JAMES ROSADO
ENGINEERING ANALYSIS
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HOMEWORK #3 PROBLEM 4

Solve the boundary value problem on the wedge,

- $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \text{for } 0 < r < \rho \text{ and } 0 < \theta < \theta_0$
- $u(r,0) = 0 \quad \text{for } 0 < r < \rho$
- $u(r,\theta_0) = 0 \quad \text{for } 0 < r < \rho$
- $u(\rho,\theta) = f(\theta) \quad \text{for } 0 < \theta < \theta_0$

We will use separation of variables and let $u(r,\theta) = \phi(\theta)G(r)$; therefore, $u_{rr} = \phi(\theta)G''(r)$, $u_r = \phi(\theta)G'(r)$, and $u_{\theta\theta} = \phi''(\theta)G(r)$. Make the substitutions into the differential equation to get,

$$\phi(\theta)G''(r) + \frac{1}{r}\phi(\theta)G'(r) + \frac{1}{r^2}\phi''(\theta)G(r) = 0$$

Notice in the first two terms we can factor out a ϕ and get,

$$\phi\left(G'' + \frac{1}{r}G'\right) + \frac{1}{r^2}\phi''G = 0$$

Move terms to either side of the equality to get,

$$\phi\left(G'' + \frac{1}{r}G'\right) = -\frac{1}{r^2}\phi''G$$

Divide out to get,

$$\frac{G'' + \frac{1}{r}G'}{\frac{1}{r^2}G} = -\frac{\phi''}{\phi} = \lambda$$

The far left side becomes,

$$\frac{G'' + \frac{1}{r}G'}{\frac{1}{r^2}G} = \frac{r^2}{G}\left(G'' + \frac{1}{r}G'\right) = \frac{r}{G}(rG'' + G') = \frac{r}{G}\frac{d}{dr}(rG')$$

So we have two implied differential equations:

- $-\frac{\phi''}{\phi} = \lambda \ \rightarrow \ \phi'' + \lambda\phi = 0$
- $\frac{r^2}{G}\left(G'' + \frac{1}{r}G'\right) = \lambda \ \rightarrow \ r^2G'' + rG' - \lambda G = 0$

Let us examine the first differential equation, it has characteristic equation,

$$r^2 + \lambda = 0 \rightarrow r_{1,2} = \pm i\sqrt{\lambda}$$

Therefore, our solution for ϕ has the form,

$$\phi(\theta) = A\cos\sqrt{\lambda}\theta + B\sin\sqrt{\lambda}\theta$$

The boundary conditions have the following implication: $u(r,0) = u(r,\theta_0) = 0 \rightarrow \ \phi(0) = \phi(\theta_0) = 0$. The first boundary condition yields $\phi(0) = A \cdot 1 + B \cdot 0 = 0 \rightarrow A = 0$. So our solution to ϕ will be of the form $\phi(\theta) = B\sin\sqrt{\lambda}\theta$. If we use the second boundary condition we get,

$$\phi(\theta_0) = B\sin\sqrt{\lambda}\theta_0 = 0$$

This means,

$$\sqrt{\lambda}\theta_0 = n\pi \rightarrow \lambda_n = \left(\frac{n\pi}{\theta_0}\right)^2$$

So the corresponding functions are,

$$\phi(\theta) = \sin\left(\frac{n\pi}{\theta_0}\theta\right)$$

Let us now analyze the second differential equation,

$$r^2G'' + rG' - \lambda G = 0$$

If we let $G(r) = r^p$ and we substitute we get,

$$r^2p(p-1)r^{p-2} + rpr^{p-1} - \lambda r^p = 0$$

Which simplifies to the following,

$$p(p-1)r^p + pr^p - \lambda r^p = 0$$

We can factor and divide out the common term and we are left with,

$$p(p-1) + p - \lambda = 0$$
$$p^2 - p + p - \lambda = 0$$
$$p^2 - \lambda = 0 \rightarrow \ p = \ \pm\sqrt{\lambda} = \pm\frac{n\pi}{\theta_0}$$

Therefore, we have the following solution,

$$G(r) = c_1r^{\frac{n\pi}{\theta_0}} + c_2r^{-\frac{n\pi}{\theta_0}}$$

But our solutions need to be bounded when r = 0, $|G(r)| < \infty$; therefore, we will disregard the second term:

$$G(r) = c_1r^{\frac{n\pi}{\theta_0}}$$

We have established the following,

- $G(r) = c_1 r^{\frac{n\pi}{\theta_0}}$
- $\phi(\theta) = \sin\left(\frac{n\pi}{\theta_0}\theta\right)$
- $\lambda_n = \left(\frac{n\pi}{\theta_0}\right)^2$

A solution to our problem is,

$$u_n(r,\theta) = B_n r^{\frac{n\pi}{\theta_0}} \sin\left(\frac{n\pi}{\theta_0}\theta\right)$$

Our general solution by the superposition principle is,

$$u(r,\theta) = \sum_{n=1}^{\infty} B_n r^{\frac{n\pi}{\theta_0}} \sin\left(\frac{n\pi}{\theta_0}\theta\right)$$

Therefore, by the definition of Fourier Series we can find a closed expression for 'B' coefficients when $r = \rho$,

$$u(\rho,\theta) = \sum_{n=1}^{\infty} B_n \rho^{\frac{n\pi}{\theta_0}} \sin\left(\frac{n\pi}{\theta_0}\theta\right) = f(\theta)$$

Therefore,

$$B_n \rho^{\frac{n\pi}{\theta_0}} = \frac{2}{\theta_0}\int_0^{\theta_0} f(\theta) \sin\left(\frac{n\pi}{\theta_0}\theta\right) d\theta$$

Hence,

$$B_n = \rho^{-\frac{n\pi}{\theta_0}} \frac{2}{\theta_0}\int_0^{\theta_0} f(\theta) \sin\left(\frac{n\pi}{\theta_0}\theta\right) d\theta$$

If we let $\rho = 1 \text{ and } \theta_0 = \frac{\pi}{3}$ we get,

$$B_n = \frac{6}{\pi}\int_0^{\frac{\pi}{3}} 5\theta e^{-2\theta} \sin(3n\theta)\, d\theta$$