NUMERICAL APPROXIMATIONS
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Let us begin with the following two polynomial approximations to a function $f(x)$:

1. $f(x) \approx p_1(x) = f(x_0) + f'(x_0)(x - x_0)$
2. $f(x) \approx p_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$

If we let $x - x_0 = h$ then $x = x_0 + h$ and substitute into the two equations above, keep in mind that these approximations are fairly accurate when x_0 is close to x.

3. $f(x_0 + h) \approx p_1(x_0 + h) = f(x_0) + f'(x_0)h$
4. $f(x_0 + h) \approx p_2(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2$

We can calculate the remainders and errors with these approximations using Taylors Theorem, where R_1, E_1 and R_2, E_2 correspond to the remainders and errors of the approximations.

$$R_1 = \frac{(x - x_0)^2}{2!} f''(c_x) = \frac{h^2}{2!} f''(c_x) \Rightarrow E_1 = \frac{h^2}{2!} M_1$$
$$R_2 = \frac{(x - x_0)^3}{3!} f''(c_x) = \frac{h^3}{3!} f''(c_x) \Rightarrow E_2 = \frac{h^3}{3!} M_2$$

Where $c_x \in [x_0, x]$, $M_1 = \max\ |f''(t)|$, and $M_2 = \max\ |f'''(t)|$, $t \in [x_0, x]$ or $[x_0, x]$. Now take equation (3) and isolate the derivative part to get,

5. $f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$

If we replace *h* with *–h* in (5) we get,

6. $f'(x_0) \approx \frac{f(x_0) - f(x_0-h)}{h}$

Equation (5) is called the **FORWARD** difference approximation and (6) is called the **BACKWARD** difference approximation. If we add equations (5) and (6) we get the **CENTERED** difference approximation:

$$2f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h} + \frac{f(x_0) - f(x_0 - h)}{h} = \frac{f(x_0 + h) - f(x_0 - h)}{h}$$

Therefore,

7. $f'(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h}$

Suppose now we consider the polynomial for $n = 3$,

8. $f(x_0 + h) \approx p_3(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3$
9. $f(x_0 - h) \approx p_3(x_0 - h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2!}h^2 - \frac{f'''(x_0)}{3!}h^3$

If we subtract the (9) from (8) we get,

$$f(x_0 + h) - f(x_0 - h) \approx 2f'(x_0)h + \frac{2f'''(x_0)}{3!}h^3$$

If we isolate $f'(x_0)$, we obtain an approximation to the derivative at a given point.

$$f'(x_0) \approx \frac{f(x_0+h)-f(x_0-h)}{2h} - \frac{f'''(x_0)}{3!}h^2$$

Now let us find the second order approximation by adding equations (8) and (9),

$$f(x_0+h)+f(x_0-h) \approx 2f(x_0)+f''(x_0)h^2$$

Isolate the second derivative term to get,

10. $f''(x_0) \approx \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2}$

SUMMARY:

F] Forwd-Difference-Approx. ➔ $f'(x_0) \approx \frac{f(x_0+h)-f(x_0)}{h}$

B] Backwd-Difference-Approx. ➔ $f'(x_0) \approx \frac{f(x_0)-f(x_0-h)}{h}$

C] Centered-Diff.-Approx. ➔ $f'(x_0) \approx \frac{f(x_0+h)-f(x_0-h)}{2h}$

D] Second-Order Approx. ➔ $f''(x_0) \approx \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2}$

Let us now consider a numerical approximation to the heat equation. Recall,

$$u_t = ku_{xx}$$

With boundary conditions: $u(0,t) = u(L,t) = 0$ and initial condition: $u(x,0) = f(x)$

Let us apply the forward difference approximation to the derivative on t, we will let $h = \Delta x$:

$$\frac{\partial u}{\partial t}(x_0,t_0) \approx \frac{u(x_0,t_0+\Delta t)-u(x_0,t_0)}{\Delta t}$$

And let us find the second-order approximation on x:

$$\frac{\partial^2 u}{\partial x^2}(x_0,t_0) \approx \frac{u(x_0+\Delta x,t_0)-2u(x_0,t_0)+u(x_0-\Delta x,t_0)}{\Delta x^2}$$

If we take the two equation above and substitute their right sides into the heat equation we get,

$$\frac{u(x_0,t_0+\Delta t)-u(x_0,t_0)}{\Delta t} \approx \frac{k}{\Delta x^2}\frac{u(x_0+\Delta x,t_0)-2u(x_0,t_0)+u(x_0-\Delta x,t_0)}{1}$$

$$u(x_0,t_0+\Delta t)-u(x_0,t_0) \approx \frac{k\Delta t}{\Delta x^2}(u(x_0+\Delta x,t_0)-2u(x_0,t_0)+u(x_0-\Delta x,t_0))$$

Therefore,

11. $u(x_0,t_0+\Delta t) \approx u(x_0,t_0)+s[u(x_0+\Delta x,t_0)-2u(x_0,t_0)+u(x_0-\Delta x,t_0)]$

Where,

$$s = \frac{k\Delta t}{\Delta x^2}$$

We have now found a relation that uses the start position x_0, start time t_0, next time $t_0 + \Delta t$, and next position $x_0 + \Delta x$. So we will make the following substitutions into (11):

$$x_j = x_0 \implies x_{j+1} = x_0 + \Delta x$$

$$t_m = t_0 \implies t_{m+1} = t_0 + \Delta t$$

Now (11) becomes,

$$u(x_j, t_{m+1}) = u(x_j, t_m) + s[u(x_{j+1}, t_m) - 2u(x_j, t_m) + u(x_{j-1}, t_m)]$$

If we consider the short notation, $u_j^{(m)} = u(x_j, t_m)$ then the above line becomes,

$$u_j^{(m+1)} = u_j^{(m)} + s\left[u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}\right]$$

Since our derivation is based on the 1 dimensional case of a rod $\Delta x = \frac{L}{N}$ where L is the length of the rod and N is the number of partitions we make. And our samples will be indexed from $j \in \{1,2,3, \dots, N-1\}$ and our time samples will be indexed from $m \in \mathbb{Z}_+ = \{0,1,2,3 \dots\}$