abstract

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Part I

Research Proposal

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Introduction

Chapter 2

Objectives

2.1 Assymetric VRH

If the sites have different potentials, then the site occupation probabilites change. This can be acommodated for by modifying the transition rates by Boltzmann's factor. This is called *VRH- Variable Range Hopping*[**?**], and has been widely studied. The common practice is to treat the system as a symmetric resistor network, but we want to ask if there are cases where this reduction to a symmetrical network is not valid.

2.2 Stotland's errata model

2.3 Geometrical implications

The geometric properties of the system are reflected in the statistics of the distances, and by extension in the statistics of the transition rates. However, there is more information in the W_{nm} matrix than just its statistics. The question arises: Are the statistics all that is needed to understand the physics of the system? See the preliminary results for farther discussion **??**.

2.4 Banded sparse matrices

The conductance of quasi-1d banded sparse matrices was studied numerically in [**?**]. There, they use *Variable-Range-Hopping* in the sparse regime, where (sparsity $\cdot$ bandwidth) $\ll 1$. However, in this work it is not clear what are the limits on either sparsity or bandwidth, and in particular where is the cross between the validity regime of VRH and that of SLRT. We will try to understand this issue analytically.

2.5 Diffusion or Subdiffusion in $2d$

The question of diffusion in most $1d$ systems was analytically solved in [**?**]. The $2d$ case is, as for as we know, not yet analytically solved, and is much less clear. In [**?**] it is claimed that in low density systems subdiffusion of order $\sim log^d$ should occur. We wish to use the block-renormalization-group method to find out if this is indeed the case. We also wish to see if there is a transition between low densities and high densities.

2.6 The spectral properties of Sinai diffusion in $1d$

In a symetric random rate system, there is regular diffusion if the harmonic average is bounded, i.e. if

$$\sum_n \frac{1}{w_n} < \infty$$

On the other hand, for an asymetric system, an activation energy builds up, and a Sinai diffusion behaviour is followed. The conductance is exponentially small in the length, and the spreading is anomalic. The differnce between those two models is clearly highly significant. We wish to investigate the difference between the models, and the relation of this difference to the spectral properties of the model.

2.7 Sinai diffusion in $2d$

The spectral properties, and spreading behaviour of a $2d$ asymetric system.

2.8 The charge carrier discreteness

Our work so far considered probability rate equations, which are valid for a single particle in the network. If we have multiple interacting particles in the network, additional factors may contribute. For example, a single occupancy rule (i.e. each site may have up to one particle) may cause Fermi blocking.

2.9 Banded Random Matrices

Chapter 3

Background

3.1 Transition matrices

Our system is a d-dimensional network, and it is described by its nodes and edges. The nodes represent either *sites* in real space or *levels* in energy space, and our particle (or particles, see 2.8) may occupy them. The edges of the network represent transition rates between the nodes. The dynamics of the model are described by a rate equation:

$$\frac{dp_n}{dt} = \sum_m w_{nm} p_m$$

Where the transition matrix

$$\mathbf{W} = [w_{nm}]_{N \times N}$$

Is the crux of this work.

Because of conservation of probability, we set the diagonal elements to

$$w_{nn} = -\sum_{m(\neq n)} w_{mn}$$

3.2 Definition of the model

The proposal regards a spatial network where each node has a defined location x_n. The transition rates w_{nm} are constructed by the expression

$$w_{nm} = w_0 \mathrm{e}^{-\epsilon_{nm}} B(x_n - x_m) \tag{3.1}$$

where $B(r)$ describes the dependence of the coupling on the distance between sites, and ϵ represent activation energies required for a transition.

The $B(r)$ matrix is a function a euclidean distance matrix, a class a matrices that is widely researched ..... TODO

3.2.1 The random site hopping model

For presentation purpose we regard the nodes of the network as *sites*, each having a location x_n. In particular (but not exclusively) we are interested in a model where the rates depend exponentially on the distance between randomly distributed sites, namely $w_{nm} \propto \exp(|x_n - x_m|/\xi)$. One can characterize such a system by a sparsity parameter s that reflects the connectivity of the network. For a random site model the natural definition is $s = \xi/r_0$, where r_0 is the average distance between neighboring sites.

The models that we address are related and motivated by various physical problems, for example: phonon propagation in disordered solids [**?** **?** **?**]; Mott hopping conductance [**?** **?** **?** **?** **?** **?**]; transport in oil reservoirs [**?** **?**]; conductance of ballistic rings [**?**]; and energy absorption by trapped atoms [**?**]. Optionally these models can be fabricated by combining oscillators: say mechanical springs or electrical RC elements.

3.3 The study of networks

In all these examples the issue is to understand how the *transport* is affected by the *sparsity* of a network. If the rates are induced by a driving source, this issue can be phrased as going *beyond* the familiar framework of Linear Response Theory (LRT), as explained below.

