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ENGINEERING ANALYSIS
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HOMEWORK #3 PROBLEM 5

Solve Laplace's equation: $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$

We will examine the general solution for the periodicity condition: $u(r, \pi) = u(r, -\pi)$. If we apply separation of variables, we will substitute $u(r, \theta) = \phi(\theta)G(r)$. Therefore,

- $u_{rr} = \phi(\theta)G''(r)$
- $u_r = \phi(\theta)G'(r)$
- $u_{\theta\theta} = \phi''(\theta)G(r)$

If we substitute into the differential equation we get,

$$\phi G'' + \frac{1}{r}\phi G' + \frac{1}{r^2}\phi'' G = 0$$

We factor and move terms to get,

$$\phi\left(G'' + \frac{1}{r}G'\right) = -\frac{1}{r^2}\phi'' G$$

By division we get,

$$\frac{G'' + \frac{1}{r}G'}{\frac{1}{r^2}G} = -\frac{\phi''}{\phi} = \lambda$$

We get two differential equations,

1. $-\frac{\phi''}{\phi} = \lambda \to \phi'' + \lambda\phi = 0$
2. $\frac{G'' + \frac{1}{r}G\prime}{\frac{1}{r^2}G} = \lambda \;\to\; r^2 G'' + rG' - \lambda G = 0$

The first differential equation has characteristic equation: $r^2 + \lambda = 0$ and has solutions $r_{1,2} = \pm i\sqrt{\lambda}$. Therefore, the solution will be of the form: $\phi(\theta) = A\cos\sqrt{\lambda}\theta + B\sin\sqrt{\lambda}\theta$. We will now utilize the periodicity condition $u(r, \pi) = u(r, -\pi) \to \phi(\pi) = \phi(-\pi)$.

$$\phi(\pi) = A\cos\sqrt{\lambda}\pi + B\sin\sqrt{\lambda}\pi = A\cos\sqrt{\lambda}\pi - B\sin\sqrt{\lambda}\pi = \phi(-\pi)$$

The cosine terms subtract out, but we can combine the sine terms to get,

$$2B\sin\sqrt{\lambda}\pi = 0$$

Again we arrive at that $\sqrt{\lambda}\pi = n\pi$; hence, $\lambda_n = n^2$ for n = 0,1,2,3,4… Our solution for ϕ will be,

$$\phi_n(\theta) = A\cos n\theta + B\sin n\theta$$

Let us now examine the second differential equation: $r^2G'' + rG' - \lambda G = 0$. But we determine λ and we substitute it in to get,

$$r^2G'' + rG' - n^2G = 0$$

We will let $G(r) = r^p$ and substitute to get,

$$r^2p(p-1)r^{p-2} + rpr^{p-1} - n^2r^p = 0$$

This equation can be simplified to,

$$p(p-1) + p - n^2 = 0$$
$$p^2 - p + p - n^2 = 0$$
$$p = \pm n$$

Therefore, one part of the solution to G(r) is: $G(r) = c_1r^n + c_2r^{-n}$

But let us examine the differential equation when n = 0, we get,

$$r^2G'' + rG' = 0 \rightarrow rG'' + G' = 0$$

The second equation is the same as (using product rule): $\frac{d}{dr}(rG') = 0$. If integrate this once we get,

$$rG' = a \rightarrow G' = \frac{a}{r}$$

Integrate a second time to get,

$$G(r) = b + a\ln r$$

So by superposition our final solution to G(r) is:

$$G_n(r) = c_1r^n + c_2r^{-n} + b + a\ln r$$

Thus we have established the following,

1. $\lambda_n = n^2$
2. $\phi_n(\theta) = A\cos n\theta + B\sin n\theta$
3. $G_n(r) = c_1r^n + c_2r^{-n} + b + a\ln r$

