# Fourier Series

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## Contents

*Abstract*—This manual provides a simple introduction to Fourier Series
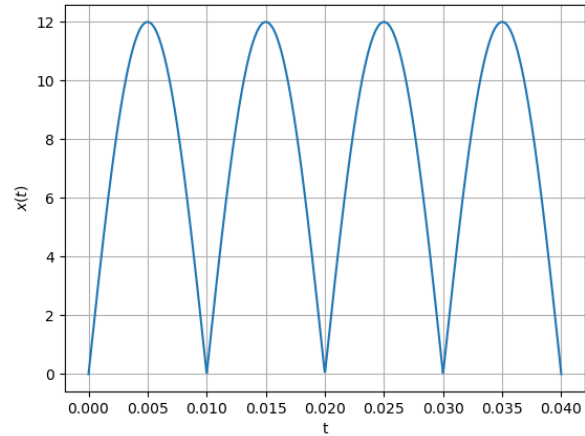


Fig. 1.1: $x(t) = A_0 \left| \sin\left(2\pi f_0 t\right)\right|$

## 1 Periodic Function

Let

$$x(t) = A_0 \left| \sin\left(2\pi f_0 t\right)\right| \tag{1.1}$$

1.1 Plot $x(t)$.
**Solution:** The following code will plot the graph in fig (1.1)

```
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/charger/codes/1.1.py
```

run the above code using the command

```
python3 1.1.py
```

1.2 Show that $x(t)$ is periodic and find its period.
**Solution:** From fig (1.1), we see that $x(t)$ is periodic. Further,

$$\text{period of } \sin(at) \quad \text{given by } \frac{2\pi}{a} \tag{1.2}$$

Now, period of $x(t)$ is

$$A_0 \left|\sin\left(2\pi f_0 t\right)\right| \implies \frac{\pi}{2\pi f_0} \tag{1.3}$$

$$\implies \frac{1}{2 f_0} \tag{1.4}$$

Verification

$$x\left(t + \frac{1}{2 f_0}\right) = A_0 \left|\sin\left(2\pi f_0 \left(t + \frac{1}{2 f_0}\right)\right)\right| \tag{1.5}$$

$$= A_0 \left|\sin\left(2\pi f_0 t + \pi\right)\right| \tag{1.6}$$

$$= A_0 \left|-\sin\left(2\pi f_0 t\right)\right| \tag{1.7}$$

$$= A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.8}$$

Hence the period of $x(t)$ is $\frac{1}{2 f_0}$.

## 2 Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.
2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t}\, dt \tag{2.2}$$

**Solution:** We have for some $n \in \mathbb{Z}$,

$$x(t)e^{-j2\pi n f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(k-n)f_0 t} \tag{2.3}$$

But we know from the periodicity of $e^{\jmath 2\pi k f_0 t}$,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{\jmath 2\pi k f_0 t} \, dt = \frac{1}{f_0} \delta(k) \qquad (2.4)$$

Thus,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi n f_0 t} \, dt = \frac{c_n}{f_0} \qquad (2.5)$$

$$\implies c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi n f_0 t} \, dt \qquad (2.6)$$

2.2 Find $c_k$ for (1.1)

**Solution:** Using (2.2),

$$c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left| \sin(2\pi f_0 t) \right| e^{-\jmath 2\pi n f_0 t} \, dt \qquad (2.7)$$

$$= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left| \sin(2\pi f_0 t) \right| \cos(2\pi n f_0 t) \, dt$$

$$+ \jmath f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left| \sin(2\pi f_0 t) \right| \sin(2\pi n f_0 t) \, dt$$

$$\qquad (2.8)$$

$$= 2f_0 \int_0^{\frac{1}{2f_0}} A_0 \sin(2\pi f_0 t) \cos(2\pi n f_0 t) \, dt$$

$$\qquad (2.9)$$

$$= f_0 A_0 \int_0^{\frac{1}{2f_0}} \left( \sin(2\pi(n+1) f_0 t) \right) dt$$

$$- f_0 A_0 \int_0^{\frac{1}{2f_0}} \left( \sin(2\pi(n-1) f_0 t) \right) dt \quad (2.10)$$

$$= A_0 \frac{1 + (-1)^n}{2\pi} \left( \frac{1}{n+1} - \frac{1}{n-1} \right) \qquad (2.11)$$

$$= \begin{cases} \frac{2A_0}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \qquad (2.12)$$