3.4 Diffusion

Our interest is focused on the diffusion coefficient D that characterizes the long time dynamics of a spreading distribution. It can be defined or deduced either from the variance $S(t) \sim Dt$ or from the decay of the survival probability $\mathcal{P}(t) \sim (Dt)^{-d/2}$. Hence it is related to the spectral properties of the transition rate matrix

$$\boldsymbol{w} \;=\; \{w_{nm}\} \tag{3.2}$$

Exploiting the formal analogy with a resistor network calculation [**?**], namely w_{nm} are like connectors and D is like conductivity, one realizes that D is given by a semi-linear functional $D[\boldsymbol{w}]$ that has the property $D[\lambda \boldsymbol{w}] = \lambda D[\boldsymbol{w}]$, while in general $D[\boldsymbol{w}^a + \boldsymbol{w}^b] > D[\boldsymbol{w}^a] + D[\boldsymbol{w}^b]$ instead of equality.

3.5 Subdiffusion

In the 1D (d=1) case, it is well known [**?**] that D can display an abrupt percolation-like transition from diffusive ($D > 0$) to sub-diffusive ($D = 0$) behavior, as the sparsity parameter drops below the critical value $s_c = 1$. Similar anomalies are found for fractal structures with $d < 2$, see [**?** **?**]. A question arises whether such a transition might happen in higher dimensions. In [**?**] the spectral properties in the 2D (d=2) case have been investigated: on the basis of the renormalization group (RG) procedure it has been deduced that $\mathcal{P}(t)$ decays in a logarithmic way, indicating anomalous (sub) diffusion.

3.6 Variable range hopping

It should be clear that there are two major routes in developing a theory for D. Instead of deducing it from spectral properties as in [**?**], one can try to find ways to evaluate it directly via a resistor network calculation [**?** **?** **?** **?** **?**], leading in the standard Mott problem to the Variable Range Hopping (VRH) estimate for D. In [**?** **?** **?**] this approach has been extended to handle "sparse" banded matrices whose elements have log-wide distribution, leading to a generalized VRH estimate. In what follows we pursue the same direction and obtain an improved estimate for D that we call Effective Range Hopping (ERH). Using this approach we show that in the 2D case, as s becomes small, the functional $D[\boldsymbol{w}]$ exhibits a smooth crossover from "linear" behavior to "semi-linear" VRH-type dependence.

3.7 Sparsity vs percolation

The problem that we consider is a variant of the percolation problem [**?** **?**]: Instead of considering a bi-modal distribution ("zeros" and "ones") we consider a log-wide distribution

of rates [1], for which the median is much smaller than the mean value. We call such network "sparse" (with quotation marks) because the large elements constitute a minority.

3.8 Anderson localization

Disregarding the "sparsity" issue, the model that we are considering is a close relative of the Anderson localization problem. In the problem that we discuss here all the off diagonal elements are positive, while the negative diagonal elements compensate them. Since all the off diagonal elements are positive numbers, it is clear that we cannot have "destructive interference", and therefore we do not have genuine Anderson localization. Therefore in general we might have diffusion, even in 1D. In 2D we have a percolation threshold, which is again not like Anderson localization. For further discussion see for example the discussion of fractons in [2].

3.9 Debye law

In the standard Anderson model the eigenvalues form a band $\lambda \in [-\lambda_c, \lambda_c]$. The states at the edge of the band are always localized. The states in the middle of the band might be de-localized if $d > 2$. The localization in a disordered elastic medium had been studied [**?**]. Disregarding the "sparsity", it is the same problem that we are considering here. It has been found that the spectrum is $\lambda \in [0, \lambda_c]$. The ground state is always the $\lambda = 0$ uniform state. The localization length diverges in the limit $\lambda \to 0$. Consequently the Debye density of states is not violated: the spectrum is asymptotically the same as that of a diffusive (non-disordered) lattice. It follows that the survival probability should be like that of diffusive system, and therefore we also expect diffusive behavior for the transport: spreading that obeys a diffusion equation.

3.10 Analogy to an electrical circuit

We may think of a "ladder" of capacitors and resistors as portrayed in the diagram.

3.11 Analogy to a masses and springs mesh

The same network may describe equal masses (which we can simply define as 1) distributed in space, connected by springs. The locations of the masses will obey a second degree differential equation:

correct this! Where k_{nm} is the spring

3.12 msc

There are diverse examples for extended systems whose dynamics is described by a rate equation. This include in particular the analysis of "random walk" where the transitions are between sites of a network. Another example is the study of energy absorption due to Fermi-Golden-Rule transitions between energy levels. In both cases there are two common questions that arise: (1) What is the survival probability of a particle that has been prepared in a given site; (2) Is there normal diffusion or maybe only sub-diffusion, and how it is related to the survival probability.

