Random Numbers

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Contents

***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/exrand
    .c
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/coeffs.
    h
```

Compile and execute the above C program using command

```
$ gcc exrand.c −lm −o exrand.out
$ ./exrand.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/
    cdf_plot.py
```

The above code is executed using command

```
$ python3 cdf_plot.py
```

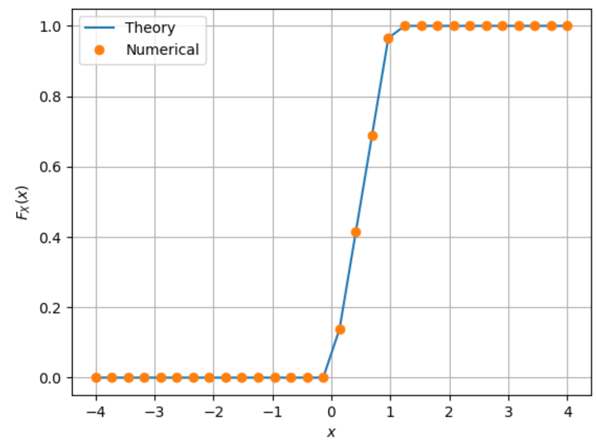

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Since U is a uniformly distributed in [0,1]
We have three cases:

- $x < 0$: $P_X(x) = 0$, and hence $F_U(x) = 0$.
- $0 \leq x < 1$: Here,

$$F_U(x) = \int_0^x du = x \tag{1.2}$$

- $x \geq 1$: Put $x = 1$ in above eqn we get $F_U(x) = 1$.

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \tag{1.3}$$

This can be verified from Fig. 1.2

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.4}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.5}$$

Write a C program to find the mean and variance of U.
Solution: Download the following files and execute the C program

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/exrand
    .c
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/coeffs.
    h
```

Compile and execute the above C program using command

```
$ gcc exrand.c −lm −o exrand.out
$ ./exrand.out
```

The mean of U is 0.500007
The varience of U is 0.083301

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x)dx \tag{1.6}$$

Solution: Verifying result theoritically
Given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.7}$$

Mean is given by

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.8}$$

$$= \int_{-\infty}^{\infty} x dx \tag{1.9}$$

$$= \left[\frac{x^2}{2}\right]_0^1 \tag{1.10}$$

$$= \frac{1}{2} \tag{1.11}$$

Varaience is given by

$$E[U - E[U]]^2 = E[U^2] - E[U]^2 \tag{1.12}$$

$$E[U]^2 = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{1.13}$$

$$= \int_{-\infty}^{\infty} x^2 dx \tag{1.14}$$

$$= \left[\frac{x^3}{3}\right]_0^1 \tag{1.15}$$

$$= \frac{1}{3} \tag{1.16}$$

$$E[U - E[U]]^2 = \frac{1}{3} - (\frac{1}{2})^2 \tag{1.17}$$

$$= \frac{1}{3} - \frac{1}{4} \tag{1.18}$$

$$= \frac{1}{12} \tag{1.19}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files and execute the C program

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/exrand
    .c
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/coeffs.
    h
```

Compile and execute the above C program using command

```
$ gcc exrand.c −lm −o exrand.out
$ ./exrand.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2
properties of cdf

- $F_X(x)$ is a nondecreasing function of x for $-\infty < x < \infty$.

