Homework 1

1) Determine the equilibrium temperature distribution for a one- dimensional rod with constant thermal properties with the following sources and boundary conditions:

$$(a)\quad Q=0,\ u(0)=0,\ u(L)=T \qquad (b)\quad Q=0,\ u_x(0)=0,\ u(L)=T$$
$$(c)\quad Q=x^2,\ u(0)=T, u_x(L)=0 \qquad (d)\quad Q=0,\ u_x(0)-u(0)=-T,\ u_x(L)+u(L)=0$$

2) For the following problems, determine an equilibrium temperature distribution (if one exists). For what values of β are there solutions? Explain physically.

$$(a)\qquad u_t=u_{xx}+1,\qquad u(x,0)=f(x),\quad u_x(0,t)=1,\quad u_x(L,t)=\beta$$
$$(b)\qquad u_t=u_{xx}+x-\beta,\quad u(x,0)=f(x),\quad u_x(0,t)=0,\quad u_x(L,t)=0$$

3) For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

$$(a)\qquad \frac{\partial u}{\partial t}=\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\qquad\qquad (b)\qquad \frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}-m\frac{\partial u}{\partial x}.$$

4) Consider the differential equation

$$\frac{d^2\phi}{dx^2}+\lambda\phi=0.$$

Determine the eigenvalues λ (and corresponding eigenfunctions) if f satisfies the following boundary conditions. Analyze three cases ($\lambda>0,\ \lambda=0,\ \lambda<0$). You may assume that the eigenvalues are real.

$$(a)\qquad \phi(0)=0,\ \ \phi(L)=0 \qquad\qquad (b)\quad \phi'(0)=0,\ \ \phi'(L)=0$$

$$(c)\qquad \phi(0)=0,\ \ \phi'(L)=0 \qquad\qquad (d)\quad \phi'(0)=0,\ \ \phi(L)=0.$$

5) Consider the diffusion equation with homogeneous Dirichlet-Neumann boundary conditions:

$$\begin{aligned} u_t &= ku_{xx}, & 0 < x < L, \;\; t > 0, \\ u(0,t) &= u_x(L,t) = 0, & t > 0, \\ u(x,0) &= f(x), & 0 < x < L. \end{aligned}$$

(a) Find the eigenvalues and eigenfunctions for the spatial problem.

(b) For the eigenvalues in (a), solve the temporal problem.

(c) Use the Superposition Principle to obtain the general solution of the given initial-boundary value problem as an infinite series for each of the following initial temperature $u(x,0) = f(x)$:

$$(i) \quad f(x) = 2\cos\left(\frac{3\pi x}{L}\right) \qquad (ii) \quad f(x) = \begin{cases} 1 & 0 < x < L/2 \\ 2 & L/2 < x < L. \end{cases}$$

6) Solve the heat equation

$$\begin{aligned} u_t &= ku_{xx}, & 0 < x < L, \;\; t > 0, \\ u_x(0,t) &= u_x(L,t) = 0, & t > 0, \\ u(x,0) &= f(x), & 0 < x < L, \end{aligned}$$

where

$$(i) \quad f(x) = 6 + 4\cos\left(\frac{5\pi x}{L}\right) \qquad (ii) \quad f(x) = \begin{cases} 0 & 0 < x < L/2 \\ 1 & L/2 < x < L. \end{cases}$$

7) For the following functions, sketch the Fourier series of $f(x)$ (on the interval $-L \le x \le L$). Compare $f(x)$ to its Fourier series:

$(a)\;\; f(x) = 1, \qquad (b)\;\; f(x) = x^2, \qquad (c)\;\; f(x) = \begin{cases} x & -L < x < 0 \\ 2x & 0 < x < L, \end{cases} \qquad (d)\;\; f(x) = \begin{cases} 0 & -L < x < 0 \\ 1 & 0 < x < L. \end{cases}$

8) For the following functions, sketch $f(x)$, the Fourier series of $f(x)$, the Fourier sine series of $f(x)$, and the Fourier cosine series of $f(x)$:

$(a)\;\; f(x) = 1, \qquad (b)\;\; f(x) = 1+x, \qquad (c)\;\; f(x) = \begin{cases} x & -L < x < 0 \\ 1+x & 0 < x < L, \end{cases} \qquad (d)\;\; f(x) = \begin{cases} 0 & -L < x < 0 \\ 1 & 0 < x < L. \end{cases}$

9) a) Sketch the Fourier cosine series of $f(x) = \sin\left(\frac{\pi x}{L}\right)$.

b) Sketch the Fourier sine series of $f(x) = \cos\left(\frac{\pi x}{L}\right)$.