7. Exp a logaritmus

7.1 Exponenciála

Def: funkce tvaru

$$f: y = a^x \quad a \in \mathbb{R}^+ \setminus \{1\}$$

"exponenciální funkce"

a ... základ

$D_f = \mathbb{R}$

$H_f = (0, \infty)$

· zdola omezená

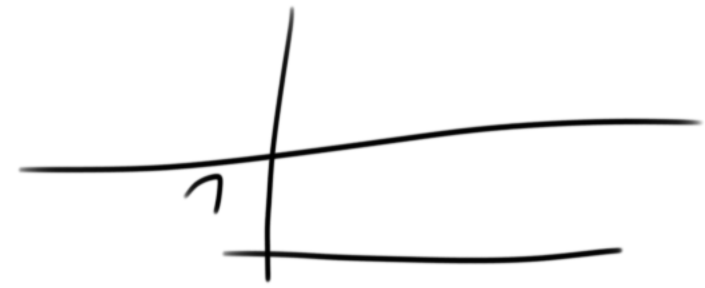
· rostoucí na D_f $a>1$

· klesající na D_f $0<a<1$

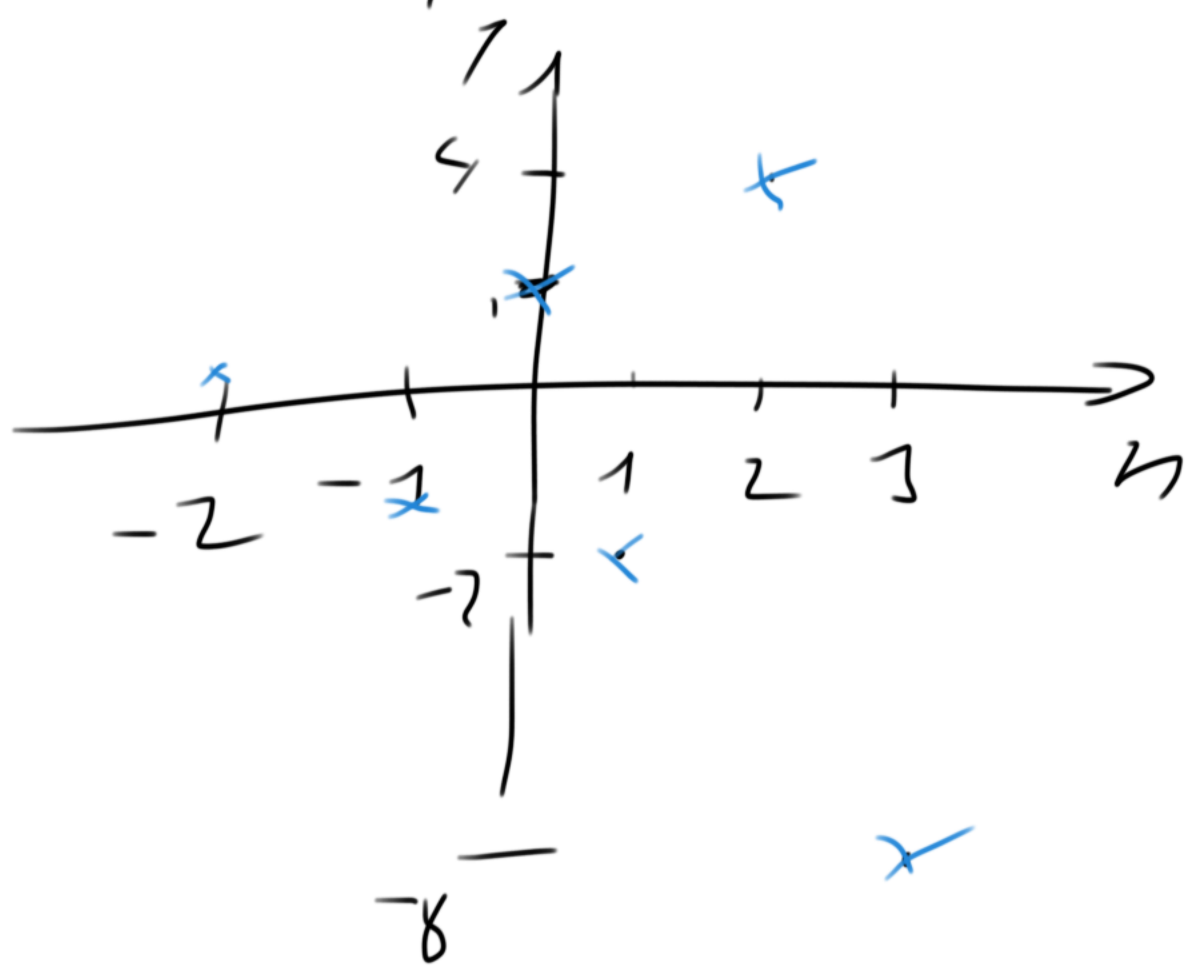
· ani sudá ani lichá

Poznámka

· $a=1$ $\quad y = 1^x = 1 \quad \forall x \in \mathbb{R}$



