Paul's Online Notes

Section 4-5 : Solving IVP's With Laplace Transforms

It's now time to get back to differential equations. We've spent the last three sections learning how to take Laplace transforms and how to take inverse Laplace transforms. These are going to be invaluable skills for the next couple of sections so don't forget what we learned there.

Before proceeding into differential equations we will need one more formula. We will need to know how to take the Laplace transform of a derivative. First recall that $f^{(n)}$ denotes the n^{th} derivative of the function f. We now have the following fact.

Fact

Suppose that f, f', f'',… $f^{(n-1)}$ are all continuous functions and $f^{(n)}$ is a piecewise continuous function. Then,

$$\mathcal{L}\left\{f^{(n)}\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

Since we are going to be dealing with second order differential equations it will be convenient to have the Laplace transform of the first two derivatives.

$$\begin{aligned} \mathcal{L}\{y'\} &= sY(s) - y(0) \\ \mathcal{L}\{y''\} &= s^2 Y(s) - sy(0) - y'(0) \end{aligned}$$

Notice that the two function evaluations that appear in these formulas, $y(0)$ and $y'(0)$, are often what we've been using for initial condition in our IVP's. So, this means that if we are to use these formulas to solve an IVP we will need initial conditions at $t = 0$.

While Laplace transforms are particularly useful for nonhomogeneous differential equations which have Heaviside functions in the forcing function we'll start off with a couple of fairly simple problems to illustrate how the process works.

Example 1 Solve the following IVP.

$$y'' - 10y' + 9y = 5t, \qquad y(0) = -1 \qquad y'(0) = 2$$

Hide Solution ▾

The first step in using Laplace transforms to solve an IVP is to take the transform of every term in the differential equation.

$$\mathcal{L}\{y''\} - 10\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{5t\}$$

Using the appropriate formulas from our **table of Laplace transforms** gives us the following.

$$s^2Y(s) - sy(0) - y'(0) - 10(sY(s) - y(0)) + 9Y(s) = \frac{5}{s^2}$$

Plug in the initial conditions and collect all the terms that have a $Y(s)$ in them.

$$\left(s^2 - 10s + 9\right)Y(s) + s - 12 = \frac{5}{s^2}$$

Solve for $Y(s)$.

$$Y(s) = \frac{5}{s^2(s-9)(s-1)} + \frac{12-s}{(s-9)(s-1)}$$

At this point it's convenient to recall just what we're trying to do. We are trying to find the solution, $y(t)$, to an IVP. What we've managed to find at this point is not the solution, but its Laplace transform. So, in order to find the solution all that we need to do is to take the inverse transform.

Before doing that let's notice that in its present form we will have to do partial fractions twice. However, if we combine the two terms up we will only be doing partial fractions once. Not only that, but the denominator for the combined term will be identical to the denominator of the first term. This means that we are going to partial fraction up a term with that denominator no matter what so we might as well make the numerator slightly messier and then just partial fraction once.

This is one of those things where we are apparently making the problem messier, but in the process we are going to save ourselves a fair amount of work!

Combining the two terms gives,

$$Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)}$$

The partial fraction decomposition for this transform is,

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

Setting numerators equal gives,

$$5 + 12s^2 - s^3 = As(s-9)(s-1) + B(s-9)(s-1) + Cs^2(s-1) + Ds^2(s-9)$$

Picking appropriate values of s and solving for the constants gives,

$$\begin{array}{llll} s = 0 & 5 = 9B & \Rightarrow & B = \frac{5}{9} \\ s = 1 & 16 = -8D & \Rightarrow & D = -2 \\ s = 9 & 248 = 648C & \Rightarrow & C = \frac{31}{81} \\ s = 2 & 45 = -14A + \frac{4345}{81} & \Rightarrow & A = \frac{50}{81} \end{array}$$

Plugging in the constants gives,

$$Y(s) = \frac{\frac{50}{81}}{s} + \frac{\frac{5}{9}}{s^2} + \frac{\frac{31}{81}}{s-9} - \frac{2}{s-1}$$

Finally taking the inverse transform gives us the solution to the IVP.

$$y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}\mathbf{e}^{9t} - 2\mathbf{e}^{t}$$

That was a fair amount of work for a problem that probably could have been solved much quicker using the techniques from the previous chapter. The point of this problem however, was to show how we would use Laplace transforms to solve an IVP.

There are a couple of things to note here about using Laplace transforms to solve an IVP. First, using Laplace transforms reduces a differential equation down to an algebra problem. In the case of the last example the algebra was probably more complicated than the straight forward approach from the last chapter. However, in later problems this will be reversed. The algebra, while still very messy, will often be easier than a straight forward approach.

Second, unlike the approach in the last chapter, we did not need to first find a general solution, differentiate this, plug in the initial conditions and then solve for the constants to get the solution. With Laplace transforms, the initial conditions are applied during the first step and at the end we get the actual solution instead of a general solution.

In many of the later problems Laplace transforms will make the problems significantly easier to work than if we had done the straight forward approach of the last chapter. Also, as we will see, there are some differential equations that simply can't be done using the techniques from the last chapter and so, in those cases, Laplace transforms will be our only solution.

Let's take a look at another fairly simple problem.

Example 2 Solve the following IVP.

$$2y'' + 3y' - 2y = t\mathbf{e}^{-2t}, \qquad y(0) = 0 \quad y'(0) = -2$$

Show Solution

Example 3 Solve the following IVP.

$$y'' - 6y' + 15y = 2\sin(3t), \qquad y(0) = -1 \quad y'(0) = -4$$

Show Solution

To this point we've only looked at IVP's in which the initial values were at $t = 0$. This is because we need the initial values to be at this point in order to take the Laplace transform of the derivatives. The problem with all of this is that there are IVP's out there in the world that have initial values at places other than $t = 0$. Laplace transforms would not be as useful as it is if we couldn't use it on these types of IVP's. So, we need to take a look at an example in which the initial conditions are not at $t = 0$ in order to see how to handle these kinds of problems.

Example 4 Solve the following IVP.

$$y'' + 4y' = \cos(t-3) + 4t, \qquad y(3) = 0 \quad y'(3) = 7$$

Show Solution ▸

So, we can now do IVP's that don't have initial conditions that are at $t = 0$. We also saw in the last example that it isn't always the best to combine all the terms into a single partial fraction problem as we have been doing prior to this example.

The examples worked in this section would have been just as easy, if not easier, if we had used techniques from the previous chapter. They were worked here using Laplace transforms to illustrate the technique and method.

 Page Last Modified : 6/3/2018