

Fourier Series

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Abstract—This manual provides a simple introduction to Fourier Series

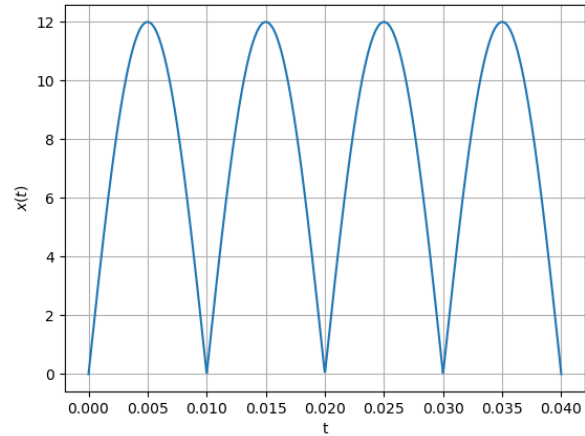


Fig. 1.1: $x(t) = A_0 |\sin(2\pi f_0 t)|$

1 PERIODIC FUNCTION

Let

$$x(t) = A_0 |\sin(2\pi f_0 t)| \quad (1.1)$$

1.1 Plot $x(t)$.

Solution: The following code will plot the graph in fig (1.1)

```
wget https://github.com/jarpula-Bhanu/EE3900/blob/main/charger/codes/1.1.py
```

run the above code using the command

```
python3 1.1.py
```

1.2 Show that $x(t)$ is periodic and find its period.

Solution: From fig (1.1), we see that $x(t)$ is periodic. Further,

$$\text{period of } \sin(at) \text{ given by } \frac{2\pi}{a} \quad (1.2)$$

Now, period of $x(t)$ is

$$A_0 |\sin(2\pi f_0 t)| \Rightarrow \frac{\pi}{2\pi f_0} \quad (1.3)$$

$$\Rightarrow \frac{1}{2f_0} \quad (1.4)$$

Verification

$$x\left(t + \frac{1}{2f_0}\right) = A_0 \left| \sin\left(2\pi f_0 \left(t + \frac{1}{2f_0}\right)\right) \right| \quad (1.5)$$

$$= A_0 |\sin(2\pi f_0 t + \pi)| \quad (1.6)$$

$$= A_0 |-\sin(2\pi f_0 t)| \quad (1.7)$$

$$= A_0 |\sin(2\pi f_0 t)| \quad (1.8)$$

Hence the period of $x(t)$ is $\frac{1}{2f_0}$.

2 FOURIER SERIES

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad (2.1)$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t} dt \quad (2.2)$$

Solution: We have for some $n \in \mathbb{Z}$,

$$x(t) e^{-j2\pi n f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(k-n)f_0 t} \quad (2.3)$$

But we know from the periodicity of $e^{j2\pi k f_0 t}$,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi k f_0 t} dt = \frac{1}{f_0} \delta(k) \quad (2.4)$$

Thus,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi n f_0 t} dt = \frac{c_n}{f_0} \quad (2.5)$$

$$\Rightarrow c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi n f_0 t} dt \quad (2.6)$$

2.2 Find c_k for (1.1)

Solution: Using (2.2),

$$\begin{aligned} c_n &= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 |\sin(2\pi f_0 t)| e^{-j2\pi n f_0 t} dt \quad (2.7) \\ &= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 |\sin(2\pi f_0 t)| \cos(2\pi n f_0 t) dt \\ &\quad + j f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 |\sin(2\pi f_0 t)| \sin(2\pi n f_0 t) dt \quad (2.8) \end{aligned}$$

$$= 2f_0 \int_0^{\frac{1}{2f_0}} A_0 \sin(2\pi f_0 t) \cos(2\pi n f_0 t) dt \quad (2.9)$$

$$\begin{aligned} &= f_0 A_0 \int_0^{\frac{1}{2f_0}} (\sin(2\pi(n+1)f_0 t)) dt \\ &\quad - f_0 A_0 \int_0^{\frac{1}{2f_0}} (\sin(2\pi(n-1)f_0 t)) dt \quad (2.10) \end{aligned}$$

$$= A_0 \frac{1 + (-1)^n}{2\pi} \left(\frac{1}{n+1} - \frac{1}{n-1} \right) \quad (2.11)$$

$$= \begin{cases} \frac{2A_0}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \quad (2.12)$$

