Assignment: Random numbers

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I. Uniform Random Numbers

Let U be a uniform random varaible between 0 and 1.

- **1.1** Generate 10^6 samples of U using a C program and save into a file called uni.dat .
  Solution: Download the following files and execute the C program
  The C code - exrand.c
  The Header - coeffs.h

  Compile and execute the above C program using command
  - gcc exrand.c -lm -o exrand.out
  - ./exrand.out

- **1.2** Load the uni.dat file into python and pot the empirical CDF of U using the samples in uni.dat . The CDF is defined as

$$F_U(x) = Pr(U \leq x) \tag{1}$$

  Solution: The following code plots Fig. 1
  Python code - cdf_plot.py
  The above code is executed using command
  - python3 cdf_plot.py

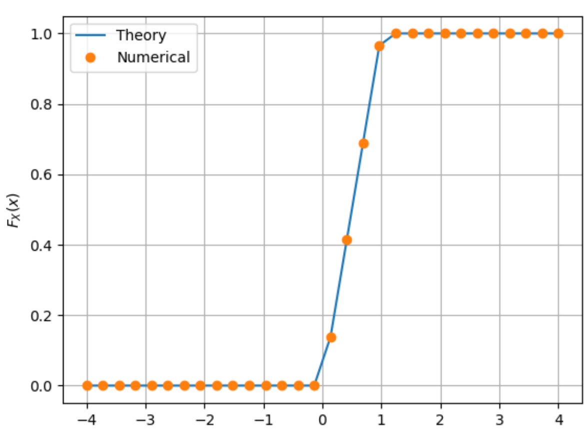

Fig. 1. The CDF of U

- **1.3** Find the theoritical expression for $F_U(x)$.
  Solution: Since U is a uniformly distributed in [0,1]
  We have three cases:
  - $x < 0$: $P_X(x) = 0$, and hence $F_U(x) = 0$.
  - $0 \leq x < 1$: Here,

$$F_U(x) = \int_0^x du = x \tag{2}$$

  - $x \geq 1$: Put $x = 1$ in above eqn we get $F_U(x) = 1$.

  Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \tag{3}$$

  This can be verified from Fig. 1

- **1.4** The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{4}$$

  and its varaience as

$$var[U] = E[U - E[U]]^2 \tag{5}$$

  Write a C program to find the mean and varaience of U

  Solution: Download and run the following C code.
  Mean and varaience - mean-var1-4.c
  The Header - coeffs.h

  Compile and execute the above C program using command
  - gcc mean-var1-4.c -lm -o mean-var1-4.out
  - ./mean-var1-4.out

  The mean of U is 0.500007
  The varience of U is 0.083301

- **1.5** Verify your result theotically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{6}$$

Solution: Verifying result theoritically
Given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{7}$$

Mean is given by

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{8}$$
$$= \int_{-\infty}^{\infty} x dx \tag{9}$$
$$= \left[\frac{x^2}{2}\right]_0^1 \tag{10}$$
$$= \frac{1}{2} \tag{11}$$

Varaience is given by

$$E[U - E[U]]^2 = E[U^2] - E[U]^2 \tag{12}$$
$$E[U]^2 = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{13}$$
$$= \int_{-\infty}^{\infty} x^2 dx \tag{14}$$
$$= \left[\frac{x^3}{3}\right]_0^1 \tag{15}$$
$$= \frac{1}{3} \tag{16}$$
$$E[U - E[U]]^2 = \frac{1}{3} - (\frac{1}{2})^2 \tag{17}$$
$$= \frac{1}{3} - \frac{1}{4} \tag{18}$$
$$= \frac{1}{12} \tag{19}$$

II. Central Limit Theorem

- **2.1** Generate 10^6 samples of the random varaible

$$X = \sum_{i=1}^{12} U_i - 6 \tag{20}$$

Using a C program, where U_i = 1,2,.........12 are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files and execute the C program
The C code - exrand.c
The Header - coeffs.h

Compile and execute the above C program using command
  - gcc exrand.c -lm -o exrand.out
  - ./exrand.out

- **2.2** Load gau.dat in python and plot the empirical CDF of X using the sample in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2
properties of cdf

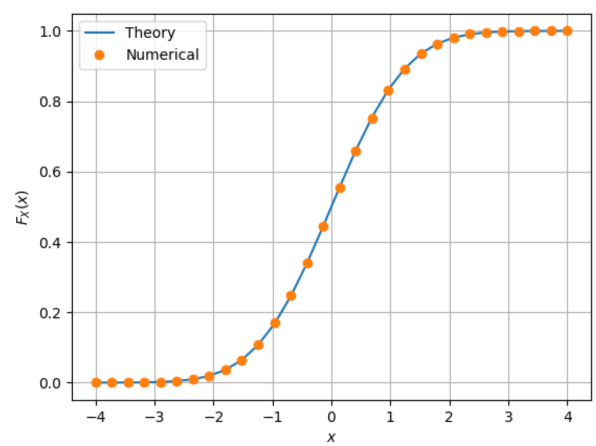

Fig. 2. The CDF of X

  - $F_X(x)$ is a nondecreasing function of x for -∞ $< x < \infty$.
  - The CDF, $F_X(x)$ ranges from 0 to 1. This makes sense since $F_X(x)$ is a probability.
  - If the maximum value of X is b, then $F_X(b)$ = 1