Part a) Inside the circular region 0 < r < 1 and 0 < θ <2π with boundary condition:

$$u(1,\theta) = 3 - \sin\theta + \frac{1}{2}\cos 4\theta$$

Notice that as r → 0 our function G(r) is unbounded; therefore, we will disregard the other 3 terms so that $|G(0)| < \infty$. Hence,

$$G_n(r) = c_1 r^n$$

Therefore, a solution to part A conditions inside a circular region is:

$$u_n(r,\theta) = (A\cos n\theta + B\sin n\theta)r^n$$

By superposition we have,

$$u(r,\theta) = \sum_{n=1}^{\infty}(A_n\cos n\theta + B_n\sin n\theta)r^n$$

Now we need to find the coefficients of this expansion, given the boundary condition:

$$u(1,\theta) = \sum_{n=0}^{\infty}(A_n\cos n\theta + B_n\sin n\theta) = 3 - \sin\theta + \frac{1}{2}\cos 4\theta = f(\theta)$$

Therefore we arrive at the following integrals:

$$A_n = \frac{2}{\pi}\int_0^{\pi}\left(3 - \sin\theta + \frac{1}{2}\cos 4\theta\right)\cos n\theta\, d\theta = \frac{2(1+(-1)^n)}{\pi(n^2-1)}$$

$$B_n = \frac{2}{\pi}\int_0^{\pi}\left(3 - \sin\theta + \frac{1}{2}\cos 4\theta\right)\sin n\theta\, d\theta = \frac{(1-(-1)^n)(7n^2-96)}{n\pi(n^2-16)}$$

Thus,

$$u(r,\theta) = \sum_{n=0}^{\infty}\left(\frac{2r^n(1+(-1)^n)}{\pi(n^2-1)}\cos n\theta + \frac{r^n(1-(-1)^n)(7n^2-96)}{n\pi(n^2-16)}\sin n\theta\right)$$

Part b) <u>**Outside**</u> the circular region r >1 and 0 < θ <2π with boundary condition:

$$u(1,\theta) = 3 - \sin\theta + \frac{1}{2}\cos 4\theta$$

Since we are working outside the circular region we need to consider all of the terms for G, since we no longer have to worry about singularities at the origin when r = 0,

$$G_n(r) = c_1 r^n + c_2 r^{-n} + b + a\ln r$$

Thus our solution becomes,

$$u(r,\theta) = \sum_{n=0}^{\infty}(A_n\cos n\theta + B_n\sin n\theta)(r^n + r^{-n} + \ln r)$$

Notice that,

$$u(1,\theta) = \sum_{n=0}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)(1^n + 1^{-n} + \ln 1)$$

$$u(1,\theta) = \sum_{n=0}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)(1 + 1 + 0)$$

$$u(1,\theta) = \sum_{n=0}^{\infty} (2A_n \cos n\theta + 2B_n \sin n\theta) = f(\theta) = 3 - \sin\theta + \frac{1}{2}\cos 4\theta$$

Thus we get,

$$2A_n = \frac{2}{\pi}\int_0^{\pi} \left(3 - \sin\theta + \frac{1}{2}\cos 4\theta\right) \cos n\theta \, d\theta = \frac{2(1 + (-1)^n)}{\pi(n^2 - 1)}$$

$$2B_n = \frac{2}{\pi}\int_0^{\pi} \left(3 - \sin\theta + \frac{1}{2}\cos 4\theta\right) \sin n\theta \, d\theta = \frac{(1 - (-1)^n)(7n^2 - 96)}{n\pi(n^2 - 16)}$$

Interestingly the only difference is a multiple of 2, hence,

$$A_n = \frac{(1 + (-1)^n)}{\pi(n^2 - 1)}$$

$$B_n = \frac{(1 - (-1)^n)(7n^2 - 96)}{2n\pi(n^2 - 16)}$$

Finally we arrive at,

$$u(r,\theta) = \sum_{n=0}^{\infty} \left(\frac{(1 + (-1)^n)}{\pi(n^2 - 1)}\cos n\theta + \frac{(1 - (-1)^n)(7n^2 - 96)}{2n\pi(n^2 - 16)}\sin n\theta\right)(r^n + r^{-n} + \ln r)$$