· $a<0$ $\quad y = -2^n \quad n \in \mathbb{Z}$



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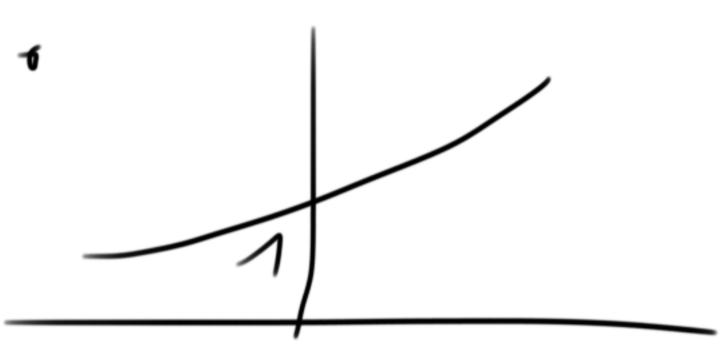

$x=0 : \quad y = a^0 = 1$

$\forall a \in \mathbb{R}^+ \setminus \{1\}$

· $a^x \cdot a^x = a^{x+x} \qquad a^{-x} = \frac{1}{a^x}$

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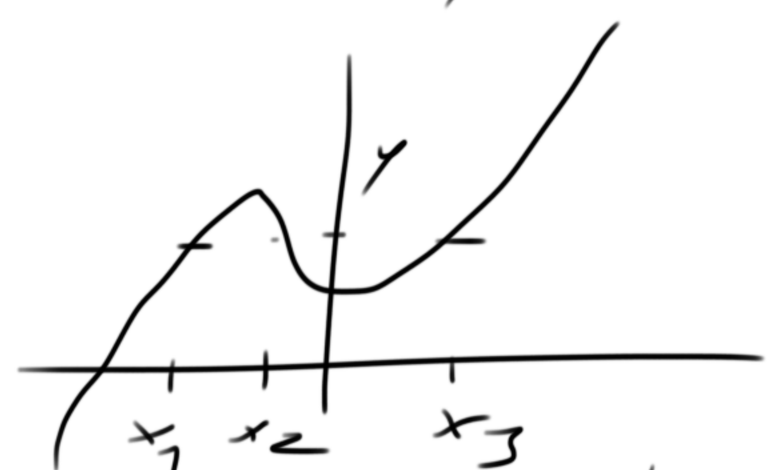
x^a ... mocninná $\qquad a^x$ ... exp.

