Random Numbers

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Contents

***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/exrand
    .c
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/coeffs.
    h
```

Compile and execute the above C program using command

```
$ gcc exrand.c −lm −o exrand.out
$ ./exrand.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/
    cdf_plot.py
```

The above code is executed using command

```
$ python3 cdf_plot.py
```

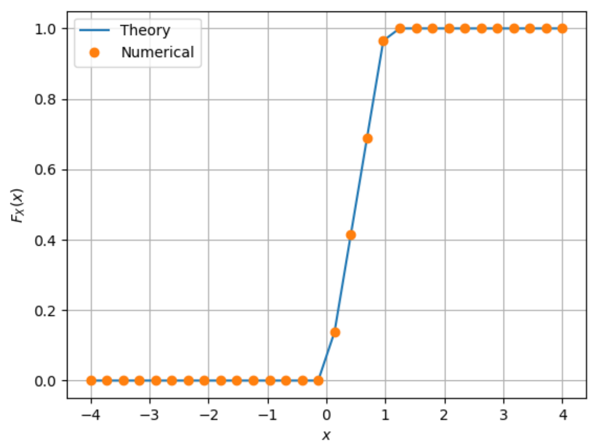

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Since U is a uniformly distributed in [0,1]
We have three cases:

- $x < 0$: $P_X(x) = 0$, and hence $F_U(x) = 0$.
- $0 \leq x < 1$: Here,

$$F_U(x) = \int_0^x du = x \tag{1.2}$$

- $x \geq 1$: Put $x = 1$ in above eqn we get $F_U(x) = 1$.

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \tag{1.3}$$

This can be verified from Fig. 1.2

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.4}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.5}$$

Write a C program to find the mean and variance of U.
Solution: Download the following files and execute the C program

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/exrand
    .c
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/coeffs.
    h
```

Compile and execute the above C program using command

```
$ gcc exrand.c −lm −o exrand.out
$ ./exrand.out
```

The mean of U is 0.500007
The varience of U is 0.083301

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x)dx \tag{1.6}$$

Solution: Verifying result theoritically
Given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.7}$$

Mean is given by

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.8}$$

$$= \int_{-\infty}^{\infty} x dx \tag{1.9}$$

$$= \left[\frac{x^2}{2}\right]_0^1 \tag{1.10}$$

$$= \frac{1}{2} \tag{1.11}$$

Varaience is given by

$$E[U - E[U]]^2 = E[U^2] - E[U]^2 \tag{1.12}$$

$$E[U]^2 = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{1.13}$$

$$= \int_{-\infty}^{\infty} x^2 dx \tag{1.14}$$

$$= \left[\frac{x^3}{3}\right]_0^1 \tag{1.15}$$

$$= \frac{1}{3} \tag{1.16}$$

$$E[U - E[U]]^2 = \frac{1}{3} - (\frac{1}{2})^2 \tag{1.17}$$

$$= \frac{1}{3} - \frac{1}{4} \tag{1.18}$$

$$= \frac{1}{12} \tag{1.19}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files and execute the C program

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/exrand
    .c
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/coeffs.
    h
```

Compile and execute the above C program using command

```
$ gcc exrand.c −lm −o exrand.out
$ ./exrand.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2
properties of cdf

- $F_X(x)$ is a nondecreasing function of x for $-\infty < x < \infty$.

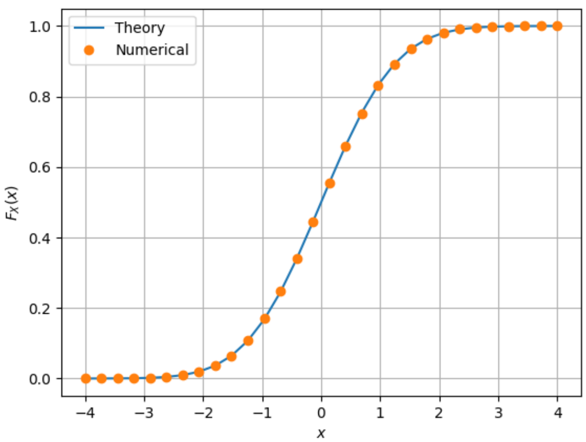

Fig. 2.2: The CDF of X

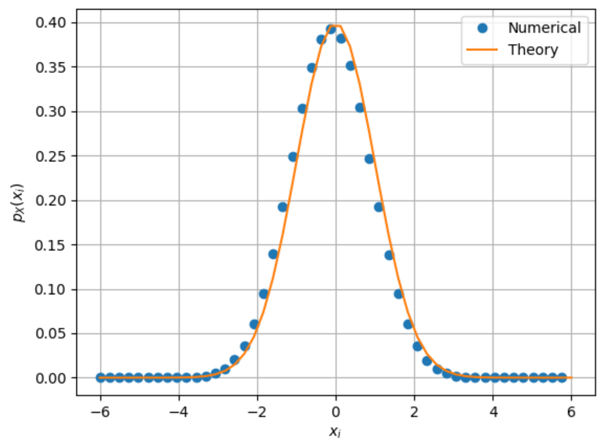

Fig. 2.3: The PDF of X

- The CDF, $F_X(x)$ ranges from 0 to 1. This makes sense since $F_X(x)$ is a probability.
- If the maximum value of X is b, then $F_X(b) = 1$

