Circuits and Transforms

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Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\,dt \tag{1.2}$$

2 Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.

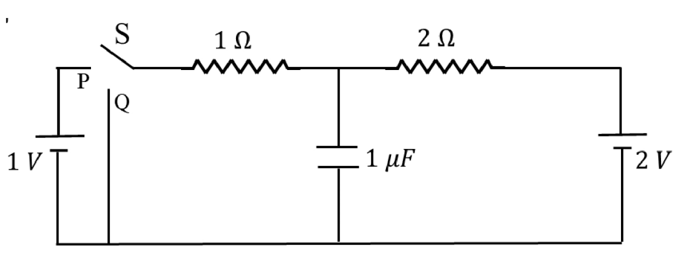

Fig. 2.1

2. Draw the circuit using latex-tikz.
Solution: The circuit drawn using the latex-tikz is Fig.(2.2)
When switch is closed at position P

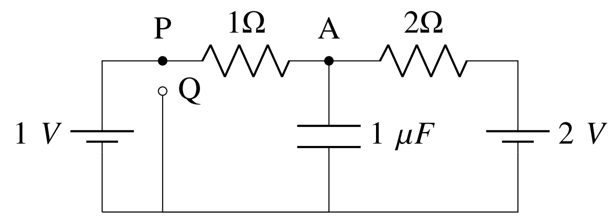

Fig. 2.2

3. Find q_1.
Solution: When the switch is closed for long time the circuit achives steady state condition. Then the equivalent circuit is given by

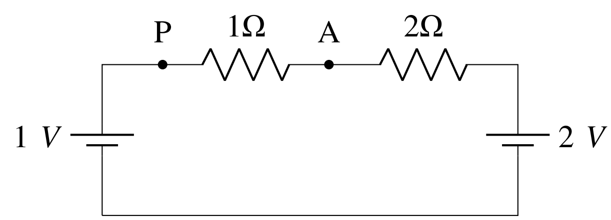

Fig. 2.3

consider the circuit as cells with internal resistors connected in series.

$$i = \frac{V}{R_{eq}} = \frac{1}{1+2} = \frac{1}{3}A \tag{2.1}$$

Now voltage across 2Ω resistro is given by

$$\implies \quad 2 - V_A = \frac{1}{3} \times 2 = \frac{2}{3} \tag{2.2}$$

$$V_A = 2 - \frac{2}{3} \tag{2.3}$$

$$= \frac{4}{3} \tag{2.4}$$

Now, charge

$$q_1 = C\Delta V \tag{2.5}$$

$$= 1 \times \frac{4}{3} \tag{2.6}$$

$$= \frac{4}{3} \tag{2.7}$$

charge on q_1 is $\frac{4}{3}\mu C$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution:

$$\mathcal{L}[u(t)] = U(s) = \int_{-\infty}^{\infty} u(t)e^{-st}dt \tag{2.8}$$

$$U(s) = \int_0^0 \frac{1}{2}e^{-st}dt + \int_0^{\infty} 1e^{-st}dt \tag{2.9}$$

$$= -\frac{1}{s}\left|e^{-st}\right|_0^{\infty} \tag{2.10}$$

$$= -\frac{1}{s}\left[e^{-s\infty} - e^{-s\times 0}\right] \tag{2.11}$$

The above term converges if and only if $e^{-st} \to 0$ as $t \to \infty$ only possible if $Re\{s\} > 0$.
hence

$$U(s) = \frac{1}{s} \qquad Re\{s\} > 0 \tag{2.12}$$

5. Show that

$$e^{-at}u(t) \overset{\mathcal{H}}{\longleftrightarrow} L\frac{1}{s+a}, \quad a > 0 \tag{2.13}$$

and find the ROC.
Solution: Let $x(t) = e^{-at}u(t)$

$$\mathcal{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt \tag{2.14}$$

$$= \int_{-\infty}^{\infty} e^{-(s+a)t}u(t)dt \tag{2.15}$$

$$X(s) = \int_0^0 \frac{1}{2}e^{-(s+a)t}dt + \int_0^{\infty} 1e^{-(s+a)t}dt \tag{2.16}$$

$$= -\frac{1}{s+a}\left|e^{-(s+a)t}\right|_0^{\infty} \tag{2.17}$$

$$= -\frac{1}{s+a}\left[e^{-(s+a)\infty} - e^{-(s+a)\times 0}\right] \tag{2.18}$$

The above term converges if and only if $e^{-(s+a)t} \to 0$ as $t \to \infty$ only possible if $Re\{s\} > -a$.
hence

$$X(s) = \frac{1}{s+a} \qquad Re\{s\} > -a \tag{2.19}$$

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

$$u(t) \overset{\mathcal{H}}{\longleftrightarrow} LV_1(s) \tag{2.20}$$

$$2u(t) \overset{\mathcal{H}}{\longleftrightarrow} LV_2(s) \tag{2.21}$$

Find the voltage across the capacitor $V_{C_0}(s)$.