7.2 Inverzní funkce

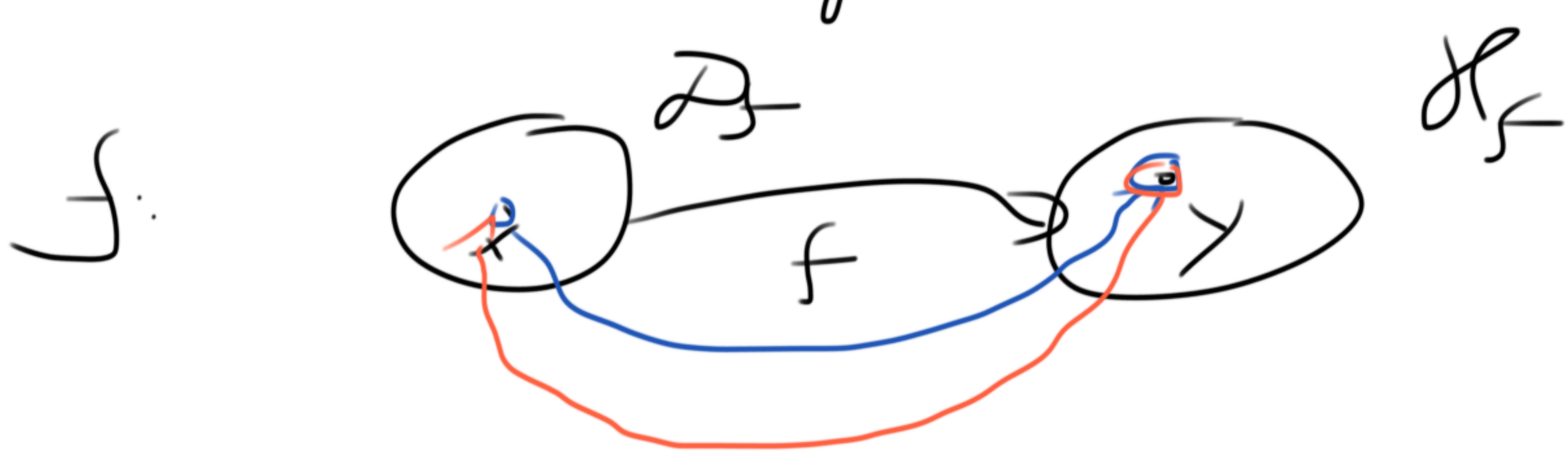
prostá fce: každému $x \in D_f$ přiřadí právě jedno $y \in H_f$. Různým $x \in D_f$ přiřadí různá y.

