ASSIGNMENT #3 Problem 1
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Consider the boundary value problem:

$$\boldsymbol{u_{xx} + u_{yy} = 0} \qquad \boldsymbol{x \in (0, L)} \textbf{ and } \boldsymbol{y \in (0, H)}$$
$$\boldsymbol{u(0, y) = u(L, y) = 0} \qquad \textbf{for } \boldsymbol{y \in (0, H)}$$
$$\boldsymbol{u(x, 0) = f(x)} \qquad \textbf{for } \boldsymbol{x \in (0, L)}$$
$$\boldsymbol{u(x, H) = g(x)} \qquad \textbf{for } \boldsymbol{x \in (0, L)}$$

a) Let $u(x, y) = \phi(x)\psi(y)$, show that φ and ψ satisfy the differential equations:

- $\phi''(x) + \lambda\phi(x) = 0$ and $\phi(0) = \phi(L) = 0$
- $\psi''(y) - \lambda\psi(y) = 0$

We will make the substitution $u(x, y) = \phi(x)\psi(y)$ into our homogeneous solution. The second derivatives are, $u_{xx} = \phi''(x)\psi(y)$ and $u_{yy} = \phi(x)\psi''(y)$.

$$u_{xx} + u_{yy} = 0 \rightarrow \phi''(x)\psi(y) + \phi(x)\psi''(y) = 0$$

Bring terms to either side of the equal sign to get,

$$\phi''(x)\psi(y) = -\phi(x)\psi''(y)$$

Isolate the x and y functions by division to get,

$$\frac{\phi''(x)}{\phi(x)} = -\frac{\psi''(y)}{\psi(y)} = -\lambda$$

So we get two second order differential equations:

- $\phi''(x) = -\lambda\phi(x) \rightarrow \phi''(x) + \lambda\phi(x) = 0$
- $\psi''(y) = \lambda\psi(y) \rightarrow \psi''(y) - \lambda\psi(y) = 0$

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b) Show that $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$, and $\psi_n(y) = c_n \cosh\left(\frac{n\pi y}{L}\right) + d_n \sinh\left(\frac{n\pi y}{L}\right)$.

Let us examine the first 2nd ODE from part a):

$$\phi''(x) + \lambda\phi(x) = 0$$

If we let $\phi(x) = e^{rx}$, then $\phi''(x) = r^2 e^{rx}$, substitute this back into our differential equation to get,

$$r^2 + \lambda = 0$$

This will yield imaginary solutions for λ > 0: $r_{1,2} = \pm\sqrt{\lambda}$. Therefore, our solution to the φ equation will be of the form:

$$\phi(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

Now we will use the boundary conditions: $u(0, y) = u(L, y) = 0$ for $y \in (0, H) \to \phi(0) = \phi(L) = 0$
If we substitute 0 into our φ function we get:

$$\phi(0) = A \cdot 1 + B \cdot 0 = 0 \to A = 0$$

So our solution will be of the form:

$$\phi(x) = B \sin \sqrt{\lambda} x$$

If we utilize the second boundary condition we get,

$$\phi(L) = B \sin \sqrt{\lambda} L = 0 \to \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

So our eigen-functions are: $\phi_n(x) = B \sin \frac{n\pi x}{L}$.

Hence,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \text{ and } \phi_n(x) = B \sin \frac{n\pi x}{L}$$

Now let us consider the second differential equation:

$$\psi''(y) - \lambda\psi(y) = 0$$

If we make a similar substitution $\psi(y) = e^{ry}$ we get,

$$r^2 - \lambda = 0 \to r_{1,2} = \pm\sqrt{\lambda}$$

Therefore, our solutions will be of the form: $\psi_1 = e^{\sqrt{\lambda}y}$ and $\psi_2 = e^{-\sqrt{\lambda}y}$. But recall the following identities for hyperbolic functions:

$$\sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2}$$

If we add the above two equations and subtract we get,

$$e^x = \sinh x + \cosh x$$
$$e^{-x} = -\sinh x + \cosh x$$

Therefore, our solutions to ψ are:

$$\psi_1 = \sinh \sqrt{\lambda} y + \cosh \sqrt{\lambda} y$$
$$\psi_2 = -\sinh \sqrt{\lambda} y + \cosh \sqrt{\lambda} y$$

But remember linear combinations of our solutions will work as well so,

$$\psi(y) = c \cosh \sqrt{\lambda} y + d \sinh \sqrt{\lambda} y$$

We already know λ:

$$\psi_n(y) = c_n \cosh\frac{n\pi y}{L} + d_n \sinh\frac{n\pi y}{L}$$

So our solution for a particular *n* is:

$$u_n(x,y) = \phi_n(x)\psi_n(y) = \left(c_n \cosh\frac{n\pi y}{L} + d_n \sinh\frac{n\pi y}{L}\right)\sin\frac{n\pi x}{L}$$

By superposition we get,

$$u(x,y) = \sum_{n=1}^{\infty}\left(c_n \cosh\frac{n\pi y}{L} + d_n \sinh\frac{n\pi y}{L}\right)\sin\frac{n\pi x}{L}$$

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c) We will now find the coefficients for c_n and d_n. We will begin with the first initial condition:

$$u(x,0) = f(x)$$

We get,

$$u(x,0) = \sum_{n=1}^{\infty}(c_n \cosh 0 + d_n \sinh 0)\sin\frac{n\pi x}{L} = f(x)$$

$$u(x,0) = \sum_{n=1}^{\infty}(c_n \cdot 1 + d_n \cdot 0)\sin\frac{n\pi x}{L} = f(x)$$

Therefore we have the following series:

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}$$

Hence,

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad \text{for } n = 1,2,3,4\ldots$$

Next we will let y = h into our function u(x,y) to help obtain the 'd' coefficients:

$$u(x,H) = \sum_{n=1}^{\infty}\left(c_n \cosh\frac{n\pi H}{L} + d_n \sinh\frac{n\pi H}{L}\right)\sin\frac{n\pi x}{L} = g(x)$$

Notice that the entire term in parentheses is the coefficient for the Fourier Sine series expansion of g(x):

$$c_n \cosh\frac{n\pi H}{L} + d_n \sinh\frac{n\pi H}{L} = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

If we subtract and then divide we can isolate the 'd' coefficient:

$$d_n \sinh\frac{n\pi H}{L} = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx - c_n \cosh\frac{n\pi H}{L}$$

This becomes,

$$d_n = \frac{1}{\sinh \frac{n\pi H}{L}} \left(\frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx - c_n \cosh \frac{n\pi H}{L} \right)$$

for n = 1,2,3,4…

d) See mathematica File