Fourier Series

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Contents

***Abstract*—This manual provides a simple introduction to Fourier Series**

1 Periodic Function

Let

$$x(t) = A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.1}$$

1.1 Plot $x(t)$.
Solution: The following code will plot the graph in fig (1.1)

```
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/charger/codes/1.1.py
```

run the above code using the command

```
python3 1.1.py
```

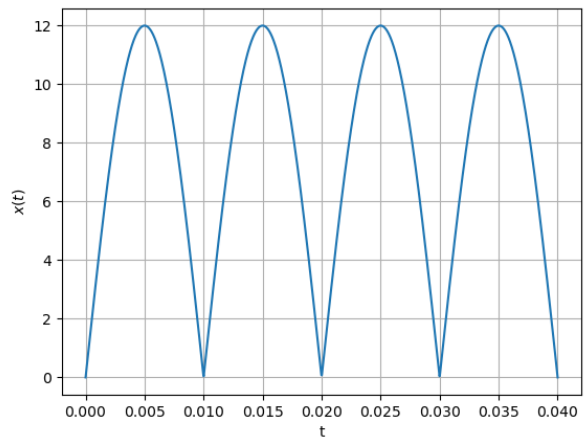

Fig. 1.1: $x(t) = A_0 \left|\sin\left(2\pi f_0 t\right)\right|$

1.2 Show that $x(t)$ is periodic and find its period.
Solution: From fig (1.1), we see that $x(t)$ is periodic. Further,

$$\text{period of } \sin(at) \quad \text{given by } \frac{2\pi}{a} \tag{1.2}$$

Now, period of $x(t)$ is

$$A_0 \left|\sin\left(2\pi f_0 t\right)\right| \implies \frac{\pi}{2\pi f_0} \tag{1.3}$$

$$\implies \frac{1}{2f_0} \tag{1.4}$$

Verification

$$x\left(t+\frac{1}{2f_0}\right) = A_0 \left|\sin\left(2\pi f_0\left(t+\frac{1}{2f_0}\right)\right)\right| \tag{1.5}$$

$$= A_0 \left|\sin\left(2\pi f_0 t + \pi\right)\right| \tag{1.6}$$

$$= A_0 \left|-\sin\left(2\pi f_0 t\right)\right| \tag{1.7}$$

$$= A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.8}$$

Hence the period of $x(t)$ is $\frac{1}{2f_0}$.

2 Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.2}$$

Solution: We have for some $n \in \mathbb{Z}$,

$$x(t)e^{-\jmath 2\pi n f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi (k-n) f_0 t} \tag{2.3}$$

But we know from the periodicity of $e^{\jmath 2\pi k f_0 t}$,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{\jmath 2\pi k f_0 t}\, dt = \frac{1}{f_0}\delta(k) \tag{2.4}$$

Thus,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi n f_0 t}\, dt = \frac{c_n}{f_0} \tag{2.5}$$

$$\implies c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi n f_0 t}\, dt \tag{2.6}$$

2.2 Find c_k for (1.1)
Solution: Using (2.2),

$$c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin(2\pi f_0 t)\right| e^{-\jmath 2\pi n f_0 t}\, dt \tag{2.7}$$

$$= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin(2\pi f_0 t)\right| \cos(2\pi n f_0 t)\ dt$$
$$+ \jmath f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin(2\pi f_0 t)\right| \sin(2\pi n f_0 t)\ dt \tag{2.8}$$

$$= 2f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin(2\pi f_0 t) \cos(2\pi n f_0 t)\ dt \tag{2.9}$$

$$= f_0 A_0 \int_{0}^{\frac{1}{2f_0}} \left(\sin(2\pi (n+1) f_0 t)\right)\ dt$$
$$- f_0 A_0 \int_{0}^{\frac{1}{2f_0}} \left(\sin(2\pi (n-1) f_0 t)\right)\ dt \tag{2.10}$$

$$= A_0 \frac{1+(-1)^n}{2\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) \tag{2.11}$$

$$= \begin{cases} \frac{2A_0}{\pi\left(1-n^2\right)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{2.12}$$

