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ENGINEERING ANALYSIS
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Homework #3 Problem 3

Consider the initial boundary value problem for the two-dimensional wave equation:

- $u_{tt} = c^2\left(u_{xx} + u_{yy}\right)$ for $0 < x < L$ & $0 < y < H$ & $t > 0$
- $u(0, y, t) = u(L, y, t) = 0$ for $0 < y < H$ & $t > 0$
- $u(x, 0, t) = u(x, H, t) = 0$ for $0 < x < L$ & $t > 0$
- $u(x, y, 0) = f(x, y)$ for $0 < x < L$ & $0 < y < H$
- $u_t(x, y, 0) = g(x, y)$ for $0 < x < L$ & $0 < y < H$

We will make the following substitutions into the differential equation, using the separation of the variables:

$u(x, y, t) = \phi(x)\psi(y)h(t)$,

$$u_{tt} = \phi\psi h''$$

$$u_{xx} = \phi''\psi h$$

$$u_{yy} = \phi\psi'' h$$

This yields the following equation,

$$\phi\psi h'' = c^2(\phi''\psi h + \phi\psi'' h)$$

Factor out the 'h' term on the right side to get,

$$\phi\psi h'' = c^2 h(\phi''\psi + \phi\psi'')$$

Divide out both sides by common factors to get,

$$\frac{h''}{c^2 h} = \frac{\phi''\psi + \phi\psi''}{\phi\psi} = -\lambda$$

Therefore, we obtain the following differential equations:

- $h'' = -c^2\lambda h \rightarrow \; h'' + c^2\lambda h = 0$
- $\phi''\psi + \phi\psi'' = -\lambda\phi\psi \rightarrow \; \phi''\psi + \phi\psi'' + \lambda\phi\psi = 0$

Let us examine the second differential equation:

$$\phi''\psi + \phi\psi'' + \lambda\phi\psi = 0 \rightarrow \psi(\phi'' + \lambda\phi) + \phi\psi'' = 0$$

Now we can move terms to opposite sides of the equal sign and divide out common terms:

$$\phi\psi'' = -\psi(\phi'' + \lambda\phi) \rightarrow \; -\frac{\psi''}{\psi} = \frac{\phi'' + \lambda\phi}{\phi} = \mu$$

Now from the ψ & φ differential equation we obtain two more separate differential equations:

- $\psi'' + \mu\psi = 0$
- $\phi'' + (\lambda - \mu)\phi = 0$

Let us analyze the first differential equation that involves ψ, the characteristic equation is:

$$r^2 + \mu = 0 \rightarrow r_{1,2} = \pm i\sqrt{\mu}$$

So the solution to the ψ equation is of the form: $\psi(y) = A\cos\sqrt{\mu}y + B\sin\sqrt{\mu}y$. The boundary conditions have the following implication: $u(x, 0, t) = u(x, H, t) = 0 \ \rightarrow \ \psi(0) = \psi(H) = 0$. As from problem #2 we obtain a similar result for our function ψ: $\psi(0) = A = 0$; therefore, $\psi(y) = B\sin\sqrt{\mu}y$. From the second part of the boundary condition we get: $\psi(H) = B\sin\sqrt{\mu}H = 0$; therefore,

$$\sqrt{\mu}H = m\pi \rightarrow \mu_m = \left(\frac{m\pi}{H}\right)^2$$

And the corresponding function is:

$$\psi_m(y) = \sin\left(\frac{m\pi y}{H}\right)$$

Now let us consider the differential equation involving φ:

$$\phi'' + (\lambda - \mu)\phi = 0$$

The characteristic equation is,

$$r^2 + (\lambda - \mu) = 0 \rightarrow r_{1,2} = \pm i\sqrt{\lambda - \mu}$$

Again we will have solutions in the form, $\phi(x) = A\cos\sqrt{\lambda - \mu}\,x + B\sin\sqrt{\lambda - \mu}\,x$. The boundary conditions have the following implication: $u(0, y, t) = u(L, y, t) = 0 \rightarrow \phi(0) = \phi(L) = 0$. When x = 0 we get, $\phi(0) = A = 0$; therefore, $\phi(x) = B\sin\sqrt{\lambda - \mu}\,x$. If we consider the second boundary condition when x = L we get, $\phi(L) = B\sin\sqrt{\lambda - \mu}\,L = 0$. Hence,

$$\sin\sqrt{\lambda - \mu}\,L = n\pi \ \rightarrow \ \lambda = \left(\frac{n\pi}{L}\right)^2 + \mu$$

But previously we found μ, if we substitute we get,

$$\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

The corresponding functions we be the following,

$$\phi(x) = \sin\sqrt{\lambda - \mu}\,x = \sin\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 - \left(\frac{m\pi}{H}\right)^2}\,x = \sin\left(\frac{n\pi x}{L}\right)$$

Therefore,

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

Thus we have established the following:

- $\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$
- $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$
- $\psi_m(y) = \sin\left(\frac{m\pi y}{H}\right)$

Let us now examine the temporal equation: $h'' + c^2\lambda h = 0$. It has the following characteristic equation,

$$r^2 + c^2\lambda = 0 \rightarrow r_{1,2} = \pm ic\sqrt{\lambda}$$

So our solution to the temporal equation will be of the form, recall we have found λ_{mn},

$$h_{mn}(t) = A\cos\left(c\sqrt{\lambda_{mn}}t\right) + B\sin\left(c\sqrt{\lambda_{mn}}t\right)$$

Therefore the solution to our wave equation will be of the form,

$$u_{mn}(x,y,t) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\left(A\cos\left(c\sqrt{\lambda_{mn}}t\right) + B\sin\left(c\sqrt{\lambda_{mn}}t\right)\right)$$