In recent years there is a growing interest in systems where the transition rates have log-wide distribution. This means that the values of the rates are distributed over many decades as in the case of log-normal or log-box distributions. Such "glassy" or "sparse" systems can be regarded as a "resistor network", and the analysis might be inspired by percolation theory, variable range hopping phenomenology, and the renormalization-group methods.

While the theory of 1D networks with near-neighbour transitions is quite complete, the more general case of quasi-1D / 2D and possibly higher dimensions lacks a unifying framework, and there are numerous open questions that we would like to study as outlined below.

Modeling .– N interconnected sites constitue a network. A single particle is bound to the network. We denote by $p_n(t)$ the probability to find the particle on site n at time t, so that $\sum_n p_n(t) = 1$. The dynamics of the system are described by the rate equation:

$$\frac{dp_n(t)}{dt} = \sum_m W_{nm} p_m(t) \tag{3.3}$$

Where W_{nm} is the *transition rate*, i.e. the rate at which probability moves from site m to site n. Because of probability conservation, we want to have $\sum_m W_{nm} = 0 \ \ \forall n$, which we can achieve by setting $W_{nn} = -\sum_{m \neq n} W_{nm}$, meaning that for each site the sum of incoming

transition rates negates the outgoing transitions. The rate equation can also be written as a vectorial equation:$\dot{\boldsymbol{p}} = \boldsymbol{W}\boldsymbol{p}$. In its basic form, the matrix $\boldsymbol{W}$ is symmetric (at the moment, see "Assymetric VRH" further down in this section), except for the main diagonal, which has values that ensure that each row's sum is zero.

The network (the values of W_{nm}) can be defined arbitrarily, but we wish to focus on networks that represent geometric systems, by defining the transition rates to depend on the distance betweeen randomly scattered points[**?**]. One such network, with the rates defined as:

$$W_{nm} = w_0 e^{(r_0 - r_{nm})/\xi} \tag{3.4}$$

Where r_{nm} is the distance between site n and m, r_0 is the typical distance between points, w_0 is the transition rate between points at distance r_0, and ξ is a scaling coefficient. In general, we are going to define $r_0 = n^{-1/d}$, where $n = \frac{N}{V}$ is the site denisty, and we are going to rescale time by setting $w_0 = 1$. This model was studied in [**?**], and is of particular interest for us.

Dimensions.– The system may be in $1d$, $2d$, $3d$ etc., or in quasi-$1d$. The $1d$ system has been studied, among others, in[**?**][**?**][**?**]. Quasi-$1d$ relates either to $1d$ systems where there are bonds between sites beyond the nearest neighbors, or to $2d$ systems with finite width (strip). Both these systems have banded matrices, with bandwidth b.

Survival probability - $\mathcal{P}(t)$.– The survival probability is the probability to remain in the starting site. If the initial condition was $p_0(0) = 1$, $p_i(0) = 0$ for $i \neq 0$, then $\mathcal{P}(t) = p_0(t)$. The survival probability is directly related to the spectral properties of the transition matrices, and it can be shown that

$$\mathcal{P}(t) = \frac{1}{N}\sum_{\lambda} e^{\lambda t} \rightarrow \frac{1}{N}\int e^{\lambda t} g(\lambda) d\lambda \tag{3.5}$$

where the λs are the eigenvalues of the matrix, namely that the survival probability is the Laplace transform of the eigenvalue density.

Transport and Spreading.– A particle can be transmitted through the system from one end to the other. This transport can be characterized in different ways. One way is to calculate the spreading $S(t)$, which is the variance (second moment) of the particle location, i.e:

$$S(t) = \sum_{n} \left(r_n(t)\right)^2 p_n \tag{3.6}$$

Where r_n is the location of the nth site. The survival probability is related to the diffusion because of scaling considerations by:

$$\mathcal{P}(t) = \left(2\pi\frac{S(t)}{r_0^2}\right)^{-d/2} \tag{3.7}$$

By definition, diffusive systems obey:

$$S(t) = 2Dt \tag{3.8}$$

$$\mathcal{P}(t) = \left(2\pi\frac{S(t)}{r_0^2}\right)^{-d/2} = \left(4\pi\frac{Dt}{r_0^2}\right)^{-d/2} \tag{3.9}$$

We can combine this result with 3.5 to obtain a relation between $g(\lambda)$ and D:

$$g(\lambda) = \mathcal{L}^{-1}[\mathcal{P}(t)] = \mathcal{L}^{-1}\left[\left(4\pi\frac{Dt}{r_0^2}\right)^{-d/2}\right] = \frac{\lambda^{\frac{d}{2}-1}}{\Gamma\left(\frac{d}{2}\right)\left(4\pi\frac{D}{r_0^2}\right)^{d/2}} \tag{3.10}$$

$$C(\lambda) = \int_{\infty}^{\lambda} g(\lambda')d\lambda' = \frac{\lambda^{\frac{d}{2}}}{\frac{d}{2}\Gamma\left(\frac{d}{2}\right)\left(4\pi\frac{D}{r_0^2}\right)^{d/2}} \tag{3.11}$$

