Homework 3

1. Consider the boundary value problem

$$\begin{array}{ll} u_{xx}+u_{yy}=0, & 0<x<L,\quad 0<y<H, \\ u(0,y)=0, & 0<y<H, \\ u(L,y)=0, & 0<y<H, \\ u(x,0)=f(x), & 0<x<L, \\ u(x,H)=g(x), & 0<x<L. \end{array}$$

a) Letting $u(x,y)=\phi(x)\psi(y)$, show that ϕ and ψ satisfy the differential equations

$$\phi''(x)+\lambda\phi(x)=0, \quad \phi(0)=0 \ \ \phi(L)=0$$

and

$$\psi''(y)-\lambda\psi(y)=0.$$

b) Show that

$$\lambda_n=\left(\frac{n\pi}{L}\right)^2 \quad \textit{and} \quad \phi_n(x)=\sin\left(\frac{n\pi x}{L}\right)$$

and

$$\psi_n(y)=c_n\cosh\left(\frac{n\pi y}{L}\right)+d_n\sinh\left(\frac{n\pi y}{L}\right).$$

c) We then use the superposition principle to get the solution

$$u(x,y)=\sum_{n=1}^{\infty}\left[c_n\cosh\left(\frac{n\pi y}{L}\right)+d_n\sinh\left(\frac{n\pi y}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right).$$

Use the initial initial conditions $u(x,0)=f(x)$ to show that

$$c_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \qquad n=1,2,\ \cdots$$

Next use $u(x,H)=g(x)$ and the c_n as above to show that

$$d_n=\frac{1}{\sinh\left(\frac{n\pi H}{L}\right)}\left(\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \ - \ c_n\cosh\left(\frac{n\pi H}{L}\right)\right), \quad n=1,\ 2,\ \cdots$$

d) Use $L=1,\ H=2,\ f(x)=x(1-x),\ g(x)=100$ to plot the solution surface using the first 10 terms of the Fourier series solution.

2. Consider the initial-boundary value problem for the 2D heat equation:

$$\begin{array}{ll} u_t=k\left(u_{xx}+u_{yy}\right), & 0<x<L,\quad 0<y<H,\quad t>0, \\ u(0,y,t)=u(L,y,t)=0, & 0<y<H,\quad t>0, \\ u(x,0,t)=u(x,H,t)=0, & 0<x<L,\quad t>0, \\ u(x,y,0)=f(x,y), & 0<x<L,\quad 0<y<H. \end{array}$$

a) Use the separation of variables $u(x,y,t)=\phi(x)\psi(y)h(t)$ and the boundary conditions to prove that

$$\phi_n(x)=\sin\left(\frac{n\pi x}{L}\right), \qquad \psi_m(x)=\sin\left(\frac{m\pi y}{H}\right), \qquad h_{mn}=e^{-\lambda_{mn}kt},$$

where

$$\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{n\pi}{H}\right)^2.$$

b) Use the initial condition $u(x, y, 0) = f(x, y)$ and the superposition principle to show that

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) e^{-kt\lambda_{mn}},$$

where

$$c_{mn} = \frac{2}{L}\frac{2}{H} \int_0^H \int_0^L f(x, y) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) dxdy.$$

c) Suppose $k = 0.5$, $L = H = 10$, and

$$f(x, y) = \begin{cases} 1, & 4 < x < 6, \quad 4 < y < 6, \\ 0, & \text{otherwise.} \end{cases}$$

Using the first 10 terms of the series solution, animate the dynamics by plotting the solution surface for

$$t = 0,\ 0.1,\ 0.2,\ 1,\ 10,\ 25,\ 50,\ 100,\ 500.$$

3. Consider the initial-boundary value problem for the 2D wave equation:

$$\begin{array}{ll} u_{tt} = c^2\left(u_{xx} + u_{yy}\right), & 0 < x < L, \quad 0 < y < H, \quad t > 0, \\ u(0, y, t) = u(L, y, t) = 0, & 0 < y < H, \quad t > 0, \\ u(x, 0, t) = u(x, H, t) = 0, & 0 < x < L, \quad t > 0, \\ u(x, y, 0) = f(x, y), & 0 < x < L, \quad 0 < y < H. \\ u_t(x, y, 0) = g(x, y), & 0 < x < L, \quad 0 < y < H. \end{array}$$

a) Use the separation of variables and the superposition principle as in Problem 5 above to find a double Fourier series solution of the 2D wave equation.

b) Suppose

$$c = 0.5, \quad L = H = 10, \quad f(x, y) = 10x(10 - x)(10 - y)\cos x \cos y, \quad g(x, y) = -1.$$

Using the first 5 terms of the series solution, animate the dynamics by plotting the solution surface for t values between $t = 0$ and $t = 3.6$ in increments of 0.3.

4. a) Solve the boundary value problem on the wedge

$$\begin{array}{ll} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, & 0 < r < \rho, \;\; 0 < \theta < \theta_0, \\ u(r,0) = 0, & 0 < r < \rho, \\ u(r,\theta_0) = 0, & 0 < r < \rho, \\ u(\rho,\theta) = f(\theta), & 0 < \theta < \theta_0 \end{array} .$$

(b) Suppose $\rho = 1$, $\theta_0 = \pi/3$, and $f(\theta) = 5\theta e^{-2\theta}$. Plot the solution surface and polar contour plot for using the first 10 terms.

5. Solve Laplace's equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_\theta = 0.$$

a) Inside a circular region ($0 < r < 1 \quad 0 < \theta < 2\pi$) subject to the following boundary conditions: BC: $u(1,\theta) = 3 - \sin\theta + \frac{1}{2}\cos 4\theta$.

b) Outside a circular region ($r > 1 \;\; 0 < \theta < 2\pi$) (think of there being an infinitely large metal surface with a hole of radius greater than 1) subject to the following boundary conditions: BC: $u(1,\theta) = 3 - \sin\theta + \frac{1}{2}\cos 4\theta$.

(Note: you may assume that the temperature stays bounded as $r \to \infty$.)

6. Solve the wave equation

$$u_{tt} = u_{xx} + u_{yy}, \quad 0 < x < 2, \quad 0 < y < 3,$$

for a vibrating rectangular membrane subject to the following boundary and initial conditions:

BC: $u(0,y,t) = 0$; $u(2,y,t) = 0$, $u(x,0,t) = 0$, $u(x,3,t) = 0$.

IC: $u(x,y,0) = 4\cos\pi x \cos\pi 3\pi y + 4\sin 8\pi x \sin\pi y, \quad u_t(x,y,0) = 0.$

7. Solve the one-dimensional heat equation below with a time-dependent heat source and non-homogeneous boundary conditions:

$$\begin{array}{ll} u_t = u_{xx} + Q(x,t) & \\ BC: & u(0,t) = 1, \quad u(\pi,t) = 0 \\ IC: & u(x,0) = x(2\pi - x), \end{array}$$

where

a) $Q(x,t) = e^{-t}\sin 3x - e^{-2t}\sin 4t$ b) $Q(x,t) = e^{-3t}\sin 5x$.