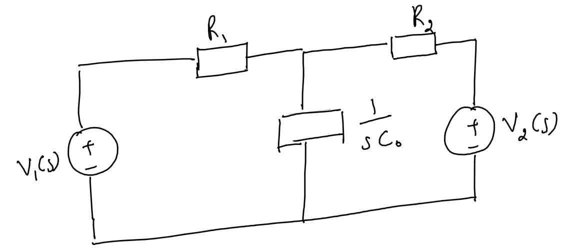

Fig. 2.4

Solution: We see that

$$V_1(s) = \frac{1}{s} \qquad V_2(s) = \frac{2}{s} \tag{2.22}$$

In Fig. 2.2, we use KCL at A.

$$\frac{V - \frac{1}{s}}{R_1} + \frac{V - \frac{2}{s}}{R_2} + sC_0V = 0 \tag{2.23}$$

$$V\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{1}{s}\left(\frac{1}{R_1} + \frac{2}{R_2}\right) \tag{2.24}$$

$$V(s) = \frac{\frac{1}{R_1} + \frac{2}{R_2}}{s\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right)} \tag{2.25}$$

$$= \frac{\frac{1}{R_1} + \frac{2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) \tag{2.26}$$

7. Find $v_{C_0}(t)$. Plot using python.
Solution: Taking the inverse Laplace transform in (2.26),

$$V(s) \overset{\mathcal{H}}{\longleftrightarrow} L\frac{2R_1 + R_2}{R_1 + R_2}u(t)\left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}\right) \tag{2.27}$$

$$= \frac{4}{3}\left(1 - e^{-\left(1.5\times 10^6\right)t}\right)u(t) \tag{2.28}$$

The following code will plot the graph Fig.(2.5)

```
wget https://github.com/jarpula-Bhanu/
    EE3900/blob/main/Cktsig/codes/2.7.py
```

run the above code using the command

```
python3 2.7.py
```

8. Verify your result using ngspice.
Solution: The follwoing code will generate the text file of stimulation.

```
wget https://github.com/jarpula-Bhanu/
    EE3900/blob/main/Cktsig/codes/2.8.cir
```

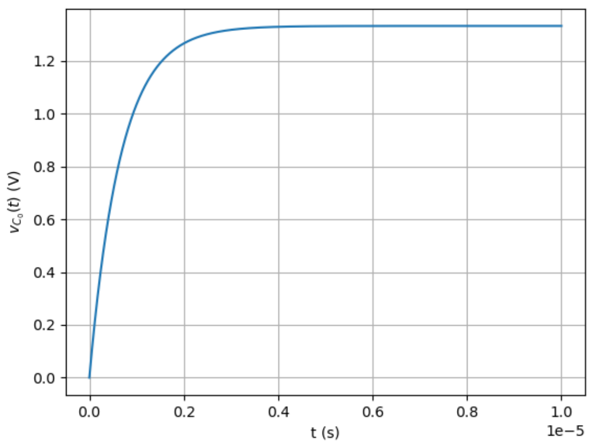

Fig. 2.5: $v_{C_0}(t)$ before the switch is flipped

The following code will plot the graph Fig.(2.6)

```
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Cktsig/codes/2.8.py
```

