Atma Ram Sanatan Dharma College, University of Delhi
February 26, 2021

Simpson’s Integration Rule

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Overview

Simpson's 1/3 Rule:

1. Introduction – Kunal Giri (19/78090)
2. Solved Example – Pyush Deep (19/78096)
3. Practical – Shubhang Gupta (19/78098)

Simpson's 3/8 Rule:

1. Introduction – Abhishek Patel (19/78106)
2. Solved Example – Priyanshu Singh (19/78082)
3. Practical 1 – Joshita Sood (19/78086)
4. Practical 2 – Harsh Yadav (19/78092)

Simpson's 1/3 Rule:

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INTRODUCTION

- Kunal Giri (19/78090)

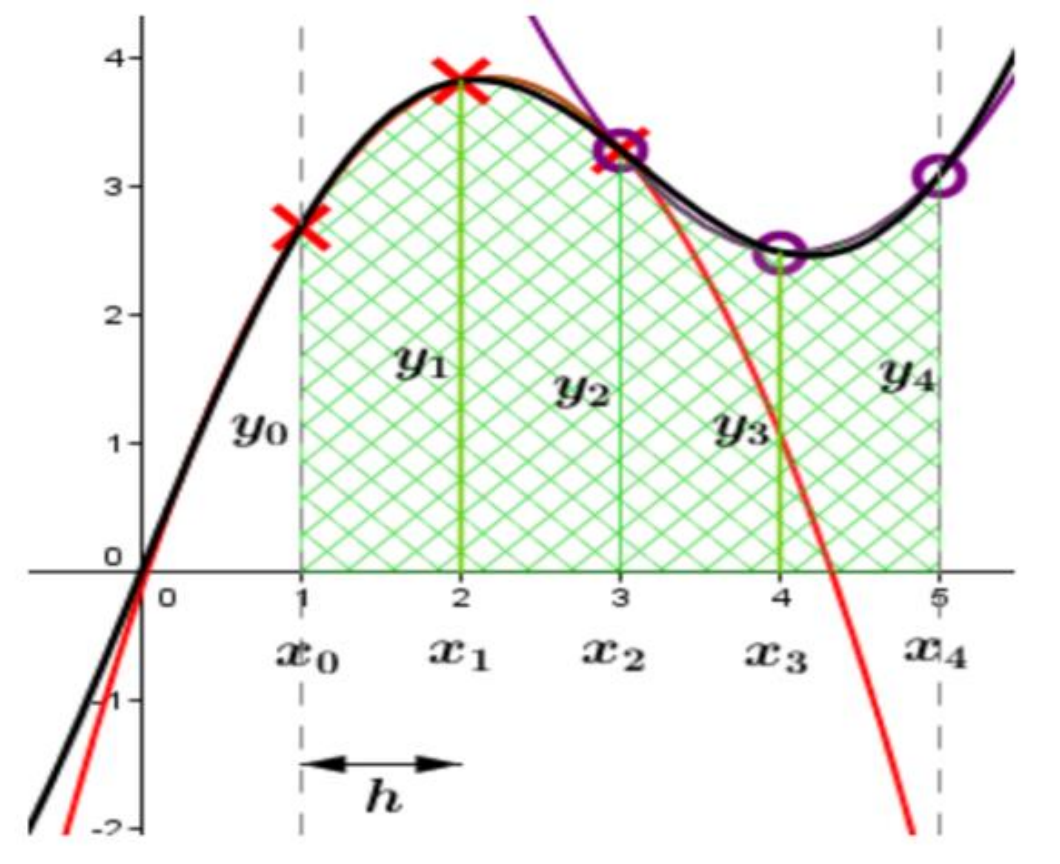
Simpson's Rule is one of the numerical methods which we use to evaluate the definite integral. Usually, to find the definite integral, we use the fundamental theorem of calculus, where we have to apply the antiderivative techniques of integration. But sometimes it is difficult to find the antiderivative of an integral. Therefore, the numerical methods are used to approximate the integral in such conditions. Other numerical methods used are Trapezoidal rule, midpoint rule, and left or right approximation using Riemann sums. Here, we are going to discuss Simpson's rule formula, 1/3 rule, 3/8 rule and examples.