3.13 Spreading of wavepackets in the quantum case

3.14 Localization

Part II

Completed Work

Chapter 4

Completed Work

4.1 The $d = 1$ random site model with nearest neighbor hopping

In $d = 1$, a network with randomly distributed sites is equivalent to an ordered lattice with random transition rates. To make this mapping we have to know the spacing distribution for this system. We find the spacing distribution in **??**. A $1d$ lattice with a known transition rates distribution is exactly solvable [3]. Basically, it is analogous to adding connectors in series, and the diffusion coefficient is the inverse of the combined resistivity

$$D = \left(\frac{1}{N} \sum_{n} \frac{1}{w_{n,n-1}} \right)^{-1} \tag{4.1}$$

Plugging in the transition rate distribution from **??**, (with the same notation of $s \equiv \xi / r_0$ and converting the sum to an integral, we obtain

$$D = \left(\frac{1}{N} \int_0^{w_0} \frac{s w^{s-1} dw}{w_0^s} \right)^{-1} = [s > 1] \quad \frac{s-1}{s} w_0 \tag{4.2}$$

This result is valid only for $s > 1$. For $s < 1$, $D = 0$ and we have subdiffusion, for which the survival probability and spreading are also exactly solvable [3], leading to:

$$S(t) \sim t^{2s/(1+s)} \tag{4.3}$$

$$\mathcal{P}(t) \sim t^{-s/(1+s)} \tag{4.4}$$

The exact results are depicted in Figure 4.2a. As explained in **??**, the cumulative spectral distribution can be deduced from $\mathcal{P}(t)$:

$$\mathcal{N}(\lambda) \sim \lambda^{s/(1+s)} \tag{4.5}$$

Our numerics agree with these results, as seen in the left upper panel of Figure 4.1.

4.2 The ERH procedure

The diffusion coefficient is defined by the dependence of the variance in time:

$$\mathrm{Var}(n) = 2d\ D\ t \tag{4.6}$$

Within the transient diffusion stage, assuming we start at a specific node n, the variance after time t will be

$$\mathrm{Var}(n) = \sum_m w_{nm} t (x_m - x_n)^2 \tag{4.7}$$

Therefore, the diffusion coefficient for evolution starting at node n is

$$D_n = \frac{1}{2d} \sum_m (x_m - x_n)^2 w_{nm} \tag{4.8}$$

Averaging over the starting node, and converting the sum to an integral, we have

$$D_{\text{linear}} = \frac{1}{2d} \iint w(r,\epsilon)\ r^2\ \rho(r,\epsilon)\ d\epsilon dr \tag{4.9}$$

This expression describes the spreading in absence of disorder correctly for arbitrary long time. However, in a sparse disordered system such as ours, the possibility of transport is a bit less trivial. Therefore, we suggest a method to approximate the diffusion, which takes percolation into account.

The basic idea behind *ERH* is that in the linear expression, nearby sites with $r \ll 1$ (and therefore $w \gg 1$) are over represented in the diffusion coefficient calculation. While the transition to these sites is indeed high, the distance covered is not enough to form a percolating cluster. Therefore, we use a threshold based on percolation theory and flat-down the rates higher then this threshold.

The rate threshold w_c is determined using the following expression

$$\iint\limits_{w(r,\epsilon)>w_c} \rho(r,\epsilon) dr d\epsilon = n_c \tag{4.10}$$