2.3 Verify (2.1) using python.

**Solution:** The following code will plot the graph in fig (2.3)

wget https://github.com/jarpula−Bhanu/
EE3900/blob/main/charger/codes/2.3.py
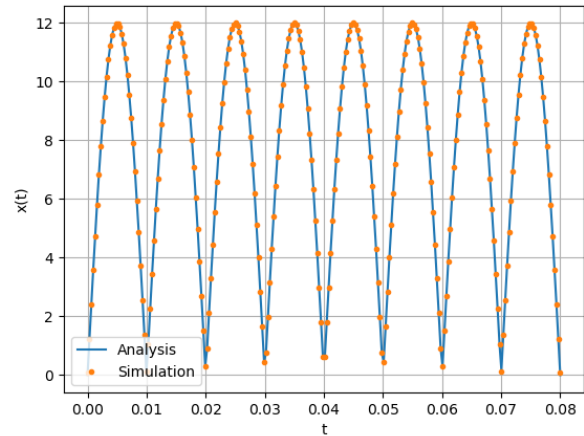
run the above code using the command

python3 2.3.py



Fig. 2.3: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t}$

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} \left( a_k \cos \jmath 2\pi k f_0 t + b_k \sin \jmath 2\pi k f_0 t \right)$$

$$\qquad (2.13)$$

and obtain the formulae for $a_k$ and $b_k$.

**Solution:** From (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \qquad (2.14)$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{\jmath 2\pi k f_0 t} + c_{-k} e^{-\jmath 2\pi k f_0 t} \quad (2.15)$$

$$= c_0 + \sum_{k=1}^{\infty} (c_k + c_{-k}) \cos(2\pi k f_0 t)$$

$$+ \sum_{k=0}^{\infty} (c_k - c_{-k}) \sin(2\pi k f_0 t) \qquad (2.16)$$

Hence, for $k \geq 0$,

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \qquad (2.17)$$

$$b_k = c_k - c_{-k} \qquad (2.18)$$

2.5 Find $a_k$ and $b_k$ for (1.1)

**Solution:** From (2.1), we see that since $x(t)$ is

even,

$$x(-t) = \sum_{k=-\infty}^{\infty} c_k e^{-\jmath 2\pi k f_0 t} \tag{2.19}$$

$$= \sum_{k=-\infty}^{\infty} c_{-k} e^{\jmath 2\pi k f_0 t} \tag{2.20}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.21}$$

where we substitute $k \mapsto -k$ in (2.20). Hence, we see that $c_k = c_{-k}$. So, from (2.18) and for $k \geq 0$,

$$a_k = \begin{cases} \dfrac{2A_0}{\pi} & k = 0 \\ \dfrac{4A_0}{\pi(1-k^2)} & k > 0, \ k \text{ even} \\ 0 & \text{otherwise} \end{cases} \tag{2.22}$$

$$b_k = 0 \tag{2.23}$$

2.6 Verify (2.13) using python.
**Solution:** The following code will plot the graph in fig (2.6)

> wget https://github.com/jarpula−Bhanu/
> EE3900/blob/main/charger/codes/2.6.py

run the above code using the command

> python3 2.6.py



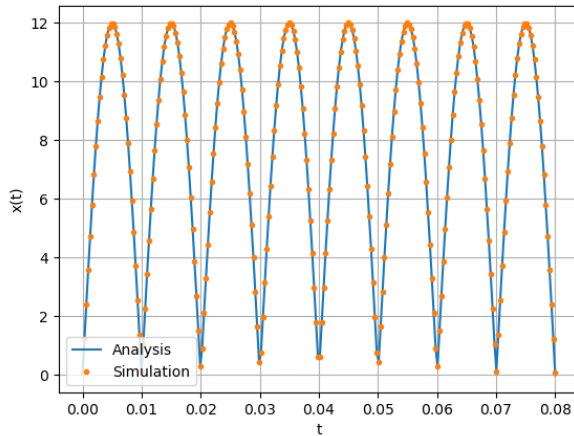Fig. 2.6: $x(t) = \sum_{k=0}^{\infty} \left(a_k \cos \jmath 2\pi k f_0 t + b_k \sin \jmath 2\pi k f_0 t\right)$