$F_X(x) = P(X \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$
The Q function is defined as:

$$Q(x) = 1 - F_X(x) \tag{2.2}$$

Hence,we can use eqn(2.2) to calculate $F_X(x)$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.3}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
$ wget https://github.com/jarpula-Bhanu/
    Random-numbers/blob/main/codes/
    pdf_plot.py
```

The above code is executed using command

```
$ wget python3 pdf_plot.py
```

properties of pdf

- PDF is symmetric about $X = 0$
- Graph is bell shaped
- Mean of graph is situated at the apex point of the bell.

2.4 Find the mean and variance of X by writing a C program.
Solution: Download and run the following C code.
The C program can be downloaded using

```
$ wget https://github.com/jarpula-Bhanu/
    Random-numbers/blob/main/codes/mean
    -var2-4.c
$ wget https://github.com/jarpula-Bhanu/
    Random-numbers/blob/main/codes/coeffs.
    h
```

Compile and execute the above C program using command

```
$ gcc mean-var2-4.c -lm -o mean-var2-4.
    out
$ ./mean-var2-4.out
```

The mean of X is 0.000326
The varience of X is 1.000907

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.4}$$

repeat the above exercise theoretically.
Solution: Verifying theoritically

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{2.5}$$

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx \tag{2.6}$$

Taking $\frac{x^2}{2} = t \rightarrow xdx = dt$

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t} dt = 0 \tag{2.7}$$

$$E[X^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx \tag{2.8}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(xe^{-\frac{x^2}{2}}) dx \tag{2.9}$$

$$= \frac{1}{\sqrt{2\pi}} \Big[-xe^{\frac{x^2}{2}} + \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \Big]_{-\infty}^{\infty} \tag{2.10}$$

$$= 1 \tag{2.11}$$

$$\textbf{variance} = E[X^2] - E[X]^2 = 1 \tag{2.12}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution: Download the following files and execute the C program

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/exrand
    .c
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/coeffs.
    h
```

Compile and execute the above C program using command

```
$ gcc exrand.c −lm −o exrand.out
$ ./exrand.out
```

The above C program will save the values of V in log.dat

and the CDF is plotted in Figure (3.1).

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/
    cdf_plot3−1.py
```

The above code is executed using command

```
$ python3 cdf_plot3−1.py
```

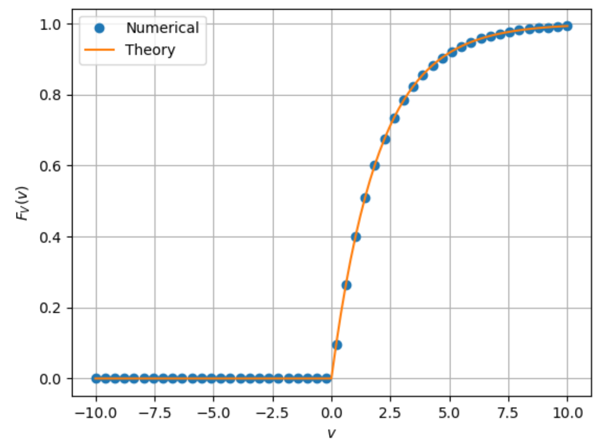

Fig. 3.1: The CDF of V

3.2 Find a theoretical expression for $F_V(x)$.
Solution:

$$F_V(x) = Pr(V \leq x) \tag{3.2}$$

$$= Pr(-2ln(1-U) \leq x) \tag{3.3}$$

$$= Pr(U \leq 1 - e^{-\frac{x^2}{2}}) \tag{3.4}$$

$$Pr(U < x) = \int_0^x dx = x \tag{3.5}$$

$$\therefore Pr(U \leq 1 - e^{-\frac{x^2}{2}}) = 1 - e^{-\frac{x^2}{2}} \tag{3.6}$$

$$\rightarrow F_V(x) = 1 - e^{-\frac{x^2}{2}} \tag{3.7}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following files and execute the C program

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/
    exrand.c
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/
    coeffs.h
```

Compile and execute the above C program using command

```
$ gcc exrand.c −lm −o exrand.out
$ ./exrand.out
```

The above code will generate T.dat file

4.2 Find the CDF of T.
Solution: The CDF is plotted in Figure (4.2).

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes
    /4−2_cdf.py
```

The above code is executed using command

