Non-Standard *n*-Sided Dice

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Abstract

Dice have existed since early human culture, being a simple random number generator. However, when two dice are rolled together, they create different sum probabilities. For example, on a six-sided die, the chances of a two or a twelve are slim, at only 1/36, while the chances of a 6 or 7 are 5/36 and 1/6, respectively. Rolling dice creates a sort of bell curve probability distribution, for any number of any-sided dice. In 1978, a German Colonel by the name of George Sicherman posted a question as to whether there were any other pair of six-sided dice that are non-standard, or not having the traditional 1,2,3,4,5, and 6 on them, that created the same probability distribution as two normal six-sided die. A pair of dice was found to have this unique property, numbered 1,2,2,3,3,4 and 1,2,3,4,5,8. These dice were discovered using polynomial factoring, where the polynomial $x^1 + x^2 + x^3 + \ldots + x^n$ represents an *n-sided* dice, and when multiplying two standard dice together, the result is: $x^2 + 2x^3 + 3x^4 \ldots + 3x^{10} + 2x^{11} + 3x^{12}$, where the coefficients represent the probability ratio of the power. By factoring the dice polynomials, and rearranging the factors, non-standard dice can be found that still give the same probability distribution as the standard dice. A computer program was written in java, in order to find all the possible non-standard, *n*-sided die.

Standard dice labeled 1-*n* provide a simple random number generator, used for many games, such as Monopoly, Risk, and Yahztee. When two or more dice are rolled together, and their faces summed, the result is a bell curved probability distribution, with the tails at 2 and 12 being much less likely to be rolled, and the peak being more likely to be rolled. However, other six-sided dice present the same probability models as normal dice, called the Sicherman dice. A computer program was written in the java programming language in order to find other non-standard dice with the same probability distributions as their same sided normal dice.

Standard Dice and Polynomials

In order to solve this problem, polynomials were used to represent the dice and their corresponding probability distributions. Dice were represented by polynomials of the form:

$$x^1 + x^2 + x^3 + x^4 + x^5 + x^6$$

for a standard six-sided die. To "roll" two of these dice, the polynomials representing the two dice are multiplied. For example, rolling two six-sided dice would yield:

$$(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)(x^1 + x^2 + x^3 + x^4 + x^5 + x^6) =$$

$$x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}$$

Where the coefficients represent the ratio of rolling the sum, and the power represents the sum of the faces of the dice. As you can see, this probability polynomial represents the odds of rolling a sum of 2-12 with two six-sided dice. In order to have the same probability model, the non-standard dice must multiply to give the same probability polynomial.

Factoring and Regrouping

In order for the two dice to roll the same probability model, they must multiply to give the same probability model as the same dice above. In order do this, the dice polynomial's factors are rearranged and regrouped, so that they still multiply to give the probability polynomial, yet they are two separate dice. To do this, the dice polynomials are first factored:

$$x^1 + x^2 + x^3 + x^4 + x^5 + x^6 = (x)(x + 1)(x^2 + x + 1)(x^2 - x + 1))$$

This represents two six-sided dice, being factored into the four polynomial factors. Given that two dice are multiplied together, the rolling of two six-sided dice can be represented by multiplying the two six dice polynomial factors:

$$((x)(x + 1)(x^2 + x + 1)(x^2 - x + 1)) ((x)(x + 1)(x^2 + x + 1)(x^2 - x + 1))$$

The associative property of multiplication says that we can move these factors around and group them with parentheses differently without changing the product. However, not all possible groupings of one or more factors will result in six-sided dice. Therefore, the two resulting groupings must each multiply to a sum of six coefficients, thus the six sides. To ensure this, the coefficients of each of the factors are all added:

$$x \to 1$$
$$(x + 1) \to 2$$
$$(x^2 + x + 1) \to 3$$
$$(x^2 - x + 1) \to 1$$

The sums of the coefficients represent the sides, but instead of them adding to equal six, they must multiply to equal six. Since the non-standard dice cannot have a zero, the (x) factor must remain on each side, so that the smallest power is x^1, and thus the smallest side on the die is 1. Since (x + 1) and ($x^2 + x + 1$) multiply to give you a sum of six coefficients, they too cannot be moved to a different side, but must stay together. Thus, the only factor that can be regrouped is the ($x^2 - x + 1$), which can be moved to the other side. This creates two possibilities for the dice: one with both ($x^2 - x + 1$) factors on one side, and another with one ($x^2 - x + 1$) factor on each side. Therefore, the two resulting dice pairs are:

$$(x(x + 1)(x^2 + x + 1)(x^2 - x + 1))(x(x + 1)(x^2 + x + 1)(x^2 - x + 1)) =$$

$$(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$$

and

$$((x)(x + 1)(x^2 + x + 1))\ \ ((x^2 - x + 1)(x)(x + 1)(x^2 + x + 1)(x^2 - x + 1)) =$$

$$(x + 2x^2 + 2x^3 + x^4)(x^1 + x^3 + x^4 + x^5 + x^6 + x^8)$$

The first pair is the standard set of dice, since we did not rearrange any of the factors. The second pair of dice, however, represent a completely different set of dice. These dice are called the Sicherman Dice, after Colonel George Sicherman, who first posed the question of non-standard dice in 1978. These two dice, when multiplied, give you the same probability polynomial, and therefore, when rolled together, give the same probability for each sum 2-12 as two standard dice. The Sicherman dice are the only non-standard, six-sided dice that mimic the probability of two standard six-sided dice.

Finding Other *n*-Sided, Non-Standard Dice

The method above works to prove whether or not there can be two *n-sided* non-standard dice that give the same probability model for their standard counterparts. For example, an 11 sided die would give the polynomial:

$$x^1 + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11}$$

Which, when factored, results in:

$$(x)(1 + x^1 + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10})$$

Since both the (x) and the longer factor must stay on each side, the factors cannot be regrouped, and therefore, a pair of 11-sided standard dice does not have a counterpart. This is because the longer factor is a prime cyclotomic polynomial, one that has positive integer coefficients, but cannot be factored or reduced. Any prime number *n-sided* dice will be factored to the factors:

$$(x)(1 + x + x^2 + x^3\ldots + x^{n-1})$$

with the latter factor being irreducible. Since the (x) factor must stay on each side, so there are no zeros on any of the dice. Also, there must be one longer factor on each side, to keep the dice equal. Thus, any prime number sided dice can only be reduced to two factors, and these factors must stay on their respective side, and so it does not have a non-standard set with the same probability models.

Other *n*-Sided, Non-Standard Dice

Other polynomials of the same pattern were factored to find other possible non-standard dice. From factoring the polynomials, a few patterns were found in the factors of any die polynomial:

1. Each *n*-sided die polynomial, there is a "special factor", seen first in that specific dice polynomial, and not in any previous *n*-sided die.

2. The dice polynomial with *n* sides contains all the "special factors" of every factor of the integer *n*.

3. Every prime sided dice polynomial *n* is factorable only to two factors: $(x)(1 + x^1 + x^2 + x^3\ldots + x^{n-1})$.

For example, these rules show that:

1. The dice polynomial $(x^1 + x^2 + x^3 + x^4)$ has the special factor $(x^2 + 1)$, unique to itself.

2. The dice polynomial $(x^1 + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12})$ representing 12 sides, contains the special factors from 1, 2, 3, 4, and 6. When the polynomial is divided by each of these special factors, the special factor for 12 is found to be $(x^4 – x^2 + 1)$

3. The dice polynomial $(x^1 + x^2 + x^3 + x^4 + x^5 + x^6 + x^7)$ is only factorable to $(x)(1 + x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$.

These rules allow for any dice polynomial to be broken down into all of its factors. Its own special factor can be found by dividing itself by all of its factors' special factors.