run the above code using the command

```
python3 2.8.py
```

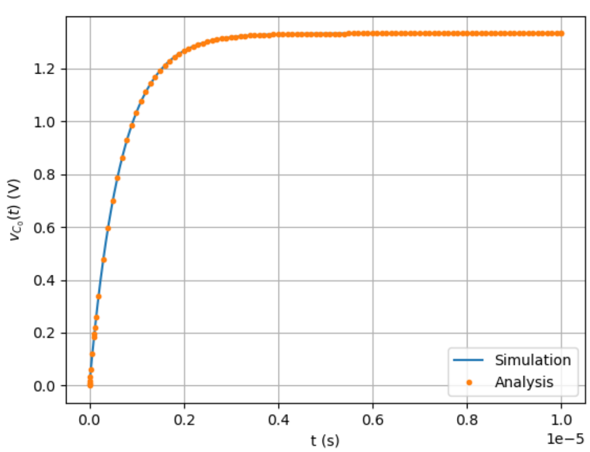

Fig. 2.6: $v_{C_0}(t)$ before the switch is flipped

9. Obtain Fig. 2.4 using the equivalent differential equation.

3 Initial Conditions

1. Find q_2 in Fig. 2.1.
Solution: When the switch is closed for long time the circuit achives steady state condition.

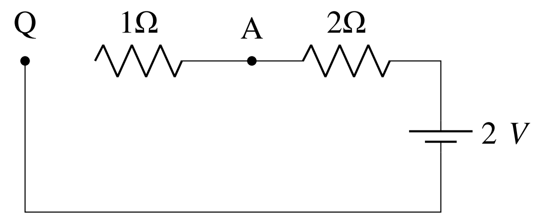

Fig. 3.1

Then the equivalent circuit is given by Since capacitor behaves as an open circuit, we use KCL at A.

$$\frac{V-0}{1}+\frac{V-2}{2}=0 \implies V=\frac{2}{3}V \tag{3.1}$$

and hence, $q_2=\frac{2}{3}\mu C$.

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.
Solution:

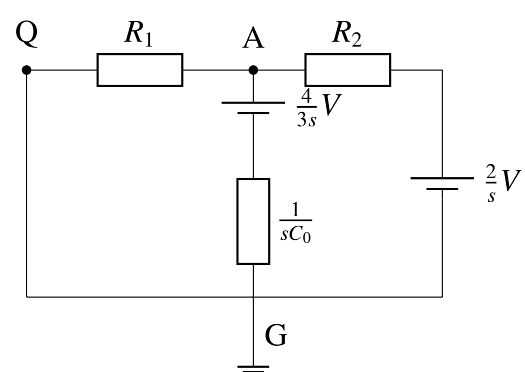

Fig. 3.2

3. $V_{C_0}(s)=?$
Solution: Using KCL at node A in Fig. 3.2

$$\frac{V-0}{R_1}+\frac{V-\frac{2}{s}}{R_2}+sC_0\left(V-\frac{4}{3s}\right)=0 \tag{3.2}$$

$$\implies V_{C_0}(s)=\frac{\frac{2}{sR_2}+\frac{4C_0}{3}}{\frac{1}{R_1}+\frac{2}{R_2}+sC_0} \tag{3.3}$$

4. $v_{C_0}(t)=?$ Plot using python.

Solution: From (3.3),

$$V_{C_0}(s) = \frac{4}{3}\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(\frac{1}{s}-\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) \tag{3.4}$$

Taking an inverse Laplace Transform,

$$v_{C_0}(t) = \frac{4}{3}e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(1-e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \tag{3.5}$$

Substituting values gives

$$v_{C_0}(t) = \frac{2}{3}\left(1+e^{-\left(1.5\times10^6\right)t}\right)u(t) \tag{3.6}$$

The following code will plot the graph Fig.(3.3)

```
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Cktsig/codes/3.4.py
```

run the above code using the command

```
python3 3.4.py
```

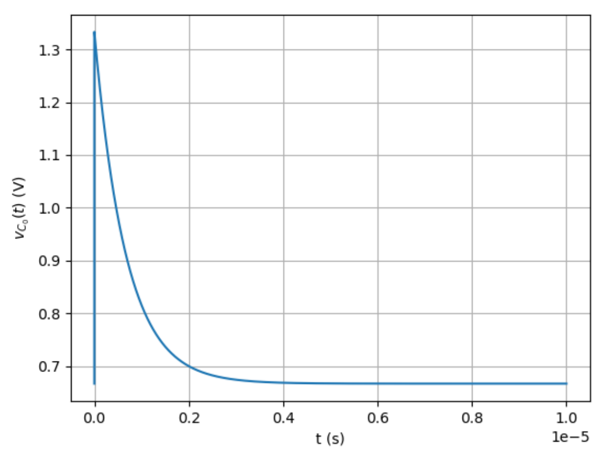

Fig. 3.3: $v_{C_0}(t)$ after the switch is flipped

5. Verify your result using ngspice.
Solution: The follwoing code will generate the text file of stimulation.

```
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Cktsig/codes/3.5.cir
```