2.3 Verify (2.1) using python.

Solution: The following code will plot the graph in fig (2.3)

```
wget https://github.com/jarpula-Bhanu/EE3900/blob/main/charger/codes/2.3.py
```

run the above code using the command

```
python3 2.3.py
```

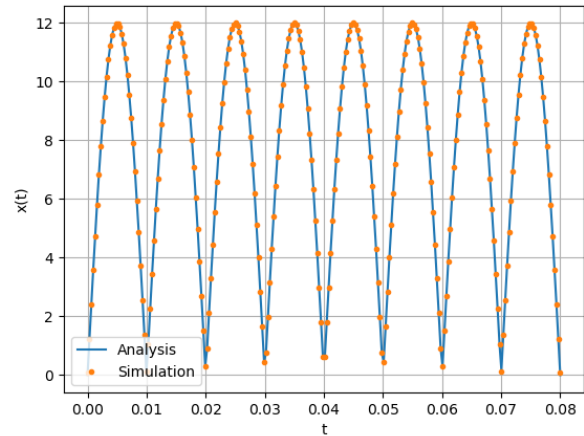


Fig. 2.3: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} (a_k \cos j2\pi k f_0 t + b_k \sin j2\pi k f_0 t) \quad (2.13)$$

and obtain the formulae for a_k and b_k .

Solution: From (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad (2.14)$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{j2\pi k f_0 t} + c_{-k} e^{-j2\pi k f_0 t} \quad (2.15)$$

$$\begin{aligned} &= c_0 + \sum_{k=1}^{\infty} (c_k + c_{-k}) \cos(2\pi k f_0 t) \\ &\quad + \sum_{k=0}^{\infty} (c_k - c_{-k}) \sin(2\pi k f_0 t) \quad (2.16) \end{aligned}$$

Hence, for $k \geq 0$,

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \quad (2.17)$$

$$b_k = c_k - c_{-k} \quad (2.18)$$

2.5 Find a_k and b_k for (1.1)

Solution: From (2.1), we see that since $x(t)$ is

even,

$$x(-t) = \sum_{k=-\infty}^{\infty} c_k e^{-j2\pi k f_0 t} \quad (2.19) \quad 3.1$$

$$= \sum_{k=-\infty}^{\infty} c_{-k} e^{j2\pi k f_0 t} \quad (2.20)$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad (2.21)$$

where we substitute $k \mapsto -k$ in (2.20). Hence, we see that $c_k = c_{-k}$. So, from (2.18) and for $k \geq 0$,

$$a_k = \begin{cases} \frac{2A_0}{\pi} & k = 0 \\ \frac{4A_0}{\pi(1-k^2)} & k > 0, k \text{ even} \\ 0 & \text{otherwise} \end{cases} \quad (2.22)$$

$$b_k = 0 \quad (2.23)$$

2.6 Verify (2.13) using python.

Solution: The following code will plot the graph in fig (2.6)

```
wget https://github.com/jarpula-Bhanu/EE3900/blob/main/charger/codes/2.6.py
```

run the above code using the command

```
python3 2.6.py
```

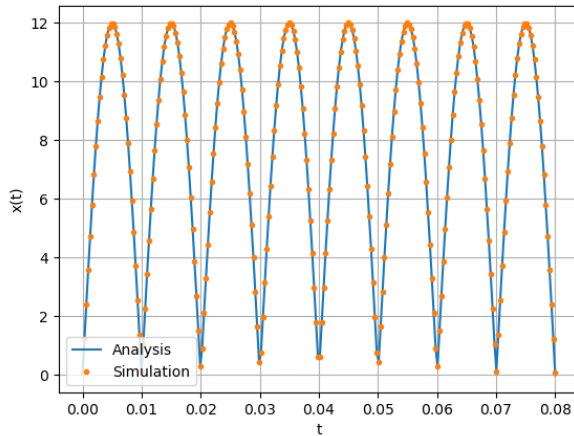


Fig. 2.6: $x(t) = \sum_{k=0}^{\infty} (a_k \cos j2\pi k f_0 t + b_k \sin j2\pi k f_0 t)$

3 FOURIER TRANSFORM

$$\delta(t) = 0, \quad t \neq 0 \quad (3.1)$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (3.2)$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt \quad (3.3)$$

3.3 Show that

$$g(t - t_0) \xleftrightarrow{\mathcal{H}} FG(f) e^{-j2\pi f t_0} \quad (3.4)$$

Solution: We write, substituting $u := t - t_0$,

$$g(t - t_0) \xleftrightarrow{\mathcal{H}} F \int_{-\infty}^{\infty} g(t - t_0) e^{-j2\pi f t} dt \quad (3.5)$$

$$= \int_{-\infty}^{\infty} g(u) e^{-j2\pi f (u+t_0)} du \quad (3.6)$$

$$= G(f) e^{-j2\pi f t_0} \quad (3.7)$$

where the last equality follows from (3.3).

