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ENGINEERING ANALYSIS
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HOMEWORK #3 Problem 5

Solve the wave equation: $u_{tt} = u_{xx} + u_{yy}$ for $0 < x < 2$ and $0 < y < 3$. For a vibrating rectangular membrane subject to the following boundary and initial conditions:

B.C.: $u(0, y, t) = u(2, y, t) = 0$ and $u(x, 0, t) = u(x, 3, t) = 0$

I.C.: $u(x, y, 0) = 4\cos \pi x \cos 3\pi y + 4 \sin 8\pi x \sin \pi y$ and $u_t(x, y, 0) = 0$

We will utilize separation of variables and let $u(x, y, t) = \phi(x)\psi(y)h(t)$ we get the new differential equation when we substitute,

$$\phi(x)\psi(y)h''(t) = \phi''(x)\psi(y)h(t) + \phi(x)\psi''(y)h(t)$$

Or for short hand,

$$\phi\psi h'' = \phi''\psi h + \phi\psi'' h$$

Factor and then divide we obtain the following,

$$\frac{h''}{h} = \frac{\phi''\psi + \phi\psi''}{\phi\psi} = -\lambda$$

We have obtained two new simpler differential equations,

- $h'' + \lambda h = 0$
- $\phi''\psi + \phi\psi'' + \lambda\phi\psi = 0$

We will begin with the second equation, factor and move terms to either side of the equality and we get,

$$\psi(\phi'' + \lambda\phi) = -\phi\psi''$$

Then divide to get,

$$\frac{\phi'' + \lambda\phi}{\phi} = -\frac{\psi''}{\psi} = \mu$$

We yet again obtain two more differential equations,

1. $\phi'' + (\lambda - \mu)\phi = 0$
2. $\psi'' + \mu\psi = 0$

The second equation has characteristic equation: $r^2 + \mu = 0 \rightarrow \ r_{1,2} = \pm i\sqrt{\mu}$. The solutions will be of the form,

$$\psi(y) = A\cos\sqrt{\mu}y + B\sin\sqrt{\mu}y$$

The boundary conditions imply: $u(x,0,t) = u(x,3,t) = 0 \;\rightarrow\; \psi(0) = \psi(3) = 0$. By the first condition we get, $\psi(0) = A \cdot 1 + B \cdot 0 = 0 \rightarrow A = 0$. Hence $\psi(y) = B\sin\sqrt{\mu}y$, if we consider the second condition we get,

$$\psi(3) = B\sin 3\sqrt{\mu} = 0$$

This means that $3\sqrt{\mu} = m\pi \rightarrow \;\mu_m = \left(\frac{m\pi}{3}\right)^2$ and the corresponding solutions will be,

$$\psi_m(y) = \sin\left(\frac{m\pi y}{3}\right)$$

Let us now find $\phi(x)$. We established,

$$\phi'' + (\lambda - \mu)\phi = 0$$

The solutions to the characteristic equation are of the form, $r_{1,2} = \pm i\sqrt{\lambda - \mu}$. Therefore, the solution to ϕ will be of the form: $\phi(x) = A\cos\sqrt{\lambda - \mu}x + B\sin\sqrt{\lambda - \mu}x$. With the first set of boundary conditions we get: $u(0,y,t) = u(2,y,t) = 0 \rightarrow \phi(0) = \phi(2) = 0$. For $\phi(0)$ we arrive at the same result as before A = 0 and $\phi(x) = B\sin\sqrt{\lambda - \mu}x$. With the second boundary conditions we have,

$\phi(2) = B\sin 2\sqrt{\lambda - \mu} = 0$. Therefore,

$$2\sqrt{\lambda - \mu} = n\pi \rightarrow \lambda_n = \left(\frac{n\pi}{2}\right)^2 + \mu$$

But we have determine μ,

$$\lambda_{mn} = \left(\frac{n\pi}{2}\right)^2 + \left(\frac{m\pi}{3}\right)^2$$

The solution for ϕ (x) we be,

$$\phi(x) = \sin\sqrt{\left(\frac{n\pi}{2}\right)^2 + \left(\frac{m\pi}{3}\right)^2 - \left(\frac{m\pi}{3}\right)^2}\,x$$

$$\phi_n(x) = \sin\left(\frac{n\pi x}{2}\right)$$

Let us finally analyze the temporal part of our solution: $h'' + \lambda h = 0$. The solutions to its characteristic equation are $r_{1,2} = \pm i\sqrt{\lambda_{mn}}$. Remember we have found λ_{mn} earlier. The solutions to the temporal equation will be of the form,

$$h(t) = A\cos\sqrt{\lambda_{mn}}t + B\sin\sqrt{\lambda_{mn}}t$$

If we combine all of our solutions a particular solution to the original 2-d wave equation is of the form,

$$u_{mn}(x,y,t) = \left(A\cos\sqrt{\lambda_{mn}}t + B\sin\sqrt{\lambda_{mn}}t\right)\sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{m\pi y}{3}\right)$$

Let us look at m = 1,2,3:

$$u_{1n}(x,y,t) = \left(A\cos\sqrt{\lambda_{1n}}t + B\sin\sqrt{\lambda_{1n}}t\right)\sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{\pi y}{3}\right)$$

$$u_{2n}(x,y,t) = \left(A\cos\sqrt{\lambda_{2n}}t + B\sin\sqrt{\lambda_{2n}}t\right)\sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{2\pi y}{3}\right)$$

$$u_{3n}(x,y,t) = \left(A\cos\sqrt{\lambda_{3n}}t + B\sin\sqrt{\lambda_{3n}}t\right)\sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{3\pi y}{3}\right)$$