Simpson's 1/3rd rule is an extension of the trapezoidal rule where we put n = 1 in the Newton's formula. Simpson rule can be derived from the various way using Newton's divided difference polynomial. Simpson's 1/3 rule is defined by:

$$\int_a^b f(x)\,dx = h/3[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \ldots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \ldots + y_{n-2})]$$

This rule is known as Simpson's One-third rule.

Derivation of Simpson's ⅓ Rule



Let the interval $[a,b]$ be divided into n equal subintervals such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. Clearly $x_n = x_0 + nh$. Hence the integral becomes as

$$I = \int_{x_0}^{x_n} y\,dx$$

Approximating y by Newton's forward difference formula, we get

$$I = \int_{x_0}^{x_n} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 \cdots \right] dx$$

$\because$ $x = x_0 + ph$, $dx = h\,dp$ & hence the above integral becomes

$$I = h \int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + \cdots \right] dp$$

which gives on simplification

$$\int_{x_0}^{x_p} y\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right] \longrightarrow (1)$$

from this general formula (eqn (1)), we can get different integration formulae by putting $n = 1, 2, 3, \ldots$.

This rule is obtained by putting $n=2$ in the eqn →

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right] \cancel{n(n}$$

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left[y_0 + \Delta y_0 + \frac{\cancel{2}\times 1}{\cancel{12}\ 6}\Delta^2 y_0\right]$$

$$\Rightarrow \int_{x_0}^{x_0+2h} f(x)\,dx = \frac{2h}{6}\left[6y_0 + 6\Delta y_0 + \Delta^2 y_0\right]$$

$$\Rightarrow \int_{x_0}^{x_0+2h} = \frac{h}{3}\left[6y_0 + 6(y_1 - y_0) + (y_2 - 2y_1 + y_0)\right]$$

$$\Rightarrow \int_{x_0}^{x_0+2h} = \frac{h}{3}\left[(y_0 + y_2) + 4y_1\right]$$

Similarly $\rightarrow$ putting $n = 4 \rightarrow$

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}\left[y_2 + y_4 + 4y_3\right]$$

for $\underline{n = n} \rightarrow$

$$\int_{x_0+(n-2)h}^{x_0+nh} = \frac{h}{3}\left[y_{n-2} + y_n + 4y_{n-1}\right]$$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+2h} f(x)\,dx + \int_{x_0+2h}^{x_0+4h} f(x)\,dx + \cdots \int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx$$

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + 4y_1 + y_2) + (y_2 + y_4 + 4y_3) + \cdots \cdots + y_{n-2} + y_n + 4y_{n-1}\right]$$

$$\Rightarrow \boxed{\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots y_{n-1})\right]}$$

This is Simpson's $\frac{1}{3}$ Rule.

Advantages of Simpson’s 1/3 rule:

1. This rule gives us more accurate result than the Trapezoidal rule.
2. Error rate is less than the other rules.
3. It can be used for more complex problems like in the case of Scientific Experiments, where the function has to be determined from the observed readings.

Disadvantages of Simpson's 1/3 rule:

1. Simpson's 1/3 rule always use even number of subintervals/ segments.
2. It always deals with evenly spaced data points.
3. It's derivation and conceptualization is more difficult than the Trapezoidal rule.
4. Just like the other rules, it also contains error which comes out to be

   |Error| = |Actual value – Approximate value|

Error in Simpson's 1/3 rule:

1. Just like the other rules, it also contains error which comes out to be

   |Error| = |Actual value – Approximate value|

Although in Simpson's rule method we get a more accurate approximation for definite integral, still the error occurs which is defined as when n = 2;

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f(\tfrac{a+b}{2}) + f(b)\right] \text{ with error as } \frac{1}{90}\left(\frac{b-a}{2}\right)^5 |f^4(z)|$$

SOLVED EXAMPLE

- Pyush Deep (19/78096)

Ques – Evaluate $I = \int_1^2 \frac{dx}{5+3x}$, using the Simpson's $\frac{1}{3}$ rule with 4 and 8 subintervals. Compare with the exact solution and find the absolute errors in the solutions.

Sol – With $N = 2N = 4, 8$

we have the following step lengths and nodal points.