prostá

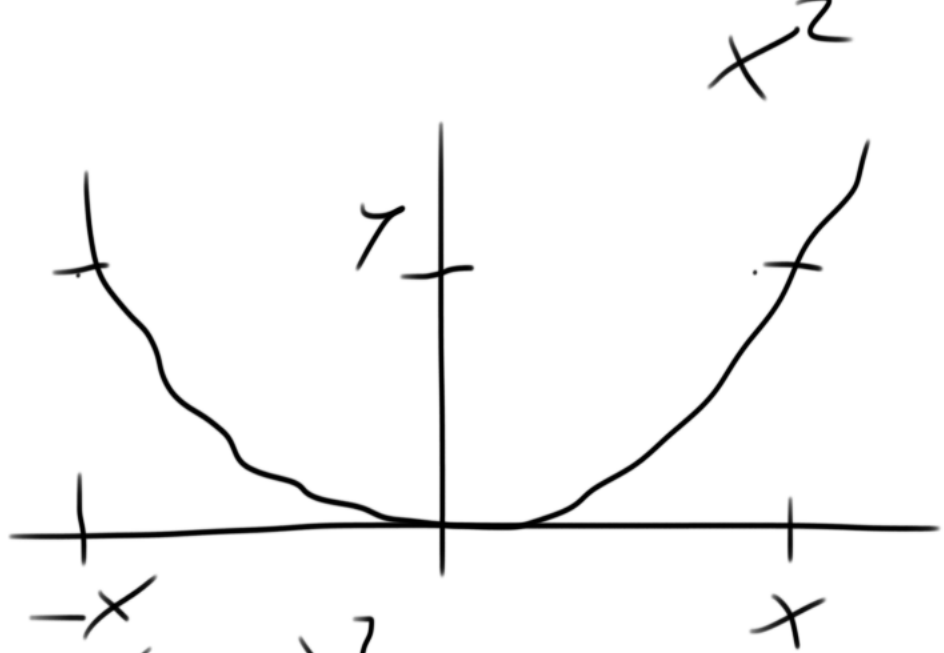


není prostá

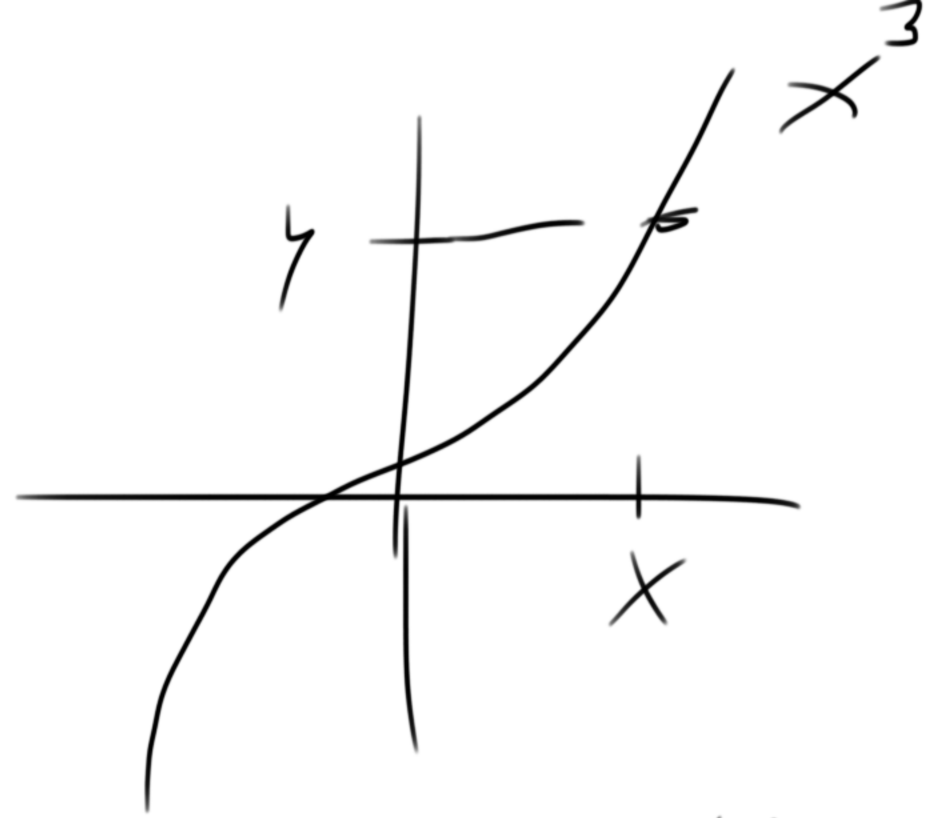
f rostoucí/klesající na D_f $\Rightarrow$ prostá



