Filter-Sounds

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Contents

Abstract—**This manual provides a simple introduction to digital signal processing.**

1 Software installation

1.1 Run the following commands.

```
sudo apt-get update
sudo apt-get install libffi-dev libsndfile1
    python3-scipy python3-numpy python3-
    matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/jarpula-Bhanu/
    EE3900/blob/main/Filter/soundfiles/
    Sound_Noise.wav
```

2.2 You will find a spectogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in problem 2.1 in the spectrogram and play. Observe the spectogram. What do you find?

Solution: There are a lot of yellow lines betweeen 440Hz to 5.1KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band nosie and execute the code.
Solution: Download and run the following code.

```
wget https://github.com/jarpula-Bhanu/
    EE3900/blob/main/Filter/Codes/2.3_noise
    .py
```

run the above code using the command

```
python3 2.3_noise.py
```

2.4 The output of the python scripy in problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectogram in problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also the signal is blank for frequencies above 5.1KHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \qquad (3.1)$$

Sketch x(n).
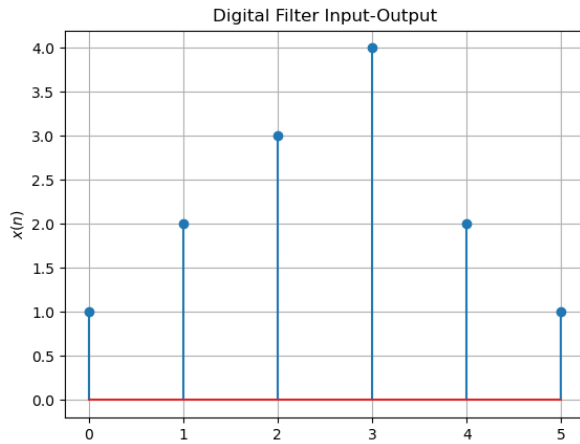**Solution:** Download and run the following code.Below code plots fig(3.1)

```
wget https://github.com/jarpula-Bhanu/
    EE3900/blob/main/Filter/Codes/3.1.py
```

run the above code using the command

```
python3 3.1.py
```

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2), y(n) = 0, n < 0 \qquad (3.2)$$
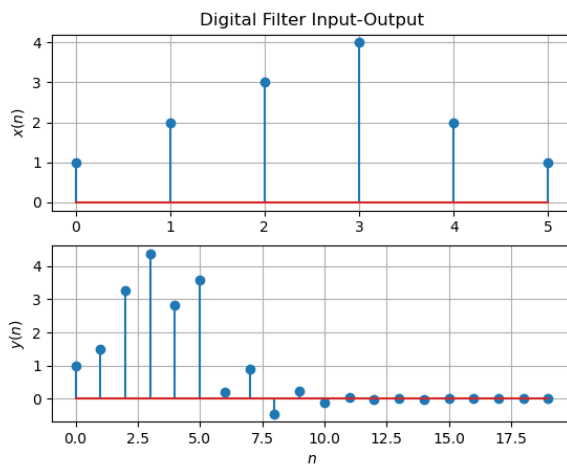
Fig. 3.1: Sketch of $x(n)$

Sketch y(n).
Solution: Download and run the following code.Below code plots fig(3.2)

wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/3.2.py

run the above code using the command

python3 3.2.py

Fig. 3.2: Sketch of $x(n)$ and $y(n)$
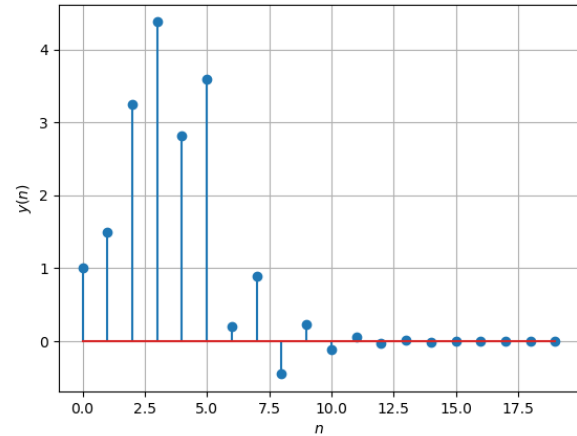
3.3 Repeat the above exercise using C code.
    Solution: Download and run the following code.Below code plots fig(3.3)

wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/3.3.c

wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/3.3_plot.
    py

run the above code using the command

gcc 3.3.c −o 3.3.out
./3.3.out
python3 3.3_plot.py

Fig. 3.3: Sketch of $y(n)$

4 Z-TRANSFORM

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \qquad (4.1)$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \qquad (4.2)$$

and find

$$\mathcal{Z}\{x(n-k)\} \qquad (4.3)$$

Solution: From (4.1)

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \qquad (4.4)$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n}$$
$$(4.5)$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \qquad (4.6)$$

$\mathcal{Z}$- transform of $x(n)$ is

$$\mathcal{Z}\{x(n)\} = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5}$$
(4.7)

$$\mathcal{Z}\{x(n-k)\} = z^{-k}(1 + 2z^{-1} + 3z^{-2}+ \quad (4.8)$$
$$4z^{-3} + 2z^{-4} + z^{-5}) \quad (4.9)$$
$$\mathcal{Z}\{x(n-k)\} = z^{-k} + 2z^{-(k+1)} + 3z^{-(k+2)} \quad (4.10)$$
$$+ 4z^{-(k+3)} + 2z^{-(k+4)} + z^{-(k+5)}$$
(4.11)