2.3 Verify (2.1) using python.
Solution: The following code will plot the graph in fig (2.3)

```
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/charger/codes/2.3.py
```

run the above code using the command

```
python3 2.3.py
```

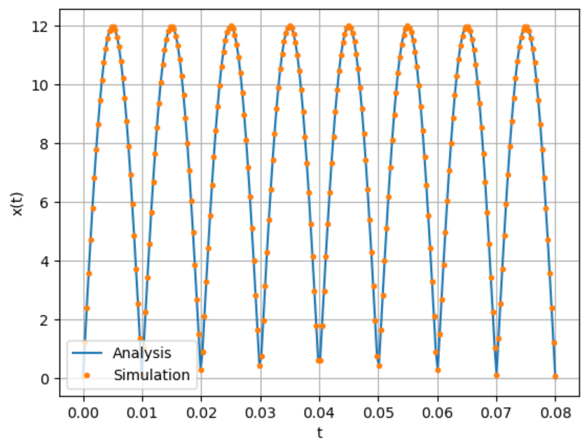

Fig. 2.3: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t}$

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} \left(a_k \cos \jmath 2\pi k f_0 t + b_k \sin \jmath 2\pi k f_0 t\right) \tag{2.13}$$

and obtain the formulae for a_k and b_k.
Solution: From (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.14}$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{\jmath 2\pi k f_0 t} + c_{-k} e^{-\jmath 2\pi k f_0 t} \tag{2.15}$$

$$= c_0 + \sum_{k=1}^{\infty} \left(c_k + c_{-k}\right) \cos(2\pi k f_0 t)$$
$$+ \sum_{k=0}^{\infty} \left(c_k - c_{-k}\right) \sin(2\pi k f_0 t) \tag{2.16}$$

Hence, for $k \geq 0$,

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \tag{2.17}$$

$$b_k = c_k - c_{-k} \tag{2.18}$$

2.5 Find a_k and b_k for (1.1)
Solution: From (2.1), we see that since $x(t)$ is

even,

$$x(-t) = \sum_{k=-\infty}^{\infty} c_k e^{-\jmath 2\pi k f_0 t} \tag{2.19}$$

$$= \sum_{k=-\infty}^{\infty} c_{-k} e^{\jmath 2\pi k f_0 t} \tag{2.20}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.21}$$

where we substitute $k \mapsto -k$ in (2.20). Hence, we see that $c_k = c_{-k}$. So, from (2.18) and for $k \geq 0$,

$$a_k = \begin{cases} \frac{2A_0}{\pi} & k = 0 \\ \frac{4A_0}{\pi\left(1-k^2\right)} & k > 0,\ k \text{ even} \\ 0 & \text{otherwise} \end{cases} \tag{2.22}$$

$$b_k = 0 \tag{2.23}$$

2.6 Verify (2.13) using python.
Solution: The following code will plot the graph in fig (2.6)

```
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/charger/codes/2.6.py
```

run the above code using the command

```
python3 2.6.py
```

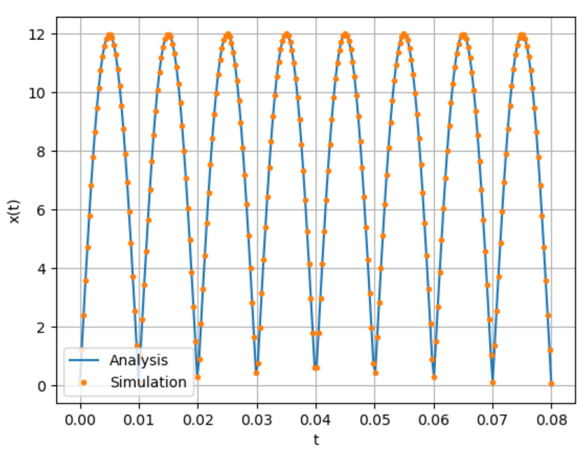

Fig. 2.6: $x(t) = \sum_{k=0}^{\infty} \left(a_k \cos \jmath 2\pi k f_0 t + b_k \sin \jmath 2\pi k f_0 t\right)$