The following code will plot the graph Fig.(3.4)

```
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Cktsig/codes/3.5.py
```

run the above code using the command

```
python3 3.5.py
```

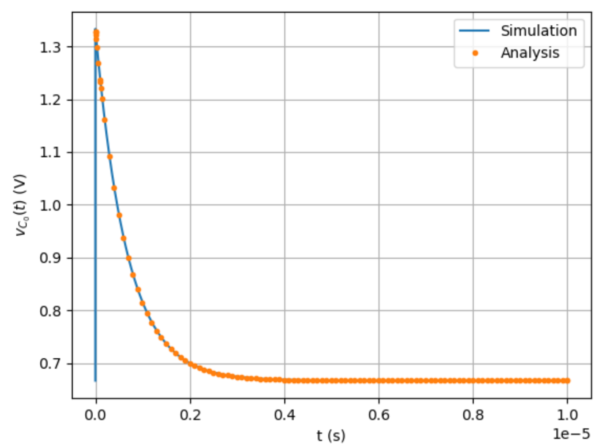

Fig. 3.4: $v_{C_0}(t)$ after the switch is flipped

6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
Solution: From the initial conditions,

$$v_{C_0}(0-) = \frac{q_1}{C} = \frac{4}{3}V \tag{3.7}$$

$$v_{C_0}(0+) = \lim_{t\to0+} v_{C_0}(t) = \frac{4}{3}V \tag{3.8}$$

Using (3.6),

$$v_{C_0}(\infty) = \lim_{t\to\infty} v_{C_0}(t) = \frac{2}{3}V \tag{3.9}$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equation.
Solution: Using Kirchoff's junction law

$$\frac{v_c(t)-0}{R_1} + \frac{v_c(t)-v_2(t)}{R_2} + \frac{\mathrm{d}q}{\mathrm{d}t} = 0 \tag{3.10}$$

where $q(t)$ is the charge on the capacitor
On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s)-0}{R_1} + \frac{V_c(s)-V_2(s)}{R_2} + \left(sQ(s)-q(0^-)\right) = 0 \tag{3.11}$$

But $q(0^-) = \frac{4}{3}C_0$ and

$$q(t) = C_0 v_c(t) \tag{3.12}$$

$$\implies Q(s) = C_0 V_c(s) \tag{3.13}$$

Thus

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \left(sC_0 V_c(s) - \frac{4}{3}C_0\right) = 0 \tag{3.14}$$

$$\implies \frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \frac{V_c(s) - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \tag{3.15}$$

which is the same equation as the one we obtained from Fig. 3.2

4 Bilinear Transform

4.1. In Fig. 2.1, consider the case when S is switched to Q right in the beginning. Formulate the differential equation
Solution: The equivalent circuit in the t-domain is shown below.

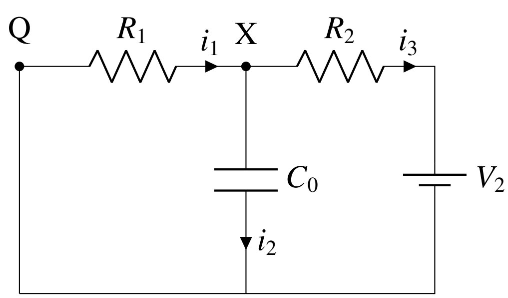

Fig. 4.1

The differential equation is the same as before

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{\mathrm{d}q}{\mathrm{d}t} = 0 \tag{4.1}$$

$$\text{i.e., } \frac{v_c(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + C_0\frac{\mathrm{d}v_c}{\mathrm{d}t} = 0 \tag{4.2}$$

but with a different initial condition

$$q(0^-) = q(0) = 0 \tag{4.3}$$

4.2. Find $H(s)$ considering the ouput voltage at the capacitor
Solution: Transforming Fig. 4.1 to the s-domain, On taking the Laplace transform on