$\Rightarrow N = 2$ and 4

$N=2$: $h = \frac{b-a}{2N} = \frac{2-1}{4} = \frac{1}{4} = 0.25$

The nodes are 1, 1.25, 1.5, 1.75, 2.0

$N=4$: $h = \frac{b-a}{2N} = \frac{1}{8} = 0.125$

The nodes are 1, 1.125, 1.25, 1.375, 1.5, 1.675, 1.75, 1.875, 2

$$f(x) = \frac{1}{5+3x}$$

Now, we have the following tables of values

$n=2N=4$:

x	1.0	1.25	1.5	1.75	2.0
$f(x)$	0.125	0.11429	0.10526	0.09756	0.09091

$n=2N=8$: we require the above values. The additional values required are the following.

x	1.125	1.375	1.625	1.875
$f(x)$	0.11940	0.10959	0.10127	0.09412

Simpson's 1/3 Rule $\Rightarrow$

$$\int_a^b f(x)\,dx = \frac{h}{3}\Big[f(x_0) + 4\{f(x_1) + f(x_3) ---- + f(x_{2N-1})\} + 2\{f(x_2) + f(x_4) + --- f(x_{2N-2})\} + f(x_{2N})\Big]$$

Now, we compute the value of the integral for $n=4$

$$I_1 = \frac{h}{3}[f(1) + 4(f(1.25) + f(1.75)) + 2f(1.5) + f(2)]$$

$$= \frac{0.25}{3}[0.125 + 4(0.11429 + 0.09756) + 2(0.10526) + 0.09091]$$

$$= \frac{0.25}{3}[0.125 + 4(0.21185) + 0.21052 + 0.09091]$$

$$= \frac{0.25}{3}[1.27383]$$

$$= 0.1061525$$

Now, we compute the value of integral for $n=8$

$$I_2 = \frac{h}{3}[f(1) + 4(f(1.125) + f(1.375) + f(1.625) + f(1.875)) + 2(f(1.25) + f(1.5) + f(1.75)) + f(2.0)]$$

$$I_2 = \frac{0.125}{3}[0.125 + 4(0.11940 + 0.10959 + 0.10127 + 0.09412) + 2(0.11429 + 0.10526 + 0.09756) + 0.09091]$$

$$\boxed{I_2 = 0.10615}$$

Now the exact value of integral

$$\int_1^2 \frac{dx}{5+3x} = \left[\frac{1}{3}\log|5+3x|\right]_1^2$$

$$= \frac{1}{3}[\log 11 - \log 8]$$

$$= 0.10615.$$

Therefore, the results obtained with $n=4$ and $n=8$ are accurate to all the places.

PRACTICAL

- Shubhang Gupta (19/78098)

Refer Maxima

Simpson's 3/8 Rule:

1. Introduction – Abhishek Patel (19/78106)
2. Solved Example – Priyanshu Singh (19/78082)
3. Practical 1 – Joshita Sood (19/78086)
4. Practical 2 – Harsh Yadav (19/78092)

