2.2 Parallel-Plate Capacitor

Part IB Electromagnetism provides background but is not necessary: the project is self-contained.

1 Introduction

According to elementary electromagnetic theory, a parallel-plate capacitor [without dielectric spacer] has plate charge q proportional to potential difference V, the constant of proportionality –'capacitance' – being given by $q/V = \epsilon_0 A/d$ where A is the area of the plates, d their separation, and ϵ_0 the 'permittivity of free space' (a constant). This assertion is usually justified by arguing that the electric field $\mathbf{E}$ is approximately uniform between the conducting plates and much weaker elsewhere, except near the plate edges whose effects may be ignored.

In this project it is assumed that the plates are rectangles of length l_z, width l_x and negligible thickness, occupying $-l_x/2 < x < l_x/2$, $y = \pm d/2$, $-l_z/2 < z < l_z/2$. If in addition $l_z \gg l_x$, then away from the ends $z = \pm l_z/2$ the electric field should be approximately two-dimensional: $\mathbf{E} \approx -\nabla\phi(x, y)$ where the potential ϕ satisfies the two-dimensional Laplace equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{1}$$

with boundary conditions

$$\begin{array}{ll} \phi(x,y) = +V/2 & \text{on the top plate: } y = d/2,\ -l_x/2 \leqslant x \leqslant l_x/2 \\ \phi(x,y) = -V/2 & \text{on the bottom plate: } y = -d/2,\ -l_x/2 \leqslant x \leqslant l_x/2 \end{array} \tag{2}$$

and

$$\phi(x,y) \to 0 \text{ as } |x| \to \infty \text{ and/or } |y| \to \infty \tag{3}$$

Then from Gauss's Law, the charge per unit length in z on the top $(+)$ or bottom $(-)$ plate is

$$\hat{q}_\pm \equiv \epsilon_0 \oint_{\gamma_\pm} \mathbf{E}\,.\,\mathbf{n}\,\mathrm{d}s = \epsilon_0 \oint_{\gamma_\pm} -\nabla\phi\,.\,(\mathrm{d}y, -\mathrm{d}x) = -\epsilon_0 \oint_{\gamma_\pm} \frac{\partial\phi}{\partial x}\,\mathrm{d}y - \frac{\partial\phi}{\partial y}\,\mathrm{d}x$$

where γ_+ (γ_-) is any simple closed contour in the x-y-plane encircling the top (bottom) plate (and not the other); note that $\phi(x, y)$ will be [even in x and] odd in y, implying that $\hat{q}_- = -\hat{q}_+$. Solutions of this two-dimensional problem give some idea of the validity of the approximation that the electric field is uniform between the plates [$\phi \approx Vy/d$, $\mathbf{E} \approx (0, -V/d, 0)$] and ignorable elsewhere, which leads to

$$\hat{q}_\pm \approx \pm\frac{\epsilon_0\, l_x}{d}\, V\ . \tag{4}$$

In terms of dimensionless variables defined by

$$x = \tfrac{1}{2}dX\ , \quad y = \tfrac{1}{2}dY\ , \quad \phi(x,y) = V\Phi(X,Y)\ , \quad \hat{q}_\pm = \epsilon_0 V \hat{Q}_\pm$$

the problem becomes

$$\frac{\partial^2\Phi}{\partial X^2} + \frac{\partial^2\Phi}{\partial Y^2} = 0\ , \tag{5}$$

$$\Phi(X,1) = +\tfrac{1}{2}\ , \quad \Phi(X,-1) = -\tfrac{1}{2} \quad \text{for } -L \leqslant X \leqslant L, \tag{6}$$

$$\Phi(X,Y) \to 0 \text{ as } |X| \to \infty \text{ and/or } |Y| \to \infty \tag{7}$$

where $L = l_x/d$ is the width-to-separation ratio of the plates. The non-dimensionalised charge per unit length on the top $(+)$ or bottom $(-)$ plate is given by

$$\hat{Q}_\pm = -\oint_{\Gamma_\pm} \frac{\partial\Phi}{\partial X}\,\mathrm{d}Y - \frac{\partial\Phi}{\partial Y}\,\mathrm{d}X \quad [\, = \pm\hat{Q}_+\,] \tag{8}$$

for any simple closed contour Γ_+ (Γ_-) in the X-Y-plane encircling only the top (bottom) plate.

