Survival and transport in glassy sparse systems

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1. Background

There are diverse examples for extended systems whose dynamics is described by a rate equation. This include in particular the analysis of "random walk" where the transitions are between sites of a network. Another example is the study of energy absorption due to Fermi-Golden-Rule transitions between energy levels. In both cases there are two common questions that arise: (1) What is the survival probability of a particle that has been prepared in a given site; (2) Is there normal diffusion or maybe only sub-diffusion, and how it is related to the survival probability.

In recent years there is a growing interest in systems where the transition rates have log-wide distribution. This means that the values of the rates are distributed over many decades as in the case of log-normal or log-box distributions. Such "glassy" or "sparse" systems can be regarded as a "resistor network", and the analysis might be inspired by percolation theory, variable range hopping phenomenology, and the renormalization-group methods.

While the theory of 1D networks with near-neighbour transitions is quite complete, the more general case of quasi-1D / 2D and possibly higher dimensions lacks a unifying framework, and there are numerous open questions that we would like to study as outlined below.

Modeling .– N interconnected sites constitue a network. A single particle is bound to the network. We denote by $p_n(t)$ the probability to find the particle on site n at time t, so that $\sum_n p_n(t) = 1$. The dynamics of the system are described by the rate equation:

$$\frac{dp_n(t)}{dt} = \sum_m W_{nm} p_m(t) \tag{1}$$

Where W_{nm} is the *transition rate*, i.e. the rate at which probability moves from site m to site n. Because of probability conservation, we want to have $\sum_m W_{nm} = 0 \ \forall n$, which we can achieve by setting $W_{nn} = -\sum_{m\neq n} W_{nm}$, meaning that for each site the sum of incoming transition rates negates the outgoing transitions. The rate equation can also be written as a vectorial equation:$\dot{\boldsymbol{p}} = \boldsymbol{W}\boldsymbol{p}$. In its basic form, the matrix $\boldsymbol{W}$ is symmetric (at the moment, see "Assymetric VRH" further down in this section), except for the main diagonal, which has values that ensure that each row's sum is zero.

The network (the values of W_{nm}) can be defined arbitrarily, but we wish to focus on networks that represent geometric systems, by defining the transition rates to depend on the distance betweeen randomly scattered points[4]. One such network, with the rates defined as:

$$W_{nm} = w_0 e^{(r_0 - r_{nm})/\xi} \tag{2}$$

Where r_{nm} is the distance between site n and m, r_0 is the typical distance between points, w_0 is the transition rate between points at distance r_0, and ξ is a scaling coefficient. In general, we are going to define $r_0 = n^{-1/d}$, where $n = \frac{N}{V}$ is the site denisty, and we are going to rescale time by setting $w_0 = 1$. This model was studied in [5], and is of particular interest for us.

Dimensions.– The system may be in $1d$, $2d$, $3d$ etc., or in quasi-$1d$. The $1d$ system has been studied, among others, in[6][7][8]. Quasi-$1d$ relates either to $1d$ systems where there are bonds between sites beyond the nearest neighbors, or to $2d$ systems with finite width (strip). Both these systems have banded matrices, with bandwidth b.

Survival probability - $\mathcal{P}(t)$.– The survival probability is the probability to remain in the starting site. If the initial condition was $p_0(0) = 1$, $p_i(0) = 0$ for $i \neq 0$, then $\mathcal{P}(t) = p_0(t)$. The survival probability is directly related to the spectral properties of the transition matrices, and it can be shown that

$$\mathcal{P}(t) \;=\; \frac{1}{N}\sum_{\lambda} e^{\lambda t} \;\rightarrow\; \frac{1}{N}\int e^{\lambda t} g(\lambda) d\lambda \tag{3}$$

where the λs are the eigenvalues of the matrix, namely that the survival probability is the Laplace transform of the eigenvalue density.

Transport and Spreading.– A particle can be transmitted through the system from one end to the other. This transport can be characterized in different ways. One way is to calculate the spreading $S(t)$, which is the variance (second moment) of the particle location, i.e:

$$S(t) \;=\; \sum_{n} \left(r_n(t)\right)^2 p_n \tag{4}$$

Where r_n is the location of the nth site. The survival probability is related to the diffusion because of scaling considerations by:

$$\mathcal{P}(t) \;=\; \left(2\pi\frac{S(t)}{r_0^2}\right)^{-d/2} \tag{5}$$

By definition, diffusive systems obey:

$$S(t) \;=\; 2Dt \tag{6}$$

$$\mathcal{P}(t) \;=\; \left(2\pi\frac{S(t)}{r_0^2}\right)^{-d/2} \;=\; \left(4\pi\frac{Dt}{r_0^2}\right)^{-d/2} \tag{7}$$

