Assignment 8
AI1110: Probability and Random Variables

Indian Institute of Technology, Hyderabad

JARUPULA SAI KUMAR
CS21BTECH11023

Question 10.9 [Papoulis Textbook]: The position of a particle in underdamped harmonic motion is a normal process with autocorrelation as in (10-60). Show that its conditional density assuming $x(O) = X_o$ and $x'(O) = v(O) = V_o$ equals

$$f\left(x|x_o, v_o\right) = \frac{1}{\sqrt{2\pi P}} e^{-\frac{(x-ax_O-bv_O)^2}{2}} \tag{1}$$

Find the constants a, b, and P.

$$R_x(\tau) = \frac{kT}{c} e^{-\alpha|\tau|}\left(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta|\tau|\right) \tag{2}$$

1 Solution

Let us try to use the given conditions to solve the constants a,b and P

$$R_x(\tau) = \frac{kT}{c} e^{-\alpha|\tau|}\left(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta|\tau|\right) \tag{3}$$

$$\underset{\sim}{x}(\tau) - a\underset{\sim}{x}(0) - b\underset{\sim}{v}(0) \perp \underset{\sim}{x}(0), \underset{\sim}{v}(0) \tag{4}$$

$$\tag{5}$$

We know that,

$$R_{xx}(\tau) = aR_{xx}(0) + bR_{xv}(0) \tag{6}$$

$$R_{xv}(\tau) = aR_{xv}(0) + bR_{vv}(0) \tag{7}$$

$\mathrm{R}_{xx}(\tau) = Ae^{-\alpha\tau}\left(\cos B\tau + \frac{\alpha}{B}\sin B\tau\right)$
$R_{xv}(\tau) = -R'_{xx}(\tau) = Ae^{-\alpha\tau}(\sin B\tau)\frac{\alpha^2+3^2}{\beta}$
$R_{vv}(\tau) = R'_{xv}(\tau) = Ae^{-\alpha\tau}\left(\cos B\tau - \frac{\alpha}{B}\sin B\tau\right)\frac{\alpha^2+\beta^2}{\beta^2}$

Inserting into (i) and solving, we obtain
a=$\mathrm{e}^{-\alpha\tau}\left(\cos B\tau + \frac{\alpha}{B}\sin B\tau\right)$
$b = \frac{1}{B}e^{-\alpha\tau}\sin B\tau$

$$P = E\left\{\left[x(t) - ax(0) - bv(0)\right]x(t)\right\} = R_{xx}(0) - aR_{xx}(t) - bR_{xv}(t)$$
$$= \frac{2kTf}{m^2}\left[1 - e^{-2\alpha t}\left(1 + \frac{2\alpha^2}{B}\sin^2 Bt + \frac{\alpha}{B}\sin 2\beta t\right)\right]$$