Jean le Rond D'Alembert's solution to the wave equation.

Given the wave equation: $u_{tt} = c^2 u_{xx}$ we can express the following way,

$$u_{tt} - c^2 u_{xx} = 0$$

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Notice that the above line can be treated as a difference of squares. Recall that a difference of squares can be written as:

$$x^2 - y^2 = (x - y)(x + y)$$

Therefore,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0$$

Or,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right) = 0$$

Let us now make the following substitutions into the equations above.

$$\boxed{v = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}}$$

$$\boxed{w = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}}$$

We obtain the following,

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) w = 0$$

Distribute the *v* and *w* and you will get the following differential equations:

$$\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0 \rightarrow v_t - cv_x = 0$$

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0 \rightarrow w_t + cw_x = 0$$

Remember that the solution to the differential equation of the form $au_x + bu_y = 0$ is $u(x, y) = f(-ay + bx)$. If we apply this to the two differential equations above we get,

$$v(x, t) = \phi(-x - ct)$$

$$w(x, t) = \psi(-x + ct)$$

If we add the two boxed equations describing *v* and *w* we get:

$$v + w = 2u_t = \phi(-x - ct) + \psi(-x + ct)$$

Therefore,

$$\frac{\partial u}{\partial t} = \frac{1}{2}(\phi(-x - ct) + \psi(-x + ct))$$

If we integrate both sides of the above equation with respect to *t* we get,

$$u(x,t) = \frac{1}{2}\int \phi(-x-ct)\, dt + \frac{1}{2}\int \psi(-x+ct)\, dt$$

If we let $z = x + ct$ and $r = x - ct$, then $dz = c\, dt$ and $dr = -c\, dt$. Utilize these substitutions in the integrals above to obtain,

$$u(x,t) = \frac{1}{2c}\int \phi(-z)\, dz + \frac{-1}{2c}\int \psi(-r)\, dr$$

$$= \Phi(z) + \Psi(r)$$

So we get,

$$u(x,t) = F(z) + G(r) = F(x+ct) + G(x-ct)$$

Where *F* and *G* can be any functions of one variable. Now suppose we have our initial conditions, how can we use those initial conditions to find these functions *F* and *G*?

$$u(x,0) = f(x)$$

$$u_t(x,0) = g(x)$$

Recall,

$$u(x,t) = F(x+ct) + G(x-ct)$$

$$u_t(x,t) = cF'(x+ct) - cG'(x-ct)$$

Let $t = 0$ to get,

$$u(x,0) = F(x) + G(x) = f(x)$$

$$\boxed{u_t(x,0) = cF'(x) - cG'(x) = g(x)}$$

Let us find $u_x(x,0)$,

$$u_x(x,0) = F'(x) + G'(x)$$

Multiply by *c* on both sides of the equation above and get (recall $u(x,0) = f(x) \rightarrow u_x(x,0) = f'(x)$),

$$\boxed{cu_x(x,0) = cF'(x) + cG'(x) = cf'(x)}$$

Add the two boxed equations to get,

$$u_t(x,0) + cu_x(x,0) = 2cF'(x) = g(x) + cf'(x)$$

Isolate $F'(x)$,

$$F'(x) = \frac{1}{2}f'(x) + \frac{1}{2c}g(x)$$

Remember the Fundamental Theorem of Calculus

$$F(x) = \int_0^x F'(s)\, ds = \int_0^x \frac{1}{2}f'(s)\, ds + \int_0^x \frac{1}{2c}g(s)\, ds$$

Hence,

$$\boxed{F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\, ds}$$

Remember that $u(x,0) = F(x) + G(x) = f(x)$ so,

$$G(x) = f(x) - F(x)$$

But we just found $F(x)$, we will substitute this into $G(x)$ to get,

$$G(x) = f(x) - \left(\frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\,ds\right)$$

$$= f(x) - \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds$$

$$G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds$$

So now we have found two equations

1. $F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\,ds$

2. $G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds$

Now let us use the above equations in our definition of $u(x,t) = F(x+ct) + G(x-ct)$.

$$u(x,t) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds + \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)\,ds$$

$$= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

Therefore given any two forcing functions *f* and *g* we can find a solution to the wave equation the following way,

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$