We can combine this result with Equation 3 to obtain a relation between $g(\lambda)$ and D:

$$g(\lambda) \;=\; \mathcal{L}^{-1}[\mathcal{P}(t)] \;=\; \mathcal{L}^{-1}\left[\left(4\pi\frac{Dt}{r_0^2}\right)^{-d/2}\right] \;=\; \frac{\lambda^{\frac{d}{2}-1}}{\Gamma\left(\frac{d}{2}\right)\left(4\pi\frac{D}{r_0^2}\right)^{d/2}} \tag{8}$$

$$C(\lambda) \;=\; \int_{\infty}^{\lambda} g(\lambda')d\lambda' \;=\; \frac{\lambda^{\frac{d}{2}}}{\frac{d}{2}\Gamma\left(\frac{d}{2}\right)\left(4\pi\frac{D}{r_0^2}\right)^{d/2}} \tag{9}$$

Quantum spreading and transport.– is another subject we wish to tackle, following the lead of [1][2][3].

2. Work plan

Geometrical implications.– The geometric properties of the system are reflected in the statistics of the distances, and by extension in the statistics of the transition rates. However, there is more information in the W_{nm} matrix than just its statistics. The question arises: Are the statistics all that is needed to understand the physics of the system? See the preliminary results for farther discussion III.

Banded sparse matrices.– The conductance of quasi-1d banded sparse matrices was studied numerically in [9]. There, they use *Variable-Range-Hopping* in the sparse regime, where (sparsity $\cdot$ bandwidth) $\ll 1$. However, in this work it is not clear what are the limits on either sparsity or bandwidth, and in particular where is the cross between the validity regime of VRH and that of SLRT. We will try to understand this issue analytically.

Diffusion or Subdiffusion in 2d.– The question of diffusion in most 1d systems was analytically solved in [7]. The 2d case is, as for as we know, not yet analytically solved, and is much less clear. In [5] it is claimed that in low density systems subdiffusion of order $\sim log^d$ should occur. We wish to use the block-renormalization-group method to find out if this is indeed the case. We also wish to see if there is a transition between low densities and high densities.

Assymetric VRH.– If the sites have different potentials, then the site occupation probabilites change. This can be acommodated for by modifying the transition rates by Boltzmann's factor. This is called *VRH- Variable Range Hopping*[10], and has been widely studied. The common practice is to treat the system as a symmetric resistor network, but we want to ask if there are cases where this reduction to a symmetrical network is not valid.

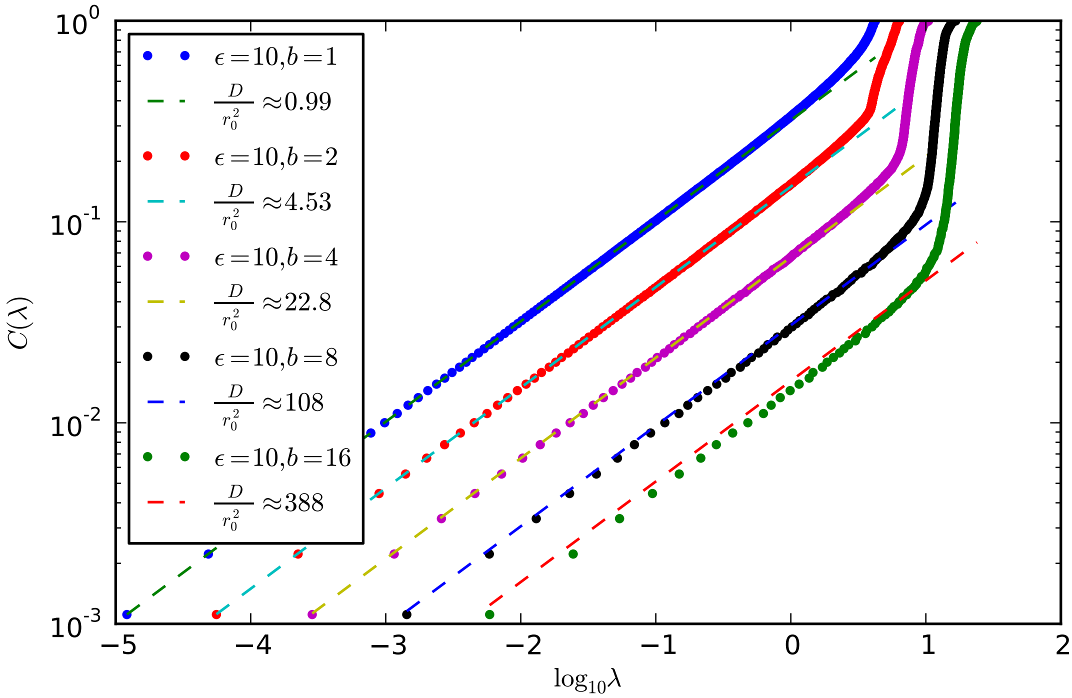

FIG. 1: Quasi-1d. No fitting parameters were used. $N = 800$ points were randomly scattered on a $1d$ line. The transition rates between them are according to Equation 10, with $\epsilon = \frac{\xi}{r_0} = 10$. Each system was truncated at a different bandwidth, as stated in the legend. The dashed line marks Equation 9, with D calculated using the resistor network method Equation 19.

