PHS LAUNCH

Making Friends With Mathematical Notation

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Get the slides

- **Mathematical notation** uses symbols to represent mathematical objects and ideas.
- Becoming familiar with mathematical notation helps you to read and understand the methodological literature and to communicate efficiently and precisely.

For population health scientists working with quantitative data, when mathematical notation works well:

- it helps us to specify in precise terms how we use the data at hand to learn about population quantities and relationships
- it explicitly encodes the assumptions we need to draw inferences
- it illuminates relationships between the different kinds of quantities we observe
- it makes it possible to communicate our ideas and methods to our colleagues in a transparent way
- it makes it possible for others to replicate our work

When mathematical notation does not work well:

- it obscures
  relationships and creates confusion
- it assumes conventions and usage
  that are not explicitly stated, and
  therefore not universally intelligible
- it makes it difficult
  for others to replicate our work
- it makes us frustrated and makes us
  question our understanding

Unfortunately, there is no such thing as a “perfect” notational system.

- notational conventions differ across disciplines
- often we find ourselves having to amend our notation in order to highlight particular aspects of the specific problem we are working on
- it is always important to be able to translate back and forth between our conceptual understanding in words and our formal notation in mathematical symbols

Today and Friday, we'll be reviewing some of the key concepts you'll need for PHS2000. In the course of this, you'll see many examples of mathematical notation, including many of the conventions that are used in biostatistics, epidemiology, econometrics, and the quantitative social sciences.

When you see notation that is confusing to you, please ask us about it!

Vectors, Scalars, & Matrices

Objectives

- See how vectors, scalars, and matrices can be used to refer to sets of numerical values
- Learn the notational conventions for vectors, scalars, and matrices.

Conventions: Vectors, Scalars, & Matrices

As quantitative population health scientists, we find ourselves spending a lot of time thinking about

- numbers (the “quantitative“ part)
- not just single numbers but sets of numbers (the “population” part)
- different sets of numbers referring to different variables of interest

Conventions: Vectors, Scalars, & Matrices

We need an efficient way of referring to these sets of numbers, e.g.

- all of our study participants' ages, instead of just one subject's age
- all of the variables measured on a single subject (e.g. age, race/ethnicity, income, cholesterol, body mass index)
- all of the variables measured on all of the subjects in our study

This becomes very important when thinking about how we can manipulate these sets of numbers to learn something about the population from which they come.

Conventions: Vectors, Scalars, & Matrices

- A **vector** is a structured set of inputs (e.g., numbers), arranged in a list.
  - e.g. $(1, 2, 4, 3, 5)$ (row vector)
  - e.g. $\begin{pmatrix} 2 \\ 5 \\ 13 \\ 9 \\ 4 \end{pmatrix}$ (column vector)
- By convention, vectors are often represented by lower case Roman letters in **boldface**, e.g.

$$\mathbf{x} = (1, 2, 4, 3, 5)$$

- A **scalar** is a vector with just one element, usually represented by a non-boldface lower case Roman letter, e.g. $x = 7$.

Conventions: Vectors, Scalars, & Matrices

- A **matrix** is a structured set of *vectors* all with the same length (number of elements)
- e.g.

$$\mathbf{X} = \begin{bmatrix} 5 & 3 & 6 & 9 \\ 4 & 12 & 3 & 13 \\ 8 & 2 & 0 & 19 \end{bmatrix}$$

- A matrix is usually represented by an upper case Roman letter in **boldface**.

Don't worry! We will review vectors and matrices in more detail when they come up in class.

Conventions: Vectors, Scalars, & Matrices

Note: Sometimes when working with a lot of vectors and matrices, some authors will suppress the boldface notation for simplicity. Usually this will be obvious from how vectors and matrices are defined in their notation.

Indexing

Objectives

- See how indexing can help us refer to specific elements within a vector or a matrix.
- Appreciate how multiple subscripts can refer to elements of a matrix.
- Note the conventions for referring to the number of subjects in a dataset (n) and the number of variables in a regression model (p).
- See how indexing works as part of the summation and product operators.

Conventions: Indexing

When working with population data, we will often observe multiple values of the same variable, e.g. over subjects, over time, etc. We will often use indexing notation to represent multiple values of the same variable, e.g.

- X is height in cm, which we observe in a sample of 300 women. We can represent the value for person i as x_i, for $i = 1, \ldots, 300$.
- All of the observed x's in the study form a vector $\mathbf{x} = (x_1, \ldots, x_{300})$.
- x_{241} refers to the 241st element in the vector $\mathbf{x}$.
- x_i refers generically to the ith element of $\mathbf{x}$.