$f^{-1}: y \longrightarrow x$ inverzní funkce



$(\pm x)^2 = y$



f je prostá $\Longrightarrow \exists$ inverzní funkce

$f: y = 2x + 3$

$f(2) = 2 \cdot 2 + 3 = 7$

$f^{-1}: 2x = y - 3$

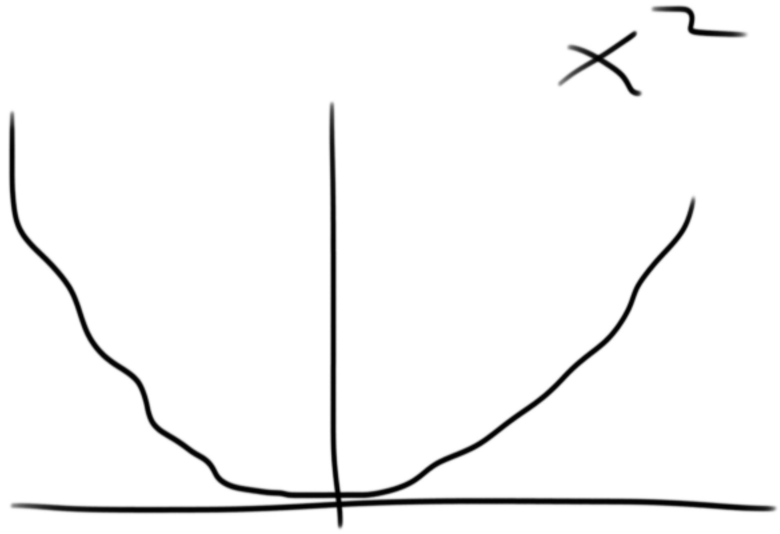
$f^{-1}(7) = \frac{1}{2}(7-3) = 2$

$x = \frac{1}{2}(y-3)$

$f: y = 2x + 3$

$f^{-1}: y = \frac{x-3}{2}$

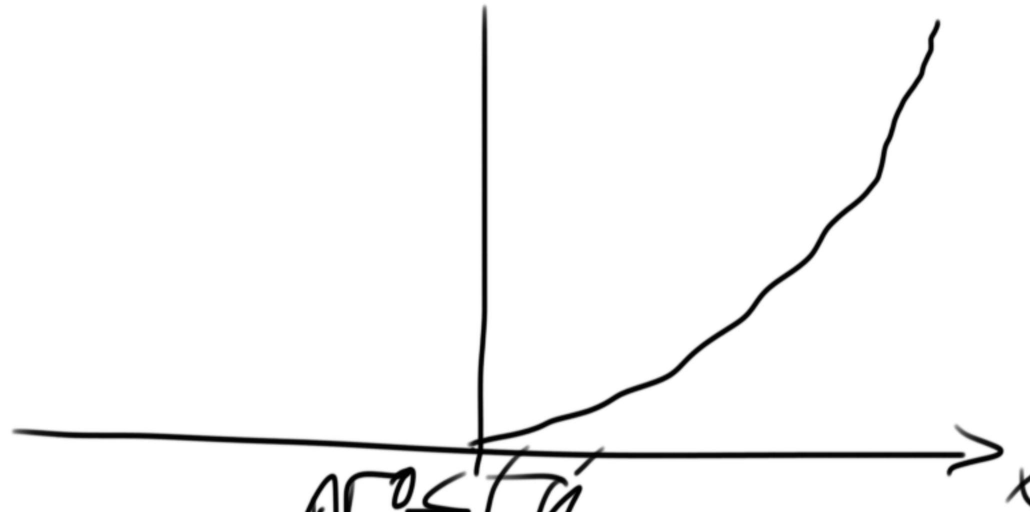
$H_{f^{-1}} = D_f \quad D_{f^{-1}} = H_f$



není prostá

$f: y = x^2 \quad D_f = \mathbb{R}$

$H_f = \mathbb{R}_0^+$



prostá

$g: y = x^2 \quad D_g = \mathbb{R}_0^+$

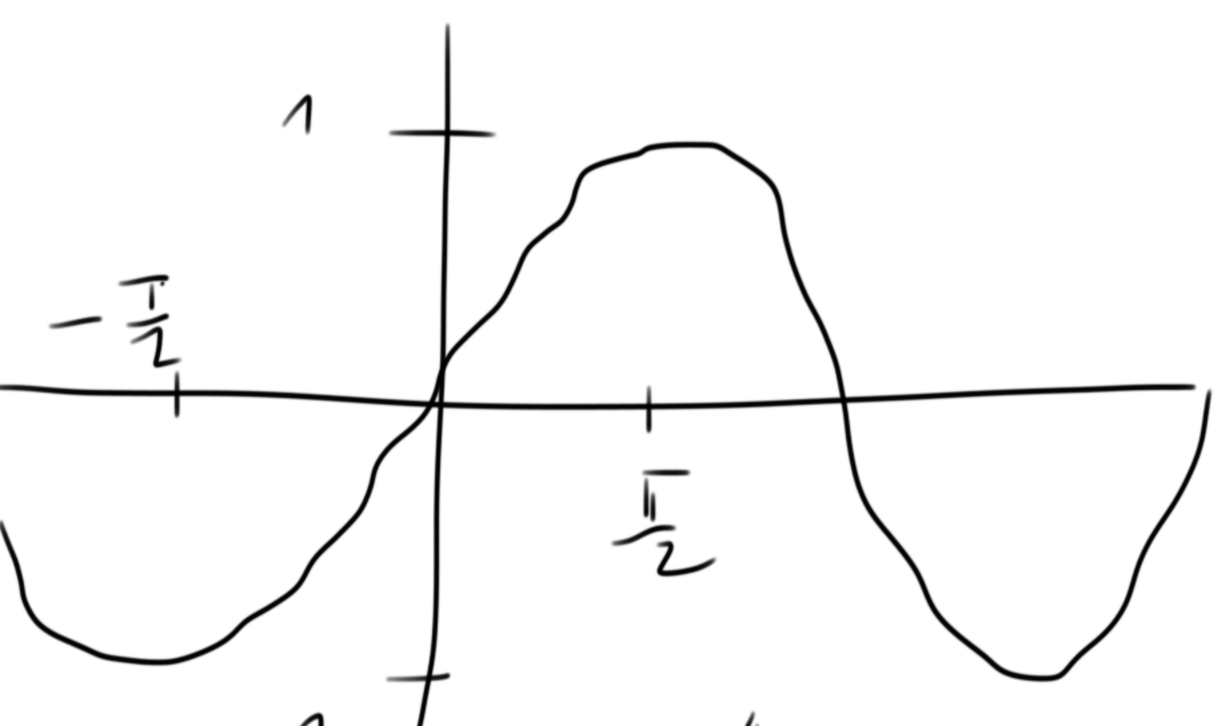
$H_g = \mathbb{R}_0^+$

existuje inverzní fce

$g^{-1}: x = \sqrt{y}$

$$\boxed{g^{-1}: y = \sqrt{x}}$$

$D_{g^{-1}} = \mathbb{R}_0^+ \quad H_{g^{-1}} = \mathbb{R}_0^+$



na $\langle -\frac{\pi}{2}; \frac{\pi}{2} \rangle$

je $\sin x$ prostá

$\longrightarrow$ inverzní

$(\sin x)^{-1} = $ arcsin x "