## 3 FOURIER TRANSFORM

3.1

$$\delta(t) = 0, \quad t \neq 0 \tag{3.1}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{3.2}$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft}\, dt \tag{3.3}$$

3.3 Show that

$$g(t-t_0) \xleftrightarrow{\mathcal{H}} FG(f)e^{-j2\pi f t_0} \tag{3.4}$$

**Solution:** We write, substituting $u := t - t_0$,

$$g(t-t_0) \xleftrightarrow{\mathcal{H}} F \int_{-\infty}^{\infty} g(t-t_0) e^{-\jmath 2\pi ft}\, dt \tag{3.5}$$

$$= \int_{-\infty}^{\infty} g(u) e^{-\jmath 2\pi f(u+t_0)}\, du \tag{3.6}$$

$$= G(f) e^{-j2\pi f t_0} \tag{3.7}$$

where the last equality follows from (3.3).

3.4 Show that

$$G(t) \xleftrightarrow{\mathcal{H}} Fg(-f) \tag{3.8}$$

**Solution:** Using the definition of the Inverse Fourier Transform,

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{\jmath 2\pi ft}\, df \tag{3.9}$$

Hence, setting $t := -f$ and $f := t$, which implies $df = dt$,

$$g(-f) = \int_{-\infty}^{\infty} G(t) e^{-\jmath 2\pi ft}\, dt \tag{3.10}$$

$$\implies G(t) \xleftrightarrow{\mathcal{H}} Fg(-f) \tag{3.11}$$

3.5 $\delta(t) \xleftrightarrow{\mathcal{H}} F$?
**Solution:** By applying the defination fo fourier transformation for $\delta(t-t_0)$ {$t_0$ be time shifting}.

$$\mathcal{F}\{\delta(t-t_0)\}(f) = \mathcal{F}(t) = \int_{-\infty}^{\infty} \delta(t-t_0) e^{-j2\pi ft} dt. \tag{3.12}$$

By applying the time shifting property of impulse we get

$$\mathcal{F}(f) = e^{-j2\pi f t_0} \tag{3.13}$$

$$i.e., \quad \delta(t-t_0) \xleftrightarrow{\mathcal{H}} Fe^{-j2\pi f t_0} \tag{3.14}$$

Now, substitute $t_0 = 0$ we get

$$\delta(t) \xleftrightarrow{\mathcal{H}} F1 \qquad (3.15)$$

3.6 $e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{H}} F?$

**Solution:** Applying the defination of inverse fourier transformation.

$$\mathcal{F}^{-1}\{\delta(f + f_0)\}(t) = f(t) = \int_{-\infty}^{\infty} \delta(f + f_0)e^{j2\pi ft}df \qquad (3.16)$$

By applying the shifting property of impulse

$$f(t) = e^{-j2\pi f_0 t} \qquad (3.17)$$

$$i.e., \quad e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{H}} F\delta(f + f_0) \qquad (3.18)$$

3.7 $\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{H}} F?$

**Solution:**

$$\cos(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \qquad (3.19)$$

$$\mathcal{F}[\cos(2\pi f_0 t)] = \int_{-\infty}^{\infty} \cos(2\pi f_0 t)e^{-j2\pi ft}dt \qquad (3.20)$$

$$= \int_{-\infty}^{\infty} \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}e^{-j2\pi ft}dt \qquad (3.21)$$

$$= \frac{1}{2}\left[ \int_{-\infty}^{\infty} e^{j2\pi f_0 t}e^{-j2\pi ft}dt + \int_{-\infty}^{\infty} e^{-j2\pi f_0 t}e^{-j2\pi ft}dt \right] \qquad (3.22)$$

$$= \frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)] \qquad (3.23)$$

$$\therefore \cos(2\pi f_0 t) \xleftrightarrow{\mathcal{H}} F\frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)] \qquad (3.24)$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python.