Conventions: Indexing

- We ask 100 subjects in our study to record the number of hours they sleep each night for a month. Thus, each of the 100 subjects in our study has 31 records of the number of hours they slept each night, which we could index as y_{it} where i indexes subject ($i = 1, \ldots, 100$) and t indexes night ($t = 1, \ldots, 31$).
- The observations for each individual i are a vector of length 31, $\mathbf{y}_i = (y_{i1}, \ldots, y_{i,31})$
- All of the observations in the study form a matrix Y of dimension 100×31,

$$\mathbf{Y} = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1,31} \\ y_{21} & y_{22} & \cdots & y_{2,31} \\ \vdots & \vdots & \ddots & \vdots \\ y_{100,1} & y_{100,2} & \cdots & y_{100,31} \end{bmatrix}$$

Conventions: Indexing

When working with a dataset,

- often we will use n to represent the number of subjects in our sample, so we might use i to index subjects with $i = 1, \ldots, n$.
- often we will use p to represent the number of variables included in a model, e.g. the model includes variables $X_1, X_2, \ldots, X_p$.

Obs	X_1	X_2	...	X_p
1	x_{11}	x_{12}	...	x_{1p}
2	x_{21}	x_{22}	...	x_{2p}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	x_{n1}	x_{n2}	...	x_{np}

Conventions: Summation & Products

We use indexing when using the summation operator:

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

means sum all of the values of x_i from $i = 1$ to $i = n$.

We can also sum over multiple dimensions, e.g.

$$\sum_{i=1}^{n} \sum_{j=1}^{p} x_{ij}$$

means sum over all the values of x_{ij} (from $i = 1$ to $i = n$ and $j = 1$ to $j = p$).

Conventions: Summation & Products

Similarly,

$$\prod_{i=1}^{n} x_i = x_1 \times x_2 \times \cdots \times x_n$$

means take the product of all of the values of x_i from $i = 1$ to $i = n$.

Conventions: Summation & Products

Sometimes, when there are multiple dimensions, to keep the number of letters being used under control, people will represent the maximum index with a capital letter, e.g.

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} X_{ijk}$$

for $i = 1, \ldots, I$, $j = 1, \ldots J$, and $k = 1, \ldots, K$.

Note that in this case, the capital letter is **not** denoting a random variable!

Conventions: Summation & Products

An alternative to specifying summation from a starting point to a stopping point is to use set notation, e.g.

$$\sum_{i \in S} x_i$$

means sum all values of x_i where i is in the set S.

See the Supplemental Slides for additional information about set notation

Conventions: Summation & Products

Sometimes, you will see the notation $\sum_i x_i$ when it's clear that summation is over all possible values of i.

Note that this enables us to write the double summation more compactly as

$$\sum_{i,\,j} x_{ij}$$

Another example (from probability, the definition of expected value):

$$\mathbb{E}[X] = \sum_{x} x\, f(x)$$

Here the subscript x on the summation symbol means "over the range of x."

On Friday we'll talk more about the definition of *expected value!*

Questions?

Probability

Objectives

- Learn basic probability notation
- Learn the conventions for random variables and their realizations.
- Note the notation for common operators for random variables (expectation, variance, etc.)

Conventions: Probability

- $P(A)$ is the probability of event A occurring. Sometimes written as $\mathbb{P}(A)$ or $Pr(A)$.
- $P(A \cap B)$ is the probability that events A and B both occur.
- $P(A \cup B)$ is the probability of either event A **or** event B occurring (where "or" means one or the other or both)
- $P(A|B)$ is the conditional probability of event A occurring **given** that B has occurred.

Conventions: Probability

- **Random variables** are usually written with upper case roman letters: X, Y, etc.
- Particular **realizations** of a random variable are written in corresponding lower case letters. e.g. $P(X = x)$ is the probability that the random variable X is equal to the specific value x.
- We will often assume that a random variable X follows a particular **probability distribution**, e.g.
  - $X \sim Normal(\mu, \sigma^2)$
  - $X \sim Bernoulli(\pi)$
  - $X \sim Poisson(\lambda)$
- Note that each of these distributions has one or more **parameters** associated with it.

Conventions: Probability

- Some common operators:
  - $E(X)$ or $\mathbb{E}(X)$ is the **expected value** of X.
  - $\text{Var}(X)$ is the **variance** of X.
  - $sd(X)$ is the **standard deviation** of X.
  - $\text{Cov}(X, Y)$ is the **covariance** of X and Y.
- X is independent of Y is often written $X \perp\!\!\!\perp Y$ or $X \perp Y$.