4.2 Obtain $X(z)$ for $x(n)$ in problem (3.1)
Solution:

$$\mathcal{Z}\{x(n)\} = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5}$$
(4.12)

$$\mathcal{Z}\{x(n-k)\} = z^{-k}(1 + 2z^{-1} + 3z^{-2}+ \quad (4.13)$$
$$4z^{-3} + 2z^{-4} + z^{-5}) \quad (4.14)$$
$$\mathcal{Z}\{x(n-k)\} = z^{-k} + 2z^{-(k+1)} + 3z^{-(k+2)} \quad (4.15)$$
$$+ 4z^{-(k+3)} + 2z^{-(k+4)} + z^{-(k+5)}$$
(4.16)

4.3 Find
$$H(z) = \frac{Y(z)}{X(z)} \quad (4.17)$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \quad (4.18)$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (4.19)$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.20)$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.21)$$

is
$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (4.22)$$

Solution: It is easy to show that

$$\delta(n) \stackrel{\mathcal{Z}}{\rightleftharpoons} 1 \quad (4.23)$$

and from (4.21),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \quad (4.24)$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (4.25)$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \stackrel{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \quad (4.26)$$

Solution:

$$a^n u(n) \stackrel{\mathcal{Z}}{\rightleftharpoons} \sum_{n=0}^{\infty} (az^{-1})^n \quad (4.27)$$

$$= \frac{1}{1 - az^{-1}} \quad |z| > |a| \quad (4.28)$$

4.6 Let
$$H\left(e^{j\omega}\right) = H\left(z = e^{j\omega}\right). \quad (4.29)$$

Plot $\left|H\left(e^{j\omega}\right)\right|$. Comment. $H(e^{j\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.
Solution: Download and run the following code. The following code plots Fig. 4.6.

```
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/4.5.py
```

run the above code using the command

```
python3 4.5.py
```

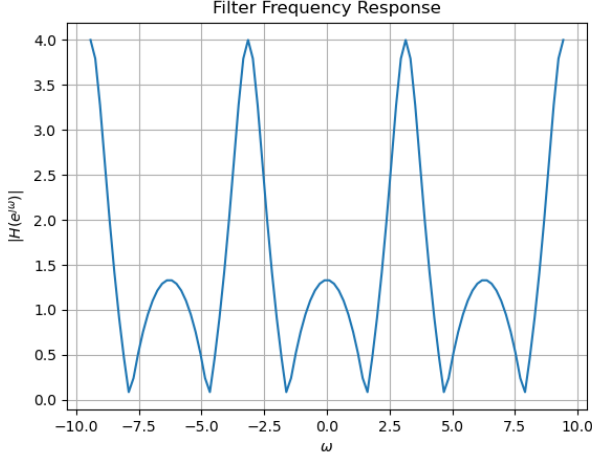We observe that $\left|H\left(e^{j\omega}\right)\right|$ is periodic with fundamental period 2π.

Fig. 4.6: $\left|H\left(e^{j\omega}\right)\right|$

$$H\left(e^{j\omega}\right) = \frac{1 + e^{-2j\omega}}{1 + \frac{1}{2}e^{-j\omega}} \tag{4.30}$$

$$\implies \left|H\left(e^{j\omega}\right)\right| = \frac{|1 + \cos 2\omega - j\sin 2\omega|}{\left|1 + \frac{1}{2}\cos\omega - \frac{1}{2}\sin\omega\right|} \tag{4.31}$$

$$= \sqrt{\frac{(1 + \cos 2\omega)^2 + (\sin 2\omega)^2}{(1 + \frac{1}{2}\cos\omega)^2 + (\frac{1}{2}\sin\omega)^2}} \tag{4.32}$$

$$= \sqrt{\frac{2 + 2\cos 2\omega}{\frac{5}{4} + \cos\omega}} \tag{4.33}$$

$$= \sqrt{\frac{2(2\cos^2\omega)4}{5 + 4\cos\omega}} \tag{4.34}$$

$$= \frac{4|\cos\omega|}{\sqrt{5 + 4\cos\omega}} \tag{4.35}$$

period of $|cos\omega|$ is π and period of $\sqrt{5 + 4\cos\omega}$ is 2π.
Now period of $\left|H\left(e^{j\omega}\right)\right|$ is $\frac{LCM(\pi,2\pi)}{HCF(\pi,2\pi)} = \frac{2\pi}{1} = 2\pi$

4.7 Express $h(n)$ in terms of $H(e^{j\omega})$
Solution: $h(n)$ is given by the inverse DTFT

of $H(e^{j\omega})$

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \tag{4.36}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} h(k)e^{-j\omega k}e^{j\omega n}d\omega \tag{4.37}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} e^{j\omega(n-k)}d\omega \tag{4.38}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} h(k)2\pi\delta[n - k] \tag{4.39}$$

$$= h(n) \tag{4.40}$$

Since

$$\int_{-\pi}^{\pi} e^{j\omega n}d\omega = 2\pi\delta[n] \tag{4.41}$$

$$and \tag{4.42}$$

$$\delta[n - k] = \begin{cases} 1 & n = k \\ 0 & \text{otherwise} \end{cases} \tag{4.43}$$

5 IMPULSE RESPONSE

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for $H(z)$ in (4.19)
Solution:

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.2}$$

Substitute $z^{-1} = x$

$$
\begin{array}{r}
2x - 4 \\
\frac{1}{2}x + 1 \overline{)\ x^2 \qquad + 1} \\
\underline{-x^2 - 2x} \\
-2x + 1 \\
\underline{2x + 4} \\
5
\end{array}
$$

$$\implies 1 + z^{-2} = \left(1 + \frac{1}{2}z^{-1}\right)\left(-4 + 2z^{-1}\right) + 5 \tag{5.3}$$

$$\implies H(z) = -4 + 2z^{-1} + \frac{5}{1 + \frac{1}{2}z^{-1}} \tag{5.4}$$

$$\frac{5}{1 + \frac{1}{2}z^{-1}} = 5\left(1 + \frac{1}{2}z^{-1}\right)^{-1} \quad (5.5)$$

$$= 5\sum_{n=0}^{\infty}\left(-\frac{z^{-1}}{2}\right)^{n} \quad (5.6)$$

$$H(z) = -4 + 2z^{-1} + 5 - \frac{5}{2}z^{-1} + \frac{5}{4}z^{-2}$$
$$- \frac{5}{8}z^{-3} + \frac{5}{16}z^{-4} - \frac{5}{32}z^{-5} + \cdots \quad (5.7)$$

$$H(z) = -4 + 2z^{-1} + 5\left(-\frac{1}{2}\right)^{n-2} \text{ for } n \geq 2 \quad (5.8)$$

Therefore, by comparing coefficients

$$h(n) = \begin{cases} 1 & n = 0 \\ -\dfrac{1}{2} & n = 1 \\ \dfrac{5}{4} & n = 2 \\ -\dfrac{5}{8} & n = 3 \\ \dfrac{5}{16} & n = 4 \end{cases} \quad (5.9)$$

Alternatively, on applying the inverse Z-transform on both sides of the equation

$$H(z) \stackrel{\mathcal{Z}}{\rightleftharpoons} h(n) \quad (5.10)$$

$$-4 \stackrel{\mathcal{Z}}{\rightleftharpoons} -4\delta(n) \quad (5.11)$$

$$2z^{-1} \stackrel{\mathcal{Z}}{\rightleftharpoons} 2\delta(n-1) \quad (5.12)$$

$$\frac{5}{1 + \frac{1}{2}z^{-1}} \stackrel{\mathcal{Z}}{\rightleftharpoons} 5\left(-\frac{1}{2}\right)^{n}u(n) \quad (5.13)$$

$$(5.14)$$

Therefore,

$$h(n) = -4\delta(n) + 2\delta(n-1) + 5\left(-\frac{1}{2}\right)^{n}u(n) \quad (5.15)$$

Download the following Python code that plots Fig. 5.1.

wget https://github.com/jarpula−Bhanu/
     EE3900/blob/main/Filter/Codes/5.1.py
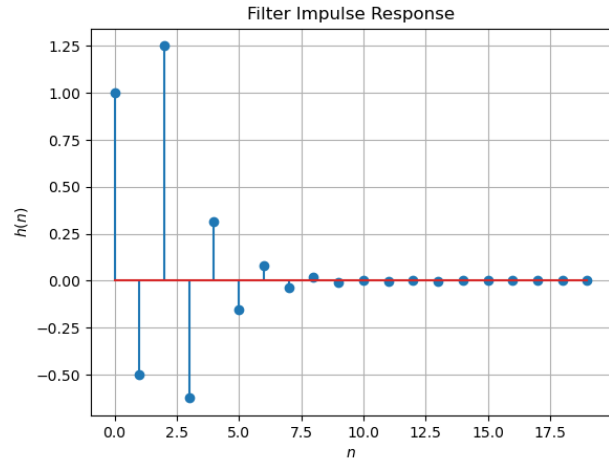
Run the code by executing

python 5.1.py

Fig. 5.1: Plot of $h(n)$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \stackrel{\mathcal{Z}}{\rightleftharpoons} H(z) \quad (5.16)$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.19),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (5.17)$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^{n}u(n) + \left(-\frac{1}{2}\right)^{n-2}u(n-2) \quad (5.18)$$

using (4.26) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution: Download and run the following code.The following code plots Fig. 5.3.

wget https://github.com/jarpula−Bhanu/
     EE3900/blob/main/Filter/Codes/5.2.py

run the above code using the command.

python3 5.2.py
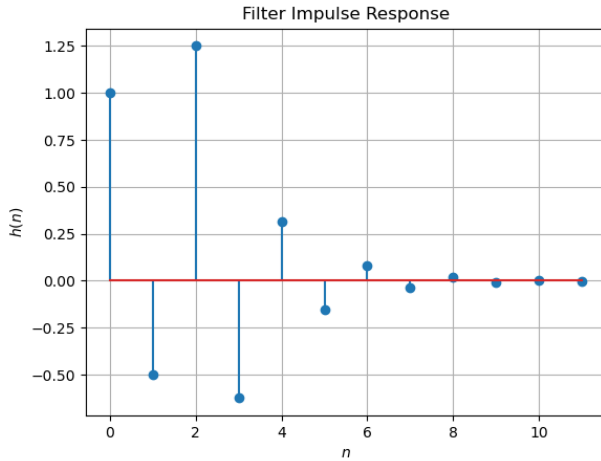
From the plot, it is clear that $h(n)$ is bounded.

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

Theoretically,

$$|u(n)| \leq 1 \qquad (5.19)$$

$$\left|\left(-\frac{1}{2}\right)^n\right| \leq 1 \qquad (5.20)$$

$$\implies \left|\left(-\frac{1}{2}\right)^n u(n)\right| \leq 1 \qquad (5.21)$$

Similarly,

$$\left|\left(-\frac{1}{2}\right)^{n-2} u(n-2)\right| \leq 1 \qquad (5.22)$$

$$\implies h(n) \leq 2 \qquad (5.23)$$

Therefore $h(n)$ is bounded.