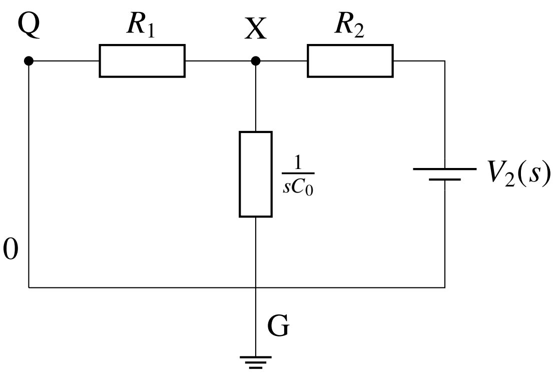

Fig. 4.2

both sides of this equation

$$\frac{V_c(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + sQ(s) - 0 = 0 \tag{4.4}$$

$$\implies V_c(s)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + sC_0 V_c(s) = \frac{V_2(s)}{R_2} \tag{4.5}$$

$$\implies \frac{V_c(s)}{V_2(s)} = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \tag{4.6}$$

The transfer function is thus

$$H(s) = \frac{\frac{1}{R_2 C_0}}{s + \frac{1}{R_1 C_0} + \frac{1}{R_2 C_0}} \tag{4.7}$$

On substituting the values, we get

$$H(s) = \frac{5 \times 10^5}{s + 1.5 \times 10^6} \tag{4.8}$$

4.3. Plot $H(s)$. What kind of filter is it?
Solution: Download the following Python code that plots Fig. 4.3

```
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Cktsig/codes/4.3.py
```

Run the codes by executing

```
python 4.3.py
```

Consider the frequency-domain transfer function by putting $s = \jmath\omega$

$$H(\jmath\omega) = \frac{5 \times 10^5}{\jmath\omega + 1.5 \times 10^6} \tag{4.9}$$

$$\implies \left|H(\jmath\omega)\right| = \frac{5 \times 10^5}{\sqrt{\omega^2 + 2.25 \times 10^{12}}} \tag{4.10}$$

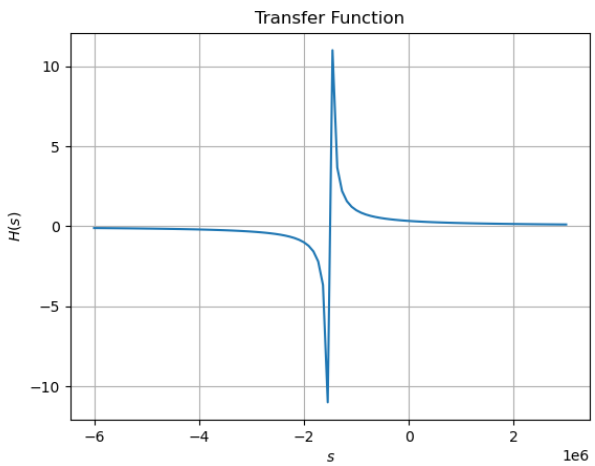

Fig. 4.3: Plot of $H(s)$

As ω increases, $\left|H(\jmath\omega)\right|$ decreases
In other words, the amplitude of high-frequency signals gets diminished and they get filtered out
Therefore, this is a low-pass filter

4.4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \tag{4.11}$$

Solution:

$$\frac{v_c(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + C_0 \frac{\mathrm{d}v_c}{\mathrm{d}t} = 0 \tag{4.12}$$

$$\implies C_0 \frac{\mathrm{d}v_c}{\mathrm{d}t} = \frac{2u(t) - v_c(t)}{R_2} - \frac{v_c(t)}{R_1} \tag{4.13}$$

$$\implies v_c(t)|_{t=n}^{n+1} = \int_n^{n+1} \left(\frac{2u(t) - v_c(t)}{R_2 C_0} - \frac{v_c(t)}{R_1 C_0} \right) \mathrm{d}t \tag{4.14}$$

By the trapezoidal rule of integration

$$\int_a^b f(t)\mathrm{d}t \approx \frac{b-a}{2}(f(a) + f(b)) \tag{4.15}$$

Consider $y(t) = v_c(t)$

$$y(n+1) - y(n) = \frac{1}{R_2 C_0}(u(n) + u(n+1)) - \frac{1}{2}(y(n+1) + y(n)) \left(\frac{1}{R_1 C_0} + \frac{1}{R_2 C_0} \right) \tag{4.16}$$

Thus, the difference equation is

$$y(n+1)\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = y(n)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right) + \frac{1}{R_2C_0}(u(n) + u(n+1)) \tag{4.17}$$