3. Preliminary results

The exponential dependence model in various dimensions.– In this model we define the rates as

$$W_{nm} = w_0 e^{(r_0 - r_{nm})/\xi} \tag{10}$$

Where r_0 is the typical distance between points, w_0 is the transition rate between points at distance r_0, and ξ is a scaling coefficient. It can be shown that the distances of randomly scattered points have a distribution described by

$$f(r)dr = e^{-\frac{\Omega_d}{2d}\left(\frac{r}{r_0}\right)^d} \frac{\Omega_d}{2} \left(\frac{r}{r_0}\right)^{d-1} \frac{dr}{r_0} \tag{11}$$

where n is the density, d the dimension, and Ω_d is the d dimensional solid angle. The distribution of the rates can be found by replacing the random variable ($|f_w(w)dw| = |f_r(r)dr|$), and we find it to be

$$f_w(w)dw = \frac{\Omega_d}{2} \left(\frac{\xi}{r_0}\right)^d \left|\ln \frac{w}{w_0}\right|^{d-1} \exp\left[-\left|\frac{\Omega_d}{2d} \frac{\xi}{r_0} \ln \frac{w}{w_0}\right|^d\right] \frac{dw}{w} \tag{12}$$

In $1d$ resistor networks, the average resistance of a system is the mean of the individual resistances, and the average conductance is the harmonic mean of the individual conductances. Following the $1d$ solution presented in [7], if the harmonic mean is finite, then the system will be diffusive with diffusion coefficient equal to the inverse of the harmonic mean:

$$D = \left(\overline{(w^{-1})}\right)^{-1} r_0^2 = \left[\int f(w) \frac{dw}{w}\right]^{-1} r_0^2 \tag{13}$$

If $D > 0$ then there is diffusion, else sub diffusion is implied. It is a known property of harmonic means, that because of their inverse dependence on values, the values closer to zero contribute most of the sum. Therefore, we are interested in the distribution of transition rates in the limit $w \to 0$. The $1d$ case of Equation 12 is:

$$f(w)dw = \frac{\xi}{r_0}\left(\frac{w}{w_0}\right)^{\frac{\xi}{r_0}}\frac{dw}{w} \tag{14}$$

$$D = \left[\int_0^{\infty}\frac{\xi}{r_0}\left(\frac{w}{w_0}\right)^{\frac{\xi}{r_0}}\frac{dw}{w^2}\right]^{-1} r_0^2 = \begin{cases}\frac{\frac{\xi}{r_0}-1}{\frac{\xi}{r_0}} w_0^{\frac{\xi}{r_0}} r_0^2 & \frac{\xi}{r_0} > 1 \\ 0 & \frac{2\xi}{r_0} < 1\end{cases} \tag{15}$$

Resistor networks.– There is an analogy between the diffusion problem we have laid out, to electrical conductance. According to Fick's first law of diffusion, the mass flux is proportional to the gradient of the density:

$$J = -D\nabla\phi = -D\cdot\frac{\rho_2-\rho_1}{L} \tag{16}$$

On the other hand, electrical conductance relates current density to potential difference:

$$J = \sigma E = -\sigma\frac{\rho_2-\rho_1}{L} \tag{17}$$

Therefore, we can treat the system as a network of resistors with the known rules of electrodynamics. We can create current through our system by applying incoming current (+1 in suitable units) in one site, outgoing in another (-1). Then, we can compute the site occupation probability difference between them. In other words, we try to solve

$$G = \frac{I}{V} \tag{18}$$

For $1d$,

$$\sigma = \frac{G}{r} \tag{19}$$

For $2d$, instead of solving the problem with both current connections, we can think of each of them seperately, and then superpose them. If a current goes from a single spot to infinity, the current should have radial symmetry. If so, the voltage at distance r will be :

$$E = \frac{J}{\sigma} = \frac{I}{2\pi r\sigma} \tag{20}$$

$$V = \int E dr = \frac{I}{2\pi\sigma}\ln\frac{r}{r_0} \tag{21}$$

The superposition of an outgoing current just doubles the result, and we obtain:

$$G = \frac{I}{V} = \frac{\pi\sigma}{\ln\frac{r}{r_0}} \tag{22}$$

$$\sigma = \frac{1}{\pi}G\ln\frac{r}{r_0} \tag{23}$$

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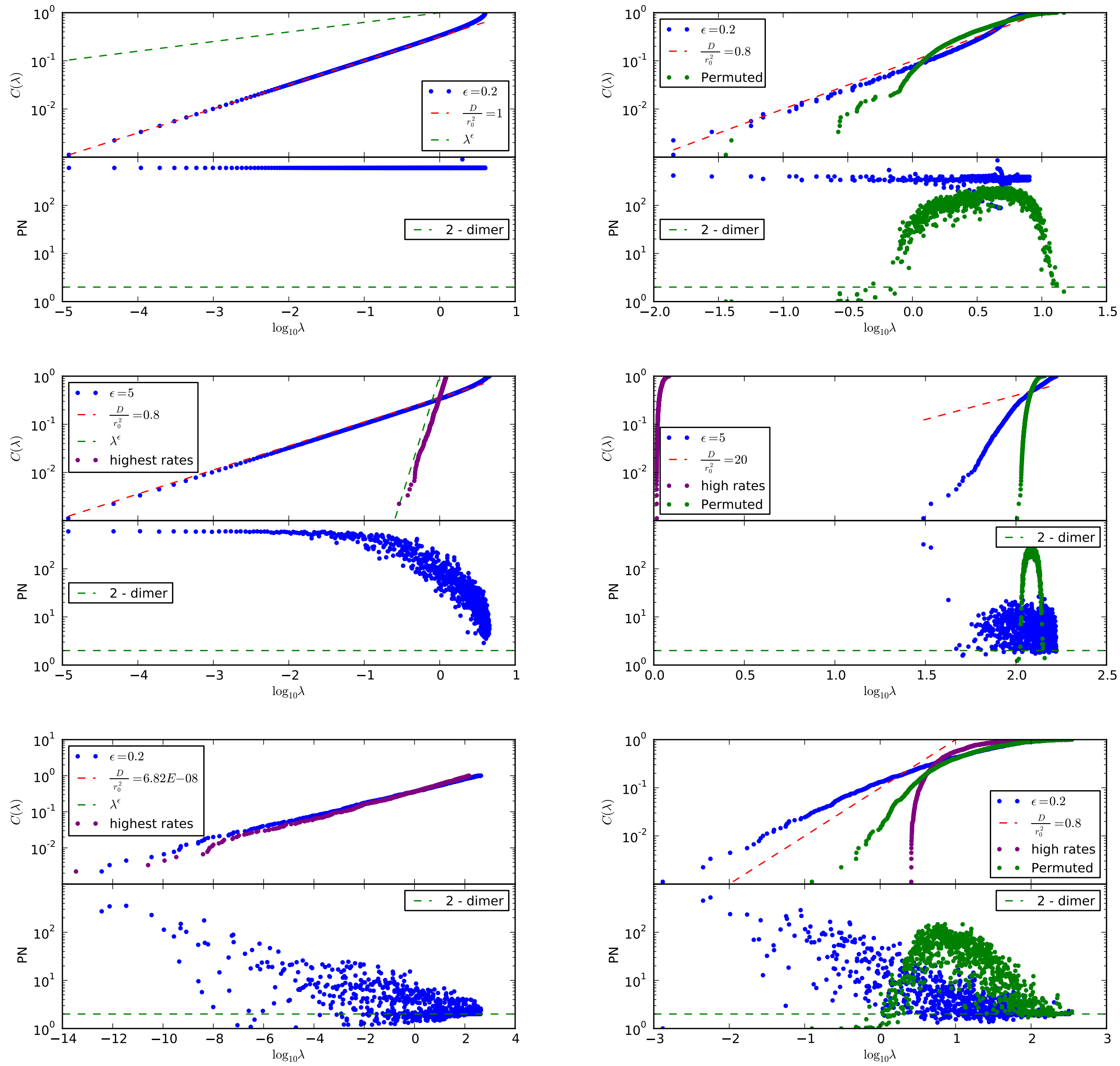

FIG. 2: A matrix W_{nm}, as defined in Equation 10, has been generated and diagonalized. Each panel describes the results for a single realization, with the indicated value of ξ. The left panels are for a 1D system, and the right panels are for a 2D system. The upper panels are for a non-random rectangular network, while the middle and lower panels are for a random network, with large and small values of ξ respectively. Each panel contains two sub-panels: the upper one is the cumulative histogram of the eignevalues, and the lower one is the participation number of each eigenstate. For sake of comparison we have added lines that describe the "raw" distribution of the rates , and the expectation that is based on a diffusion hypothesis Equation 9 Equation 14, and the result for a randomly permuted version of the matrix. See text for further discussion.