5.4 Convergent? Justify using the ratio test.
Solution: The ratio test for convergence

$$\lim_{n\to\infty} \left|\frac{h(n+1)}{h(n)}\right| = \lim_{n\to\infty} \left|\frac{\left(-\frac{1}{2}\right)^{n-1}\left(\frac{1}{4}+1\right)}{\left(-\frac{1}{2}\right)^{n-2}\left(\frac{1}{4}+1\right)}\right| \qquad (5.24)$$

$$= \lim_{n\to\infty} \left|-\frac{1}{2}\right| \qquad (5.25)$$

$$= \frac{1}{2} < 1 \qquad (5.26)$$

Therefore, $h(n)$ is convergent which implies that it is bounded.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \qquad (5.27)$$

Is the system defined by (3.2) stable for the impulse response in (5.16)?
Solution: Note that

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \qquad (5.28)$$

$$= 2\left(\frac{1}{1+\frac{1}{2}}\right) = \frac{4}{3} \qquad (5.29)$$

Thus, the given system is stable.

5.6 Verify the above result using a python code.
Solution: Download the following code.

wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/5.6.py

Run the code by executing

python 5.6.py

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \qquad (5.30)$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/5.4.py

run the above code using the command.
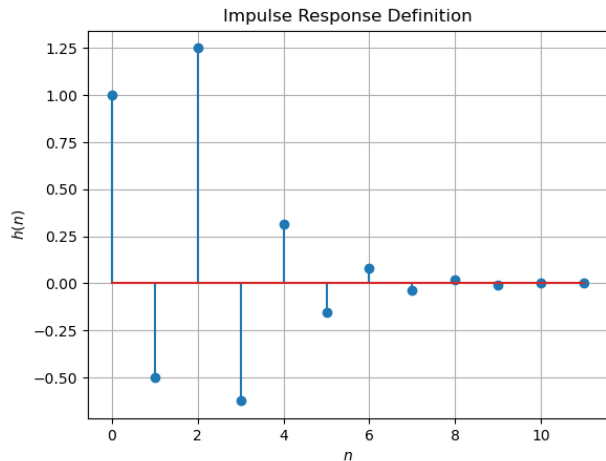
python3 5.4.py

Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad (5.31)$$

Comment. The operation in (5.31) is known as *convolution*.

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/5.5.py

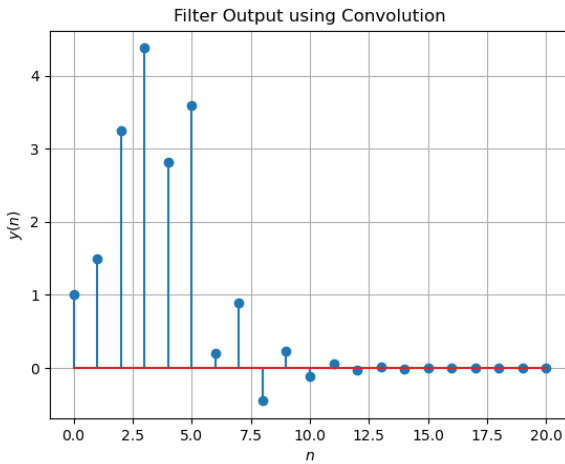run the above code using the command.

python3 5.5.py

Fig. 5.8: $y(n)$ from the definition of convolution

5.9 Express the above convolution using a Teoplitz matrix.

Solution: Let

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \qquad \mathbf{h} = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ -0.62 \\ 0.31 \\ -0.16 \end{pmatrix} \quad (5.32)$$

Their convolution is given by the product of the following Toeplitz matrix $\mathbf{T}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 \\ 1.25 & -0.5 & 1 & 0 & 0 & 0 \\ -0.62 & 1.25 & -0.5 & 1 & 0 & 0 \\ 0.31 & -0.62 & 1.25 & -0.5 & 1 & 0 \\ -0.16 & 0.31 & -0.62 & 1.25 & -0.5 & 1 \\ 0 & -0.16 & 0.31 & -0.62 & 1.25 & -0.5 \\ 0 & 0 & -0.16 & 0.31 & -0.62 & 1.25 \\ 0 & 0 & 0 & -0.16 & 0.31 & -0.62 \\ 0 & 0 & 0 & 0 & -0.16 & 0.31 \\ 0 & 0 & 0 & 0 & 0 & -0.16 \end{pmatrix}$$
$$(5.33)$$

and $\mathbf{x}$

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} = \mathbf{Tx} = \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ 4.38 \\ 2.81 \\ 3.59 \\ 0.12 \\ 0.78 \\ -0.62 \\ 0 \\ -0.16 \end{pmatrix} \quad (5.34)$$

Download the following code .

wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/5.9.py

Run the code by executing

python 5.9.py

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \quad (5.35)$$

Solution: from 5.31, we substitute $k := n - k$ to get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.36)$$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \quad (5.37)$$

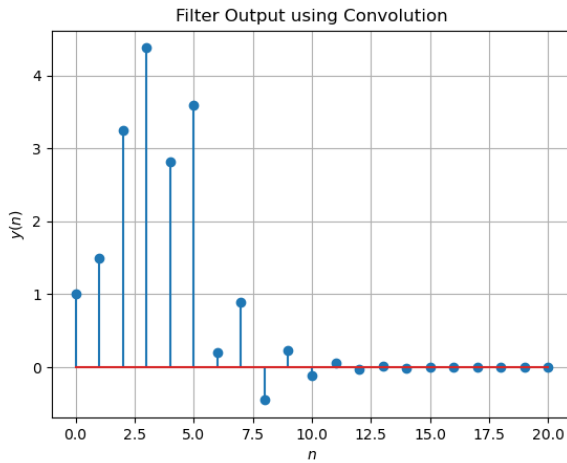$$= \sum_{k=-\infty}^{\infty} x(n-k)h(k) \quad (5.38)$$

Fig. 5.9: Plot of the convolution of $x(n)$ and $h(n)$

6 DFT AND FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1$$

(6.1)

and $H(k)$ using $h(n)$.