3 Fourier Transform

3.1

$$\delta(t) = 0, \quad t \neq 0 \tag{3.1}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{3.2}$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt \tag{3.3}$$

3.3 Show that

$$g(t - t_0) \stackrel{\mathcal{H}}{\longleftrightarrow} F G(f) e^{-j2\pi f t_0} \tag{3.4}$$

$$\tag{3.5}$$

Solution:

3.4 Show that

$$G(t) \stackrel{\mathcal{H}}{\longleftrightarrow} F g(-f) \tag{3.6}$$

Solution:

3.5 $\delta(t) \stackrel{\mathcal{H}}{\longleftrightarrow} F?$
Solution: By applying the defination fo fourier transformation for $\delta(t-t_0)$ {t_0 be time shifting}.

$$\mathcal{F}\left\{\delta(t - t_0)\right\}(f) = \mathcal{F}(t) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi f t} dt. \tag{3.7}$$

By applying the time shifting property of impulse we get

$$\mathcal{F}(f) = e^{-j2\pi f t_0} \tag{3.8}$$

$$i.e., \quad \delta(t - t_0) \stackrel{\mathcal{H}}{\longleftrightarrow} F e^{-j2\pi f t_0} \tag{3.9}$$

Now, substitute $t_0 = 0$ we get

$$\delta(t) \stackrel{\mathcal{H}}{\longleftrightarrow} F 1 \tag{3.10}$$

3.6 $e^{-j2\pi f_0 t} \stackrel{\mathcal{H}}{\longleftrightarrow} F?$
Solution: Applying the defination of inverse fourier transformation.

$$\mathcal{F}^{-1}\left\{\delta(f + f_0)\right\}(t) = f(t) = \int_{-\infty}^{\infty} \delta(f + f_0) e^{j2\pi f t} df \tag{3.11}$$

By applying the shifting property of impulse

$$f(t) = e^{-j2\pi f_0 t} \tag{3.12}$$

$$i.e., \quad e^{-j2\pi f_0 t} \stackrel{\mathcal{H}}{\longleftrightarrow} F \delta(f + f_0) \tag{3.13}$$

3.7 $\cos(2\pi f_0 t) \stackrel{\mathcal{H}}{\longleftrightarrow} F?$

Solution:

$$\cos(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \tag{3.14}$$

$$\mathcal{F}\left[\cos(2\pi f_0 t)\right] = \int_{-\infty}^{\infty} \cos(2\pi f_0 t) e^{-j2\pi f t} dt \tag{3.15}$$

$$= \int_{-\infty}^{\infty} \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} e^{-j2\pi f t} dt \tag{3.16}$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{j2\pi f_0 t e^{-j2\pi f t} dt} + \int_{-\infty}^{\infty} e^{-j2\pi f_0 t} e^{-j2\pi f t} dt\right] \tag{3.17}$$

$$= \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right] \tag{3.18}$$

$$\therefore \cos(2\pi f_0 t) \overset{\mathcal{H}}{\longleftrightarrow} F \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right] \tag{3.19}$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python.

3.9 Show that

$$\text{rect}\,(t) \overset{\mathcal{H}}{\longleftrightarrow} F \text{sinc}\,(t) \tag{3.20}$$

Verify using python.

3.10 $\text{sinc}\,(t) \overset{\mathcal{H}}{\longleftrightarrow} F$?. Verify using python.

4 Filter

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5V.
4.2 Find $h(t)$.
4.3 Verify your result using through convolution.

5 Filter Design

5.1 Design a Butterworth filter for $H(f)$.
5.2 Design a Chebyschev filter for $H(f)$.
5.3 Design a circuit for your Butterworth filter.
5.4 Design a circuit for your Chebyschev filter.