HOMEWORK 2: PROBLEM #5
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**Problem:**
**Consider a slightly damped vibrating string that satisfies:**

**PDE:** $\boldsymbol{u_{tt} = c^2 u_{xx} - \beta u_t}$

**Boundary Conditions:** $\boldsymbol{u(0,t) = u(L,t) = 0}$

**Initial Conditions:** $\boldsymbol{u(x,0) = f(x)}$ **and** $\boldsymbol{u_t(x,0) = g(x)}$

Use separation of variables to determine the solutions. You may also assume that $\beta < \frac{2\pi c}{L}$.

The first step in separation of variables is to let $u(x,t) = \phi(x)G(t)$, so we have separated our solution function into a product of two functions: the spatial function $\phi(x)$ and the temporal function $G(t)$. Now let us find the first and second order derivatives with respect to *x* and *t* for our solution function:

1. $u_t = \phi G'$
2. $u_{tt} = \phi G''$
3. $u_x = \phi' G$
4. $u_{xx} = \phi'' G$

We will substitute (1) - (4) into the PDE to get,

$$\phi G'' = c^2 \phi'' G - \beta \phi G'$$

Rearrange the terms and factor we get,

$$\phi G'' + \beta \phi G' = c^2 \phi'' G$$

$$\phi(G'' + \beta G') = c^2 \phi'' G$$

By division of the above line we can separate our temporal and spatial quantities,

$$\frac{G'' + \beta G'}{c^2 G} = \frac{\phi''}{\phi} = -\lambda$$

From the above extended equality statement we get two ODE's:

i) $G''(t) + \beta G'(t) + c^2 \lambda G(t) = 0$

ii) $\phi''(x) + \lambda \phi(x) = 0$

Let us begin with the spatial differential equation. If we let $\phi(x) = e^{rx}$ and substitute into the spatial equation we get the following characteristic equation for *r*:

$$r^2 + \lambda = 0 \implies r_{1,2} = \pm\sqrt{-\lambda}$$

There are three possible cases we need to consider: λ < 0 , λ = 0, λ > 0.

CASE 1) λ < 0 implies $r_{1,2} = \pm\sqrt{-\lambda}$ so our possible solution will take the form,

$$\phi(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$$

We will now consider our boundary conditions to help find the values of A and B. Since $u(0,t) = 0 \Longrightarrow \phi(0) = 0$ and $u(L,t) = 0 \Longrightarrow \phi(L) = 0$. Therefore for case 1:

$$\phi(0) = A + B = 0 \Longrightarrow A = -B$$

Then our spatial function becomes,

$$\phi(x) = -Be^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$$

Now let us consider $\phi(L) = 0$,

$$\phi(L) = -Be^{\sqrt{-\lambda}L} + Be^{-\sqrt{-\lambda}L} = B\left(e^{\sqrt{-\lambda}L} + e^{-\sqrt{-\lambda}L}\right) = 0$$

If we analyze the right half of the equation, $B\left(e^{\sqrt{-\lambda}L} + e^{-\sqrt{-\lambda}L}\right) = 0$, there are two possibilities either B = 0 or $e^{\sqrt{-\lambda}L} + e^{-\sqrt{-\lambda}L} = 0$, the latter is impossible unless L = 0 which contradicts the whole premise of having a 1-dimensional string; therefore, the formed B = 0 is true and for the case λ < 0 yields the trivial spatial solution $\phi(x) = 0$.

CASE 2) λ < 0 implies $r_1 = r_2 = 0$; therefore, we have repeated roots and our spatial function may take the form: $\phi(x) = Ax + B$. We will follow the same steps as in case 1 and use the boundary conditions to determine A and B. For $\phi(0) = B = 0 \Longrightarrow \phi(x) = Ax$, Now if we consider the other boundary condition: $\phi(L) = AL = 0 \Longrightarrow A = 0$; therefore, we arrive again at the trivial solution for $\phi(x)$.

CASE 3) λ > 0 implies $r_{1,2} = \pm i\sqrt{\lambda}$; therefore, we have imaginary roots and our spatial function will take the form: $\phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$. For the first boundary condition we get: $\phi(0) = A \cdot 1 + B \cdot 0 = A = 0$; therefore, $\phi(x) = B\sin(\sqrt{\lambda}x)$. If we consider the second boundary condition we arrive at a different conclusion: $\phi(L) = B\sin(\sqrt{\lambda}L) = 0$. Here we can conclude that the sine term is only 0 when $\sqrt{\lambda}L = n\pi$; now we have arrived at a relationship for our λ 's:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

This leads to our eigen-functions:

$$\phi_n(x) = B\sin\left(\frac{n\pi}{L}x\right)$$

We will now examine the temporal differential equation:

$$G''(t) + \beta G'(t) + c^2\lambda G(t) = 0$$

The characteristic equation for the above differential equation is,

$$r^2 + \beta r + c^2\lambda = 0$$

If we utilize the quadratic formula we will obtain the following roots of the equation above,

$$r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4c^2\lambda}}{2}$$

But remember we may assume that $\beta < \frac{2\pi c}{L}$, but notice that our roots, with the eigenvalues incorporated is,

$$r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4c^2\left(\frac{n\pi}{L}\right)^2}}{2}$$