```
$ python3 4−2_cdf.py
```

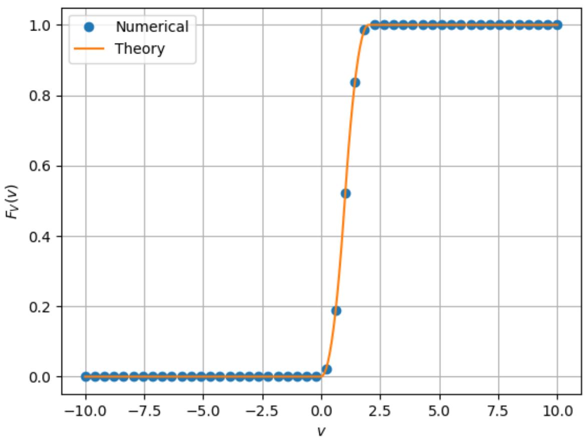

Fig. 4.2: The CDF of T

4.3 Find the PDF of T.
Solution: The PDF is plotted in Figure (4.3).

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes
    /4−3_pdf.py
```

The above code is executed using command

```
$ python3 4−3_pdf.py
```

4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: Let $T = U_1 + u_2$ where U_1 and U_2 are the uniform independent random varaible.We know ,pdf of T is defined as

$$f_T(t) = \int_{-\infty}^{\infty} f_{U_1}(x) f_{U_2}(t-x)dx \tag{4.2}$$

$$= \int_{-\infty}^{\infty} f(x) f(t-x)dx \tag{4.3}$$

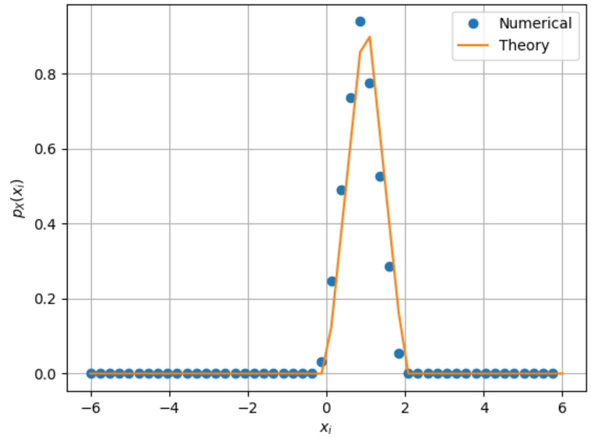

Fig. 4.3: The PDF of T

Since U_1 and U_2 are uniform random varaible between (0,1) they have the same density i.e. $f_{U_1} = f_{U_2} = f$

$$f = \begin{cases} 1 & 0 < x < 1 \\ 0 & otherwise \end{cases} \tag{4.4}$$

The integrand $f(x)f(t-x)$ hence will have value either 0 or 1.
It is 1 when
$0 < x < 1$ and $0 < t - x < 1$

- case 1:when $0 < x < 1$, the limits run from $x = 0$ to $x = t$,so

$$f_T(t) = \int_0^t 1dx = t \tag{4.5}$$

- case 2:when $1 < x < 2$, the limits run from $x = t - 1$ to $x = 1$,so

$$f_T(t) = \int_{t-1}^1 1dx = 2 - t \tag{4.6}$$

- case 3:when $x < 0$ and $x > 2$,The integrand is 0 so,

$$f_T(t) = 0 \tag{4.7}$$

In terms of T and t we can Write

$$\therefore f_T(t) = \begin{cases} t & 0 < t < 1 \\ 2 - t & 1 < t < 2 \\ 0 & otherwise \end{cases} \tag{4.8}$$

Now the theoretical expression of cdf of T

$$F_T(t) = \int_0^t f_T(t)dx \tag{4.9}$$

$$= \begin{cases} \int_0^t dt & 0 < t < 1 \\ \int_0^t (2-t)dt & 1 < t < 2 \\ \int_0^2 (2-t)dt & t > 2 \end{cases} \tag{4.10}$$

$$= \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 < t < 1 \\ 2t - \frac{t^2}{2} & 1 < t < 2 \\ 1 & t > 2 \end{cases} \tag{4.11}$$

4.5 Verify your results through a plot.
Solution: The cdf i.e. eqn4.11 can be verified from Fig.4.2
and The pdf i.e. eqn4.8 can be verified from Fig.4.3

5 Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.
Solution: Download the following files and execute the C program

```
$ wget https://github.com/jarpula-Bhanu/
    Random-numbers/blob/main/codes/exrand
    .c
$ wget https://github.com/jarpula-Bhanu/
    Random-numbers/blob/main/codes/coeffs.
    h
```

The above code will generate equiprobable.dat file
Compile and execute the above C program using command

```
$ gcc exrand.c -lm -o exrand.out
$ ./exrand.out
```

5.2 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.
Solution: Download the following files and execute the C program

```
$ wget https://github.com/jarpula-Bhanu/
    Random-numbers/blob/main/codes/exrand
    .c
$ wget https://github.com/jarpula-Bhanu/
    Random-numbers/blob/main/codes/coeffs.
    h
```

The above code will generate Y.dat file
Compile and execute the above C program using command

```
$ gcc exrand.c -lm -o exrand.out
$ ./exrand.out
```