Questions?

Notation in regression models

Objectives

- Remind ourselves of the conventional (generic) notation for linear regression model
- Appreciate how we might amend notation to highlight particular aspects of a model.
- Appreciate why bucking convention in notation can become confusing
- A few notes on potentially confusing situations
- Note the trade-offs between using letters vs. names for variables.

Conventions: Regression models

You are probably familiar with seeing β's in a regression model, e.g.

$$Y_i = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p + \epsilon_i$$

These β's are unknown population parameters which we will estimate by fitting a model, yielding a set of estimates $\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_p$.

We'll talk more about hat notation next week!

Conventions: Regression models

Imagine that we are writing the methods section of a paper exploring social and environmental predictors of low birthweight. In a sample of births, we have collected data on

- low birthweight (Y_i)
- 4 variables on maternal socioeconomic position (e.g. X_1=education, X_2=household income, X_3=wealth, and X_4=neighborhood poverty)
- 3 environmental variables (Z_1=PM2.5, Z_2=black carbon, and Z_3 =water quality)
- 3 demographic variables (W_1=maternal age, W_2=marital status, W_3=maternal race/ethnicity)

After extensive analysis, we decide to present the results of a regression model that includes all 10 of these variables.

Conventions: Regression models

We could write our model generically, using $p = 4 + 3 + 3 = 10$ to index all of the variables we included:

$$Y_i = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p + \epsilon_i$$

Or, for more clarity and to draw the reader's attention to the different sets of socioeconomic, environmental, and demographic variables we are including in the model, we could write

$$Y_i = \alpha + \sum_{j=1}^{4} \beta_j x_{ij} + \sum_{k=1}^{3} \gamma_k z_{ik} + \sum_{l=1}^{3} \lambda_l w_{il} + \epsilon_i$$

Notice how we now have different Greek letters representing different parameters of the model: α is the overall intercept, β's represent the effects of the socioeconomic variables, γ's represent the effects of the environmental variables, and λ's represent the effects of the demographic variables.

Conventions: Regression models

Consider the pros and cons of how we amended the conventional notation here:

Pros:

- Our notation highlights the conceptual distinctions between socioeconomic, environmental, and demographic variables by using different letters to represent the variables and different Greek letters to represent the regression parameters.
- The notation is actually more specific than the $\beta_0, \ldots, \beta_p$ notation (which is highly generic).

Cons:

- The reader has to mentally orient to three different variable types (X, Z, and W), three different indexes (j, k, and l), and five different Greek letters (α, β, γ, λ, and ϵ)!

Conventions: Regression models

Consider what we **didn't** do here, e.g.

$$N_z = \sigma_0 + \sigma_1 d_1 + \sigma_2 d_2 + \cdots + \sigma_r d_q + \zeta_z$$

where N_z is the birthweight for subject $z = 1, \ldots, Z$, $d_1, \ldots, d_q$ are the covariates, $\sigma_0, \ldots, \sigma_q$ are the regression coefficients, and ζ_z are the error terms.

There is nothing stopping us from defining and using this notation to describe our model, but the notation here is so removed from convention for regression models that it would be virtually unintelligible to most readers!

Conventions: Regression models

In general:

- Y used to represent the outcome
- X used to represent predictors
- Sometimes X used to represent exposures of particular interest and Z used to represent covariates ('control' variables)
- In time to event data (survival analysis), sometimes T is used to represent event times
- Usually β used to represent regression coefficients (occasionally we will also see α, γ, δ, λ, τ...)

A few notes

Things to watch out for:

- π is often used in biostatistics to represent a probability, (e.g. $Y \sim Bernoulli(\pi)$ or in a logistic regression model, $\text{logit}(\pi) = \beta_0 + \beta_1 x_1$)
- You might also see notation like $\pi(x)$, where $\pi(\cdot)$ is being used to denote a function that returns the probability of an outcome for a given value of x, i.e.

$$\pi(x) = \mathbb{P}(Y = 1 | X = x)$$

- **BUT** in the normal probability density function,

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(y-\mu)^2}{2\sigma^2} \right\}$$

Here, π is the mathematical constant, $\pi \approx 3.14159$!