So are these roots real, imaginary? Well let us examine the following,

If $\beta < \frac{2\pi c}{L} \implies \beta < \frac{2\pi c}{L} < \frac{2n\pi c}{L}$ then if we square both sides of this inequality we get,

$$\beta^2 < \left(\frac{2n\pi c}{L}\right)^2 = 4c^2\left(\frac{n\pi}{L}\right)^2 = 4c^2\lambda_n$$

Therefore,

$$\beta^2 < 4c^2\lambda_n$$

Hence the discriminant of our roots will be negative, we will denote $\mu_n = |\beta^2 - 4c^2\lambda_n|$. We will have the following roots of our characteristic equation to the temporal equation:

$$r_{1,2} = \frac{\beta}{2} \pm i\frac{\sqrt{\mu_n}}{2}$$

So we obtain the following solutions to our temporal equation:

$$G_1(t) = e^{\frac{\beta}{2}t}\left(\cos\frac{\sqrt{\mu_n}}{2}t + i\sin\frac{\sqrt{\mu}}{2}t\right)$$

$$G_2(t) = e^{\frac{\beta}{2}t}\left(\cos\frac{\sqrt{\mu_n}}{2}t - i\sin\frac{\sqrt{\mu_n}}{2}t\right)$$

So the homogeneous solution to our temporal equation will be a linear combination of our two separate solutions above:

$$G_n(t) = e^{\frac{\beta}{2}t}\left(A_n\cos\frac{\sqrt{\mu_n}}{2}t + B_n\sin\frac{\sqrt{\mu_n}}{2}t\right)$$

Where,

$$\mu_n = |\beta^2 - 4c^2\lambda_n| \text{ and } \lambda_n = \frac{n\pi}{L}$$

If we take our spatial solution and temporal we can form our solution to the wave equation,

$$u_n(x,t) = \phi_n(x)G_n(t)$$

$$u_n(x,t) = e^{\frac{\beta}{2}t}\left(A_n\cos\frac{\sqrt{\mu_n}}{2}t + B_n\sin\frac{\sqrt{\mu_n}}{2}t\right)\sin\left(\frac{n\pi}{L}x\right)$$

Then by superposition we have the general series form,

$$u(x,t)=\sum_{n=1}^{\infty} e^{\frac{\beta}{2}t}\left(A_n\cos\frac{\sqrt{\mu_n}}{2}t+B_n\sin\frac{\sqrt{\mu_n}}{2}t\right)\sin\left(\frac{n\pi}{L}x\right)$$

Now we have to find the coefficients utilizing our initial conditions. We will begin with the first initial condition:

$$u(x,0)=f(x)$$

If we make the above substitution we get,

$$u(x,0)=\sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi}{L}x\right)=f(x)$$

We can obtain the coefficients $A_n$ by utilizing the definition of Fourier Series,

$$A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Now we will use the second initial condition to find the $B_n$ coefficient. First take the derivative and not we are going to use the product rule with respect to *t*. As you can see the differentiation yields a lengthy equation.

$$u_t(x,t)=\sum_{n=1}^{\infty}\left(\frac{\beta}{2}e^{\frac{\beta}{2}t}\left(A_n\cos\frac{\sqrt{\mu_n}}{2}t+B_n\sin\frac{\sqrt{\mu_n}}{2}t\right)+\frac{\sqrt{\mu_n}}{2}e^{\frac{\beta}{2}t}\left(-A_n\sin\frac{\sqrt{\mu_n}}{2}t+B_n\cos\frac{\sqrt{\mu_n}}{2}t\right)\right)\sin\left(\frac{n\pi}{L}x\right)$$

Let t → 0 and we get,

$$u_t(x,0)=\sum_{n=1}^{\infty}\left(\frac{\beta}{2}A_n+\frac{\sqrt{\mu_n}}{2}B_n\right)\sin\left(\frac{n\pi}{L}x\right)=g(x)$$

This is again the Fourier Sine series for g(x); therefore,

$$\frac{\beta}{2}A_n+\frac{\sqrt{\mu_n}}{2}B_n=\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

If we isolate the $B_n$ we get,

$$B_n=-\frac{\beta}{\sqrt{\mu_n}}A_n+\frac{4}{L\sqrt{\mu_n}}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

But we already know what $A_n$ is, so let us substitute,

$$B_n=-\frac{\beta}{\sqrt{\mu_n}}\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx+\frac{4}{L\sqrt{\mu_n}}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Or simply,

$$B_n=\int_0^L\frac{4g(x)-2\beta f(x)}{L\sqrt{\mu_n}}\sin\left(\frac{n\pi}{L}x\right)dx$$

Therefore, our final solution to the lightly damped vibrating string is the following,

$$u(x,t) = \sum_{n=1}^{\infty} e^{\frac{\beta}{2}t} \left( A_n \cos \frac{\sqrt{\mu_n}}{2} t + B_n \sin \frac{\sqrt{\mu_n}}{2} t \right) \sin\left(\frac{n\pi}{L} x\right)$$

Where,

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

$$B_n = \int_0^L \frac{4g(x) - 2\beta f(x)}{L\sqrt{\mu_n}} \sin\left(\frac{n\pi}{L} x\right) dx$$

$$\mu_n = |\beta^2 - 4c^2 \lambda_n|$$

$$\lambda_n = \frac{n\pi}{L}$$

It is interesting to note that the $B_n$ depend on f(x) and g(x) and $A_n$