**Solution:**

$$x(t) = A_0 |\sin(2\pi f_0 t)| \qquad (3.25)$$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \qquad (3.26)$$

$$= \int_{-\infty}^{\infty} A_0 |\sin(2\pi f_0 t)| e^{-j2\pi ft}dt \qquad (3.27)$$

$$= A_0 \int_{0}^{\infty} \sin(2\pi f_0 t)e^{-j2\pi ft}dt + K \qquad (3.28)$$

$$= \frac{A_0}{2j} \int_{0}^{\infty} (e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})e^{-j2\pi ft}dt + K \qquad (3.29)$$

$$= \frac{A_0}{2j} \int_{0}^{\infty} (e^{j2\pi(f_0-f)t} + e^{-j2\pi(f_0-f)t})dt + K \qquad (3.30)$$

$$= \frac{A_0}{2j}\left[0 - \frac{1}{j2\pi(f_0 - f)}\right] - \frac{A_0}{2j}\left[0 - \frac{1}{j2\pi(f_0 + f)}\right] \qquad (3.31)$$

$$= \frac{A_0}{2\pi(f_0^2 - f^2)} + k \qquad (3.32)$$

By symmetry we get

$$k = \frac{A_0}{2\pi(f_0^2 - f^2)} \qquad (3.33)$$

$$\therefore \quad X(f) = \frac{A_0}{\pi(f_0^2 - f^2)} \qquad (3.34)$$

The following code will plot the graph in fig (3.8)

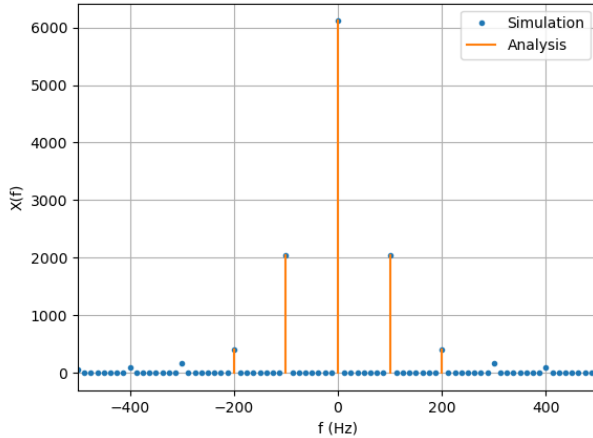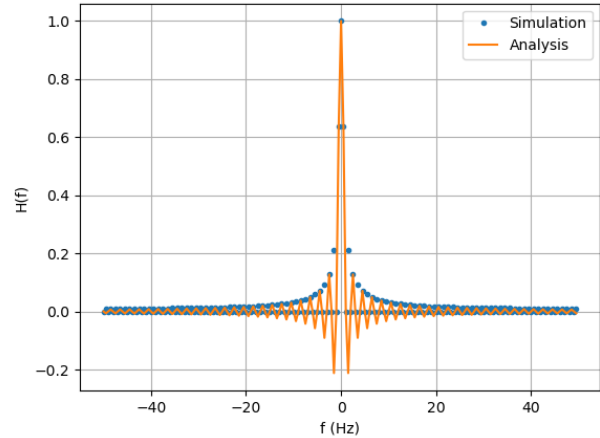wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/charger/codes/3.8.py

run the above code using the command

python3 3.8.py

3.9 Show that

$$\text{rect}(t) \xleftrightarrow{\mathcal{H}} F\text{sinc}(t) \qquad (3.35)$$

Verify using python.

Fig. 3.8: $\mathcal{F}[x(t)]$



Fig. 3.9: $\mathcal{F}[\text{rect}(t)]$

**Solution:**

$$\text{rect}(t) = \begin{cases} 1 & |t| < T \\ 0 & \text{otherwise} \end{cases} \quad (3.36)$$

$$\mathcal{F}[\text{rect}(t)] = \int_{-\infty}^{\infty} \text{rect}(t)\, e^{-j2\pi ft} dt \quad (3.37)$$

$$= \int_{-T}^{T} e^{-j2\pi ft} dt \quad (3.38)$$

$$= \frac{1}{-j2\pi f} \left[ e^{-j2\pi fT} - e^{j2\pi fT} \right] \quad (3.39)$$

$$= \frac{1}{\pi f} \left[ \frac{e^{j2\pi fT} - e^{-j2\pi fT}}{2j} \right] \quad (3.40)$$

$$= \frac{\sin \pi f}{\pi f} \quad (3.41)$$

$$= \text{sinc}(f) \quad (3.42)$$

The following code will plot the graph in fig (3.9)

