Homework 2

1) Consider the one dimensional heat equation with homogeneous Dirichlet conditions and initial condition:

$$\text{PDE: } u_t = ku_{xx}, \qquad \text{BC: } u(0,t) = u(L,t) = 0, \qquad \text{IC: } u(x,0) = f(x)$$

(a) Suppose $k = 0.2, L = 1$, and $f(x) = 180x(1-x)^2$. Using the first 10 terms in the series, plot the solution surface and enough time snapshots to display the dynamics of the solution

(b) What happens to the solution as $t \to \infty$? Explain your answer in light of (a) and the physical interpretation of the problem. Does (b) reflect this?

(c) Redo parts (b) and (c) for $k = 01, L = \pi$, and $f(x) = 2\cos 3x$

2) Consider the one dimensional wave equation with boundary conditions and initial conditions:

$$\text{PDE: } u_{tt} = c^2 u_{xx}, \qquad \text{BC: } u(0,t) = u(L,t) = 0, \qquad \text{IC: } u(x,0) = f(x) \quad u_t(x,0) = g(x).$$

(a) Suppose $c = 1, L = 1$, $f(x) = 180x^2(1-x)$, and $g(x) = 0$. Using the first 10 terms in the series, plot the solution surface and enough time snapshots to display the dynamics of the solution

(b) What happens to the solution as $t \to \infty$? Explain your answer in light of (a) and the physical interpretation of the problem. Does (b) reflect this?

(c) Redo parts (b) and (c) for $c = 1, L = \pi$, $f(x) = 2\sin 3x$, and $g(x) = 1 - x$

3) (a) Show that $u(x,t) = (x+t)^3$ is a solution of the wave equation $u_{tt} = u_x x$.

(b) What initial condition does u satisfy?

(c) Plot the solution surface.

(d) Using (c), discuss the difference between the conditions $u(x,0)$ and $u(O,t)$.

4) In class (see page 138 of your text), we have proved that the solution of the one-dimensional wave equation

$$\text{PDE: } u_{tt} = c^2 u_{xx}, \qquad \text{BC: } u(0,t) = u(L,t) = 0, \qquad \text{IC: } u(x,0) = f(x) \quad u_t(x,0) = g(x)$$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right) \right).$$

Use trigonometric identities to show that this can be expressed as

$$u(x,y) = R(x-ct) + S(x+ct),$$

where R and S are functions of one variable.

5) Consider a slightly damped vibrating string that satisfies

$$\text{PDE: } u_{tt} = c^2 u_{xx} - \beta u_t, \qquad \text{BC: } u(0,t) = u(L,t) = 0, \qquad \text{IC: } u(x,0) = f(x) \quad u_t(x,0) = g(x).$$

Use separation of variables to determine the solutions. You may assume that $\beta < \frac{2\pi c}{L}$.

6) Write a computer program that numerically computes solutions of the heat equation with initial temperature given in Fig 6.3.4 (page 228 of your text). We may assume $k = 1$ and $L = 1$. Use your program to compute solution out to 10 steps in time, i.e. $m = 0, 1, \cdots 10$ for different values of s:

$$(a) \quad s = 0.49 \qquad (b) \quad s = .50 \qquad (c) \quad s = .52.$$

Discuss stability based on these values of s.

7) Write a computer program that numerically computes solutions of the wave equation with

$$\text{BC: } u(0,t) = u(1,t) = 0, \qquad \text{IC: } u(x,0) = \begin{cases} 1 & \frac{1}{4} < x < \frac{3}{4} \\ 0 & \text{else} \end{cases} \qquad u_t(x,0) = 0.$$

We may assume $c = 1$. Use your program to compute solution out to 10 steps in time, i.e. $m = 0, 1, \cdots 10$ for different values of Δx and Δt:

$$(a) \quad \Delta t = \frac{\Delta x}{2} \qquad (b) \quad \Delta t = \Delta x \qquad (c) \quad \Delta t = 2\Delta x.$$