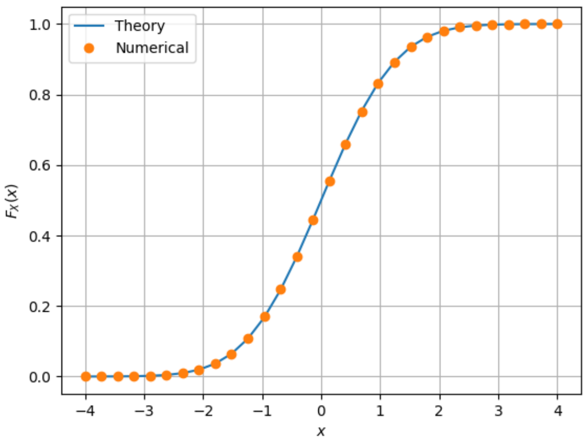

Fig. 2.2: The CDF of X

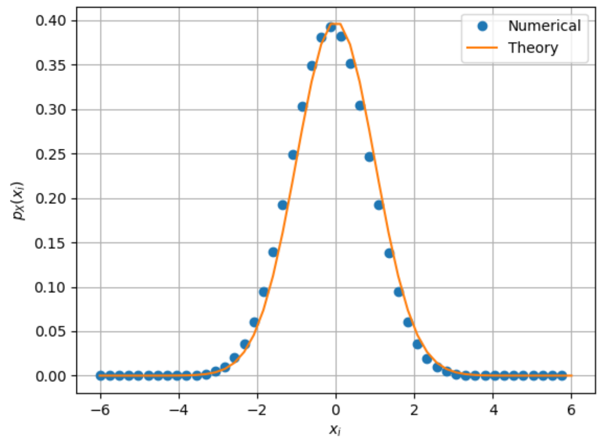

Fig. 2.3: The PDF of X

- The CDF, $F_X(x)$ ranges from 0 to 1. This makes sense since $F_X(x)$ is a probability.
- If the maximum value of X is b, then $F_X(b)$ = 1

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below
Python code - pdf_plot.py
The above code is executed using command

- python3 pdf_plot.py

properties of pdf

- PDF is symmetric about $X = 0$
- Graph is bell shaped
- Mean of graph is situated at the apex point of the bell.

2.4 Find the mean and variance of X by writing a C program.
Solution: Download and run the following C code.
The C program can be downloaded using

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/mean
    −var2−4.c
```

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/coeffs.
    h
```

Compile and execute the above C program using command

```
$ gcc mean−var2−4.c −lm −o mean−var2−4.
    out
$ ./mean−var2−4.out
```

The mean of X is 0.000326
The varience of X is 1.000907

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.
Solution: Verifying theoritically

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{2.4}$$

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx \tag{2.5}$$

Taking $\frac{x^2}{2} = t \rightarrow xdx = dt$

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t} dt = 0 \tag{2.6}$$

$$E[X^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx \tag{2.7}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(xe^{-\frac{x^2}{2}}) dx \tag{2.8}$$

$$= \frac{1}{\sqrt{2\pi}} \Big[-xe^{\frac{x^2}{2}} + \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \Big]_{-\infty}^{\infty} \tag{2.9}$$

$$= 1 \tag{2.10}$$

$$\textbf{variance} = E[X^2] - E[X]^2 = 1 \tag{2.11}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution: Download the following files and execute the C program

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/exrand
    .c
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/coeffs.
    h
```

Compile and execute the above C program using command

```
$ gcc exrand.c −lm −o exrand.out
$ ./exrand.out
```

The above C program will save the values of V in log.dat

and the CDF is plotted in Figure (3.1).

```
$ wget https://github.com/jarpula−Bhanu/
    Random−numbers/blob/main/codes/
    cdf_plot3−1.py
```

The above code is executed using command

```
$ python3 cdf\_plot3−1.py
```

3.2 Find a theoretical expression for $F_V(x)$.

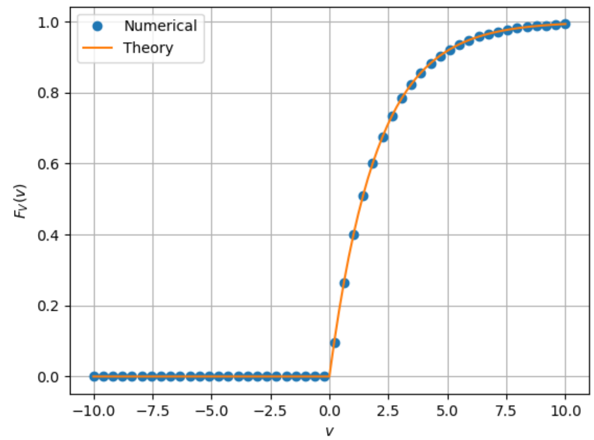

Fig. 3.1: The CDF of V

$$F_V(x) = Pr(V \le x) \tag{3.2}$$

$$= Pr(-2ln(1-U) \le x) \tag{3.3}$$

$$= Pr(U \le 1 - e^{-\frac{x^2}{2}}) \tag{3.4}$$

$$Pr(U < x) = \int_0^x dx = x \tag{3.5}$$

$$\therefore Pr(U \le 1 - e^{-\frac{x^2}{2}}) = 1 - e^{-\frac{x^2}{2}} \tag{3.6}$$

$$\rightarrow F_V(x) = 1 - e^{-\frac{x^2}{2}} \tag{3.7}$$