7.3 Logaritmus

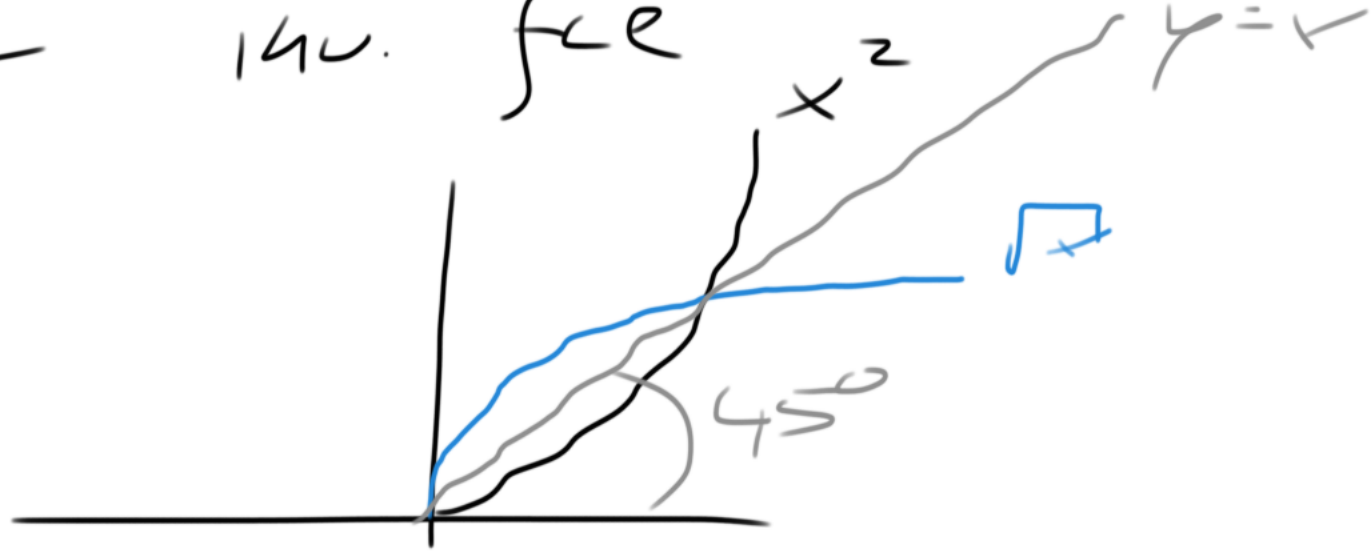
exp. je prostá na D_f $\forall a \in \mathbb{R}^+ \setminus \{1\}$