2 Analytic Solution

The problem (5)–(7) can be solved analytically by conformal mapping, but only in awkward implicit form involving elliptic integrals: see for example reference [1]*. However, if L is 'large' it may be expected that (*i*) far from the plates, the solution can be approximated by that for zero plate separation, i.e. with (6) replaced by

$$\lim_{Y\downarrow 0}\Phi(X,Y)=+\tfrac{1}{2}\ ,\quad \lim_{Y\uparrow 0}\Phi(X,Y)=-\tfrac{1}{2}\quad \text{for } -L\leqslant X\leqslant L, \tag{9}$$

which is exactly

$$\begin{aligned}\Phi &= -(2\pi)^{-1}\,\mathrm{Re}\,[\mathrm{i}\ln(X-L+\mathrm{i}Y)-\mathrm{i}\ln(X+L+\mathrm{i}Y)]\\ &= (2\pi)^{-1}\left[\cos^{-1}\left(\frac{X-L}{\sqrt{(X-L)^2+Y^2}}\right)-\cos^{-1}\left(\frac{X+L}{\sqrt{(X+L)^2+Y^2}}\right)\right]\mathrm{sgn}\,Y\end{aligned} \tag{10}$$

and has far-field 'dipole' behaviour†

$$\Phi\sim\frac{L}{\pi}\frac{Y}{X^2+Y^2}\quad \text{as } |X|\to\infty \text{ and/or } |Y|\to\infty\ ; \tag{11}$$

(*ii*) the 'fringing field' near the plate ends, say those at $x=+L$, can be approximated by that for semi-infinite plates, with (6)–(7) replaced by

$$\Phi(X,1)=+\tfrac{1}{2}\ ,\quad \Phi(X,-1)=-\tfrac{1}{2}\quad \text{for } -\infty\leqslant X\leqslant L, \tag{12}$$

$$\Phi(X,Y)\to 0 \text{ as } X\to\infty \text{ and/or } |Y|\to\infty\ . \tag{13}$$

For this, conformal mapping yields the simple, albeit implicit, analytic solution

$$Z\equiv X+\mathrm{i}Y=L-2\mathrm{i}W+\pi^{-1}\left(1+\mathrm{e}^{-2\pi\mathrm{i}W}\right)\ ,\quad W=-\Phi+\mathrm{i}\Psi \tag{14}$$

where $\Psi(X,Y)$ is a 'harmonic conjugate' of $(-\Phi)$ satisfying the Cauchy-Riemann equations $(-\Phi)_X=\Psi_Y$, $(-\Phi)_Y=-\Psi_X$; physically it is a 'flux function' ‡, lines of constant Ψ being everywhere tangent to the (non-dimensionalised) electric field $\nabla(-\Phi)\equiv\nabla\times(0,0,\Psi)$.

> **Question 1** For the solution (14) for semi-infinite plates, plot a few contours of constant $\Phi\in[-1/2,1/2]$ (equipotentials) and of constant $\Psi\in(-\infty,\infty)$ (electric field lines) in the $(X–L,Y)$ plane.
>
> Using
> $$-\frac{\partial\Phi}{\partial X}+\mathrm{i}\frac{\partial\Phi}{\partial Y}=\frac{\mathrm{d}W}{\mathrm{d}Z}=\frac{1}{\mathrm{d}Z/\mathrm{d}W} \tag{15}$$
> in conjunction with (14), plot the [Y-component – the only non-zero component – of] electric field $-\partial\Phi/\partial Y$ against $X–L$ on the midplane $Y=0$ ($\Phi=0$) and on both lower ($\Psi<0$) and upper ($\Psi>0$) surfaces of the top plate at $Y=1$ ($\Phi=1/2$). How does $-\partial\Phi/\partial Y$ behave as $X\to-\infty$ in each case, as $X\to L$ on the plate surfaces and as $X\to\infty$ on the midplane?

*Note that this reference uses 'capacitance' to mean what in the project is capacitance *per unit length* (i.e. the ratio of charge *per unit length* to potential difference), and has $1/4\pi$ in place of ϵ_0.

