PROBLEM #4

The solution to the one dimensional wave equation with given boundary conditions and initial conditions:

PDE: $u_{tt} = c^2 u_{xx}$

BC: $u(0,t) = u(L,t) = 0$

IC: $u(x,0) = f(x), u_t(x,0) = g(x)$

Is given by:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)$$

Use trigonometric identities to show this solution can be expressed in the following way:

$$u(x,t) = R(x-ct) + S(x+ct)$$

Where *R* and *S* are functions of one variable.

Firs let us begin with the sum and difference identities for sine and cosine:

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

If we let, $\alpha = \frac{n\pi x}{L}$ and $\beta = \frac{n\pi ct}{L}$, the four equations above become:

1) $$\sin\left(\frac{n\pi x}{L} + \frac{n\pi ct}{L}\right) = \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) + \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)$$

2) $$\sin\left(\frac{n\pi x}{L} - \frac{n\pi ct}{L}\right) = \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) - \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)$$

3) $$\cos\left(\frac{n\pi x}{L} + \frac{n\pi ct}{L}\right) = \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) - \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)$$

4) $$\cos\left(\frac{n\pi x}{L} - \frac{n\pi ct}{L}\right) = \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) + \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)$$

Add lines 1) and 2) together to get:

$$\sin\left(\frac{n\pi x}{L} + \frac{n\pi ct}{L}\right) + \sin\left(\frac{n\pi x}{L} - \frac{n\pi ct}{L}\right) = 2\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

Take the line above and divide both sides by two and then multiply by A_n to get,

$$\frac{A_n}{2}\left[\sin\left(\frac{n\pi x}{L} + \frac{n\pi ct}{L}\right) + \sin\left(\frac{n\pi x}{L} - \frac{n\pi ct}{L}\right)\right] = A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

If we rearrange the terms to a format similar to the problem solution we get:

5)
$$A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) = \frac{A_n}{2}\left[\sin\left(\frac{n\pi}{L}(x+ct)\right) + \sin\left(\frac{n\pi}{L}(x-ct)\right)\right]$$

Take line 3) and subtract it from line 4) to obtain,

$$\cos\left(\frac{n\pi x}{L} - \frac{n\pi ct}{L}\right) - \cos\left(\frac{n\pi x}{L} + \frac{n\pi ct}{L}\right) = 2\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)$$

If we take the line above and multiply by B_n, divide by 2, and rearrange the terms we get,

6)
$$B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right) = \frac{B_n}{2}\left[\cos\left(\frac{n\pi}{L}(x-ct)\right) - \cos\left(\frac{n\pi}{L}(x+ct)\right)\right]$$

Add lines 5) and 6) to get,

$$A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right) =$$

$$\frac{A_n}{2}\left[\sin\left(\frac{n\pi}{L}(x+ct)\right) + \sin\left(\frac{n\pi}{L}(x-ct)\right)\right] + \frac{B_n}{2}\left[\cos\left(\frac{n\pi}{L}(x-ct)\right) - \cos\left(\frac{n\pi}{L}(x+ct)\right)\right]$$

Notice that the left hand of the equation equals the terms being summed in solution to the waved equation. If we groups terms differently we get,

$$= \frac{1}{2}\left[A_n \sin\left(\frac{n\pi}{L}(x+ct)\right) - B_n \cos\left(\frac{n\pi}{L}(x+ct)\right)\right] + \frac{1}{2}\left[A_n \sin\left(\frac{n\pi}{L}(x-ct)\right) + B_n \cos\left(\frac{n\pi}{L}(x-ct)\right)\right]$$

Let $X = x + ct$ and $\bar{X} = x - ct$ and $\omega = \frac{n\pi}{L}$, then the above line becomes:

7)
$$= \frac{1}{2}\left[A_n \sin(\omega X) - B_n \cos(\omega X)\right] + \frac{1}{2}\left[A_n \sin(\omega \bar{X}) + B_n \cos(\omega \bar{X})\right]$$

Let us consider the following, if you are given $A\sin(\omega t) + B\cos(\omega t)$ we can express as a single sinusoid:

$$A\sin(\omega t) + B\cos(\omega t) = R\cos(\omega t - \delta)$$

where,

$$R = \sqrt{A^2 + B^2}, \qquad \text{and} \qquad \delta = \tan^{-1}\left(\frac{B}{A}\right)$$

If we apply this to line 7) we get,

$$R_n = \sqrt{\left(\frac{A_n}{2}\right)^2 + \left(\frac{B_n}{2}\right)^2} = \frac{1}{2}\sqrt{A_n^2 + B_n^2}$$

For the first half of line 7) it can becomes,

$$\boxed{\frac{1}{2}[A_n \sin(\omega X) - B_n \cos(\omega X)] = R_n \cos(\omega X + \delta)}$$

Here we have +δ because, $\delta = \tan^{-1}\left(-\frac{B_n}{A_n}\right) = -\tan^{-1}\left(\frac{B_n}{A_n}\right)$. Repeat this for the second half of line 7) to get,

$$\boxed{\frac{1}{2}[A_n \sin(\omega \bar{X}) + B_n \cos(\omega \bar{X})] = R_n \cos(\omega \bar{X} - \delta)}$$

Substitute the boxed equations into line 7) we get,

8) $\frac{1}{2}[A_n \sin(\omega X) - B_n \cos(\omega X)] + \frac{1}{2}[A_n \sin(\omega \bar{X}) + B_n \cos(\omega \bar{X})] = R_n \cos(\omega X + \delta) + R_n \cos(\omega \bar{X} - \delta)$

Where,

$$R_n = \frac{1}{2}\sqrt{A_n^2 + B_n^2} \quad \text{and} \quad \delta = \tan^{-1}\left(\frac{B_n}{A_n}\right)$$

Recall, $X = x + ct$ and $\bar{X} = x - ct$ and $\omega = \frac{n\pi}{L}$, so line 8) becomes,

$$= R_n \cos\left(\frac{n\pi}{L}(x + ct) + \delta\right) + R_n \cos\left(\frac{n\pi}{L}(x - ct) - \delta\right)$$

From when we added lines 5) and 6) we can conclude;

$$A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) = R_n \cos\left(\frac{n\pi}{L}(x + ct) + \delta\right) + R_n \cos\left(\frac{n\pi}{L}(x - ct) - \delta\right)$$

Or simply put it,

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) \\ &= \sum_{n=1}^{\infty} R_n \cos\left(\frac{n\pi}{L}(x + ct) + \delta\right) + \sum_{n=1}^{\infty} R_n \cos\left(\frac{n\pi}{L}(x - ct) - \delta\right) \end{aligned}$$

The two summations are cosine series expansions acting on $(x + ct)$ and $(x - ct)$; therefore,

$$u(x,t) = S(x + ct) + R(x - ct)$$

Where,

$$S(x + ct) = \sum_{n=1}^{\infty} R_n \cos\left(\frac{n\pi}{L}(x + ct) + \delta\right) \quad \text{and} \quad R(x - ct) = \sum_{n=1}^{\infty} R_n \cos\left(\frac{n\pi}{L}(x - ct) - \delta\right)$$

And,

$$R_n = \frac{1}{2}\sqrt{A_n^2 + B_n^2} \quad \text{and} \quad \delta = \tan^{-1}\left(\frac{B_n}{A_n}\right)$$