5.3 Plot Y using a scatter plot.
Solution:

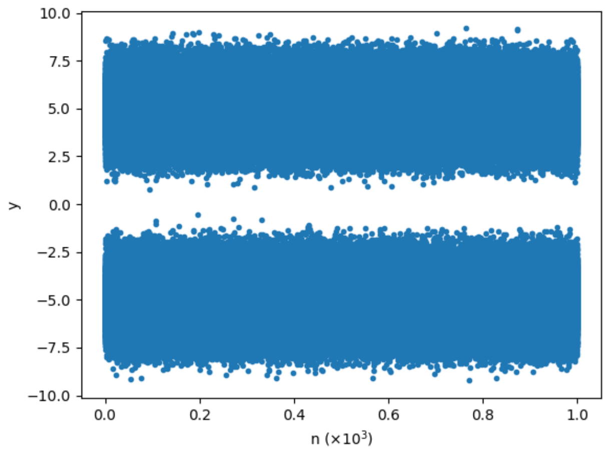

Fig. 5.3: Scatter plot

5.4 Guess how to estimate X from Y.
Solution: From the graph we can see that X = 1 usually corelates to $Y > 0$ and X = -1 corelates to $Y < 0$

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.3}$$

Solution: The estimathed value $\hat{X}$ is given by

$$\hat{X} = \begin{cases} 1 & Y > 0 \\ -1 & Y < 0 \end{cases} \tag{5.4}$$

For $X = 1$

$$Y = A + N \tag{5.5}$$

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.6}$$

$$= \Pr(Y < 0 | X = 1) \tag{5.7}$$

$$= \Pr(A < -N) \tag{5.8}$$

$$= F_A(-N) \tag{5.9}$$

$$= \int_{-\infty}^{-N} f_A(x)dx \tag{5.10}$$

If $N > 0, f_A(x) = 0$. then

$$P_{e|0} = 0 \tag{5.11}$$

The estimathed value $\hat{X}$ is given by

$$\hat{X} = \begin{cases} -1 & Y > 0 \\ 1 & Y < 0 \end{cases} \tag{5.12}$$

For X = -1

$$Y = -A + N \tag{5.13}$$
$$P_e = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.14}$$
$$= \Pr\left(Y < 0 | X = -1\right) \tag{5.15}$$
$$= \Pr\left(N < A\right) \tag{5.16}$$
$$= F_A(N) \tag{5.17}$$
$$= \int_{-\infty}^{N} f_A(x)dx \tag{5.18}$$

If $N < 0, f_A(x) = 0$. then

$$P_{e|1} = 0 \tag{5.19}$$

5.6 Find P_e assuming that X has equiprobable symbols.
Solution: Since X is equiprobable we assume $Pr(X = 1)$ and $Pr(\hat{X} = -1)$

$$P_e = \frac{(P_{e|0} + P_{e|1})}{2} = 0 \tag{5.20}$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution:

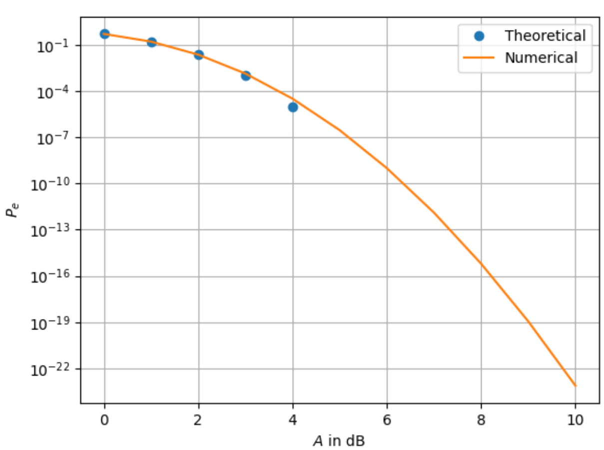

Fig. 5.7: Theoritical P_e

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.
Solution: We note that

$$P_{e|0} = Pr(\hat{X} = 1 | X = -1) \tag{5.21}$$
$$= Pr(Y > 0 | X = -1) \tag{5.22}$$
$$= Pr(AX + N > 0 | X = -1) \tag{5.23}$$

Replacing 0 in eqn(5.23) with δ and performing similar operation for $P_{e|1}$ we get

$$P_{e|0} = Pr(A - N > \delta | X = -1) \tag{5.24}$$
$$P_{e|1} = Pr(A + N > \delta | X = 1) \tag{5.25}$$

Now

$$P_e = Pr(X = -1)Q(A + \delta) + Pr(X = 1)Q(A - \delta) \tag{5.26}$$
$$= \frac{1}{2}(Q(A + \delta) + Q(A - \delta)) \tag{5.27}$$

Differentiating with respect to δ gives the following equation

$$f_N(A + \delta) = f_N(A - \delta) \tag{5.28}$$

which implies that for $A \neq 0$,$\delta = 0$ and for $A = 0$, $\delta \in \mathbb{R}$

5.9 Repeat the above exercise when

$$p_X(0) = p \tag{5.29}$$

Solution:

$$p f_N(A + \delta) = (1 - p) f_N(A - \delta) \tag{5.30}$$
$$p e^{-\frac{(A+\delta)^2}{2}} = (1 - p) e^{-\frac{(A-\delta)^2}{2}} \tag{5.31}$$
$$\implies \delta = \frac{1}{2A} \log\left[\frac{p}{1 - p}\right] \tag{5.32}$$

