ENGINEERING ANALYSIS
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HOMEWORK #3 PROBLEM 2

Consider the initial boundary value problem for the two dimensional heat equation:

$$
\begin{array}{ll}
u_t = k\left(u_{xx} + u_{yy}\right) & \text{for } 0 < x < L \quad \& \ 0 < y < H \quad \& \ t > 0 \\
u(0, y, t) = u(L, y, t) = 0 & \text{for } 0 < y < H \quad \& \ t > 0 \\
u(x, 0, t) = u(x, H, t) = 0 & \text{for } 0 < x < L \quad \& \ t > 0 \\
u(x, y, 0) = f(x, y) & \text{for } 0 < x < L \quad \& \ 0 < y < H
\end{array}
$$

a) Use separation of variables $u(x, y, t) = \phi(x)\psi(y)h(t)$ and the given boundary conditions to prove that

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad \psi_n\left(\frac{n\pi y}{L}\right) \quad \text{and} \quad h_{mn} = e^{-\lambda_{mn}kt}$$
$$\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

First the derivatives are:

$$u_t = \phi(x)\psi(y)h'(t)$$
$$u_{xx} = \phi''(x)\psi(y)h(t)$$
$$u_{yy} = \phi(x)\psi''(y)h(t)$$

Substitute them into the differential equation to get,

$$\phi(x)\psi(y)h'(t) = k\left(\phi''(x)\psi(y)h(t) + \phi(x)\psi''(y)h(t)\right)$$

We can factor out the $kh(t)$ on the right hand side of the above equation:

$$\phi(x)\psi(y)h'(t) = kh(t) \cdot \left(\phi''(x)\psi(y) + \phi(x)\psi''(y)\right)$$

By division of common terms we get,

$$\frac{h'(t)}{kh(t)} = \frac{\phi''(x)\psi(y) + \phi(x)\psi''(y)}{\phi(x)\psi(y)} = -\lambda$$

Therefore, we have the following equations:

- $h' + \lambda kh = 0$
- $\phi''\psi + \phi\psi'' = -\lambda\phi\psi$

Let us examine the second differential equation: $\phi''\psi + \phi\psi'' = -\lambda\phi\psi$. Bring all terms to one side to get,

$$\phi''\psi + \phi\psi'' + \lambda\phi\psi = 0$$

If we factor ψ we get,

$$\psi(\phi'' + \lambda\phi) + \phi\psi'' = 0$$

Then isolate terms we get,

$$-\frac{\psi''}{\psi} = \frac{\phi'' + \lambda\phi}{\phi} = \mu$$

We get the following differential equations:

- $\psi'' = -\mu\psi \rightarrow \psi'' + \mu\psi = 0$
- $\phi'' + \lambda\phi = \mu\phi \rightarrow \phi'' + (\lambda - \mu)\phi = 0$

If we examine the first equation and let $\psi = e^{ry}$ we get the following characteristic equation:

$$r^2 + \mu = 0$$

It has the following solutions for μ > 0:

$$r_{1,2} = \pm\sqrt{\mu}$$

This means that the solution for ψ will be of the form: $\psi(y) = A\cos\sqrt{\mu}y + B\sin\sqrt{\mu}y$. If we utilize the boundary conditions: $u(x, 0, t) = u(x, H, t) = 0 \ \rightarrow \psi(0) = \psi(H) = 0$. So at $\psi(0) = A \cdot 1 + B \cdot 0 = 0$, this means that A = 0; hence, $\psi(y) = B\sin\sqrt{\mu}y$. If we consider the next part of the boundary conditions we get: $\psi(H) = 0 \rightarrow \ \ B\sin\sqrt{\mu}H = 0$. Therefore,

$$\mu_m = \left(\frac{m\pi}{H}\right)^2$$

Therefore, the corresponding functions for ψ are:

$$\psi_m(y) = \sin\left(\frac{m\pi y}{H}\right)$$

Let us now consider the second equation for φ:

$$\phi'' + (\lambda - \mu)\phi = 0$$

If we make the substitution: $\phi = e^{rx}$, we will get the following characteristic equation:

$$r^2 + (\lambda - \mu) = 0$$

With roots: $r_{1,2} = \pm i\sqrt{\lambda - \mu}$, so our solution to φ will have the form:

$$\phi(x) = A\cos\sqrt{\lambda - \mu}\,x + B\sin\sqrt{\lambda - \mu}\,x$$

From the boundary conditions we get the following: $u(0, y, t) = u(L, y, t) = 0 \rightarrow \phi(0) = \phi(L) = 0$. If we substitute x = 0 into φ we get: $\phi(0) = A \cdot 1 + B \cdot 0 = 0 \rightarrow A = 0$. Therefore, φ will be of the form:

$$\phi(x) = B\sin\sqrt{\lambda - \mu}\,x$$

If we utilize the second boundary condition and substitute x = L we get:

$$\phi(L) = B\sin\sqrt{\lambda-\mu}\,L = 0$$

For the above case to be true we must have:

$$\sqrt{\lambda-\mu}L = n\pi$$

$$\sqrt{\lambda-\mu} = \frac{n\pi}{L}$$

Hence:

$$\lambda = \left(\frac{n\pi}{L}\right)^2 + \mu$$

But recall, $\mu_m = \left(\frac{m\pi}{H}\right)^2$, if we substitute we get:

$$\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

And the corresponding functions of φ are:

$$\phi_n(x) = \sin\sqrt{\lambda-\mu}\,x = \sin\left(x\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 - \left(\frac{m\pi}{H}\right)^2}\right) = \sin\left(\frac{n\pi x}{L}\right)$$

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

So in summary we have the following relationships:

- $\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$
- $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$
- $\psi_m(y) = \sin\left(\frac{m\pi y}{H}\right)$

Let us now solve for the solution to the temporal equation $h' + \lambda k h = 0$. If we isolate the 'h' terms we get:

$$\frac{h'}{h} = -\lambda k \to \ln|h| = -\lambda k t + C$$

Hence,

$$h(t) = Ce^{-\lambda k t}$$

But we already know λ: $\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$

We have: $h_{mn}(t) = e^{-\lambda_{mn} k t}$

So a solution to our heat equation is: $u_{mn}(x,y,t) = C_{mn}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)e^{-\lambda_{mn} k t}$

Part b)

Let us consider m = 1, then: $u_{1n}(x,y,t) = C_{1n}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{\pi y}{H}\right)e^{-\lambda_{1n}kt}$, but remember for m = 1 we can also have every linear combination of this solution so:

$$u_1(x,y,t) = \sum_{n=1}^{\infty} C_{1n}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{\pi y}{H}\right)e^{-\lambda_{1n}kt}$$

For m = 2 we get:

$$u_2(x,y,t) = \sum_{n=1}^{\infty} C_{2n}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{2\pi y}{H}\right)e^{-\lambda_{2n}kt}$$

For m = 3 we get:

$$u_3(x,y,t) = \sum_{n=1}^{\infty} C_{3n}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{3\pi y}{H}\right)e^{-\lambda_{3n}kt}$$

We can continue this indefinitely and the sum of these solutions $u_1, u_2, u_3, \ldots u_m$ is also a solution to the heat equation:

$$u(x,y,t) = u_1 + u_2 + u_3 + \cdots + u_m$$

Therefore,

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)e^{-\lambda_{mn}kt}$$

Let us now consider the initial condition $u(x,y,0) = f(x,y)$

$$u(x,y,0) = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} C_{mn}\sin\left(\frac{n\pi x}{L}\right)\right)\sin\left(\frac{m\pi y}{H}\right)$$

Let us call the inner summation S_m since it is independent of y:

$$u(x,y,0) = \sum_{m=1}^{\infty} S_m \sin\left(\frac{m\pi y}{H}\right)$$

Notice that this is a Fourier Sine series; hence,

$$S_m = \frac{2}{H}\int_0^H f(x,y)\sin\left(\frac{m\pi y}{H}\right)dy$$

But S_m is the following:

$$S_m = \sum_{n=1}^{\infty} C_{mn}\sin\left(\frac{n\pi x}{L}\right)$$

Another Fourier Sine series expansion; thus,

$$C_{mn} = \frac{2}{L}\int_0^L S_m \sin\left(\frac{n\pi x}{L}\right)dx$$

But recall,

$$S_m = \frac{2}{H}\int_0^H f(x,y)\sin\left(\frac{m\pi y}{H}\right)dy$$

We can substitute this into C_{mn} to get,

$$C_{mn} = \frac{2}{L}\int_0^L \left(\frac{2}{H}\int_0^H f(x,y)\sin\left(\frac{m\pi y}{H}\right)dy\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

Rearrange the integral a little and get,

$$C_{mn} = \frac{2}{H}\frac{2}{L}\int_0^L\int_0^H f(x,y)\sin\left(\frac{m\pi y}{H}\right)\sin\left(\frac{n\pi x}{L}\right)dy\,dx$$