^{-1})$$

As we are in S^N_{++}, we know that the eigenvalue decomposition will be of the form $Q \Lambda Q^T$ (here of $X^{-1/2} V X^{-1/2}$)

$$g(t) = \mathrm{tr}(X^{-1}(I + tX^{-1/2} V X^{-1/2})^{-1}) \quad \text{to show the decomposition}$$

$$= \mathrm{tr}(Q^T X^{-1} Q (I + t\Lambda)^{-1})$$

$$= \sum_{i=1}^{N} Q^T X^{-1} Q (1 + t\lambda_i)^{-1}$$

This is a sum of convex function and so it's convex!

3] We know that $\sigma_i(X) = \sup_{\substack{\|u\|_2 = 1 \\ \|v\|_2 = 1}} u^T X v$, u, v unit vectors

$$f(X) = \sum_{i=1}^{N} \sigma_i(X) = \sum_{i=1}^{N} \sup_{\substack{\|u\|_2 = 1 \\ \|v\|_2 = 1}} u_i^T X v_i = \sup_{\substack{\|u_1\|_2 \\ \|v_1\|_2}} u_1^T X v_1 + \sum_{i=2}^{N} \sup_{\substack{\|u_i\|_2 \\ \|v_i\|_2}} u_i^T X v_i$$

↳ Each term in the sum is a supremum of linear functions and supremum of linear function are convex.

So f is convex

Exercise 4 :

1] • K_{M+} is not empty, for example $(N, N-1, \dots, 1)$ is in K_{M+} → has non empty interior

• K_{M+} is closed because it's a cone defined with homogenous inequalities (a finite number)

• K_{M+} is pointed because

And so K_{M+} is a proper cone.

2] $K^*_{M+} = \{ y \in \mathbb{R}^N, \ y^T x \geq 0 \text{ for all } x \in K$

To characterize y, we can consider the components one by one :

→ For y_1, it should satisfy $y^T x \geq 0$ for all x with $x_1 \geq \dots \geq x_N$

y_1 must be nonnegative to hold this inequality.

Understanding this pattern, we understand that K^*_{M+} is simply :

$$K^*_{M+} = \{ y \in \mathbb{R}^N, \ y_1 \geq 0, y_2 \geq 0 \dots, y_N \geq 0 \}$$