5.10 Repeat the above exercise using the MAP criterion.
Solution: Using Baye's Theorem
$Pr(X = 1 | Y = y)$

$$= \frac{Pr(N = y - A | X = 1) Pr(X = 1)}{Pr(Y = y)} \tag{5.33}$$
$$= \frac{(1 - p) f_N(y - A)}{p f_N(y + A) + (1 - p) f_N(y - A)} \tag{5.34}$$
$$= \frac{(1 - p)}{(1 - p) + p e^{-2yA}} \tag{5.35}$$

and

$$Pr(X = -1|Y = y)$$
$$= \frac{Pr(N = y + A|X = -1)Pr(X = -1)}{Pr(Y = y)} \tag{5.36}$$
$$= \frac{(p)f_N(y + A)}{pf_N(y + A) + (1 - p)f_N(y - A)} \tag{5.37}$$
$$= \frac{(p)}{(p) + (1 - p)e^{2yA}} \tag{5.38}$$

Hence

$$\frac{(1 - p)}{(1 - p) + pe^{-2yA}} \gtrless \frac{(p)}{(p) + (1 - p)e^{2yA}} \tag{5.39}$$
$$(1 - p)e^{2Ay} \gtrless p \tag{5.40}$$
$$\implies y \gtrless \frac{1}{2A} \log\left[\frac{p}{1 - p}\right] \tag{5.41}$$

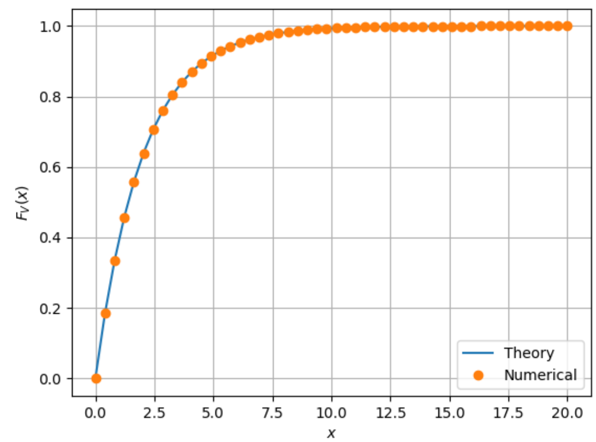

Fig. 6.1: The CDF of V

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution: The following codes plots Fig.6.1 - CDF of V and Fig.6.1 - PDF of V

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/
    cdf_plot_6−1.py
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/
    pdf_plot_6−1.py
```

run above python code using the command

```
$ python3 cdf_plot_6−1.py
$ python3 pdf_plot_6−1.py
```

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.

Solution: We transform the variables X_1 and X_2 as:

$$X_1 = R\cos\Theta \tag{6.3}$$
$$X_2 = R\sin\Theta \tag{6.4}$$

where $R \in [0, \infty), \Theta \in [0, 2\pi)$. The Jacobian

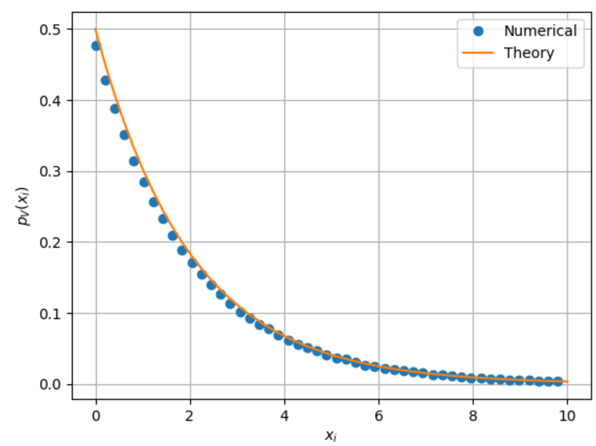

Fig. 6.1: The PDF of V

Matrix for this transformation is given by

$$\mathbf{J} = \begin{pmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_2}{\partial R} \\ \frac{\partial X_1}{\partial \Theta} & \frac{\partial X_2}{\partial \Theta} \end{pmatrix} \tag{6.5}$$
$$= \begin{pmatrix} \cos\Theta & \sin\Theta \\ -R\sin\Theta & R\cos\Theta \end{pmatrix} \tag{6.6}$$
$$\implies |\mathbf{J}| = R \tag{6.7}$$

We also know that

$$|\mathbf{J}|p_{X_1,X_2}(x_1, x_2) = p_{R,\Theta}(r, \theta) \tag{6.8}$$
$$\implies p_{R,\Theta}(r, \theta) = Rp_{X_1}(x_1)p_{X_2}(x_2) \tag{6.9}$$
$$= \frac{R}{2\pi}\exp\left(-\frac{X_1^2 + X_2^2}{2}\right) \tag{6.10}$$
$$= \frac{R}{2\pi}\exp\left(-\frac{R^2}{2}\right) \tag{6.11}$$

where (6.9) follows as X_1, X_2 are iid random variables. Thus,

$$p_R(r) = \int_0^{2\pi} p_{R,\Theta}(r,\theta)d\theta \tag{6.12}$$

$$= R\exp\left(-\frac{R^2}{2}\right) \tag{6.13}$$

However, $V = X_1^2+X_2^2 = R^2 \geq 0$, thus $F_V(x) = 0$ for $x \geq 0$.

$$F_V(x) = F_R(\sqrt{x}) \tag{6.14}$$

$$= \int_0^{\sqrt{x}} r\exp\left(-\frac{r^2}{2}\right)dr \tag{6.15}$$

$$= \int_0^{\frac{x}{2}} e^{-t}dt = 1 - e^{-\frac{x}{2}} \tag{6.16}$$

where $t = \frac{r^2}{2}$ and so, for $x \geq 0$,

$$p_V(x) = \frac{1}{2}e^{-\frac{x}{2}} \tag{6.17}$$

Hence,

$$F_V(x) = \begin{cases} 1 - e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{6.18}$$

$$p_V(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{6.19}$$

The value $\alpha = 0.5$,
the theory matches the stimulation
The following code plots the Fig.6.1

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.20}$$

Solution: The following codes plots Fig.6.3 - CDF of A and Fig.6.3 - PDF of A

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes
    /6−3_cdf.py
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes
    /6−3_pdf.py
```