3.4 Show that

$$G(t) \xleftrightarrow{\mathcal{H}} Fg(-f) \quad (3.8)$$

Solution: Using the definition of the Inverse Fourier Transform,

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df \quad (3.9)$$

Hence, setting $t := -f$ and $f := t$, which implies $df = dt$,

$$g(-f) = \int_{-\infty}^{\infty} G(t) e^{-j2\pi f t} dt \quad (3.10)$$

$$\implies G(t) \xleftrightarrow{\mathcal{H}} Fg(-f) \quad (3.11)$$

3.5 $\delta(t) \xleftrightarrow{\mathcal{H}} F?$

Solution: By applying the definition of fourier transformation for $\delta(t - t_0)$ $\{t_0$ be time shifting $\}$.

$$\mathcal{F} \{ \delta(t - t_0) \} (f) = \mathcal{F}(t) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi f t} dt. \quad (3.12)$$

By applying the time shifting property of impulse we get

$$\mathcal{F}(f) = e^{-j2\pi f t_0} \quad (3.13)$$

$$\text{i.e., } \delta(t - t_0) \xleftrightarrow{\mathcal{H}} F e^{-j2\pi f t_0} \quad (3.14)$$

Now, substitute $t_0 = 0$ we get

$$\delta(t) \xleftrightarrow{\mathcal{H}} F1 \quad (3.15)$$

$$3.6 \quad e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{H}} F?$$

Solution: Applying the definition of inverse fourier transformation.

$$\mathcal{F}^{-1}\{\delta(f + f_0)\}(t) = f(t) = \int_{-\infty}^{\infty} \delta(f + f_0) e^{j2\pi f t} df \quad (3.16)$$

By applying the shifting property of impulse

$$f(t) = e^{-j2\pi f_0 t} \quad (3.17)$$

$$i.e., \quad e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{H}} F\delta(f + f_0) \quad (3.18)$$

$$3.7 \quad \cos(2\pi f_0 t) \xleftrightarrow{\mathcal{H}} F?$$

Solution:

$$\cos(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \quad (3.19)$$

$$\mathcal{F}[\cos(2\pi f_0 t)] = \int_{-\infty}^{\infty} \cos(2\pi f_0 t) e^{-j2\pi f t} dt \quad (3.20)$$

$$= \int_{-\infty}^{\infty} \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} e^{-j2\pi f t} dt \quad (3.21)$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{j2\pi f_0 t} e^{-j2\pi f t} dt + \int_{-\infty}^{\infty} e^{-j2\pi f_0 t} e^{-j2\pi f t} dt \right] \quad (3.22)$$

$$= \frac{1}{2} [\delta(f - f_0) + \delta(f + f_0)] \quad (3.23)$$

$$\therefore \cos(2\pi f_0 t) \xleftrightarrow{\mathcal{H}} F \frac{1}{2} [\delta(f - f_0) + \delta(f + f_0)] \quad (3.24)$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python.

Solution:

$$x(t) = A_0 |\sin(2\pi f_0 t)| \quad (3.25)$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \quad (3.26)$$

$$= \int_{-\infty}^{\infty} A_0 |\sin(2\pi f_0 t)| e^{-j2\pi f t} dt \quad (3.27)$$

$$= A_0 \int_0^{\infty} \sin(2\pi f_0 t) e^{-j2\pi f t} dt + K \quad (3.28)$$

$$= \frac{A_0}{2j} \int_0^{\infty} (e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}) e^{-j2\pi f t} dt + K \quad (3.29)$$

$$= \frac{A_0}{2j} \int_0^{\infty} (e^{j2\pi(f_0 - f)t} + e^{-j2\pi(f_0 + f)t}) dt + K \quad (3.30)$$

$$= \frac{A_0}{2j} \left[0 - \frac{1}{j2\pi(f_0 - f)} \right] - \frac{A_0}{2j} \left[0 - \frac{1}{j2\pi(f_0 + f)} \right] \quad (3.31)$$

$$= \frac{A_0}{2\pi(f_0^2 - f^2)} + k \quad (3.32)$$

By symmetry we get

$$k = \frac{A_0}{2\pi(f_0^2 - f^2)} \quad (3.33)$$

$$\therefore X(f) = \frac{A_0}{\pi(f_0^2 - f^2)} \quad (3.34)$$

The following code will plot the graph in fig (3.8)

```
wget https://github.com/jarpula-Bhanu/EE3900/blob/main/charger/codes/3.8.py
```