†In fact, for any L the far-field can be approximated by that for two line charges of strength $\pm\hat{Q}_+$ per unit length at $(X,Y)=\pm(0,1)$, namely

$$\Phi=-\frac{\hat{Q}_+}{4\pi}\ln\left[\frac{X^2+(Y-1)^2}{X^2+(Y+1)^2}\right]\ ,\ \text{which}\ \sim\frac{\hat{Q}_+}{\pi}\frac{Y}{X^2+Y^2}\quad \text{as } |X|\to\infty \text{ and/or } |Y|\to\infty\ ,$$

consistent with (11) insofar as the approximation (4) is valid.

‡It is the analogue of a 'streamfunction' in fluid dynamics, where a two-dimensional incompressible velocity field is customarily represented as $\nabla\times(0,0,\Psi)$, and if irrotational as $\nabla\Phi$ for a 'velocity potential' Φ (note the different sign convention).

3 Numerical Method

The problem (5)–(7) is to be restricted to a finite domain $|X| < D_X$, $|Y| < D_Y$, with the final boundary condition (7) replaced by $\Phi(X,Y) = 0$ on the boundaries of the domain, i.e. for $X = \pm D_X$ or $Y = \pm D_Y$, and solved numerically using the following *finite-difference method.*

Consider the mesh of grid points $(X,Y) = (mh, nh)$ for integers $m \in [-N_X, N_X]$ and $n \in [-N_Y, N_Y]$, where the mesh spacing $h = 1/N_h$ for some integer N_h, and $D_X = hN_X$, $D_Y = hN_Y$. The discretised approximation to $\Phi\,(mh, nh)$ will be denoted by $\Phi_{m,n}$. [If using MATLAB, this might be stored in the (N_X+m+1, N_Y+n+1) element of an array, or in the $(m+1, n)$ element if solving only in the first quadrant - see later.]

Question 2 Show that for 'small' h

$$\frac{1}{h^2}\left[\Phi_{i-1,j} + \Phi_{i+1,j} + \Phi_{i,j-1} + \Phi_{i,j+1} - 4\Phi_{i,j}\right] \;=\; 0 \tag{16}$$

is a discretised approximation to the Laplace equation (5).

Equation (16), together with the boundary conditions

$$\Phi_{i,N_h} = \tfrac{1}{2}\ , \quad \Phi_{i,-N_h} = -\tfrac{1}{2} \quad \text{when } \; -L/h \leqslant i \leqslant L/h$$

and

$$\Phi_{i,j} = 0 \quad \text{when } i = \pm N_X \text{ and/or } j = \pm N_Y,$$

can be solved using an *iteration scheme*: an initial guess $\Phi^{(0)}_{i,j}$ is made for $\Phi_{i,j}$ (consistent with the boundary conditions, of course) and a sequence of approximations $\left\{\Phi^{(k)}_{i,j}\ ,\ k \geqslant 0\right\}$ is computed iteratively, the $\Phi^{(k+1)}_{i,j}$ being determined (collectively) from the $\Phi^{(k)}_{i,j}$. One such scheme, known as *Jacobi iteration*, is

$$\Phi^{(k+1)}_{i,j} \;=\; \frac{1}{4}\left[\Phi^{(k)}_{i-1,j} + \Phi^{(k)}_{i+1,j} + \Phi^{(k)}_{i,j-1} + \Phi^{(k)}_{i,j+1}\right] \tag{17}$$

A better method is *Gauss-Seidel iteration* which replaces old values with new as soon as they become available. Assuming that $\Phi_{i,j}$ is updated in order of increasing i and j, the scheme becomes

$$\Phi^{(k+1)}_{i,j} \;=\; \frac{1}{4}\left[\Phi^{(k+1)}_{i-1,j} + \Phi^{(k)}_{i+1,j} + \Phi^{(k+1)}_{i,j-1} + \Phi^{(k)}_{i,j+1}\right]. \tag{18}$$

Further improvement may be obtained via *Successive Overrelaxation* (SOR):

$$\Phi^{(k+1)}_{i,j} = (1-\omega)\Phi^{(k)}_{i,j} \;+\; \frac{\omega}{4}\left[\Phi^{(k+1)}_{i-1,j} + \Phi^{(k)}_{i+1,j} + \Phi^{(k+1)}_{i,j-1} + \Phi^{(k)}_{i,j+1}\right]. \tag{19}$$

where $\omega \in (0,2)$ is the 'relaxation parameter' (less than 2 for stability reasons). Typically $\omega > 1$ (*over*relaxation) will improve convergence significantly compared to (18). More details can be found in introductory numerical methods texts, such as references [2]–[5].