Solution: The following code plots Fig. 6.1.

```
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/6.1.py
```

run the above code using the command.

```
python3 6.1.py
```


Fig. 6.1: Plots of the real parts of the DFT of $x(n)$ and $h(n)$

6.2 Compute

$$Y(k) = X(k)H(k)$$

(6.2)

Solution: Download and run the following code.

```
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/6.2.py
```

run the above code using the command.

```
python3 6.2.py
```

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \ldots, N-1$$

(6.3)

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/6.3.py
```

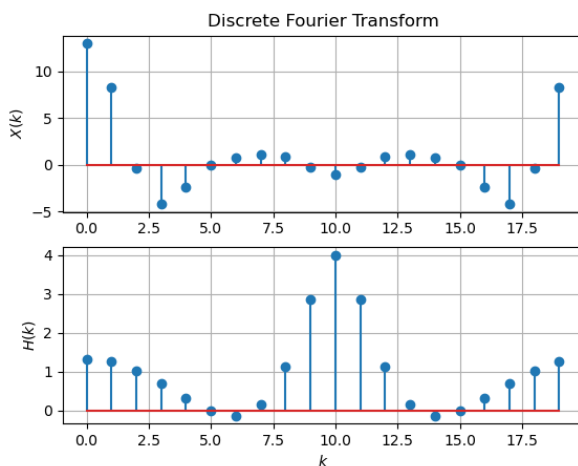run the above code using the command.
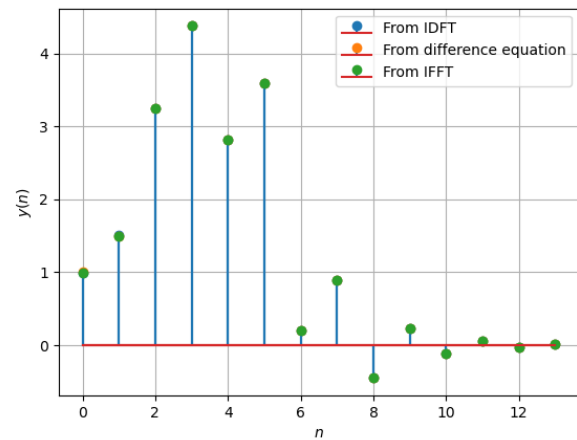
```
python3 6.3.py
```
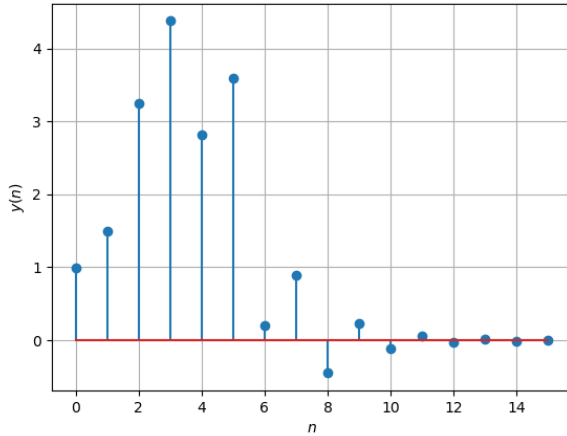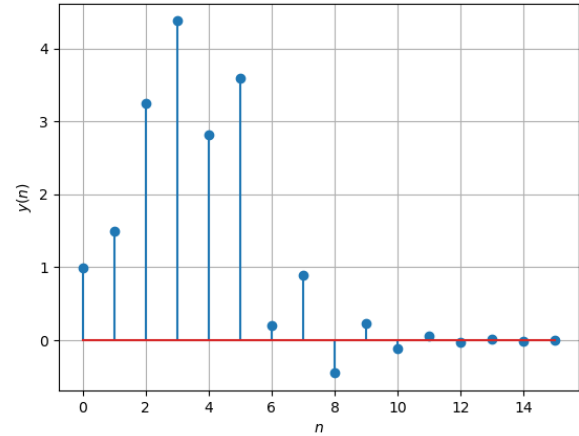

Fig. 6.3: $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: Download the code from

```
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/6.4.py
```

and execute it using

```
$ python3 6.4.py
```

Fig. 6.4: $y(n)$ using FFT and IFFT

Fig. 6.6: $y(n)$ using DFT matrix

Observe that Fig. (6.4) is the same as $y(n)$ in Fig. (3.2).

6.5 Wherever possible, express all the above equations as matrix equations.

Solution: We use the DFT Matrix, where $\omega = e^{-\frac{j2k\pi}{N}}$, which is given by

$$\mathbf{W} = \begin{pmatrix} \omega^0 & \omega^0 & \ldots & \omega^0 \\ \omega^0 & \omega^1 & \ldots & \omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^0 & \omega^{N-1} & \ldots & \omega^{(N-1)(N-1)} \end{pmatrix} \quad (6.4)$$

i.e. $W_{jk} = \omega^{jk}$, $0 \le j, k < N$. Hence, we can write any DFT equation as

$$\mathbf{X} = \mathbf{W}\mathbf{x} = \mathbf{x}\mathbf{W} \quad (6.5)$$

where

$$\mathbf{x} = \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(n-1) \end{pmatrix} \quad (6.6)$$

Using (6.3), the inverse Fourier Transform is given by

$$\mathbf{x} = \mathcal{F}^{-1}(\mathbf{X}) = \mathbf{W}^{-1}\mathbf{X} = \frac{1}{N}\mathbf{W}^{\mathbf{H}}\mathbf{X} = \frac{1}{N}\mathbf{X}\mathbf{W}^{\mathbf{H}} \quad (6.7)$$

$$\implies \mathbf{W}^{-1} = \frac{1}{N}\mathbf{W}^{\mathbf{H}} \quad (6.8)$$

where H denotes hermitian operator. We can rewrite (6.2) using the element-wise multiplication operator as

$$\mathbf{Y} = \mathbf{H} \cdot \mathbf{X} = (\mathbf{W}\mathbf{h}) \cdot (\mathbf{W}\mathbf{x}) \quad (6.9)$$

6.6 Verify the above equation by generating the DFT matrix in python.

Solution: Download the code from

