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ENGINEERING ANALYSIS
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HOMEWORK #3 PROBLEM 7

Solve the one dimensional heat equation below with a time-dependent heat source and non-homogeneous boundary conditions:

$$u_t = u_{xx} + Q(x,t)$$

Boundary Conditions: $u(0,t) = 1 = A(t)$and $u(\pi,t) = 0 = B(t)$
Initial Conditions: $u(x,0) = x(2\pi - x)$

We need to make the boundary conditions homogeneous first, we will introduce the following new function,

$$r(x,t) = A(t) + \frac{x}{\pi}\big(B(t) - A(t)\big)$$

$$r(x,t) = 1 + \frac{x}{\pi}(0 - 1)$$

$$r(x,t) = 1 - \frac{x}{\pi}$$

And we will use this in our displacement function v(x,t):

$$v(x,t) = u(x,t) - r(x,t) = u(x,t) - \left(1 - \frac{x}{\pi}\right)$$

Notice,

$$v(0,t) = u(0,t) - (1 - 0) = 1 - 1 = 0$$

$$v(\pi,t) = u(\pi,t) - \left(1 - \frac{\pi}{\pi}\right) = 0 - (1 - 1) = 0$$

$$v(x,0) = u(x,0) - \left(1 - \frac{x}{\pi}\right) = x(2\pi - x) - \left(1 - \frac{x}{\pi}\right)$$

So we have achieved our new homogeneous boundary conditions and a new initial condition for v(x,t) satisfying,

$$v_t = v_{xx} + Q(x,t)$$

The expansion of v(x,t) for the homogeneous differential equation: $v_t = v_{xx}$ is,

$$v_h(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} A_n \sin nx$$

For the non-homogeneous problem we will utilize variation of parameters to get a similar eigenfunction expansion,

$$v(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin nx$$

Let us now use this in our displaced differential equation for v(x,t):

$$v_t = \sum_{n=1}^{\infty} \frac{dA_n}{dt} \sin nx$$

$$v_{xx} = -\sum_{n=1}^{\infty} A_n(t) n^2 \sin nx$$

Substitute these above two lines into the differential equation we get,

$$\sum_{n=1}^{\infty} \frac{dA_n}{dt} \sin nx = -\sum_{n=1}^{\infty} A_n(t) n^2 \sin nx + Q(x,t)$$

Or

$$\sum_{n=1}^{\infty} \frac{dA_n}{dt} \sin nx + A_n(t) n^2 \sin nx = Q(x,t)$$

$$\sum_{n=1}^{\infty} \left(\frac{dA_n}{dt} + A_n(t) n^2\right) \sin nx = Q(x,t)$$

Part a) Let us consider the case when $\boldsymbol{Q(x,t) = e^{-t} \sin 3x - e^{-2t} \sin 4x}$,

$$\sum_{n=1}^{\infty} \left(\frac{dA_n}{dt} + A_n(t) n^2\right) \sin nx = e^{-t} \sin 3x - e^{-2t} \sin 4x$$

Notice that we are ‘filtering’ out only two terms from the series on the left when n = 3 and n = 4, thus;

$$\left(\frac{dA_3}{dt} + A_3(t) 3^2\right) \sin 3x - \left(\frac{dA_4}{dt} + A_4(t) 4^2\right) \sin 4x = e^{-t} \sin 3x - e^{-2t} \sin 4x$$

Or in a more general format,

$$\frac{dA_n}{dt} + A_n(t) n^2 = \begin{cases} 0 & n \neq 3, n \neq 4 \\ e^{-t} & n = 3 \\ -e^{-2t} & n = 4 \end{cases}$$

We have three differential equations implied and for each we will need to solve the homogeneous problem and non-homogeneous problem

$A_n' + A_n n^2 = 0$	$A_n(t) = A_n(0) e^{-n^2 t}$
$A_3' + 9A_3 = e^{-t}$	$A_3(t) = A_{3h}(t) + A_p(t) = \frac{1}{8} e^{-t} + A_3(0) e^{-9t}$
$A_4' + 16A_4 = -e^{-2t}$	$A_4(t) = A_{4h}(t) + A_p(t) = -\frac{1}{14} e^{-2t} + A_4(0) e^{-16t}$

The coefficients can be found,

$$A_n(0) = \frac{2}{\pi}\int_0^{\pi}\left[x(2\pi - x) - \left(1 - \frac{x}{\pi}\right)\right]\sin nx\,dx = \frac{-2(-2 + n^2 + (-1)^n(2 + n^2\pi^2))}{n^3\pi}$$

$$A_3(0) = \frac{2(9\pi^2 - 5)}{27\pi}$$

$$A_4(0) = \frac{-1 + \pi^2}{2\pi}$$

Thus,

$$v(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin nx$$

Where,

$$A_n(t) = \begin{cases} \dfrac{-2(-2 + n^2 + (-1)^n(2 + n^2\pi^2))}{n^3\pi}e^{-n^2t} & n \neq 3, n \neq 4 \\ \dfrac{1}{8}e^{-t} + \dfrac{2(9\pi^2 - 5)}{27\pi}e^{-9t} & n = 3 \\ -\dfrac{1}{14}e^{-2t} + \dfrac{-1 + \pi^2}{2\pi}e^{-16t} & n = 4 \end{cases}$$

Our solution to the original differential equation will be,

$$u(x,t) = v(x,t) + r(x,t)$$

$$u(x,t) = 1 - \frac{x}{\pi} + \sum_{n=1}^{\infty} A_n(t)\sin nx$$

Part b) Let us consider the case when $\boldsymbol{Q(x,t) = e^{-3t}\sin 5x}$,

$$\sum_{n=1}^{\infty}\left(\frac{dA_n}{dt} + A_n(t)n^2\right)\sin nx = e^{-3t}\sin 5x$$

Again, we are ‘filtering’ out our solution when n = 5, hence,

$$\frac{dA_n}{dt} + A_n(t)n^2 = \begin{cases} 0 & n \neq 5 \\ e^{-3t} & n = 5 \end{cases}$$

Thus we have two differential equations,

$A_n' + A_n n^2 = 0$	$A_n(t) = A_n(0)e^{-n^2 t}$
$A_5' + 25A_5 = e^{-3t}$	$A_5(t) = A_{5h}(t) + A_p(t) = \frac{1}{22}e^{-3t} + A_5(0)e^{-25t}$

Where,

$$A_n(0) = \frac{-2(-2 + n^2 + (-1)^n(2 + n^2\pi^2))}{n^3\pi}$$

Thus,

$$v(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin nx$$

Where,

$$A_n(t) = \begin{cases} \frac{-2(-2 + n^2 + (-1)^n(2 + n^2\pi^2))}{n^3\pi} e^{-n^2 t} & n \neq 5 \\ \frac{1}{22}e^{-3t} + \left(-\frac{42}{125\pi} + \frac{2\pi}{5}\right)e^{-25t} & n = 5 \end{cases}$$

Thus,

$$u(x,t) = 1 - \frac{x}{\pi} + \sum_{n=1}^{\infty} A_n(t) \sin nx$$