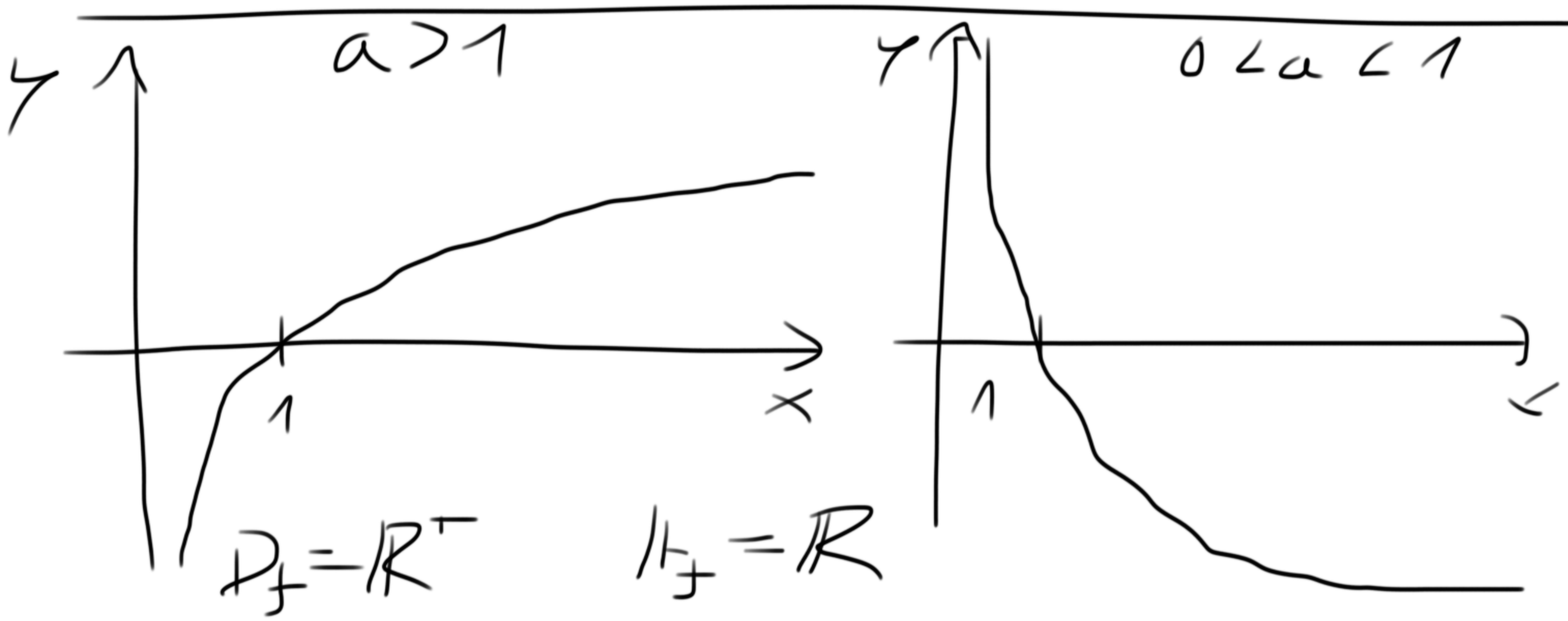
$\longrightarrow$ existuje inverzní fce.