```
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/6.6.py
```

and execute it using

```
$ python3 6.6.py
```

The above code plots (6.6)

6.7 Compute the 8-point FFT in C.

Solution: Download the following C code

```
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/6.7.c
```

run the above C code using the command

```
        $ gcc 6.7.c −lm −o 6.7.out
        $ ./6.7.out
```

The above code generates the text files that are loaded in the following code
Download the following code

```
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/6.7.1
    _plot.py
wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/6.7.2
    _plot.py
```
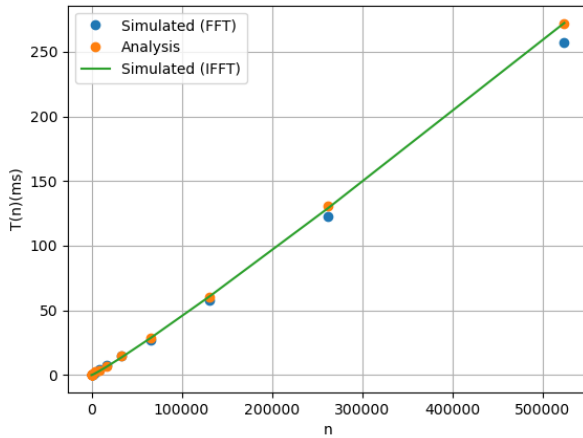
run the above code using the command

Fig. 6.7: Complexity of FFT/IFFT is O(n log n)

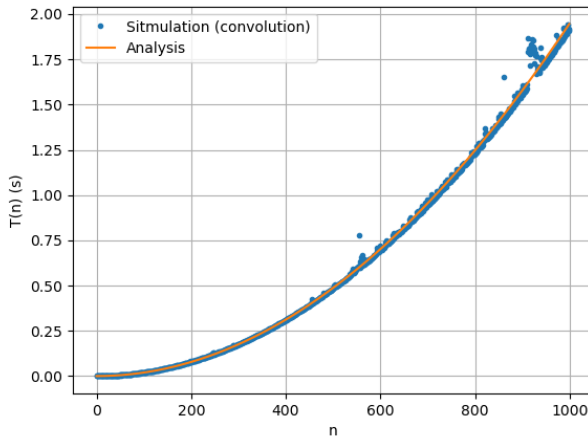Fig. 6.7: Complexity of convolution is O(n^2)