In fact, since for this configuration $\Phi(X,Y)$ is even in X and odd in Y, it is only necessary to solve (19) in the first quadrant, i.e. for $0 \leqslant i < N_X$ and $0 < j < N_Y$, with $\Phi_{i,0} = 0$ and $\Phi_{-1,j}$ equated to $\Phi_{1,j}$.

Programming Task: Write a program to implement the SOR method (19). As a criterion for when to stop iterations, define the residual

$$r_k = \frac{1}{N_{pts}} \sum_i \sum_j \left| \Phi_{i,j}^{(k)} - \Phi_{i,j}^{(k-1)} \right| \tag{20}$$

where the sum is over all interior points, and N_{pts} is the number of them. The k-th iteration $\Phi^{(k)}$ is deemed sufficiently accurate if r_k is less than a given tolerance, e.g. $10^{-2}, 10^{-4}, \ldots$.

Your program should accommodate a more general configuration with the plates at $Y = \pm Mh$ for an arbitrary integer $M \in [0, N_Y)$, so that as a preliminary test, you can set $M = 0$ and compare the numerical results with the analytic solution (10) for plates with zero separation. Since this is an exact solution for any L, the comparison can be done with smallish values of L, D_X and D_Y if $\Phi_{i,j}$ is set equal to (10), rather than to zero, on the outer boundaries. Do so with $L = 1$, $D_X = D_Y = 2$ and successively smaller values of h, recording the value of Φ at $(X, Y) = (0, 1)$ to demonstrate convergence to the exact solution (10) as $h \to 0$. [What is the order of accuracy?]

Before beginning coding, please read below to have in mind what else your program(s) will be expected to do.

The remainder of the project is concerned exclusively with the problem (5)–(7), i.e. with plates at $Y = \pm 1$.

Question 3 Run your program with each of the following five sets of inputs:

set	L	D_x	D_y	h
A	1	2	2	0.5
B	2	4	4	0.5
C	2	4	4	0.25
D	2	4	4	0.125
E	2	8	8	0.5

First confirm that for set A, correct to three decimal places,

$$\Phi_{0,1} = \Phi_{0,3} = 0.244 \ , \quad \Phi_{1,1} = \Phi_{1,3} = 0.238 \ , \quad \Phi_{2,1} = \Phi_{2,3} = 0.208 \ ,$$

$$\Phi_{3,1} = \Phi_{3,3} = 0.095 \ , \quad \Phi_{3,2} = 0.173 \ .$$

Then for each of the sets B, C, D and E, (i) state the values of $\Phi_{i,j}$ at $(X, Y) = (0, 2)$ and at $(X, Y) = (2, 2)$ correct to at least three decimal places, and (ii) plot

- the approximation $\Phi_{0,j}$, $0 \leqslant j \leqslant N_Y$ to $\Phi(0, Y = jh)$ against Y $(\geqslant 0)$,
- the approximation to $\Phi(2, Y)$ against Y $(\geqslant 0)$,
- an approximation to the [Y-component – the only non-zero component –of] electric field $-\partial\Phi/\partial Y$ against X $(\geqslant 0)$ on the midplane $Y = 0$ and on both surfaces of the top plate at $Y = 1$, e.g. using

$$\frac{\partial \phi}{\partial Y}(X, Y) \approx \frac{\Phi(X, Y + h) - \Phi(X, Y - h)}{2h}$$

or

$$\frac{\partial \phi}{\partial Y}(X, Y) \approx \frac{-\Phi(X, Y + 2h) + 4\Phi(X, Y + h) - 3\Phi(X, Y)}{2h} \ .$$

The corresponding plots for sets B, C and D, and those for sets B and E, should be superposed to display the variation with grid spacing and with domain size. How have you ensured the required three-decimal-place accuracy?