Each one of these solutions can have linear combinations of sines and cosines and by the superposition principle we get,

$$u_{1n}(x,y,t) = \sum_{n=1}^{\infty}\left(A_n\cos\sqrt{\lambda_{1n}}t + B_n\sin\sqrt{\lambda_{1n}}t\right)\sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{\pi y}{3}\right)$$

$$u_{2n}(x,y,t) = \sum_{n=1}^{\infty}\left(A_n\cos\sqrt{\lambda_{2n}}t + B_n\sin\sqrt{\lambda_{2n}}t\right)\sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{2\pi y}{3}\right)$$

$$u_{3n}(x,y,t) = \sum_{n=1}^{\infty}\left(A_n\cos\sqrt{\lambda_{3n}}t + B_n\sin\sqrt{\lambda_{3n}}t\right)\sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{3\pi y}{3}\right)$$

But then the sum of each of these solutions is also a solution: $u_{1n} + u_{2n} + u_{3n}$ ... is a solution. Hence we arrive at the double summation,

$$u_{mn}(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(A_n\cos\sqrt{\lambda_{mn}}t + B_n\sin\sqrt{\lambda_{mn}}t\right)\sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{m\pi y}{3}\right)$$

Where,

$$\lambda_{mn} = \left(\frac{n\pi}{2}\right)^2 + \left(\frac{m\pi}{3}\right)^2$$

We will use the initial condition, $u(x,y,0) = 4\cos\pi x\cos 3\pi y + 4\sin 8\pi x\sin\pi y$ to find the coefficients.

$$u_{mn}(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(A_n\cdot 1 + B_n\cdot 0)\sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{m\pi y}{3}\right)$$

$$u_{mn}(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{m\pi y}{3}\right) = 4\cos\pi x\cos 3\pi y + 4\sin 8\pi x\sin\pi y$$

We will call the inner summation on 'n' S_n:

$$\sum_{m=1}^{\infty}S_n\sin\left(\frac{m\pi y}{3}\right) = 4\cos\pi x\cos 3\pi y + 4\sin 8\pi x\sin\pi y = f(x,y)$$

Thus,

$$S_n = \frac{2}{3}\int_0^3(4\cos\pi x\cos 3\pi y + 4\sin 8\pi x\sin\pi y)\sin\left(\frac{m\pi y}{3}\right)dy$$

But recall that,

$$S_n = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) = \frac{2}{3}\int_0^3 (4\cos\pi x\cos 3\pi y + 4\sin 8\pi x\sin\pi y)\sin\left(\frac{m\pi y}{3}\right) dy$$

Hence,

$$A_n = \frac{2}{2}\int_0^2 \frac{2}{3}\int_0^3 (4\cos\pi x\cos 3\pi y + 4\sin 8\pi x\sin\pi y)\sin\left(\frac{m\pi y}{3}\right) dy \sin\left(\frac{n\pi x}{2}\right) dx$$

Upon rearranging and simplifying we get,

$$A_n = \frac{2}{3}\int_0^2 \int_0^3 (4\cos\pi x\cos 3\pi y + 4\sin 8\pi x\sin\pi y)\sin\left(\frac{m\pi y}{3}\right)\sin\left(\frac{n\pi x}{2}\right) dy\, dx$$

Therefore,

$$A_{mn} = \frac{16mn(1+(-1)^m)(1-(-1)^n)}{(m^2-81)(n^2-4)\pi^2}$$

We will find the ‘B’ coefficients in a similar way except we will find the derivative of our solution,

$$\frac{\partial u_{mn}}{\partial t} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(-A_n\sqrt{\lambda_{mn}}\sin\sqrt{\lambda_{mn}}t + B_n\sqrt{\lambda_{mn}}\cos\sqrt{\lambda_{mn}}t\right)\sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{m\pi y}{3}\right)$$

If we let t = 0, we get,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(-A_n\sqrt{\lambda_{mn}}\cdot 0 + B_n\sqrt{\lambda_{mn}}\cdot 1\right)\sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{m\pi y}{3}\right)$$

Or simply,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_n\sqrt{\lambda_{mn}}\sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{m\pi y}{3}\right) = 0$$

We will follow the same procedure as before and let the inner sum be S_n,

$$\sum_{m=1}^{\infty} S_n \sin\left(\frac{m\pi y}{3}\right) = 0$$

Therefore,

$$S_n = \frac{2}{3}\int_0^3 0\sin\left(\frac{m\pi y}{3}\right) dy = 0$$

So $S_n = 0$; and,

$$S_n = \sum_{n=1}^{\infty} B_n\sqrt{\lambda_{mn}}\sin\left(\frac{n\pi x}{2}\right) = 0$$

Again we can find the ‘B’ coefficients using integrals,

$$B_n\sqrt{\lambda_{mn}} = \frac{2}{2}\int_0^2 0\sin\left(\frac{n\pi x}{2}\right) dx = 0$$

Thus, $B_{mn} = 0$. This means our solution depends entirely on the ‘A’ coefficients,

$$u_{mn}(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_n \cos\sqrt{\lambda_{mn}}t \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{m\pi y}{3}\right)$$

The complete solution is,

$$u_{mn}(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{16mn(1+(-1)^m)(1-(-1)^n)}{(m^2-81)(n^2-4)\pi^2} \cos\sqrt{\lambda_{mn}}t \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{m\pi y}{3}\right)$$