A few notes

- We saw before that $\sum$ is used as the summation operator, e.g. $\sum_{i=1}^{n} x_i$, and it looks like a capital Σ. But note that $\boldsymbol{\Sigma}$ is sometimes used to represent a variance-covariance matrix, e.g.

$$\mathbf{X} \sim MVN(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

- Don't worry about understanding the details of these usages right now; the message is that sometimes the same symbol is used to represent different things, depending on the context. We'll try to point out when this happens, in order to head off any confusion!

Questions?

Notation survival tips

Notation survival tips

- We realize that learning new notation can feel daunting.
- We hope that you'll come to see that there is elegance and even beauty in being able to represent quantitative concepts in rigorous mathematical symbology
- Notation is always evolving: we often have to revise our notation when we realize there is a new feature or nuance that needs to be communicated clearly

Notation survival tips

- When learning new notation, try to take notes on all the different parts of the notation so that you understand what each symbol means.
- When in doubt, look for the part of the lecture slides (or the textbook or the journal article) where the notation is defined. (Most good scientific writing will include this *somewhere*, even if it's just in an appendix).
- Familiarize yourself with the most common conventions used in your field(s).
- When in doubt, ASK!

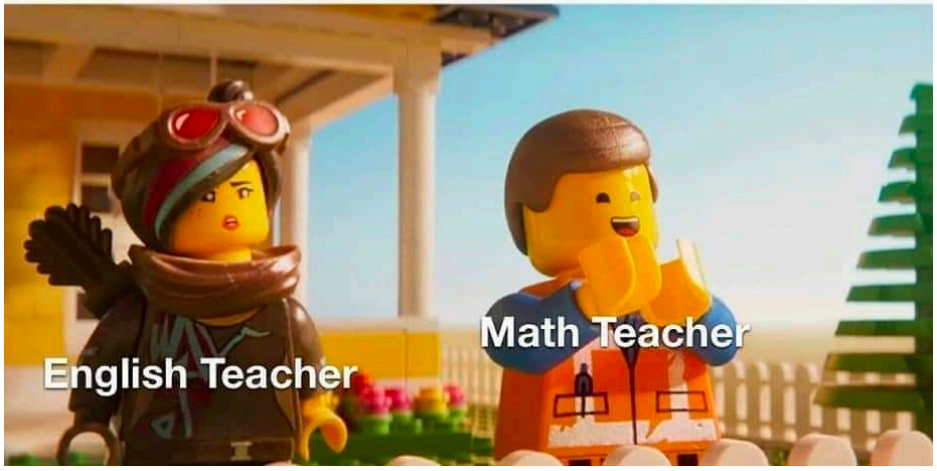
Different ways of knowing

Mathematical representation is one important tool that we use for organizing our knowledge, but it is not the only way of knowing.

Past tense	Past Participle
Grew	Grown
Flew	?

$$\frac{\text{grew}}{\text{grown}} = \frac{\text{flew}}{x}$$

$$x = \frac{\text{flew} \cdot \text{grown}}{\text{grew}} = \text{flown}$$



Writing math notation

Writing math notation

In this course, you will often have to write out math when completing problem sets or take home exams or even when taking notes. What are some options for streamlining your workflow in writing math?

Writing math notation

- You always have the option of writing out math by hand, taking a photo with your phone, and submitting the photo along with your problem set or take home exam.
    - **Pros**: easy to do
    - **Cons**: tedious to have to take a photo every time you need to show math. Image files can be large. Doesn't look elegant.

Writing math notation

- Another option is to use Microsoft Equation Editor in MS Word.
  - **Pros**: most people are already familiar with MS Word and Equation Editor.
  - **Cons**: very “fiddly”; takes a long time to typeset math in a WYSIWYG environment.

Writing math notation

- Another option is to use LaTeX.
  - LaTeX(pronounced *LAH-* or *LAY-tek*) is a syntax language that is used to typeset documents. It's great for creating reports and presentations that look professional and is particularly useful when typesetting mathematical expressions. LaTeX is also great for reproducibility, since there is a "script" file that documents how the output file (generally a .pdf) was created and styled.
  - LaTeXis WYSIWYM (what you see is what you mean)
  - **Pros**: very powerful: you can typeset just about anything. Looks super elegant. LaTeX is the preferred method of typesetting used in any math-related field (e.g. biostatistics), so if you will be working in a related space it's probably worth it to learn.
  - **Cons**: the learning curve can be a bit steep at first. Proficiency in LaTeX is a lifelong journey

Options for using LaTeX

Overleaf

- Overleaf is an online LaTeX edition tool that allows you to create LaTeX documents directly in your web browser.
- Harvard University is providing free Overleaf Professional accounts to all students, faculty, and staff. Overleaf Professional accounts provide real-time track changes, unlimited collaborators, and full document history. You can claim your Overleaf Professional account at `https://overleaf.com/edu/harvard`
- Harvard Library offers a guide to help you use Overleaf at `https://guides.library.harvard.edu/overleaf`
- Also check out this guide by Overleaf.