run above python code using the command

```
$ python3 6−3_cdf.py
$ python3 6−3_pdf.py
```

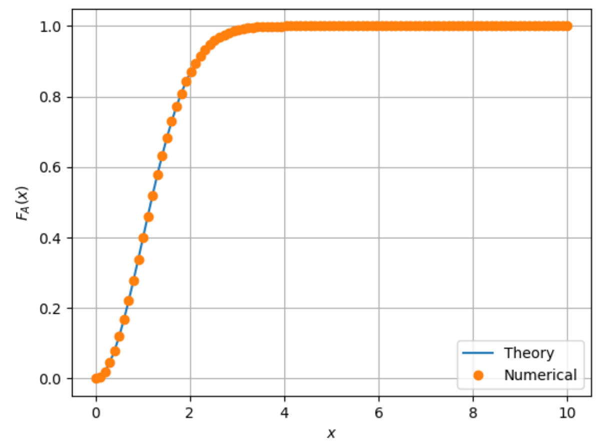

Fig. 6.3: The CDF of A

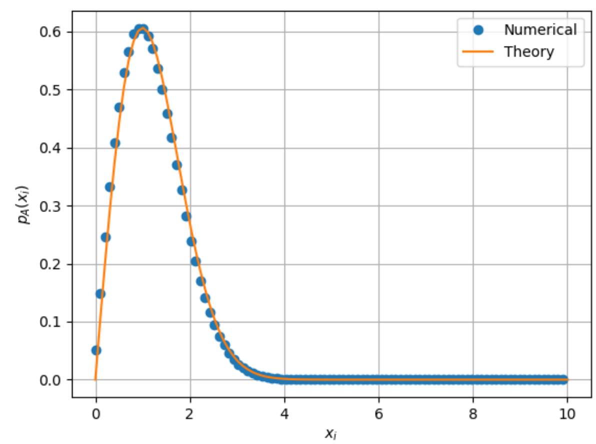

Fig. 6.3: The PDF of A

7 Conditional Probability

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \leq \gamma \leq 10$ dB.
Solution: see Fig. 7.4

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$
Solution: The estimathed value $\hat{X}$ is given by

$$\hat{X} = \begin{cases} 1 & Y > 0 \\ -1 & Y < 0 \end{cases} \tag{7.3}$$

For $X = 1$

$$Y = A + N \tag{7.4}$$

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{7.5}$$

$$= \Pr(Y < 0|X = 1) \tag{7.6}$$

$$= \Pr(A < -N) \tag{7.7}$$

$$= F_A(-N) \tag{7.8}$$

$$= \int_{-\infty}^{-N} f_A(x)dx \tag{7.9}$$

By definition

$$f_A(x) = \begin{cases} \frac{x}{\sigma^2} exp(-\frac{x^2}{2\sigma^2}) & x \geq 0 \\ 0 & otherwise \end{cases} \tag{7.10}$$

If $N > 0, f_A(x) = 0$. then

$$P_e = 0 \tag{7.11}$$

if $N < 0$, then

$$P_e(N) = \int_{-\infty}^{-N} f_A(x)dx \tag{7.12}$$

$$= \int_{-\infty}^{0} 0dx + \int_{0}^{-N} f_A(x)dx \tag{7.13}$$

$$= \int_{0}^{-N} \frac{x}{\sigma^2} exp(-\frac{x^2}{2\sigma^2})dx \tag{7.14}$$

$$= 1 - exp(-\frac{N^2}{2\sigma^2}) \tag{7.15}$$

$$\therefore P_e(N) = \begin{cases} 1 - exp(-\frac{N^2}{2\sigma^2}) & N < 0 \\ 0 & otherwise \end{cases} \tag{7.16}$$