The Computer Program

In order to find the number of, and all non-standard, *n*-sided die that mimic the probability models of two standard *n*-sided die. The program was written in the java programming language, and came to a total of around to about 500 lines of code. The Computer program used the same theory demonstrated above, factoring polynomials and then finding all possible regroupings that give valid dice. The computer program uses five main steps:

First, the computer program finds the *n*-sided dice polynomial's special factor. It does this by finding all of the factors of the user-specified integer *n*-sided dice. For each of these factors, it finds their special factor of their die polynomial. It does this in either one of two ways. The first way involves if the factor follows one of three patterns, based upon if the integer *n* is:

1. Prime: the special factor is $(1 + x^1 + x^2 + x^3 \ldots + x^{n-1})$
2. A power of two (ie 8,64): the special factors is $(x^{\log_2 n-1})$
3. $n/2$ is Prime: with $n/2$'s special factor being $(1 + x + x^2 + x^3 \ldots + x^{n/2-1})$, then the special factor is $(1 + x - x^2 + x^3 \ldots - x^{n/2-1})$, with alternating addition and subtraction.

If *n* does not satisfy these requirements, the program then finds each of *n*'s special factors' special factors, by recursively calling itself. For example, for 24, the program could not find any special factor for 12 using these methods, so it then finds the special factors of 2,3,4, and 6. Once it has these factors, it divides 12's dice polynomial by each of them, resulting in 12's special factor.

Secondly, With *n*'s special factors found, the computer program then finds the rest of the polynomial factors of *n*'s dice polynomial, and adds them to a list. For each of the factors, it finds the sum of its coefficients, by plugging in one for each of them. With these sums, it then groups all of the factors into groups by their sum. This way, it can reorganize the factors while still keeping the same number of each factor on both sides.

Next, once the factors are found for each sum, it then finds each possible combination of each of the groups. For example, if *n* had three factors summing to two, it would find all combinations of those six factors, three on each side, where there are three on each side, so they have equal products.

Next, the computer program combines each factor with each other factor. It does this by combining the first two sum factor groups, and then combining that combination with the next combination of sum factor groups. It combines until it has run out of factor sum groups.

Once all the combination are found, the program then converts the polynomial to a dice. Once it has all the possible dice, it checks to make sure that none of the dice have negative sides, and that all of them roll to give the same probability as their standard counterparts.

This program was tested by finding the six-sided counterpart, the Sicherman dice, and then finding a four-sided counterpart, which was checked and proven by hand. In both cases, the program found the alternative pair of dice.

Findings

Using this program, non-standard dice were found for all *n* sides up to 50. Along with these non-standard dice, three patterns were found:

1. Prime sided dice have exactly one set of dice that give the same probability polynomial, the standard die.

2. Dice whose only factors are one, themselves, and two prime numbers have exactly two sets of dice that give the same probability function, the standard set, and one other not-standard set.

3. In general, the greater number of distinct factors a number has, the more alternative dice.

These two patterns were the only discernible patterns to the number of possible dice related to the number of sides. The reason for this was another pattern on the dice polynomial:

For a dice polynomial n, where $n/2$ is prime with the special factor $(1 + x + x^2 + x^3 \ldots + x^{n/2-1})$, n's dice polynomial's special factor is $(1 + x - x^2 + x^3 \ldots - x^{n/2-1})$.

Given this, some combinations of factors that worked in theory ended up creating negative dice faces, which contradicted the concept of a positive integer-labeled dice. These negative coefficients could only be counteracted by combining the factors with one or more other factors. Therefore, this constricted the possibilities considerably, and also made them much harder to determine without a brute force attempt involving all possible combinations.

In conclusion, the computer program was run for all numbers up to 100, and below are the first twenty dice's special factors and possible combinations, including the standard pair of die.

Table 1

Sides	Special Factor	Possible Dice
1	x	1
2	$x + 1$	1
3	$x^2 + x + 1$	1
4	$x^2 + 1$	2

5	$x^4 + x^3 + x^2 + x + 1$	1
6	$x^2 - x + 1$	2
7	$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$	1
8	$x^4 + 1$	4
9	$x^6 + x^3 + 1$	2
10	$x^4 - x^3 + x^2 - x + 1$	2
11	$x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$	1
12	$x^4 - x^2 + 1$	7
13	$x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$	1
14	$x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$	2
15	$x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$	2
16	$x^8 + 1$	10
17	$x^{16} + x^{15} + x^{14} + \ldots + x^3 + x^2 + x + 1$	1
18	$x^6 - x^3 + 1$	7
19	$x^{18} + x^{17} + x^{16} + \ldots + x^3 + x^2 + x + 1$	1
20	$x^8 - x^6 + x^4 - x^2 + 1$	7

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