- **2.3** Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$P_X(x) = \frac{d}{dx} F_X(x) \tag{21}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 3 using the code below
Python code - pdf_plot.py
The above code is executed using command

– python3 pdf_plot.py

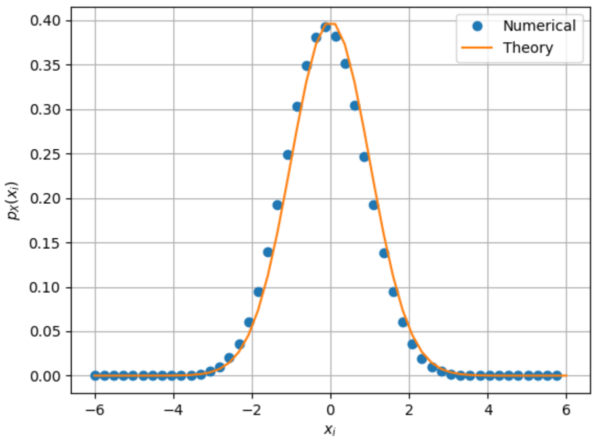

Fig. 3. The PDF of X

properties of pdf

– PDF is symmetric about X = 0
– Graph is bell shaped
– Mean of graph is situated at the apex point of the bell.

- **2.4** Find the mean and varaience of X by writing a C program.

  Solution: Download and run the following C code.
  Mean and varaience - mean-var2-4.c
  The Header - coeffs.h

  Compile and execute the above C program using command
  – gcc mean-var2-4.c -lm -o mean-var2-4.out
  – ./mean-var2-4.out

  The mean of X is 0.000326
  The varience of X is 1.000907

- **2.5** Given that

$$P_X(x) = \frac{1}{\sqrt{2\pi}} e^{(-\frac{x^2}{2})}, -\infty < x < \infty \tag{22}$$

  repeat the above exercise theoritically.

  Solution: Verifying theoritically

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{23}$$

$$= \frac{1}{2} erf\left(\frac{x}{\sqrt{2}}\right) \tag{24}$$

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx \tag{25}$$

  Taking $\frac{x^2}{2} = t \rightarrow xdx = dt$

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t} dt = 0 \tag{26}$$

$$E[X^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx \tag{27}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(xe^{-\frac{x^2}{2}}) dx \tag{28}$$

$$= \frac{1}{\sqrt{2\pi}} \left[-xe^{\frac{x^2}{2}} + \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \right]_{-\infty}^{\infty} \tag{29}$$

$$= 1 \tag{30}$$

$$\textbf{variance} = E[X^2] - E[X]^2 = 1 \tag{31}$$

III. From uniform to other

- **3.1** Generate samples of

$$V = -2ln(1-U) \tag{32}$$

  and plot its CDF.
  Solution: Download the following files and execute the C program
  The C code - exrand.c
  The Header - coeffs.h

  Compile and execute the above C program using command
  – gcc exrand.c -lm -o exrand.out
  – ./exrand.out

  The above C program will save the values of V in log.dat

  The CDF of X is plotted in Fig. 4 using the code
  Python code - cdf_plot3-1.py
  The above code is executed using command
  – python3 cdf_plot3-1.py

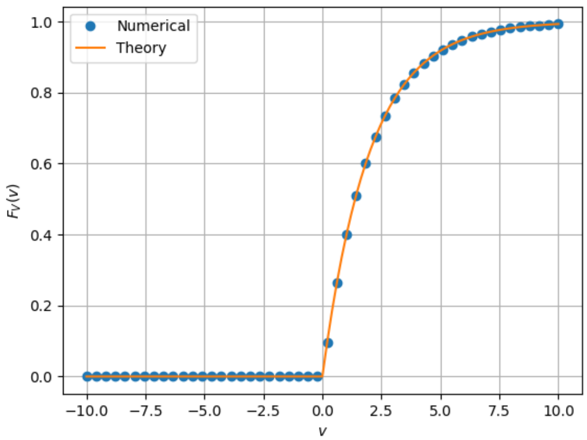

Fig. 4. The CDF of V

- **3.2** Find a theoritical expression for $F_V(x)$.
  Solution:

$$F_V(x) = Pr(V \leq x) \tag{33}$$

$$= Pr(-2ln(1-U) \leq x) \tag{34}$$

$$= Pr(U \leq 1 - e^{-\frac{x^2}{2}}) \tag{35}$$

$$Pr(U < x) = \int_0^x dx = x \tag{36}$$

$$\therefore Pr(U \leq 1 - e^{-\frac{x^2}{2}}) = 1 - e^{-\frac{x^2}{2}} \tag{37}$$

$$\rightarrow F_V(x) = 1 - e^{-\frac{x^2}{2}} \tag{38}$$