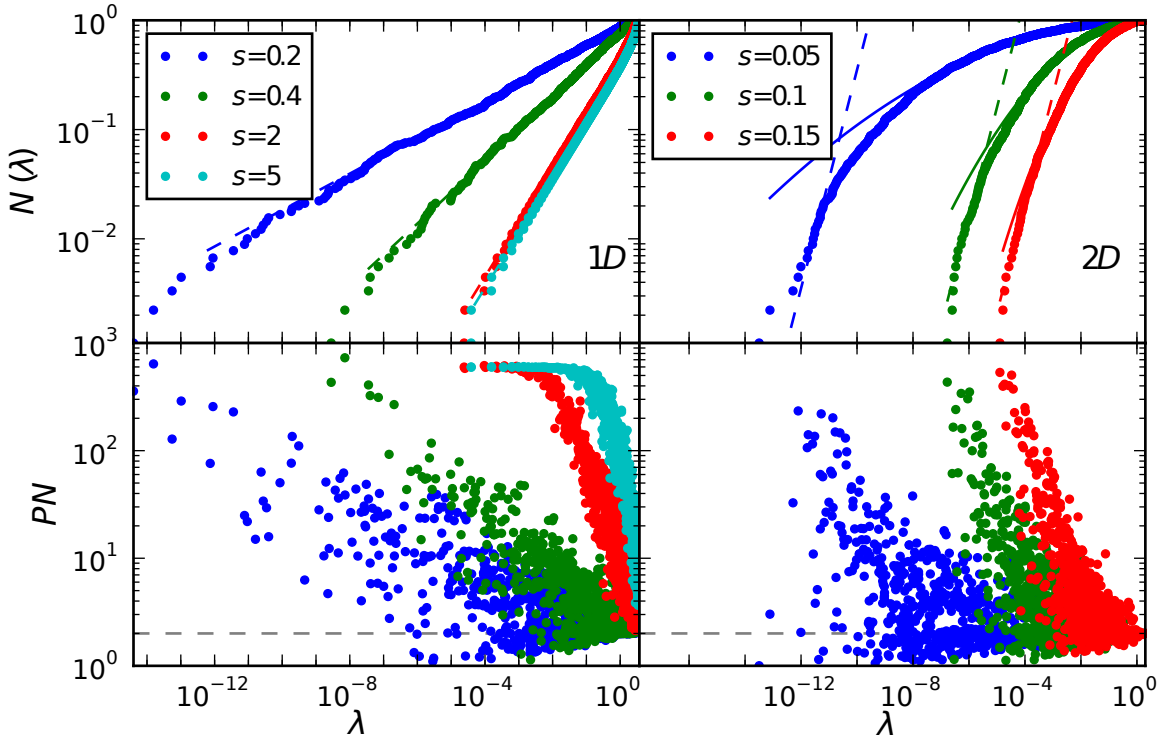


Figure 4.1: The cumulative eigenvalue distributions $\mathcal{N}(\lambda)$ for the d=1 (1D) and for the d=2 (2D) degenerate models, and the respective PN of the eigenstates (lower panels). Several representative values of s are considered. The dots are determined via numerical diagonalization. The dashed lines in the upper plots correspond to diffusive behavior. The solid lines are according to the RG analysis of [4], namely **??** . There is a striking difference between the d=1 and the d=2 cases. For d=1, the log-log slope for sparse models ($s < 1$) is less than $d/2$, meaning that we have sub-diffusion. In the d=2 case the small-λ log-log slope is always $d/2$, which corresponds to normal diffusion. The horizontal dashed line in the lower panels indicates the special value PN= 2 that corresponds to dimer formation.

Where n_c is the number of connected sites necessary to form a percolation cluster (see TODO :ref to section explaining this better).

Next, we suppress the rates higher than w_c to have the value w_c, and use the linear expression 4.9, to obtain:

$$D_{\text{ERH}} = \frac{1}{2d} \iint \min\{w(r,\epsilon), w_c\}\ r^2\ \rho(r,\epsilon)\ d\epsilon dr \tag{4.11}$$

4.3 The ERH calculation for several types of network

4.3.1 The $d = 2$ ordered lattice model, with n.n. hopping

The model is a two dimensional grid, with only nearest neighbor hopping possible. Each site has 4 neighbors of equal distance, meaning that the distribution of sites surrounding each site is simply

$$\rho(r,\epsilon) \ = \ 4\ f(\epsilon)\delta(r - r_0) \tag{4.12}$$

It is well known from percolation theory [**?**] that in this model the number of necessary connections per site is $n_c = 2$. We can find w_c from 4.10,an put this into the ERH expression 4.11 to have:

$$D_{\text{ERH}} = \left[\frac{1}{2}w_c + \frac{1}{2}\int_0^{w_c} w\tilde{f}(w)dw\right] r_0^2 \tag{4.13}$$

Where we converted the $f(\epsilon)d\epsilon$ integral to a $\tilde{f}(w)dw$ integral. In absence of the disorder (i.e. all the rates are equal), the known result $D = w_0 r_0^2$ is restored.

4.3.2 The degenerate hopping model

In this model the sites are randomly distributed in a d dimensional space, all with $\epsilon = 0$. Every site is connected to all of the other sites, with the transition rates according to **??**.

Since w depends only on r, we may define the ERH threshold by r_c instead of w_c. Rephrasing equation 4.10, and using $\rho(r,\epsilon)$ from **??**, r_c is determined according to

$$\int_0^{r_c} \frac{\Omega_d r^{d-1} dr}{r_0^2} \ = \ n_c \tag{4.14}$$

With the simple solution:

$$r_c \equiv \left(\frac{d}{\Omega_c} n_c\right)^{1/d} r_0 \tag{4.15}$$

$$w_c = w_0 \exp\left(-\frac{r_c}{\xi}\right) \tag{4.16}$$

Now we shall put all of the above into the ERH expression **??**. The solution involves the incomplete gamma function [5],

$$\Gamma(\ell{+}1, x) = \int_0^x r^\ell \mathrm{e}^{-r} dr = \ell!\ \mathrm{EXP}_\ell(x)\ \mathrm{e}^{-x} \tag{4.17}$$