wget https://github.com/jarpula−Bhanu/
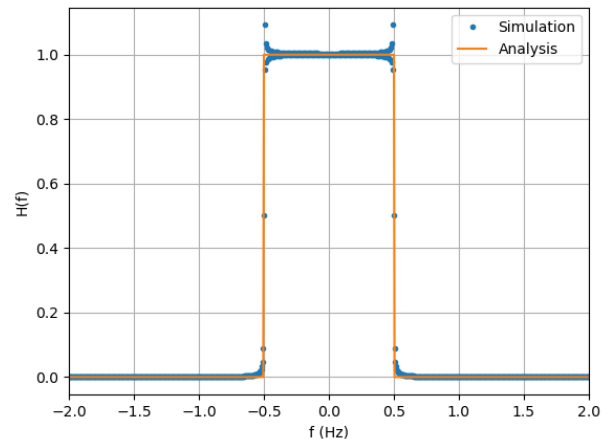EE3900/blob/main/charger/codes/3.9.py

run the above code using the command

python3 3.9.py

3.10 $\text{sinc}(t) \overset{\mathcal{H}}{\longleftrightarrow} F$?. Verify using python.
**Solution:** from (3.8) we can say

$$\text{sinc}(t) \overset{\mathcal{H}}{\longleftrightarrow} F\text{rect}(-f) \quad (3.43)$$

$$\text{rect}(f) \quad (3.44)$$

The following code will plot the graph in fig (3.10)

wget https://github.com/jarpula−Bhanu/
EE3900/blob/main/charger/codes/3.10.py

run the above code using the command

python3 3.10.py



Fig. 3.10: $\mathcal{F}[\text{sinc}(t)]$

## 4 FILTER

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5V.
**Solution:** The function $H(f)$ is a low pass filter which filters out even harmonics and leaves the zero frequency component behind. The rectangular function represents an ideal

low pass filter. Suppose the cutoff frequency is $f_c = 50$ Hz, then

$$H(f) = \text{rect}\left(\frac{f}{2f_c}\right) = \begin{cases} 1 & |f| < f_c \\ 0 & \text{otherwise} \end{cases} \quad (4.1)$$

Multiplying by a scaling factor to get DC 5V,

$$H(f) = \frac{\pi V_0}{2A_0}\text{rect}\left(\left(\frac{f}{2f_c}\right)\right) \quad (4.2)$$

where $V_0 = 5$ V.

4.2 Find $h(t)$.

**Solution:** Suppose $g(t) \xleftrightarrow{\mathcal{H}} FG(f)$. Then, for some nonzero $a \in \mathbb{R}$

$$g(at) \xleftrightarrow{\mathcal{H}} F \int_{-\infty}^{\infty} g(at)e^{-\jmath 2\pi ft}\,dt \quad (4.3)$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} g(u)e^{\left(-\jmath 2\pi \frac{f}{a}t\right)}\,dt \quad (4.4)$$

$$= \frac{1}{a}G\left(\frac{f}{a}\right) \quad (4.5)$$

where we have substituted $u := at$. Using (4.5) of the Fourier Transform in (4.1),

$$h(t) = \frac{2\pi V_0}{A_0}f_c\text{sinc}\,(2f_ct) \quad (4.6)$$

4.3 Verify your result using through convolution.
**Solution:** The following code will plot the graph in fig (4.3)

wget https://github.com/jarpula−Bhanu/
EE3900/blob/main/charger/codes/4.3.py

run the above code using the command
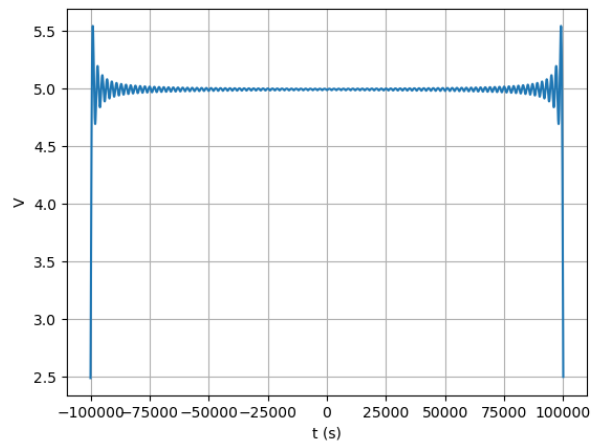
python3 4.3.py



Fig. 4.3: convolution of two signals

## 5 Filter Design

5.1 Design a Butterworth filter for $H(f)$.
5.2 Design a Chebyschev filter for $H(f)$.
5.3 Design a circuit for your Butterworth filter.
5.4 Design a circuit for your Chebyschev filter.