Question 4 Try different values of $\omega \in (1, 2)$. Investigate how the optimal ω, for which convergence to the stopping condition takes the fewest iterations, varies with the size and shape of the grid. (It may be instructive to carry out computations with other values of h.) Does it also depend on the tolerance, and/or the initial guess $\Phi^{(0)}_{i,j}$?

Question 5 Explain carefully why (8) can be discretised as

$$\begin{aligned}\hat{Q}_{\pm} \approx & \sum_{i=M_L}^{M_R} \left[\Phi_{i,N_B} - \Phi_{i,N_B-1} + \Phi_{i,N_T} - \Phi_{i,N_T+1}\right] \\ & + \sum_{j=N_B}^{N_T} \left[\Phi_{M_L,j} - \Phi_{M_L-1,j} + \Phi_{M_R,j} - \Phi_{M_R+1,j}\right] \end{aligned} \qquad (21)$$

for appropriate values of M_L, M_R, N_B and N_T.

Programming Task: Write code to compute an numerical approximation to $\hat{Q}_+$ using (21), given a solution $\Phi_{i,j}$ from your SOR program. Use your judgement about whether to have a separate program which does this or just add this functionality to code already written. It may be helpful to maintain some flexibility so that you can also compute the charge elsewhere.

Question 6 Compute a numerical approximation to $\hat{Q}_+$ for sets B, C and D in Question 3, stating clearly the values of M_L, M_R, N_B and N_T used. What do these computations indicate about the 'discretisation errors', i.e. the discrepancies between what is obtained using a discrete lattice of points and what would be measured in the continuum in a laboratory experiment? [For example, can you identify the order of accuracy as $h \to 0$?] You may wish to try smaller values of h, and/or extrapolation to $h = 0$. Keep in mind that since precision of 2-3 significant figures should be sufficient, you may be able to increase the tolerance in order to speed up the computations, especially as you experiment. Mention any checks done to ensure the correctness of the computations.

Question 7 What is the effect of using the boundary condition $\Phi = 0$ on the sides of the finite domain compared to the original boundary condition that Φ vanishes at infinity? You should investigate this by observing what happens as the boundaries at $X = \pm D_X$ and/or those at $Y = \pm D_Y$ are moved closer to the plates. It would be appropriate to include a few more plots to illustrate your answer.

Question 8 Investigate numerically the dependence of $\hat{Q}_+$, the (top) plate charge per unit length non-dimensionalised by $\epsilon_0 V$ (or the capacitance per unit length non-dimensionalised by ϵ_0), on the plate-width-to-separation ratio L, displaying results in table and/or graphical form. You should use values of D_X and D_Y which give, subject to reasonable computing time, as good an approximation as possible to an infinite domain, while still having h sufficiently small for acceptable accuracy [or maybe use more than one h and extrapolate to $h = 0$]. Assess the extent to which this has been achieved, e.g.

by observing the effect of varying these parameters.§ You might also consider using (11), or some other appropriate condition, instead of $\Phi = 0$ on the outer boundaries.

How do your results compare with the approximation (4)? Comment on, and try to explain, deviations which cannot be attributed to discrete-lattice or finite-domain effects.

References

1. H. B. Palmer, The Capacitance of a Parallel-Plate Capacitor by the Schwartz-Christoffel¶ Transformation, *Transactions of the American Institute of Electrical Engineers*, vol.56(3), pp.363-366 (1937). See http://dx.doi.org/10.1109/T-AIEE.1937.5057547.

2. S. D. Conte and C. de Boor, *Elementary Numerical Analysis*, McGraw-Hill, 1980.

3. C.-E. Fröberg, *Numerical Mathematics*, Benjamin-Cummings, 1985.

4. A. Iserles, *A First Course in the Numerical Analysis of Differential Equations*, CUP, 1996.

5. W. H. Press *et al.*, *Numerical Recipes: The Art of Scientific Computing*, CUP.

§A titbit of information: according to reference [1], for $L = 2$ the exact analytic solution (for an infinite domain) gives $\tilde{Q}_+ = 3.266$ [actually 3.263] to four figures.

¶Schwarz-Christoffel actually, after the German mathematician Karl Hermann Amandus Schwarz (1843-1921) – not to be confused with the French mathematician Laurent-Moïse Schwartz (1915-2002).