7.3 For a function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x)\,dx \tag{7.17}$$

Find $P_e = E[P_e(N)]$.
Solution: since $N \sim \mathcal{N}(0, 1)$,

$$P_N(x) = \frac{1}{\sqrt{2\pi}} exp(-\frac{x^2}{2}) \tag{7.18}$$

and form eqn7.16

$$P_e(N) = \begin{cases} 1 - exp(-\frac{N^2}{2\sigma^2}) & N < 0 \\ 0 & otherwise \end{cases} \tag{7.19}$$

$$P_e(N) = E[P_e(N)] = \int_{-\infty}^{\infty} P_e(x)P_N(x)dx \tag{7.20}$$

If $x < 0, P_e(x) = 0$ and using the fact that for an even function

$$\int_{-\infty}^{\infty} f(x) = 2\int_{-\infty}^{0} f(x) \tag{7.21}$$

we get

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} exp\left(-\frac{x^2}{2}\right)\left(1 - exp\left(-\frac{x^2}{2\sigma^2}\right)\right)dx \tag{7.22}$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} exp\left(-\frac{x^2}{2}\right)dx \tag{7.23}$$

$$- \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} exp\left(-\frac{(1+\sigma^2)x^2}{2\sigma^2}\right)dx \tag{7.24}$$

$$= \frac{\sqrt{2\pi} - \sqrt{\frac{\pi(2\sigma^2)}{1+\sigma^2}}}{2\sqrt{2\pi}} \tag{7.25}$$

$$= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\sigma^2}{1+\sigma^2}} \tag{7.26}$$

$$\tag{7.27}$$

For a Rayleigh Distribution with scale = σ,

$$E[A^2] = 2\sigma^2 \tag{7.28}$$

$$\gamma = 2\sigma^2 \tag{7.29}$$

$$\therefore P_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{2+\gamma}} \tag{7.30}$$

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ. Comment.
Solution: The following code plots Fig. 7.4
P_e is plotted w.r.t γ

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/7−4
    _gamma.py
```

The above code is executed using command

```
$ python3 7−4_gamma.py
```

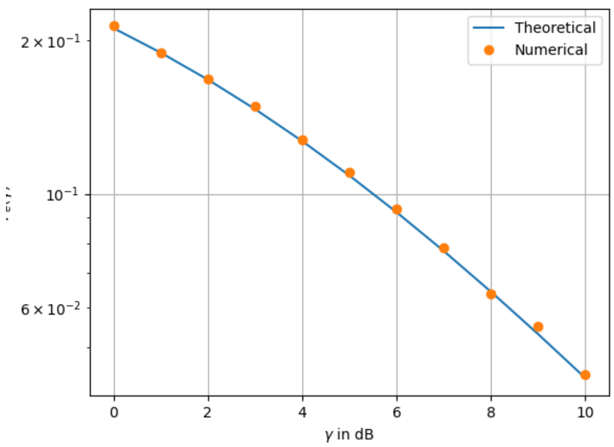

Fig. 7.4: P_e w.r.t γ

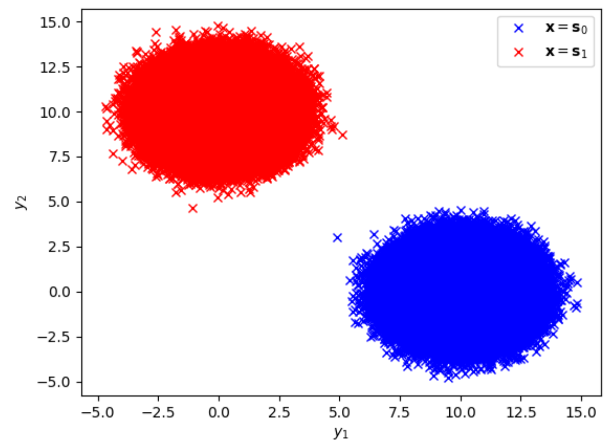

Fig. 8.1: Scatter plot of Y for $A = 10$

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix}0\\1\end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix}n_1\\n_2\end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0,1). \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.
Solution: The following code plots the scatter plot when $\mathbf{x} = \mathbf{s}_0$ and $\mathbf{x} = \mathbf{s}_1$ in Fig.8.1

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes
    /8−1_scatter_plot.py
```

The above code is executed using command

```
$ python3 8−1_scatter_plot.py
```

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.
Solution: The multivariate Gaussian distribution is defined as

$$p_{\mathbf{x}}(x_1,\dots,x_k) = \frac{1}{\sqrt{(2\pi)^k|\mathbf{\Sigma}|}}\exp\left\{-\frac{1}{2}(\mathbf{x}-\mu)^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)\right\} \tag{8.5}$$

where μ is the mean vector, $\mathbf{\Sigma} = E\left[(\mathbf{x}-\mu)(\mathbf{x}-\mu)^T\right]$ is the covariance matrix and $|\mathbf{\Sigma}|$ is the determinant of $\mathbf{\Sigma}$. For a bivariate gaussian distribution,