INTRODUCTION

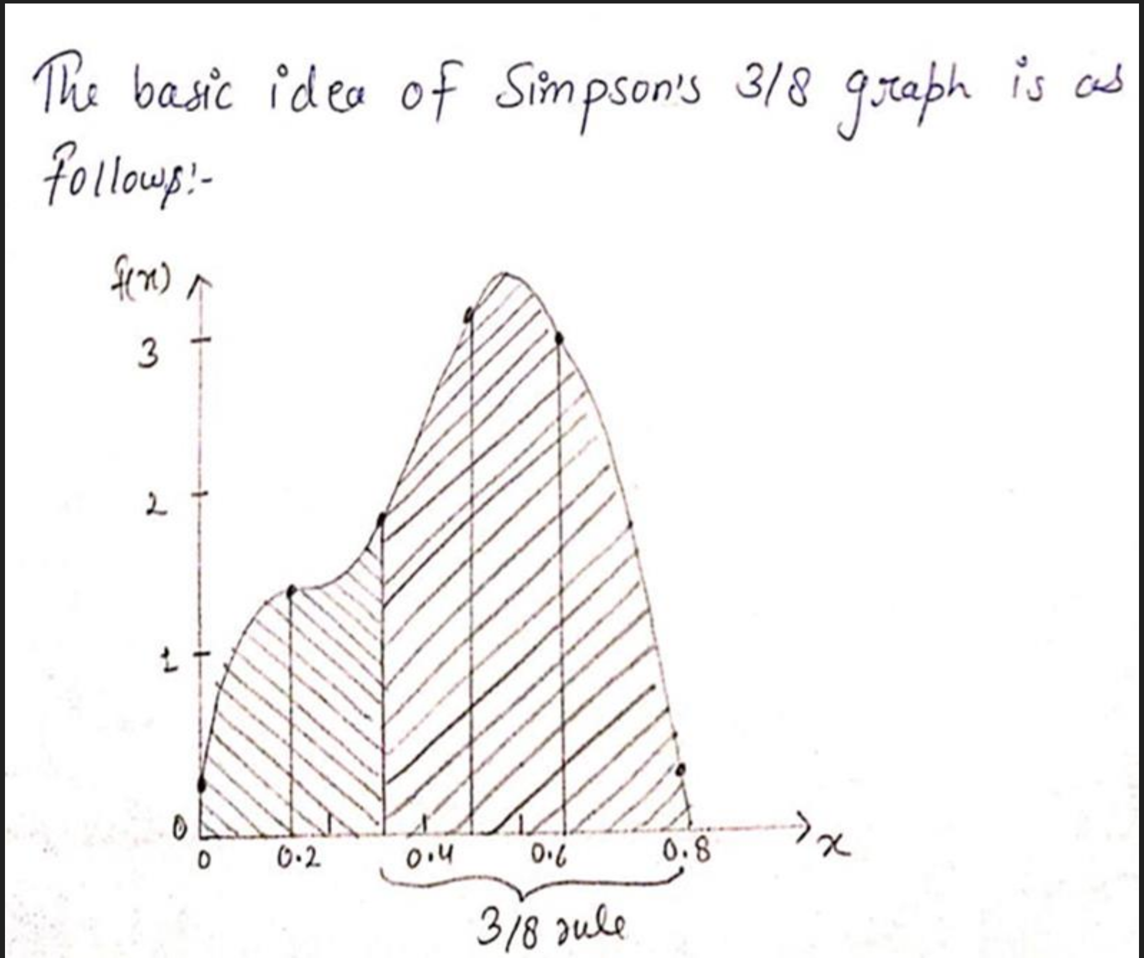
- Abhishek Patel (19/78106)

Simpson's 3/8-rule is a numerical method that approximates the value of a definite integral by evaluating the integrand at finitely many points and based upon a cubic interpolation rather than a quadratic interpolation.

Simpson's 3/8 rule is given by:

$$\int_a^b f(x)\,d(x) = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_2+\cdots y_{n-1})+2(y_3+y_6+y_9+\cdots+y_{n-3})\right]$$

Derivation of Simpson’s 3/8 Rule



Let f(x) be continous on [a, b]. we partition the interval [a, b] into n equal subinterval each of width 'h'

$$h = \frac{\text{upper limit} - \text{Lower limit}}{\text{Interval}}$$

$$h = \frac{b-a}{n}$$

Simpson's 3/8 rule can be used with the number of segments equal to 3, 6, 9, ...
(can be certain integers that are multiple of 3)

The general quadratic formula is given by

$$\int_{x_0}^{x_n} f(x)\,d(x) = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{2}\Delta^2 y_0 + \frac{n(n-2)^3}{24}\Delta^3 y_0 + \cdots \text{upto}(n+1) \text{ terms}.\right.$$

Put $n=3$ into above formula and taking the curve $y=f(x)$ between the points (x_0, y_0), (x_1, y_1), (x_2, y_2) (x_3, y_3) as a polynomial of degree three, So that the difference of the order greater than three becomes zero, we get,

$$\int_{x_0}^{x_3} f(x) = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3(6-3)}{12}\Delta^2 y_0 + \frac{3(3-2)^2}{24}\Delta^3 y_0\right]$$

$$= 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 + y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 2y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}\left[8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + y_3 - 3y_2 + 3y_1 - y_0\right]$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Similarly, taking the curve $y=f(x)$ between the point (x_3, y_3) and (x_6, y_6)

$$\int_{x_3}^{x_6} f(x)\,dx = \frac{3h}{8}\left(y_3 + 3y_4 + 3y_5 + y_6\right)$$

Similarly,

$$\int_{x_6}^{x_9} f(x)\,dx = \frac{3h}{8}\left(y_6 + 3y_7 + 3y_8 + y_9\right)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\int_{x_{n-3}}^{x_n} f(x)\,dx = \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

Now adding this 'n' interrals we get,

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\Big[(y_0 + y_n) + 2\left(y_3 + y_6 + y_9 + \cdots y_{n-3}\right) + 3\left(y_1 + y_2 + y_4 + \cdots y_{n-1}\right)\Big]$$

Advantages of Simpson’s 3/8 rule:

1. First, the error term is smaller than Simpson's rule.
2. We can use this formula for large scientific experiments also.