Definice: inverzní funkce k a^x se nazývá logaritmus o základu a

$$f: y = \log_a x$$

graf inv. fce





$D_f = \mathbb{R}^+$ $\quad H_f = \mathbb{R}$

rostoucí na D_f $a > 1$

klesající na D_f $0 < a < 1$

neomezená neperiodická

ani sudá ani lichá

$\underline{\log_a x} = y \iff a^y = x$

na co musíme umocnit a, abychom dostali x? $a^y = x$

$\log_{10} 10 = 1$ $10^1 = 10$

$\log_{10} 100 = 2$ $10^2 = 100$

$\log_2 16 = 4$ $2^4 = 16$

$\log_3 \frac{1}{9} = -2$ $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

$\log_2 16 = \log_2 2^4 = 4$

$\log_3 \frac{1}{9} = \log_3 3^{-2} = -2$

$$\boxed{\log_a a^y = y}$$

Počítání s logaritmy

$$\log_a(r\cdot s) = \log_a r + \log_a s$$

$$\log_a\left(\frac{r}{s}\right) = \log_a r - \log_a s$$

$$\log_a r^s = s\cdot \log_a r \quad s\in\mathbb{R}$$

$$\log_a a^r = r \quad r\in\mathbb{R}$$

nutné znát

$$\forall a\in\mathbb{R}^+\setminus\{1\},\ \forall r,s\in\mathbb{R}^+$$

$$\log_b x = \frac{\log_a x}{\log_a b}$$

$$f: x \longrightarrow y \qquad f^{-1}: y \longrightarrow x$$

$$f^{-1}(f(x)) = f^{-1}(y) = x$$

$$f(f^{-1}(x)) = x$$

$$\log_a a^x = x \qquad a^{\log_a x} = x$$

$\log_a a = 1 \qquad \log_a 1 = 0$

$\log_3 27 = \log_3 3^{3} = 3 \cdot \log_3 3$

$= 3 \cdot 1 = 3$

$\log_{10} 1000 = \log_{10}(10 \cdot 10 \cdot 10) = 3 \log_{10} 10 = 3$

$\log_{10} 10^6 = 6 \qquad 1000000$

$\log_{10} \underline{10^2} = 2 \qquad \log_{10} 50 = \underline{1,6989\ldots}$

$\log_{10} \underline{10} = 1$

$21 = 2 \cdot 10^1 + 1 \cdot 10^0$

$(6)_{10} = 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = (110)_2$

$2^0 = 1 \qquad (8)_{10} = (\underline{1000})_2$

$2^1 = 2$

$2^2 = 4 \qquad \log_2 8 = 3$

$2^3 = 8$

7.4 Eulerovo číslo

$$\left(1+\frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} e$$

$$e \doteq 2{,}718281828459045\ldots$$

iracionální

$$a = e \longrightarrow \boxed{\boxed{e^x} \quad \log_e x = \ln x}$$

$$\left(1+\frac{t}{n}\right)^n \longrightarrow e^t$$

$$N(t) = e^{R\cdot t}$$

statistika $\quad \dfrac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$\psi(x,t) = e^{-\frac{i}{\hbar}\hat{H}\cdot t}\,\psi(x)$$

$$e = \sum_{i=0}^{\infty} \frac{1}{i!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$$

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{\ddots}}}}}}$$

7.5 Exp a log rovnice

Rovnice: $\underline{f(x) = 0}$ $\quad x \in M$

$$f \begin{cases} \exp \\ \log \end{cases}$$

7.5.1 Exp. rovnice

I, II, III

I) $a^{f(x)} = a^{g(x)}$

$\Leftrightarrow \quad f(x) = g(x)$

$3^{x+3} = 3^2$

$x+3 = 2$

$\underline{x = -1}$

$5^{3x-1} = 1$

$5^{3x-1} = 5^0$

$3x - 1 = 0$

$x = \frac{1}{3}$

$\underbrace{5^x \cdot 2^x} = 100^{x-1}$

$(5 \cdot 2)^x = (10^2)^{x-1}$

$10^x = 10^{2x-2}$

$x = 2x - 2$

$x = 2$

Zk. $LS = 5^2 \cdot 2^2 = 100$

$PS = 100^{2-1} = 100$

$\underline{LS = PS}$

II $$a^{f(x)} = b^{g(x)}$$
$$\cancel{\Leftrightarrow} \quad f(x) = g(x)$$

složeni LS i PS sa istom fci $h(x)$
h prosta

$$a^{f(x)} = b^{g(x)}$$

$$h\left(a^{f(x)}\right) = h\left(b^{g(x)}\right)$$

ekvivalentni uprava

$h(x) = \log_c x \longrightarrow$ „zlogaritmovat"

$$a^{f(x)} = b^{g(x)} \quad / \log_c$$

$$\log_c a^{f(x)} = \log_c b^{g(x)}$$

$$f(x) \cdot \underbrace{\log_c a} = g(x) \cdot \underbrace{\log_c b}$$ čísla

$$f(x) = g(x) \cdot \frac{\log_c b}{\log_c a} = \underline{\log_a b}$$

$$2^x \cdot 5^{2x} = 3^{x-2} \quad / \log$$

$$\log 2^x + \log 5^{2x} = \log 3^{x-2}$$

$$x \cdot \log 2 + 2x \log 5 = (x-2) \cdot \log 3$$

$$x\underbrace{(\log 2 + 2 \cdot \log 5 - \log 3)}_{\text{číslo}} = -2 \cdot \log 3$$

$$x = \frac{-2 \cdot \log 3}{\log 2 + 2 \cdot \log 5 - \log 3}$$

IV) substituce

$$7^{2x} + 7^x - 6 = 0$$

$$(7^x)^2 + 7^x - 6 = 0$$

$$y = 7^x$$

$$y^2 + y - 6 = 0$$

$$(y+3)(y-2) = 0$$

$$-3, \; 2$$

x_1:

$$7^{x_1} = -3$$

NŘ

$$7^x > 0 \; \forall x \in \mathbb{R}$$

x_2:

$$7^{x_2} = 2$$

$$\boxed{x_2 = \frac{\log 2}{\log 7}}$$

7.5.2 Log. rce

I $\log_a f(x) = \log_a g(x)$

$\Leftrightarrow f(x) = g(x)$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

II) ... substituce

$\log_4 (2x+3) = \underline{2}$

$\log_4 (2x+3) = \log_4 16$

$2x+3 = 16$

$x = \frac{13}{2}$

$\log_2 x - \log_4 64 = 0$

$\log_2 x = \log_4 64$

$\log_2 x = \frac{\log_2 64}{\log_2 4}$

$\log_2 x = \frac{1}{2} \log_2 64^{\frac{1}{2}}$

$\log_2 x = \log_2 8$

$x = 8$

Pozor na def. obor.