And the polynomial

$$\mathrm{EXP}_\ell(x) \quad = \quad \sum_{k=0}^{\ell} \frac{1}{k!}\, x^k \tag{4.18}$$

The integral is only over dr, and it is split into the domains $0 < r < r_c$ and $r > r_c$. Namely,

$$\begin{aligned} D_{\mathrm{ERH}} &= \frac{w_0 \Omega_d}{2d} \int_0^{r_c} \mathrm{e}^{-r_c/\xi} \frac{r^{d+1}}{r_0^d} dr \quad + \quad \frac{w_0 \Omega_d}{2d} \int_{r_c}^{\infty} \mathrm{e}^{-r/\xi} \frac{r^{d+1}}{r_0^d} dr \\ &= \frac{w_0 \Omega_d}{2d} \mathrm{e}^{-r_c/\xi} \frac{r_c^{d+2}}{d+2} \frac{1}{r_0^d} \quad + \quad \frac{w_0 \Omega_d}{2d} \frac{\xi^{d+2}}{r_0^d} \Gamma\left(d+2, \frac{r_c}{\xi}\right) \\ &= \frac{w_0 \Omega_d \xi^{d+2}}{2d(d+2) r_0^d} \Gamma\left(d+3, \frac{r_c}{\xi}\right) \qquad (4.19) \\ &= \mathrm{EXP}_{d+2}\left(\frac{1}{s_c}\right)\ \mathrm{e}^{-1/s_c}\ D_{\mathrm{linear}} \qquad (4.20) \end{aligned}$$

Where $s_c = \xi / r_c$, and D_{linear} is formally obtained by setting $r_c = 0$ in D_{ERH} i.e.:

$$D_{\mathrm{linear}} \quad = \quad \frac{(d{+}1)!\, \Omega_d}{2d}\, s^{d+2}\, w_0 r_0^2 \tag{4.21}$$

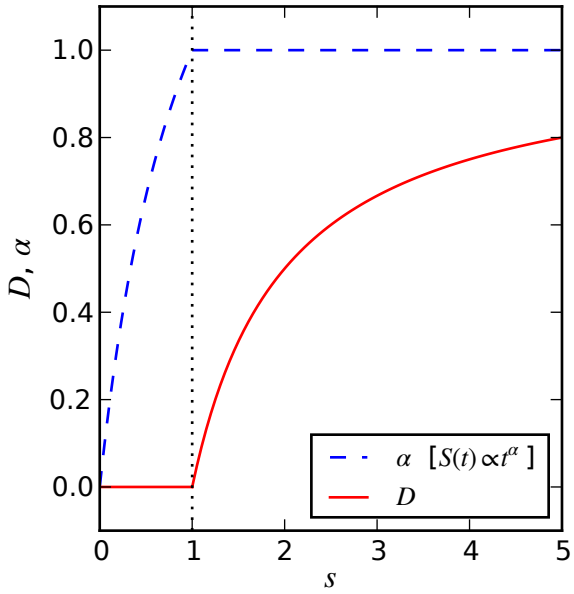
The ERH result depends on n_c, the number of bonds required for percolation. For $d = 2$, we have used the $n_c = 4.5$ result from [6], and compared it with our numerics in figure 4.2b.

4.3.3 The Mott hopping model

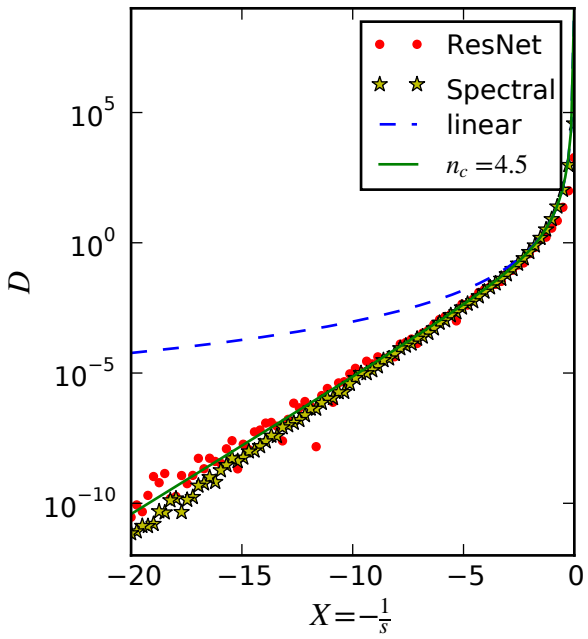
For the Mott hopping model, with rates according to **??**, and rate distribution according to **??**, the ERH threshold is determined using Equation 4.10:

$$\epsilon_c \equiv \left(\frac{d}{\Omega_d} \frac{n_c}{s^d}\right)^{1/(d+1)} \tag{4.22}$$