If we let m = 1,2,3 then by superposition our solutions we be of the form,

$$u_{1n}(x,y,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{\pi y}{H}\right)\left(A\cos\left(c\sqrt{\lambda_{1n}}t\right) + B\sin\left(c\sqrt{\lambda_{1n}}t\right)\right)$$

$$u_{2n}(x,y,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{2\pi y}{H}\right)\left(A\cos\left(c\sqrt{\lambda_{2n}}t\right) + B\sin\left(c\sqrt{\lambda_{2n}}t\right)\right)$$

$$u_{3n}(x,y,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{3\pi y}{H}\right)\left(A\cos\left(c\sqrt{\lambda_{3n}}t\right) + B\sin\left(c\sqrt{\lambda_{3n}}t\right)\right)$$

Since each of these are solutions to the wave equation, then their sum is also a solution: $u_{1n} + u_{2n} + u_{3n}$, therefore,

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\left(A_{mn}\cos\left(c\sqrt{\lambda_{mn}}t\right) + B_{mn}\sin\left(c\sqrt{\lambda_{mn}}t\right)\right)$$

Let us now use the first initial condition $u(x,y,0) = f(x,y)$ to get,

$$u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right) = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}A_{mn}\sin\left(\frac{n\pi x}{L}\right)\right)\sin\left(\frac{m\pi y}{H}\right) = f(x,y)$$

We will denote the inner summation as S_m:

$$\sum_{m=1}^{\infty}S_m\sin\left(\frac{m\pi y}{H}\right) = f(x,y)$$

Notice that is a Fourier Sine series of f(x,y) and we can determine the coefficient S_m by the following:

$$S_m = \frac{2}{H}\int_0^H f(x,y)\sin\left(\frac{m\pi y}{H}\right)dy$$

Recall, that we denoted S_m the following,

$$S_m = \sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{n\pi x}{L}\right)$$

Set both of these equalities equal to each other to get,

$$\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{n\pi x}{L}\right) = \frac{2}{H}\int_0^H f(x,y)\sin\left(\frac{m\pi y}{H}\right)dy$$

The expression on the right is a function on 'x' and the left is the Fourier Sine series of it; therefore, we can find an expression for the coefficients A_{mn} by,

$$A_{mn} = \frac{2}{L}\int_0^L \frac{2}{H}\int_0^H f(x,y)\sin\left(\frac{m\pi y}{H}\right)dy\,\sin\left(\frac{n\pi x}{L}\right)dx$$

If we rearrange terms we get,

$$A_{mn} = \frac{2}{L}\frac{2}{H}\int_0^L\int_0^H f(x,y)\sin\left(\frac{m\pi y}{H}\right)\sin\left(\frac{n\pi x}{L}\right)dy\,dx$$

We will now use the second initial condition to find the 'B' coefficients: $u_t(x,y,0) = g(x,y)$. First let us differentiate the function u(x,y,t) with respect to 't.'

$$u_t(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\left(-A_{mn}c\sqrt{\lambda_{mn}}\sin\left(c\sqrt{\lambda_{mn}}t\right) + B_{mn}c\sqrt{\lambda_{mn}}\cos\left(c\sqrt{\lambda_{mn}}t\right)\right)$$

If we substitute t = 0 we get,

$$u_t(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}c\sqrt{\lambda_{mn}}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right) = g(x,y)$$

We will follow the same steps as we did for the first initial condition,

$$\sum_{m=1}^{\infty} S_m\sin\left(\frac{m\pi y}{H}\right) = g(x,y)$$

Therefore,

$$S_m = \frac{2}{H}\int_0^H g(x,y)\sin\left(\frac{m\pi y}{H}\right)dy$$

But here,

$$S_m = \sum_{n=1}^{\infty} B_{mn}c\sqrt{\lambda_{mn}}\sin\left(\frac{n\pi x}{L}\right)$$

Equate the two expressions for S_m and we obtain,

$$\sum_{n=1}^{\infty} B_{mn} c\sqrt{\lambda_{mn}} \sin\left(\frac{n\pi x}{L}\right) = \frac{2}{H}\int_0^H g(x,y)\sin\left(\frac{m\pi y}{H}\right) dy$$

Therefore,

$$B_{mn} c\sqrt{\lambda_{mn}} = \frac{2}{L}\int_0^L \frac{2}{H}\int_0^H g(x,y)\sin\left(\frac{m\pi y}{H}\right) dy \sin\left(\frac{n\pi x}{L}\right) dx$$

If we rearrange the terms we get,

$$B_{mn} = \frac{4}{LHc\sqrt{\lambda_{mn}}}\int_0^L \int_0^H g(x,y)\sin\left(\frac{m\pi y}{H}\right)\sin\left(\frac{n\pi x}{L}\right) dy\, dx$$

We have established the solution to the wave equation with the given B.C. and I.C.:

- $u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\left(A_{mn}\cos\left(c\sqrt{\lambda_{mn}}t\right) + B_{mn}\sin\left(c\sqrt{\lambda_{mn}}t\right)\right)$

- $B_{mn} = \frac{4}{LHc\sqrt{\lambda_{mn}}}\int_0^L \int_0^H g(x,y)\sin\left(\frac{m\pi y}{H}\right)\sin\left(\frac{n\pi x}{L}\right) dy\, dx$

- $A_{mn} = \frac{2}{L}\frac{2}{H}\int_0^L \int_0^H f(x,y)\sin\left(\frac{m\pi y}{H}\right)\sin\left(\frac{n\pi x}{L}\right) dy\, dx$

- $\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$