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\times\left\{\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}-\frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right\}\right] \tag{8.6}$$

where

$$\mu = \begin{pmatrix}\mu_x\\\mu_y\end{pmatrix}, \mathbf{\Sigma} = \begin{pmatrix}\sigma_x^2 & \rho\sigma_x\sigma_y\\ \rho\sigma_x\sigma_y & \sigma_y^2\end{pmatrix}, \tag{8.7}$$

$$\rho = \frac{E\left[(x-\mu_x)(y-\mu_y)\right]}{\sigma_x\sigma_y}. \tag{8.8}$$

$$\mathbf{y}|s_0 = \begin{pmatrix}A+n_1\\n_2\end{pmatrix} \tag{8.9}$$

$$\mathbf{y}|s_1 = \begin{pmatrix}n_1\\A+n_2\end{pmatrix} \tag{8.10}$$

Substituting these values in (8.6),

$$p(\mathbf{y}|s_0) = \frac{1}{2\pi\sigma_{y_1}\sigma_{y_2}\sqrt{1-\rho_1^2}}\exp\left[-\frac{1}{2\left(1-\rho_1^2\right)}\times\left\{\frac{(y_1-A)^2}{\sigma_{y_1}^2}+\frac{(y_2)^2}{\sigma_{y_2}^2}-\frac{2\rho_1(y_1-A)(y_2)}{\sigma_{y_1}\sigma_{y_2}}\right\}\right] \tag{8.11}$$

$$p(\mathbf{y}|s_1) = \frac{1}{2\pi\sigma_{y_1}\sigma_{y_2}\sqrt{1-\rho_2^2}} \exp\left[-\frac{1}{2\left(1-\rho_2^2\right)} \times \left\{\frac{(y_1)^2}{\sigma_{y_1}^2} + \frac{(y_2-A)^2}{\sigma_{y_2}^2} - \frac{2\rho_2(y_1)(y_2-A)}{\sigma_{y_1}\sigma_{y_2}}\right\}\right] \tag{8.12}$$

where,

$$\rho_1 = E[(y_1 - A)(y_2)] = E[n_1 n_2] = 0,$$
$$\rho_2 = E[(y_1)(y_2 - A)] = E[n_1 n_2] = 0,$$
$$\sigma_{y_1} = \sigma_{y_2} = 1 \tag{8.13}$$

For equiprobably symbols, the MAP criterion is defined as

$$p(\mathbf{y}|s_0) \underset{s_1}{\overset{s_0}{\gtrless}} p(\mathbf{y}|s_1) \tag{8.14}$$

Using (8.11) and (8.12) and substituting the values from (8.13), we get

$$(y_1 - A)^2 + y_2^2 \underset{s_0}{\overset{s_1}{\gtrless}} y_1^2 + (y_2 - A)^2 \tag{8.15}$$

On simplifying, we get the decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{8.16}$$

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{8.17}$$

with respect to the SNR from 0 to 10 dB.
Solution: The following code plots Fig. 8.3

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/8−3
    _SNR.py
```

The above code is executed using command

```
$ python3 8−3_SNR.py
```

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.
Solution:

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{8.18}$$

Given that $\mathbf{s}_0$ was transmitted, the received signal is

$$\mathbf{y}|\mathbf{s}_0 = \begin{pmatrix} A \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \tag{8.19}$$

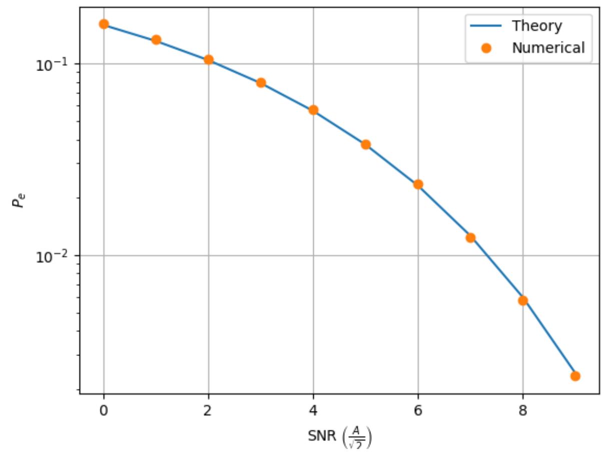

Fig. 8.3: P_e w.r.t SNR from 0 to 10 dB

From (8.16), the probability of error is given by

$$P_e = \Pr(y_1 < y_2 | \mathbf{s}_0) = \Pr(A + n_1 < n_2) \tag{8.20}$$
$$= \Pr(n_2 - n_1 > A) \tag{8.21}$$

Note that $n_2 - n_1 \sim \mathcal{N}(0, 2)$. Thus,

$$P_e = \Pr\left(\sqrt{2}w > A\right) \tag{8.22}$$
$$\Pr\left(w > \frac{A}{\sqrt{2}}\right) \tag{8.23}$$
$$\Rightarrow P_e = Q\left(\frac{A}{\sqrt{2}}\right) \tag{8.24}$$