ENG EK 103: Computational Linear Algebra: Spring 2021

Exam 1

Name: ____

BU Student ID number: ____________________________________

Here are a few important ground rules.

- The exam is open notes, open book, but not open internet.
- **Show all your work. Answers with no work to support them will receive no credit.**
- Be sure you write your name in the given space above.
- Did you write your name up there? Really, please do so.
- Every page (except this one) has a space in the header to write your name. Please do so!
- Point assignments on problems do not necessarily reflect problem difficulty.
- There are a lot of empty pages after each problem for your work. Please start each problem on the page of that problem!
- For most problems, there are multiple versions given. **You should only do the one determined by the given rule (based on your BU ID number). There is no extra credit for solving more than one version!**

1. (20 points) Consider the system of equations

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & - & 3x_3 & = & b_1, \\ 2x_1 & + & x_2 & - & x_3 & = & b_2, \\ 3x_1 & + & 2x_2 & - & (6+\alpha)x_3 & = & b_3 \end{array}$$

For the first three parts of this question, pick $\alpha = 0$.

Select the target vector $\boldsymbol{b}$ *for this problem based on the* ***first*** *digit of your BU ID number: U****X****x-xx-xxxx. Please circle this choice below.*

Note that you only need to do work for one of the choices below and no additional credit will be given for work on a second (or third!) choice.

First digit is 0, 1, 2, 3:	**First digit is 4, 5, 6:**	**First digit is 7, 8, 9:**
$\boldsymbol{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$	$\boldsymbol{b} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$	$\boldsymbol{b} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$

(a) (5 points) Write the system of equations as $Ax = b$.

(b) (5 points) Is A a symmetric matrix? Compute the angle between the first and second column of A.

(c) (5 points) Using Gauss-Jordan eliminations, please reduce the system $Ax = b$ down into its reduced row-echelon equivalent $Rx = d$. When expressed in augmented matrix form, this means:

$$[A|b] \overset{\text{Gauss-Jordan}}{\Longrightarrow} [R|d].$$

Show work required in going from $A \longrightarrow U \longrightarrow R$, circle all the pivots while going from $A \longrightarrow U$, and label the pivot columns and the free variable columns within R.

(d) (5 points) For this part, assume α is no longer zero. Compute the value of α for which the matrix is not invertible. Is this value of α unique? Justify your answer.

(e) (Bonus - 5 points) Pick a value of α in part 4, so that A is not invertible. Are there values for b_1, b_2, b_3 for which there is a solution to $Ax = b$.

SP22: Matrix inverse is not on exam 1. Only look at (a,b,c)

3. (20 points) (Solving $\boldsymbol{Ax} = \boldsymbol{b}$ problems using Gaussian elimination and LU factorization.)

Consider the system

$$\begin{array}{ccccccc} 4x_1 & + & 4x_2 & + & 8x_3 & = & b_1 \\ 2x_1 & + & 3x_2 & + & 10x_3 & = & b_2 \\ 2x_1 & + & x_2 & + & 6x_3 & = & b_3 \end{array}$$

Select the target vector $\mathbf{b}$ *for this problem based on the* ***third*** *digit of your BU ID number: Uxx-**X**x-xxxx. Please circle this choice below.*

Note that you only need to do work for one of the choices below and no additional credit will be given for work on a second (or third!) choice.

Third digit is 0, 1, 2, 3:	**Third digit is 4, 5, 6:**	**Third digit is 7, 8, 9:**
$\boldsymbol{b} = \begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix}$	$\boldsymbol{b} = \begin{bmatrix} 4 \\ 9 \\ 3 \end{bmatrix}$	$\boldsymbol{b} = \begin{bmatrix} 0 \\ 7 \\ 1 \end{bmatrix}$

(a) (2 points) Rewrite the problem in matrix form $\boldsymbol{Ax} = \boldsymbol{b}$.

(b) (5 points) Using Gaussian elimination, reduce the matrix A into an upper triangular matrix U where

$$A \underset{\boxed{}\, E_1}{\xrightarrow{\text{Elimination step \#1}}} \underset{\boxed{}\, E_2}{\xrightarrow{\text{Elimination step \#2}}} U$$

We have not really done elimination matrices this year

Stipulation: You must show work! This includes writing out the row operations in elimination steps #1 and #2, as well as writing out the elimination matrices E_1 and E_2.

(c) (3 points) Using your answers from (b), identify the pivots of A.

(d) (5 points) Using your answers from (b), write out the L matrix in the LU factorization of A where L has to satisfy $A = LU$.

(e) (5 points) Solve the problem $\boldsymbol{Ax} = \boldsymbol{b}$ by solving the two sub-problems in the LU factorization routine, namely

$$\begin{aligned} L\boldsymbol{c} &= \boldsymbol{b} \\ U\boldsymbol{x} &= \boldsymbol{c} \end{aligned}$$

We have not done LU yet.

5. (25 points) The goal of this problem is to find the complete solution of $A\boldsymbol{x} = \boldsymbol{b}$. along the way you must determine a sequence of things as described below

   Choose the A and $\boldsymbol{b}$ *corresponding to the* ***fifth*** *digit of your BU ID number: Uxx-xx-**X**xxx. Please circle this choice below.*

   Note that you only need to do work for one of the choices below and no additional credit will be given for work on a second (or third!) choice.

Fifth digit is 0, 1, 2, 3:	**Fifth digit is 4, 5, 6:**	**Fifth digit is 7, 8, 9:**
$A = \begin{bmatrix} 1 & 2 & -3 & 1 \\ -2 & -3 & 1 & -1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$	$A = \begin{bmatrix} 2 & -1 & -4 & 0 \\ -4 & 5 & 2 & 6 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$	$A = \begin{bmatrix} 2 & 4 & -6 & 2 \\ -4 & 6 & 2 & -2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} -8 \\ 2 \end{bmatrix}$

   (a) (5 points) Find the row-reduced form $\begin{bmatrix} U & \boldsymbol{c} \end{bmatrix}$ of the augmented matrix.

   (b) (5 points) Determine the columns with pivots and write the subspace $C(A)$ as a linear combination of vectors.

   (c) (5 points) Determine the free variables and all special vectors $\boldsymbol{s}$ for the null space $N(A)$.

   (d) (5 points) Determine the particular solution $\boldsymbol{x}_p$ in which all the free variables are set to zero.

   (e) (5 points) Write down the complete solution $\boldsymbol{x}_o$.