Notes on Physics from Symmetry

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This document contains my personal notes on Jakob Schwichtenberg's Physics from Symmetry (Schwichtenberg, 2015), with a sprinkling of notes from my undergraduate physics course in quantum field theory (and, to a lesser extent, general relativity).

1. Special relativity

1.1. Definitions and postulates

In special relativity, **inertial frames of reference** are coordinate systems moving with constant velocity relative to each other. Special relativity has two basic postulates:

1. **The principal of relativity**: The laws of physics are the same in all inertial frames of reference.
2. **The invariance of the speed of light**: The velocity of light has the same value c in all inertial frames of reference.

Theorem 1.1 (Invariant of special relativity). *Consider two events A and B in an inertial observer O's frame of reference. Let the time interval measured by O between the two events be (Δt), and the three spatial intervals be (Δx), (Δy), (Δz). Then, the quantity*

$$(\Delta s)^2 \coloneqq (\Delta ct)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta y)^2 \tag{1.1}$$

is **invariant** *between all frames of reference. I.e.*

$$(\Delta s') = (\Delta s) \tag{1.2}$$

for any inertial frame of reference O'.

Theorem 1.1 follows directly from the invariance of the speed of light (consider a pair of mirrors, for two observers with relative velocity).

Definition 1.1 (Proper time). *Proper time, τ, is the time measured by an observer in the special frame of reference where the object in question is at rest. In this frame of reference,*

$$(\Delta s)^2 = (c\Delta\tau)^2. \tag{1.3}$$

In the infinitesimal limit

$$(\mathrm{d}s)^2 = (c\,\mathrm{d}\tau)^2. \tag{1.4}$$

Physically, Defn. 1.1 means that all observers agree on the time interval between events for an observer who travels with the object in question. However, different observers **do not** in general agree on the time interval between events generally: $(\Delta t) \neq (\Delta t')$ – this is called **time dilation**.

1.2. c is an upper speed limit

All observers agree on the value of $(\mathrm{d}s)^2 = (c\,\mathrm{d}\tau)^2$. Furthermore, we commonly assume that there exists a minimal proper time of $\tau = 0$ for two events if $\Delta s^2 = 0$. We can therefore write that when $\tau = 0$

$$c^2 = \frac{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2}{(\mathrm{d}t)^2} \tag{1.5}$$

between two events with an infinitesimal distance. We can equate the right-hand side with a squared velocity, and hence

$$\tau = 0 \implies c^2 = v^2 \tag{1.6}$$

so

$$(\mathrm{d}s)^2 \geq 0 \implies c^2 \geq v^2 \tag{1.7}$$

for **any** pair of events (which are causally connected, although how this follows is not immediately clear to me right now).

1.3. Minkowski Notation

Definition 1.2 (Four-vector (contravariant)). *A position four-***vector** *is defined as*

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}. \tag{1.8}$$

Definition 1.3 (Minkowski metric). *The Minkowski metric is defined as*

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1). \tag{1.9}$$

We define $\eta^{\mu\nu}$ through the relation

$$\eta^{\mu\nu}\eta_{\nu\sigma} = \delta^\mu_\sigma \tag{1.10}$$

where we have appled the **Einstein summation convention**, where a repeated Greek index implies a summation from 0 to 3 (where the zeroth index is time), and a repeated Roman index is summed from 1 to 3. Hence, for a matrix multiplication between two 3×3 matricies A and B, $(AB)_{ij} = A_{ik}B_{kj}$, and $(A^T)_{ij} = A_{ji}$.

Definition 1.4 (One-form (covariant vector)). *We define a one-form as*

$$x_\mu = \eta_{\mu\nu}x^\nu. \tag{1.11}$$

Thus,

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu. \tag{1.12}$$

Definition 1.5 (Scalar product). *A scalar product between four-vectors x and y is defined as*

$$x \cdot y \coloneqq x^\mu y^\nu \eta_{\mu\nu} = x_\mu y_\nu \eta^{\mu\nu} = x^\mu y_\mu = x_\nu y^\nu \tag{1.13}$$

due to the symmetry of the metric: $\eta_{\mu\nu} = \eta_{\nu\mu}$.

In general, a (q, r) tensor field can have q upper indicies and r lower indicies, defined through the following transformation law:

Definition 1.6 ((q, r) tensor). *A (q, r) tensor is defined through the transformation law*

$$T'^{i'_1 i'_2 \ldots i'_q}_{j'_1 j'_2 \ldots j'_r}(x') = T^{i_1 i_2 \ldots i_q}_{j_1 j_2 \ldots j_r}(x)(M^{i'_1}_{i_1} \ldots M^{i'_q}_{i_q})(M^{j'_1}_{j_1} \ldots M^{j'_r}_{j_r}). \tag{1.14}$$

1.4. Lorentz transformations

From the invariant of SR (Theorem 1.1), we have

$$\mathrm{d}s'^2 = \mathrm{d}x'_\mu \, \mathrm{d}x'_\nu \eta^{\mu\nu} = \mathrm{d}x_\mu \, \mathrm{d}x_\nu \eta^{\mu\nu} \tag{1.15}$$

for all reference frames. We denote Λ as a (1,1) tensor field, which transforms a four-vector from one reference frame to another:

$$\mathrm{d}x'^\mu = \Lambda^\mu_\nu \, \mathrm{d}x^\nu \tag{1.16}$$

which leaves the $\mathrm{d}s^2$ invariant, i.e. $\mathrm{d}s'^2 = ds^2$. It follows that

$$\begin{aligned} \eta_{\mu\nu} &= \Lambda^\sigma{}_\mu \Lambda^\delta{}_\nu \eta_{\sigma\delta} \\ \eta &= \Lambda^T \eta \Lambda \end{aligned} \tag{1.17}$$

References

Schwichtenberg, J., 2015 *Physics from symmetry*. Springer.