```
$ python3 6.7.1_plot.py
$ python3 6.7.2_plot.py
```

The above code plots the graphs 6.7 and 6.7

7 FFT

7.1 Definitions

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\mathbf{F}_N = \left[W_N^{mn}\right] \tag{7.3}$$

where W_N^{mn} are the elements of $\mathbf{F}_N$.

3. Let

$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$

4. The 4 point *DFT diagonal matrix* is defined as

$$\mathbf{D}_4 = diag\begin{pmatrix} W_N^0 & W_N^1 & W_N^2 & W_N^3 \end{pmatrix} \tag{7.6}$$

7.2 Problems

1. Show that
$$W_N^2 = W_{N/2} \tag{7.7}$$

   Solution: From defination

$$W_N = e^{-j2\pi/N} \tag{7.8}$$
$$W_N^2 = \left(e^{-j2\pi/N}\right)^2 \tag{7.9}$$
$$= e^{-j2\pi/N/2} \tag{7.10}$$
$$= W_{N/2} \tag{7.11}$$

2. Find $\mathbf{P}_6$.
   Solution: $\mathbf{P}_6 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^5 & \mathbf{e}_4^2 & \mathbf{e}_4^4 & \mathbf{e}_4^6 \end{pmatrix}$

$$\mathbf{P}_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.12}$$

3. Find $\mathbf{D}_3$.
   Solution:

$$\mathbf{D}_3 = diag\begin{pmatrix} W_3^0 & W_3^1 & W_3^2 \end{pmatrix} \tag{7.13}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-j2\pi/3} & 0 \\ 0 & 0 & (e^{-j2\pi/3})^2 \end{pmatrix} \tag{7.14}$$

4. Show that

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.15}$$

Solution:

$$\mathbf{F}_4\mathbf{P}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix}\begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \qquad (7.16)$$

$$= \begin{bmatrix} \mathbf{I}_2\mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{I}_2\mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix} \qquad (7.17)$$

$$= \begin{bmatrix} \mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix} \qquad (7.18)$$

$$= \begin{bmatrix} W_2^0 & W_2^0 & W_2^0 & W_2^0 \\ W_2^0 & W_2^1 & W_2^1 & W_2^2 \\ W_2^0 & W_2^0 & -W_2^0 & -W_2^0 \\ W_2^0 & W_2^1 & -W_2^1 & -W_2^2 \end{bmatrix} \qquad (7.19)$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_2^1 & W_2^1 & W_2^2 \\ 1 & W_2^0 & -W_2^0 & -W_2^0 \\ 1 & W_2^1 & -W_2^1 & -W_2^2 \end{bmatrix} \qquad (7.20)$$

from eqn(7.7) we get $W_2 = W_4^2$

$$\mathbf{F}_4\mathbf{P}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^2 & W_4^1 & W_4^3 \\ W_4^0 & W_4^4 & W_4^2 & W_4^6 \\ W_4^0 & W_4^6 & W_4^3 & W_4^9 \end{bmatrix} \qquad (7.21)$$

5. Show that

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix}\begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix}\mathbf{P}_N \quad (7.22)$$

Solution: Observe that for even N and letting $\mathbf{f}_N^i$ denote the ith column of $\mathbf{F}_N$,

$$\begin{pmatrix} \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ -\mathbf{D}_{N/2}\mathbf{F}_{N/2} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_N^2 & \mathbf{f}_N^4 & \dots & \mathbf{f}_N^N \end{pmatrix} \qquad (7.23)$$

and

$$\begin{pmatrix} \mathbf{I}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{I}_{N/2}\mathbf{F}_{N/2} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_N^1 & \mathbf{f}_N^3 & \dots & \mathbf{f}_N^{N-1} \end{pmatrix} \qquad (7.24)$$

Thus,

$$\begin{pmatrix} \mathbf{I}_2\mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{I}_2\mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{pmatrix}\begin{pmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{f}_N^1 & \dots & \mathbf{f}_N^{N-1} & \mathbf{f}_N^2 & \dots & \mathbf{f}_N^N \end{pmatrix} \qquad (7.25)$$

and so,

$$\begin{pmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{pmatrix}\begin{pmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{pmatrix}\mathbf{P}_N$$

$$= \begin{pmatrix} \mathbf{f}_N^1 & \mathbf{f}_N^2 & \dots & \mathbf{f}_N^N \end{pmatrix} = \mathbf{F}_N \qquad (7.26)$$

6. Find

$$\mathbf{P}_4\mathbf{x} \qquad (7.27)$$

Solution: We have,

$$\mathbf{P}_4\mathbf{x} = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix}\begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{pmatrix} = \begin{pmatrix} x(0) \\ x(2) \\ x(1) \\ x(3) \end{pmatrix} \quad (7.28)$$

7. Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \qquad (7.29)$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.

Solution: Writing the terms of X,

$$X(0) = x(0) + x(1) + \dots + x(N-1) \qquad (7.30)$$

$$X(1) = x(0) + x(1)e^{-\frac{j2\pi}{N}} + \dots + $$
$$+ x(N-1)e^{-\frac{j2(N-1)\pi}{N}} \qquad (7.31)$$

$$\vdots$$

$$X(N-1) = x(0) + x(1)e^{-\frac{j2(N-1)\pi}{N}} + \dots + $$
$$+ x(N-1)e^{-\frac{j2(N-1)(N-1)\pi}{N}} \qquad (7.32)$$

Clearly, the term in the mth row and n^{th} column is given by ($0 \le m \le N-1$ and $0 \le n \le N-1$)

$$T_{mn} = x(n)e^{-\frac{j2mn\pi}{N}} \qquad (7.33)$$

and so, we can represent each of these terms as a matrix product

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \qquad (7.34)$$

where $\mathbf{F}_N = \left[e^{-\frac{j2mn\pi}{N}} \right]_{mn}$ for $0 \le m \le N-1$ and $0 \le n \le N-1$.

8. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix}\begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \quad (7.35)$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix}\begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \quad (7.36)$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix}\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \quad (7.37)$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \quad (7.38)$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \quad (7.39)$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \quad (7.40)$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \quad (7.41)$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \quad (7.42)$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \quad (7.43)$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \quad (7.44)$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \quad (7.45)$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \quad (7.46)$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \quad (7.47)$$

9. For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \quad (7.48)$$

compte the DFT using (7.29)
Solution: Download the code from

wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/7.9.py

and execute it using

$ python3 7.9.py

10. Repeat the above exercise using the FFT after zero padding **x**.
    Solution: Download the code from

wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/7.10.py

and execute it using

$ python3 7.10.py

11. Write a C program to compute the 8-point FFT.
    Solution: Download the code from

wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/7.11.c

compile it using

$ gcc 7.11.c −lm −o 7.11.out

and execute it using

$ ./7.11.out

8 EXERCISES

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

    output_signal = signal.lfilter(b, a,
        input_signal)

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m) y(n-m) = \sum_{k=0}^{N} b(k) x(n-k) \quad (8.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.
Solution: Download the code from

wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/7.1.py

and execute it using

$ python3 7.1.py

8.2 Repeat all the exercises in the previous sections for the above a and b.

Solution: For the given values, the difference equation is

$$y(n) - (4.44)\,y(n-1) + (8.78)\,y(n-2)$$
$$- (9.93)\,y(n-3) + (6.90)\,y(n-4)$$
$$- (2.93)\,y(n-5) + (0.70)\,y(n-6)$$
$$- (0.07)\,y(n-7) = \left(5.02 \times 10^{-5}\right) x(n)$$
$$+ \left(3.52 \times 10^{-4}\right) x(n-1) + \left(1.05 \times 10^{-3}\right) x(n-2)$$
$$+ \left(1.76 \times 10^{-3}\right) x(n-3) + \left(1.76 \times 10^{-3}\right) x(n-4)$$
$$+ \left(1.05 \times 10^{-3}\right) x(n-5) + \left(3.52 \times 10^{-4}\right) x(n-6)$$
$$+ \left(5.02 \times 10^{-5}\right) x(n-7) \tag{8.2}$$

From (8.1), we see that the transfer function can be written as follows

$$H(z) = \frac{\sum_{k=0}^{N} b(k)z^{-k}}{\sum_{k=0}^{M} a(k)z^{-k}} \tag{8.3}$$

$$= \sum_{i} \frac{r(i)}{1 - p(i)z^{-1}} + \sum_{j} k(j)z^{-j} \tag{8.4}$$

where $r(i)$, $p(i)$, are called residues and poles respectively of the partial fraction expansion of $H(z)$. $k(i)$ are the coefficients of the direct polynomial terms that might be left over. We can now take the inverse z-transform of (8.4) and get using (4.26),

$$h(n) = \sum_{i} r(i)[p(i)]^{n} u(n) + \sum_{j} k(j)\delta(n-j) \tag{8.5}$$

Substituting the values,

$$h(n) = [(2.76)\,(0.55)^{n}$$
$$+ (-1.05 - 1.84j)\,(0.57 + 0.16j)^{n}$$
$$+ (-1.05 + 1.84j)\,(0.57 - 0.16j)^{n}$$
$$+ (-0.53 + 0.08j)\,(0.63 + 0.32j)^{n}$$
$$+ (-0.53 - 0.08j)\,(0.63 - 0.32j)^{n}$$
$$+ (0.20 + 0.004j)\,(0.75 + 0.47j)^{n}$$
$$+ (0.20 - 0.004j)\,(0.75 - 0.47j)^{n}]u(n)$$
$$+ \left(-6.81 \times 10^{-4}\right)\delta(n) \tag{8.6}$$

The values $r(i)$, $p(i)$, $k(i)$ and thus the impulse response function are computed and plotted at

wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/7.2.1.py

The filter frequency response is plotted at

wget https://github.com/jarpula−
    Bhanu/EE3900/blob/main/Filter/
    Codes/7.2.2.py

Observe that for a series $t_n = r^n$, $\frac{t_{n+1}}{t_n} = r$. By the ratio test, t_n converges if $|r| < 1$. We observe that for all i, $|p(i)| < 1$ and so, as $h(n)$ is the sum of many convergent series, we see that $h(n)$ converges and is bounded. From (**??**),

$$\sum_{n=0}^{\infty} h(n) = H(1) = \frac{\sum_{k=0}^{N} b(k)}{\sum_{k=0}^{M} a(k)} = 1 < \infty \tag{8.7}$$

Therefore, the system is stable. From Fig. (8.2), $h(n)$ is negligible after $n \geq 64$, and we can apply a 64-bit FFT to get y(n). The following code uses the DFT matrix to generate $y(n)$ in Fig. (8.2).

wget https://github.com/jarpula−Bhanu/
    EE3900/blob/main/Filter/Codes/7.2.3.py

The codes can be run all at once by typing
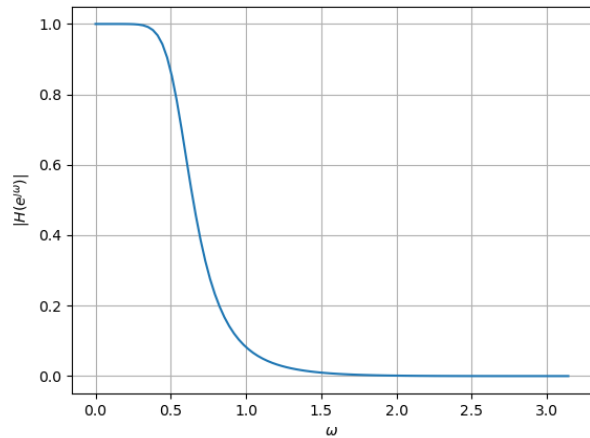
```
$ python3 7.2.*.py
```


Fig. 8.2: Plot of $h(n)$

8.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=4 and cutoff-frequency=4kHz.
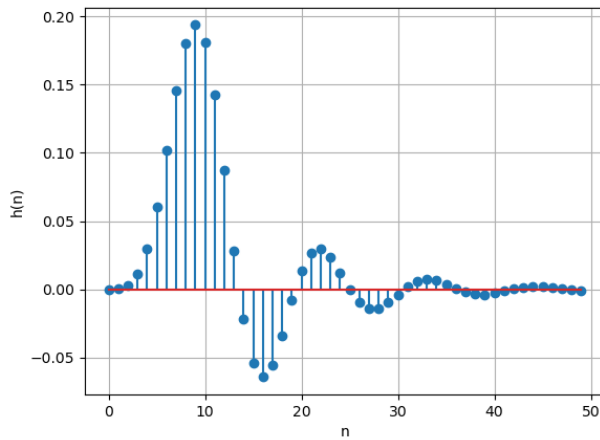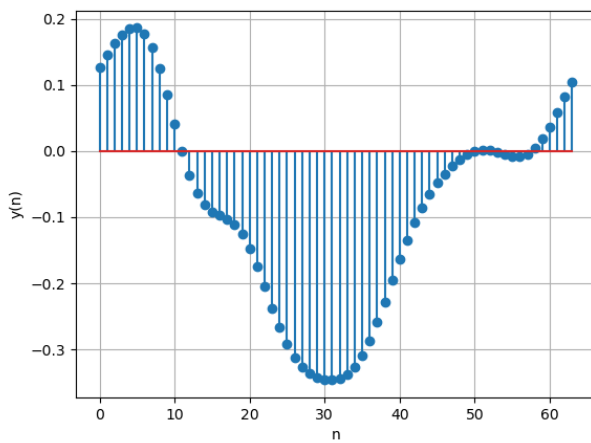
Fig. 8.2: Filter frequency response

Fig. 8.2: Plot of $y(n)$

8.5 Modifying the code with different input parameters and to get the best possible output.
Solution: A better filtering was found on setting te order of the filter to be 7