$$w_c = w_0 \exp(-\epsilon_c) \tag{4.23}$$



(a) Spreading in d=1



(b) Spreading in d=2

Figure 4.2: long

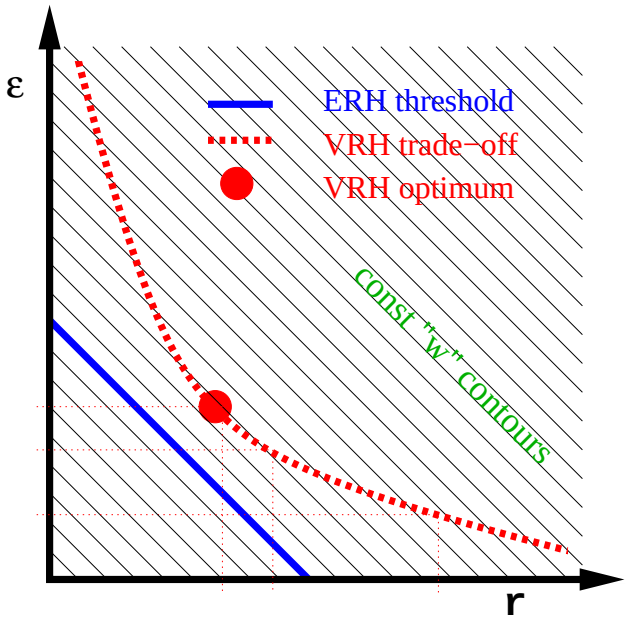


Figure 4.3: Comparing VRH with ERH. The thin black lines correspond to equal rates. The solid blue line corresponds to the ERH threshold w_c, while the red dashed line corresponds to the VRH trade-off. The red dot is the VRH optimum point, used for the VRH calculation.

This integral has two dimensions, dr and $d\epsilon$, split into two domains $w > w_c$ and $0 < w < w_c$. The two domains are separated by the line $\epsilon + (r/\xi) = \epsilon_c$ which is illustrated in Figure 4.3.

It is therefore natural to change variables:

$$x = \epsilon + (r/\xi) \tag{4.24}$$

$$y = \frac{1}{2}(-\epsilon + (r/\xi)) \tag{4.25}$$

hence

$$\begin{aligned} D_{\text{ERH}} &= \frac{w_0\Omega_d}{2dr_0^d}\int_0^{\epsilon_c}\xi dx\int_{-x/2}^{x/2}dy\,\mathrm{e}^{-\epsilon_c}\left(\xi y+\xi\frac{x}{2}\right)^{d+1} \\ &+ \frac{w_0\Omega_d}{2dr_0^d}\int_{\epsilon_c}^{\infty}\xi dx\int_{-x/2}^{x/2}dy\,\mathrm{e}^{-x}\left(\xi y+\xi\frac{x}{2}\right)^{d+1} \\ &= \frac{w_0\Omega_d}{2dr_0^d}\xi^{d+2}\mathrm{e}^{-\epsilon_c}\frac{\epsilon_c^{d+3}}{(d+2)(d+3)} \\ &+ \frac{w_0\Omega_d}{2dr_0^d(d+2)}\xi^{d+2}\Gamma\left(d+3,\epsilon_c\right) \\ &= \frac{w_0\Omega_d\xi^{d+2}}{2d(d+2)(d+3)r_0^d}\Gamma\left(d+4,\epsilon_c\right) \\ &= \mathrm{EXP}_{d+3}\left(\epsilon_c\right)\ \mathrm{e}^{-\epsilon_c}\ D_{\text{linear}} \end{aligned} \tag{4.26}$$

With EXP_ℓ and D_{linear} the same as in Equation 4.18 and Equation 4.21.

4.3.4 The Quasi one dimensional model

This model is a Banded one dimensional lattice model. Each site is connected to b other sites, the disorder arising from variations in ϵ. $\rho(r,\epsilon)$ is given in **??**. We find the ERH threshold from Equation 4.10:

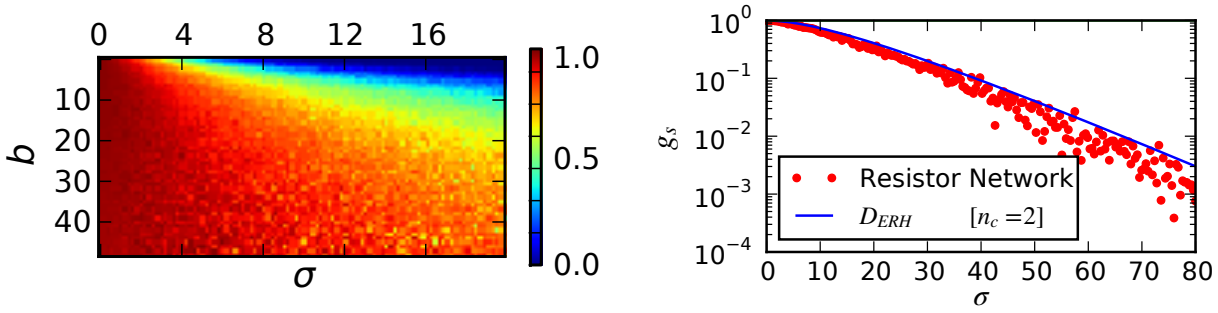
$$\int_0^\infty \frac{\Omega_d\, r^{d-1}dr}{r_0^d}\ F\left(\log\left(\frac{w_0}{w_c}B(r)\right)\right) = n_c \tag{4.27}$$

Where $F(\epsilon)$ is the cumulative distribution function, or the integral of $f(\epsilon)$.