Options for using LaTeX

R Markdown

- R Markdown provides an authoring framework for data science. You can use a single R Markdown file to both
  - save and execute code, and
  - generate high quality reports that can be shared with an audience.
- R Markdown enhances reproducibility since both the computing code and narratives are in the same document, and results are automatically generated from the source code.
- It is worth learning how to use R Markdown to streamline your workflow.
- The best reference source is: `https://bookdown.org/yihui/rmarkdown/`
- Some additional resources are available here.

R Markdown

- To use R Markdown, you should have installed R (`https://www.r-project.org`) and the RStudio IDE (`https://www.rstudio.com`). Next, you can install the `rmarkdown` package in R:

```
# Install from CRAN
install.packages('rmarkdown')
```

- If you want to generate PDF output, you will need to install LaTeX. For R Markdown users who have not installed LaTeX before, you can install TinyTeX (`https://yihui.name/tinytex/`). TinyTeX is a lightweight, portable, cross-platform, and easy-to-maintain LaTeX distribution.

```
install.packages('tinytex')
tinytex::install_tinytex()  # install TinyTeX
```

- With the `rmarkdown` package, RStudio/Pandoc, and LaTeX you should be able to compile most R Markdown documents.

Takehome Message

- It's worth taking the time to learn how to use to typeset math, particularly if you are going to use tools like R Markdown to create a reproducible workflow.
- However, the learning curve can be steep.
- If your code doesn't work, you can often Google a solution, but also consider posting questions to the Canvas discussion board (or ask ChatGPT)
- Take your time learning to use these new tools – remember that there are always other options.
- **Ask for help when you need it.**

Questions?

Supplemental Slides

Set Notation

Objectives

- Learn about special symbols for important sets of numbers.
- See how we can enumerate the elements of a set explicitly using $\{\}$.
- See how the ellipsis $(\ldots)$ can be used to denote elements in a regular sequence.
- Review the meaning of common operators in set notation.

Conventions: Set Notation

A **set** is a collection of elements or members. Typically, we represent a set with a capital letter, e.g. A, B, C, etc. By convention, particular symbols are reserved for the most important sets of numbers:

- $\emptyset$ – empty set
- **N** or $\mathbb{N}$ – natural numbers (non-negative integers)
- **Z** or $\mathbb{Z}$ – integers
- **Q** or $\mathbb{Q}$ – rational numbers
- **R** or $\mathbb{R}$ – real numbers

Conventions: Set Notation

A set can be described by enumerating all of its elements between curly brackets, e.g.

- $\{7, 3, 15, 31\}$ is a set holding the four numbers 3, 7, 15, and 31.
- $\{a, c, b\}$ is the set containing 'a', 'b', and 'c'.

When denoting a set that contains elements from a regular sequence, we sometimes use an ellipsis, e.g.

- $\{1, 2, 3, \ldots, 100\}$ is the set of integers between 1 and 100 inclusive.

Conventions: Set Notation

A few important symbols:

Symbol	Symbol Name	Meaning
$A \cap B$	intersection	objects that belong to set A **and** set B
$A \cup B$	union	objects that belong to set A **or** set B
$A \subseteq B$	subset	A is a subset of B. Set A is included in set B.
$A \subset B$	strict subset	A is a subset of B, but A is not equal to B
$A \nsubseteq B$	not subset	set A is not a subset of set B
$a \in A$	element of	set membership
$x \notin A$	not element of	no set membership

For reference: lower case Greek letters

α	alpha	ξ	xi
β	beta	π	pi
γ	gamma	ρ	rho
δ	delta	σ	sigma
ϵ	epsilon	τ	tau
ζ	zeta	υ	upsilon
η	eta	ϕ	phi
θ	theta	χ	chi
ι	iota	ψ	psi
κ	kappa	ω	omega
λ	lambda		
μ	mu		
ν	nu		

For reference: upper case Greek letters

(In practice, Greek letters that have Latin look-alikes are not used to avoid confusion).

Γ	Gamma
Δ	Delta
Λ	Lambda
Φ	Phi
Π	Pi
Ψ	Psi
Σ	Sigma
Θ	Theta
ϒ	Upsilon
Ξ	Xi
Ω	Omega