Disadvantages of Simpson’s 3/8 rule:

1. The number of subintervals must be divisible by 3.
2. It is obviously not accurate, there will always be an error between it and the actual integral.

Error in Simpson's 3/8 rule:

Just like other rules, it also contains error which comes out to be

$$\int_a^b f(x)dx \approx \frac{(b-a)}{8}\left[f(a) + 3f(\tfrac{2a+b}{3}) + 3f(\tfrac{a+2b}{3}) + f(b)\right] \text{ with error as } \left|\frac{(b-a)^5}{6480} f^4(z)\right|$$

SOLVED EXAMPLE

- Priyanshu Singh (19/78082)

Question 1 : Find Solution of an equation $\int_0^6 \frac{dx}{1+x^2}$ using Simpson's $\frac{3}{8}$ rule .

Sol ---> Formula

$$\int_a^b f(x)dx \approx 3\,h/8\,[(y_0 + y_n) + 2(y_3 + y_6 + \cdots) + 3(y_1 + y_2 + y_4 + \cdots)]$$

Where h = (b-a)/n and n must be the "multiple of 3"

Taking n=6

h = (6-0)/6 = 1

- At x_0=0 , y_0=1
- x_1= 1 , y_1 = 0.5
- x_2= 2 , y_2= 0.2
- x_3= 3 , y_3=0.1
- $x_4 = 4$, $y_4 = 0.588$
- $x_5 = 5$, $y_5 = 0.0385$
- $x_6 = 6$, $y_6 = 0.027$

Putting the value in formula:

$$\int_0^6 f(x)dx \approx 3h/8[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

$$\int_0^6 \frac{dx}{1+x^2} \approx 3/8[\ (1+0.027)\ + 2(0.1) + 3(0.5 + 0.2 + 0.588 + 0.0385)\ ]$$

$$\approx 1.3571$$

We are taking the decimal value up to 5 decimal places.

To check the error

$$\int_0^6 \frac{dx}{1+x^2} = \tan^{-1} 6 - \tan^{-1} 0 = 1.4056$$

Error $\approx$ 1.4056 -1.3571

$\approx$ 0.0485

Question 2: Find Solution of an equation $\int_0^{\pi/2} e^{sinx}\ dx$ using Simpson's $\frac{3}{8}$ rule .

Sol ---> Formula

$$\int_a^b f(x)dx \approx 3\,h/8\,[(y_0 + y_n) + 2(y_3 + \ y_6 + \cdots) + 3(y_1 + y_2 + y_4 + \cdots)]$$

Where h = (b-a)/n and n must be the "multiple of 3"

Taking n = 3

h= $(\frac{\pi}{2} - 0)/3 = \pi/6$

- At $x_0 = 0$, $y_0 = 1$
- $x_1 = \pi/6$, $y_1 = 1.648$
- $x_2 = \pi/3$, $y_2 = 2.362$
- $x_3 = \pi/2$, $y_3 = 2.718$

Putting the value in formula

$$\int_0^{\pi/2} f(x)dx \approx 3h/8[(y_0 + y_3) + 3(y_1 + y_2)]$$

$$\int_0^{\pi/2} e^{sinx}\ dx \approx \frac{3(\pi/6)}{8}[(1 + 2.718) + 3(1.64 + 2.36)]$$

$$\approx 3.08622$$

We are taking the decimal value up to 5 decimal places.

To check the error

$$\int_0^{\pi/2} e^{sinx}\ dx = 3.10438$$

Error ≈ 3.10438 - 3.08622

≈ 0.01816

PRACTICAL - 1

- Joshita Sood (19/78086)

Refer Maxima

PRACTICAL - 2

- Harsh Yadav (19/78092)

Refer Maxima

THE END

GROUP 5