4.5. Find $H(z)$
Solution: Let $\mathcal{Z}\{y(n)\} = Y(z)$
On taking the Z-transform on both sides of the difference equation

$$zY(z)\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = Y(z)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right) + \frac{1}{R_2C_0}\left(\frac{1}{1-z^{-1}} + \frac{z}{1-z^{-1}}\right) \tag{4.18}$$

$$Y(z)\left(z + \frac{z}{2R_1C_0} + \frac{z}{2R_2C_0} - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = \frac{1}{R_2C_0}\frac{1+z}{1-z^{-1}} \tag{4.19}$$

Also

$$v_2(t) = 2 \qquad \forall t \geq 0 \tag{4.20}$$

$$\implies x(n) = 2u(n) \tag{4.21}$$

$$\implies X(z) = \frac{2}{1-z^{-1}} \qquad |z| > 1 \tag{4.22}$$

Thus, the transfer function in z-domain is

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.23}$$

$$= \frac{\frac{1+z}{2R_2C_0}}{z + \frac{z}{2R_1C_0} + \frac{z}{2R_2C_0} - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}} \tag{4.24}$$

$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0} - z^{-1} + \frac{z^{-1}}{2R_1C_0} + \frac{z^{-1}}{2R_2C_0}} \tag{4.25}$$

On substituting the values

$$H(z) = \frac{2.5 \times 10^5(1 + z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \tag{4.26}$$

with the ROC being

$$|z| > \max\left(1, \left|\frac{7.5\times10^5 - 1}{7.5\times10^5 + 1}\right|\right) \tag{4.27}$$
$$\implies |z| > 1 \tag{4.28}$$

4.6. How can you obtain $H(z)$ from $H(s)$?
Solution: The Z-transform can be obtained from the Laplace transform by the substitution

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{4.29}$$

where T is the step size of the trapezoidal rule (1 in our case)
This is known as the bilinear transform
Thus

$$H(z) = \frac{\frac{1}{R_2C_0}}{2\frac{1-z^{-1}}{1+z^{-1}} + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \tag{4.30}$$
$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 - z^{-1} + \left(\frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right)(1 + z^{-1})} \tag{4.31}$$
$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0} - z^{-1} + \frac{z^{-1}}{2R_1C_0} + \frac{z^{-1}}{2R_2C_0}} \tag{4.32}$$
$$= \frac{2.5\times10^5(1 + z^{-1})}{7.5\times10^5 + 1 + (7.5\times10^5 - 1)z^{-1}} \tag{4.33}$$

which is the same as what we obtained earlier

4.7. Find $v(n)$. Verify using ngspice and the differential equation.
Solution: We have,

$$V(z) = H(z)V_i(z) \tag{4.34}$$
$$= \frac{TV_2\tau\left(1 + z^{-1}\right)}{C_0R_2\left(1 - z^{-1}\right)\left((2\tau + T) - (2\tau - T)z^{-1}\right)} \tag{4.35}$$
$$= \frac{V_2\tau(z+1)}{2C_0R_2}\sum_{k=-\infty}^{\infty}\left(1 - p^k\right)u(k)z^{-k} \tag{4.36}$$

where $p := \frac{2\tau-T}{2\tau+T}$. Thus,

$$v(n) = \frac{V_2\tau}{C_0R_2}\left[u(n)\left(1 - p^n\right) + u(n+1)\left(1 - p^{n+1}\right)\right] \tag{4.37}$$

where $p := \frac{2\tau-1}{2\tau+1}$. We take $T = 10^{-7}$ as the sampling interval. these equalities.
Download the following Python code that plots Fig. 4.4

```
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Cktsig/codes/4.7.cir
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Cktsig/codes/4.7.py
```

Run the codes by executing

```
ngspice 4.7.cir
python 4.7.py
```

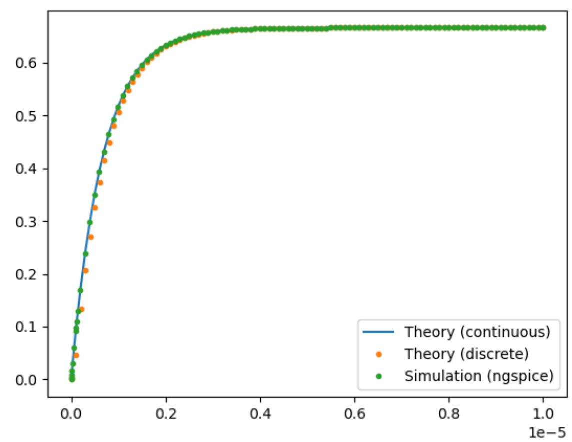

Fig. 4.4: Representation of output across C_0.