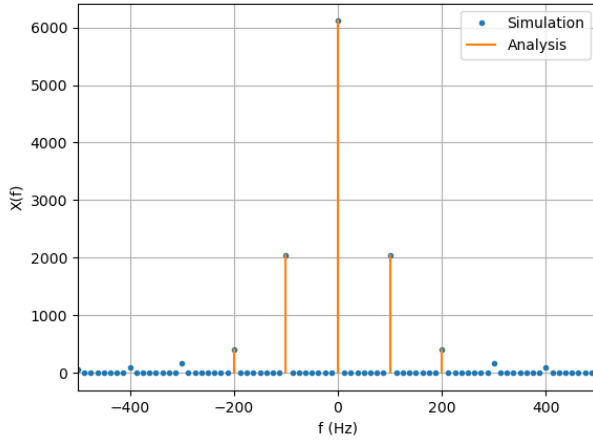
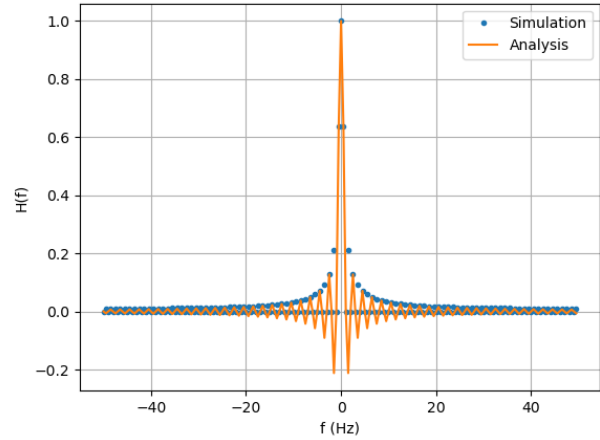
run the above code using the command

```
python3 3.8.py
```

3.9 Show that

$$\text{rect}(t) \xleftrightarrow{\mathcal{H}} F \text{sinc}(t) \quad (3.35)$$

Verify using python.

Fig. 3.8: $\mathcal{F}[x(t)]$ Fig. 3.9: $\mathcal{F}[\text{rect}(t)]$

Solution:

$$\text{rect}(t) = \begin{cases} 1 & |t| < T \\ 0 & \text{otherwise} \end{cases} \quad (3.36)$$

$$\mathcal{F}[\text{rect}(t)] = \int_{-\infty}^{\infty} \text{rect}(t) e^{-j2\pi ft} dt \quad (3.37)$$

$$= \int_{-T}^T e^{-j2\pi ft} dt \quad (3.38)$$

$$= \frac{1}{-j2\pi f} [e^{-j2\pi fT} - e^{j2\pi fT}] \quad (3.39)$$

$$= \frac{1}{\pi f} \left[\frac{e^{j2\pi fT} - e^{-j2\pi fT}}{2j} \right] \quad (3.40)$$

$$= \frac{\sin \pi f T}{\pi f} \quad (3.41)$$

$$= \text{sinc}(f) \quad (3.42)$$

The following code will plot the graph in fig (3.9)

```
wget https://github.com/jarpula-Bhanu/EE3900/blob/main/charger/codes/3.9.py
```

run the above code using the command

```
python3 3.9.py
```

3.10 $\text{sinc}(t) \xleftrightarrow{\mathcal{H}} F?$. Verify using python.

Solution: from (3.8) we can say

$$\text{sinc}(t) \xleftrightarrow{\mathcal{H}} F \text{rect}(-f) \quad (3.43)$$

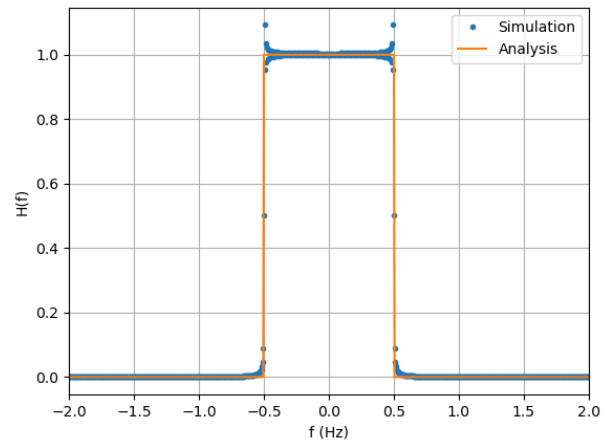
$$\text{rect}(f) \quad (3.44)$$

The following code will plot the graph in fig (3.10)

```
wget https://github.com/jarpula-Bhanu/EE3900/blob/main/charger/codes/3.10.py
```

run the above code using the command

```
python3 3.10.py
```

Fig. 3.10: $\mathcal{F}[\text{sinc}(t)]$

4 FILTER

- 4.1 Find $H(f)$ which transforms $x(t)$ to DC 5V.
- 4.2 Find $h(t)$.
- 4.3 Verify your result using through convolution.

5 FILTER DESIGN

- 5.1 Design a Butterworth filter for $H(f)$.
- 5.2 Design a Chebyshev filter for $H(f)$.

- 5.3 Design a circuit for your Butterworth filter.
- 5.4 Design a circuit for your Chebyshev filter.