For now, we shall assume a flat profile,

$$B(r) = \begin{cases} 1 & \text{if } r <= b \\ 0 & \text{if } r > b \end{cases} \tag{4.28}$$

Together with a uniform distribution of ϵ over $[0,\sigma]$, which means that the rates are log-uniform distributed. Large σ corresponds to a very wide distribution of rates, meaning a "sparse" system (see **??**).



(a) Several values of b and σ

(b) For $b = 10$ only.

Figure 4.4: Numerics Vs ERH estimation for the quasi one dimensional network. $g_s = D/D_{\text{linear}}$ is plotted on both graphs. On b, the blue line is the ERH estimation. If the linear result was correct, g_s would have been 1 for all values of σ.

Note that even with large σ, the distribution has a lower non zero bound, meaning that the system is still diffusive.

We continue our threshold seeking from Equation 4.27, using this flat profile, and find ϵ_c to be:

$$2b\ F(\epsilon_c) \ = \ n_c \tag{4.29}$$

$$\epsilon_c \ = \ \frac{n_c}{2b}\sigma \tag{4.30}$$

The ERH integral (Equation 4.11)is replaced by a sum, leading to

$$D_{\text{ERH}} \ = \ \frac{1}{\sigma}\left[\left(1+\frac{n_c}{2b}\sigma\right)\mathrm{e}^{-\frac{n_c}{2b}\sigma} - \mathrm{e}^{-2\sigma}\right]\ \tilde{b}w_0 \tag{4.31}$$

Where we have defined

$$\tilde{b} \ \equiv \ \sum_{r=1}^{b} r^2 \ = \ = \frac{1}{6}b(b+1)(2b+1) \tag{4.32}$$

As before, the linear result is formally obtained by setting $n_c = 0$.

This result is compared to numerical computation in Figure 4.4.

Part III

Preliminary Analysis

Part IV

Appendix

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Appendix A

Published Papers

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Appendix B

Spacing Statistics in d-dimensions

Some clarifing points regarding the statistics of uniformly distributed sites will be made in this section.

There are N sites distributed randomly within a d-dimensional hypercube of volume L^d. We define a typical length r_0 by:

$$\frac{L^d}{r_0^d} \;=\; N \tag{B.1}$$

From now on we shall ignore boundary conditions, assuming N is large enough. In the numerics we have used periodic boundary conditions.

If we choose some arbitrary point as the origin, the distribution of sites around this point will be:

$$\rho(r)dr = \frac{\Omega_d r^{d-1}}{r_0^d} dr \tag{B.2}$$

Where Ω_d is the d dimensional solid angle:

$$\Omega_d \;=\; 2,\; 2\pi,\; 4\pi, \ldots \tag{B.3}$$

The distribution of the nearest neighbor distance can be derived [**?** **?** **?**], using a differential equation. (Apparently my previous extension of the $1d$ solution to d dimensions was just a good guess, the proof is less trivial then I've thought).

Let us denote by $P(r)$ the probability that the first near neighbor will be between r and $r + dr$. The probability of having no neighbors up to distance r is

$$P_0(r) \;=\; 1 - \int_0^r P(r)dr \tag{B.4}$$

$P(r)$ must equal the probability of having no neighbors up to distance r times the probabilty of finding a neighbor between r and $r + dr$. So $P(r)$ must satisfy:

$$P(r) = P_0(r) \times \rho(r) = \left[1 - \int_0^r P(r)dr\right] \rho(r) \tag{B.5}$$

$$\frac{P(r)}{\rho(r)} = 1 - \int_0^r P(r)dr \tag{B.6}$$

$$\frac{d}{dr}\left(\frac{P(r)}{\rho(r)}\right) = -\rho(r)\frac{P(r)}{\rho(r)} \tag{B.7}$$

Which has the solution:

$$P(r) = \rho(r)\mathrm{e}^{-\int_0^r \rho(r)dr} \tag{B.8}$$

$$= \frac{\Omega_d r^{d-1}}{r_0^d}\mathrm{e}^{-\frac{\Omega_d}{d}\left(\frac{r}{r_0}\right)^d} \tag{B.9}$$

Where in the last step we have plugged in $\rho(r)$ from Equation B.2

Appendix C

Resistor Network Calculation